ABSTRACT

Title of dissertation: MATCHING ISSUES:
AN AUCTION WITH EXTERNALITIES
AND
UNRAVELING IN MATCHING MECHANISMS

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This dissertation examines two problems that may arise in matching problems. The first two chapters deal with auctions for multiple units where bidders exhibit externalities. The third chapter links risk aversion and information to unraveling in labor markets.

Auctions can lead to efficient allocations in a wide class of assignment problems. In the presence of externalities, however, efficiency may no longer be guaranteed. This dissertation shows that a modification of Ausubel & Milgrom (2002)'s generalized ascending price auction can be used to allocate multiple items to bidders in this case. Despite the presence of externalities, the resulting auction possesses an efficient Nash equilibrium in pure strategies leading to a core allocation. Furthermore, under certain restrictions on bidder valuations, truthful revelation of valuations is found to a dominant strategy. The auction is augmented to include explicitly the auctioneer's preferences over final outcomes. Externalities affecting non-participants can thus be accounted for straightforwardly.

In Cournot game where capacity constraints are determined in an auction prior to the market interaction, the valuations for capacity in the auction will exhibit externalities. Using the generalized ascending price auction allows the bidding firms to reach a joint profit maximizing capacity allocation below the Cournot equilibrium level. Since this comes at the expense of consumer surplus the auctioneer may have an incentive to specify its own valuation taking into account total surplus maximization. Then, the final capacity allocation is bounded by the profit maximizing and the Cournot equilibrium level.

Unraveling labor markets, that is periodic labor markets where appointments are made earlier and earlier often leading to a break-down of the market, have been linked to risk-averse workers attempting to reduce the variability of the outcome. In many cases, early contracts are used to fix a wage when the relative supply and demand of workers in the market and hence the division of surplus is uncertain. This chapter represents a different approach. Both workers and firms have preferences over matchings and uncertainty is introduced through the quality of workers. Risk averse workers or risk-loving firms are found to be necessary for early contracting. Further research strategies are suggested.

MATCHING ISSUES: AN AUCTION WITH EXTERNALITIES AND UNRAVELING IN MATCHING MECHANISMS

by

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To my parents.

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Chapter 1

Introduction

This dissertation consists of two part. In the next two chapters an auction for multiple units is described that can lead to an efficient allocation even when bidder valuations are characterized by externalities. The fourth chapter looks briefly at mechanisms matching workers to firms and seeks to identify conditions under which such labor markets begin to unravel. A number of different assumptions about information structures are made.

It might seem strange at first to find two papers on auctions and a chapter on matching workers to firms combined in a single dissertation. After all, a certain coherence within a dissertation is normally expected. There does exist a connection, however, in that both an auction and the labor market described are mechanisms to coordinate supply and demand on a two-sided market. In the auction, a seller attempts to allocate several objects to potential buyers in a way that will maximize its objective function. The firms and workers in the labor market, similarly, try to be matched to their preferred agent on the other side of the market. In both cases, not all agents can achieve their most preferred allocation and the market has to adjucate the competing claims of the participating agents.

But more than just the presence of market mechanisms, it is a focus on conditions in which they may fail to produce the desired outcome that unites the two parts of the dissertation. Externalities in auctions, that is where bidders care about the objects they *and* their competitors win, have been shown to be problematic for many types of auctions.[1] In many cases an efficient allocation can not be found. The next chapter traces this fact to their inability to elicit information about the magnitude of the externalities from the bidders and to allow bidders to subsidize each others' purchases. From this observation, the analysis proceeds to propose an alternative auction mechanism that generates an efficient final allocation. It is based on Ausubel & Milgrom (2002)'s ascending proxy auction but defines bids in terms of allocations rather than bundles of goods. This also allows the auctioneer to specify its own preferences over allocations and take it into account explicitly when determining the final allocation. The terminal payoffs of all bidders are shown to be in the core. Furthermore, the strategic properties of the auction are explored and a condition for the existence of an equilibrium in which bidders bid truthfully is established.

The third chapter implements the auction for externalities in the context of market game where firms compete in quantities (Cournot game). In a first stage, firms acquire capacity through the auction; in the second stage they produce output subject to a capacity constraint. By allowing firms to limit each other's purchase of capacity, the ascending proxy auction leads to a capacity allocation that is below the Cournot equilibrium level. Industry profits are maximized. Since this comes at the price of reduced consumer surplus in the market for final output, the auctioneer may have an incentive to increase the capacity allocation to all firms. By specifying

[1] For an overview of the literature, see chapter 2.

its preferences accordingly, the auctioneer can trade off industry profit and the revenue it obtains from the auction for increased consumer surplus. Unless firms can be induced to produce above their unconstrained equilibrium level, total surplus cannot be maximized, however.

The fourth chapter takes another look at unraveling labor markets. Markets are said to unravel when transactions take place at ever earlier stages, possibly leading to the breakdown of the market. Roth & Xing (1994) have identified unraveling in a large number of (mainly) labor markets and provide a classification of various stages of unraveling. Early contracting often takes place before all information about workers or firms is available and is thus not efficient. The reasons that it takes place nevertheless have been linked to risk averse agents and uncertain final market outcomes. By contracting early, workers can negotiate a wage outside of the market where fluctuations in supply and demand lead to considerable uncertainty over their remuneration.[2]

Rather than require uncertain wages, this chapter adopts the approach of Niederle & Roth (2003) where uncertainty stems directly from preferences over the quality of agents. In a two-period model, the final quality of workers is not known in the first period, but some information is available in the form of a signal. By contracting early, workers can avoid an uncertain match depending on their realized quality. While risk aversion is found to be a sufficient condition for early contracting, labor markets can unravel even when workers are risk neutral as long as firms are sufficiently risk-loving. The conditions for unraveling are shown to depend on the

[2] Chapter 4 provides a more detailed summary of the argument.

availability of the signal in the first period.

Despite their common focus on matching issues in a wider sense, the two parts of the dissertation are different methodologically. The general ascending proxy auction provides the solution to a problem in certain markets; the fourth chapter examines a different problem without, however, going beyond a description of conditions responsible for the issue. Further research is necessary to find a mechanism that solves early contracting in unraveling markets.

Chapter 2

The generalized ascending proxy auction in the presence of externalities

2.1 Introduction

Consider an auction for possibly multiple objects in which bidders do not only care about which items *they* win, but also about the bundles obtained by their competitors. In other words, bidder valuations for sets of items depend on the identity of the winners of the remaining objects, and all bidders – winning and non-winning – impose externalities on each other. The presence of externalities is likely to affect non-trivially the bidding behavior and the outcome of an auction. This chapter attempts to address the resulting issues and proposes an auction mechanism that takes explicit account of externalities in valuations. The auction will be shown to be efficient regardless of the number of participants or objects and without restrictions on the allowable valuations. It extends the existing literature by providing an efficient mechanism for the most general case of valuations with externalities.

Although externalities in the context of auctions have not been at the forefront of academic interest, their presence in real-world auctions is almost certain. Whenever firms competing in the same markets bid for items, they would have an interest in ensuring that the objects they do not win are allocated to those firms that pose the least severe competitive threat to them. Moldovanu & Ewerhart (2001), for example, recount that mobile phone operators in the recent European spectrum auctions had an interest in minimizing the total number of firms winning licenses. In particular, each new firm winning the right to operate a mobile phone network was thought to reduce the profit for the firms already in the market by more than a competing incumbent winning an additional license. As a consequence the existing incumbents tried to prevent the success of potential entrants.

This example illustrates nicely three key characteristics of externalities in valuations. First, externalities are often non-anonymous, that is, the magnitude of the externality depends on the identity of the agent that imposes it. Second, externalities can be found in the context of multi-unit auctions. And third, externalities may be imposed even on agents that are not interested in winning anything and therefore do not appear in traditional auction formats. In the case of mobile phone licenses, consumers – while not interested in obtaining a license themselves – benefit from increased competition. More generally, the identity of the firm winning an auction might affect buyers in downstream markets. Including externalities into the auction design does thus provide a more complete approach to modeling auction efficiency.

Most practical auction design, especially when involving government agencies, has acknowledged the possible impact bidder strategies can have on competition and consumer surplus, and bidding and participation rules are generally designed to guarantee a minimal acceptable level of competition in the downstream market. This paper attempts to go further by proposing the generalized ascending price auction (g-APA) due to Ausubel & Milgrom (2002) as a mechanism that can allow bidders to influence the outcome without necessarily winning any objects themselves. At least in theory, consumer representatives or downstream firms thus have the ability to affect the allocation of items to bidders. In cases where bidders not interested in winning any items are unable to participate in the auction, the auctioneer itself can incorporate the preferences of the non-participants into the determination of the final outcome. Governments concerned with consumer surplus can therefore replace bidding and participation rules designed to increase competition with clearly stated auctioneer preferences, increasing the transparency of the process and reducing the scope for criticism by non-winning firms.

Section 2.2 will provide some background to auctions for multiple units and externalities in valuations. Readers familiar with the subject can go directly to section 2.3 where the generalized ascending price auction will be introduced. In subsection 2.3.1 a simple example illustrates some of the problems associated with the presence of externalities and introduces the generalized ascending price auction. The formal model will be developed in section 2.3.2. In essence, Ausubel & Milgrom (2002)'s generalized ascending proxy auction is adapted as a dynamic extension of the menu auction proposed by Bernheim & Whinston (1986). Section 2.4 shows that despite the presence of externalities the strategic properties of Ausubel & Milgrom (2002)'s original auction are still in place. In particular, as long as the auction leads to Vikrey-Clarke-Groves payoffs for the bidders, it induces truth-telling, thereby solving the problem of finding equilibrium strategies for the bidders.

2.2 Background

The objective of an auction is generally twofold: first, to maximize revenues for the seller and secondly to allocate the objects for sale efficiently to the bidder with the highest valuation. While these objectives need not coincide, Ausubel & Cramton (1999) have shown that, if resale after the auction is permitted, efficiency is conducive to revenue maximization.[1] Optimal – in the sense of seller revenue maximizing – and efficient auctions for single objects without externalities have been examined from a large number of angles and are by now fairly well understood. The allocation of multiple units and the presence of externalities in valuations pose additional problems, however.

2.2.1 Externalities in valuations

If bidders care about who obtains the items they don't win, the identity of the bidders participating in an auction is payoff-relevant.[2] Due to the inherent constraints on bidding behavior and permissible outcomes, traditional auctions will in general not lead to an efficient allocation in this context. With negative externalities, for example, bidders might be willing to pay the seller not to sell (Jehiel et al. 1999). While this could indeed be efficient, most standard auctions do not allow

[1] Efficiency in Ausubel & Cramton (1999), as in almost all auction research, is defined solely with respect to the participants in the auction.

[2] The implications of externalities in valuations were first explored in a series of articles by Phillippe Jehiel, Benny Moldovanu and Ennio Stacchetti (Jehiel & Moldovanu 1996, 2000, Jehiel et al. 1996, 1999).

such an outcome. Jehiel & Moldovanu (1996) show that under certain assumptions bidders may take the strategic decision not to participate in an auction, thereby committing not to impose an externality on a competitor. This in turn can keep another agent from bidding aggressively, possibly winning the auction, and causing a negative externality on the non-participating bidder. Slightly paradoxically, perhaps, both losing and winning bidders may withdraw from the auction.[3] Nevertheless, even a credible commitment by some bidders not to participate does not guarantee an efficient allocation.

Furthermore, since bidders cannot express the magnitude of externalities other than implicitly by bidding in excess of their valuation, the auctioneer lacks information necessary to find an efficient allocation. Unless externalities are symmetric (Das Varma 2002), the allocation resulting from an auction may thus not be efficient (Cornet & Laan 2001, Brocas 2002, Aseff & Chade 2002). Allowing bidders to collude does not remedy the problem (Caillaud & Jehiel 1998).

