ABSTRACT

Title of Dissertation: TRACE DIAGRAMS, REPRESENTATIONS, AND LOW-DIMENSIONAL TOPOLOGY

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This thesis concerns a certain basis for the coordinate ring of the character variety of a surface. Let G be a connected reductive linear algebraic group, and let Σ be a surface whose fundamental group π is a free group. Then the coordinate ring $\mathbb{C}[\mathrm{Hom}(\pi, G)]$ of the homomorphisms from π to G is isomorphic to $\mathbb{C}[G^{\times r}] \cong \mathbb{C}[G]^{\otimes r}$ for some $r \in \mathbb{N}$. The coordinate ring $\mathbb{C}[G]$ may be identified with the ring of matrix coefficients of the maximal compact subgroup of G. Therefore, the coordinate ring on the character variety, which is also the ring of invariants $\mathbb{C}[\mathrm{Hom}(\pi, G)]^G$, may be described in terms of the matrix coefficients of the maximal compact subgroup.

This correspondence provides a basis $\{\chi_\alpha\}$ for $\mathbb{C}[\mathrm{Hom}(\pi, G)]^G$, whose constituents will be called *central functions*. These functions may be expressed as labelled graphs called trace diagrams. This point-of-view permits diagram manipulation to be used to construct relations on the functions.

In the particular case $G = \mathsf{SL}(2,\mathbb{C})$, we give an explicit description of the central functions for surfaces. For rank one and two fundamental groups, the diagrammatic approach is used to describe the symmetries and structure of the central function basis, as well as a product formula in terms of this basis. For $\mathsf{SL}(3,\mathbb{C})$, we describe how to write down the central functions diagrammatically using the Littlewood-Richardson Rule, and give some examples. We also indicate progress for $\mathsf{SL}(n,\mathbb{C})$.

TRACE DIAGRAMS, REPRESENTATIONS, AND LOW-DIMENSIONAL TOPOLOGY

by

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DEDICATION

I dedicate this work to Jenni and to the One who created her.

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Chapter 1 **Introduction**

The purpose of this work is to explore the use of diagrammatic techniques in studying the structure of certain character varieties. The space of representations is a useful tool for studying a particular group, even when restricting to the finite-dimensional irreducible representations. It should come as no surprise that the space of representations of the fundamental group of a surface encodes a lot of information about that surface. Indeed, this set of representations in some sense actually encodes the possible geometries on the surface. This thesis examines the algebraic structure of a particular basis of functions on the space of representations of the fundamental group.

Let G be a connected reductive linear algebraic group. If $U < G$ is the maximal compact subgroup of G, then the coordinate ring $\mathbb{C}[G]$ may be identified with $C_{alg}(U)$, the algebra of matrix coefficients of finite-dimensional unitary representations of U. Moreover, for the action of G on $\mathbb{C}[G]$ by simultaneous conjugation, the ring of invariants $\mathbb{C}[G]^G$ is generated by the *characters* of such representations [CSM].

Let Σ be a compact surface with boundary and consider

$$\mathcal{R} = \mathrm{Hom}(\pi_1(\Sigma, x_0), G),$$

the space of homomorphisms from the fundamental group of Σ into G. The *G-character variety of* Σ is defined as the categorical quotient $\mathfrak{X} = \mathcal{R}//G$. This space may be identified with *conjugacy classes of completely reducible representations*

[Dol]. Since the fundamental group of Σ is a free group $\mathcal{F}_r$ of rank $r \in \mathbb{N}$, the space $\mathcal{R}$ of homomorphisms is isomorphic to G^r. Hence $\mathbb{C}[\mathcal{R}] \cong \mathbb{C}[G^r]$. The coordinate ring of the character variety consists of the G-invariant functions on this space:

$$\mathbb{C}[\mathfrak{X}] \cong \mathbb{C}[\mathcal{R}]^G \cong \mathbb{C}[G^r]^G \cong (\mathbb{C}[G]^{\otimes r})^G \cong (C_{alg}(U)^{\otimes r})^G. \tag{1.1}$$

An application of the Peter-Weyl Theorem gives a decomposition

$$C_{alg}(U) = \bigoplus_{\lambda} V_\lambda^* \otimes V_\lambda,$$

where $\{V_\lambda\}$ is the set of all irreducible finite-dimensional representations of U [CSM]. An additive basis for $\mathbb{C}[\mathfrak{X}]$ is obtained by inserting this decomposition into (1.1) and decomposing the resulting tensors into irreducibles. This construction is described in detail in Chapter 5.

The constituents of this basis are called *central functions*, and are the central object studied in this thesis. They may be described explicitly as *spin networks*, which are special types of labelled graphs. Spin networks may be identified canonically with functions in $\mathbb{C}[\mathfrak{X}]$, and provide enough algebraic horsepower to give explicit descriptions of central functions and some of their properties.

This point-of-view was originated by mathematical physicist John Baez, who interprets these spin networks as quantum mechanical "state vectors." In [Ba], he shows that the space of square integrable functions on a certain space of smooth connections modulo gauge transformations is spanned by graphs similar to the ones given here. More recently, the work of Florentino [FMN] uses a similar basis to produce distributions related to geometric quantization of moduli spaces of flat connections on a surface. The application of spin network bases to the Fricke-Klein-Vogt problem, and in particular to character varieties, was considered by Adam Sikora [Sik]. The core problem, as described in Chapter 5, was first introduced to me by my advisor Bill Goldman. Notes based on his correspondences with Nicolai Reshetikhin [Res], Charles Frohman, and Joanna Kania-Bartoszyńska provided the foundation for the explicit description of central functions for $\mathsf{SL}(2,\mathbb{C})$ given in Chapter 6.

Outline

This thesis describes in detail the case $G = \mathsf{SL}(2,\mathbb{C})$ and rank $r = 2$. To a lesser extent, higher rank $\mathsf{SL}(2,\mathbb{C})$ cases and the $G = \mathsf{SL}(3,\mathbb{C})$ case are considered. There is also some discussion of the most general case.

Chapter 2 gives necessary background from representation theory, including the classification of $\mathsf{SU}(n)$-representations.

In Chapters 3 and 4, spin networks and trace diagrams are formally introduced, with special emphasis on $\mathsf{SL}(2,\mathbb{C})$ and $\mathsf{SL}(3,\mathbb{C})$.

Chapter 5 describes in detail the construction of the central functions of a surface, and explicitly demonstrates how spin networks may be used to construct a basis for $\mathbb{C}[\mathfrak{X}]$. The role of the topology of the surface in this construction is strongly emphasized.

Chapter 6 describes results for the case $G = \mathsf{SL}(2,\mathbb{C})$. The algebraic structure of $\mathbb{C}[\mathfrak{X}]$ is described in detail for the rank one and two cases. In particular, for the rank two case, there is a theorem describing the symmetry of central functions, a recurrence formula which may be used to compute an arbitrary central function, and a formula for the product of two central functions. Finally, there is a computation expressing an arbitrary polynomial in terms of this basis, which may be inverted to find an explicit formula for central functions. Finally, the general rank case for $G = \mathsf{SL}(2, \mathbb{C})$ is briefly discussed.

Chapter 7 describes progress for $G = \mathsf{SL}(3, \mathbb{C})$. Computations are more difficult in this case, and the irreducible representations are much harder to describe. The primary result is an explicit diagrammatic description of intertwiners, allowing for the central functions to be written down in terms of diagrams. A few examples are given, and diagrams for general groups are also discussed.

Some closing remarks about possible further applications of spin networks are given in Chapter 8. A new proof of the Fricke-Klein-Vogt Theorem is given, and there is speculation about how the computation of central functions may proceed in the general case.

Chapter 2 **Background from Representation Theory**

This chapter describes some basic facts about the representation theory of Lie groups. For more details, see [CSM, Ful, FH].

A *representation* of a given Lie group G is a pair (π, V), where V is a vector space over $\mathbb{C}$ and π is a continuous homomorphism $\pi : G \to \mathsf{GL}(V)$. Here, $\mathsf{GL}(V)$ is the Lie group comprised of invertible linear transformations. The action of an element $g \in G$ on V is denoted $\pi_g : V \to V$. For a matrix group $G \subset \mathsf{GL}(n, \mathbb{C})$, the *standard representation* of G is the vector space $V = \mathbb{C}^n$ with $\pi_g(v) = gv$, the matrix product.

An *irreducible representation* V has no nontrivial invariant subspaces, meaning there is no U such that $\pi_g(u) \in U$ for all $u \in U$. A finite-dimensional representation is *completely reducible* if it can be decomposed into irreducible invariant subrepresentations. In this case, $V = V_1 \oplus \cdots \oplus V_k$ and

$$\pi_g(v) = \pi_g((v_1, \ldots, v_k)) = \pi_{g_1}(v_1) \cdots \pi_{g_k}(v_k).$$

As a general rule, representations are not completely reducible. However, all *unitary representations* are completely reducible. To be unitary, the representation must be invariant under a non-degenerate *Hermitian inner product* $\langle \cdot, \cdot \rangle$, so that $\langle \pi_g(v), \pi_g(w) \rangle = \langle v, w \rangle$. If G is finite or compact, this inner product may be constructed by adding or integrating over an arbitrary non-degenerate Hermitian inner product. The compact case requires additionally a translation-invariant measure on G called the *Haar measure*.

A *G-map* between representations (π, V) and (ϖ, W) is an invariant linear map $A : V \to W$ satisfying $A(\pi_g(v)) = \varpi_g(Av)$. The set of such maps will be denoted $\mathrm{Hom}_G(V, W)$. If this map is also a vector space isomorphism, then (π, V) and (ϖ, W) are *equivalent representations*. The following classical lemma indicates that G-invariance is a very rigid structure:

Lemma 2.1 (Schur's Lemma)**.** *Let G be a Lie group and let $A \in \mathrm{Hom}_G(V, W)$, where V and W are irreducible. If V and W are equivalent, then*

$$\dim_{\mathbb{C}}[\mathrm{Hom}_G(V, W)] = 1,$$

and A is a multiple of the identity with respect to appropriate bases. Otherwise, $A = 0$.

Thus, the possible G-maps between representations are determined by the equivalences of their irreducible components.

Given a G-representation (π, V) and an H-representation (ϖ, W), the *tensor representation* $(\pi \otimes \varpi, V \otimes W)$ is the $G \times H$-representation with

$$(\pi \otimes \varpi)_{(g,h)}(v \otimes w) = \pi_g(v) \otimes \varpi_h(w).$$

If $G = H$, the result is also a G-representation with $(\pi \otimes \varpi)_g(v \otimes w) = \pi_g(v) \otimes \varpi_g(w)$.

Given a G-representation (π, V), the *dual representation* $(\check{\pi}, V^*)$ is defined for $f \in V^*$ by $(\check{\pi}_g(f))(v) = f(\pi_{g^{-1}}(v))$.

For any subgroup $H < G$, a G-representation (π, V) restricts to an H-representation. Moreover, an H-representation (ϖ, W) gives rise to an *induced representation* $(\pi, \bigoplus_{\sigma \in G/H} W)$ on G, where if $g \cdot g_0 = g_\tau h$ then

$$\pi_g\Big(\sum g_\sigma w_\sigma\Big) = \sum g_\tau \pi_h(w_\sigma).$$

2.1 Functions on Compact Lie Groups

Let G act by conjugation on G^*, the linear space of functions on G:

$$g \cdot f(x) = f(gxg^{-1}).$$

A *class function* is a function which is invariant under this action, and may be interpreted as a function on the space of conjugacy classes. Given a finite-dimensional G-representation (π, V), the *character* of the representation is the trace map $\chi_\pi(g) = \mathrm{tr}(\pi_g)$. Characters are automatically conjugation-invariant, hence class functions. The characters of direct sum, tensor, and dual representations satisfy:

$$\chi_{\pi \oplus \varpi} = \chi_\pi + \chi_\varpi; \qquad \chi_{\pi \otimes \varpi} = \chi_\pi \chi_\varpi; \qquad \chi_{\check{\pi}}(g) = \chi_\pi(g^{-1}).$$

Let G be compact, and assume all representations are unitary. In this case, the classical Peter-Weyl Theorem relates representations to functions on G. The *matrix elements* or *representative functions* of a representation are the functions

$$g \mapsto v^*(\pi_g(w))$$

for some $v^* \in V^*$ and $w \in V$. The space of such functions is a subalgebra $C_{\mathrm{alg}}(G)$ of the algebra $C(G)$ of continuous functions on G. It is also contained in $L^2(G)$, the space of square-integrable functions on G.

Theorem 2.2 (Peter-Weyl Theorem)**.** *Let G be a compact group, and suppose $\{(\lambda, V^\lambda)\}_{\lambda \in \Lambda}$ is a complete set of inequivalent finite-dimensional representations of G. Then, an arbitrary G-representation V may be constructed as the completion of a direct sum of copies of V^λ for $\lambda \in \Lambda$. Consequently, with respect to uniform convergence, $C_{\mathrm{alg}}(G)$ is a dense subring of $C(G)$.*

This theorem has important consequences for the structure of $C_{\mathrm{alg}}(G)$. Note that $C_{\mathrm{alg}}(G)$ is a $G \times G$-representation with $\pi_{(g,h)}$ taking $f(x) \mapsto f(gxh^{-1})$.

Theorem 2.3. *There is a $G \times G$-equivalence*

$$\bigoplus_{\lambda \in \Lambda} V_\lambda^* \otimes V_\lambda \cong C_{\mathrm{alg}}(G),$$

where the isomorphism takes $v^* \otimes w \in V_\lambda^* \otimes V_\lambda$ to the function $g \mapsto v^*(\lambda_g(w))$. This isomorphism takes the direct sum inner product on the left to the L^2-inner product on the right. Moreover, the characters χ_λ form an orthonormal basis for $L^2(G)^G$, the Hilbert space of class functions on G.

These theorems also show that all compact Lie groups G are matrix groups and that all irreducible G-representations are finite-dimensional and determined up to isomorphism by their characters. Finally, an arbitrary class functions can be expanded into a convergent sequence $\sum_\lambda \langle f, \chi_\lambda \rangle \chi_\lambda$, with respect to the L^2 inner product.

2.2 Lie Algebra Representations

A *Lie algebra representation* is a pair (Π, V) where V is a vector space over $\mathbb{C}$ and $\Pi : \mathfrak{g} \to \mathfrak{gl}(V)$ satisfies $\Pi_{[x,y]} = [\Pi_x, \Pi_y] = \Pi_x \Pi_y - \Pi_y \Pi_x$. Recall that $\mathfrak{gl}(V)$ is the Lie algebra of endomorphisms on V. Every representation $\pi : G \to \mathsf{GL}(V)$ on a Lie group induces a map $\pi_* : T_e(G) \to T_e(\mathsf{GL}(V))$ on the tangent spaces, which is a Lie algebra representation $\pi_* : \mathfrak{g} \to \mathfrak{gl}(V)$.

In particular, the *commutator representation* $\Psi : G \to \mathrm{Aut}(G)$ consisting of inner automorphisms $\Psi_g(h) = ghg^{-1}$ induces the *adjoint representation of* $\mathfrak{g}$:

$$\mathrm{ad} : \mathfrak{g} \to \mathrm{Der}(\mathfrak{g}) \subset \mathfrak{gl}(V),$$

where $\mathrm{Der}(\mathfrak{g})$ is the *derivation algebra* of $\mathfrak{g}$. Moreover, the induced map of the automorphism $\Psi_g : G \to G$ is an automorphism $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$, giving the *adjoint representation of* G.

Lie algebra and Lie group representations are closely related. In fact, there is a *one-to-one* correspondence between representations of connected, simply-connected Lie groups and representations of their Lie algebras, which is induced by the differential/exponential maps.

The Unitary Trick

There is a one-to-one correspondence between compact connected Lie groups U and connected, reductive linear algebraic groups G over $\mathbb{C}$. The correspondence is constructed via respect to their Lie algebras $\mathfrak{u}$ and $\mathfrak{g}$, with $\mathfrak{g}$ being the *complexification* of $\mathfrak{u}$ and $\mathfrak{u}$ the *compact real form* of $\mathfrak{g}$. Of particular interest is the $\mathbb{C}$-algebra equivalence

$$\mathbb{C}[G] = C_{\mathrm{alg}}(U)$$

between the ring of matrix coefficients of U and the *coordinate ring* of G. Using the Peter-Weyl Theorem, this equivalence implies:

Theorem 2.4. *Let G be a connected, reductive linear algebraic group with maximal compact subgroup U. Then*

$$\mathbb{C}[G] \cong \bigoplus_{\lambda \in \Lambda} V_\lambda^* \otimes V_\lambda,$$

where $\{V_\lambda\}_{\lambda \in \Lambda}$ is the set of finite-dimensional irreducible representations of U.

2.3 Classification of $\mathsf{SU}(n)$-Representations

This section describes the classification of finite-dimensional irreducible representations of the unitary group $\mathsf{SU}(n)$. This will have consequences for the representations of $\mathsf{SL}(n, \mathbb{C})$, since $\mathsf{SU}(n)$ is the maximal compact subgroup of $\mathsf{SL}(n, \mathbb{C})$.

Under *Weyl's correspondence*, $\mathsf{GL}(n, \mathbb{C})$-representations are closely related to representations of symmetric groups. Let $V = \mathbb{C}^n$ denote the standard representation of $\mathsf{GL}(n, \mathbb{C})$, and let V_λ be a representation of the symmetric group Σ_d. There is a natural injection $V_\lambda \hookrightarrow V^{\otimes d}$, whose image will be denoted $\mathbb{S}_\lambda V$. Then the actions of Σ_d and $\mathsf{GL}(n, \mathbb{C})$ on $V^{\otimes d}$ commute. Both direct sums and irreducibility pass through this construction. In particular, there is a one-to-one correspondence between irreducible representations of $\mathsf{GL}(n, \mathbb{C})$ and those of Σ_d contained in $V^{\otimes d}$ [CSM, Ful, FH].

The irreducible representations of Σ_d may be indexed by partitions of the integer d, hence the irreducible representations of $\mathsf{GL}(n, \mathbb{C})$ arising in this manner are indexed by integer sequences

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n.$$

These representations restrict to $\mathsf{SU}(n)$, although there is an equivalence of representations in this case for $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $\lambda_1 + i \geq \lambda_2 + i \geq \cdots \geq \lambda_n + i$ since the determinant is fixed. Hence, the irreducible representations of $\mathsf{SU}(n)$ are indexed by integer partitions

$$\lambda = (\lambda_1, \ldots, \lambda_{n-1}) \quad \text{with} \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq 0.$$

A dominant weight argument on the Lie algebra $\mathfrak{sl}(n, \mathbb{C})$ can be used to show that these comprise the *entire* list of finite-dimensional irreducible representations of $\mathsf{SU}(n)$. The next section describes these representations explicitly.

Young Projectors

Integer partitions are commonly represented by *Young diagrams*, or collections of boxes. For example, the partitions of 4 are represented by

$$(4) = \begin{array}{|c|c|c|c|}\hline \; & \; & \; & \; \\ \hline \end{array}\,, \quad (3,1) = \begin{array}{|c|c|c|}\hline \; & \; & \; \\ \hline \; \\ \cline{1-1} \end{array}\,, \quad (2,2) = \begin{array}{|c|c|}\hline \; & \; \\ \hline \; & \; \\ \hline \end{array}\,, \quad (2,1,1) = \begin{array}{|c|c|}\hline \; & \; \\ \hline \; \\ \cline{1-1} \; \\ \cline{1-1} \end{array}\,, \quad (1,1,1,1) = \begin{array}{|c|}\hline \; \\ \hline \; \\ \hline \; \\ \hline \; \\ \hline \end{array}\,.$$

For a given partition $\lambda = (\lambda_1, \dots, \lambda_k)$ of $d \in \mathbb{N}$, define $a_\lambda, b_\lambda : V^{\otimes d} \to V^{\otimes d}$ by

$$a_\lambda = s_{\lambda_1} \otimes s_{\lambda_2} \otimes \cdots \otimes s_{\lambda_k}, \qquad b'_\lambda = t_{\lambda_1} \otimes t_{\lambda_2} \otimes \cdots \otimes t_{\lambda_k},$$

where $s_i : V^{\otimes i} \to V^{\otimes i}$ is the *symmetrizer* on i factors and $t_j : V^{\otimes j} \to V^{\otimes j}$ is the *anti-symmetrizer* on j factors. The anti-symmetrizer maps a given element to the sum of positive permutations minus the sum of negative permutations; its image is isomorphic to the exterior power $\bigwedge^j V$. Every partition λ also has a *conjugate partition* λ^T given by transposing the diagram. For example, the partitions $(3,1)$ and $(2,1,1)$ are conjugate. Define the map $b_\lambda : V^{\otimes d} \to V^{\otimes d}$ by $b_\lambda = b'_{\lambda^T}$, which is therefore the anti-symmetrizer on the *columns* of the diagram.

For a fixed diagram, a *Young tableau* is an assignment of the integers $1, \dots, d$ to the boxes of the diagram in such a way that numbers are increasing in each column and row. For example, the $(2,2)$-partition has Young tableaux

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}\,.$$

Two numbering schemes are obvious: number rows first then columns, or number columns first then rows. The first is called the *standard row tableau*, and the second the *standard column tableau*. Given a Young tableau Y, let the permutation $\sigma_Y \in \Sigma_d$ be that taking the standard row tableau to Y, and let $\tau_Y \in \Sigma_d$ be that taking the standard column tableau to Y. Then, the *Young symmetrizer* of the tableau is

$$c_Y = \tau_Y^{-1} b_\lambda \tau_Y \circ \sigma_Y^{-1} a_\lambda \sigma_Y,$$

and the *Young projector* of the partition λ is

$$c_\lambda = \sum_Y c_Y.$$

The representation V_λ above is precisely the image of $c_\lambda : V^{\otimes d} \to V^{\otimes d}$. Hence, these are also representations of the group of interest $\mathsf{SU}(n)$, and they form a complete set of finite-dimensional irreducible representations. Examples of this construction are given in Chapter 7.

2.4 Representations of $\mathsf{SL}(2, \mathbb{C})$

For the case $\mathsf{SL}(2, \mathbb{C})$, the admissible Young diagrams have just one row, hence are indexed by the natural numbers $\mathbb{N}$. For each diagram, there is just one Young tableau, and b_λ is trivial since it represents permutations on columns. Therefore, the irreducible representations are the images of

$$c_n = \sum_{\sigma \in \Sigma_n} \sigma : V^{\otimes n} \to V^{\otimes n}.$$

This image consists of the elements of $V^{\otimes n}$ which are invariant under all permutations. It is commonly called the nth *symmetric power* of $V = \mathbb{C}^2$, and will be denoted by $V_n \equiv \mathrm{Sym}^n(V)$. This is also identified with the space of degree-n homogeneous polynomials in $\mathbb{C}[e_1, e_2]$, where $e_1 = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ and $e_2 = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$ comprise the standard basis of V. For example, V_0 is the trivial representation $\mathbb{C}$, while V_1 is the standard representation $V = \mathbb{C}^2$.

It will be important later to specify a basis for both $V_n = \mathrm{Sym}^n(V)$ and the "dual" space $V_n^* \equiv \mathrm{Sym}^n(V^*)$. The dual V^* may be identified with row vectors and the basis $e_1^* = e_1^T$ and $e_2^* = e_2^T$. Then $V_1 = \mathbb{C}e_1 \oplus \mathbb{C}e_2$ and $V_1^* = \mathbb{C}e_1^* \oplus \mathbb{C}e_2^*$. Denote the symmetric powers of these representations by

$$V_n = \mathrm{Sym}^n(V) \quad \text{and} \quad V_n^* = \mathrm{Sym}^n(V^*).$$

Note that $(V_n)^* \cong V_n \cong V_n^*$, but the spaces $(V_n)^*$ and V_n^* are not quite the same. To see the difference, pair elements in V_n with elements in V_n^*. Denote the projection of $v_1 \otimes v_2 \otimes \cdots \otimes v_n \in V^{\otimes n}$ to V_n by $v_1 v_2 \cdots v_n$. Then, bases for V_n and V_n^* are given by the elements

$$\mathsf{n}_{n-k} = e_1^{n-k} e_2^k \quad \text{and} \quad \mathsf{n}_{n-k}^* = (e_1^*)^{n-k}(e_2^*)^k, \quad k = 0, \ldots, n.$$

In these terms, the pairing is

$$\mathsf{n}_{n-k}^*(v_1 v_2 \cdots v_n) = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} (\mathsf{n}_{n-k})^*(v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)}),$$

where Σ_n is the symmetric group. In particular,

$$\mathsf{n}_{n-k}^*(\mathsf{n}_{n-l}) = \frac{(n-k)!k!}{n!}\delta_{kl} = \frac{1}{\binom{n}{k}}\delta_{kl}.$$

Thus, V_n and $(V_n)^*$ pair in the normal way, while V_n and V_n^* pair with an extra binomial factor.

Explicitly, the action of $g = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \in \mathsf{SL}(2, \mathbb{C})$ on V_n is

$$\begin{aligned} g \cdot \mathsf{n}_{n-k} &= (g_{11}e_1 + g_{21}e_2)^{n-k}(g_{12}e_1 + g_{22}e_2)^k \\ &= \sum_{\substack{0 \le j \le n-k \\ 0 \le i \le k}} \binom{n-k}{j}\binom{k}{i} \left(g_{11}^{n-k-j} g_{12}^{k-i} g_{21}^{j} g_{22}^{i} \right) \mathsf{n}_{n-(i+j)}. \end{aligned}$$

Hence, the matrix elements with respect to this pairing are

$$\begin{aligned} \mathsf{n}_{n-l}^*(g \cdot \mathsf{n}_{n-k}) &= \sum_{\substack{0 \le j \le n-k \\ 0 \le i \le k}} \binom{n-k}{j}\binom{k}{i} \left(g_{11}^{n-k-j} g_{12}^{k-i} g_{21}^{j} g_{22}^{i} \right) \mathsf{n}_{n-l}^*(\mathsf{n}_{n-(i+j)}) \\ &= \sum_{i+j=l} \frac{\binom{n-k}{j}\binom{k}{i}}{\binom{n}{k}} \left(g_{11}^{n-k-j} g_{12}^{k-i} g_{21}^{j} g_{22}^{i} \right). \end{aligned}$$

Similarly, g acts on the dual V_n^* in the usual way: $(g \cdot \mathsf{n}_{n-k}^*)(v) = \mathsf{n}_{n-k}^*(g^{-1}(v))$ for $v \in V_n$.

The tensor product $V_a \otimes V_b$, where $a, b \in \mathbb{N}$, is also a representation of $\mathsf{SL}(2, \mathbb{C})$ and decomposes into irreducible representations as follows:

Proposition 2.5 (Clebsch-Gordan formula)**.**

$$V_a \otimes V_b \cong \bigoplus_{j=0}^{\min(a,b)} V_{a+b-2j}.$$

Proof. An irreducible representation V_a has weights $\{a, a-2, \ldots, -a+2, -a\}$ [FH]. The weights of $V_a \otimes V_b$ consist of all possible sums of weights of V_a with weights of V_b. With multiplicity, these are

$$\begin{aligned}
&\{a+b, a-2+b, \ldots, -a+b\} \sqcup \{a+(b-2), a-2+(b-2), \ldots, -a+(b-2)\} \\
&\qquad\qquad\qquad\qquad \sqcup \cdots \sqcup \{a-b, a-2-b, \ldots, -a-b\} \\
&= \{a+b, a+b-2, \ldots, -(a+b)\} \sqcup \{a+b-2, \ldots, 2-(a+b)\} \\
&\qquad\qquad\qquad\qquad \sqcup \cdots \sqcup \{|a-b|, \ldots, -|a-b|\}.
\end{aligned}$$

The decomposition follows by noting that these are the only possible irreducible representations which give this set of weights. $\square$

We will denote the set of admissible representations by

$$\lceil a, b \rfloor \equiv \{a+b-2j : 0 \le j \le \min(a,b)\} = \{a+b, a+b-2, \ldots, |a-b|\}.$$

Hence $V_a \otimes V_b = \bigoplus_{c \in \lceil a,b \rfloor} V_c$.

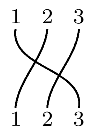
Representations of more general $\mathsf{SL}(n, \mathbb{C})$ are discussed in Chapter 7.

Chapter 3 **Spin Networks**

This chapter is a self-contained introduction to spin networks and the spin network calculus. Most of the material here can be found in the literature [CFS, Kau, Pen, St]. It seems prudent to include a full treatment here because we give a nonstandard definition of spin networks, which is more natural when working with traces. This definition leads to different versions of the usual spin network relations. Additionally, we place a greater emphasis on functorial properties and the symmetry of certain spin network functions.

Motivation for Diagrammatics

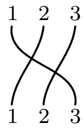
One motivation for the theory of spin networks is the use of diagrams to perform calculations that can be extremely tedious using traditional methods. Diagrammatic techniques are useful for maps as simple as permutations. For example, the diagram



says as much about a permutation as the traditional cycle notation $(1\,2\,3)$. When it comes to composing permutations, it can be easier to compute a result using diagrams than using cycle notation. For example, computing

$$(1\ 2\ 3) \circ (1\ 2) \circ (2\ 3) \circ (1\ 3\ 2)$$

directly is messy and unenlightening compared to stacking diagrams, which gives:

 $=$ 

$= (1\ 2\ 3).$

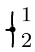
Diagrams allow for more natural "non-linear" algebraic manipulations. Parentheses are usually unnecessary, and the full strength of topological invariance can be leveraged.

The diagrams used in this thesis are most compatible with the language of representation theory, since they can be interpreted as maps between irreducible representations of a specified group. The structure of the diagrams will vary depending on the group G.

3.1 Basic Definitions

This chapter is concerned entirely with spin networks suitable for working with $G = \mathsf{SL}(2, \mathbb{C})$, or more generally any 2×2 matrix group. In this case, a spin network is a graph that is identified with a specific function between tensor powers of $V = \mathbb{C}^2$, the standard $\mathsf{SL}(2, \mathbb{C})$-representation.

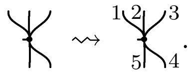
In order for this function to be well-defined, the edges incident to each vertex of the spin network must have a cyclic ordering. This ordering is often called a *ciliation*, since it may be represented on paper by a small mark drawn between two of the edges. The edges adjacent to a ciliated vertex are ordered by proceeding in a clockwise fashion from this mark. For example, in the degree 2 case, there are two possible ciliations:

 $\rightsquigarrow$ 

and  $\rightsquigarrow$ 

.

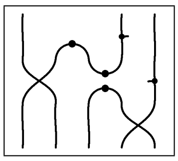
Another example follows:



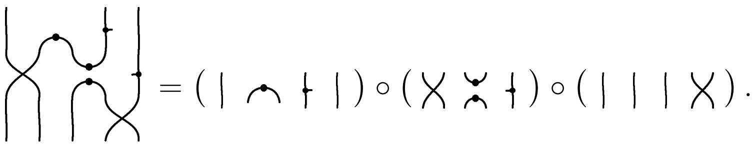
Definition 3.1. A *spin network* s is a graph with vertex set $\mathsf{s}_i \sqcup \mathsf{s}_o \sqcup \mathsf{s}_v$ consisting of degree 1 *inputs* s_i, degree 1 *outputs* s_o and degree 2 *ciliated vertices* s_v. The graph need not be connected, and the graph $\bigcirc$ with no vertices is permitted.

Denote the set of spin networks by S and the set of spin networks with exactly I inputs and O outputs by S_I^O. For fixed I and O, the vector space $\mathbb{C}\{S_I^O\}$ will be denoted by $[\mathsf{S}]_I^O$, or sometimes $[\mathsf{S}^2]_I^O$. Denote by S or S^2 the union of all such vector spaces.

Spin networks are usually drawn in general position inside an oriented square with inputs at the bottom and outputs at the top. This convention permits a definition of the *composition* of two spin networks. If $\mathsf{s}_1 \in [\mathsf{S}]_{I_1}^{O_1}$, $\mathsf{s}_2 \in [\mathsf{S}]_{I_2}^{O_2}$, and $O_1 = I_2$, then the composition $\mathsf{s}_2 \circ \mathsf{s}_1$ is defined to be the graph obtained by placing s_2 on top of s_1. Thus, the output vertices of s_1 are identified with the input vertices of s_2, and the new spin network has I_1 inputs and O_2 outputs. For example, the diagram

$\mathsf{s} =$  $\in [\mathsf{S}]_5^3$

could be represented as a composition of three spin networks:



(The marks on the local extrema here are a notational convenience and do not indicate vertices of the graph.)

Since spin networks are just graphs with ciliations, it does not matter how the graph is represented inside the square. Strands may be moved about freely and ciliations may “slide” along the strands. As long as the endpoints remain fixed, the underlying spin network does not change.

3.2 Spin Network Component Maps

This section describes how spin networks may be identified with functions.

In the language of category theory, $(\mathbb{N}, \mathsf{S})$ forms a category, where the objects are the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$ and the morphisms are spin networks. A spin network $\mathsf{s} \in [\mathsf{S}]_I^O$ is a morphism from I to O. There is a natural functor from this category to $(\mathbb{N}, \mathsf{F})$, where F is the set of linear maps between tensor powers of $V = \mathbb{C}^2$. The image of $\mathsf{s} \in [\mathsf{S}]_I^O$ under this functor will be denoted by $f_{\mathsf{s}} : \{V^{\otimes I} \to V^{\otimes O}\}$, and a function $V^{\otimes 0} \to V^{\otimes 0}$ will be interpreted as a constant. The function is computed by decomposing s into four simple maps, which are defined, given $v, w \in V = \mathbb{C}^2$ and the standard basis $\{e_1, e_2\}$ for V, by the following:

$$
\begin{array}{rcl}
\vert \in [\mathsf{S}]_1^1 & \longrightarrow & v \mapsto v \quad (\text{the } \textit{identity}); \\
\frown \in [\mathsf{S}]_2^0 & \longrightarrow & v \otimes w \mapsto v^* w \quad (\text{the } \textit{cap}, \text{ or inner product}); \\
\smile \in [\mathsf{S}]_0^2 & \longrightarrow & 1 \mapsto e_1 \otimes e_1 + e_2 \otimes e_2 \quad (\text{the } \textit{cup}); \\
\frown \in [\mathsf{S}]_2^0 & \longrightarrow & v \otimes w \mapsto \det[v\, w] \quad (\text{the } \textit{ciliated cap});
\end{array}
$$

This decomposition assumes a certain monoidal structure on S. If $\mathsf{s}_i \in [\mathsf{S}]_{I_i}^{O_i}$, then there is a map $(\mathsf{s}_1, \mathsf{s}_2) \mapsto \mathsf{s}_1\mathsf{s}_2 \in [\mathsf{S}]_{I_1+I_2}^{O_1+O_2}$ defined by placing two spin networks side by side. In this case, $f_{\mathsf{s}_1\mathsf{s}_2} = f_{\mathsf{s}_1} \otimes f_{\mathsf{s}_2}$. Hence, the identity function on $V^{\otimes I}$ is the image of $\underbrace{\vert\,\vert \cdots \vert}_{I \text{ strands}}$. Moreover, permutations are given by spin networks with no local extrema:

$$f : v \otimes w \to w \otimes v.$$

For example, the cap with opposite ciliation may be decomposed = = ∘ , and so its function is

$$f\,(v \otimes w) = f \circ f\,(v \otimes w) = f\,(w \otimes v) = \det[w\; v] = -\det[v\; w].$$

The definition given here differs from the literature [CFS, Kau, Pen]. In particular, we omit the $i = \sqrt{-1}$ factor in the definition of to gain an advantage in trace calculations. Also, the maps and are included in order to simplify the following proof.

Functorial Properties

Theorem 3.2. *Every spin network* s *may be decomposed into the four above maps, giving a function* f_{s}*. This construction provides a functor* $\mathsf{s} \to f_{\mathsf{s}}$*.*

Proof. Position s in such a way that all ciliations occur at local maxima, with ciliation pointing up. The remainder of the diagram is a collection of arcs and loops without vertices. Each arc may be taken to be a cap , cup , or line , while each loop may be decomposed = ∘ .

For the second statement, it must be shown that every decomposition of s gives the same function. Because the vertices occur at local maxima in the decompositions, it suffices to show that the function of any arc or loop is well-defined. Two different decompositions of an arc or loop can only differ by the insertion or deletion of a number of 'kinks' of the form . Since =

$\;\circ\;$, the kink's function is computed:

$$f(v) = f\, f \circ f\, f\,(v)$$
$$= f\, f\,(v \otimes e_1 \otimes e_1 + v \otimes e_2 \otimes e_2)$$
$$= (v \cdot e_1)e_1 + (v \cdot e_2)e_2 = v = f\,(v).$$

Thus, these kinks do not change the overall function f_{s}. Alternate proofs may be found in [CFS, Kau]. $\square$

Given this theorem, there is no difficulty in interpreting a spin network s itself as a function. From now on, the notation f_{s} will only be used to highlight this difference in categories. It will also be convenient to expand the meaning of 'decomposition' to include the following maps:

Proposition 3.3. *In the spin network sense,*

1. *the* swap, $: V \otimes V \to V \otimes V$ *takes* $v \otimes w \mapsto w \otimes v$;
2. *the vertex on a straight line,* $: V \to V$ *takes* $v \mapsto \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right] v$;
3. *the vertex on a cup,* $: \mathbb{C} \to V \otimes V$ *takes* $1 \mapsto e_1 \otimes e_2 - e_2 \otimes e_1$;
4. *with opposite ciliations,* $= -$, $= -$, *and* $= -$.

Proof. The first statement requires no proof. For the second and third statement, use the decompositions

$$= \;=\; \circ \qquad \text{and} \qquad = \;=\; \circ\;.$$

Thus (v) for an arbitrary vector $v = \left[\begin{smallmatrix} v^1 \\ v^2 \end{smallmatrix}\right]$ is computed by

$$(v) = (v) = \circ\,(v) = (v \otimes e_1 \otimes e_1 + v \otimes e_2 \otimes e_2)$$
$$= \det[v\; e_1]e_1 + \det[v\; e_2]e_2 = -v^2 e_1 + v^1 e_2 = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right] v.$$

Similarly, $\smile(1)$ is computed by

$$\begin{aligned}\smile(1) &= \vert \frown \vert \circ \smile\smile(1)\\ &= \vert \frown \vert\,(e_1\otimes e_1\otimes e_1\otimes e_1 + e_1\otimes e_1\otimes e_2\otimes e_2\\ &\qquad\qquad + e_2\otimes e_2\otimes e_1\otimes e_1 + e_2\otimes e_2\otimes e_2\otimes e_2)\\ &= \det[e_1\, e_2]e_1\otimes e_2 + \det[e_2\, e_1]e_2\otimes e_1 = e_1\otimes e_2 - e_2\otimes e_1.\end{aligned}$$

The final statement follows from the observation $\frown = \bigcirc = -\frown$, which has already been demonstrated. $\square$

Assumptions for Local Extrema

At this point, the maps $\frown$ and $\smile$ will not be needed, and for all spin networks in $\mathsf{s} \in \mathsf{S}$ we make the following assumption:

Convention 3.4. The set of ciliated vertices coincides *exactly* with the set of local extrema. The ciliations are usually omitted, with the understanding that

$$\smile \;\equiv\; \smile : 1 \mapsto e_1\otimes e_2 - e_2\otimes e_1 \quad\text{and}\quad \frown \;\equiv\; \frown : v\otimes w \mapsto \det[v\, w].$$

Under this assumption, there are just three component maps: $\vert$, $\smile$ and $\frown$. Vertices are usually omitted, and diagrams are not topologically invariant since

$$\begin{aligned}\mathcal{S}(v) = \frown \vert \circ \vert \smile(v) &= \frown \vert\,(v\otimes e_1\otimes e_2 - v\otimes e_2\otimes e_1)\\ &= \det[v\, e_1]e_2 - \det[v\, e_2]e_1 = -v^2e_2 - v^1e_1 = -v = -\vert(v).\end{aligned}$$

Thus, each straightened kink introduces a sign and in general

$$(\mathcal{S}\cdots\mathcal{S})^n = (-1)^n\, (/\cdots/)^n.$$

This problem is commonly avoided by tacking on a factor of $i = \sqrt{-1}$ to all ciliated vertices. Unfortunately, this fix makes trace calculations difficult, so we choose instead to keep track of signs introduced by straightening kinks.

3.3 Symmetry Relations

Spin networks exhibit considerable symmetry, which can be exploited for calculations. For example:

Proposition 3.5. *Let* $\mathsf{s} \in [\mathsf{S}]_I^O$ *be a spin network with function* $f_{\mathsf{s}} : V^{\otimes I} \to V^{\otimes O}$. *Denote its images under reflection through vertical and horizontal lines by* $\mathsf{s}^{\leftrightarrow}$ *and* $\mathsf{s}^{\updownarrow}$, *respectively. Then*

$$f_{\mathsf{s}^{\leftrightarrow}} = (-1)^{|\mathsf{s}_v|}(f_{\mathsf{s}})^{\leftrightarrow} : V^{\otimes I} \to V^{\otimes O},$$

where $|\mathsf{s}_v|$ *is the number of local extrema in the diagram and* $f^{\leftrightarrow}$ *is the same as* f *but with the ordering of inputs and outputs reversed. The function* $f_{\mathsf{s}^{\updownarrow}} : V^{\otimes O} \to V^{\otimes I}$ *is exactly the dual of* f_{s} *with respect to the standard inner product on* V. *In other words, if* $\mathcal{B}_I$ *is the standard basis for* $V^{\otimes I}$ *then*

$$f_{\mathsf{s}^{\updownarrow}}(v_1 \otimes \cdots \otimes v_O) = (f_{\mathsf{s}})^*(v_1 \otimes \cdots \otimes v_O) = \sum_{e_b \in \mathcal{B}_I} \left(f_{\mathsf{s}}(e_b) \cdot (v_1 \otimes \cdots \otimes v_O)\right) e_b.$$

Proof. This only needs to be demonstrated for the component maps, since both $f_{\mathsf{s}^{\leftrightarrow}}$ and $f_{\mathsf{s}^{\updownarrow}}$ respect composition:

$$\begin{aligned}
f_{\mathsf{s}^{\leftrightarrow} \circ \mathsf{t}^{\leftrightarrow}} &= f_{(\mathsf{s} \circ \mathsf{t})^{\leftrightarrow}} = (-1)^{|\mathsf{s}_v|+|\mathsf{t}_v|}(f_{\mathsf{s}\circ\mathsf{t}})^{\leftrightarrow} = (-1)^{|\mathsf{s}_v|}(f_{\mathsf{s}})^{\leftrightarrow} \circ (-1)^{|\mathsf{t}_v|}(f_{\mathsf{t}})^{\leftrightarrow} = f_{\mathsf{s}^{\leftrightarrow}} \circ f_{\mathsf{t}^{\leftrightarrow}};\\
f_{\mathsf{s}^{\leftrightarrow}\mathsf{t}^{\leftrightarrow}} &= f_{(\mathsf{ts})^{\leftrightarrow}} = (-1)^{|\mathsf{t}_v|+|\mathsf{s}_v|}(f_{\mathsf{ts}})^{\leftrightarrow} = (-1)^{|\mathsf{s}_v|}(f_{\mathsf{s}})^{\leftrightarrow}(-1)^{|\mathsf{t}_v|}(f_{\mathsf{t}})^{\leftrightarrow} = f_{\mathsf{s}^{\leftrightarrow}} \otimes f_{\mathsf{t}^{\leftrightarrow}};\\
f_{\mathsf{s}^{\updownarrow} \circ \mathsf{t}^{\updownarrow}} &= f_{(\mathsf{t} \circ \mathsf{s})^{\updownarrow}} = (f_{\mathsf{t}\circ\mathsf{s}})^* = (f_{\mathsf{t}} \circ f_{\mathsf{s}})^* = (f_{\mathsf{s}})^* \circ (f_{\mathsf{t}})^* = f_{\mathsf{s}^{\updownarrow}} \circ f_{\mathsf{t}^{\updownarrow}};\\
f_{\mathsf{s}^{\updownarrow}\mathsf{t}^{\updownarrow}} &= f_{(\mathsf{st})^{\updownarrow}} = (f_{\mathsf{st}})^* = (f_{\mathsf{s}} \otimes f_{\mathsf{t}})^* = (f_{\mathsf{s}})^* \otimes (f_{\mathsf{t}})^* = f_{\mathsf{s}^{\updownarrow}} \otimes f_{\mathsf{t}^{\updownarrow}}.
\end{aligned}$$

For the component maps, consider first $|$, which is invariant under both reflections. Its functions satisfy $f_{|} = \mathbb{I} = (f_{|})^* = (f_{|})^{\leftrightarrow}$. For the local extrema, reflecting  through a vertical line gives  $= -$ , hence a sign is introduced in $f_{\mathsf{s}^{\leftrightarrow}}$ for every local extremum. For the reflection $f_{\mathsf{s}^{\updownarrow}}$, consider $\mathsf{s} = \smile$:

$$(f_{\mathsf{s}})^*(v \otimes w) = \smile(1) \cdot (v \otimes w) = (e_1 \otimes e_2 - e_2 \otimes e_1) \cdot (v \otimes w)$$
$$= v^1 w^2 - v^2 w^1 = \det[v\ w] = \frown(v_1 \otimes v_2).$$

Thus $f_{\mathsf{s}^{\updownarrow}} = f_{\frown} = (f_{\mathsf{s}})^*$ as expected. Similar identities hold for the other types of local extrema. $\square$

When applied to relations, these symmetries give:

Theorem 3.6 (Spin Network Reflection Theorem)**.** *A relation* $\sum_m \alpha_m \mathsf{s}_m = 0$ *among some collection of spin networks* $\{\mathsf{s}_m\}$ *is equivalent to the same relation for the vertically reflected set* $\{\mathsf{s}_m^{\updownarrow}\}$ *and up to sign for the horizontally reflected set* $\{\mathsf{s}_m^{\leftrightarrow}\}$*. More precisely,*

$$\sum_m \alpha_m \mathsf{s}_m = 0 \qquad \Longleftrightarrow \qquad \sum_m \alpha_m \mathsf{s}_m^{\updownarrow} = 0 \qquad \Longleftrightarrow \qquad \sum_m \alpha_m (-1)^{|\mathsf{s}_v^m|} \mathsf{s}_m^{\leftrightarrow} = 0.$$

There is a similar relation for rotation through π, and later sections give formulae for other types of reflections and rotations.

3.4 The Spin Network Calculus

Proposition 3.7 (Loop and Fundamental Binor Identities)**.** *Any spin network can be expressed as a sum of diagrams with no crossings or loops. In particular,*

$$\times = |\,| - \asymp\,; \qquad \bigcirc\ \mathsf{s} = \mathrm{tr}(\mathbb{I})\mathsf{s} = 2\mathsf{s}. \tag{3.1}$$

Proof. Evaluate on an arbitrary vector:

$$
\begin{aligned}
\big(\,\mid\,\mid - \times\big)(v \otimes w) &= v \otimes w - w \otimes v \\
&= (v^1 e_1 + v^2 e_2) \otimes (w^1 e_1 + w^2 e_2) \\
&\qquad\qquad - (w^1 e_1 + w^2 e_2) \otimes (v^1 e_1 + v^2 e_2) \\
&= (v^1 w^1 - w^1 v^1) e_1 \otimes e_1 + (v^1 w^2 - w^1 v^2) e_1 \otimes e_2 \\
&\qquad\qquad + (v^2 w^1 - w^2 v^1) e_2 \otimes e_1 + (v^1 w^2 - w^2 v^1) e_2 \otimes e_2 \\
&= (v^1 w^2 - v^2 w^1)(e_1 \otimes e_2 - e_2 \otimes e_1) \\
&= \det[v\, w] \smile \;=\; \smile \circ \frown (v \otimes w) = \overset{\smile}{\frown} (v \otimes w).
\end{aligned}
$$

For the loop:

$$\bigcirc\,(1) = \frown \circ \smile (1) = \frown (e_1 \otimes e_2 - e_2 \otimes e_1) = \det[e_1\, e_2] - \det[e_2\, e_1]) = 2. \qquad \square$$

The first of these relations is called the *Fundamental Binor Identity*, and represents a fundamental type of structure in mathematics; it is the core concept in defining both the *Kauffman Bracket Skein Module* in knot theory [BFK] and the *Poisson bracket* on the set of loops on a surface, which Goldman describes in [Gol1]. It can also be identified with the *characteristic polynomial* for 2×2 matrices (3.2).

$\mathsf{SL}(2, \mathbb{C})$-Invariance

Since 2×2 matrices act on V, the definition of spin networks may be extended to allow matrices to act on the strands. We distinguish this case by calling such graphs *trace diagrams*. The action $v \mapsto \mathbf{x} \cdot v$ is represented by inserting a polygon on a strand and identifying $\Diamond \leftrightarrow \mathbf{x}$. The corresponding action on the tensor product $V^{\otimes n}$ is represented by

$$\Diamond\Diamond\cdots\Diamond\,(v_1 \otimes \cdots \otimes v_n) = \mathbf{x}v_1 \otimes \cdots \otimes \mathbf{x}v_n.$$

The matrices $\mathbf{x} \in \mathsf{SL}(2,\mathbb{C})$ satisfy the following special property:

Proposition 3.8. *The spin network component maps* $\mid$, $\cup = \underset{\downarrow}{\cup}$, *and* $\cap = \overset{\downarrow}{\cap}$ *are* $\mathsf{SL}(2,\mathbb{C})$*-invariant.*

Proof. This is clearly true for $\mid$. For the local extrema,

$$\begin{aligned} \cap(v \otimes w) &= \det[\mathbf{x}v\ \mathbf{x}w] = \det(\mathbf{x}[v\ w]) \\ &= \det(\mathbf{x}) \cdot \det[v\ w] = 1 \cdot \det[v\ w] = \cap(v \otimes w) \end{aligned}$$

indicates that $\overset{\downarrow}{\cap} \circ \mathbf{x} = \mathbf{x} \circ \overset{\downarrow}{\cap}$. The proof for $\underset{\downarrow}{\cup}$ follows by reflection. $\square$

Given the decomposition into component maps, the previous proposition implies that *all* spin networks are $\mathsf{SL}(2,\mathbb{C})$-invariant. Note that the condition required for invariance is *exactly* $\det(\mathbf{x}) = 1$, so there is no general invariance outside $\mathsf{SL}(2,\mathbb{C})$. Moreover, all $\mathsf{SL}(2,\mathbb{C})$-invariant maps between tensor powers of V occur as spin networks:

Proposition 3.9. *The image of* $[\mathsf{S}]_I^O$ *in* $\{f : V^{\otimes I} \to V^{\otimes O}\}$ *is exactly the set of* $\mathsf{SL}(2,\mathbb{C})$*-invariant linear maps* $V^{\otimes I} \to V^{\otimes O}$*. Moreover, since the restriction of* $[\mathsf{S}]_I^O$ *to diagrams without crossings is the* Temperley-Lieb Algebra $\mathcal{TL}_I^O$*, the basis of* $\mathcal{TL}_I^O$ *is also a basis for the* $\mathsf{SL}(2,\mathbb{C})$*-invariant linear functions.*

Proof. By duality and the fact that $I+O$ must be even, the statement for $[\mathsf{S}]_I^O$ is equivalent to the statement for $[\mathsf{S}]_n^n$, where $n = \frac{1}{2}(I+O)$. By Schur's Lemma, the $\mathsf{SL}(2,\mathbb{C})$-invariant maps $V^{\otimes n} \to V^{\otimes n}$ are generated by permutations on n letters. Using the binor identity, such permutations may be realized as spin networks with crossings, hence spin networks generate the set of $\mathsf{SL}(2,\mathbb{C})$-invariant maps.

For the second statement, it only needs to be shown that the basis of $\mathcal{TL}_n$ is linearly independent, as a set of functions. It is well-known that

$$\dim(\mathcal{TL}_n) = \frac{1}{n+1}\binom{2n}{n},$$

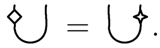
the nth *Catalan Number*. It can be shown that the space of G-invariant maps $V^{\otimes n} \to V^{\otimes n}$ has the same dimension. See [CFS] for details. $\square$

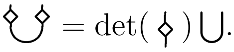
As examples, the first few Temperley-Lieb Algebras have bases:

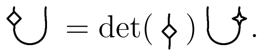
Consequently, invariant functions $V^{\otimes 1} \to V^{\otimes 1}$ are multiples of the identity, while any invariant function $f : V^{\otimes 2} \to V^{\otimes 2}$ may be expressed as a linear combination $f = \alpha_1 \,||\, + \alpha_2$  .

3.5 Trace Diagram Interpretation

The $\mathsf{SL}(2,\mathbb{C})$-invariance of diagrams also means that matrices in such a diagram can "slide across" a vertex (local extremum) by simply inverting the matrix, so that

if  $= \mathbf{x}^{-1} \in \mathsf{SL}(2,\mathbb{C})$, then 

For a general matrix $\mathbf{x} \in M_{2\times 2}$, the determinant is introduced in such relations since . If $\mathbf{x}$ is invertible, this implies



Including matrices in spin networks leads to the following definition:

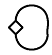
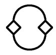
Definition 3.10. A *trace diagram* is a spin network s whose edges may be labelled by $M_{2\times 2}$ matrix variables. The algebra of trace diagrams is denoted by T or T^2, or by $[\mathsf{T}]_I^O$ or $[\mathsf{T}^2]_I^O$ if the number of inputs and outputs is specified.

Trace diagrams satisfy the same categorical properties of spin networks, and may be interpreted as maps

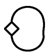
$$G \times \cdots \times G \longrightarrow \{f : V^{\otimes I} \to V^{\otimes O}\}.$$

Just as closed spin networks are interpreted as constants, so closed trace diagrams are interpreted as functions $G \times \cdots \times G \to \mathbb{C}$. For example,

Proposition 3.11. *For* $\mathbf{x} \in M_{2\times 2}$ *and* $\mathbb{I} = \left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]$,

 $= 2 = \operatorname{tr}(\mathbb{I})$;  $= \operatorname{tr}(\mathbf{x}) =$ ;  $= \det(\mathbf{x}) \cdot \operatorname{tr}(\mathbb{I})$.

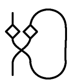
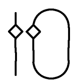
Proof. The loop value has already been calculated, while the last relation is implied by this and by  $= \det(\mathbf{x})$ . For $\mathbf{x} = [\mathbf{x}_1\ \mathbf{x}_2] = \begin{bmatrix}\mathbf{x}_{11} & \mathbf{x}_{12}\\ \mathbf{x}_{21} & \mathbf{x}_{22}\end{bmatrix}$, the second is given by:

 $(1) =$  $\circ (\mathbf{x} \otimes \mathbb{I}) \circ$  $(1) =$  $\circ (\mathbf{x} \otimes \mathbb{I})(e_1 \otimes e_2 - e_2 \otimes e_1)$

$=$  $(\mathbf{x}_1 \otimes e_2 - \mathbf{x}_2 \otimes e_1) = \det[\mathbf{x}_1\ e_2] - \det[\mathbf{x}_2\ e_1]$

$= \mathbf{x}_{11} - (-\mathbf{x}_{22}) = \operatorname{tr}(\mathbf{x}).$ □

As another example, the binor identity  $=$  $-$  gives

 $=$  $-$  $\Longrightarrow \quad \mathbf{x}^2 = \mathbf{x} \cdot \operatorname{tr}(\mathbf{x}) - \det(\mathbf{x})\mathbb{I}, \qquad (3.2)$

which is just the characteristic polynomial.

Trace diagrams are usually not $\mathsf{SL}(2,\mathbb{C})$-invariant, but they do satisfy the following:

Theorem 3.12. *Closed trace diagrams are invariant under simultaneous conjugation by G. In other words, for every* $\mathsf{t} \in [\mathsf{T}]_0^0$ *and* $g \in \mathsf{SL}(2,\mathbb{C})$,

$$f_{\mathsf{t}}(g\mathbf{x}_1 g^{-1}, g\mathbf{x}_2 g^{-1}, \ldots, g\mathbf{x}_n g^{-1}) = f_{\mathsf{t}}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n).$$

Proof. A closed trace diagram consists of a collection of loops marked by elements of G. By the previous proposition, each such loop is a trace of the product of matrices along the loop. Such functions are necessarily invariant under simultaneous conjugation. $\square$

Relations among trace diagrams are preserved under reflection, as in Proposition 3.5. However, since the dual of a matrix is its inverse, the matrices in a diagram $\mathtt{t}^{\updownarrow}$ are the inverses of the corresponding matrices in $\mathtt{t}$.

Aside from invariance, there is a similar theory for any group acting on a finite-dimensional complex vector spaces. These more general trace diagrams are the topic of the next chapter.

3.6 Symmetrizers and Representations

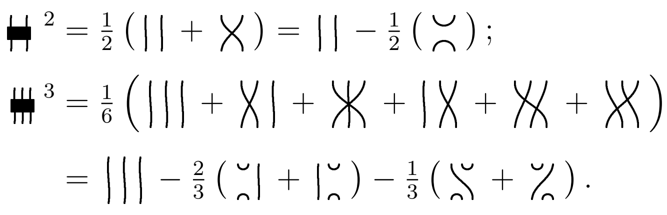
Another important $\mathsf{SL}(2,\mathbb{C})$-invariant map is the symmetrizer, defined by:

Definition 3.13. The *symmetrizer*  $^n : V^{\otimes n} \to V^{\otimes n}$ is the map taking

$$v_1 \otimes v_2 \otimes \cdots \otimes v_n \mapsto \frac{1}{n!} \sum_{\sigma \in \Sigma_n} v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)}, \tag{3.3}$$

where $v_i \in V$ and Σ_n is the group of permutations on n elements.

For example,

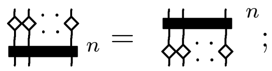


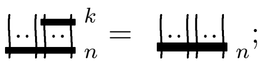
The crossings are removed by applying the Fundamental Binor Identity.

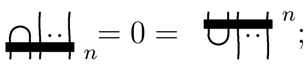
The image of ^{n} is a subspace of $V^{\otimes n}$ isomorphic to the nth symmetric power $\mathrm{Sym}^n V$, and thus it can be thought of as either the projection $\pi : V^{\otimes n} \to \mathrm{Sym}^n V$ or as the inclusion $\iota : \mathrm{Sym}^n V \to V^{\otimes n}$.

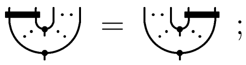
Thus, if a diagram $\mathsf{s} \in [\mathsf{S}]_I^O$ has symmetrizers at its top and bottom, it can be identified with a unique map between irreducible representations $V_I \to V_O$. We will freely interpret such spin networks as maps between tensor powers of these irreducible representations.

Proposition 3.14 (Basic Symmetrizer Properties)**.**

Invariance:  (3.4)

stacking relation:  (3.5)

capping/cupping:  (3.6)

symmetrizer sliding:  (3.7)

Proof. Invariance is a general property of spin networks.

After applying the n-symmetrizer in the *stacking relation*, the elements are all symmetric, and so an additional k-symmetrizer has no effect.

For the *capping* and *cupping relations*, notice that

$$\frown \circ \; (v \otimes w) = \frown \left(\tfrac{1}{2}(v \otimes w + w \otimes v)\right) = \tfrac{1}{2}(\det[v\ w] + \det[w\ v]) = 0.$$

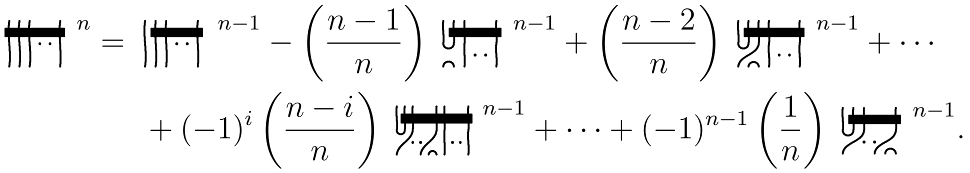
This implies the general case because, by the stacking relation, one may insert between $\frown$ and ^{n}. The other case follows by reflection.

There are a number of ways to demonstrate (3.7). It follows by reflection (Proposition 3.5) or as a special case of $\mathsf{SL}(2, \mathbb{C})$-invariance, since $=$ $=$ for $= g = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right] \in \mathsf{SL}(2, \mathbb{C})$. More directly, expand the symmetrizer into a sum of permutations. Since each permutation is a product of transpositions, then the statement is an extension of the relation $=$. $\square$

Recurrence Properties

Although it is easy to write down an arbitrary  ^{n} in terms of permutations, it is usually rather difficult to write it down in terms of diagrams without crossings (the Temperley-Lieb algebra). The next two propositions give recurrence relations which simplify this process.