Most authors have only examined externalities in the context of single-unit auctions or bidders with unit demand. Allowing externalities in a multi-unit framework greatly complicates the matter and makes a traditional mechanism design approach impractical. An exception to this claim is Bernheim & Whinston (1986)'s menu auction which can be reinterpreted as an auction with externalities. The mechanism proposed in section 2.3 takes a menu auction approach and places it into the dynamic context of Ausubel & Milgrom (2002)'s generalized ascending price auc-

[3] While the set-up seems at first slightly *ad hoc*, Jehiel et al. (1996) provide real-world illustrations of strategic non-participation.

tion. Rather than allowing bidders to withdraw and be ignored in the allocation process, the auction proposed encourages agents to participate. The auctioneer can then use the additional information on valuations to find an efficient allocation. A menu auction combined with techniques from the matching literature not only simplifies the analysis of externalities it also provides a good framework to account for multiple objects.[4]

2.2.2 Multi-unit auctions

The allocation of multiple objects to multiple bidders involves two separate tasks. First, bidders have to be induced to reveal their valuations truthfully. And second, an efficient allocation and prices supporting this allocation have to be found given the reported valuations. Assuming information about valuations to be common knowledge, Bikhchandani & Mamer (1997) treat the efficient allocation as the solution to a linear program whose dual provides the supporting prices. Competitive prices thus exist if and only if an efficient allocation can be found. In the case that objects are indivisible, the solution to the linear program has to be integer valued.

The existence of Walrasian equilibrium prices for multiple items is by no means guaranteed unless valuations are restricted in a way that all goods are substitutes (Kelso & Crawford 1982, Gul & Stacchetti 1999, Milgrom 2000). This rules out the

[4]These issues are not restricted to auctions. More generally, any allocation of objects to agents in a General Equilibrium framework is more or less severely complicated by the presence of externalities. Shapley & Shubik (1969), for example, show that negative externalities can lead to an empty core and thus to the non-existence of a stable allocation.

interesting case where synergies between items raise the value of a bundle above the sum of the values of the respective items, that is where values are super-additive. Bikhchandani & Ostroy (2002) have shown that equilibrium prices exist nonetheless. These prices, however, are both non-linear and non-anonymous. By relaxing the constraints that the price of a bundle of items equals the sum of the prices of its components (linear pricing) and that this price is independent of the identity of the buyer (anonymity), they can solve a linear program and its dual, yielding an efficient allocation and the corresponding prices.

For a bidder to reveal its valuation truthfully through its bidding behavior, the price for a bundle of objects it finally win must be independent of its own bids; otherwise, it have an incentive to reduce its demand. The intuition is as follows: if the price paid for a bundle of objects is the outcome of a bid placed by the winning bidder, demand reduction will reduce the price of all inframarginal objects won at the cost of the potential loss of a marginal unit. In two common multi-unit auction formats, the uniform price auction and the pay-as-you bid auction, the gains exceed this loss, and demand reduction will lead to inefficient allocations with strictly positive probability (Ausubel & Cramton 1995, 2002).[5]

An efficient, incentive compatible mechanism to allocate multiple items to bidders has been developed in Clarke (1971) and Groves (1973)'s generalization of Vickrey (1961)'s second price auction. Bidders report their marginal values to the auctioneer who allocates the objects efficiently according to the reports. The price paid by the winner is then calculated as the highest reported valuation of the losing

[5]For a summary of various traditional multi-unit auction formats, see Krishna (2002).

bidders. If bidders are plausibly assumed to be willing to pay up to their values, this corresponds to the highest price any of the losing bidders would be prepared to pay. Hence, agents' bids only affect the final allocation and the price others might have to pay, but not the price they pay themselves. Bidding true valuations is a dominant strategy.

In spite of this desirable property, the Vickrey-Clarke-Groves (VCG) mechanism is not widely used in practice (Rothkopf et al. 1990). This may be due to several reasons. First, bidders may be wary to reveal their valuations for fear that they will become public after the auction. Competitors could then use this information in strategic interactions after the auction or in future auction games. Furthermore, if bidders are uncertain about their values, Milgrom & Weber (1982) have demonstrated the advantage of an ascending, dynamic auction. By observing the bidding behavior of their competitors, as long as values are affiliated, bidders can adjust their expectation about the true value of an object over the course of the auction. Although Milgrom & Weber (1982) have originally shown this for the single-unit case, the same principle applies to auctions for multiple units.

Generalizing the concept of an English auction to multiple homogeneous units, Ausubel (1995, 2004) proposes an ascending price clock auction where at any point in time all bidders indicate the number of units they demand at the corresponding price. Whenever the number of units demanded by a bidder's opponent falls below the number of items still available, that bidder wins – "clinches" – the difference between the number of units available and the residual demand. If bidders have weakly declining marginal independent values, the ascending price clock auction leads to an efficient allocation and VCG-prices and payoffs.[6]

An early ascending price auction for multiple heterogeneous goods is suggested by Demange et al. (1986). They find that with truthful bidding, the auction terminates at the lowest competitive equilibrium prices. Although their set-up is restricted to bidders with unit demand, and thus to goods being substitutes, their concept of raising the price on all goods that are *necessarily* overdemanded provides the basis for Gul & Stacchetti (2000) and Ausubel (2000, 2002)'s dynamic multi-unit auctions where bidders demand multiple objects. Demange et al. (1986) do not examine the strategic incentives facing bidders. A result by Leonard (1983), however, indicates that minimal equilibrium prices are incentive compatible.

With heterogeneous objects and multi-unit demand, the link between competitive prices and incentive compatibility becomes more problematic. Gul & Stacchetti (2000) develop an auction where bidders report their most preferred bundle or bundles at the given price vector to the auctioneer. Prices are increased on all items that are in excess demand until each item is demanded only once. If goods are substitutes, the auction will terminate at the smallest Walrasian prices. Unlike in the case of unit demand, Gul & Stacchetti (2000) find that no ascending price auction with a *single* price trajectory leads to VCG-payoffs for all possible preferences with substitutes. They conclude that incentive compatibility will be an issue for ascending auctions. While retaining the principle of raising prices on overdemanded objects, Ausubel (2000, 2002) considerably simplifies the auction mechanism. More importantly, however, he achieves incentive compatibility by changing the price path

[6] Perry & Reny (2001) generalize Ausubel (1995, 2004)'s auction to allow for affiliated values.

to lead to modified VCG-payoffs. Moreover, a true VCG-outcome can be reached if multiple auctions are run parallel.

The almost necessity of the substitute condition for the existence of Walrasian equilibrium (Milgrom 2000) precludes the use of linear price ascending auctions for many practical applications with efficiency as the goal. In a recent paper, Ausubel & Milgrom (2002) implement an ascending auction where bids are placed on bundles of objects rather than individual items and the auctioneer determines the final allocation as a feasible matching between bidders and packages. It is shown that the ascending price package auction terminates in an efficient allocation relative to the valuations underlying bidding behavior and that a semi-truthful Nash equilibrium exists.

In a way, Ausubel & Milgrom (2002) is also a dynamic application of Bernheim & Whinston (1986)'s menu auction. Rather than bidding on items, agents place bids on outcomes, defined in terms of packages allocated to them. This gives bidders a more expressive language to communicate their preferences to the auctioneer who is then able to find an efficient allocation and supporting prices.

2.3 The generalized ascending proxy auction for allocations

Most of the issues arising from the presence of externalities in valuations can be traced to the fact that the value of the outcome of an auction to a certain bidder depends on the identity and the composition of the packages of all winners. The two main consequences are difficult to address using the standard auction formats.

If the value of a set of items is a function of the complete allocation of objects, the maximal price a bidder is willing to pay depends on its expectation about the outcome of the auction. Unless an agent bids the minimum values for each package there is no guarantee that it will not have *ex-post* regrets. While this problem is most severe in sealed-bid auctions where bidders have no information about other bidders strategies, being able to observe interim allocations in an ascending auction cannot completely solve the uncertainty about the final outcome. Moreover, since bidders cannot communicate valuations contingent on the final outcome, the auctioneer will generally have insufficient information to find an efficient allocation.

In addition, with bidders possibly obtaining a benefit from the externalities caused by other agents winning certain items, they have an interest in subsidizing each others bids. Take, for example, a bidder which would like to prevent its closest competitor winning an item. Paying a third agent to bid on the object – or paying the auctioneer to keep it, as in Jehiel et al. (1999) – might not only be an optimal strategy, it might also lead to an efficient outcome. In theory, nothing, apart possibly from legal restraints, would keep bidders from colluding in this manner. Nevertheless, in practice, even abstracting from problems of communication and bargaining, it is rather unlikely to succeed as the size of the subsidy would in itself be contingent on the final allocation, which would be unknown at the time the contract is drawn up. Contracts would indeed have to be contingent themselves.

An alternative would be to allow agents to place bids on *allocations* rather than on bundles of goods. This allows both the communication of contingent valuations to the auctioneer and the implicit subsidization of other agents' bids. Bernheim

& Whinston (1986)'s menu auction provides a static mechanism for this approach. Although much more general in scope, the generalized ascending proxy auction of Ausubel & Milgrom (2002) with outcomes defined in terms of allocations can be interpreted as the dynamic counterpart of a menu auction. The following simple example will illustrate some of its key aspects.

2.3.1 An example with externalities

Consider the case where three agents, $n = 1, 2, 3$ bid for one item. The auctioneer is assumed to be interested only in revenue maximization. Although the generalized ascending proxy auction is specifically proposed to account for multiple items, restricting the example to a single unit saves on notation while providing the intuition for subsequent results. Let μ_n denote the outcome in which bidder n wins the object. In allocation μ_0, the auctioneer retains the item.

Example 1. Bidder 1 has no interest in winning the object itself, but would like to prevent its competitors from winning. The negative externality imposed on bidder 1 is particularly large in the case that bidder 3 obtains the object. Valuations for each bidder, v_n, are given in table 2.1.

The efficient allocation is μ_2 with a combined utility of 25. Any auction in which winner determination is based on finding the agent with the highest valuation would allocate the object to bidder 3. The inefficiency is caused by the inability of bidders 1 and 2 to coordinate their preferences.

In the generalized ascending proxy auction, agents submit their valuations for

Table 2.1: Auction Example: Bidder valuations

	μ_0	μ_1	μ_2	μ_3
v_1	20	20	15	0
v_2	0	0	10	0
v_3	0	0	0	23

allocations to a proxy bidder. In each round, the proxy bidder increases its bid by the minimum allowable increment on the allocation that would yield the highest payoff taking into account the bid. The auction ends when no new bids are placed and agents pay their bids.[7] Agents can thus place bids on the same allocation making it possible to collaborate without the need for communication.

In round 1, bidder 1 places the minimum bid (through its proxy) on allocations μ_0 and μ_2. Bidders 2 and 3 bid on μ_2 and μ_3, respectively. In case of draws, assume the auctioneer selects an assignment randomly from the revenue maximizing ones. Bidding continues until round 5. By then, the bids on the allocations are as follows.

In round 6, bidder 1 starts bidding on allocation μ_2, effectively supporting bidder 2's bid. Now, μ_2 is the winning allocation and bidder 3 will increase its bids on μ_3. For the next several rounds bidder 3 and the coalition for bidders 1 and 2 will alternate increasing their bids while the interim allocation will alternate between μ_2 and μ_3.

Once bidder 2 has reached a bid of 10, it drops out of the auction, as increas-

[7]A more detailed description of the auction will be given in the next section.

Table 2.2: Auction Example: Bids after round 5

	μ_0	μ_1	μ_2	μ_3
bidder 1	5	5	0	0
bidder 2	0	0	5	0
bidder 3	0	0	0	5
	5	5	5	5

Table 2.3: Auction Example: Bids in round 6

	μ_0	μ_1	μ_2	μ_3
bidder 1	6	6	1	0
bidder 2	0	0	6	0
bidder 3	0	0	0	6
	6	6	7	6

ing its bid might lead to a negative payoff. While bidder 3 continues bidding on μ_3, bidder 1 raises its bid on all other allocations. The process of alternating bid increases between bidder 1 and 3 continues until the final round.

Bidding continues until bidder 3 has bid up to its valuation and drops out of the auction. No new bids will be placed and the auctioneer picks μ_2 or μ_3 randomly. In case μ_2 is chosen, the auction ends. If μ_3 is the interim assignment, bidder 1 increases its bids one last time and μ_2 will be the revenue maximizing allocation. In both cases, the auction ends at the efficient allocation.