Proposition 3.15. *The symmetrizer*  ^{n} *satisfies:*



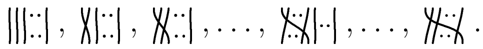
(3.8)

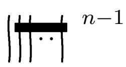
Proof. If Σ_n is the group of permutations on the set $N_n = \{1, 2, \ldots, n\}$, then

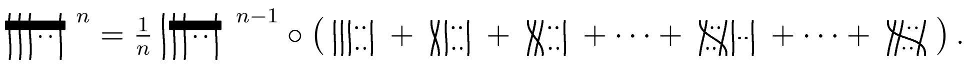
$$|\Sigma_n| = |N_n|\,|\Sigma_{n-1}|.$$

This statement has a combinatorial proof. Interpret $|\Sigma_n|$ as the number of ways to arrange n people in a line. To do this, one may first select someone to be at the front of the line ($|N_n|$ choices), and then rearrange the remaining $n-1$ people ($|\Sigma_{n-1}|$ choices).

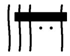
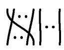
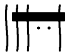
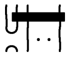
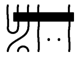
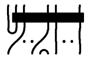
In diagram form, the selection of someone to head the line corresponds to one of the diagrams

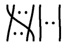
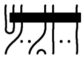


The arrangement of the remaining people corresponds to . The diagrammatic form of the above reasoning is:

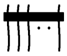
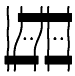
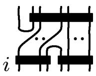


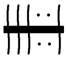
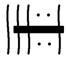
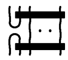
Now, use the binor identity to remove crossings. Most of the resulting terms disappear, since any term whose cups are not in the 'first position' on top will vanish due to the *capping relation*. In particular:

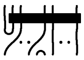
 $\circ$  $=$  $-$  $+$  $+ \cdots + (-1)^i$  ,

where i is the number of 'kinks'  in  or 1 plus the number of kinks in  . Finally, group the number of terms on the righthand side with the same number of kinks together: there will be $n-i-1$ terms with i kinks. $\square$

Proposition 3.16.  ^{n} *also satisfies the recurrence relations:*

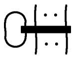
 $^n =$  $_{i}\,{}^{n-1}_{n-i} + (-1)^i \left(\frac{n-i}{n}\right)$  $_{i}\,{}^{n-1}_{n-i}$; (3.9)

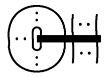
 $^n =$  $^{n-1} - \left(\frac{n-1}{n}\right)$  ${}^{n-1}_{n-1}$. (3.10)

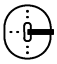
Proof. The second relation is a special case of the first. For the first, compose (3.8) with  $^i \otimes$  $^{n-i}$. This has no effect on the lefthand side, by the *stacking relation*. On the righthand side, all but one of the terms with a cap on the bottom vanish, due to the *capping relation*, since they will cap off either the  ^{i} or the  $^{n-i}$. The one term which remains 'caps between' these two symmetrizers. The coefficient is $(-1)^i \left(\frac{n-i}{n}\right)$ since in recurrence (3.8), i is equal to one more than the number of kinks  in  . $\square$

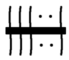
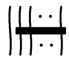
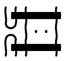
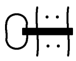
The next relations follow directly from these recurrences:

Proposition 3.17 (Looping Relations)**.**

 $^n = \left(\frac{n+1}{n}\right)$  $^{n-1}$; (3.11)

$k\Big\{$  $^n = \left(\frac{n+1}{n-k+1}\right)$  $^{n-k}$; (3.12)

 $^n = n+1.$ (3.13)

Proof. Close off the left strand in (3.10) above. Then,  ^{n},  $^{n-1}$, and  $^{n-1}_{n-1}$ become  ^{n},   $^{n-1} = 2$  $^{n-1}$ and  $^{n-1}$, respectively. Now collect terms to get (3.11), and proceed to (3.12) or (3.13) by applying the first relation k or n times. □

3.7 Trivalent Spin Networks

Recall the Clebsch-Gordon decomposition (Proposition 2.5):

$$V_a \otimes V_b \cong \bigoplus_{c \in \lceil a,b \rfloor} V_c, \qquad \lceil a,b \rfloor = \{a+b, a+b-2, \dots, |a-b|\}.$$

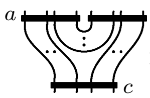
The requirement $c \in \lceil a,b \rfloor$ is equivalent to the following symmetric condition:

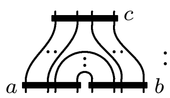
Definition 3.18. A triple (a,b,c) of nonnegative integers is *admissible* when

$$\tfrac{1}{2}(-a+b+c), \quad \tfrac{1}{2}(a-b+c), \quad \tfrac{1}{2}(a+b-c) \in \mathbb{N}. \tag{3.14}$$

Thus, $c \in \lceil a,b \rfloor$ is equivalent to requiring $\{a,b,c\}$ to be admissible.

Two maps arise from the Clebsch-Gordon decomposition: an injection $i_c^{a,b} : V_c \to V_a \otimes V_b$ and a projection $P_{a,b}^c : V_a \otimes V_b \to V_c$. Both have simple diagrammatic depictions [CFS]:

$i_c^{a,b} =$ 

$: V_c \to V_a \otimes V_b;$ $\qquad P_{a,b}^c =$ 

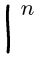
$: V_a \otimes V_b \to V_c.$

The admissibility condition (3.14) is the requirement that there is a nonnegative number of strands connecting each pair of symmetrizers. These "strand numbers" appear frequently in diagram manipulations, and will be referenced by the Greek letters α, β, γ:

Convention 3.19. Given an admissible triple (a,b,c), denote by α, β, and γ the total number of strands connecting V_b to V_c, V_a to V_c, and V_a to V_b, respectively. Also, denote by δ the total number of strands in the diagram. Therefore

$$\alpha \equiv \tfrac{1}{2}(-a+b+c); \quad \beta \equiv \tfrac{1}{2}(a-b+c); \quad \gamma \equiv \tfrac{1}{2}(a+b-c); \quad \delta \equiv \tfrac{1}{2}(a+b+c).$$

Note that (a,b,c) is admissible if and only if $\alpha,\beta,\gamma \in \mathbb{N}$.

Because the maps $i_c^{a,b}$ and $P_{a,b}^c$ are so important, they are commonly depicted using thicker, labelled lines. We represent n lines with a symmetrizer by one thick line labelled n, so that  $^n \equiv$  ^{n}. Such lines lead to a new notation for spin networks:

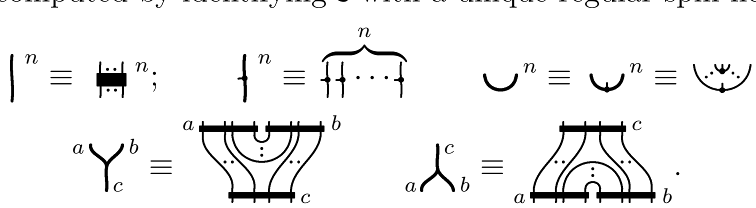
Definition 3.20. A *trivalent spin network* s is a graph drawn in the plane with vertices of degree ≤ 3 and edges labelled by positive integers such that:

- 2-vertices are ciliated and coincide with local extrema;
- 3-vertices are drawn 'up'  or 'down'  ;
- any two edges meeting at a 2-vertex have the same label;
- the three labels adjacent to any vertex form an admissible triple.

If there are m input edges with labels l_i for $i = 1,\ldots,m$ and n output edges with labels l'_i for $i = 1,\ldots,n$, the network is identified with a map between tensor products of irreducible $\mathsf{SL}(2,\mathbb{C})$ representations,

$$f_{\mathsf{s}} : V_{l_1} \otimes \cdots \otimes V_{l_m} \to V_{l'_1} \otimes \cdots \otimes V_{l'_n}.$$

The map is computed by identifying s with a unique regular spin network by:



Vertices of degree 2 are normally chosen to coincide with local extrema, and degree-3 vertices, when expanded, also have a number of ciliated vertices. The need to keep track of these ciliations makes diagram manipulation a more delicate operation.

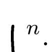
Trivalent Diagram Manipulations

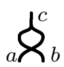
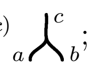
For the remainder of this paper, we assume that all sets of labels incident to a common vertex in a diagram are admissible. Moreover, whenever we sum over a label in a diagram, the sum is taken over all possible values of that label for which the requisite triples in the diagram are admissible.

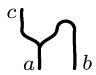
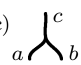
Most of the proofs in this section are simplified by recognizing that spin networks are topologically invariant apart from a factor of $(-1)^{\frac{1}{2}c}$, where c is the number of ciliations in a diagram. More direct arguments are included here for consistency with the trace diagram interpretation.

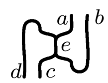
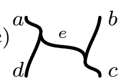
The identity  $= -$ gives rise to the following compendium of sign changes through diagram manipulations:

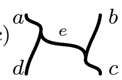
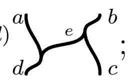
Proposition 3.21.

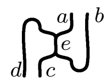
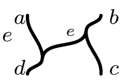
 $= (-1)^n$ ; (3.15)

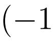
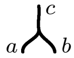
 $= (-1)^{\frac{1}{2}(a+b-c)}$ ; (3.16)

 $= (-1)^{\frac{1}{2}(-a+b+c)}$ ; (3.17)

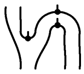
 $= (-1)^{\frac{1}{2}(a+b+c+d-2e)}$ ; (3.18)

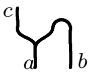
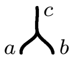
$(-1)^{\frac{1}{2}(a+c)}$  $= (-1)^{\frac{1}{2}(b+d)}$ ; (3.19)

 $= (-1)^{b+d-e}$ . (3.20)

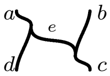
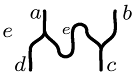
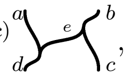
Proof. First, (3.15) is a restatement of  $^n = (-1)^n$  ^{n} and (3.16) follows by reflection, since  contains $\gamma = \frac{1}{2}(a+b-c)$ local extrema.

For (3.17), notice that in the simplest case

 $= -$  ,

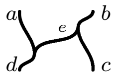
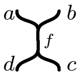
the negative sign comes from the strand on top of the diagram. Similarly, the general case for transforming  into  has a sign for each strand between b and c, giving $(-1)^\alpha = (-1)^{\frac{1}{2}(-a+b+c)}$. This identity is used twice to give (3.18).

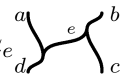
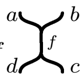
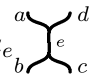
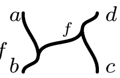
Finally, (3.19) follows from:

 $= (-1)^e$  $= (-1)^{e+\frac{1}{2}(d+e-a+b+e-c)}$  ,

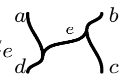
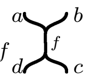
and (3.20) is given by combining (3.18) and (3.19). $\square$

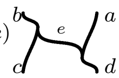
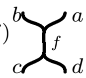
The above relations permit the definition of a "$\frac{\pi}{4}$-reflection" on certain types of diagrams, which will be important later:

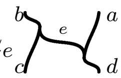
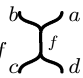
Proposition 3.22. *If a relation consists entirely of terms of the form*  *and*  *, then one may "reflect about the line through a and c" in the following sense:*

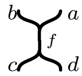
$\sum_e \alpha_e$  $= \sum_f \beta_f$  $\iff \sum_e \alpha_e$  $= \sum_f \beta_f$  .

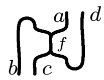
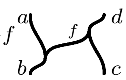
Proof. By horizontally reflecting the first relation, using Theorem 3.6,

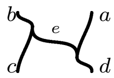
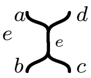
$\sum_e \alpha_e$  $= \sum_f \beta_f$ 

$\iff \sum_e \alpha_e (-1)^{\frac{1}{2}(a+b+c+d-2e)}$  $= \sum_f \beta_f (-1)^{\frac{1}{2}(a+b+c+d-2f)}$ 

$\iff \sum_e \alpha_e$  $= \sum_f \beta_f$  , where the signs cancel due to the admissibility conditions.

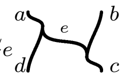
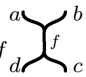
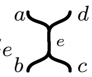
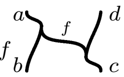
Now, add strands to both sides, so that the right side  becomes

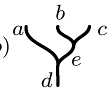
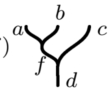
 $= (-1)^{b+d-f}$ .

Likewise, on the left side,  becomes $(-1)^{b+d-e}$ . Once again, admissibility implies that e and f must have the same parity, so these signs cancel. $\square$

Two alternate versions of this proposition follow.

Corollary 3.23.

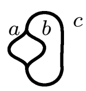
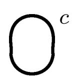
$\sum_e \alpha_e$  $= \sum_f \beta_f$  $\iff \sum_e \alpha_e$  $= \sum_f \beta_f$ 

$\iff \sum_e \alpha_e (-1)^{\frac{1}{2}(e-b)}$  $= \sum_f \beta_f (-1)^{\frac{1}{2}(d-f)}$ .

Proof. The first statement is equivalent to that in the previous proposition, aside from a factor of $(-1)^{\frac{1}{2}(a+c-b-d)}$ on the  terms. But this factor cancels since it is independent of the summation and occurs in both equations.

For the second equivalence, compose the diagrams on the left with a , and apply a reflected version of (3.17). $\square$

As for regular spin networks, any closed trivalent spin network may be interpreted as a constant. The simplest such diagrams are computed next.

Proposition 3.24. *Let* $\Theta(a,b,c) =$  *and* $\Delta(c) =$ . *Then* $\Theta(a,b,c)$ *is symmetric in* $\{a,b,c\}$ *and*

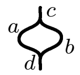
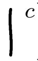
$$\Delta(c) = c+1 = \dim(V_c); \tag{3.21}$$

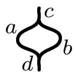
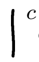
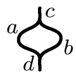
$$\Theta(a,b,c) = \frac{\left(\frac{-a+b+c}{2}\right)!\left(\frac{a-b+c}{2}\right)!\left(\frac{a+b-c}{2}\right)!\left(\frac{a+b+c+2}{2}\right)!}{a!b!c!} = \frac{\alpha!\beta!\gamma!(\delta+1)!}{a!b!c!}; \tag{3.22}$$

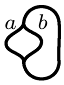
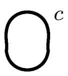
$$\Theta(1,a,a+1) = \Delta(a+1) = a+2. \tag{3.23}$$

Proof. The first equation (3.21) is a consequence of the *Looping Relation* (3.11). That $\Theta(1, a, a+1) = \Delta(a+1)$ is a consequence of the stacking relation. The formula for $\Theta(a, b, c)$ may be verified using the recurrence formula in Corollary 6.10. See [CFS] for a more direct proof. $\square$

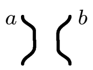
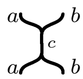
Ratios of Δ and Θ show up in the next two propositions, which tell us how to "pop bubbles" and how to "fuse together" two thick edges. The first demonstrates the usefulness of Schur's Lemma in diagrammatic techniques.

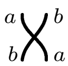
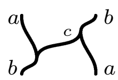
Proposition 3.25 (Bubble Identity)**.**  $= \left(\frac{\Theta(a,b,c)}{\Delta(c)}\right.$  $\left.\right) \delta_{cd}$*, where* δ_{cd} *is the Kronecker delta.*

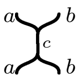
Proof. Schur's Lemma requires  $= C$  δ_{cd} for some constant C, since  is a map between irreducible representations. This equation remains true if we "close off" the diagrams, giving:

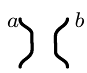
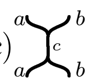
 $= C$  $\quad \Longrightarrow \quad C = \dfrac{\Theta(a,b,c)}{\Delta(c)}.$ $\square$

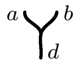
Proposition 3.26 (Fusion Identities)**.**

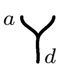
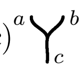
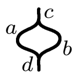
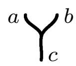
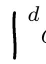
 $= \displaystyle\sum_{c \in \lceil a,b \rfloor} \left(\frac{\Delta(c)}{\Theta(a,b,c)}\right)$ 

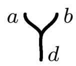
 $= \displaystyle\sum_{c \in \lceil a,b \rfloor} (-1)^{\frac{1}{2}(a-b+c)} \left(\frac{\Delta(c)}{\Theta(a,b,c)}\right)$ .

Proof. Maps of the form  for $c \in \lceil a, b \rfloor$ form a basis for the space of $\mathsf{SL}(2, \mathbb{C})$-invariant maps $V_a \otimes V_b \to V_a \otimes V_b$ [CFS]. Thus, the first diagram may be expressed as a linear combination:

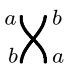
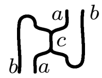
 $= \displaystyle\sum_{c \in \lceil a,b \rfloor} C(c)$ .

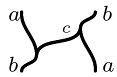
Given a specific $d \in \lceil a, b \rfloor$, the constant $C(d)$ is computed by composing this expression with , giving:

 $= \sum_{c \in \lceil a,b \rfloor} C(c)$  $\circ$  $= \sum_{c \in \lceil a,b \rfloor} C(c) \left(\frac{\Theta(a,b,c)}{\Delta(c)} \right)$  $\circ$  δ_{cd}

$= C(d) \left(\frac{\Theta(a,b,d)}{\Delta(d)} \right)$  $\implies \quad C(d) = \frac{\Delta(d)}{\Theta(a,b,d)}.$

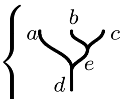
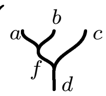
For the second equation:

 $= \sum_{c \in \lceil a,b \rfloor} (-1)^{\frac{1}{2}(-a+b+c)} \left(\frac{\Delta(c)}{\Theta(a,b,c)} \right)$ 

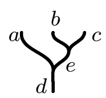
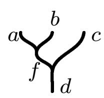
$= \sum_{c \in \lceil a,b \rfloor} (-1)^{\frac{1}{2}(a-b+c)} \left(\frac{\Delta(c)}{\Theta(a,b,c)} \right)$ . $\square$

3.8 6j-Symbols

There are two natural bases for the $\mathsf{SL}(2, \mathbb{C})$-invariant maps $V_d \to V_a \otimes V_b \otimes V_c$:

$\Big\{$  $\Big\}_{e \in \lceil a,d \rfloor \cap \lceil b,c \rfloor}$ and $\Big\{$  $\Big\}_{f \in \lceil a,b \rfloor \cap \lceil c,d \rfloor}$.

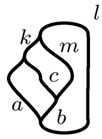
Definition 3.27. The coefficients used to switch between these bases are called *6j-symbols* and defined by:

 $= \sum_{f \in \lceil a,b \rfloor \cap \lceil c,d \rfloor} \left[\begin{smallmatrix} a & b & f \\ c & d & e \end{smallmatrix} \right] \cdot$ .

This differs slightly from the usual definition in the literature [CFS, Kau].

These coefficients are closely related to the value of the following closed spin network:

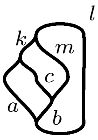
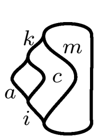
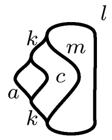
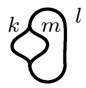
Definition 3.28. Given $a, b, c, k, l, m \in \mathbb{N}$ with the triples $\{a, b, m\}$, $\{a, c, k\}$, and $\{b, c, l\}$ all admissible, the *tetrahedral coefficient* is

$$\mathrm{Tet}(a, b, c, k, l, m) \equiv$$  .

Proposition 3.29. *The tetrahedral coefficient may be expressed in terms of* $6j$*-symbols by*

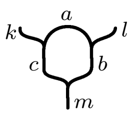
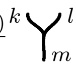
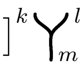
$$\mathrm{Tet}(a, b, c, k, l, m) = \left(\frac{\Theta(a, c, k)\Theta(k, l, m)}{\Delta(k)}\right) \begin{bmatrix} a & c & k \\ m & l & b \end{bmatrix}. \tag{3.24}$$

Proof. Move one strand and apply the bubble identity:

 $= \sum_i \left[\begin{smallmatrix} a & c & i \\ m & l & b \end{smallmatrix}\right]$  $= \left[\begin{smallmatrix} a & c & k \\ m & l & b \end{smallmatrix}\right]$  $= \left[\begin{smallmatrix} a & c & k \\ m & l & b \end{smallmatrix}\right] \frac{\Theta(a,c,k)}{\Delta(k)}$  . □

Another use of the tetrahedral coefficient is:

Proposition 3.30 (Triple Bubble Identity)**.**

 $= \frac{\mathrm{Tet}(a,b,c,k,l,m)}{\Theta(k,l,m)}$  $= \frac{\Theta(a,c,k)}{\Delta(k)} \left[\begin{smallmatrix} a & c & k \\ l & m & b \end{smallmatrix}\right]$  .

Proof. Close off strands on both sides of the equation, as in Proposition 3.25. □

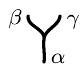
Chapter 4 **Trace Diagrams**

This chapter focuses on properties of trace diagrams for more general groups. As for $\mathsf{SL}(2,\mathbb{C})$, there are generally two ways to represent such diagrams: as graphs with edges corresponding to the standard representation, and as trivalent diagrams with edges labelled by finite-dimensional representations. Outside of a few cases, not much is known about the general theory of such diagrams.

4.1 General Spin Networks

The advantage of the definition for spin networks given in the previous section is that it easily generalizes to other cases. The more general definition follows.

Definition 4.1. Let G be a group. A *G-spin network* s is a ciliated, directed graph drawn in the plane with vertices of degree ≤ 3 and edges labelled by finite-dimensional irreducible representations of G such that:

- all vertices are either sources or sinks and are ciliated, giving adjacent edges a well-defined ordering;
- the degree 1 edges are partitioned into *inputs* and *outputs*;
- both edges incident to a 2-vertex have the same label;
- the representations V_α, V_β, and V_γ are allowed to meet at a 3-vertex 

only if there is a nonzero G-invariant map $V_\alpha \to V_\beta \otimes V_\gamma$;

- trivalent vertices are labelled by specific maps between the representations at the corresponding edges, called *intertwiners*.

If there are m inputs with adjacent edges labelled V_{l_i} and n outputs with adjacent edges labelled $V_{l'_i}$, the diagram is identified with a map

$$f_{\mathsf{s}} : V_{l_1} \otimes \cdots \otimes V_{l_m} \to V_{l'_1} \otimes \cdots \otimes V_{l'_n}.$$

If there are markings along specific edges corresponding to matrix variables, then the diagram is called a *G-trace diagram* and represents a function

$$G \times \cdots \times G \longrightarrow \{f : V_{l_1} \otimes \cdots \otimes V_{l_m} \to V_{l'_1} \otimes \cdots \otimes V_{l'_n}\}.$$

As in the previous chapter, the function is computed by decomposing the diagram into its smallest pieces. There are two things to clarify about a trace diagram's function: first, when an edge's orientation is opposite the 'direction' of a function, the function uses the *dual* of the representation. For example, $\uparrow^{a} : V_a \to V_a$, while $\downarrow^{a} : (V_a)^* \to (V_a)^*$. Second, the degree 2 vertices encode vector space isomorphisms $V_a \cong V_a^*$. For this to work, the resulting function must be well-defined.

Diagrams without inputs or outputs are called *closed diagrams*. They may be interpreted as a function $V^{\otimes 0} \to V^{\otimes 0}$, and therefore as a constant in the base field. Such functions, being linear, are determined by their value at 1.

4.2 Trace Diagrams for Matrix Groups

A broad discussion of the properties of general trace diagrams is outside the scope of this thesis. However, we will mention how such diagrams for matrix groups may be represented as unlabelled diagrams. The next section will cover in detail the case $G = \mathsf{SL}(3, \mathbb{C})$, from which most of the results generalize.

In the case where G is an $n \times n$ matrix group, spin networks may be defined in terms of unlabelled graphs. In this case, $V = \mathbb{C}^n$ is the standard representation, and the conjugate transpose map $v \mapsto v^*$ gives a vector space isomorphism $V \cong V^*$. Note that V and V^* may not be isomorphic as G-representations.

Definition 4.2. An *n-spin network* is a directed, *ciliated* graph with vertices of degree 1,2, and n and the following additional structure:

- all vertices are either sources or sinks;
- degree n vertices are *ciliated*, giving adjacent edges a well-defined ordering;
- degree 1 vertices are partitioned into *inputs* and *outputs*.

If $V = \mathbb{C}^n$ is the standard representation, then such diagrams may be interpreted as functions from $\breve{V} \otimes \cdots \otimes \breve{V} \longrightarrow \breve{V} \otimes \cdots \otimes \breve{V}$, where $\breve{V}$ represents either V or V^*. The number and type of factors corresponds to the number and type of inputs and outputs.

If there are markings present, the diagram is an *n-trace diagram* and interpreted as a map from $G \times \cdots \times G$ to the space of such functions.

Certain parts of this definition are irrelevant in some cases. The ciliation in particular is only needed to give a well-defined sign to the maps for each vertex. When n is odd, all that matters is a *cyclic* ordering, and so the ciliation need not be drawn when the diagrams are represented in the plane. Even when n is even, only two types of ciliations are necessary.

Component Maps

There are two ways to compute the general function of a trace diagram. The first parallels the *component map* model in the previous chapter:

Proposition 4.3. *Any n-trace diagram can be subdivided into the following basic maps, where $v, w, v_i \in V$, $f \in V^*$, $\{e_i\}$ form a basis for V, and represents any $n \times n$ matrix $\mathbf{x}$:*

- $: V \to V$ *where* $v \mapsto v$, *the identity;*
- $: V \otimes V^* \to \mathbb{C}$ *where* $v \otimes f \mapsto f(v)$*;*
- $: \mathbb{C} \to V^* \otimes V$ *where* $1 \mapsto \sum_{i=1}^{n} e_i^T \otimes e_i$*;*
- $: V \otimes V \to \mathbb{C}$ *where* $v \otimes w \mapsto w^* v$*;*
- $: V \otimes \cdots \otimes V \to \mathbb{C}$ *where* $v_1 \otimes v_2 \otimes \cdots \otimes v_n \mapsto \det[v_1 \ \cdots \ v_n]$*;*
- $: V \to V$ *where* $v \mapsto \mathbf{x}v$*;*
- *the diagrams , , and defined similarly on the dual.*

Note that the vertices in and should be ciliated. The proof that such functions are well-defined is similar to the case $G = \mathsf{SL}(2, \mathbb{C})$ considered previously.

Combinatorial Method

A second method which may be used to compute these diagrams is combinatorial in nature, and only applies to *spin networks*. It requires the following definition:

Definition 4.4. A *labelling* of an n-spin network is an assignment of one of the basis elements $\{e_i\}_{i=1}^n$ to each edge, such that (i) at each 2-vertex both edges have *the same* label, and (ii) at each n-vertex the three edges have *different* labels.

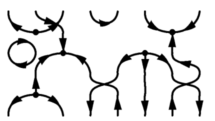
Given a labelling and a ciliated vertex v, the permutation on edge labels induced by the ciliation is denoted σ_v. Define the *sign* of v to be $\text{sign}(\sigma_v)$ if v is a source, or $-\text{sign}(\sigma_v)$ if v is a sink. Given a labelled spin network s, the *sign* of s is the product of signs at its n-vertices.

Proposition 4.5. *Let* s *be a spin network with map* $f_{\mathsf{s}} : \check{V}^{\otimes m_1} \to \check{V}^{\otimes m_2}$*. Then the coefficient of the basis element* $e_{j_1} \otimes \cdots \otimes e_{j_{m_2}}$ *in the expansion of* $f_{\mathsf{s}}(e_{i_1} \otimes \cdots \otimes e_{i_{m_1}})$ *is equal to the sum of the signs of all possible labellings of* s *which respect the label sets* $e_{i_1}, \ldots, e_{i_{m_1}}$ *and* $e_{j_1}, \ldots, e_{j_{m_2}}$ *of the input and output edges.*

In section 8.2, this proposition may be restated in terms of the *signed pre-chromatic index* of a graph.

4.3 3-Spin Networks

This section describes the practical application of the above to the case $G \subset M_{3\times 3}$. When drawing the diagrams, we place the input vertices on the bottom of some "box" and the output vertices on the top. For simplicity, we assume the diagrams do not contain degree 2 vertices. The ciliation may be omitted since only a cyclic ordering will be necessary at 3-vertices; such an ordering is implicit in drawing the diagram in the plane. For example, the diagram



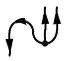
maps $(V^*)^{\otimes 3} \otimes V \otimes V^* \otimes V \otimes V^* \longrightarrow V \otimes V^* \otimes V \otimes V^* \otimes V \otimes (V^*)^{\otimes 2}$.

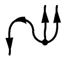
Component Maps

It will be helpful to restate the component maps in this case. Keep in mind that V may be thought of as column vectors and V^* as row vectors. Then, for $v, v_i \in V$ and $w^T \in V^*$, the component maps are

- $: V \to V$ where $v \mapsto v$, the identity;
-  $: V \otimes V^* \to \mathbb{C}$ where $v \otimes w^T \mapsto w^T v$;
-  $: \mathbb{C} \to V^* \otimes V$ takes $1 \mapsto e_1^T \otimes e_1 + e_2^T \otimes e_2 + e_3^T \otimes e_3$;
-  $: V \otimes V \otimes V \to \mathbb{C}$ where $v_1 \otimes v_2 \otimes v_3 \mapsto \det[v_1\, v_2\, v_3]$.

The components of opposite orientations are also necessary. But the dual diagram of a network s, formed by reversing the directions of all arrows, is computed by interchanging V and V^*. For example, $: V^* \to V^*$ is the identity on V^* rather than V.

As an example of this decomposition, the map  $: V^* \to V \otimes V$ is the same graph as , and therefore its function is computed via

 $= ($  $\otimes$ $\otimes$ $) \circ ($ $\otimes$  $).$

The next proposition lists additional simple maps whose explicit formulae will be useful. These are given with respect to the standard bases $\{e_1, e_2, e_3\}$ of V and $\{e_1^T, e_2^T, e_3^T\}$ of V^*.

Proposition 4.6 (Properties of 3-Spin Networks)**.** *As maps,*

-  $: \mathbb{C} \to V \otimes V \otimes V$ *takes*

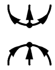
$$1 \mapsto e_1 \otimes e_2 \otimes e_3 - e_1 \otimes e_3 \otimes e_2 + e_2 \otimes e_3 \otimes e_1 - e_2 \otimes e_1 \otimes e_3 + e_3 \otimes e_1 \otimes e_2 - e_3 \otimes e_2 \otimes e_1.$$

-  $: V^* \to V \otimes V$ *takes* $e_1^T \mapsto e_2 \otimes e_3 - e_3 \otimes e_2$*;*

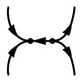
-  $: V \otimes V \to V^*$ *takes* $v_1 \otimes v_2 \mapsto (v_1 \times v_2)^T$*, the cross product;*

-  $: \mathbb{C} \to \mathbb{C}$ *takes* $1 \mapsto 3 = \dim V$ *and is identified with* 3*;*

-  $: \mathbb{C} \to \mathbb{C}$ *takes* 1 *to* $6 = 3! = 2 \dim V$ *and is identified with* 6*;*

-  $: V \otimes V \otimes V \to V \otimes V \otimes V$*, the* anti-symetrizer*, takes* $v_1 \otimes v_2 \otimes v_3 \mapsto v_1 \otimes v_2 \otimes v_3 - v_1 \otimes v_3 \otimes v_2 + v_2 \otimes v_3 \otimes v_1 - v_2 \otimes v_1 \otimes v_3 + v_3 \otimes v_1 \otimes v_2 - v_3 \otimes v_2 \otimes v_1$*;*

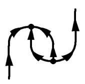
-  $: V \otimes V \to V \otimes V$ *takes* $v_1 \otimes v_2 \mapsto v_1 \otimes v_2 - v_2 \otimes v_1$*;*

-  $: V \otimes V^* \to V^* \otimes V$ *takes* $e_1 \otimes e_1^T \mapsto -(e_2^T \otimes e_2 + e_3^T \otimes e_3)$ *and* $e_1 \otimes e_2^T \mapsto e_2^T \otimes e_1$.

Proof. Either a direct computation or the labelling interpretation of a spin network's value in Proposition 4.5 may be used. For example, the coefficient of $e_{\sigma(1)} \otimes e_{\sigma(2)} \otimes e_{\sigma(3)}$ in  (1) is $\det[e_{\sigma(1)}\ e_{\sigma(2)}\ e_{\sigma(3)}] = \operatorname{sign}(\sigma)$, the sign of the permutation σ. The other maps are similarly verified. $\square$

3-Diagram Manipulations

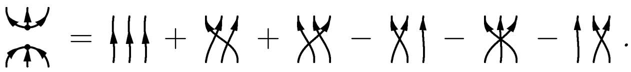
This section gives additional properties of 3-spin networks. The first proposition considers degree 3 vertices.

Proposition 4.7. *(a)*  $=$  $= 2$ *; (b)*  $= -$  .

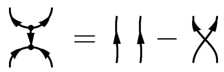
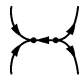
Proof. For (a), Schur's Lemma implies that this map is a multiple of the identity. Obtain the constant by evaluating on a single basis element. Alternately, there are just two colorings possible when the endpoints are fixed.

For (b), swapping edges of a 3-vertex changes the sign at that vertex. $\square$

The map  is a true anti-symmetrizer, since it evaluates to the sum of the even permutations of its inputs minus the sum of the odd permutations:

Proposition 4.8. 

The *fundamental binor identity* of 2-spin networks extends to the present case, provided the cap and cup in  are "attached." The resulting  is the anti-symmetrizer on two elements.

Proposition 4.9 ($\mathsf{SL}(3,\mathbb{C})$ Binor Identities). *(a)*  *; (b)*  = 

Proof. Relation (a) may be evaluated directly, and (b) follows by rotating (a). □

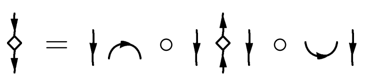
4.4 3-Trace Diagrams

Next the properties of trace diagrams, or diagrams with matrices in $M_{3\times3}$, are considered. Any 3-trace diagram with matrices may be drawn in the plane in such a way that matrices are all on upward-facing arrows, so the only additional component map, beyond that for spin networks, is

-  $: V \to V$ where $v \mapsto \mathbf{x}v$ (*trace diagrams* only).

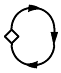
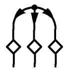
A matrix acts on "down arrows" via the contragradient representation:

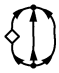
Proposition 4.10.  $: V^* \to V^*$ *takes* $w \in V^*$ *to* $\mathbf{x}^T w$.

Proof. Use the decomposition  □

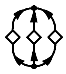
Some simple properties of trace diagrams follow.

Proposition 4.11. *Given a matrix* $\mathbf{x} \in M_{3\times3}$ *represented by* *, with* $\mathbf{x}^{-1}$ *represented by*  *if it exists, the following identities hold:*

1.  $= \mathrm{tr}(\mathbf{x})$ *and*  $= \det(\mathbf{x}) \cdot$  *;*

2.  $= \mathrm{tr}(\mathbb{I}) = 3 = \dim V = \dim V^*$ *and*  $= 2\mathrm{tr}(\mathbf{x})$*;*

3.  $= \det(\mathbf{x}) \cdot$  *and*  $= \det(\mathbf{x}) \cdot$  *;*

4.  $= 6 \det(\mathbf{x})$ *and*  $= 2 \det(\mathbf{x}) \mathrm{tr}(\mathbf{x}^{-1})$*;*

Proof. The determinant result is given by

 $\circ\, \mathbf{x}(v_1 \otimes v_2 \otimes v_3) = \det[\mathbf{x}v_1\ \mathbf{x}v_2\ \mathbf{x}v_3] = \det[\mathbf{x}] \det[v_1\ v_2\ v_3]$

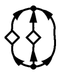
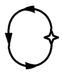
$= \det[\mathbf{x}]$  $(v_1 \otimes v_2 \otimes v_3).$

The trace calculation (2) is:

 $=$  $\circ(\mathbf{x} \otimes \mathbb{I}) \circ$  $=$  $\circ(\mathbf{x}^1 \otimes e_1 + \mathbf{x}^2 \otimes e_2 + \mathbf{x}^3 \otimes e_3) = \mathbf{x}_{11} + \mathbf{x}_{22} + \mathbf{x}_{33} = \mathrm{tr}(\mathbf{x}).$

The result is the same with two 2-vertices, since they may be ‘cancelled’.

The remaining results follow by these results and propositions in the previous section. For example, the final calculation is:

 $= \det(\mathbf{x}) \cdot$  $= 2 \det(\mathbf{x}) \cdot$  $= 2 \det(\mathbf{x}) \mathrm{tr}(\mathbf{x}^{-1}).$ □

Closed 3-Trace Diagrams

When closed 3-trace diagrams are evaluated, the result is a trace word, just as for 2-trace diagrams, since the binor identity of Proposition 4.9 allows all 3-vertices to be removed. Unfortunately, it is *not* possible to express such diagrams in terms of diagrams without crossings. To evaluate such maps, a choice has to be made between crossings and 3-vertices.

The trace word interpretation does suggest the following:

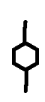
Proposition 4.12. *3-spin networks without local extrema and 2-vertices are* $\mathsf{SL}(3,\mathbb{C})$*-invariant, and closed 3-trace diagrams without local extrema and 2-vertices are invariant under simultaneous conjugation in their matrix variables.*

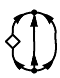
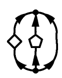
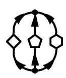
Proof. This follows from the binor identity, but here is a more direct proof. Let s be the spin network. Insert a copy of $\mathbf{x} \in \mathsf{SL}(3,\mathbb{C})$ along each edge incident to a source vertex, and a copy of $\mathbf{x}^{-1}$ along each edge incident to a sink vertex. Denote this diagram by s'. Then, $\mathsf{s}' = \mathsf{s}$ as functions, by the above relations. Moreover, all matrices on the interior edges of s' cancel, leaving copies of $\mathbf{x}$ or $\mathbf{x}^{-1}$ on the inputs and outputs. Indeed, $\mathsf{s}' = \mathbf{x} \circ \mathsf{s} \circ \mathbf{x}^{-1}$, and so $\mathbf{x} \circ \mathsf{s} = \mathsf{s}' \circ \mathbf{x} = \mathsf{s} \circ \mathbf{x}$.

If this construction is applied to a closed trace diagram t, then t' is exactly what is obtained by conjugating in the matrix variables, and therefore f_{t} is invariant under simultaneous conjugation. $\square$

Because of the trace interpretation, relations among trace diagrams give rise to trace relations. This is a very fruitful source of trace relations. Among them is the characteristic polynomial, so in a sense it contains all possible trace relations.

The following notation will be useful:

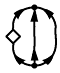
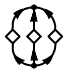
Notation 4.13. Given matrices $\mathbf{x} = \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in M_{3\times 3}$ represented by diagrams , , and , respectively, define $[\![\mathbf{x}_1]\!]$, $[\![\mathbf{x}_1, \mathbf{x}_2]\!]$, and $[\![\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3]\!]$ as follows:

$[\![\mathbf{x}_1]\!] =$ ; $\qquad [\![\mathbf{x}_1, \mathbf{x}_2]\!] =$ ; $\qquad [\![\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3]\!] =$ .

Thus, $[\![\mathbf{x}_1, \mathbf{x}_2]\!] = [\![\mathbf{x}_1, \mathbf{x}_2, \mathbb{I}]\!]$, and $[\![\mathbf{x}_1]\!] = [\![\mathbf{x}_1, \mathbb{I}]\!] = [\![\mathbf{x}_1, \mathbb{I}, \mathbb{I}]\!]$.

When the matrices are equal, this notation gives the following

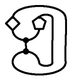
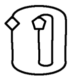
Proposition 4.14. $[\![\mathbf{x}]\!] = 2\mathrm{tr}(\mathbf{x})$*,* $[\![\mathbf{x}, \mathbf{x}]\!] = 2\det(\mathbf{x})\mathrm{tr}(\mathbf{x}^{-1})$*, and* $[\![\mathbf{x}, \mathbf{x}, \mathbf{x}]\!] = 2\det(\mathbf{x})$*.*

Proof. In diagrammatic form, these are given by $[\![\mathbf{x}]\!] =$ , $[\![\mathbf{x},\mathbf{x}]\!] =$ , and $[\![\mathbf{x},\mathbf{x},\mathbf{x}]\!] =$ , which have already been evaluated. □

The simplest trace relation comes from the binor identity  $=$  $-$ , and provides expressions for $[\![\mathbf{x}_1,\mathbf{x}_2]\!]$ and $[\![\mathbf{x},\mathbf{x}]\!]$. It also gives a formula for $\mathrm{tr}(\mathbf{x}^{-1})$ in terms of $\mathrm{tr}(\mathbf{x})$ and $\det(\mathbf{x})$:

Proposition 4.15. $[\![\mathbf{x}_1,\mathbf{x}_2]\!] = \mathrm{tr}(\mathbf{x}_1)\mathrm{tr}(\mathbf{x}_2) - \mathrm{tr}(\mathbf{x}_1\mathbf{x}_2)$.

Proof. The binor identity implies the equivalent relation

 $=$  $-$ . □

Corollary 4.16. $[\![\mathbf{x},\mathbf{x}]\!] = \mathrm{tr}(\mathbf{x})^2 - \mathrm{tr}(\mathbf{x}^2)$.

Corollary 4.17. $\mathrm{tr}(\mathbf{x}^{-1}) = \frac{1}{2\det(\mathbf{x})}(\mathrm{tr}(\mathbf{x})^2 - \mathrm{tr}(\mathbf{x}^2))$.

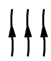
Proof. Combine previous relations to obtain

$$\mathrm{tr}(\mathbf{x})^2 - \mathrm{tr}(\mathbf{x}^2) = [\![\mathbf{x},\mathbf{x}]\!] = 2\det(\mathbf{x})\mathrm{tr}(\mathbf{x}^{-1})). \quad \square$$

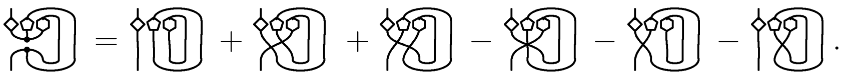
The most potent trace relations arise from expanding the anti-symmetrizer  in terms of permutations. It allows $[\![\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3]\!]$ to be expressed as a trace polynomial:

Proposition 4.18. $[\![\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3]\!] = \mathrm{tr}(\mathbf{x}_1)\mathrm{tr}(\mathbf{x}_2)\mathrm{tr}(\mathbf{x}_3) + \mathrm{tr}(\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3) + \mathrm{tr}(\mathbf{x}_1\mathbf{x}_3\mathbf{x}_2) - \mathrm{tr}(\mathbf{x}_1)\mathrm{tr}(\mathbf{x}_2\mathbf{x}_3) - \mathrm{tr}(\mathbf{x}_2)\mathrm{tr}(\mathbf{x}_1\mathbf{x}_3) - \mathrm{tr}(\mathbf{x}_3)\mathrm{tr}(\mathbf{x}_1\mathbf{x}_2)$.

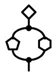
Proof. Begin with the identity

 $=$  $+$  $+$  $-$  $-$  $-$ .

Apply $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ to the top strands, and close off the last two strands to get:



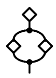
In terms of the original matrices, this equation is


$$= \mathrm{tr}(\mathbf{x}_2)\mathrm{tr}(\mathbf{x}_3)\mathbf{x}_1 + \mathbf{x}_1\mathbf{x}_2\mathbf{x}_3 + \mathbf{x}_1\mathbf{x}_3\mathbf{x}_2 - \mathrm{tr}(\mathbf{x}_3)\cdot\mathbf{x}_1\mathbf{x}_2 - \mathrm{tr}(\mathbf{x}_2\mathbf{x}_3)\cdot\mathbf{x}_1 - \mathrm{tr}(\mathbf{x}_2)\cdot\mathbf{x}_1\mathbf{x}_3. \tag{4.1}$$

Close off the final strand, or take the trace of this equation, to get the desired result. $\square$

This formula is sometimes referred to as the *polarization* of the characteristic polynomial, and indeed it is probably best thought of as a generalization of the characteristic polynomial:

Corollary 4.19. $\mathbf{x}^3 - \mathrm{tr}(\mathbf{x})\mathbf{x}^2 + \frac{1}{2}(\mathrm{tr}(\mathbf{x})^2 - \mathrm{tr}(\mathbf{x}^2))\mathbf{x} - \det(\mathbf{x})\mathbb{I} = 0.$

Proof. Set $\mathbf{x} = \mathbf{x}_1 = \mathbf{x}_2 = \mathbf{x}_3$ and use the fact that  $= 2\det(\mathbf{x})\mathbb{I}$ in (4.1) to obtain:

$$2\det(\mathbf{x})\mathbb{I} = \mathrm{tr}(\mathbf{x})^2\mathbf{x} + \mathbf{x}^3 + \mathbf{x}^3 - \mathrm{tr}(\mathbf{x})\mathbf{x}^2 - \mathrm{tr}(\mathbf{x}^2)\mathbf{x} - \mathrm{tr}(\mathbf{x})\mathbf{x}^2.$$

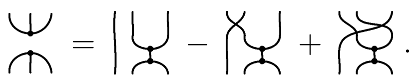
Collect terms and divide by two. $\square$

This result could also have been obtained via the following:

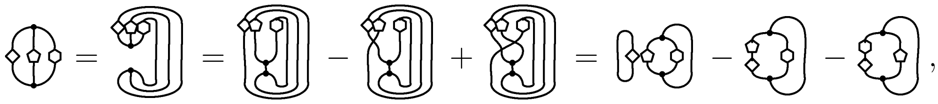
Proposition 4.20. $\mathrm{tr}(\mathbf{x}_1)[\![\mathbf{x}_2, \mathbf{x}_3]\!] = [\![\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3]\!] + [\![\mathbf{x}_2\mathbf{x}_1, \mathbf{x}_3]\!] + [\![\mathbf{x}_3\mathbf{x}_1, \mathbf{x}_2]\!].$

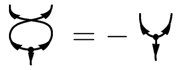
Proof. Regroup the terms of the permutation expansion of :

to obtain the alternate expression



Now, insert matrices and close off the diagram to get



which is the desired formula. (Note that the sign switches in the last term due to the extra swap which must be eliminated,  .) □

This concludes the discussion of relations among 3-trace diagrams. There is clearly a lot more to do, especially for diagrams with three or more vertices. There is evidence to suggest that certain theorems on bicubic planar graphs might give rise to methods for computation of general 3-trace diagrams without crossings. This is discussed further in section 8.2.

Adjugate Matrices

The theory of trace diagrams is closely tied to basic linear algebra. Most of the $\mathsf{SL}(3,\mathbb{C})$ maps have natural interpretations in terms of inner products, cross products, and determinants:

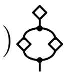
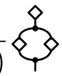
 $(v_1, v_2) = v_1 \cdot v_2;$

 $(v_1, v_2) = (v_1 \times v_2)^T;$

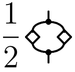
 $(v_1, v_2, v_3) = \det[v_1\; v_2\; v_3] = v_1 \cdot (v_2 \times v_3).$

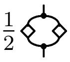
Thus, in some sense, the diagram calculus is composed entirely of the inner product, the cross product, and the triple product.

As another example, the identity

 $=$  $= \frac{1}{2}\det(\mathbf{x}^{-1})$  $= \frac{1}{2\det(\mathbf{x})}$ 

demonstrates that

$\mathbf{x}^{-1} = \frac{\mathrm{Adj}(\mathbf{x})}{\det(\mathbf{x})} =$  $= \frac{1}{\det(\mathbf{x})}\left(\frac{1}{2}\right.$  $\left.\right).$

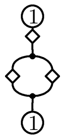
Therefore, the map $\frac{1}{2}$  is the traditional *adjugate matrix* $\mathrm{Adj}(\mathbf{x})$.

Recall that $\mathrm{Adj}(\mathbf{x})$ is constructed from the 2x2 *cofactor determinants* of the matrix $\mathbf{x}$. If the diagram  corresponds to the unit vector e_i, then multiplying a vector by a matrix can be represented using diagrams. For example,

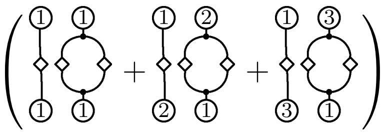
$$= e_1^T \mathbf{x} e_2 = \mathbf{x}_{12},$$

the $(1,2)$ matrix entry of $\mathbf{x}$. Using cofactor expansion across the first row of a matrix to compute the determinant corresponds to the equation

$$\det(\mathbf{x}) = \frac{1}{2}$$



$$= \frac{1}{2}$$



Moreover, the $[\![\cdot,\cdot]\!]$ and $[\![\cdot,\cdot,\cdot]\!]$ notations used earlier have the following interpretations as adjugates:

Proposition 4.21. $[\![\mathbf{x},\mathbf{x}]\!] = 2\mathrm{tr}(\mathrm{Adj}(\mathbf{x}))$ *and* $[\![\mathbf{x},\mathbf{x},\mathbf{x}]\!] = 2\mathrm{tr}(\mathbf{x}\mathrm{Adj}(\mathbf{x}))$.

4.5 Properties for General Groups

Spin networks are generically described as trivalent graphs labelled by representations, although in certain cases they have alternate representations in terms of simple, unlabelled diagrams. In the case for $\mathsf{SL}(2,\mathbb{C})$ and $\mathsf{SL}(3,\mathbb{C})$, these corresponded to graphs with vertex degrees in $\{1,2\}$ and in $\{1,2,3\}$, respectively. These diagrams generalize to the case $\mathsf{SL}(n,\mathbb{C})$, with graphs having vertex degrees in $\{1,2,n\}$.

These diagrams could just as easily be used with any matrix group in $M_{n\times n}$, although their fundamental property, $\mathsf{SL}(n,\mathbb{C})$-invariance, is lost. The following proposition describes how the diagrams behave with respect to matrix groups in general.

Proposition 4.22. *A matrix* $\mathbf{x} \in M_{n\times n}$ *acts on* n*-spin networks via the following relations:*

- *If* $\mathbf{x} \in \mathsf{SL}(n,\mathbb{C})$*, then the degree* n *vertices and are invariant;*
- *If* $\mathbf{x} \in \mathsf{O}(n)$*, the orthogonal group, then* $\circ\,(\mathbf{x}\otimes\mathbf{x}) =$ *, so the local extrema are invariant.*
- *If* $\mathbf{x} \in \mathsf{SO}(n)$*, then matrices are invariant with respect to both local extrema and the degree* n *vertices.*
- *If* $\mathbf{x} \in \mathsf{sl}(n,\mathbb{C})$*, then* $= 0$*, and a single loop with just* $\mathbf{x}$ *kills the entire diagram. Also,* $[\mathbf{x}_1, \mathbf{x}_2] = -\mathrm{tr}(\mathbf{x}_1\mathbf{x}_2)$ *for* $\mathbf{x} \in \mathsf{sl}(3,\mathbb{C})$*.*
- *If* $\mathbf{x}^k = \mathbb{I}$ *for some* k*, then the characteristic polynomial simplifies to give a simpler trace relation.*

The proofs of these statements involve applying the definitions of these groups to the diagrams. A full discussion of the properties of such matrices in diagrams is beyond the scope of this thesis. In some sense, invariance with respect to different component maps is what defines the classical Lie groups. The 2-vertices, which have been omitted in the above discussion, are $\mathsf{U}(n)$ invariant. Indeed, the 2-vertex can be arbitrarily defined as some G-invariant 2-form.

Chapter 5 **Central Functions of** $\mathrm{Hom}(\pi, G)$

This chapter introduces the central question of this thesis. The goal is to study the coordinate ring $\mathbb{C}[\mathfrak{X}]$, where $\mathfrak{X}$ is the G-character variety of a surface Σ, for some reductive group G. Many spaces of geometric structures, such as Teichmüller space and moduli space, are contained within the character variety [Gol2], and so the structure of the coordinate ring gives an abundance of information about the geometry of the surface. The approach given here analyzes a canonical basis for $\mathbb{C}[\mathfrak{X}]$ consisting of what we call *central functions*. This chapter describes the coordinate ring and the construction of these functions, while later chapters consider specific examples.

5.1 The Character Variety

Let Σ be a compact oriented surface with nonempty boundary and fundamental group π. The boundary condition permits Σ to be retracted onto a 1-complex, hence π is isomorphic to a free group of rank r:

$$\pi \cong a_1 * a_2 * \cdots * a_r \equiv \mathcal{F}_r.$$

Consider the space of homomorphisms of π into a reductive linear algebraic group G, denoted $\mathrm{Hom}(\pi, G)$. Since π is a free group, $f \in \mathrm{Hom}(\pi, G)$ is determined by its values on the generating letters of $\mathcal{F}_r$. Hence, there is a canonical isomorphism $\mathrm{Hom}(\pi, G) \cong G^r$ sending

$$f \longmapsto (f(a_1), \ldots, f(a_r)).$$

The group G acts on $\mathrm{Hom}(\pi, G)$ by simultaneous conjugation:

$$g \cdot (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r) = (g\mathbf{x}_1 g^{-1}, g\mathbf{x}_2 g^{-1}, \ldots, g\mathbf{x}_r g^{-1}).$$

The orbit of a point under this action may not be closed. For example, when $G = \mathsf{SL}(2, \mathbb{C})$ and $a \neq 0$,

$$\begin{bmatrix} b^{-1} & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} b & 0 \\ 0 & b^{-1} \end{bmatrix} = \begin{bmatrix} 1 & \frac{a}{b^2} \\ 0 & 1 \end{bmatrix} \xrightarrow[b \to 0]{} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

However, $\left[\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ are in different conjugacy classes.

The set of semistable points of $\mathrm{Hom}(\pi, G)$ is denoted $\mathrm{Hom}(\pi, G)^{ss}$. These points are the reductive homomorphisms, for which every invariant subspace has an invariant complement. Equivalently, every $f \in \mathrm{Hom}(\pi, G)^{ss}$ is *completely reducible.* The orbit space $\mathrm{Hom}(\pi, G)^{ss}/G$ has the structure of an affine algebraic variety $\mathfrak{X}$, commonly called the *G-character variety of* Σ. It may also be defined as the categorical quotient

$$\mathfrak{X} \equiv \mathrm{Hom}(\pi, G)^{ss}/G = \mathrm{Hom}(\pi, G)/\!/G.$$

Hence, the character variety is comprised of conjugacy classes of completely reducible homomorphisms in $\mathrm{Hom}(\pi, G)$ [Dol, Gol2].

The fundamental object of interest in this thesis is $\mathbb{C}[\mathfrak{X}]$, the coordinate ring of the character variety. On the level of $\mathbb{C}$-algebras, this ring is equivalent to $\mathbb{C}[\mathrm{Hom}(\pi, G)]^G$, the coordinate ring of functions on G which are invariant under simultaneous conjugation. Procesi has shown that the coordinate ring of $\mathsf{SL}(n, \mathbb{C})$-character varieties is generated by traces of products of matrices [Pro].

5.2 The Central Function Decomposition

Recall the isomorphism given in Theorem 2.4:

$$\mathbb{C}[G] \cong \bigoplus_{\lambda \in \Lambda} V_\lambda^* \otimes V_\lambda,$$

where $\{V_\lambda\}_{\lambda \in \Lambda}$ is the set of finite-dimensional irreducible representations of the maximal compact subgroup $U \subset G$. This also induces a decomposition of $\mathbb{C}[\mathrm{Hom}(\pi, G)]$, since

$$\mathbb{C}[\mathrm{Hom}(\pi, G)] \cong \mathbb{C}[G^r] \cong \mathbb{C}[G]^{\otimes r} \cong \Big(\bigoplus_{\lambda} V_\lambda^* \otimes V_\lambda\Big)^{\otimes r}. \tag{5.1}$$

The action of G by simultaneous conjugation passes through these isomorphisms, giving a decomposition of the coordinate ring $\mathbb{C}[\mathfrak{X}]$.

Consider the rank one case $\pi \cong \mathcal{F}_1$, and assume all representations are unitary. In terms of bases $\{e_i\}$ for V_λ and $\{e_i^*\}$ for V_λ^*, the above isomorphism takes $e_i^* \otimes e_j \in V_\lambda^* \otimes V_\lambda$ to the representative function $\mathbf{x} \mapsto e_i^*(\mathbf{x} \cdot e_j)$. The G-invariants are determined by the isomorphisms

$$\mathbb{C}[\mathfrak{X}] = \mathbb{C}[G]^G \cong \bigoplus_{\lambda \in \Lambda} (V_\lambda^* \otimes V_\lambda)^G \cong \bigoplus_{\lambda \in \Lambda} \mathbb{C}\, \chi_\lambda,$$

where $\chi_\lambda(\mathbf{x}) = \sum_i e_i^*(\mathbf{x} \cdot e_i) = \mathrm{tr}(\mathbf{x})$ is the *character* of the representation.

In the more general case, the decomposition continues:

$$\begin{aligned}
\mathbb{C}[\mathrm{Hom}(\pi, G)] &\cong \Big(\bigoplus_{\lambda} V_\lambda^* \otimes V_\lambda\Big)^{\otimes r} \cong \bigotimes_{i=1}^{r} \bigoplus_{\lambda_i \in \Lambda} V_{\lambda_i}^* \otimes V_{\lambda_i} \\
&\cong \bigoplus_{\lambda_1, \ldots, \lambda_r \in \Lambda} \big(V_{\lambda_1}^* \otimes \cdots \otimes V_{\lambda_r}^*\big) \otimes \big(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_r}\big)
\end{aligned}$$

At this point, the explicit isomorphism to $\mathbb{C}[\mathrm{Hom}(\pi, G)]$ takes

$$(e_{i_1}^* \otimes e_{i_2}^* \otimes \cdots \otimes e_{i_r}^*) \otimes (e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_r})$$

to the function

$$(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r) \mapsto e^*_{i_1}(\mathbf{x}_1 \cdot e_{j_1}) e^*_{i_2}(\mathbf{x}_2 \cdot e_{j_2}) \cdots e^*_{i_r}(\mathbf{x}_r \cdot e_{j_r}).$$

These functions generate the coordinate ring, although they are not necessarily irreducible. A basis of irreducibles is constructed by decomposing the tensor powers into irreducibles in a canonical way. In particular, if $\underline{\lambda} = (\lambda_1, \ldots, \lambda_r) \in \Lambda^r$ and the tensor product decomposes

$$V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_r} \cong \bigoplus_{\alpha \in A(\underline{\lambda})} V^{\underline{\lambda}}_{\alpha}, \tag{5.2}$$

then the coordinate ring becomes

$$\mathbb{C}[\mathfrak{X}] \cong \bigoplus_{\substack{\underline{\lambda} \in \Lambda^r \\ \alpha, \beta \in A(\underline{\lambda})}} \left((V^{\underline{\lambda}}_{\beta})^* \otimes V^{\underline{\lambda}}_{\alpha} \right)^G \cong \bigoplus_{\substack{\underline{\lambda} \in \Lambda^r \\ \alpha \in A(\underline{\lambda})}} \mathbb{C} \chi^{\underline{\lambda}}_{\alpha}. \tag{5.3}$$

The functions $\chi^{\underline{\lambda}}_{\alpha}$ are called the *central functions of* $\mathrm{Hom}(\pi, G)$, or the *G-central functions of* Σ. They are not well-defined, since they depend on the injection $V^{\underline{\lambda}}_{\alpha} \hookrightarrow V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_r}$. Regardless, for each choice of injection, they provide a basis for $\mathbb{C}[\mathfrak{X}]$. The next section concerns the possible choices for central function bases.

It is easiest to see how this works with a simple example. Let $G = \mathsf{SL}(2, \mathbb{C})$ and suppose $\pi = \mathcal{F}_2$. Then, the irreducible representations are indexed by the natural numbers $\mathbb{N}$, and the condition $\alpha \in A(\underline{\lambda})$ becomes the admissibility condition $c \in \lceil a, b \rfloor$ (Proposition 2.5). The choice of injection is clear, and so the central functions are parametrized by triples $\chi^{a,b,c} \equiv \chi^{(a,b)}_c$.