Table 2.4: Auction Example: Bidder 2 drops out

	μ_0	μ_1	μ_2	μ_3
bidder 1	15	15	10	0
bidder 2	0	0	10	0
bidder 3	0	0	0	20
	15	15	20	20

Table 2.5: Auction Example: Final round

	μ_0	μ_1	μ_2	μ_3
bidder 1	18	18	13	0
bidder 2	0	0	10	0
bidder 3	0	0	0	23
	18	18	23	23

The next section will provide the formal definition of the generalized ascending proxy auction over allocations.

2.3.2 The formal model

Analyzing the properties of the proposed auction can be greatly simplified by abandoning somewhat the techniques of mechanism design and interpreting it as a matching algorithm instead.[8] Essentially, an auction is a method to match bundles

[8] Much of the early multi-unit auction literature was indeed interested in finding a matching algorithm (see, for example, Kelso & Crawford (1982), Leonard (1983), Demange et al. (1986)).

of objects with bidders and although much of the matching literature is concerned with matching agents with each other, some key results can be incorporated. More precisely, the ascending price auction mimics a deferred acceptance algorithm, in which bidders propose a matching together with a transfer to the auctioneer. The final assignment is deferred until no bidder makes a further proposition.[9] Some definitions are useful.

Some Preliminaries

Let $N = \{0, \ldots, N\}$ be the participants in an auction, where the seller is denoted as $n = 0$ and the buyers are $n = 1, \ldots, N$. Let S be the finite set of objects to be sold. Objects may be homogeneous or heterogeneous.[10] An allocation of objects to bidders can then be defined as a matching which assigns the items in S to the agents in N.

Definition (Assignment). An assignment (matching) μ is a mapping from $N \cup S$ to the set of all subsets of $N \cup S$ such that for all $s \in S$ and $n \in N$,

1. $|\mu(s)| = 1$ and $\mu(s) \in N$
2. $\mu(n) \subseteq S \cup \emptyset$ and $\mu(n) = \emptyset$ if $\mu(n) \nsubseteq S$
3. $\mu(s) = n$ if and only if $s \in \mu(n)$.

Let $\mathfrak{M}$ be the set of all matchings.

More recently, Ausubel & Milgrom (2002) exploit the analogy in their package auction.

[9] An extensive treatment of matching theory can be found in Roth & Sotomayor (1990).

[10] Treating multiple units of identical items as distinct objects will not affect the results (Bikhchandani & Mamer 1997).

A matching thus links bidders and objects in a very intuitive way: each object is only allocated to a single agent and agents are matched either to a subset of the objects for sale or the empty set. This is in contrast to some of the matching literature where agents that are effectively unmatched are assumed to be matched to themselves. Furthermore, under this definition, all objects are assigned, with the unsold items being retained by the seller.

Taking matchings as the primitive of the analysis avoids having to deal with feasibility constraints in the optimal allocation program as matchings are feasible by definition. It also allows an easy definition of externalities by defining valuations for each bidder n, $v_n(\mu)$, over the set of all matchings $\mathfrak{M}$. Let $v_n = \{v_n(\mu), \forall \mu \in \mathfrak{M}\}$ be the vector of valuations for bidder n. Valuations for the auctioneer can be defined analogously as $\hat{v}_0 = \{\hat{v}_0(\mu), \forall \mu \in \mathfrak{M}\}$. The auctioneer can thus incorporate preferences of agents who cannot participate directly in the auction, but who suffer the externalities of the auction outcome.

Definition (Externalities). Valuations are said to exhibit externalities if there is at least one $n \in N$ and at least two distinct assignments, $\mu, \mu' \in \mathfrak{M}$, such that $\mu(n) = \mu'(n)$ and $v_n(\mu) \neq v_n(\mu')$.

The following assumptions simplify the analysis considerably. They are for the most part standard in the auction literature and their implications and limitations are well understood.

Assumption. Valuations are characterized by the following properties.

A1 *Independent Private Values*: for every bidder n and m, v_n and v_m are independent and privately known.

A2 *Quasi-linear Utility*: the payoff a bidder n obtains from assignment μ and payment $b_n(\mu)$ is given by $\pi_n(\mu) = v_n(\mu) - b_n(\mu)$.

A3 $\min_\mu v_n(\mu) = 0, \max v_n(\mu) \leq v_{max} < \infty, \forall n \in N, \mu \in \mathfrak{M}$

A4 *Free Disposal* For any μ and μ' such that $\mu(m) = \mu'(m)$ for all $m \in N\backslash 0, n$ and $\mu(n) \subset \mu'(n)$, $v_n(\mu) \leq v_n(\mu')$.

Remark. Assumption A4 is slightly more and slightly less innocuous than it appears at first. Since preferences are ordinal, setting the lowest valuation to zero is unproblematic. It implies, however, that the status quo in which the auctioneer is in possession of all objects may have a positive payoff, and that opting out of the auction cannot guarantee the status-quo payoff. This departure from the usual assumptions is caused by the presence of externalities. With externalities, presumably, an agent is affected by the assignment irrespective of whether it participates in the auction or not. From a non-strategic point of view, non-participation is thus equivalent to participation with only zero bids.

The payoffs to the auctioneer from a given assignment μ are defined as the weighted sum of its valuation and the combined payments by the bidders, $\hat{\pi}_0(\mu) = \alpha\hat{v}_0(\mu) + (1-\alpha)\sum_{N\backslash 0} b_n(\mu)$ for some $\alpha \in [0, 1)$. Such a formulation is general enough to allow for a wide variety of auction objectives. If revenue maximization is paramount, for example, or if the auctioneer does not have good information on its valuations v_0, a larger weight may be placed on revenue. Conversely, a low weight might be associated with revenue to stress the importance of the auctioneer's valuations. In general, the correct weight will depend on the objective of the auctioneer. In order to simplify the analysis, $\pi'_0(\mu)$ can be normalized in the following way.

Definition (Auctioneer payoffs). The payoff to the auctioneer from assignment μ and bids $b_n(\mu)$ is given by

$$\pi_0(\mu) = v_0(\mu) + \sum_{N\backslash 0} b_n(\mu) \tag{2.1}$$

where $v_0(\mu) = \frac{\alpha}{1-\alpha}\hat{v}_0(\mu)$.

Since $\alpha < 1$, assumption A3 is satisfied by the auctioneer's normalized preferences.

The generalized ascending proxy auction for allocations

As its name suggests, the generalized ascending price proxy auction introduced by Ausubel & Milgrom (2002) is based on their ascending proxy auction for packages of items. At the beginning of the auction, each agent reports the vector of its valuations to a computerized proxy agent that will bid on its behalf. Bidding proceeds in rounds. After each round the auctioneer computes its preferred, provisionally winning assignment. The proxy then checks whether this assignment maximizes its bidder's payoffs. If not, it increases the bid on the payoff-maximizing allocation.

Introducing a proxy between the participants and the actual auction has two main advantages. First, it speeds up the auction considerably as a computerized proxy together with the auction algorithm eliminates the need for communication between bidders and auctioneer after every round. It also shifts strategic considerations to before the beginning of the auction when the proxies are provided with the maximum bid amounts by every bidder. In addition to an increase in auction speed this has the further advantage of reducing uncertainty about bidding strategies. Since bidders will not have to re-evaluate their bidding strategy after every round, the scope for making mistakes should be reduced.

Let $b_n^t(\mu)$ denote the bid placed by n, through its proxy bidder, on assignment μ. The provisionally winning assignment at round t, μ^{*t} maximizes the auctioneer's payoff give the bids submitted at t,[11]

$$\mu^{*t} = \arg\max_{\mu \in \mathfrak{M}} \left\{ v_0(\mu) + \sum_{N \setminus 0} b_n(\mu) \right\}$$

Remark. An auction only makes sense if α is strictly less than one, as defined above. Otherwise, the auctioneer would simply select its preferred allocation without taking any bids into consideration. A selection of $\alpha = 0$ would reduce the auctioneer to a pure revenue maximizer as $v_0(\mu) = 0$ for all μ in that case.

The lowest bid a bidder i can place on an assignment μ at round t, $\underline{b}_n^t(\mu)$ is its bid from the previous round if the bid is on the provisionally winning assignment μ^{*t-1} of the preceding round, otherwise it has to raise the bid by a minimum bid increment ϵ,

[11]Although μ^{*t} is a function of α for the sake of simplicity this relationship will not be made notationally explicit.

$$\underline{b}_n^t(\mu) \begin{cases} b_n^{t-1}(\mu) & \text{if } \mu = \mu^{*t-1} \\ \\ b_n^{t-1}(\mu) + \epsilon & \text{otherwise} \end{cases}$$

The auction ends in round $T+1$ when no new bids are placed and the provisionally winning assignment becomes final, $\mu^{*T} = \mu^*$. Each bidder pays the auctioneer the final bid, $b_n^T(\mu^*)$ placed by its proxy agent.

The profit a bidder obtains from an assignment in any round is $\hat{\pi}_n^t(\mu) = v_n(\mu) - b_n^t(\mu)$. Let the maximum profit achievable for bidder n in round t be

$$\pi_n^t = \max\left\{0, \max_{\mu \in \mathfrak{M}}(v_n(\mu) - b_n^t(\mu))\right\}$$

Of all the possible strategies a bidder could follow in the absence of the proxy two seem particularly intuitive.[12]

Definition (Straightforward Bidding). A bidder (proxy agent) is said to bid straightforward if in any round it places a bid on the assignment that would lead to a maximum profit. In other words,

$$b_n^t(\mu) \begin{cases} \underline{b}_n^t(\mu) & \text{if } \hat{\pi}_n^t(\mu) = \pi_n^t \\ \\ b_n^{t-1}(\mu) & \text{otherwise} \end{cases}$$

Definition (Limited Straightforward Bidding). A bidder (proxy agent) is said to bid limited straightforward if in any round it places a bid on the assignment that would lead to a maximum profit, *if this is bigger than some target rate of profit,* $\tilde{\pi}_n$.

[12]The terminology follows Ausubel & Milgrom (2002) as closely as possible.

In other words,

$$b_n^t(\mu)\begin{cases}\underline{b}_n^t(\mu) & \text{if } \hat{\pi}_n^t(\mu)=\max\{\tilde{\pi}_n,\pi_n^t\}\\ \\ b_n^{t-1}(\mu) & \text{otherwise}\end{cases}$$

The proxies in the ascending price allocation proxy auction are programmed to bid straightforward relative to the valuations provided by their agents. While this may sound overly restrictive, agents are by no means forced to report their valuations truthfully and can thus – at least to some extent – influence bidding behavior. In particular, limited straightforward bidding can be achieved easily by a semi-sincere reporting strategy, that is by reporting valuations that take account of the agent's profit target.

Definition (Semi-sincere strategy). Under a semi-sincere strategy a bidder reports valuations $\tilde{v}_n$ to the proxy that take account of its profit target.

$$\tilde{v}_n(\mu)=\max\{0,v_n(\mu)-\tilde{\pi}_n\}$$

Definition (Sincere strategy). A bidder is said to follow a sincere strategy if it reports its valuations truthfully to the proxy.

Remark. While (limited) straightforward bidding describes bidding behavior, sincere and semi-sincere strategies refer to the reports given by the bidders to the proxy. With proxies constrained to bid straightforward, the two concepts coincide.

One characteristic of the (limited) straightforward bidding behavior which is potentially quite attractive is that the payoffs to a bidder from all allocations on which it has placed a positive bid are within one bid increment. This implies two things. First, a bidder is approximately indifferent between which outcomes the auctioneer chooses as long as it has placed a positive bid on the resulting allocation. And second, none of the bidders who bids a positive amount on the interim allocation the auctioneer chooses has an incentive to raise their bid. In a way straightforward bidding minimizes the risk of being surprised by the end of the auction. The following lemma and its corollary show this more formally.

Lemma 1. *For any bidder n who follows a straightforward bidding strategy, in any round t,*

$$|\max\{0, v_n(\mu) - \pi_n\} - b_n^t(\mu)| \leq \epsilon \tag{2.2}$$

Proof. Equation (2.2) is implied by

$$\pi_n^t \geq v_n(\mu) - b_n^t(\mu) \geq \pi_n^t - \epsilon$$

and the fact that $b_n^t \geq 0$.