Example 5.1. Compute the central function $\chi^{1,1,2}$ in terms of the traces $\mathrm{tr}(\mathbf{x}_1)$, $\mathrm{tr}(\mathbf{x}_2)$, and $\mathrm{tr}(\mathbf{x}_1 \mathbf{x}_2^{-1})$. The standard basis elements $\{\mathsf{n}_2, \mathsf{n}_1, \mathsf{n}_0\}$ for $V_2 = \mathrm{Sym}^2(V)$ become, after injecting into $V \otimes V$, the elements

$$\{e_1 \otimes e_1, \frac{1}{2}(e_1 \otimes e_2 + e_2 \otimes e_1), e_2 \otimes e_2\}.$$

The space $\mathrm{Hom}(\pi, G)$ is identified with $G \times G$, and a pair $(\mathbf{x}_1, \mathbf{x}_2)$ acts on this basis to give

$$\{\mathbf{x}_1 e_1 \otimes \mathbf{x}_2 e_1, \frac{1}{2}(\mathbf{x}_1 e_1 \otimes \mathbf{x}_2 e_2 + \mathbf{x}_1 e_2 \otimes \mathbf{x}_2 e_1), \mathbf{x}_1 e_2 \otimes \mathbf{x}_2 e_2\}.$$

Project this back to V_2 to obtain, for $\mathbf{x}_1 = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ and $\mathbf{x}_2 = \left[\begin{smallmatrix} e & f \\ g & h \end{smallmatrix}\right]$:

$$\begin{array}{lccc} \{ & ae \cdot \mathsf{n}_2 & +(ag+ce) \cdot \mathsf{n}_1 & +cg \cdot \mathsf{n}_0, \\ & \frac{1}{2}(af+be) \cdot \mathsf{n}_2 & +\dfrac{1}{2}(ah+cf+bg+de) \cdot \mathsf{n}_1 & +\frac{1}{2}(ch+dg) \cdot \mathsf{n}_0, \\ & bf \cdot \mathsf{n}_2 & +(bh+df) \cdot \mathsf{n}_1 & +dh \cdot \mathsf{n}_0 \quad \}. \end{array}$$

Finally, read off the trace:

$$\begin{aligned} \chi^{1,1,2}(\mathbf{x}_1, \mathbf{x}_2) &= ae + \tfrac{1}{2}(ah + cf + bg + de) + dh \\ &= (a+d)(e+h) - \tfrac{1}{2}(ah + de - bg - cf) \\ &= \mathrm{tr}(\mathbf{x}_1)\mathrm{tr}(\mathbf{x}_2) - \tfrac{1}{2}\mathrm{tr}(\mathbf{x}_1 \mathbf{x}_2^{-1}). \end{aligned}$$

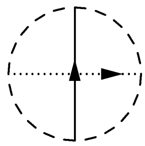
5.3 Surface Cuts and Representations

A compact surface Σ with boundary necessarily retracts onto the one-point union $\bigvee^r(S^1) \equiv S^1 \vee \cdots \vee S^1$, where r is the rank of the fundamental group. Consider a deformation retraction

$$\eta : \Sigma \to S^1 \vee \cdots \vee S^1,$$

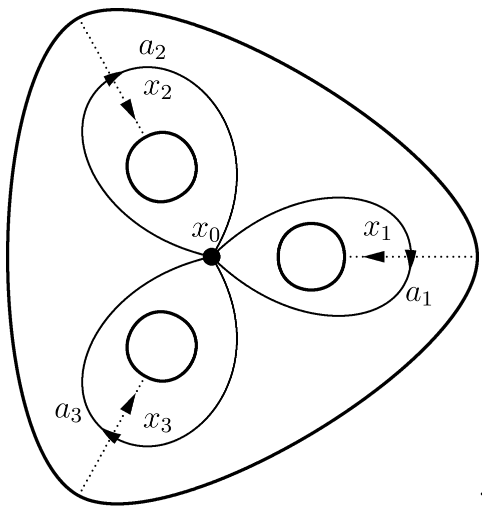
with corresponding loops a_i around the ith term in the wedge sum. Then, π is freely generated by a_i, and a function $f \in \mathrm{Hom}(\pi, G)$ is determined entirely by its values on $\{a_i\}$. This is what gives the isomorphism $\mathrm{Hom}(\pi, G) \cong G^r$.

Now, construct r pairings (x_i, d_i), where x_i is a point on the ith loop of $\bigvee^r(S^1)$ and d_i is an orientation of that same loop. Then $\bigvee^r(S^1) \setminus \{x_i\}$ is simply-connected. For a suitable choice of η, the inverse image $\eta^{-1}(\{x_i\})$ consists of several arcs with disjoint neighborhoods which "connect" boundary components of Σ. They have orientations induced by d_i, as in the figure



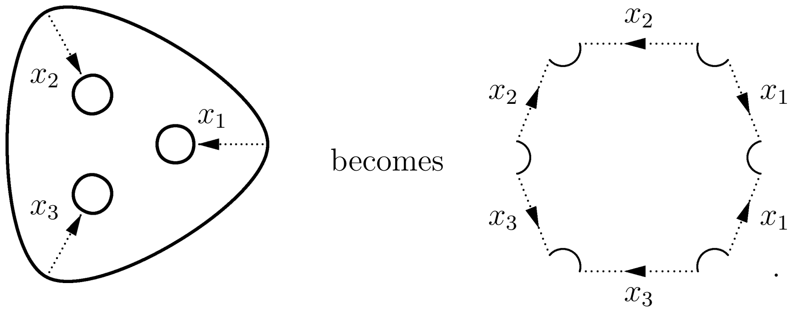
where the dotted line is the cut and the thick line is a loop of $\bigvee^r(S^1)$. These oriented arcs will be called *cuts*, and the complete set of r cuts will be called a *cut set*.

For example, the four-holed sphere retracts onto $S^1 \vee S^1 \vee S^1$, and so its fundamental group has three generators, indicated by the loops in



These generators induce the cut set $\{(x_1, d_1), (x_2, d_2), (x_3, d_3)\}$ indicated by the dotted lines.

The space $\Sigma \setminus \{\eta^{-1}(x_i)\}$ formed by removing these cuts is a simply-connected open subset of Σ. Denote its closure by Σ'. Notice that Σ' looks like a polygon with neighborhoods of its corners removed. The original surface is reconstructed from Σ' by pairing edges in some way. For example, if Σ is the four-holed sphere, then its fundamental group has rank 3 and Σ' is a "hexagon":

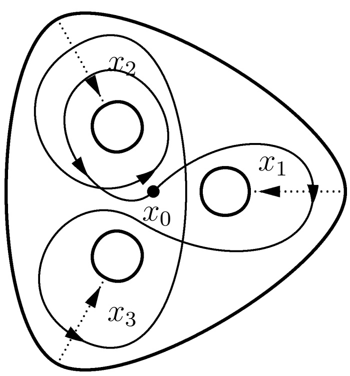


As seen in this example, every set of cuts is homotopically *equivalent* to a realization of Σ as a $2r$-gon with edges identified in some way.

An assignment of matrices to cuts induces a direct isomorphism between $\mathrm{Hom}(\pi, G)$ and G^r:

Definition 5.2. Given a cut set $\{(x_i, d_i)\}$ of a surface Σ, a *pointed cut* is a triple $(\mathbf{x}_i, x_i, d_i)$, where $\mathbf{x}_i \in G$. The collection of r such triples is a *pointed cut set.*

Given a pointed cut set, a homomorphism $f \in \mathrm{Hom}(\pi, G)$ may be defined as follows. A loop $a \in \pi_1(\Sigma, x_0)$ is homotopic to a loop a' which is transverse to the given cut set. Define $f(a) \equiv f(a')$ to be the product of elements $\mathbf{x}_i$ of the pointed cuts, written in the order they are crossed along a'. The matrix $\mathbf{x}_i$ is used for a positive crossing, while $\mathbf{x}_i^{-1}$ is used for a negative crossing. For example the loop based at x_0 in



is taken to the word $\mathbf{x}_1\mathbf{x}_3^{-1}\mathbf{x}_2^{-2}$. This construction provides a set equivalence

$$\{\text{pointed cut sets}\} = \{\text{cut sets}\} \times \operatorname{Hom}(\pi, G).$$

Therefore, there is an isomorphism $G^r \cong \operatorname{Hom}(\pi, G)$ for every fixed cut set of Σ.

Now consider the character variety $\mathfrak{X} = \operatorname{Hom}(\pi, G)//G$. The conjugacy quotient means that loops are considered *without basepoint*, while the semisimple restriction just restricts the possible $\{\mathbf{x}_i\}$. It is more interesting what happens when passing to the coordinate ring $\mathbb{C}[\mathfrak{X}] = \mathbb{C}[\operatorname{Hom}(\pi, G)//G] \cong \mathbb{C}[\operatorname{Hom}(\pi, G)]^G$. Since the *regular* invariant functions are precisely the polynomials of word traces, this is exactly what is obtained from considering the algebra of loops on the surface. In other words,

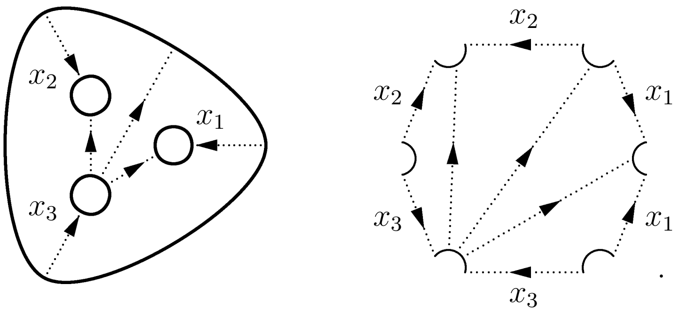
Proposition 5.3. *Let G be a reductive group and let Σ be a compact surface with boundary having fundamental group π. Then, there is an injection from the $\mathbb{C}$-algebra of invariant regular functions in $\mathbb{C}[\operatorname{Hom}(\pi, G)]^G$ into the $\mathbb{C}$-algebra of loops on Σ. The identification is obtained by labelling a cut set by an r-tuple of matrices in G.*

5.4 Cut Triangulations

The topology of the surface can be used to further the decomposition of invariant functions, by specifying the injections used to give (5.2). The chosen injection will depend on a number of additional cuts which give a "triangulation" of the surface:

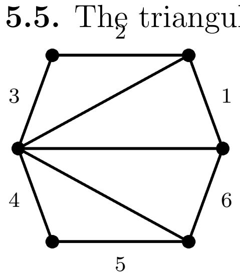
Definition 5.4. A *cut triangulation* of a surface Σ with fundamental group $\mathcal{F}_r$ is a cut set of Σ, together with a set of $2r - 3$ additional cuts which divide Σ into a set of triangles with neighborhoods of vertices removed. These additional cuts will be called *trivial cuts.*

A cut triangulation of Σ produces exactly $2(r-1)$ triangles, and $3(r-1)$ edge identifications may be used to obtain the original surface. In the case of the four-holed sphere, one triangulation is

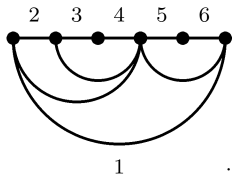


Every cut triangulation provides a canonical basis for $\mathrm{Fun}^G(\mathrm{Hom}(\pi, G))$. The correspondence is indicated by the following example.

Example 5.5. The triangulation



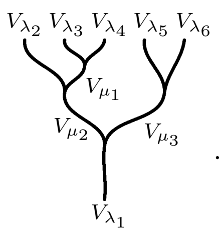
may be redrawn as



The righthand side induces the nesting $(2 \cdot (3 \cdot 4)) \cdot (5 \cdot 6)$, where each curved arc represents a set of parenthesis. This gives an injection

$$V_{\lambda_1} \hookrightarrow \big(V_{\lambda_2} \otimes (V_{\lambda_3} \otimes V_{\lambda_4})\big) \otimes \big(V_{\lambda_5} \otimes V_{\lambda_6}\big),$$

which corresponds to possible labellings of the dual graph



A complete labelling also includes an intertwiner at each vertex.

The ideas in this example are used to prove:

Theorem 5.6. *Let Σ be a compact surface with boundary. Given a cut triangulation extending a specified cut set, every spin network labelling of its dual 1-skeleton induces a trace diagram which is identified with a G-invariant function $\mathrm{Hom}(\pi, G) \to \mathbb{C}$. Moreover, for every cut triangulation, the set of such trace diagrams is a basis for $\mathrm{Fun}^G(\mathrm{Hom}(\pi, G))$.*

Proof. The 1-skeleton is a graph with vertices of valency 3. Choose one of two possible orientations for the graph which satisfy the source/sink condition. Place r matrix markings along the nontrivial cut set, in the direction induced by the cut set. This, together with a labelling, provides the requisite trace diagram.

To verify that the set of such networks forms a basis, recall (5.1). The decomposition (5.3) assumed that the injections for both $(V_{\underline{\alpha}}^{\underline{\lambda}})^*$ and $V_{\underline{\alpha}}^{\underline{\lambda}}$ were the same. This is not strictly necessary; any decomposition of

$$V_{\lambda_1}^* \otimes \cdots \otimes V_{\lambda_r}^* \otimes V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_r} \tag{5.4}$$

into tensor product pairs $V_\alpha^* \otimes V_\alpha$ is permitted. Since the summation is over all λ_1, we may assume that $V_\alpha^* = V_{\lambda_1}^*$, reducing the problem further to searching for injections

$$V_{\lambda_1} \hookrightarrow V_{\lambda_2}^* \otimes V_{\lambda_3}^* \otimes \cdots \otimes V_{\lambda_r}^* \otimes V_{\lambda_1} \otimes V_{\lambda_2} \otimes \cdots \otimes V_{\lambda_r}. \tag{5.5}$$

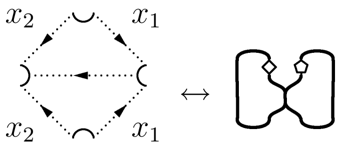
Permute the tensor powers so they occur in the same order as the cut set appears in Σ'. Then, there is a one-to-one correspondence between triangulations of Σ' and associative pairings of the righthand side of (5.5), as indicated in the previous example. Each such pairing provides a decomposition

$$\bigoplus (V_{\lambda_1}^* \otimes \cdots \otimes V_{\lambda_r}^*) \otimes (V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_r}) \cong \bigoplus V_{\lambda_1}^* \otimes V_{\lambda_1} \cong \bigoplus \mathbb{C}\chi_{\lambda_1}.$$

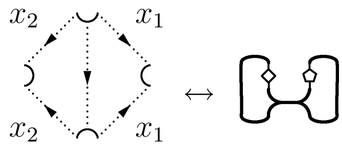
Note that χ_{λ_1} is determined not only by the representation V_{λ_1}, but also by the injection and labellings corresponding to the triangulation. $\square$

There are more general notions of "triangulation" which may be extended to give additional bases. Indeed, the ordering of tensor components is not strictly necessary, and central functions may also be defined for alternate orderings. The main point is that the selection of a particular trivalent graph with appropriate matrix markings indicates a choice of a particular central function basis.

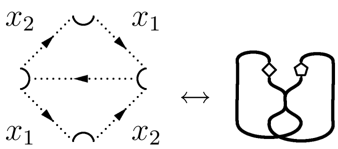
As a concrete example, the central functions induced by triangulations of the three-holed sphere are



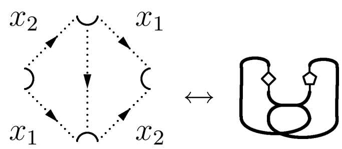
and



For the one-holed torus, which has the same fundamental group, the central functions are

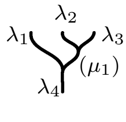


and

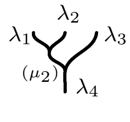


Since the fundamental groups are the same, all four function types provide bases for each surface type. The more general notion of triangulation allows for this expanded structure.

Transformations between central function bases are given by *recoupling coefficients*, which are the generalizations of $6j$-symbols given by the general change-of-basis formula



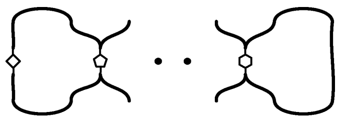
$$= \sum_{(\mu_2)} \begin{bmatrix} \lambda_1 & \lambda_2 & (\mu_2) \\ \lambda_3 & \lambda_4 & (\mu_1) \end{bmatrix}$$



The parentheses here are meant to indicate that both the diagrams and the summation must take into account the particular intertwiners chosen for each vertex. If the recoupling coefficients for a given group G are known, then a formula for one central function gives a formula for all central functions.

The computation of these central functions is not an easy task, even in the case $G = \mathsf{SL}(2, \mathbb{C})$, where the diagrammatic theory is well-known. The main ingredient required in the computation is a formula for a general injection $V_\alpha \hookrightarrow V_{\lambda_1} \otimes V_{\lambda_2}$. If the computation is approached diagrammatically, then a diagrammatic depiction of all irreducible representations is also required. Chapter 7 describes the diagrammatics for $G = \mathsf{SL}(3, \mathbb{C})$.

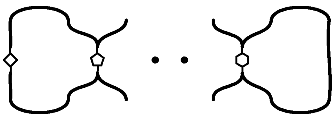
In the next chapter, we will consider the case $G = \mathsf{SL}(2, \mathbb{C})$ in detail. The corresponding central functions are chosen to be diagrams of the form

,

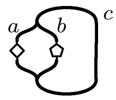
where the polygons represent matrices in $\mathsf{SL}(2, \mathbb{C})$.

Chapter 6 **Central Functions for $G = \mathsf{SL}(2, \mathbb{C})$**

In the previous chapter, it was shown that labelled diagrams of the form



comprise a basis for the central functions of a surface Σ with boundary. Here, polygons are used to represent elements of G. This chapter describes some of these functions explicitly in the case $G = \mathsf{SL}(2, \mathbb{C})$. Properties of the rank one case are described in the first section. The remainder of the chapter concerns the rank two case, for which the functions have the form

$$\chi^{a,b,c}(\mathbf{x}_1, \mathbf{x}_2) =$$ 

.

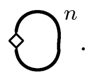
The author learned of the diagrammatic description of these functions from notes of Reshetikhin [Res], which were also the starting point for the proofs of Theorems 6.6 and 6.14. Most of the results in this chapter are also contained in [LP].

6.1 Rank One $\mathsf{SL}(2, \mathbb{C})$ Central Functions

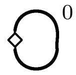
The algebraic construction of central functions in the rank one case is given directly by the isomorphisms

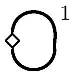
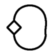
$$\mathbb{C}[\mathfrak{X}] \cong \mathbb{C}[G]^G \cong \bigoplus_{n \geq 0} (V_n^* \otimes V_n)^G \cong \bigoplus_{n \geq 0} \mathbb{C}\, \chi^n,$$

where $\chi^n \in \mathrm{End}(V_n)^G$ is a multiple of the identity on V_n. Therefore, the central functions are parametrized by the finite-dimensional irreducible $\mathsf{SL}(2,\mathbb{C})$-representations. The function χ^n corresponds to an invariant function in $\mathbb{C}[G]^G$ by

$$\chi^n = \sum_{i=0}^{n} \mathsf{n}_i(\mathsf{n}_i)^T \longmapsto \sum_{i=0}^{n} \tbinom{n}{i} \mathsf{n}_i^* \otimes \mathsf{n}_i \overset{\Upsilon}{\longmapsto} \sum_{i=0}^{n} \tbinom{n}{i} \mathsf{n}_i^*(\mathbf{x} \cdot \mathsf{n}_i) =$$ 

We will freely identify χ^n with its image in $\mathbb{C}[G]^G$.

For example, the trivial representation V_0 gives $\chi^0 = 1 =$  . The standard representation V_1 has diagonal matrix coefficients $\mathbf{x}_{11}$ and $\mathbf{x}_{22}$, hence

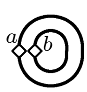
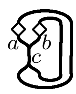
$\chi^1 =$  $=$  $= \mathbf{x}_{11} + \mathbf{x}_{22} = \mathrm{tr}(\mathbf{x}).$

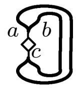
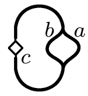
The remaining functions may be computed directly, or by using the following product formula:

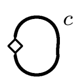
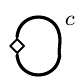
Theorem 6.1 (Rank One $\mathsf{SL}(2,\mathbb{C})$ Central Function Product Formula)**.**

$$\chi^a \chi^b = \sum_{c \in \lceil a,b \rfloor} \chi^c \tag{6.1}$$

Proof. Recall the fusion and bubble identities in Propositions 3.25 and 3.26. If the matrix $\mathbf{x}$ is represented by  , then:

$\chi^a\chi^b =$  $= \sum_{c \in \lceil a,b \rfloor} \left(\frac{\Delta(c)}{\Theta(a,b,c)} \right)$ 

$= \sum_{c \in \lceil a,b \rfloor} \left(\frac{\Delta(c)}{\Theta(a,b,c)} \right)$  $= \sum_{c \in \lceil a,b \rfloor} \left(\frac{\Delta(c)}{\Theta(a,b,c)} \right)$ 

$= \sum_{c \in \lceil a,b \rfloor} \left(\frac{\Delta(c)\Theta(a,b,c)}{\Theta(a,b,c)\Delta(c)} \right)$  $= \sum_{c \in \lceil a,b \rfloor}$  $= \sum_{c \in \lceil a,b \rfloor} \chi^c.$

There is also a direct algebraic proof using characters of the representations. From the Clebsch-Gordan decomposition,

$$(V_a \otimes V_b)^* \otimes (V_a \otimes V_b) \cong \bigoplus_{c,d \in \lceil a,b \rfloor} V_c^* \otimes V_d.$$

Hence

$$\mathrm{End}(V_a \otimes V_b)^G \cong \bigoplus_{c \in \lceil a,b \rfloor} \mathrm{End}(V_c)^G$$

and the characters satisfy

$$\chi^a \chi^b = \chi_{(V_a \otimes V_b)} = \chi_{\oplus_c V_c} = \sum_{c \in \lceil a,b \rfloor} \chi^c. \quad \square$$

Corollary 6.2. *As functions*

$$\chi^n = \mathrm{tr}(\mathbf{x})\chi^{n-1} - \chi^{n-2}. \tag{6.2}$$

Proof. The product formula (6.1) gives

$$\chi^n \chi^1 = \chi^{n+1} \chi^{n-1},$$

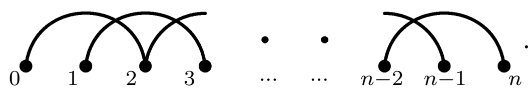
from which the recurrence relation follows since $\chi^1 = \mathrm{tr}(\mathbf{x})$. $\square$

This corollary implies that every χ^n is a polynomial in $\mathrm{tr}(\mathbf{x})$. Letting $x = \mathrm{tr}(\mathbf{x})$, the rank one central functions can be thought of as $\chi^n(x) \in \mathbb{C}[x]$.

Closed Formula for Rank One Central Functions

Given the above lemma, it is a straightforward task to find a closed formula for $\chi^n(x)$. The following lemma contains the necessary combinatorial result.

Lemma 6.3. *Suppose there are $n-1$ arcs connecting the points $\{0, 1, 2, \ldots, n\}$ with each point i connected to the two points $i \pm 2$ as in the following picture:*



Then, there are $\binom{n-r}{r}$ ways to select r non-intersecting arcs.

Proof. The proof is by induction on n. In the case $n = 1$, there are no arcs, and just one way to make a selection. As expected, $\binom{1-0}{0} = 1$. In the case $n = 2$, there is just one arc, and one way to make the selection. The base case is completed by noting that $\binom{2-0}{0} = 1$ and $\binom{2-1}{1} = \binom{1}{1} = 1$.

By way of induction, assume that there are $\binom{k-r}{r}$ choices of r arcs for any $k < n$. Now, consider the case for $\{0, 1, \ldots, n\}$ and a choice of r out of $n-1$ arcs. If a selection contains the first arc, it must not contain the second, and therefore must contain $r-1$ out of the last $n-2$. Hence, by induction, there $\binom{n-2-(r-1)}{r-1} = \binom{n-1-r}{r-1}$ such selections. Otherwise, if the first arc is not included, there are $\binom{n-1-r}{r}$ choices. Since any choice of arcs must fall into one of these two disjoint categories, there are

$$\binom{n-1-r}{r} + \binom{n-1-r}{r-1} = \binom{n-r}{r}$$

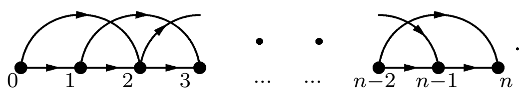
choices all together. This last identity is the basic sum in Pascal's triangle. $\square$

This relation implies:

Lemma 6.4. *The polynomial* $\chi^n(x)$ *is*

$$\chi^n(x) = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^r \binom{n-r}{r} x^{n-2r}.$$

Proof. Suppose $\chi^n(x)$ is computed by repeated application of the recurrence (6.2). If we define $\chi^{-1}(x) \equiv 0$, then this process only ends when $\chi^0(x) = 1$ is reached. Each term in the final result comes from a unique path from 0 to n in the following directed graph:



Each curved arc contributes (-1) and each straight segment x to the final term. A path with r curved arcs must have $n-2r$ straight segments, so it will contribute

$(-1)^r x^{n-2r}$ to the final sum. By the above lemma, the total contribution of the x^{n-2r} term will therefore be $(-1)^r \binom{n-r}{r} x^{n-2r}$. The limits follow from the fact that paths must have between 0 and $\lfloor \frac{n}{2} \rfloor$ curved arcs. $\square$

The coefficients of these formulae for $0 \leq i \leq n$ can be thought of as the coefficients for the *change-of-basis matrix* between the bases $\{1, x, x^2, \ldots, x^n\}$ and $\{\chi^0, \chi^1, \ldots, \chi^n\}$. The inverse formulae, which expresses x^n in terms of the central function basis, is given next.

Proposition 6.5. *The term x^n may be written in terms of $\chi^{n-2r}(x)$:*

$$x^n = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \left[\binom{n}{r} - \binom{n}{r-1} \right] \chi^{n-2r}(x),$$

where it is assumed that $\binom{n}{r} = 0$ for $r \leq 0$.

Proof. Use induction. For the base cases $n = 0, 1$, the only term is $r = 0$ since $\lfloor \frac{n}{2} \rfloor = 0$, and the formula reduces to:

$$\left[\binom{0}{0} - \binom{0}{-1} \right] \chi^0(x) = (1-0)\chi^0(x) = 1 = x^0. \checkmark$$
$$\left[\binom{1}{0} - \binom{1}{-1} \right] \chi^1(x) = (1-0)\chi^1(x) = x = x^1. \checkmark.$$

Assume by induction that the proposition holds for x^n. Then the formula

$x\chi_n = \chi^{n+1} + \chi^{n-1}$ gives:

$$
\begin{aligned}
x^{n+1} = x(x^n) &= \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \left[\tbinom{n}{r} - \tbinom{n}{r-1}\right] x\chi^{n-2r} \\
&= \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \left[\tbinom{n}{r} - \tbinom{n}{r-1}\right] (\chi^{n+1-2r} + \chi^{n-1-2r}) \\
&= \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \left[\tbinom{n}{r} - \tbinom{n}{r-1}\right] \chi^{n+1-2r} + \sum_{r=1}^{\lfloor \frac{n}{2} \rfloor +1} \left[\tbinom{n}{r-1} - \tbinom{n}{r-2}\right] \chi^{n+1-2r} \\
&= \tbinom{n}{0}\chi^{n+1} + \left[\tbinom{n}{\lfloor \frac{n}{2} \rfloor} - \tbinom{n}{\lfloor \frac{n}{2} \rfloor -1}\right] \chi^{n-1-2\lfloor \frac{n}{2} \rfloor} + \sum_{r=1}^{\lfloor \frac{n}{2} \rfloor} \left[\tbinom{n}{r} - \tbinom{n}{r-2}\right] \chi^{n+1-2r} \\
&= \left(\sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \left[\tbinom{n+1}{r} - \tbinom{n+1}{r-1}\right] \chi^{n+1-2r} \right) + \left[\tbinom{n}{\lfloor \frac{n}{2} \rfloor} - \tbinom{n}{\lfloor \frac{n}{2} \rfloor -1}\right] \chi^{n-1-2\lfloor \frac{n}{2} \rfloor}.
\end{aligned}
$$

This last step uses the fact that

$$\tbinom{n+1}{r} - \tbinom{n+1}{r-1} = \left[\tbinom{n}{r-1} + \tbinom{n}{r}\right] - \left[\tbinom{n}{r-2} + \tbinom{n}{r-1}\right] = \tbinom{n}{r} - \tbinom{n}{r-2}.$$

Finally, notice that if n is even the last term vanishes since $\chi^{n-1-2\lfloor \frac{n}{2} \rfloor} = \chi^{-1}$, and the upper index does not need to change since $\lfloor \frac{n}{2} \rfloor = \lfloor \frac{n+1}{2} \rfloor$. If n is odd, then $\chi^{n-1-2\lfloor \frac{n}{2} \rfloor} = \chi^{n+1-2\lfloor \frac{n+1}{2} \rfloor}$ and the binomials may be altered:

$$\tbinom{n}{\lfloor \frac{n}{2} \rfloor} - \tbinom{n}{\lfloor \frac{n}{2} \rfloor -1} = \tbinom{n}{\frac{n}{2}-\frac{1}{2}} - \tbinom{n}{\frac{n}{2}-\frac{3}{2}} = \tbinom{n}{\frac{n}{2}+\frac{1}{2}} - \tbinom{n}{\frac{n}{2}-\frac{3}{2}} = \tbinom{n+1}{\lfloor \frac{n+1}{2} \rfloor} - \tbinom{n+1}{\lfloor \frac{n+1}{2} \rfloor -1}. \quad \square$$

Here is a list of the first several χ^n:

$$\{1,\ x,\ x^2-1,\ x^3-2x,\ x^4-3x^2+1,\ x^5-4x^3+3x,\ x^6-5x^4+6x^2-1\}.$$

These functions satisfy some other interesting properties. For instance,

$$\chi^n(i) = i^n F_n,$$

where F_n is the *nth Fibonacci number*. For this reason, they are sometimes called *Fibonacci polynomials*. Benjamin and Quinn give a number of combinatorial results related to these polynomials in [BQ].

The rank one central functions may also be expressed as functions of eigenvalues, since χ^n is determined by its values on normal forms $\begin{bmatrix} \lambda & * \\ 0 & \lambda^{-1} \end{bmatrix} \in \mathsf{SL}(2,\mathbb{C})$. Using this fact, one can show that

$$\chi^n(\lambda + \lambda^{-1}) = \lambda^n + \lambda^{n-2} + \cdots + \lambda^{2-n} + \lambda^{-n} = \frac{\lambda^{n+1} - \lambda^{-n-1}}{\lambda - \lambda^{-1}} = [n+1]_\lambda,$$

where $[n+1]_\lambda$ is the quantized integer for $q = \lambda$.

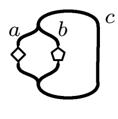
The following table gives the first several rank one $\mathsf{SL}(2,\mathbb{C})$ central functions:

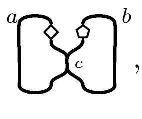
Function	Expansion for $x = \mathrm{tr}(\mathbf{x})$
χ^0	1
χ^1	x
χ^2	$x^2 - 1$
χ^3	$x^3 - 2x$
χ^4	$x^4 - 3x^2 + 1$
χ^5	$x^5 - 4x^3 + 3x$

Table 6.1: Rank One $\mathsf{SL}(2,\mathbb{C})$ Central Functions.

6.2 Rank Two $\mathsf{SL}(2,\mathbb{C})$ Central Functions

In the rank two case, central functions are computed via

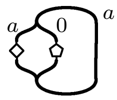
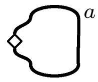
$\chi^{a,b,c}(\mathbf{x}_1, \mathbf{x}_2) =$ 

$=$ 

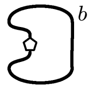
,

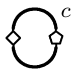
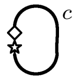
where  and  denote the matrices $\mathbf{x}_1$ and $\mathbf{x}_2$, respectively.

For example, if $b = 0$, then admissibility demands that $a = c$, and the central function is

$$\chi^{a,0,a}(\mathbf{x}_1, \mathbf{x}_2) =$$  $$=$$  $$= \chi^a(\mathbf{x}_1).$$

Likewise, if  represents $\mathbf{x}_2^{-1}$, then

$$\chi^{0,b,b}(\mathbf{x}_1, \mathbf{x}_2) =$$  $$= \chi^b(\mathbf{x}_2);$$

$$\chi^{c,0,c}(\mathbf{x}_1, \mathbf{x}_2) =$$  $$=$$  $$= \chi^c(\mathbf{x}_1\mathbf{x}_2^{-1}).$$

As special cases, the first few central functions are $\chi^{0,0,0} = 1$ and

$$\chi^{1,0,1} = \mathrm{tr}(\mathbf{x}_1) \equiv x; \qquad \chi^{0,1,1} = \mathrm{tr}(\mathbf{x}_2) \equiv y; \qquad \chi^{1,1,0} = \mathrm{tr}(\mathbf{x}_1\mathbf{x}_2^{-1}) \equiv z.$$

We will use x, y, z throughout this chapter to denote these traces.

Algebraic Construction

Without the diagrams, such functions can be calculated via the isomorphism

$$\mathbb{C}[G \times G]^G \cong \sum_{\substack{a,b \in \mathbb{N} \\ c \in \lceil a,b \rfloor}} \mathbb{C}\,\chi^{a,b,c},$$

where $\chi^{a,b,c}$ corresponds to the image of

$$\sum_{k=0}^{c} \mathsf{c}_k (\mathsf{c}_k)^T \mapsto \sum_{k=0}^{c} \binom{c}{k} \mathsf{c}_k^* \otimes \mathsf{c}_k$$

under the injection

$$V_c^* \otimes V_c \hookrightarrow V_a^* \otimes V_b^* \otimes V_a \otimes V_b.$$

determined by the Clebsch-Gordon injection $\iota : V_c \hookrightarrow V_a \otimes V_b$. We freely use $\chi^{a,b,c}$ to denote its image in $\mathbb{C}[G \times G]^G$.

An explicit formula for ι provides a means to compute $\chi^{a,b,c}$ directly. Recall the map  $: V_0 \hookrightarrow V_1 \otimes V_1$ given by

$$\mathsf{c}_0 \mapsto \mathsf{a}_0 \otimes \mathsf{b}_1 - \mathsf{a}_1 \otimes \mathsf{b}_0.$$

This generalizes to the following injection $V_0 \hookrightarrow V_a \otimes V_a$:

$$\smile^a : \mathsf{c}_0 \longmapsto \sum_{m=0}^{a} (-1)^m \binom{a}{m} \mathsf{a}_{a-m} \otimes \mathsf{b}_m. \tag{6.3}$$

As an example of this equation, $\chi^{1,1,0}$ is computed directly:

$$\begin{aligned}
\chi^{1,1,0} &\mapsto \mathsf{c}_0^* \otimes \mathsf{c}_0 \\
&\mapsto (\mathsf{a}_0^* \otimes \mathsf{b}_1^* - \mathsf{a}_1^* \otimes \mathsf{b}_0^*) \otimes (\mathsf{a}_0 \otimes \mathsf{b}_1 - \mathsf{a}_1 \otimes \mathsf{b}_0) \\
&\mapsto (\mathsf{a}_0^* \otimes \mathsf{a}_0) \otimes (\mathsf{b}_1^* \otimes \mathsf{b}_1) - (\mathsf{a}_1^* \otimes \mathsf{a}_0) \otimes (\mathsf{b}_0^* \otimes \mathsf{b}_1) \\
&\qquad - (\mathsf{a}_0^* \otimes \mathsf{a}_1) \otimes (\mathsf{b}_1^* \otimes \mathsf{b}_0) + (\mathsf{a}_1^* \otimes \mathsf{a}_1) \otimes (\mathsf{b}_0^* \otimes \mathsf{b}_0) \\
&\mapsto \mathbf{x}_{11}^1 \otimes \mathbf{x}_{22}^2 - \mathbf{x}_{12}^1 \otimes \mathbf{x}_{21}^2 - \mathbf{x}_{21}^1 \otimes \mathbf{x}_{12}^2 + \mathbf{x}_{22}^1 \otimes \mathbf{x}_{11}^2 \\
&\mapsto (\mathbf{x}_{11}^1 \mathbf{x}_{22}^2 + \mathbf{x}_{22}^1 \mathbf{x}_{11}^2) - (\mathbf{x}_{12}^1 \mathbf{x}_{21}^2 + \mathbf{x}_{21}^1 \mathbf{x}_{12}^2) = \mathrm{tr}(\mathbf{x}_1 \mathbf{x}_2^{-1}) = z.
\end{aligned}$$

The representation V_c is identified with a subset of $V^{\otimes c}$ via the equivariant maps

$$\begin{tikzcd}
V_c \arrow[r, bend left, "\mathsf{Sym}"] & V^{\otimes c} \arrow[l, bend left, "\mathsf{Proj}"]
\end{tikzcd}$$

where $\mathsf{Proj} \circ \mathsf{Sym} = \mathrm{id}$. Thus, when $c = a + b$, ι is given by the commutative diagram

$$\begin{tikzcd}
V^{\otimes c} \arrow[r, equal] \arrow[dr, phantom, "\circlearrowright"] & V^{\otimes a} \otimes V^{\otimes b} \arrow[d, "\mathsf{Proj} \otimes \mathsf{Proj}"] \\
V_c \arrow[u, "\mathsf{Sym}"] \arrow[r, "\iota"'] & V_a \otimes V_b.
\end{tikzcd}$$

In particular,

$$\binom{c}{k} \mathsf{c}_k \overset{\iota}{\longmapsto} \sum_{\substack{0 \le i \le a \\ 0 \le j \le b \\ i+j=k}} \binom{a}{i} \mathsf{a}_i \otimes \binom{b}{j} \mathsf{b}_j. \tag{6.4}$$

The general form of ι is determined by combining (6.3) and (6.4) in the following diagram:

$$\begin{tikzcd}
V_c \arrow[rr, "\iota"] \arrow[d, "\iota"'] \arrow[drr, phantom, "\circlearrowright"] & & V_\beta \otimes V_\alpha \arrow[d, "\mathrm{id} \otimes \smile^\gamma \otimes \mathrm{id}"] \\
V_a \otimes V_b & & V_\beta \otimes V_\gamma \otimes V_\gamma \otimes V_\alpha \arrow[ll]
\end{tikzcd}$$

It follows that the mapping $\iota : V_c \to V_a \otimes V_b$ is explicitly given by:

$$\begin{aligned}\binom{c}{k}\mathsf{c}_k \longmapsto & \sum_{\substack{0\le i\le\beta\\0\le j\le\alpha\\0\le m\le\gamma\\i+j=k}} \binom{\beta}{i}\mathsf{a}_i \otimes \left[(-1)^m\binom{\gamma}{m}\mathsf{a}_{\gamma-m}\otimes\mathsf{b}_m\right]\otimes\binom{\alpha}{j}\mathsf{b}_j \\ \longmapsto & \sum_{\substack{0\le i\le\beta\\0\le j\le\alpha\\0\le m\le\gamma\\i+j=k}} (-1)^m\binom{\beta}{i}\binom{\alpha}{j}\binom{\gamma}{m}\mathsf{a}_{i+\gamma-m}\otimes\mathsf{b}_{j+m}.\end{aligned}$$

Using this formula, a general central function is computed as the trace of some transformation from V_c to V_c. To obtain this transformation, inject both c_k and c_k^* using ι. Let $\mathbf{x}_1$ act on the resulting "V_a" terms, and $\mathbf{x}_2$ on the resulting "V_b" terms. Pair the two V_a terms together and the two V_b terms together to obtain the desired transformation.

In practice, it will be much easier to compute the central functions diagrammatically. There is no need in the diagrams to keep track of the binomial factors or the difference between the Clebsch-Gordon injection and projection.

6.3 Symmetries for Rank Two

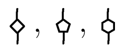
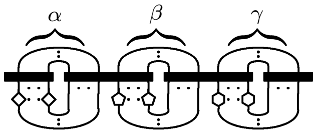
The next result is not clear from the algebraic definition of spin networks, but essentially trivial in diagram form. In the theorem, we will use $\sigma(\diamondsuit_1, \diamondsuit_2, \diamondsuit_3)$ to denote the ordered triple $(\diamondsuit_{\sigma(1)}, \diamondsuit_{\sigma(2)}, \diamondsuit_{\sigma(3)})$ obtained by applying a given permutation $\sigma \in \Sigma_3$ to the triple $(\diamondsuit_1, \diamondsuit_2, \diamondsuit_3)$.

Theorem 6.6 (Symmetry of Central Functions)**.** *Suppose a central function is expressed as a polynomial* $\mathfrak{p}$ *in the variables* $x = \mathrm{tr}(\mathbf{x}_1)$, $y = \mathrm{tr}(\mathbf{x}_2)$, *and* $z = \mathrm{tr}(\mathbf{x}_1\mathbf{x}_2^{-1})$, *so that* $\mathfrak{p}_{a,b,c}(x, y, z) = \chi^{a,b,c}(\mathbf{x}_1, \mathbf{x}_2)$ *for some admissible triple* $\{a, b, c\}$.

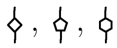
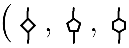
These polynomials are symmetric with respect to (x, y, z) in the following sense:

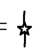
$$\mathfrak{p}_{\sigma(a,b,c)}(x, y, z) = \mathfrak{p}_{a,b,c}(\sigma^{-1}(y, x, z)).$$

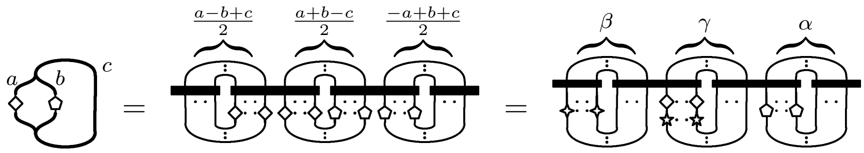
Proof. Define the following function $G \times G \times G \to \mathbb{C}$:

$\chi_{\alpha,\beta,\gamma}($  $) =$  ,

where the symmetrizer on the right is assumed to 'wrap around' to the one on the left (imagine this diagram being drawn on a cylinder). By construction this function is symmetric, in the sense that:

$\chi_{\sigma(\alpha,\beta,\gamma)}\left(\sigma\left(\right.\right.$  $\left.\left.\right)\right) = \chi_{\alpha,\beta,\gamma}\left(\right.$  $\left.\right)$.

For $\mathbf{x}_1 =$ , $\mathbf{x}_1^{-1} =$ , $\mathbf{x}_2 =$ , $\mathbf{x}_2^{-1} =$ , a central function $\chi^{a,b,c}(\mathbf{x}_1, \mathbf{x}_2)$ may be drawn as:

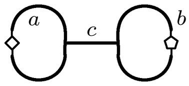


,

with the symmetrizers in the last two diagrams assumed to wrap around as before. Thus, $\mathfrak{p}_{a,b,c}(x, y, z) = \chi_{\alpha,\beta,\gamma}(\mathrm{tr}(\mathbf{x}_2), \mathrm{tr}(\mathbf{x}_1^{-1}), \mathrm{tr}(\mathbf{x}_1\mathbf{x}_2^{-1}))$ and so

$$\mathfrak{p}_{\sigma(a,b,c)}(x, y, z) = \chi_{\sigma(\alpha,\beta,\gamma)}(y, x, z) = \chi_{\alpha,\beta,\gamma}(\sigma^{-1}(y, x, z)) = \mathfrak{p}_{a,b,c}(\sigma^{-1}(y, x, z)). \quad \square$$

This symmetry was in some sense expected, given the initial definition of central functions as the basis for some space of homomorphisms from a surface Σ to G. In the rank two case, one surface under consideration is the *three-holed sphere*. If this is considered as the regular sphere with three equally spaced holes on a diameter, then the $\mathbb{Z}_3$ symmetry on the surface is clear. It is this symmetry that carries over to the central functions. This symmetry is also one reason why we prefer this basis of central functions to the alternate choice

$\Bigg\{$  $\Bigg\}_{c\in\lceil a,a\rfloor\cap\lceil b,b\rfloor}$.

The following table of central functions with parameters 1, 2, and 3 demonstrates how the symmetry of Theorem 6.6 works:

$\chi^{1,2,3} = xy^2 - \frac{2}{3}(yz+x)$	$\chi^{2,3,1} = yz^2 - \frac{2}{3}(xz+y)$	$\chi^{3,1,2} = x^2z - \frac{2}{3}(xy+z)$
$\chi^{3,2,1} = xz^2 - \frac{2}{3}(yz+x)$	$\chi^{1,3,2} = y^2z - \frac{2}{3}(xy+z)$	$\chi^{2,1,3} = x^2y - \frac{2}{3}(xz+y)$

Table 6.2: Example of Rank Two $\mathsf{SL}(2,\mathbb{C})$ Central Function Symmetry.

6.4 A Recurrence Relation for Rank Two

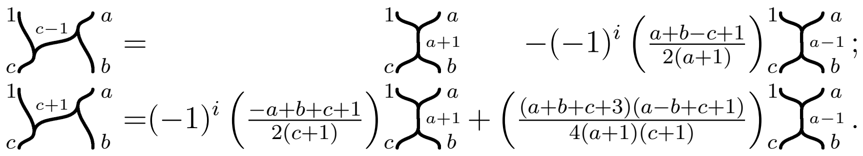
This section uses the explicit computation of four $6j$-symbols to give a recurrence relation for rank two central functions, similar to (6.2) for the rank one case.

Define the *rank* of a central function to be:

$$\delta = \operatorname{rank}(\chi^{a,b,c}) = \tfrac{1}{2}(a+b+c).$$

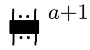
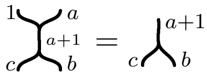
We will obtain a recurrence relation for an arbitrary central function $\chi^{a,b,c}$ by manipulating diagrams to express the product $\operatorname{tr}(\mathbf{x}_1)\cdot\chi^{a,b,c}(\mathbf{x}_1,\mathbf{x}_2)$ as a sum of central functions. This formula can be rearranged to write $\chi^{a,b,c}$ as a linear combination of central functions *with lower rank.* There are three main ingredients to the diagram manipulations: the *bubble* and *fusion* identities from Section 3.7, and the two *recoupling formulae* in the following lemma.

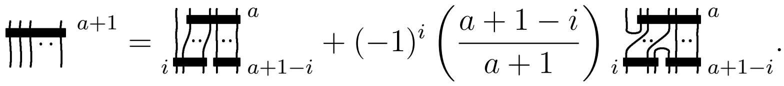
Lemma 6.7. *For* $i = \frac{1}{2}(a-b+c+1)$ *and appropriate triples admissible,*

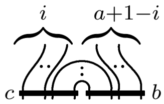


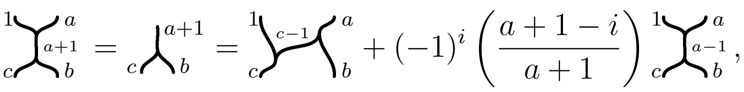
(6.5)

(6.6)

Proof. Given the formulae for the number of strands between two symmetrizers in Convention 3.19, i is the number of strands connecting  to  in . For (6.5), use $n = a+1$ and i in recurrence relation (3.9) to get:

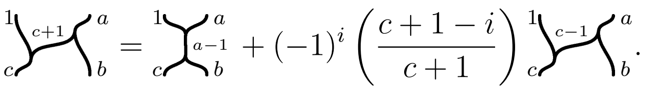


Compose this equation with  to get, via the *stacking relation*:

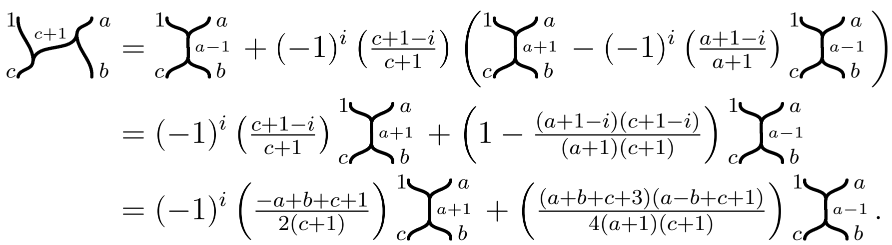


which is the desired result.

To prove (6.6), switch a and c in the previous relation and apply a "$\frac{\pi}{4}$-reflection" about the $1 \leftrightarrow b$ axis as in Proposition 3.22. Then i is unchanged and the equation becomes



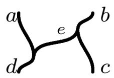
Rearrange this equation, and use (6.5) in its exact form to get:



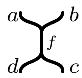
For the last step, since $a+1-i = \frac{1}{2}(a+b-c+1)$ and $c+1-i = \frac{1}{2}(-a+b+c+1)$, the numerator of the last term is

$$\begin{aligned} 4((a+1)(c+1) &- (a+1-i)(c+1-i)) \\ &= 4(a+1)(c+1) - ((b+1)+(c-a))((b+1)-(c-a)) \\ &= 4(a+1)(c+1) - (b+1)^2 + (a-c)^2 \\ &= ((a+1)-(c+1))^2 + 4(a+1)(c+1) - (b+1)^2 \\ &= ((a+1)+(c+1))^2 - (b+1)^2 \\ &= (a+1+c+1+b+1)(a+1+c+1-b-1) \\ &= (a+b+c+3)(a-b+c+1). \qquad \square \end{aligned}$$

Note that these are four coefficients in the general change-of-basis formula



$$= \sum_{f \in \lceil a,b \rfloor \cap \lceil c,d \rfloor} \left[\begin{smallmatrix} a & b & f \\ c & d & e \end{smallmatrix}\right]' \cdot$$



Up to sign, these are the same as the regular $6j$-symbols introduced in Definition 3.27. By Corollary 3.23,

$$\left[\begin{smallmatrix} a & b & f \\ c & d & e \end{smallmatrix}\right]' = (-1)^{\frac{1}{2}(b+d-e-f)} \left[\begin{smallmatrix} a & b & f \\ c & d & e \end{smallmatrix}\right].$$

Therefore the above lemma gives formulae for the following $6j$-symbols:

Corollary 6.8.

$$\left[\begin{smallmatrix} 1 & a & a+1 \\ b & c+1 & c \end{smallmatrix}\right] = 1; \qquad \left[\begin{smallmatrix} 1 & a & a-1 \\ b & c+1 & c \end{smallmatrix}\right] = (-1)^{\frac{1}{2}(a-b+c+2)} \tfrac{(a+b-c)}{2(a+1)};$$

$$\left[\begin{smallmatrix} 1 & a & a+1 \\ b & c-1 & c \end{smallmatrix}\right] = (-1)^{\frac{1}{2}(a-b+c+2)} \tfrac{(-a+b+c)}{2c}; \qquad \left[\begin{smallmatrix} 1 & a & a-1 \\ b & c-1 & c \end{smallmatrix}\right] = \tfrac{(a+b+c+2)(a-b+c)}{4(a+1)c}.$$

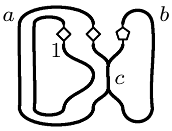
These coefficients are necessary in the proof of the following theorem:

Theorem 6.9. *When $a, c \geq 0$, the product $x \cdot \chi^{a,b,c}(x,y,z)$ is expressed in terms of central functions as*

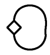
$$x \cdot \chi^{a,b,c} = \chi^{a+1,b,c+1} + \tfrac{(a+b-c)^2}{4a(a+1)} \chi^{a-1,b,c+1} \\ + \tfrac{(-a+b+c)^2}{4c(c+1)} \chi^{a+1,b,c-1} + \tfrac{(a+b+c+2)^2(a-b+c)^2}{16a(a+1)c(c+1)} \chi^{a-1,b,c-1}. \quad (6.7)$$

The equation holds for $a = 0$ or $c = 0$, provided the terms with a or c in the denominator are excluded.

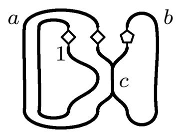
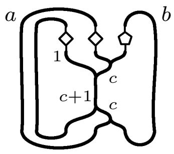
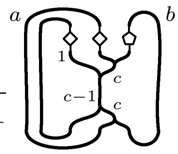
Proof. Diagrammatically, $x \cdot \chi^{a,b,c}(x,y,z)$ is represented by



,

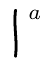
since $x = \mathrm{tr}(\mathbf{x}_1) =$  and multiplication is automatic on disjoint diagrams. Manipulate the diagram with the following three steps to obtain a sum over χ's.

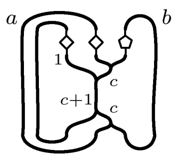
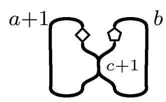
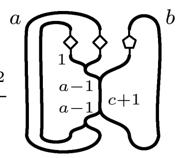
First, apply the fusion identity to connect the lone  strand to the $\chi^{a,b,c}$:

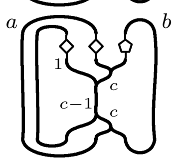
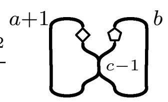
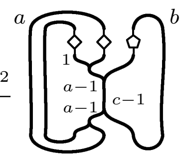
 $=$  $+ \dfrac{c}{c+1}$  , (6.8)

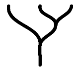
where the coefficients are evaluated from

$$\frac{\Delta(c \pm 1)}{\theta(1, c, c \pm 1)} = \frac{c \pm 1 + 1}{c + \frac{3}{2} \pm \frac{1}{2}}.$$

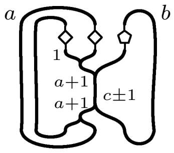
Second, use the $6j$-symbols computed in Corollary 6.8 above to move the  strand from one side of the diagram to the other:

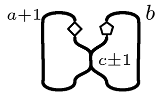
 $=$  $+ \frac{(a+b-c)^2}{4(a+1)^2}$  (6.9)

 $= \frac{(-a+b+c)^2}{4c^2}$  $+ \frac{(a+b+c+2)^2(a-b+c)^2}{16(a+1)^2c^2}$  . (6.10)

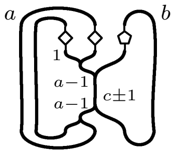
In each case, there are two recouplings: one for the top piece  and one for the corresponding bottom piece. As a consequence of Schur's Lemma, or the bubble identity, both recouplings must introduce the same coefficient $a \pm 1$.

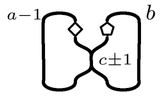
Finally, use the bubble identity to collapse the final pieces:



$= \left(\frac{\Theta(1,a,a+1)}{\Delta(a+1)}\right)$ 

$= \chi^{a+1,b,c\pm1};$



$= \left(\frac{\Theta(1,a,a-1)}{\Delta(a-1)}\right)$ 

$= \left(\frac{a+1}{a}\right) \chi^{a-1,b,c\pm1}.$

Multiply the coefficients obtained in the last few equations to obtain (6.7).

Now consider the special cases. If $a = 0$, then $b = c$ and $\frac{(-a+b+c)^2}{4c(c+1)} = \frac{c}{c+1}$, so the desired formula is exactly (6.8). Similarly, for $c = 0$, the desired formula is (6.9). □

Despite the fact that the diagrams used are not topologically invariant, this result is exactly that obtained by ignoring the signs introduced by kinks completely. In following the calculation, this is because all signs are eventually squared. As a second explanation independent of the proof, the final result is not influenced because all terms in the formula have the same number of ciliations modulo 4.

A consequence of this multiplication formula is

Corollary 6.10 (Central Function Recurrence)**.** *When $a, c > 0$, an arbitrary central function $\chi^{a,b,c}$ may be expressed*

$$\chi^{a,b,c} = x \cdot \chi^{a-1,b,c-1} - \frac{(a+b-c)^2}{4a(a-1)}\chi^{a-2,b,c} - \frac{(-a+b+c)^2}{4c(c-1)}\chi^{a,b,c-2} - \frac{(a+b+c)^2(a-b+c-2)^2}{16a(a-1)c(c-1)}\chi^{a-2,b,c-2}. \quad (6.11)$$

The relation still holds for $a = 1$ or $c = 1$, provided the terms with $a-1$ or $c-1$ in the denominator are excluded.

Proof. Rearrange and reindex the terms in (6.7) by replacing a with $a-1$ and c with $c-1$. The requirement $a, c \geq 0$ becomes $a, c > 0$, which is equivalent to requiring $\{a-1, b, c-1\}$ to be admissible. The special cases $a = 0$ or $c = 0$ become $a = 1$ or $c = 1$. □

Note that formulae for multiplication by y and z may be obtained by applying the symmetry relation of Theorem 6.6. This fact will be indispensable in the proof of Theorem 8.1.

6.5 Graded Structure for Rank Two

This section concerns the types of terms which occur in the central function basis. The majority of the content in this section was suggested by Carlos Florentino after reading an early draft of [LP].

Recall the $\alpha, \beta, \gamma, \delta$ notation used earlier, and the notation

$$\chi_{\alpha,\beta,\gamma}(y, x, z) = \chi^{a,b,c}(\mathbf{x}_1, \mathbf{x}_2)$$

introduced in the proof of Theorem 6.6. In these terms, recurrence (6.11) is

$$\chi_{\alpha,\beta,\gamma} = \chi_{0,1,0}\chi_{\alpha,\beta-1,\gamma} - \frac{\gamma^2}{a(a-1)}\chi_{\alpha+1,\beta-1,\gamma-1} - \frac{\alpha^2}{c(c-1)}\chi_{\alpha-1,\beta-1,\gamma+1} - \frac{\delta^2(\beta-2)^2}{a(a-1)c(c-1)}\chi_{\alpha,\beta-2,\gamma}.$$

Note that the symmetry theorem guarantees the interchangeability of (a, α) and (c, γ) here.

Proposition 6.11. *The polynomial $\chi^{a,b,c} = \chi_{\alpha,\beta,\gamma}$ is monic, with highest degree monomial $x^\beta y^\alpha z^\gamma$.*

Proof. Induct on the rank $\delta = \alpha + \beta + \gamma$ of central functions. The statement is clearly true for the base cases, since $\chi_{0,0,0} = 1, \chi_{0,1,0} = x, \chi_{1,0,0} = y$, and $\chi_{0,0,1} = z$.

The recurrence relation implies that the highest order term of $\chi_{\alpha,\beta,\gamma}$ is x times the highest order term of $\chi_{\alpha,\beta-1,\gamma}$, hence $x(x^{\beta-1}y^{\alpha}z^{\gamma}) = x^{\beta}y^{\alpha}z^{\gamma}$. This fact, together with the appropriate symmetric facts for y and z, completes the induction. $\square$

The basis also preserves a certain grading on $\mathbb{C}[x, y, z]$. To define this grading, partition the standard basis $\mathcal{B} = \{x^a y^b z^c\}_{a,b,c\in\mathbb{N}}$ of this space as follows. Let $f : \mathcal{B} \to \mathbb{Z}_2 \times \mathbb{Z}_2$ be defined by:

$$f(x^a y^b z^c) = (a + c, b + c) \mod 2.$$

Under multiplication, $\mathcal{B}$ is a semigroup and f is a homomorphism since

$$\begin{aligned} f(x^a y^b z^c) + f(x^{a'} y^{b'} z^{c'}) &\equiv_2 (a + c, b + c) + (a' + c', b' + c') \\ &\equiv_2 (a + a' + c + c', b + b' + c + c') \\ &\equiv_2 f(x^{a+a'} y^{b+b'} z^{c+c'}). \end{aligned}$$

Therefore, f defines a grading on this basis.

Proposition 6.12. *The basis $\{\chi^{a,b,c}\}$ respects the $\mathbb{Z}_2 \times \mathbb{Z}_2$ grading f on $\mathbb{C}[x, y, z]$ defined above, in the sense that*

$$\chi^{a,b,c} \in \mathrm{Span}(f^{-1}((a, b) \mod 2)).$$

Proof. This is another proof by induction on the rank. Clearly, $\chi^{0,0,0} = 1 \in f^{-1}(0, 0)$, and likewise $\chi^{1,0,1} = x \in f^{-1}(1, 0)$, $\chi^{0,1,1} = y \in f^{-1}(0, 1)$, and $\chi^{1,1,0} = z \in f^{-1}(1, 1)$. In the induction step, note that

$$(a, b) \equiv_2 (1, 0) + (a - 1, b) \equiv_2 (a - 2, b),$$

so all terms on the righthand side of the recurrence relation in Corollary 6.10 have the same grading. Thus $\chi^{a,b,c} \in f^{-1}(a, b)$. $\square$

This proposition means that central functions can be divided into four types corresponding to the decomposition

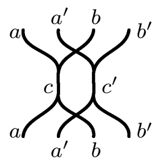
$$\{x^a y^b z^c\}_{a,b,c\in\mathbb{N}} = \{1, x, y, z\} \times \{(x^2)^i (y^2)^j (z^2)^k\}_{i,j,k\in\mathbb{N}}.$$

The four types correspond to the four choices in the first set $\{1, x, y, z\}$.