By the definition of π_n^t, $\pi_n^t \geq v_n(\mu) - b_n^t(\mu)$. Furthermore, suppose that $v_n(\mu) - b_n^t(\mu) < \pi_n^t - \epsilon$. Then, $v_n(\mu) - (b_n^t(\mu) - \epsilon) < \pi_n^t$. Let $t' < t$ be the last round at which the bid on μ was $b_n^{t'} = b_n^t - \epsilon$. Then $v_n(\mu) - b_n^{t'}(\mu) < \pi_n^t$. Since the provisional profit for bidders is weakly declining over time $\pi_n^t \leq \pi_n^{t'}$ and a straightforward bidder would not have increased its bid on μ, a contradiction to the initial supposition. □

Corollary 1. *As bid increments approach zero, the profit expected from any assignment on which a bidder places a non-zero bid is the same,*

$$\lim_{\epsilon \to 0} b_n^t = \max\left\{0, v_n(\mu) - \pi_n^t\right\}$$

Furthermore, at T, every bidder has a profit of π_n^T.

A desirable consequence of the straightforward bidding strategy is that it allows the auctioneer to maximize efficiency by assigning equal weights to revenue and its valuation. This is due to the fact that non-zero bids simply represent true valuations minus a profit demand, π_n^t, which is constant across allocations.

Corollary 2. *At T the auctioneer chooses a value-maximizing assignment,*

$$\begin{aligned}\mu^{*T} &\in \arg\max_{\mu\in\mathfrak{M}} v_0(\mu) + \sum_{N\backslash 0} b_n^T(\mu)\\ &= \arg\max_{\mu\in\mathfrak{M}} v_0(\mu) + \sum_{N\backslash 0}(v_n(\mu) - \pi_n^T)\\ &= \arg\max_{\mu\in\mathfrak{M}} \sum_{N} v_n(\mu) - \sum_{N\backslash 0}\pi_n^T\end{aligned}$$

And its payoff is $\pi_0^T = \max_{\mu\in\mathfrak{M}} \sum_N v_n(\mu) - \sum_{N\backslash 0}\pi_n^T$.

It should be noted that the efficiency property of the generalized ascending proxy auction is defined with respect to the adjusted valuations of the auctioneer and thus to α. Pure revenue maximization ($\alpha = 0$) would lead to an allocation that is efficient for the bidder in the auction in the traditional sense. Any other weight will lead to a maximum value taking into account the trade-off between revenue and pure allocative efficiency for the auctioneer, which depends on its overall objective function.[13]

Ausubel & Milgrom (2002) have suggested an alternative way of looking at an ascending price auction with straightforward bidding. By viewing the auction as a

[13]Distortions introduced by taxation, for example, might lead a government to increase the weight on revenue generation to meet its budgetary needs.

sequential coalition formation process as well as a mechanism to achieve equilibrium in a market, it can be related more broadly to general equilibrium market games such as Shapley & Shubik (1972)'s bilateral trading game. In particular, it can be described as a hybrid game in which coalitions of players cooperate, but competition exists between coalitions. Furthermore, coalitions form and re-form endogenously over the course of the auction.[14] In every round of the auction, bidders direct their profit demands, π_n^t, and a list of assignments compatible with those demands at the auctioneer. From those, the auctioneer chooses a winning coalition whose demands maximize its own payoff and implements one of the possibly multiple assignments suggested by all the members of the coalition.[15] Unlike in the case of an auction without externalities, bidders that are not in the winning coalition may receive non-zero profits and may even be allocated goods. They do, however, not realize their profit demands. If those non-winning bidders follow a (limited) straightforward strategy, they will submit reduced profit demands to the auctioneer in the subsequent round, in effect trying to block the assignment and the payoffs of the winning coalition. The auction will end when no bidder reduces its profit demands, that is when no bidder increases its bid and every bidder is part of the winning coalition. Some more notation is necessary to analyze this process more formally.

The value of a coalition of agents is commonly defined as the maximum utility

[14]Zhao (1996) examines more generally the stability of coalitions and the existence of hybrid, that is cooperative-competitive, equilibria in a game with externalities.

[15]The formation of coalitions is only implicit in the mechanism. By choosing an interim allocation, the auctioneer proposes a coalition of players who win, in the sense that they receive their profit demands.

its members can achieve by themselves. This definition may be problematic in the presence of externalities, as the utility achievable by a coalition depends on the action of agents that are non-members. For the present analysis, however, it is possible and convenient to abstract from this problem at least somewhat since the auctioneer controls all items. Coalitions that do not include the auctioneer are thus unable to affect the payoffs of the members of the winning coalition. The winning coalition, in contrast, determines the payoff of all non-winning coalitions.[16]

Definition (Coalitional Value). The coalitional value function, $V(C)$ is defined for all $C \subseteq N$ as,

$$V(C) = \begin{cases} \max_{\mu \in \mathfrak{M}} \sum_{n \in C} v_n(\mu) & \text{if } 0 \in C \\ \sum_{j \in C} v_n(\mu^*) & \text{if } 0 \notin C \end{cases}$$

where $\mu^* \in \arg\max_{\mu \in \mathfrak{M}} \sum_{m \in T} v_m(\mu)$, for $T \ni 0$.

The auction can now be described as a game (N,V) in coalition form in which the seller forms a coalition with the winning bidders to implement an assignment in exchange for payments from these bidders. The result is a vector of payoffs, π for the seller and *all* bidders. The stability of a coalition, winning or non-winning, that is the inability of at least one of its members to increase its payoffs by joining a different coalition, can now be expressed in terms of payoffs.

Definition. A payoff vector, π is feasible for coalition C if $V(C) \geq \sum_C \pi_n$.

[16]This definition implies that the auctioneer can allocate items to bidders that are not members in the winning coalition C and that all participants of the auction are denied resale or trading of units.

A payoff is feasible if it is feasible for the coalition of all players, N.

Definition. A payoff vector is unblocked if no coalition C can achieve higher payoffs by itself, $\sum_C \pi_n \geq V(C)$.

Definition (Core). The core of the coalition form game (N,V) is the set of feasible and unblocked payoff vectors, π.

The core necessarily maximizes combined payoffs in order not to be blocked by the coalition of all players. Furthermore, although the core is defined in terms of payoffs, there is at least one underlying allocation that is compatible with the core payoffs. This allocation is stable since otherwise the auctioneer could redistribute some of the items in a way that a new coalition of winners would be better off. Although it may at first seem unintuitive, the payoffs in the core may not be unique. Indeed, if utility is perfectly transferable between agents – through monetary transfers, for example – the core is a compact, convex possibly empty polyhedron in $|N| - 1$ space, defined by the feasibility and blocking constraints.[17]

The link to the general ascending proxy auction for allocations can now be made formally.

Proposition 1. *The general ascending proxy auction for allocations terminates at a payoff vector π^T that is in the core with respect to stated bidder valuations.*

Proof. Suppose the auction does not end in the core. Then there exists a coalition, C, which can achieve higher payoffs by itself. Two cases can be distinguished.

[17]For a description of general core geometry see Shapley (1972).

1. The auctioneer is part of the blocking coalition, $0 \in C$. Then,

$$\pi_0^T + \sum_{C\backslash 0} \pi_n^T < \max_{\mu\in\mathfrak{M}} \sum_C v_n(\mu)$$

$$\max_{\mu\in\mathfrak{M}} \sum_{N\backslash 0}(v_n(\mu) - \pi_n^T) + v_0(\mu) + \sum_{C\backslash 0} \pi_n^T < \max_{\mu\in\mathfrak{M}} \sum_C v_n(\mu)$$

$$\max_{\mu\in\mathfrak{M}} \sum_{N\backslash 0}(v_n(\mu) - \pi_n^T) + v_0(\mu) < \max_{\mu\ in\mathfrak{M}} \sum_C v_n(\mu) - \sum_{C\backslash 0} \pi_n^T$$

$$\max_{\mu\in\mathfrak{M}} \sum_{N\backslash 0} b_n^T(\mu) + v_0(\mu) < \max_{\mu\in\mathfrak{M}} \sum_{C\backslash 0} b_n^T(\mu) + v_0(\mu) \leq \max_{\mu\in\mathfrak{M}} \sum_{N\backslash 0} b_n^T + v_0(\mu)$$

The first inequality follows from the definition of a blocking coalition, the second line from Corollary 2. Some re-arranging and the fact that bids are weakly positive and the auctioneer does not place any bids lead to the contradiction.

2. If the auctioneer is not a member of the blocking coalition, $0 \notin C$, its payoff is determined by the value-maximizing allocation for $N\backslash C$, $\mu^{N\backslash C}$. Thus,

$$\sum_C \pi^T < \sum_C v_n(\mu^{N\backslash C})$$

$$\sum_C v_n(\mu^*) - b_n(\mu^*) < \sum_C v_n(\mu^{N\backslash C})$$

This implies that there is at least one $n \in C$ for which $v_n(\mu^*) - b_n(\mu^*) < v_n(\mu^{N\backslash C})$, contradicting lemma 1.

□

The existence of the core in the generalized ascending proxy auction becomes a much stronger result in light of prior work in the general equilibrium and matching literature. Shapley & Shubik (1969) provide an example where negative externalities lead to the non-existence of a core allocation. Roth (1984) shows that the set of stable allocations may be empty even in a one-to-one matching game. And both Sasaki & Toda (1996, 2001) and Klaus & Klijn (2003) have to impose restrictions on allowable preferences or expectations of post-blocking behavior to obtain existence.

Furthermore, Jehiel & Moldovanu (1996) observe that even in a single unit auction the core is empty unless externalities are dominated by the bidders' valuations for the item. This non-existence result seems to contradict the core property of the generalized ascending proxy auction. It can, however, be traced to two differences in the setup of Jehiel & Moldovanu (1996)'s model. First, the auctioneer in the generalized ascending proxy auction can dump objects on bidders who have not been bidding for them. Since objects are freely disposable (assumption A.4) this does not harm any bidder who obtains such an unwanted win. Moreover, while Jehiel & Moldovanu (1996) allow side-payments between bidders, in the generalized ascending proxy auction this is not permissible. That is, bidders cannot influence each other's bidding strategy and any blocking coalition has to include the auctioneer. As a result, the set of viable blocking coalitions is reduced in a way that leads to the existence of the core.

Proposition 1 does not mean, however, that bidders will report their valuations sincerely or even semi-sincerely. Thus nothing guarantees an efficient outcome with respect to true valuations. The following proposition shows that this problem is not severe. In fact, for any vector report of valuations by its opponents, any bidder has a semi-sincere best response. Given the restrictions on bidding strategies that the proxy already imposes, assuming that agents would report semi-truthfully is therefore not overly unrealistic, particularly considering the relative simplicity of such a strategy.

Proposition 2. *For any vector of valuations $\tilde{v}_{-n}$ submitted by bidder n's opponents and bidder n's report $\tilde{v}_n$, let $\mu^* = \max_{\mu \in M} \sum_N \tilde{v}_m(\mu) + \tilde{v}_n(\mu)$ be the final assignment chosen by the auctioneer. Define the highest profit n can achieve, given reports $\tilde{v}_{-n}$ as*

$$\pi_n^* = \sup \left\{ \pi_n | \mu^* \in \arg\max_{\mu \in \mathfrak{M}} [v_n(\mu) - \max(0, v_n(\mu) - \pi_n)] \right\}$$

Then, $\tilde{v}_n(\mu) = \max\{0, v_n(\mu) - \pi_n^*\}$ is a best response to $\tilde{v}_{-n}$.

Proof. Let $\hat{v}_n(\mu)$ be any other, not necessarily semi-truthful, strategy for bidder n, $\mu^0 \in \arg\max_{\mu \in \mathfrak{M}} \left\{ \sum_{N \setminus n} \tilde{v}_m(\mu) + \hat{v}_n(\mu) \right\}$ and $\hat{\pi}_n = v_n(\mu^0) - b_n(\mu^0)$. Two cases can be distinguished.

1. If $\mu^0 = \mu^*$, $\pi_n^* = \hat{\pi}_n$ by definition.

2. If $\mu^0 \neq \mu^*$ and $\hat{\pi}_n > \pi_n^*$, then

$$\pi_n^* \neq \sup \left\{ \pi_n | \mu^* \in \arg\max_{\mu \in \mathfrak{M}} [v_n(\mu) - \max(0, v_n(\mu) - \pi_n)] \right\},$$

   contradicting its definition.