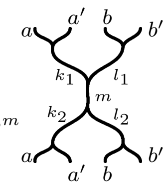
6.6 Multiplicative Structure for Rank Two

General $6j$-symbols and recoupling formulae may be used to write down a formula for the product of any two central functions. The following lemma encodes the most tedious diagram manipulations:

Lemma 6.13.



$$= \sum_{j_i,k_i,l_i,m} \mathrm{C}^{abc,a'b'c'}_{j_1k_1l_1,j_2k_2l_2,m}$$



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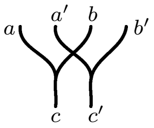
where the coefficients are given by the formula

$$\mathrm{C}^{abca'b'c'}_{j_1k_1l_1,j_2k_2l_2,m} = \frac{\Theta(c,c',m)}{\Delta(m)} \prod_{i=1,2} \frac{\Delta(j_i)}{\Theta(a',b,j_i)} \cdot \begin{bmatrix} a & a' & k_i \\ j_i & c & b \end{bmatrix} \begin{bmatrix} b' & b & l_i \\ j_i & c' & a' \end{bmatrix} \begin{bmatrix} k_i & l_i & m \\ c' & c & j_i \end{bmatrix}.$$

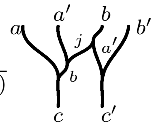
The following 15 triples are assumed to be admissible:

$$\{a', b, j_i\}, \{c, j_i, k_i\}, \{c', j_i, l_i\}, \{b, j_i, l_i\}, \{k_i, l_i, m\}, \{a, a', k_i\}, \{b, b', l_i\}, \{c, c', m\}.$$

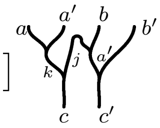
Proof. It suffices to demonstrate the diagram manipulation for the top half of the diagram, which by symmetry must be the same for the bottom half. Combining these two manipulations and applying a bubble identity will give the desired result. Signs will be watched closely throughout, but the admissible triples will be enumerated only after the manipulation.



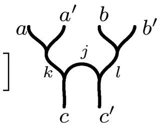
$$= \sum_{j} (-1)^{\frac{1}{2}(a'-b+j)} \frac{\Delta(j)}{\Theta(a',b,j)}$$



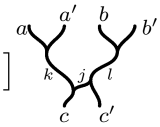
$$= \sum_{j,k} (-1)^{\frac{1}{2}(a'-b+j)+j} \frac{\Delta(j)}{\Theta(a',b,j)} \begin{bmatrix} a & a' & k \\ j & c & b \end{bmatrix}$$



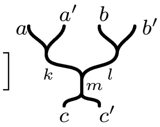
$$= \sum_{j,k,l} (-1)^{\frac{1}{2}(a'-b-j)} \frac{\Delta(j)}{\Theta(a',b,j)} \begin{bmatrix} a & a' & k \\ j & c & b \end{bmatrix} \begin{bmatrix} b' & b & l \\ j & c' & a' \end{bmatrix}$$



$$= \sum_{j,k,l} (-1)^{\frac{1}{2}(a'-b-j)+\frac{1}{2}(j+l-c')} \frac{\Delta(j)}{\Theta(a',b,j)} \begin{bmatrix} a & a' & k \\ j & c & b \end{bmatrix} \begin{bmatrix} b' & b & l \\ j & c' & a' \end{bmatrix}$$



$$= \sum_{j,k,l,m} (-1)^{\frac{1}{2}(a'-b+c-c'-j-m)+l} \frac{\Delta(j)}{\Theta(a',b,j)} \begin{bmatrix} a & a' & k \\ j & c & b \end{bmatrix} \begin{bmatrix} b' & b & l \\ j & c' & a' \end{bmatrix} \begin{bmatrix} k & l & m \\ c' & c & j \end{bmatrix}$$

 .

The (-1) terms all cancel in the end, a consequence of the fact that the following triples must be admissible:

$$\{a', b, j\}, \{c, j, k\}, \{c', j, l\}, \{b, j, l\}, \{k, l, m\}, \{a, a', k\}, \{b, b', l\}, \{c, c', m\}.$$

One computes the 13-parameter coefficients $\mathrm{C}^{abc,a'b'c'}_{j_1k_1l_1,j_2k_2l_2,m}$ by reflecting this result vertically, taking two sets of indices for the variables j, k, l, m on the two halves, and noting that the resulting bubble in the middle collapses with a factor of $\frac{\Theta(c,c',m)}{\Delta(m)}$ for $m = m_1 = m_2$. □

This lemma is used to write down the central function multiplication table. Note the symmetry with respect to k, l, m, which is guaranteed by Theorem 6.6.

Theorem 6.14 (Multiplication of $\mathsf{SL}(2, \mathbb{C})$ Rank Two Central Functions)**.** *The product of two central functions $\chi^{a,b,c}$ and $\chi^{a',b',c'}$ is*

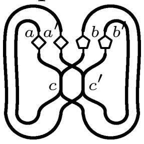
$$\chi^{a,b,c}\chi^{a',b',c'} = \sum_{j_1,j_2,k,l,m} \mathrm{C}_{j_1klm}\mathrm{C}_{j_2klm} \frac{\Theta(a,a',k)\Theta(b,b',l)\Theta(c,c',m)}{\Delta(k)\Delta(l)\Delta(m)} \chi^{k,l,m},$$

where the sum is taken over admissible triples

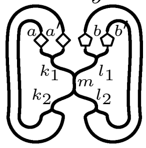
$$\{a, a', k\},\ \{b, b', l\},\ \{c, c', m\},\ \{a', b, j_i\},\ \{c, j_i, k\},\ \{c', j_i, l\},\ \{b, j_i, l\},\ \{k, l, m\}$$

and the coefficients are $\mathrm{C}_{j_iklm} = \frac{\Delta(j_i)}{\Theta(a', b, j_i)} \begin{bmatrix} a & a' & k \\ j_i & c & b \end{bmatrix} \begin{bmatrix} b' & b & l \\ j_i & c' & a' \end{bmatrix} \begin{bmatrix} k & l & m \\ c' & c & j_i \end{bmatrix}.$

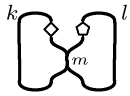
Proof. The previous lemma and the *bubble identity* imply



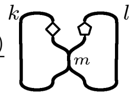
$$= \sum_{j_i, k_i, l_i, m} \mathrm{C}^{abc, a'b'c'}_{j_1k_1l_1, j_2k_2l_2, m}$$



$$= \sum_{j_i, k, l, m} \mathrm{C}^{abc, a'b'c'}_{j_1kl, j_2kl, m} \left(\frac{\Theta(a, a', k)\Theta(b, b', l)}{\Delta(k)\Delta(l)} \right)$$



$$= \sum_{j_i, k, l} \mathrm{C}_{j_1klm}\mathrm{C}_{j_2klm} \frac{\Theta(a, a', k)\Theta(b, b', l)\Theta(c, c', m)}{\Delta(k)\Delta(l)\Delta(m)}$$



. $\square$

6.7 Direct Formula for Rank Two

The computation of a direct formula for central functions is rather difficult. One step in this process is the expansion of an arbitrary polynomial in terms of central functions, thus computing the coefficients in

$$x^A y^B z^C = \sum_{k,l,m} C_{k,l,m} \chi^{k,l,m},$$

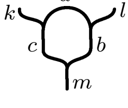
This will be done by first expressing x^A, y^B, and z^C in terms of $\chi^{a,0,a}$, $\chi^{0,b,b}$, and $\chi^{c,c,0}$, respectively, and then computing the product $\chi^{a,0,a}\chi^{0,b,b}\chi^{c,c,0}$.

Proposition 6.5 states that

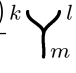
$$x^A = \sum_{r=0}^{\lfloor \frac{A}{2} \rfloor} \left[\binom{A}{r} - \binom{A}{r-1} \right] \chi^{A-2r}(x).$$

Therefore $\chi^{A-2r}(x) = \chi^{A-2r,0,A-2r}$. Since the formulae for y^B and z^C follow by symmetry, the first step is complete.

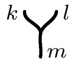
For the second step, recall the *triple bubble identity* (Proposition 3.30):



$$= \frac{\mathrm{Tet}(a, b, c, k, l, m)}{\Theta(k, l, m)}$$



$$= \frac{\Theta(a, c, k)}{\Delta(k)} \begin{bmatrix} a & c & k \\ l & m & b \end{bmatrix}$$



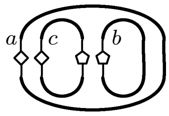
.

All that remains is to put these ingredients together to obtain the final formula. We will use the shorthand notation $\left(\!\!\binom{A}{r}\!\!\right)$ to indicate the difference $\binom{A}{r} - \binom{A}{r-1}$.

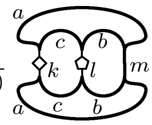
Theorem 6.15. *The term $x^A y^B z^C$ written in terms of central functions is:*

$$x^A y^B z^C = \sum_{\substack{r,s,t=0\\ k,l,m}}^{\lfloor\frac{A}{2}\rfloor,\lfloor\frac{B}{2}\rfloor,\lfloor\frac{C}{2}\rfloor} \left(\!\!\binom{A}{r}\!\!\right)\left(\!\!\binom{B}{s}\!\!\right)\left(\!\!\binom{C}{t}\!\!\right) \frac{\Delta(l)\Delta(m)\Theta(A-2r,C-2t,k)}{\Delta(k)\Theta(A-2r,B-2s,m)\Theta(B-2s,C-2t,l)} \begin{bmatrix} A-2r & C-2t & k \\ l & m & B-2s \end{bmatrix}^2 \chi^{k,l,m}.$$

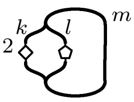
Proof. The product $\chi^{a,0,a}\chi^{0,b,b}\chi^{c,c,0}$ is computed by first fusing each pair of strands together and then applying the "triple bubble" identity in the previous lemma twice.



$$= \sum_{k,l,m} \frac{\Delta(k)\Delta(l)\Delta(m)}{\Theta(a,c,k)\Theta(b,c,l)\Theta(a,b,m)}$$



$$= \sum_{k,l,m} \frac{\Delta(k)\Delta(l)\Delta(m)}{\Theta(a,c,k)\Theta(b,c,l)\Theta(a,b,m)} Tet(a,b,c,k,l,m)^2$$



$$= \sum_{k,l,m} \frac{\Theta(a,c,k)\Delta(l)\Delta(m)}{\Delta(k)\Theta(b,c,l)\Theta(a,b,m)} \begin{bmatrix} a & c & k \\ l & m & b \end{bmatrix}^2 \chi^{k,l,m}.$$

Combine this with the expansion of the x^A, y^B, and z^C terms to complete the formula. □

It is a tedious but straightforward calculation to check that this is the same result obtained using the multiplication formula from Theorem 6.14. This formula may be inverted to obtain a direct formula for an arbitrary central function. The change-of-basis between central functions and the standard basis for $\mathbb{C}[x,y,z]$ is an (infinite) triangular matrix, so the only remaining step is to apply a formula for the inverse of a triangular matrix.

The following table lists, in order of increasing δ, several central functions which were computed with Mathematica using recurrence (6.11). Given the symmetry guaranteed by Theorem 6.6, only one function per triple of indices needs to be listed.

δ	$\chi^{a,b,c}$	$\chi_{\alpha,\beta,\gamma}$	Expansion for $x = \mathrm{tr}(\mathbf{x}_1), y = \mathrm{tr}(\mathbf{x}_2), z = \mathrm{tr}(\mathbf{x}_1\mathbf{x}_2^{-1})$
0	$\chi^{0,0,0}$	$\chi_{0,0,0}$	1
1	$\chi^{1,0,1}$	$\chi_{0,1,0}$	x
2	$\chi^{2,0,2}$	$\chi_{0,2,0}$	$x^2 - 1$
	$\chi^{1,1,2}$	$\chi_{1,1,0}$	$xy - \frac{1}{2}z$
3	$\chi^{3,0,3}$	$\chi_{0,3,0}$	$x^3 - 2x$
	$\chi^{2,1,3}$	$\chi_{1,2,0}$	$x^2y - \frac{2}{3}(xz + y)$
	$\chi^{2,2,2}$	$\chi_{1,1,1}$	$xyz - \frac{1}{2}(x^2 + y^2 + z^2) + 1$
4	$\chi^{4,0,4}$	$\chi_{0,4,0}$	$x^4 - 3x^2 + 1$
	$\chi^{3,1,4}$	$\chi_{1,3,0}$	$x^3y - \frac{3}{4}x^2z - \frac{1}{2}(3xy - z)$
	$\chi^{2,2,4}$	$\chi_{2,2,0}$	$x^2y^2 - xyz + \frac{1}{6}z^2 - \frac{1}{2}(x^2 + y^2) + \frac{1}{3}$
	$\chi^{3,2,3}$	$\chi_{1,2,1}$	$x^2yz - \frac{2}{3}(xz^2 + xy^2) - \frac{1}{2}x^3 - \frac{1}{9}(2yz - 13x)$

Table 6.3: Rank Two $\mathsf{SL}(2, \mathbb{C})$ Central Functions.

Chapter 7 **Central Functions for Other Groups**

The well-known Littlewood-Richardson Rule [Ful] describes how one may decompose the tensor product of two irreducible representations of $\mathsf{SU}(n)$, and is a necessary ingredient in the computation of arbitrary $\mathsf{SL}(n, \mathbb{C})$-central functions. This section describes how this rule is represented using spin networks. In dimensions 2 and 3, this gives a surprisingly simple description of the rule which also demonstrates its inherent symmetry. Much of the necessary background for this chapter is discussed in Chapter 2.

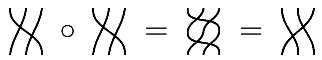
7.1 Diagrams for $\mathsf{SU}(n)$ Representations

It is first necessary to describe irreducible $\mathsf{SU}(n)$ representations diagrammatically. Beyond $\mathsf{SU}(2)$, these representations are usually described in terms of *Young projectors*, which are compositions of symmetrizers and anti-symmetrizers. The exposition that follows parallels [St]. Necessary background on Young projectors may be found in [FH, Ful].

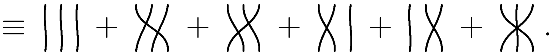
The symmetric group Σ_n is easily represented using diagrams. For example, the permutation $(1\,2\,3)$ in cycle notation, or $\left[\begin{smallmatrix}2&3&1\\1&2&3\end{smallmatrix}\right]$ in traditional notation, could just as easily be represented by the diagram

.

With this notation, the composition of permutations corresponds to the composition of two diagrams, so that

$$(1\,2\,3) \circ (1\,2\,3) =$$  $$= (1\,3\,2).$$

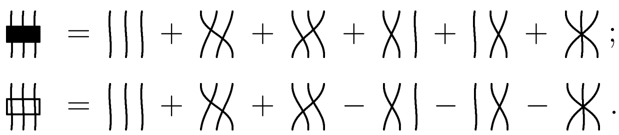
Likewise, sums of permutations in the group algebra $\mathbb{C}\Sigma_n$ can be represented by sums of diagrams. For example, the sum of all permutations on 3 elements is

$$a_{(3)} \equiv$$ 

The following notation will be used for the symmetrizer and anti-symmetrizer on Σ_n:

 $$^{n} \equiv \sum_{\sigma \in \Sigma_n} \sigma \equiv a_{(n)};$$  $$^{n} \equiv \sum_{\sigma \in \Sigma_n} \mathrm{sign}(\sigma) \cdot \sigma \equiv b_{(1,\ldots,1)}.$$

For example,



In contrast to previous chapters, the factor $\frac{1}{n!}$ is not included. This notation also varies slightly from [St].

Next, Young tableau and projectors are used to describe arbitrary representations of Σ_n.

Definition 7.1. Let an arbitrary partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ of $n \in \mathbb{N}$ be given, with conjugate partition $\mu = (\mu_1, \ldots, \mu_l)$. Then the *Young diagram* of λ is the diagram consisting of μ_1 rows of boxes, with λ_i boxes in the ith row and μ_j in the jth column. A *Young tableau* is an assignment of $\{1, 2, \ldots, n\}$ to the boxes in a Young diagram in such a way that the entries in each row and column are increasing.

There are two standard numbering schemes for any Young diagram: the *standard row numbering* counts left to right, then top to bottom, while the *standard column numbering* counts top to bottom, then left to right. For a fixed Young tableau, define $\sigma_a \in \Sigma_n$ to be the permutation taking the Young tableau to the standard row numbering, and define $\sigma_b \in \Sigma_n$ to be the permutation taking the Young tableau to the standard column numbering.

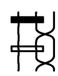
Definition 7.2. The *general symmetrizer* c_λ corresponding to an arbitrary partition λ and Young tableau is given by $c_\lambda = a_\lambda \cdot b_\lambda$, where a_λ is a "product" of symmetrizers and b_λ a "product" of anti-symmetrizers:

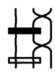
$$a_\lambda = \sigma_a^{-1} \circ \left({}^{\lambda_1} \; {}^{\lambda_2} \cdots {}^{\lambda_k} \right) \circ \sigma_a;$$
$$b_\lambda = \sigma_b^{-1} \circ \left({}^{\mu_1} \; {}^{\mu_2} \cdots {}^{\mu_l} \right) \circ \sigma_b;$$
$$c_\lambda = a_\lambda \cdot b_\lambda = \sigma_a^{-1} \circ \left({}^{\lambda_1} \; {}^{\lambda_2} \cdots {}^{\lambda_k} \right) \circ \sigma_a \sigma_b^{-1} \circ \left({}^{\mu_1} \; {}^{\mu_2} \cdots {}^{\mu_l} \right) \circ \sigma_b.$$

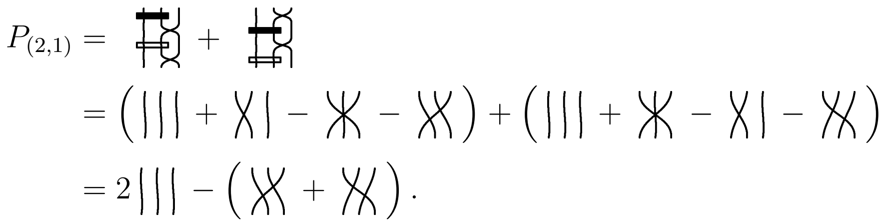
Each strand corresponds to a copy of V, the standard representation. The order of the strands is given by the Young tableau numbering. Thus, a_λ permutes the boxes labelled $\{1, \dots, \lambda_1\}$, those labelled $\{\lambda_1 + 1, \dots, \lambda_1 + \lambda_2\}$, and so on.

Definition 7.3. The *Young projector* P_λ corresponding to a given partition of n is the sum of c_λ over all possible Young tableau.

For example, the Young tableau $\begin{smallmatrix} 1 & 2 \\ 3 & \end{smallmatrix}$ contains the standard row numbering. Therefore, $\sigma_a = (1)$, $\sigma_b = (2\,3)$, and

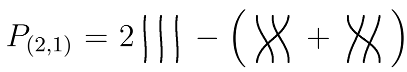
$$c_{12,3} = (1) \circ \quad \circ (1) \circ (2\,3)^{-1} \circ \quad \circ \quad \circ (2\,3) =$$  .

Likewise, the diagram for $\begin{smallmatrix} 1 & 3 \\ 2 & \end{smallmatrix}$ is $c_{13,2} =$ . Together these form the Young projector



This is a map $V^{\otimes 3} \to V^{\otimes 3}$ whose image is isomorphic to the irreducible representation $V_{(2,1)}$ of Σ_3. The coefficients in this formula could also be read off from the character table for Σ_3:

	(1)	(12)	(123)
(3)	1	1	1
(2,1)	2	0	-1
(1,1,1)	1	-1	1

The coefficients in the equation  are the entries in the row $(2,1)$ for the corresponding conjugacy class! This actually works for all Σ_n-representations [St].

Since the finite-dimensional irreducible representations of $\mathsf{SU}(n)$ are all realized as representations of Σ_d for some $d \in \mathbb{N}$ (Chapter 2), the Young projector is sufficient to describe the representations in diagrammatic form. These diagrams also satisfy idempotence and orthogonality conditions [St]. However, all that is needed to proceed is the understanding of how to write down Young symmetrizers and Young projectors.

7.2 The Littlewood-Richardson Rule

For reductive groups, every finite-dimensional representation may be decomposed into irreducibles. Hence, every tensor product may also be decomposed:

$$V_a \otimes V_b = \bigoplus_{c \in \Diamond[a,b]} V_c,$$

where $\Diamond[a,b]$ represents the set of all possible factors of $V_a \otimes V_b$. The simplest example is $\mathsf{SU}(2)$, for which the irreducibles are the symmetric powers $V_a = \mathrm{Sym}^a V$ and

$$V_a \otimes V_b = V_{a+b} \oplus V_{a+b-2} \oplus \cdots \oplus V_{|a-b|}.$$

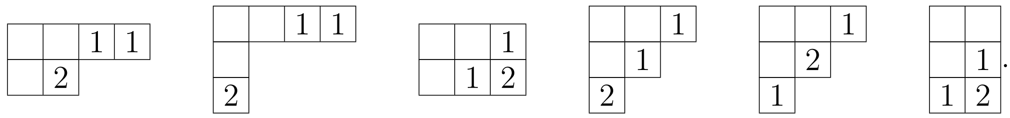
For $\mathsf{SU}(n)$, this decomposition is determined by the following rule [FH, Ful]:

Proposition 7.4 (Littlewood-Richardson Rule). *Given representations V_λ and V_μ of $\mathsf{SU}(n)$, there is a one-to-one correspondence between the irreducible factors of $V_\lambda \otimes V_\mu$ and the* strict μ-expansions *of the partition λ with n rows or less. Each expansion corresponds to an irreducible component V_ν, where ν is the partition formed from the μ-expansion by removing the columns with n boxes.*

A *strict μ-expansion* for a given $\mu = (\mu_1, \ldots, \mu_l)$ is an addition of μ_1 boxes labelled with a 1, μ_2 boxes labelled with a 2, and so on to the Young diagram for the partition λ in such a way that (i) the sequence of numbered boxes in any column is *strictly* increasing, hence no two of the same number are in the same column; (ii) in the sequence formed by reading off the numbered boxes from right to left along the top row, and then right to left along subsequent rows, one never has more i boxes than j boxes if $i > j$.

This is not an easy rule to state, even in low dimensions. Perhaps it would be better to give an example. For $\mathsf{SU}(3)$, a representation $V_{a,b}$ corresponds to the

Young diagram with $a+b$ boxes in the first row and b boxes in the second row, that is the partition $(a+b, b)$ of $a+2b$. How can one decompose the product $V_{1,1} \otimes V_{1,1}$? Take $\lambda = \mu = (2,1)$, so that the strict μ-expansions with 3 or fewer rows are:



Note the three given rules for such expansion: there are no more than 3 rows in any diagram, no number is repeated in any column, and the 1 boxes are all 'above' the 2 boxes. This gives the tensor decomposition (formed by removing the columns with three boxes):

$$V_{1,1} \otimes V_{1,1} = V_{2,2} \oplus V_{3,0} \oplus V_{0,3} \oplus V_{1,1} \oplus V_{1,1} \oplus V_{0,0}.$$

As this example shows, there may be more than one injection

$$V_{a,b} \hookrightarrow V_{c,d} \otimes V_{e,f}.$$

This was not possible for irreducible $\mathsf{SU}(2)$ representations.

The remainder of this section describes the Littlewood-Richardson rule in terms of diagrams for dimensions two and three. The key will be that for every component $c \in \diamondsuit[a,b]$, there is up to scalar multiples a unique surjective map $V_a \otimes V_b \twoheadrightarrow V_c$. Rather than finding all possible values of c, we will find all possible projections of $V_a \otimes V_b$ onto irreducible components. It turns out that diagrams work well for representing these projections, and are especially suited to demonstrating their inherent symmetry. For the most part, each column of a strict μ-expansion corresponds to a strand of a diagram for the given projection, and therefore *determining the types of possible columns will determine the types of diagrams.*

7.3 SU(2) Admissibility Condition

In dimension two, the irreducible representations are $V_a = \mathrm{Sym}^a V$ for integers $a \geq 0$, where V is the standard representation. These correspond to trivial partitions (a) and are represented by symmetrizers

$$P_{(a)} \leftrightarrow \quad {}^{a}.$$

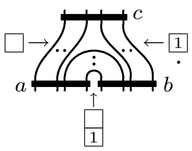
Using the Littlewood-Richardson rule, the decomposition of $V_a \otimes V_b$ corresponds to adding b boxes labelled with $\boxed{1}$ to a blank boxes $\square$. In a strict μ expansion, three types of columns may occur:

$$\begin{array}{|c|}\hline \\ \hline 1 \\ \hline\end{array} \qquad \square \qquad \boxed{1}$$

Thus, if i boxes are added to the first row, there are i $\boxed{1}$ columns, $b-i$ $\begin{array}{|c|}\hline \\ \hline 1 \\ \hline\end{array}$ columns, and $a-b+i$ $\square$ columns, giving the representation V_{a-b+2i}. Note that $b-i \leq a$, since there can be no more than a boxes labelled 1 on the second row. Thus, i can take any value between b (everything added to the first row) and $a - \min(a,b)$ (as much as possible added to the second row), which implies the usual decomposition $V_a \otimes V_b = \oplus_{c \in \lceil a,b \rfloor} V_c$.

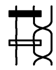
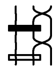
Diagrammatically, the projection $V_a \otimes V_b \to V_c$ is represented by a collection of edges connecting the symmetrizer ^{c} to a pair of symmetrizers ^{a} ^{b}. In the strict expansion, there are $\alpha = \frac{1}{2}(-a+b+c)$ $\boxed{1}$ columns, $\beta = \frac{1}{2}(a-b+c)$ $\square$ columns, and $\gamma = \frac{1}{2}(a+b-c)$ $\begin{array}{|c|}\hline \\ \hline 1 \\ \hline\end{array}$ columns. Hence, the usual *admissibility condition* is the simple fact that there are a nonnegative number of each column type.

Alternately, there is a one-to-one correspondence between columns and connecting strands in the diagram



7.4 SU(3) Admissibility Condition

Represent the irreducible representations V_λ of $\mathsf{SU}(3)$ diagrammatically as indicated earlier. Then, the diagram

$$P_{(2,1)} = $$  $$+$$ 

may be viewed as either a surjective map $V^{\otimes 3} \to V_{1,1}$ or as an injective map $V_{1,1} \to V^{\otimes 3}$. We will denote this projector by a single black bar labelled with two numbers:

 $$^{a,b} \equiv P_{a,b} : V^{\otimes a+2b} \to V^{\otimes a+2b}.$$

In a strict μ-expansion for $\mathsf{SU}(3)$ representations, there may be several types of columns. The following proposition gives an algorithm for partitioning these columns into well-defined sets.

Proposition 7.5. *Let a strict μ-expansion of the partition λ be given. Remove the columns of the resulting diagram in the following order:*

$$\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\quad \begin{array}{|c|}\hline \\\hline 1\\\hline 2\\\hline\end{array}\quad \begin{array}{|c|c|}\hline & 1\\\hline \multicolumn{1}{|c|}{} \\\cline{1-1} \multicolumn{1}{|c|}{2} \\\cline{1-1}\end{array}\quad \begin{array}{|c|}\hline 1\\\hline\end{array}\quad \begin{array}{|c|}\hline \\\hline 1\\\hline\end{array}\quad \begin{array}{|c|}\hline \\\hline \\\hline 1\\\hline\end{array}\quad \begin{array}{|c|}\hline \\\hline\end{array}\quad \begin{array}{|c|}\hline \\\hline \\\hline\end{array}.$$

Thus, all $\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}$ *columns are removed first, then all* $\begin{array}{|c|}\hline \\\hline 1\\\hline 2\\\hline\end{array}$ *columns, and so on. The diagram obtained at each step is a strict expansion, although for different μ and λ, and the process terminates with an empty diagram. Moreover, only one of the two column types* $\begin{array}{|c|c|}\hline & 1\\\hline \multicolumn{1}{|c|}{} \\\cline{1-1} \multicolumn{1}{|c|}{2} \\\cline{1-1}\end{array}$ *and* $\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}$ *will be used.*

Proof. To verify that each step gives a strict expansion, it must be shown that (i) the numbers in each column are still strictly increasing, and (ii) when sequenced in the appropriate manner (right to left, top to bottom), there will never be more $\boxed{2}$'s than $\boxed{1}$'s.

For (ii), note that whenever a $\boxed{2}$ is removed, it can be regarded as the *first* $\boxed{2}$ in the sequence of numbers. Since it is always removed with a $\boxed{1}$, its removal does not influence the strictness requirement. Once all $\boxed{2}$'s are removed, the condition (ii) is trivially satisfied.

The requirement (i) will be verified for each column type. For $\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}$, note that any $\boxed{2}$ in the second row must have a corresponding $\boxed{1}$ in the first row by the strictness condition. Assume this $\boxed{1}$ is the leftmost in the first row, and swap it with the box in $\boxed{2}$'s column. Then, remove this entire column. After this step, the first row will still have all $\boxed{}$ boxes to the left of $\boxed{1}$ boxes. The only column which remains changed is that which received a $\boxed{}$ in place of $\boxed{1}$, hence must still satisfy (i).

After the removal of all possible $\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}$, the remaining $\boxed{2}$ must be in the third row. By condition (ii), each $\boxed{2}$ must have a corresponding $\boxed{1}$ in either the first or second row. Remove box sets $\begin{array}{|c|}\hline \\ \hline 1 \\ \hline 2 \\ \hline \end{array}$ until no more are available, and then remove $\begin{array}{|c|c|}\hline & 1 \\ \hline \\ \cline{1-1} 2 \\ \cline{1-1} \end{array}$. During this process:

- There will always be $\boxed{}$ boxes when needed, since each $\boxed{1}$ in the second row is directly below a $\boxed{}$ in the first row.

- If there are no more $\boxed{1}$ in the second row, there must be enough in the first row to remove the rest of the $\boxed{2}$ boxes. But then each $\boxed{2}$ in the third row must have two empty boxes above it (otherwise it could not have been placed in the third row), allowing them to be removed as $\begin{array}{|c|c|}\hline & 1 \\ \hline \\ \cline{1-1} 2 \\ \cline{1-1} \end{array}$. This removal process ends only when there are no more $\boxed{2}$.

- At each step, the diagram may be reorganized so that an entire column is being removed, or two entire columns in the case of $\ytableaushort{{}1,{},2}$. This is only a problem if a number is inserted in the middle of a row of $\ytableaushort{{}}$'s, and this can be prevented by choosing the leftmost number in the required row.

When removing $\ytableaushort{{},1,2}$, if $\ytableaushort{2}$ run out before $\ytableaushort{1}$, then the remaining $\ytableaushort{1}$ boxes may occur in any row. If $\ytableaushort{1}$ runs out first, then the remaining $\ytableaushort{1}$ may occur anywhere but the second row, so that $\ytableaushort{{},1}$ will never be removed. Therefore, the process will not remove both $\ytableaushort{{}1,{},2}$ and $\ytableaushort{{},1}$.

At this point, all that remains are $\ytableaushort{{}}$ and $\ytableaushort{1}$ boxes, so each column corresponds uniquely to one of those above. Removal in the order above ensures that the columns are removed one by one. $\square$

Stated in other words, this proposition says that every strict μ-expansion can be written as a sum of sets of boxes of these forms. Here is an example:

$$\ytableaushort{{}{}{}{}111,{}{}112,2} = \ytableaushort{1,2} + \ytableaushort{{},1,2} + \ytableaushort{1} + \ytableaushort{1} + \ytableaushort{{},1} + \ytableaushort{{},{}} + \ytableaushort{{},{}} .$$

The next proposition concerns the uniqueness of this decomposition:

Proposition 7.6. *The decomposition of an arbitrary strict expansion into eight sets of "columns" as indicated above is unique up to the relation*

$$\ytableaushort{{}1,{},2} + \ytableaushort{{},1} \sim \ytableaushort{{},1,2} + \ytableaushort{{},{}} + \ytableaushort{1} .$$

Proof. Relations cannot exist without $\ytableaushort{2}$, since the columns are unique with respect to $\ytableaushort{{}}$ and $\ytableaushort{1}$. Moreover, a relation cannot include $\ytableaushort{1,2}$, since these are the only columns removed with $\ytableaushort{2}$ in the second row. Any relation which includes one of $\ytableaushort{{}1,{},2}$ and $\ytableaushort{{},1,2}$ must also include the other. Once these are in place, there is only one way to complete the relation. $\square$

This proposition is sufficient to describe the $\mathsf{SU}(3)$ admissibility condition, and will be used to determine the diagrammatic form of the Littlewood-Richardson Rule in this case.

In addition to the permutation maps, the $\mathsf{SU}(3)$ case requires the following component maps, which were first described in Chapter 4.

$: V^{\otimes 2} \to \mathbb{C}$ with $v \otimes w \mapsto v^* w$ (the *inner product*);

$: \mathbb{C} \to V^{\otimes 3}$ with $1 \mapsto e_1 \otimes e_2 \otimes e_3 + e_2 \otimes e_3 \otimes e_1 + e_3 \otimes e_1 \otimes e_2$
$\qquad - e_2 \otimes e_1 \otimes e_3 - e_1 \otimes e_3 \otimes e_2 - e_3 \otimes e_2 \otimes e_1;$

$: V^{\otimes 3} \to \mathbb{C}$ with $u \otimes v \otimes w \mapsto \det[u\, v\, w];$

$: V \to V^{\otimes 2}$ with $e_i \mapsto e_{i+1} \otimes e_{i+2} - e_{i+2} \otimes e_{i+1};$

$: V^{\otimes 2} \to V$ with $e_i \otimes e_i \mapsto 0, \quad e_i \otimes e_{i+1} \mapsto e_{i+2}, \quad e_{i+1} \otimes e_i \mapsto -e_{i+2}.$

In the last two cases, the indices are considered modulo 3. Compositions of these maps are anti-symmetrizers:

$$= \circ = \quad \text{and} \quad = \circ = .$$

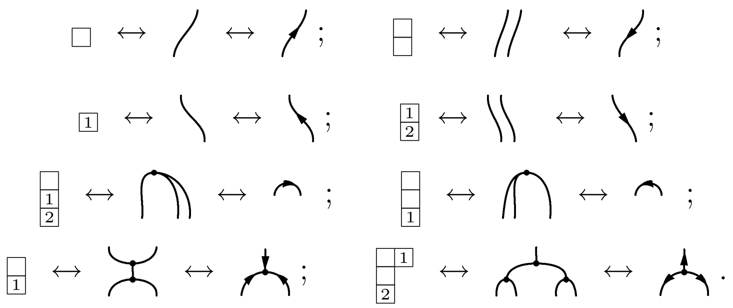
As in the $\mathsf{SU}(2)$ case, each of the columns given in Proposition 7.5 corresponds to a specific way to connect three projectors. A specific projection $V_\lambda \otimes V_\mu \to V_\nu$ is represented in trivalent spin network form by $\lambda\ \mu\ \nu$, with an appropriate intertwiner labelling the vertex. It is formed using the following connections:

- $\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}$ connects 2 strands between V_μ and V_ν using ;
- $\begin{array}{|c|}\hline \ \\\hline 1\\\hline 2\\\hline\end{array}$ connects 1 strand of V_λ to 2 strands of V_μ using ;
- $\begin{array}{|c|c|}\hline \ & 1\\\hline \ \\\cline{1-1} 2\\\cline{1-1}\end{array}$ connects 2 strands of V_λ, 2 strands of V_μ, and a strand of V_ν using ;

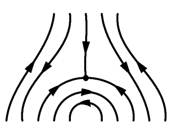
- $\boxed{1}$ connects a stand of V_μ to a strand of V_ν using ;
- $\begin{array}{|c|}\hline \\ \hline 1 \\ \hline\end{array}$ connects a strand of V_λ, a strand of V_μ, and two strands of V_ν using ;
- $\begin{array}{|c|}\hline \\ \hline \\ \hline 1 \\ \hline\end{array}$ connects two strands of V_λ with a strand of V_μ using ;
- $\boxed{}$ connects a strand of V_λ to V_ν using ;
- $\begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}$ connects two strands of V_λ to V_ν using .

This depiction forgoes the orientation originally assigned to 3-spin networks. If this orientation is reintroduced, then two strands in the same column may be identified with a single down strand

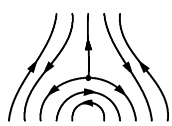
Essentially, this represents an isomorphism from V^* to the image of in $V^{\otimes 2}$. This adds much more symmetry to the diagrams above, for up to a constant the following diagrams are correlated:



Given the mutual exclusivity of and , all strand types possible for a single projector can be represented on a single diagram *without crossings*, if multiples of the same types are permitted to be placed atop one another:



or



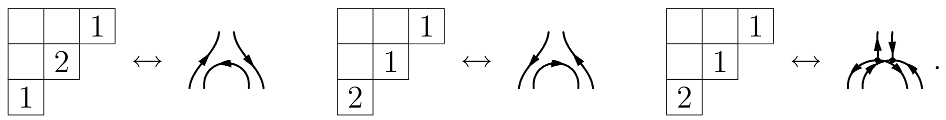
.

An advantage to this oriented approach is that the representations are easier to read off. For example, if there are four up and three down arrows at the top of a diagram, then the corresponding representation is $V_{4,3}$.

Admissibility and Multiplicity for $\mathsf{SU}(3)$

This description of the possible projections $V_\lambda \otimes V_\mu \to V_\nu$ gives a much clearer picture than the Littlewood-Richardson Rule as normally stated, especially with regard to symmetry. It can also be used as a starting point for determining if $\nu \in \Diamond[\lambda, \mu]$, and more generally, how many ways ν can occur as a strict μ-expansion of λ.

To begin, notice that the following three diagrams all have the same endpoints: they connect three $V_{1,1}$'s:



Since only the above eight diagrams may be used for a projector, the types of endpoints (up and down arrows) of V_λ, V_μ, and V_ν determine the diagram uniquely up to interchange of these three diagrams. Since the third case is not allowed under the algorithm in the previous section, *all multiplicities in the Littlewood-Richardson Rule for* $\mathsf{SU}(3)$ *arise from the interchangeability of the 'cycles'*  *and*  .

The admissibility condition for $\mathsf{SU}(3)$ may now be stated. It is only necessary to determine whether three irreducible representations can be connected by the above diagrams, and if they can, how many possible diagrams there are. This will give both *admissibility* and *multiplicity*.

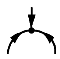
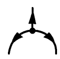
Theorem 7.7 ($\mathsf{SU}(3)$ Admissibility Condition)**.** *The multiplicity of the projection*

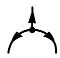
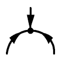
$V_{a_1,b_1} \otimes V_{a_2,b_2} \twoheadrightarrow V_{b_3,a_3}$ *(note the order of a and b on the last piece is switched) is*

$$M = 1 + \min\{a'_i, b'_i, \gamma_i\}_{i=1}^{3},$$

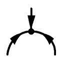
where $a'_i = a_i - \frac{1}{6}(|N| + N)$*,* $b'_i = b_i - \frac{1}{6}(|N| - N)$*, and* $\gamma_i = a'_{i+1} + a'_{i+2} - a'_i$*, with the indices in the last equation considered mod 3.*

Proof. This proof amounts to counting the number of possible *oriented* diagrams which can be used to connect the three representations. Starting with the assumption that such a diagram is possible, sets of strands are "removed" until the empty diagram is left. This process determines the strands which existed in the original diagram.

Define $N = (a_1 + a_2 + a_3) - (b_1 + b_2 + b_3)$. In order to be admissible, this must be in $3\mathbb{Z}$ since, of the eight diagram types,  and  contribute +3 and -3 to this number, while the rest contribute 0. These two diagrams are mutually exclusive according to the above algorithm, so $\mathrm{sign}(N)$ determines the type of diagram which appears and $|\frac{1}{3}N|$ the number of such diagrams.

In particular, if $N < 0$, then $-\frac{1}{3}N$ is the number of  which appear, while if $N > 0$, $\frac{1}{3}N$ is the number of  which appear. The primed constants $\{a'_i, b'_i\}_{i=1}^{3}$ are defined to be the number of each type of endpoint remaining after these diagram triples have been removed, hence:

$N < 0 \implies$ remove  $\implies a'_i = a_i, \quad b'_i = b_i + \frac{1}{3}N;$

$N > 0 \implies$ remove  $\implies a'_i = a_i - \frac{1}{3}N, \quad b'_i = b_i;$

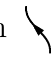
$N = 0 \implies$ none removed $\implies a'_i = a_i, \quad b'_i = b_i.$

Given that

$$\frac{1}{2}(|N| + N) = \begin{cases} |N| & N \geq 0; \\ 0 & N \leq 0, \end{cases} \qquad \text{and} \qquad \frac{1}{2}(|N| - N) = \begin{cases} 0 & N \geq 0; \\ |N| & N \leq 0, \end{cases}$$

these integers may be expressed directly as

$$a'_i = a_i - \tfrac{1}{6}(|N| + N), \qquad b'_i = b_i - \tfrac{1}{6}(|N| - N).$$

Let γ_3 be the number of strands of the form  or  , γ_2 the number of the form  or  and γ_1 the number of the form  or  . Because all strands with vertices were removed,

$$N' = (a'_1 + a'_2 + a'_3) - (b'_1 + b'_2 + b'_3) = 0.$$

It follows that

$$\gamma_3 = \tfrac{1}{2}((a'_1 + b'_1) + (a'_2 + b'_2) - (a'_3 + b'_3)) = a'_1 + a'_2 - a'_3 = b'_1 + b'_2 - b'_3.$$

Similar formulae hold for γ_2 and γ_1. Since $\{\gamma_i\}$ represent "physical" quantities, they must be nonnegative, so admissibility requires

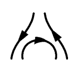
$$\gamma_i = a'_{i+1} + a'_{i+2} - a'_i = b'_{i+1} + b'_{i+2} - b'_i,$$

where the indices are considered mod 3.

It is clear at this point where the multiplicity arises, since the numbers $\{\gamma_i\}$ do not determine the numbers of strands of the types

uniquely. To determine the extent of non-uniqueness, define

$$M = 1 + \min\{a'_i, b'_i, \gamma_i\}_{i=1}^{3}.$$

The number $M - 1$ is exactly the number of *cycles* of the form  or  which occur in the diagram. In particular, for any $m \in \{0, \ldots, (M-1)\}$, a diagram using m  and $(M-1) - m$  may be constructed.

It remains to show that the remainder of the diagram may be completed once the *triple points* and *cycles* have been removed. Define

$$a_i'' = a_i' - (M-1), \qquad b_i'' = b_i' - (M_1), \qquad \gamma_i' = \gamma_i - (M_1)$$

to be the values of these constants after the cycles are removed. One of these constants must now be zero. Up to symmetry, there are two cases to check: $a_i'' = 0$ or $b_i'' = 0$, and $\gamma_i' = 0$. For the first, suppose $a_1'' = 0$. Then the number of each strand type is:

$$0 \;, \quad \gamma_3' \;, \quad 0 \;, \quad \gamma_2' \;, \quad b_3'' \;, \quad b_2'' \;.$$

On the other hand, if say $\gamma_3 = 0$, then the numbers are:

$$0 \;, \quad 0 \;, \quad a_1'' \;, \quad b_1'' \;, \quad a_2'' \;, \quad b_2'' \;.$$

The other cases may be handled similarly. Hence, the diagram may be completed, and the number of values for m is the *multiplicity*, which is M. □

The symmetry of the above condition is given by

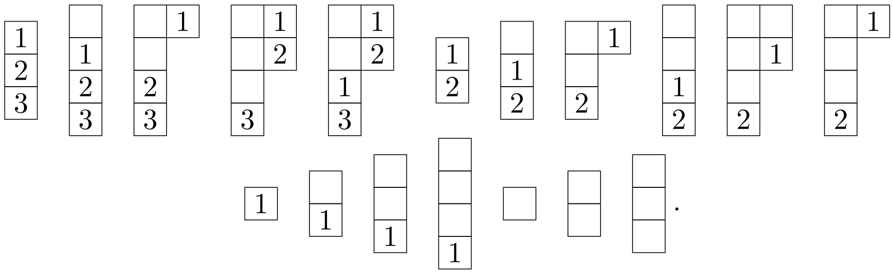
Proposition 7.8. *The* $\mathsf{SU}(3)$ *admissibility/multiplicity condition is symmetric in the following sense. If the multiplicity* M *of the projection* $V_{a_1,a_2} \otimes V_{b_1,b_2} \twoheadrightarrow V_{c_1,c_2}$ *is written as a function of a* 2×3 *matrix* D *with*

$$M\left(\left[\begin{smallmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{smallmatrix}\right]\right) = M(D),$$

then $M(D) = M(D')$ *whenever* D' *is formed from* D *by permuting its rows or columns.*

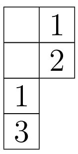
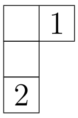
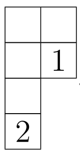
7.5 SU(4) and Beyond

The key ideas used in the previous sections can be extended to higher dimensions, although it is more difficult to describe the diagrams. However, there are some new phenomena which arise starting with $\mathsf{SU}(4)$. In this case, the columns of a strict expansion can be ordered in the following way:



It is not hard to see how this ordering works for general $\mathsf{SU}(n)$: start from n and work down to 1. For each number, remove it first from the highest (closest to the top) possible row and proceed to the lowest possible row. Give the cases for the higher numbered, or empty, boxes running out first.

However, this case breaks the pattern of the previous two somewhat. In each of those cases, there were $n^2 - 1$ types of columns corresponding to all the ways to combine columns of different lengths for V_a and V_b. In this case, there are *eighteen* columns in total. This new phenomenon occurs with the following three columns:

 and  and .

We are unsure of the implications of these “extra” columns. They are admissible under the Littlewood-Richardson Rule and cannot be decomposed in terms of the other types of columns. They represent the $\mathsf{SU}(4)$ projections

$$V_{0,1,0} \otimes V_{1,0,1} \twoheadrightarrow V_{0,1,0} \quad \text{and} \quad V_{0,1,0} \otimes V_{0,1,0} \twoheadrightarrow V_{1,0,1} \quad \text{and} \quad V_{1,0,1} \otimes V_{0,1,0} \twoheadrightarrow V_{0,1,0}.$$

Their existence will probably add another redundancy to diagram fillings, and therefore another ingredient to the computation of multiplicity.

7.6 Group Properties and Diagrammatics

It is possible that the possible strand types, or column types, which occur in the Littlewood-Richardson Rule relate somehow to the dimension of the coordinate ring $\mathbb{C}[\mathcal{X}]$ on the character variety. When $G = \mathsf{SL}(2, \mathbb{C})$, for example, the rank two coordinate ring $\mathbb{C}[G \times G]^G$ is three-dimensional, and there are three possible strand types under the Littlewood-Richardson Rule. Likewise, for $G = \mathsf{SL}(3, \mathbb{C})$, there are eight types of strands and the local dimension of $\mathbb{C}[G \times G]^G$ is eight. This correspondence is clear for $\mathsf{SL}(2, \mathbb{C})$, but it remains to be seen whether it is just a coincidence for $\mathsf{SL}(3, \mathbb{C})$. If it is not a coincidence, how does this correspondence generalize to $\mathsf{SL}(n, \mathbb{C})$?

Chapter 8 **Concluding Remarks**

This chapter describes the application of spin networks to the Fricke-Klein-Vogt Theorem, and also offers some speculation on other possible applications of spin networks. The final section concerns the next step for tackling the main problem of this thesis, as described in Chapter 5.

8.1 The Fricke-Klein-Vogt Theorem and Geometry

Spin networks can be used to prove a classical theorem of Fricke, Klein, and Vogt [FK, Vo]. While there are many proofs of this theorem, we provide a direct constructive proof. The machinery used may seem excessive, but it could offer a means of extending the theorem to more general groups and higher rank cases.

Theorem 8.1 (Fricke-Klein-Vogt Theorem). *Let $G = \mathsf{SL}(2,\mathbb{C})$ act on $G \times G$ by simultaneous conjugation. Then, every regular function $f : \mathsf{SL}(2,\mathbb{C}) \times \mathsf{SL}(2,\mathbb{C}) \to \mathbb{C}$ satisfying*

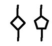
$$f(\mathbf{x}_1, \mathbf{x}_2) = f(g\mathbf{x}_1 g^{-1}, g\mathbf{x}_2 g^{-1}) \qquad \text{for all } g \in \mathsf{SL}(2,\mathbb{C}),$$

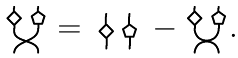
can be written as a polynomial in the three trace variables $x = \operatorname{tr}(\mathbf{x}_1)$, $y = \operatorname{tr}(\mathbf{x}_2)$, and $z = \operatorname{tr}(\mathbf{x}_1 \mathbf{x}_2^{-1})$.

Proof. Given the isomorphism

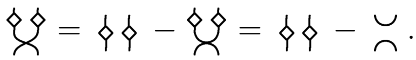
$$\mathbb{C}[G \times G]^G \cong \bigoplus_{c \in \lceil a,b \rfloor} \mathbb{C}\, \chi^{a,b,c},$$

it suffices to show that (i) every polynomial in x, y, and z can be written in terms of central functions $\chi^{a,b,c}$, and (ii) every central function may be written as a polynomial in x, y, and z. Theorem 6.15 gives an explicit formula for the first statement, and two proofs of the second statement follow:

Nonconstructive diagrammatic proof. Expanding the symmetrizers in the central function $\chi^{a,b,c}$ gives a collection of circles with matrix elements, each of which correspond to a product of traces of words in $\mathbf{x}_1, \mathbf{x}_2$, so it suffices to express the trace of any word in $\mathbf{x}_1, \mathbf{x}_2$ as a polynomial in x, y, and z. This reduction depends entirely on the binor identity, which when composed with $\mathbf{x}_1 \otimes \mathbf{x}_2 =$  gives:

 (8.1)

As special case, if  denotes $\mathbf{x}_1^{-1}$ then

 and 

By the first relation, no loop need contain both $\mathbf{x}_1$ and $\mathbf{x}_1^{-1}$, while by the second relation, no word need have more than one of a given matrix. This reduces the problem to traces of words $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_1\mathbf{x}_2$, and $\mathbf{x}_1\mathbf{x}_2^{-1}$. Closing off (8.1) gives:

$$\mathrm{tr}(\mathbf{x}_1\mathbf{x}_2) = \mathrm{tr}(\mathbf{x}_1)\mathrm{tr}(\mathbf{x}_2) - \mathrm{tr}(\mathbf{x}_1\mathbf{x}_2^{-1}).$$

Thus, the word $\mathbf{x}_1\mathbf{x}_2$ is unnecessary, leaving only $x = \mathrm{tr}(\mathbf{x}_1)$, $y = \mathrm{tr}(\mathbf{x}_2)$, and $z = \mathrm{tr}(\mathbf{x}_1\mathbf{x}_2^{-1})$.

Constructive diagrammatic proof. Proceed by induction on the rank $\delta = \frac{1}{2}(a+b+c)$ of a central function $\chi^{a,b,c}$. For the base cases $\delta = 0, 1$ recall that

$$\chi^{0,0,0} = 1,\ \chi^{1,0,1} = x,\ \chi^{0,1,1} = y,\ \chi^{1,1,0} = z.$$

For $\delta > 0$, inductively assume that all central functions with rank less than δ are in $\mathbb{C}[x, y, z]$. The admissibility conditions imply that at least two out of the triple $\{a, b, c\}$ are nonzero. By symmetry, these may be assumed to be a and c. But Corollary 6.10 permits $\chi^{a,b,c}$ to be written in terms of central functions of lower rank:

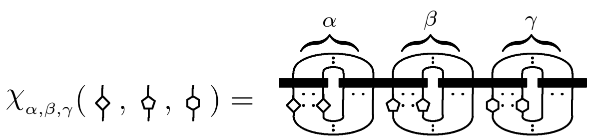
$$\chi^{a,b,c} = x \cdot \chi^{a-1,b,c-1} - \tfrac{(a+b-c)^2}{4a(a-1)}\chi^{a-2,b,c} - \tfrac{(-a+b+c)^2}{4c(c-1)}\chi^{a,b,c-2} - \tfrac{(a+b+c)^2(a-b+c-2)^2}{16a(a-1)c(c-1)}\chi^{a-2,b,c-2}.$$

By induction, each of the terms on the right must be in $\mathbb{C}[x, y, z]$. Therefore, $\chi^{a,b,c} \in \mathbb{C}[x, y, z]$, completing the proof. $\square$

This theorem is closely related to the possible hyperbolic structures on the three-holed sphere. Indeed, such a structure is uniquely determined by the lengths of the boundary curves, which can be related to the trace variables x, y, and z.

More General Coordinate Systems

It is interesting to ask whether this correspondence also holds in higher rank cases. Since spin network bases can be defined as graphs which are dual to triangulations, it seems a natural step to relate them to the *Penner coordinates* on Teichmüller space. There is also a corresponding spin network basis for each *pants decomposition* of a surface, giving a connection with the *Fenchel-Nielsen coordinates.* This basis consists of products of diagrams/functions of the rank two form



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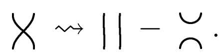
Deformations

Given these correspondences, central functions may provide insight into how the geometry of a surface changes under certain actions, such as *Dehn twists* or *earthquake deformations*. It is possible that such deformations behave in a canonical way on spin networks, allowing the geometry of the resulting surface to be neatly described. In particular, this may be a way to obtain formulae for how the Penner or Fenchel-Nielsen coordinates change under such deformations.

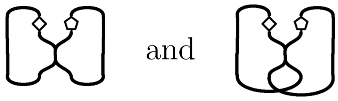
The Poisson Structure

Goldman defined a bracket on the algebra of loops on a surface in [Gol1] which has a Poisson structure and can be used to give a bracket on the function space $\mathbb{C}[\mathfrak{X}]$. The *Casimirs* of this bracket are the boundary elements because they may be taken to be disjoint from all other loops on the surface. The bracket on the remaining elements may be used to give a general formula for a symplectic form on the surface.

The action of this bracket on a minimal set of generators induces its action on all trace words. Moreover, since it is a derivation, the bracket itself is defined by the local operation



Using the derivation property of the bracket together with the recurrence formula for rank two central functions, one can obtain a formula for the Poisson bracket of two arbitrary central functions. Additional information is needed, however, since the central functions depend on the fundamental group of a surface, and not its actual shape. In particular, the distinction between



becomes very important.

Moreover, Chas and Sullivan have extended the notion of the bracket to a homology theory [CS], which begs the question: how does one define a homology theory for general spin networks?

8.2 Combinatorics of Spin Networks

It is possible that spin networks could shed some light on a number of high profile theorems and conjectures regarding graph coloring, since the values of spin networks are essentially just chromatic indices.

Recall the following theorem, proven in section 4.5:

Proposition 8.2. *Let* $\mathsf{s} \in [\mathsf{S}^{\mathsf{n}}]_I^O$ *be a spin network corresponding to a map* $\mathsf{s} : V^{\otimes I} \to V^{\otimes O}$*. Then the coefficient of the basis element* $e_{j_1} \otimes \cdots \otimes e_{j_O}$ *in the expansion of* $\mathsf{s}(e_{i_1} \otimes \cdots \otimes e_{i_I})$ *is equal to the signed sum of all possible labellings of* s *by* $\{e_1, e_2, \ldots, e_n\}$ *which respect the input and output labels* $e_{i_1}, \ldots, e_{i_I}$ *and* $e_{j_1}, \ldots, e_{j_O}$*.*

This theorem may be restated in the language of graph theory. Given a graph $G = (V, E)$ and label set N, a *Tait coloring* or *edge coloring* of G by N is an assignment $K : V \to N$ of labels to edges such that no two edges incident to the same vertex have the same label. A *edge pre-coloring* is an assignment $K' : V' \to N$ for some subset $V' \subset V$. A coloring K *extends* K' if $K|_{V'} = K'$, and this condition is written $K \succ K'$.

If G is a ciliated n-valent graph, then a coloring by $N = \{1, \ldots, n\}$ induces a permutation at each vertex, and therefore a sign. Thus every edge-coloring K has a well-defined $\text{sign}(K) = \pm 1$, defined to be the product of the signs at the vertices. With these notational conventions, the *signed pre-chromatic index* of G with respect to an edge pre-coloring K' is

$$\bar{\chi}^e_{K'}(G) \equiv \sum_{K \succ K'} \text{sign}(K).$$

The coefficients of a spin network map are exactly these signed pre-chromatic indices:

Proposition 8.3. *Let $N = \{1, \ldots, n\}$, and let $\mathsf{s} \in [\mathsf{S^n}]_I^O$ be a spin network with I inputs, O outputs, and $|\text{sink}(\mathsf{s})|$ source vertices. Then, the corresponding map $\mathsf{s} : V^{\otimes I} \to V^{\otimes O}$ is given by the mapping:*

$$e_{i_1} \otimes \cdots \otimes e_{i_I} \longmapsto (-1)^{|\text{sink}(\mathsf{s})|} \sum_{j_k \in N} \bar{\chi}^e_{K'}(\mathsf{s}) e_{j_1} \otimes \cdots \otimes e_O,$$

where K' is the pre-coloring assigning $i_1, \ldots, i_I$ to the input edges of s and $j_1, \ldots, j_O$ to the output edges of s.

Spin Networks and Bicubic Graphs

All spin networks are *bipartite* graphs, since the source/sink condition provides a natural partition of the vertices. Thus, spin networks in $\mathsf{S^3}$ are *bicubic graphs*, meaning both bipartite and cubic (trivalent). It is known that every bicubic graph is 3-edge colorable, and it has been conjectured that every 3-connected planar bicubic graph is Hamiltonian (the *Barnette Conjecture*).

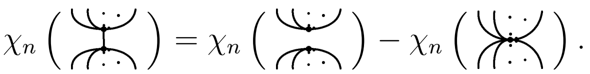
In the context of spin networks, the first statement implies that every $\mathsf{s} \in \mathsf{S^3}$ has a term with nontrivial coefficient. Because the coefficient is a *signed* index, this does not necessarily mean the spin network evaluates to a nonzero function. For the second statement, note that the *binor identity* $= \,|\,| \,-$ permits any $s \in \mathsf{S}^3$ to be expressed as a sum of planar bicubic graphs. Thus the Barnette Conjecture is equivalent to the following:

Conjecture 8.4. *Every spin network* $\mathsf{s} \in \mathsf{S}^3$ *can be expressed as a sum of planar Hamiltonian diagrams.*

It is possible that the theory of 3-spin networks, or of trace diagrams, could shed some light on the Barnette Conjecture. As a first step, it seems likely that a Hamiltonian cycle in a diagram would permit an algorithm for computing the value of a 3-spin network.

Coloring and the Binor Identity

The above propositions relating spin network values and chromatic indices would be more useful if either were easy to compute. Unfortunately, the computation of chromatic indices, for both vertices and edges, is an NP-complete problem. The vertex chromatic index $\chi_n(G)$, is most directly computed using the recursion



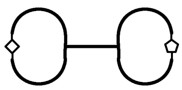
There is not an easy recurrence for edge colorings, although the binor identity $= \,|\,| \,-$ gives rise to such a recurrence for *signed* colorings. Of course, the binor identity for S^n gives rise to a recurrence for more general signed colorings.

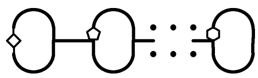
It is hoped that techniques for spin network simplification will overlap with techniques for computation of chromatic polynomials, providing for some cross-pollination. It seems likely, given the connections described here. The correspondence will might prove especially valuable for the computation of central functions beyond $\mathsf{SL}(2, \mathbb{C})$.

8.3 Computation of Central Functions

The main focus of this thesis was the structure of central function bases for a given group G and a compact surface Σ with boundary. The case $G = \mathsf{SL}(2, \mathbb{C})$ was studied extensively, especially in the rank one and two cases. A partial list of remaining questions follows:

- What are the symmetries of central function bases of rank ≥ 3? This should generalize from the rank two case and be particularly evident in diagram form.

- Is there a direct way to compute the alternate rank two central functions ? If so, this could also provide a direct way to compute the rank n central functions of the form

  .

  This also relates G to its Lie algebra $\mathfrak{g}$; since

  $$\mathbf{X} \equiv \mathbf{x} - \tfrac{1}{2}\mathrm{tr}(\mathbf{x})\mathbb{I} = \quad \in \mathfrak{sl}(2, \mathbb{C}),$$

  the matrices in these alternate bases may be replaced with their Lie algebra representatives.

- Develop an algorithm for computing central functions for $G = \mathsf{SL}(n, \mathbb{C})$. The first step would be extending the diagrammatic Littlewood-Richardson Rule in Chapter 7. In general, the structure of the coordinate ring $\mathbb{C}[\mathfrak{X}]$ is not well understood for $G = \mathsf{SL}(n, \mathbb{C})$ when $n \geq 4$. This also requires a good understanding of how general diagrams are manipulated.

- What do central functions look like for other groups? The relationship of the G-coordinate variety to geometric structures provides a strong motivation for looking at other cases.

I am most intrigued by the relationship with the Lie algebra, since this may provide a direct way to demonstrate the relationship between the *fundamental class* of a surface and the Poisson bracket. This fundamental class, combined with the Scott-Wolpert form, may be used to give the symplectic structure of a surface, which therefore induces the Poisson bracket. Is there a simple, direct way to relate this to Goldman's bracket? The cut triangulations introduced in Chapter 5 are a first step in this construction.

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