□

The existence of semi-sincere best responses naturally suggests semi-sincere Nash equilibrium behavior as well. Yet it is possible to make this link even more precise: a Nash equilibrium in limited straightforward bidding – or alternatively semi-truthful reporting – strategies will terminate the auction at the bidder optimal core point, that is the payoffs of no bidder can be improved without either violating some core constraints or making another bidder worse off.

Definition (Bidder-optimal payoff). A vector of payoffs in the core of the auction game, $\pi \in \text{core}(N, V)$, is bidder-(Pareto-)optimal if there is no other payoff vector, π', such that $\pi' \in \text{core}(N, V), \pi \neq \pi'$ and $\pi'_n \geq \pi_n$ for every bidder i.

Proposition 3. *Let π be a bidder optimal point in the core. Then, there exists a Nash equilibrium supporting π with equilibrium strategies $\tilde{v}_n(\mu) = \max\{0, v_n(\mu) - \pi_n\}$. Furthermore, if the auction game has a Nash equilibrium with semi-sincere strategies, its payoffs are bidder optimal.*

Proof. Suppose, bidder n has a profitable deviation leading to a payoff vector $\pi' \in \text{core}(N, V)$ such that

$$\pi'_n > \pi_n$$

$$\pi'_w \geq \pi_l, \forall w \in W$$

$$\pi'_l < \pi_l, \forall l \in L$$

That is, relative to π, all bidders in W gain from moving to a payoff vector π', everyone in L sees their profits reduced. Let the assignments μ^* and μ' be associated with π and π', respectively.

Now two cases can be distinguished.

1. If $\mu' = \mu^*$, the equilibrium strategies guarantee that $\pi'_m \geq \pi_m \forall j \in N \backslash 0$. So $L = \emptyset$ and since $\pi'_n > \pi_n$, π would not have been bidder optimal.

2. If $\mu' \neq \mu^*$, the bidding strategies imply that $b_l(\mu') = 0, \forall l \in L$ as $v_l(\mu') < \pi_l$. Otherwise, μ' would lead to a profit of at least π_l.

In order for bidder n's deviation to change the allocation from μ^* to μ' it has to be the case that its new bid vector b'_n compensates the auctioneer for the revenue lost due those zero bids adjusted for the difference in the auctioneer's valuations, or

$$b'_n(\mu') - b'_n(\mu^*) + \sum_W b_w(\mu') - \sum_W b_w(\mu^*) + v_0(\mu') - v_0(\mu^*) > \sum_L [v_l(\mu^*) - \pi_l] \quad (2.3)$$

Again, two cases can be distinguished. In both cases, n will not bid less than $\max\{0, v_n(\mu) - \pi'_n\}$ on any assignment.

(a) If $v_n(\mu') \geq v_n(\mu^*)$, $b_n(\mu') - b_n(\mu^*) \geq b'_n(\mu') - b'_n(\mu^*)$, and making use of equation 2.3,

$$b_n(\mu') + \sum_W b_w(\mu') + v_0(\mu') > b_n(\mu^*) + \sum_W b_w(\mu^*) + \sum_L [v_l(\mu^*) - \pi_l] + v_0(\mu^*)$$

Hence, the auctioneer would never have selected assignment μ^*, contradicting our assumption.

(b) If $v_n(\mu') < v_n(\mu^*)$, it must be the case that $b_n(\mu') = 0$. But this implies that $\pi'_n > v_n(\mu')$ and a strategy b'_n leading to a new assignment μ' is not a profitable deviation.

This completes the proof of the first part of the proposition.

For the second claim, suppose π is not a bidder optimal core point. Then there exists a point π' in the core at which $\pi'_n > \pi_n$ and $\pi'_m \geq \pi_m, \forall m \in N \backslash 0, n$. By reporting $\max\{0, \tilde{v}_n(\mu) + \pi_n - \pi'_n\}$, bidder n can increase its profit to π'_n. $\square$

The second part of proposition 3 is intuitive. If the auction terminates at a core payoff that is not bidder optimal, then there must be another vector of payoffs in the core at which all bidders are weakly and some of them strictly better off. As a consequence at least one agent has an incentive to change its strategy to make the auction terminate at that point. To explain the converse is slightly more complicated. In order for a deviating bidder to induce the auctioneer to change the assignment from the initially payoff maximizing μ^*, it would have to compensate the auctioneer for the revenue lost and the difference in the value of the allocation. That is, the deviating bidder has to reduce its bid on μ^* by more than on its desired allocation μ'. But then it can be shown, either the auction would have ended at μ' with the initial reports, or the deviation is not profitable.

A similar argument shows that there is no profitable deviation from the bidder-optimal point in the core for coalitions either: bidder optimality is associated with a coalition proof equilibrium. This result can be linked to the literature on matching algorithms. Roth & Sotomayor (1990) report that agents proposing in a one-to-one matching model have an incentive to report their (ordinal) preferences truthfully to obtain their optimal core payoffs (Theorems 4.7 and 4.10.). Since the ascending price allocation proxy auction requires the reporting of *cardinal* valuations and since monetary transfers are part of the algorithm, bidders report truthfully up to a constant. This connection between bidder optimality and a Nash equilibrium in semi-sincere strategies in an auction context has previously been made by Bernheim & Whinston (1986) whose results are more general.[18]

[18] The link between Nash equilibria and bidder-optimal core payoffs relies on semi-sincere bidding.

The fact that the semi-sincere Nash equilibrium payoffs are bidder-optimal implies that the auctioneer has an incentive to misrepresent its own valuations in order to obtain a more favorable core payoff (Roth & Sotomayor 1990). Moreover, since the auctioneer choses the final allocation based in part on its reported valuations but does not make any payments, any strategy to achieve higher core payoffs cannot be semi-sincere. Unlike the bidders, the auctioneer therefore has an incentive to distort relative valuations.[19] Bernheim & Whinston (1986)'s conclusion – as the one reached in Ausubel & Milgrom (2002) and in proposition 3 – therefore requires the auctioneer to act non-strategically.

2.4 Bidder-optimal core payoffs

Since the core is closed and bounded, existence of bidder-optimal core payoffs and therefore of a semi-sincere Nash equilibrium in the ascending price proxy auction is guaranteed. The set of bidder-optimal core payoffs is not constrained to be single-valued, however, and may indeed be a continuum of payoffs. The following example is adapted from example 1.

Example 2. Bidder 1 and 2 want to prevent their competitors obtaining the object (table 2.6). Bidder 3 is only interested in winning. The core constraints are

Other equilibria may also exist and not necessarily have associated payoffs that are bidder-optimal.

[19]A very simple strategy for the auctioneer, for example, would be to raise its reported valuation for the allocation with the second highest aggregate value.

Table 2.6: Bidder valuations

	μ_0	μ_1	μ_2	μ_3
v_0	0	0	0	0
v_1	10	10	0	0
v_2	10	0	10	0
v_3	0	0	0	10

consistent with the following restrictions on the payoff vector π:

$$\pi_0 + \pi_1 + \pi_2 = 20$$

$$\pi_0 \geq 10$$

$$\pi_3 \geq 0$$

Thus any payoff vector where $\pi_0 = 10$ and $\pi_1 + \pi_2 = 10$ is bidder optimal and in the core.

A multiplicity of bidder-optimal payoff vectors raises the question about bidder coordination on a particular Nash equilibrium. In effect, if there are more than one bidder-optimal point in the core, bidders are both cooperating and competing with each other at the same time. While all bidders agree on the desirability of terminating at a bidder-optimal point, there is necessarily disagreement about *which* of them should be chosen.

This bargaining problem does not arise when the bidder-optimal core point is unique. Without externalities in valuations, Ausubel & Milgrom (2002) find that there is a unique bidder optimal point in the core if and only if the generalized

Vickrey (VCG-)payoffs are in the core. In that case, the two coincide.

Definition (Vickrey payoffs). Let π_n^{VCG} be the vector of generalized Vickrey payoffs.

$$\pi_n^{VCG} = V(N) - V(N \backslash n), \forall n \in N \backslash 0$$

$$\pi_0^{VCG} = V(N) - \sum_{N \backslash 0} \overline{\pi}_n$$

Their conclusion is based on the observation that the highest payoff a bidder can extract from the auctioneer is bounded by its contribution to the value of the coalition of all players. Any profit demand exceeding π_n^{VCG} would be blocked by the coalition of the seller with all bidders but bidder n. Although the link between VCG-payoffs and bidder optimality still holds in the presence of externalities and non-zero valuations for the auctioneer, it is no longer as straightforward. For a bidder's VCG-payoff, π_n^{VCG}, to be its highest core payoff, two conditions must hold. First, there cannot be any payoff vector π in the core where $\pi_n > \pi_n^{VCG}$; and second, bidder n must not be able to block π_n^{VCG} itself.

The first condition is obvious. The second stems from the presence of externalities and the fact that an allocation that is value-maximizing for a coalition including the auctioneer is likely to have a positive value even for non-members. In that case, it could be possible that the payoff from not being a member of the winning coalition has a higher payoff than π_n^{VCG}. The following proposition shows that this is never the case.

Proposition 4. *For any $C \ni 0$ and $n \notin C$, let $\mu^C = \arg\max_{\mu \in \mathfrak{M}} \sum_C v_m(\mu)$. Then,*

$$v_n(\mu^C) \leq V(C \cup n) - V(C)$$

Proof. The proof follows straight from the definition of the coalitional value function $V(\cdot)$. $\square$

Ausubel & Milgrom (2002)'s description of the unique bidder optimal core point then continues to hold.[20]

Proposition 5 (Theorem 5). *A bidder's Vickrey payoff, $\pi^V CG_n$, is the highest core payoff it can achieve.*

$$\pi^V CG_n = V(N) - V(N \backslash n) = max\,\{\pi_n | \pi \in \text{core}(N, V)\}$$

Proposition 6 (Theorem 6). *The core contains a unique bidder-dominant point if and only if the Vickrey payoff vector $\pi^V CG$ is in the core. If $\pi^V CG$ is in the core, it is bidder Pareto-dominant.*

Proof. The proofs can be found in Ausubel & Milgrom (2002). $\square$

The inclusion of VCG-payoffs in the core thus solve the bargaining problem between the bidders.[21] Since no bidder can ever expect a higher core payoff, a semi-sincere bidding strategy with VCG-profit demands weakly dominates all other bidding strategies. Nonetheless, coordination on this outcome is problematic. Calculating Vickrey payoffs requires information about all bidders' valuations, information that is generally private.

[20] Reference to their theorems are provided in brackets.

[21] Bikhchandani & Ostroy (2002), amongst others, have stated technical conditions on the coalitional value function under which Vickrey payoffs are in the core. Essentially, the removal of a set of agents from a coalition is required to have a larger (negative) impact on total payoffs than the sum of their VCG-payoffs.

Ausubel & Milgrom (2002) find a restriction on valuations under which the generalized ascending proxy auction induces VCG-payoffs when bidders submit their valuations truthfully. This restriction in fact accomplishes two separate tasks. First, it ensures that Vickrey payoffs are indeed in the core. This in itself is not enough, however, as the auction does not generally terminate at the bidder-optimal core point with truthful reporting. Secondly, it leads to a price adjustment path that enters the core at the bidder-optimal point. Roughly, valuations are restricted in a way that the increase in the coalitional value from adding a bidder to an existing coalition decreases in the size of the initial coalition (buyer submodularity). If this is the case, it is worthwhile for any coalition to add a bidder who demands its Vickrey payoffs.

Definition (Buyer submodularity). The coalitional value function is buyer submodular if for all $n \in N\backslash 0$,

$$V(C' \cup n) - V(C') \geq V(C \cup n) - V(C), \forall 0 \in C' \subset C$$

Buyer submodularity can also hold if valuations are characterized by externalities. Ausubel & Milgrom (2002)'s conclusion thus still applies.

Proposition 7. *If the coalitional value function is buyer submodular, truthful reporting is a Nash equilibrium strategy in the generalized ascending proxy auction leading to Vickrey payoffs for every bidder.*

Proof. The proof follows Ausubel & Milgrom (2002) in establishing first the connection between truthful reporting and Vickrey payoffs. The only difference is the interpretation of a 'winning coalition'. For Ausubel & Milgrom (2002), winners are bidders who obtain objects; in this context, every bidder whose profit demand is met is a winner. After the final round T, let C be the set of bidders who realize their profit demands π_n. From corollary 1, it is clear that only bidders in C will pay the auctioneer. Let there be a bidder n for whom $\pi_n < \pi_n^{VCG}$. Then, bidder n must be a part of the coalition that obtains its profit demands.

$$\begin{aligned}
V(C) - \sum_C \pi_m &< V(C) - \sum_C \pi_m + \left[\pi_n^{VCG} - \pi_n\right] \\
&= V(C) - \sum_C \pi_m + [V(N) - V(N \backslash n) - \pi_n] \\
&\leq V(C) - \sum_C \pi_m + [V(C \cup n) - V(C) - \pi_n] \\
&= V(C \cup n) \sum_{C \cup n} \pi_m
\end{aligned} \tag{2.4}$$

The second line uses the definition of Vickrey payoffs, the third line follows from buyers being substitutes. Equation 2.4 shows that the auctioneer will include any bidder n whose profit demands are less than Vickrey payoffs into the coalition of bidders that determine the final outcome of the auction. Since profit demands decline over the course of the auction, n's demands will be met whenever $\pi_n \leq \pi^V CG_n$. Thus with negligible bid increments $\pi_n \geq \pi_n^{VCG}$.

By Theorem 5 of Ausubel & Milgrom (2002), payoffs are bounded from above by π^{VCG}, which is the payoff from reporting values truthfully to the proxy. □

The following example makes the connection between buyer submodularity and VCG-payoffs with truthful revelation for the generalized ascending proxy auction explicit.

Example 3. Valuations are according to table 2.7.

Table 2.7: Bidder valuations

	μ_0	μ_1	μ_2	μ_3
v_0	0	0	0	0
v_1	25	25	15	0
v_2	0	0	10	0
v_3	0	0	0	20

Vickrey payoffs are given by $\pi_0^{VCG} = 20, \pi_1^{VCG} = 5, \pi_2^{VCG} = 0, \pi_3^{VCG} = 0$. They are in the core and are the termination payoffs of the generalized ascending proxy auction with truthful reporting.

The restrictions imposed by buyer submodularity on individual preferences have been stressed by Ausubel & Milgrom (2002) for cases where valuations are defined in terms of bundles of items. They find a link between the submodularity of the coalitional value function and a substitutes condition defined for units of goods. Roughly, demand for a particular item should not fall if the price of another item is increased. This rules out complementarities between items. Conditions on valuations with externalities such that the generalized ascending proxy auction has an equilibrium in sincere strategies have not been explored to date and remain an interesting area of research.

2.5 Conclusion

Externalities in valuations have been shown to cause problems for many standard auctions, unless the type of the externalities is severely restricted. Since they do not provide a mechanism to express bidder valuations fully, the auctioneer lacks the information to allocate objects efficiently. Furthermore, in some cases efficiency would require bidders to be able to pay the auctioneer for not selling certain items at all. This is clearly not possible in a standard auction.

This paper suggests the use of a variation of Ausubel & Milgrom (2002)'s generalized ascending proxy auction for cases where valuations are thought to be characterized by externalities. Instead of bidding on bundles of goods, the bidders are allowed to bid on entire allocations and can therefore influence the composition of packages awarded to their opponents. This also encourages the participation of agents who are effected by the outcome of the auction but are not interested in winning any items themselves. Furthermore, the generalized ascending proxy auction allows the auctioneer to include its own preferences explicitly and publicly in calculating the final allocation. This may be a particularly attractive feature for governments which thus incorporate the preferences of agents unable to participate in the auction. In particular, consumer surplus considerations can be incorporated into the auction process.

At the beginning of the auction, bidders submit valuations for allocations to a computerized proxy which will bid on their behalf up to the stated values. While the proxy is constrained to bid straightforwardly on payoff maximizing allocations, the bidders are free to distort their reports. Despite the presence of externalities, the auction terminates in the core with respect to the stated valuations, that is the allocation is efficient and coalitions of bidders cannot obtain a more favorable outcome from the auctioneer.

A number of desirable strategic properties of the ascending proxy auction carry over to the case when bidding is on allocations. First, independent of their opponents' reporting strategy, bidders always have a semi-sincere best response in which the relative preference order of allocations is reported truthfully to the proxy. Furthermore, it is shown that there exists a Nash equilibrium in semi-sincere strategies that takes the auction to the bidder optimal core point despite the presence of externalities. This equilibrium is efficient. Unfortunately, the bidder-optimal core payoffs and the associated equilibria are not unique. Coordination on a particular payoff therefore involves implicit negotiation between the bidders and knowledge of all bidders' valuations, making it potentially difficult to reach the equilibrium in practice. Bargaining over the final payoff will be eliminated if the bidder-optimal payoff is unique. Ausubel & Milgrom (2002) show that this is the case when Vickrey-Clarke-Groves payoffs are in the core. They also identify a technical condition (buyer submodularity) under which the generalized ascending proxy auction terminates at Vickrey payoffs, making truthful reporting of valuations a dominant strategy. This is still the case when externalities are present.

The strategic properties of the generalized ascending proxy auction and its ability to generate an efficient allocation thus make it an attractive alternative to existing auctions when the presence of externalities is suspected. Two directions of further research suggest themselves. The restrictions on valuations necessary to make truthful revelation a dominant strategy have not been explored in the context of externalities. If these are found unlikely to be met in real-world applications, empirical tests would have to establish whether bidders can coordinate on an equilibrium in the core.

Chapter 3

Externalities in a capacity auction

3.1 Introduction

Over the last decade or so, multiple-unit auctions have become important tools both for the government and private sector firms in markets where the number of participants is small and price-taking assumptions therefore do not hold. While the sale of government assets, such as mobile phone frequency spectrum, has received most public attention – due surely to the sheer size of these transactions – auctions are also commonly used for government procurement and business-to-business sales. In many cases, the participants in these auctions are firms and the items for sale are not for final consumption but are inputs in a production process.

This has an important consequence for the valuation of objects. If the participants of an auction compete in the same final goods markets, its outcome may affect the nature of *post*-auction competition through the resulting allocation of inputs. Both winners and firms not obtaining any items are interested not only in which items they win, but also in what their competitors win as the value of a particular set of inputs may depend on the identities of all winners and their respective bundles. Thus, firms have an interest in the overall allocation of goods and each winner imposes an externality, positive or negative, on all other firms.

While the presence of externalities in an auction might at first seem slightly overly theoretical and perhaps of little practical interest, Moldovanu & Ewerhart (2001), for example, recount that mobile phone operators in the recent European spectrum auctions attempted to minimize the total number of firms winning licenses. In particular, each new firm winning the right to operate a mobile phone network was thought to reduce the profit for the firms already in the market by more than a competing incumbent winning an additional license. As a consequence the existing incumbents tried to prevent the success of potential entrants.

The effects of externalities in valuations was first explored in a series of articles by Phillippe Jehiel, Benny Moldovanu and Ennio Stacchetti [1]. They find that bidders may have an incentive to pay the auctioneer not to sell an item – an outcome not possible under most traditional auction mechanisms – or make the strategic decision not to participate in the auction. Such a commitment might prevent a competitor from bidding aggressively, winning and imposing an externality on the non-participating bidder. Even if credible, however, non-participation may not lead to an efficient outcome of an auction. More generally, unless externalities are symmetric (Das Varma 2002), efficiency cannot be guaranteed (Aseff & Chade 2002, Cornet & Laan 2001, Brocas 2002). Allowing bidders to collude in order to internalize the effects they impose on each other does not remedy the problem (Caillaud & Jehiel 1998).

Ausubel & Milgrom (2002) propose an auction mechanism that allows bidders to express bids on bundles of goods rather than forcing them to bid on items separately, thereby taking account of non-additive valuations. Furthermore, they

[1] See, for example, Jehiel & Moldovanu (1996, 2000), Jehiel et al. (1996, 1999)

suggest a generalization in which bidders report preferences over outcomes of the auction in a way similar to Bernheim & Whinston (1986)'s menu auction.[2] Unlike the first-price static auction of Bernheim & Whinston (1986), however, computerized proxies bid on behalf of each bidder in an ascending price auction. The resulting allocations are shown to be stable with payoffs in the core of the auction game.

If the outcomes of the generalized ascending price auction are defined as complete allocations of items to bidders, Ausubel & Milgrom (2002)'s framework can be adapted to the case of externalities. Bids on bundles of goods are then contingent on the winning allocation, bidders can fully express valuations and the generalized ascending price auction terminates in its core with respect to reported valuations. Moreover, under certain conditions, there exists a Nash equilibrium in which bidders report their valuations truthfully to the proxy agent.

In general, the literature on auctions with externalities has treated valuations as exogenous, and the properties of various auctions in the presence of externalities has been the exclusive focus of analysis. Authors who deal with endogenous valuations – that is valuations that are determined within the context of an auction-market game – tend not to focus explicitly on the effect of externalities.[3]

The aim of this paper is thus twofold. First it attempts to incorporate the generalized multiple-unit auction of Ausubel & Milgrom (2002) into a more general

[2]By defining outcomes to include payments by the bidders, Ausubel & Milgrom (2002) in fact permit more general preferences orderings than the traditional menu auction which assumes quasi-linearity.

[3]See, for example Gale & Stegman (2001), Krishna (1999, 1993).

market game. In the first stage of the game, capacity is auctioned to a number of firms who will compete with each other in a second-stage final-goods market. Competition in the market game determines valuations for allocations in the auction, while the capacity won by each firm constrains their second-stage output decisions. Furthermore, alternative assumptions about the auction mechanism will be shown to affect the nature of competition in the final-goods market.

Secondly, the paper shows how the auctioneer's preferences can be included explicitly in obtaining the final allocation of the general ascending proxy auction. This is not only a novel theoretical feature in the auction literature, it also has important practical implications. Since the efficiency property of an auction is generally defined with respect to its participants, whenever the outcome of the auction affects non-participants, such as the consumers in an oligopolistic market, any 'efficient auction' is unlikely to maximize overall surplus. Recognizing this fact, governments have tended to include bidding and participation rules in order to stimulate competition in the final goods market. Although these rules influence the final allocation by restricting bidding behavior, they may not be as transparent and obvious in their motivation as clearly stated auctioneer preferences and may be denounced as *ad hoc* by losing bidders.

The first part of the paper, section 3.2.1, describes the Cournot game played by firms in the second period where output is limited by a capacity constraint. This constraint is the outcome of an auction run prior to the market interaction. The generalized ascending proxy auction is introduced in section 3.2.2. Unlike traditional auctions, it will lead to a capacity allocation that is efficient with respect to the valuations of both the firms and the auctioneer. Section 3.3 proposes two sets of valuations for the auctioneer that seem particularly intuitive and briefly examines their consequence on bidding strategies. It will be shown that for a wide class of auctioneer preferences, bidding behavior is unchanged. Section 3.4 concludes.

3.2 The Cournot Auction Game

Suppose there exists a market in which $N = \{1, \ldots, N\}$ firms potentially compete with each other. In the first stage of the game, each firm takes part in an auction for productive capacity. Although not necessary for the subsequent analysis, it is convenient to assume that firms do not own any capacity prior to the auction, so that the vector of capacity $k = \{k_n\}_{n \in N}$ resulting from the auction acts as a constraint on second-stage production. For simplicity, firms are assumed to engage in quantity (Cournot) competition in the final goods market. The analysis is standard.

3.2.1 Stage 2: The Cournot Game

Each firm, n, can produce perfectly divisible output, q_n, up to the firm-specific capacity constraint, k_n, at a cost of $C_n(q_n)$. All output is assumed to be homogeneous and is sold at a market price, $P(Q)$, determined by the combined output of all firms, $Q = \sum_N q_n$. Both the inverse demand function $P(Q)$ and the individual cost functions $C_n(q_n)$ are common knowledge. Firms simultaneously make their output decision $q_n \leq k_n$, production takes place and all output is sold. Profit v_n for each firm is thus a function of its own output decision and the combined output of its competitors $Q_{-n} = \sum_{m \in N \setminus n} q_m$:

$$v_n(q_n, Q_{-n}) = P\left(q_n, Q_{-n}\right) q_n - C_n\left(q_n\right) \tag{3.1}$$

Each firm maximizes its profit given the output of its competitors and the capacity constraint. The resulting reaction function $q_n^*(Q_{-n})$ is obtained from the first order condition of the constrained profit maximization.

Lemma 2. *Given the combined output of its competitors Q_{-n}, firm n's reaction function $q_n^*(Q_{-n}, k_n)$ is given by*

$$q_n^*(Q_{-n}, k_n) = \min\left\{\tilde{q}_n, k_n\right\}$$

where $\tilde{q}_n$ solves

$$\tilde{q}_n : \frac{\partial}{\partial Q} P\Big(\tilde{q}_n, Q_{-n}\Big)\tilde{q}_n + P\Big(\tilde{q}_n, Q_{-n}\Big) = \frac{\partial}{\partial q_n} C_n(\tilde{q}_n)$$

The Nash equilibrium of the Cournot game is found as the intersection of all firms' reaction functions. It can be described by the first order conditions of each firm's profit maximizing exercise.

Lemma 3. *The constrained Cournot Nash equilibrium output levels $q^*(k) = \{q_n^*\}_{n \in N}(k)$ are determined by the following conditions:*

$$q_n^*(k) = \min\left\{\tilde{q}_n, k_n\right\} \tag{3.2}$$

where $\tilde{q}_n$ solves

$$\tilde{q}_n : \frac{\partial}{\partial Q} P\Big(\sum_{N \setminus n} q_m^* + \tilde{q}_n\Big) q_n + P\Big(\sum_{N \setminus n} q_m^* + \tilde{q}_n\Big) = \frac{\partial}{\partial q_n} C_n(\tilde{q}_n) \tag{3.3}$$

The unconstrained Cournot Nash equilibrium is given by $q_n^* = \tilde{q}_n$.

The following assumptions guarantee that a finite q^* exists and that it is indeed a maximum. While common in the analysis of Cournot games, it is not necessarily obvious that they would hold in practice.

Assumption. Market demand $P(Q)$ and the individual cost functions $C_n(q_n)$ are continuous and characterized by the following properties.

A1.1 $\overline{P} \geq P(0) \geq C_n(0) \, \forall n \in N$ and $\infty > \overline{P}$.

A1.2 Concave demand: $\frac{\partial}{\partial Q} P(Q) \leq 0$ and $\frac{\partial^2}{\partial Q^2} P(Q) \leq 0$.

A1.3 Convex cost: $\frac{\partial}{\partial q_n} C_n(q_n) \geq 0$ and $\frac{\partial^2}{\partial q_n^2} C_n(q_n) \geq 0 \, \forall n \in N, q_n \leq k_n$.

It is a well known result that the unconstrained Cournot Nash equilibrium is unlikely to maximize efficiency measured as the combined profit of all firms. By lowering the market price, an increase in the production of any firm lowers the profit of all other firms. Profit maximizing firms do not account for this negative externality when making their output decision and therefore produce above the efficient level. For any coalition of firms $L \subseteq N$ there exist an output vector q^L which maximizes the combined profit of this coalition, the coalition-efficient output level. Due to the externality, coalition-efficiency involves fixing the output for both members and non-members. Ignoring capacity constraints, coalition efficiency is defined as follows.

Definition (Coalition-efficient output). For any coalition $L \subseteq N$ the coalition-efficient (L-efficient) output q^L is the vector of output levels $\{q_m\}_{m \in N}$ that maximizes combined coalitional profit.

$$q^L = \arg \max_{q_1,\ldots,q_N} \left\{ \sum_L P\left(\sum_N q_m\right) q_m - C(q_m) \right\}$$

$$\text{subject to } 0 \leq q_n$$

Lemma 4. *The coalition-efficient output levels $\{q_n^L\}_{n \in N}$ are determined by the following first order conditions:*

$$q_n^L : \frac{\partial}{\partial Q} P\left(\sum_L q_n^L\right) \sum_L q_n^L + P\left(\sum_L q_n^L\right) - \frac{\partial}{\partial q_n} C_n(q_n^L) = 0, \qquad \forall n \in L \qquad (3.4)$$

$$q_m^L : q_m^L = 0 \qquad \forall m \notin L \qquad (3.5)$$

Comparing the first-order conditions for the unconstrained Cournot equilibrium with the coalition-efficient outcome (equations (3.2) and (3.4)) shows that any coalition of firms with more than a single member would like to reduce output relative to the Cournot outcome. The following propositions further characterize the coalition-efficient output.

Proposition 8. *For any two coalitions L and L' such that $L \subset L'$, $\sum_{L'} q_n^{L'} \geq \sum_L q_n^L$.*

Proof. Suppose $\sum_{L'} q_n^{L'} < \sum_L q_n^L$. Then assumptions A1.1 to A1.3 and Lemma 4 would require that for each firm $n \in L'$ $q_n^{L'} \geq q_n^L$, generating a contradiction. □

Proposition 9. *Unless costs are linear, for any two coalitions L and L' such that $L \subset L'$, and all $n \in L$, $q_n^L \geq q_n^{L'}$.*

Proof. The proof is again by contradiction. Suppose that there exists a firm $n \in L$ for which $q_n^L < q_n^{L'}$. Then, by assumption A1.3 and ruling out linear cost functions,

$\frac{\partial}{\partial q_n} C_n(q_n)|_{q_n=q_n^L} > \frac{\partial}{\partial q_n} C_n(q_n)|_{q_n=q_n^{L'}}$. Lemma 4 then requires that $\sum'_L q_m^{L'} \leq \sum_L q_m^L$. But this is impossible as Proposition 8 has shown. $\square$

Thus adding firms to a coalition will increase its coalition-efficient output and reduce the production and profits of each member of the smaller initial coalition unless costs are linear. With linear costs, the coalition-efficient output is invariant with coalition size and any division of production between its members is equivalent from the coalition perspective.[4] In both cases, the average profit for members of the initial coalition falls.

Of course, coalition-efficient output levels are not an equilibrium of the unconstrained Cournot market game. Firms which are not members of the coalition and would therefore be required not to produce at all clearly have an incentive to increase output. Furthermore, in the absence of binding contracts members of the coalition, too, would profit from raising production.[5] As a consequence not even a coalition of size $|N|$ can achieve efficiency without some external commitment mechanism.

The capacity auction in the first stage of the game can act as such a commitment mechanism. By determining the capacity for each firm, it effectively imposes a constraint upon output in the market game. Potentially at least, firms could thus use the first-period auction to allocate capacity in a way such that the constrained Nash equilibrium of the Cournot game would be equivalent to some coalition-efficient

[4]If costs are strictly convex, increasing the coalition size in effect reduces coalition marginal costs, thereby increasing the optimal output level.

[5]See equations (3.2) and (3.4).

outcome. The following analysis shows that with any traditional auction where the bidder with the highest valuation obtains an object such hopes are unfounded.

Suppose the total capacity available for sale is equal to the unconstrained Cournot output. Since there exists no equilibrium in pure strategies with aggregate output greater than Cournot output, this assumption ensures that no exogenous capacity limit influences the results.

Assumption. The total level of capacity available in the first-round auction K is weakly higher than the combined unconstrained Nash equilibrium output of the Cournot game: $K \geq \sum_N q_n^*$.

In order for any coalition of size smaller than $|N|$ to attain efficiency it has to be able to withhold capacity from all non-members. Since total capacity available exceeds the efficient level of any size coalition, this would require a coalition to win more than required for efficiency and commit itself not to exceed efficient production in the second period. Credibility issues and collective property rights aside, this is not possible. The combined loss to any coalition from the initiation of production by a non-member is lower than the benefit of producing to the latter. Therefore, no coalition can profitably prevent production by a non-member.

Proposition 10. *For any coalition $L \subset N$ and any $m \notin L$,*

$$\left| \sum_{n \in L} \frac{\partial \pi_n}{\partial q_m} |_{q_n^L} \right| \leq \left| \frac{\partial \pi_m}{\partial q_m} |_{q_m^L} \right|$$

Proof. From equation (3.1) and Lemma 4, the change in profit from an incremental increase in production for firm m at q_m^L is

$$\frac{\partial \pi_m}{\partial q_m}|_{q^L} = P\left(\sum_{n \in L} q_n^L\right) - \frac{\partial}{\partial q_m} C_m(0)$$

The loss for coalition L is

$$\sum_{n \in L} \frac{\partial \pi_n}{\partial q_m}|_{q^L} = \frac{\partial}{\partial Q} P\left(\sum_L q_n^L\right) \sum_L q_n^L$$

From assumption [A1.3] and the first order condition equation (3.4) it follows that $\left|\frac{\partial \pi_m}{\partial q_m}|_{q^L}\right| \geq \left|\sum_{n \in L} \frac{\partial \pi_n}{\partial q_m}|_{q^L}\right|$. □

Moreover, the following proposition shows that the gain to any firm from increasing production at the efficient level is greater than the loss to any other single firm. That is, no single firm can profitably prevent an expansion of production by any other member of the coalition – at least not without itself raising production. This is true for coalitions of all size.

Proposition 11. *At the vector of coalition-efficient output levels q^L,*

$$\left|\frac{\partial \pi_n}{\partial q_m}|_{q^L}\right| < 0 < \left|\frac{\partial \pi_m}{\partial q_m}|_{q^L}\right|$$

Proof. From equation (3.1),

$$\begin{aligned}\frac{\partial \pi_m}{\partial q_m}|_{q^L} &= \frac{\partial}{\partial Q} P\left(\sum_N q_l\right) q_m^L + P\left(\sum_N q_m\right) - \frac{\partial}{\partial q_m} C(q_m^L) \\ &\geq 0\end{aligned}$$

The loss to a firm $n \in N \backslash n$ is obtained from the same equation,

$$\frac{\partial \pi_n}{\partial q_m}|_{q^L} = \frac{\partial}{\partial Q} P\left(\sum_N q_l\right) q_n^L$$

The proof then follows from the first-order condition equation (3.4). □

As a consequence, a traditional auction in the first period of the game cannot induce an allocation of capacity that would guarantee efficient output levels in the second period market game. This is the result of the particular nature of the externalities in the Cournot game and the limitations of traditional auction mechanisms. Since firms have an incentive to raise output beyond the efficient level, in order for production to be limited, the auctioneer would have to retain some capacity. A voluntary commitment by firms not to bid for this capacity is not credible. Rather, bidders would have to pay the auctioneer not to sell, an outcome not possible in traditional auctions, as Jehiel et al. (1999) observe.

Moreover, even if the auctioneer could be paid not to sell a certain number of units, Proposition 11 shows that no single bidder could do so profitably. Although an increase in production by a firm reduces aggregate profits, the fall in profit for any *individual* firm is smaller than the increase in profit of the deviating firm at the efficient level of output. In order to outbid a firm wishing to acquire additional capacity, its competitor would have to pay a price exceeding the loss it would sustain were the capacity sold. Since traditional auctions do not allow bidders to bid jointly – on the auctioneer not selling, in this case – they generally cannot lead to efficient capacity constraints for the Cournot game.

This inability of traditional auctions to lead to efficiency can be traced to a constraint on communication between bidders and the auctioneer in traditional auctions. Since each unit of capacity sold to a firm – up to its optimal output given by Lemma 2 – allows it to increase production in the second period, each unit sold also reduces the profits of all other firms. Unless bidders are able to communicate the magnitude of this externality to the auctioneer, it lacks the information necessary to find the efficient allocation. Stated slightly differently, while firms' valuations are in terms of the capacity allocation to *all* firms, they can only communicate the value of capacity *they* win.

Allowing bidders to express externalities by making bids on capacity contingent on the entire allocation of capacity overcomes this shortcoming. First, firms can quantify the value of unused capacity, either retained by the auctioneer or allocated to bidders whose capacity constraint is already non-binding. And second, by combining the externalities reported by all firms through their bids, the auctioneer can find the efficient allocation. Ausubel & Milgrom (2002)'s generalized ascending proxy auction, a dynamic generalization of Bernheim & Whinston (1986)'s menu auction, can be used to this end. The following section outlines the auction mechanism briefly. A more detailed description of the auction in the context of externalities can be found in Ranger (2004).

3.2.2 Stage 1: The Generalized Ascending Proxy Auction

The generalized ascending proxy auction due to Ausubel & Milgrom (2002) allows firms to place bids on entire allocations rather than simply units of capacity. It is therefore directly concerned with the final allocation of capacity, with bidders being able to influence the units allocated to their competitors as well as to themselves. This has a further implication. Since bidders can directly bid on the levels of capacity to all other participants in the auction, participation may be profitable even to agents that are not interested in winning capacity themselves. A bidder representing consumer interests, for example, could try to bid in a way that would slacken the capacity constraints for the second-period market game.

Taking allocations as the primitive of the auction also allows the inclusion of seller preferences in the winner determination. Rather than use bidding and participation rules that could be criticized as *ad hoc* by losing bidders, the auctioneer can thus explicitly achieve an objective other than pure revenue maximization. This may be particularly attractive for governments who – due to consumer surplus considerations – are interested in fostering competition in final goods markets and the more direct approach of inviting consumer representatives to bid is not practical.

The generalized ascending proxy auction can be described as a mechanism in which bidders in subsequent rounds suggest an allocation together with a payment to the auctioneer who selects its profit maximizing allocation. The auction ends when no bidder is making a further proposal.[6] All firms $N = \{1, \ldots, N\}$ participate in the auction. Denote the auctioneer by $n = 0$. Then, an allocation of capacity can be defined as a vector, k, which assigns to each participant, including the auctioneer, a level of capacity k_n.

Definition. An allocation k is a vector of capacity constraints $k = \{k_0, k_1, \ldots, k_N\}$ such that

1. $k_n \geq 0 \; \forall n \in N$

[6]For a detailed description of the more general auction mechanisms see Ausubel & Milgrom (2002). Ranger (2004) applies the auction to the special case where bids are on allocations.

2. $\sum_{n \in N} k_n = K$

where K is the predetermined level of total capacity available.

Let $\mathfrak{K}$ be the set of all possible allocations.

The value of a particular allocation k to a firm n can be found by combining its profit function from the market game (equation (3.1)) with all firms' optimal output decisions constrained by the capacity constraint k as derived from Lemma 2. Unlike the profit function in a standard auction, it is defined over the set of all allocations $\mathfrak{K}$: $v_n = \{v_n(k), \forall k \in \mathfrak{K}\}$. Since firms are not required to produce up to the capacity constraints in the second period and final output is assumed to be homogeneous, the valuations for different allocations may be the same.

Definition. The value $v_n(k)$ to firm n of an allocation $k \in \mathfrak{K}$ is given by

$$v_n(k) = P\left(\sum_N q_m^*(k)\right) q_n^*(k) - C_n(q_n^*(k)) \tag{3.6}$$

The value for the auctioneer is $v_0(k)$.

At the beginning of the auction, each bidder submits a vector of values to a computerized proxy that will bid in a predetermined way on its behalf.[7] Let $b_n^t(k)$ denote the bid placed by bidder n's proxy in round t on assignment k. The auctioneer then selects a provisionally winning assignment k^{*t} which maximizes its

[7]Using a proxy has two main advantages. First, it speeds up the auction as the interaction of the proxy with the auction algorithm makes communication between the auctioneer and the bidders after each round redundant. Secondly, since the bidding behavior of the proxy is known, it shifts strategic considerations to the beginning of the auction, thereby potentially reducing the scope for mistakes.

own payoffs. This payoff includes the valuation of the auctioneer and the sum of the submitted bids.[8]

$$k^{*t} = \arg\max_{k \in \mathfrak{K}} \sum_{N \setminus 0} b_n^t(k) + v_0(k)$$

In each round, the payoff a bidder obtains from a particular allocation is given by the value this allocation provides in the second period and the bid placed, $\hat{\pi}_n^t(k) = v_n(k) - b_n^t(k)$. The maximum payoff bidder n can achieve in round t is then

$$\pi_n^t = \max \left\{ 0, \max_{k \in \mathfrak{K}} (v_n(k) - b_n^t(k)) \right\}.$$

The computerized proxy is programmed to bid straightforwardly (myopic bidding) relative to the values give by the firms. That is, the proxy raises the bid on those allocations that achieve the maximum payoff for its bidder unless they were provisionally winning allocations in the previous round. In other words,

$$b_n^t(k) = \begin{cases} \underline{b}_n^t(k) & \text{if } \hat{\pi}_n^t(k) = \pi_n^t \\ \\ b_n^{t-1}(k) & \text{otherwise} \end{cases}$$

[8] Not specifying $v_0(k)$ in terms of second-period output allows the auctioneer have more general preferences.

where

$$\underline{b}_n^t(k) = \begin{cases} b_n^{t-1}(k) & \text{if } k = k^{*t-1} \\ \\ b_n^{t-1}(k) + \epsilon & \text{otherwise} \end{cases}$$

and ϵ is the minimum bid increment in the auction.

The auction ends in round $T+1$ when no new bids are placed and the provisionally winning assignment becomes final, $k^{*T} = k^*$. Each bidder pays the auctioneer the final bid, $b_n^T(k^*)$.

While straightforward bidding by the proxy restricts the price path of the auction, bidders can influence the point at which their proxy stops bidding through their value reports. A particularly simple strategy is for a firm n to require the proxy to demand at least a profit target π_n by shading the maximum bids uniformly. This semi-sincere bidding strategy involves submitting valuations $\tilde{v}_n(k) = \max\{0, v_n(k) - \tilde{\pi}_n\}$. Ausubel & Milgrom (2002) have shown that bidders always have a semi-sincere best response, that is, the analysis of the auction can be restricted to semi-sincere strategies without loss of generality. If bidders report semi-sincerely, the auction terminates at an allocation that generates the highest profits in the second-stage constrained Cournot game consistent with the auctioneer's preferences.

Proposition 12. *At T the auctioneer chooses a capacity allocation that maximizes* *the combined payoffs of all firms and the auctioneer.*

$$\begin{aligned} k^{*T} &\in \arg\max_{k \in \mathfrak{K}} v_0(k) + \sum_{N\backslash 0} b_n^T(k) \\ &= \arg\max_{k \in \mathfrak{K}} v_0(k) + \sum_{N\backslash 0} (v_n(k) - \pi_n^T) \\ &= \arg\max_{k \in \mathfrak{K}} \sum_N v_n(k) - \sum_{N\backslash 0} \pi_n^T \end{aligned}$$

where

$$v_n(k) = P\left(\sum_N q_m^*(k)\right) q_n^*(k) - C_n\left(q_n^*(k)\right)$$

And its payoff is $\pi_0^T = \max_{k \in \mathfrak{M}} \sum_N v_n(k) - \sum_{N\backslash 0} \pi_m^T$.

Proof. The first line of the argument is the auctioneer's payoff maximization problem. The proposition then follows from the straightforward bidding of the proxies and semi-sincere reporting. □

Moreover, at T no coalition of firms can offer an allocation-bid combination to the auctioneer that would improve the payoffs of all of its members weakly and for at least one of them strongly and that would be accepted by the auctioneer. In other words, the outcome of the auction is stable and its associated payoffs are in the core of the auction game, core(N, V) where $V(L) = \max \sum_L v_n(k)$.[9]

Proposition 13. *The capacity auction terminates at a payoff vector π^T that is in the core with respect to the reported valuations.*

[9]For a more detailed analysis of the core of the auction see Ausubel & Milgrom (2002) and Ranger (2004).

Proof. Suppose the auction does not lead to the core payoffs. Then there exists a coalition of firms L which can achieve higher payoffs for itself and the auctioneer.

$$\pi_0^T + \sum_{L\backslash 0} \pi_n^T < \max_{k\in\mathfrak{K}} \sum_{L} v_n(k)$$

$$\max_{k\in\mathfrak{K}} \sum_{N\backslash 0} (v_n(k) - \pi_n^T) + v_0(k) + \sum_{L\backslash 0} \pi_n^T < \max_{k\in\mathfrak{K}} \sum_{L} v_n(k)$$

$$\max_{k\in\mathfrak{K}} \sum_{N\backslash 0} (v_n(k) - \pi_n^T) + v_0(k) < \max_{k \ in\mathfrak{K}} \sum_{C} v_n(k) - \sum_{L\backslash 0} \pi_n^T$$

$$\max_{k\in\mathfrak{K}} \sum_{N\backslash 0} b_n^T(k) + v_0(k) < \max_{k\in\mathfrak{K}} \sum_{L\backslash 0} b_n^T(k) + v_0(k)$$

$$\leq \max_{k\in\mathfrak{K}} \sum_{N\backslash 0} b_n^T + v_0(k)$$

The first inequality follows from the definition of a blocking coalition, the second line from Proposition 12. Some re-arranging and the fact that bids are weakly positive and the auctioneer does not place any bids lead to the contradiction. □

Proposition 13 limits the payoff a firm can achieve in the auction. Since final payoffs have to be in the core, no reporting strategy can lead to a payoff that exceeds the highest payoff compatible with the core constraints. Otherwise, there would be a coalition of firms that can block the outcome by making new bids which would raise both their own and the auctioneer's payoffs. Since the point at which the auction terminates depends on the amount π_n by which each bidder n shades its true valuations, a link between the highest bidder payoffs and Nash equilibrium in semi-sincere strategies can now be established.

Definition (Bidder-optimal payoff). A vector of payoffs in the core of the auction game, $\pi \in \text{core}(N, V)$, is bidder-(Pareto-)optimal if there is no other payoff vector, π', such that $\pi' \in \text{core}(N, V), \pi \neq \pi'$ and $\pi'_n \geq \pi_n$ for every bidder n.

Proposition 14. *Let π be a bidder optimal point in the core. Then, there exists a Nash equilibrium supporting π with equilibrium strategies $\tilde{v}_n(k) = \max\{0, v_n(k) - \pi_n\}$. Furthermore, if the auction game has a Nash equilibrium with semi-sincere strategies, its payoffs are bidder optimal.*

Proof. See Ranger (2004) for detailed proof. □

This result is stronger than it might appear at first. Despite the presence of externalities, the generalized ascending proxy auction possesses a Nash equilibrium that is both efficient and relatively simple from a strategic point of view. Unlike in traditional auctions, by reporting their valuations semi-truthfully firms can bid on capacity constraints that reflect the externalities cause by increased production in the second period. Together with the ability of the auctioneer to express its own valuations this permits the mechanism to find a value-maximizing capacity allocation.

Furthermore, the auction offers firms the highest overall payoffs that are compatible with the auctioneer's preferences and a stable capacity allocation. The link between the Nash equilibrium in semi-sincere strategies and bidder optimality is intuitive. At a bidder-optimal point in the core no bidder can demand higher profits without reducing the payoffs to at least one opponent. Since this opponent demands its own bidder-optimal payoff, however, this is not possible.[10] Although

[10]Optimal core payoffs and semi-sincere reporting of preferences have been examined in the

bidder-optimality cannot be linked to a single pessimal payoff for the auctioneer, it is clear that since the maximum value of the capacity allocations is fixed bidder and seller optimality are mutually exclusive.

Although semi-sincere bidding is strategically simple, the informational requirements are non-trivial. In order for firms to find a bidder optimal core payoff, knowledge of the auctioneer's and all other firms' value functions are required. Transparency and an imposed non-strategic behavior of the auctioneer suggest that the auctioneer's value function be published before the auction. Furthermore, firms might be able to arrive at good estimates of each others' value functions. The demand function in the second period is identical for each firm and assumed to be common knowledge, so that the only uncertainty arises from different cost functions. Since the firms operate in the same market, they might have relatively precise ideas about inter-firm cost differences and can approximate the resulting value functions, making it possible to estimate each others' value function with some precision.

A more serious problem, perhaps, is the possible multiplicity of bidder-optimal core points and the resulting multiplicity of Nash equilibria in semi-sincere strategies. Coordinating on a particular equilibrium strategy then involves both cooperation of the firms against the auctioneer to terminate at a bidder-optimal payoff and competition over *which* of them to select. The auction mechanism does not give an *a priori* answer to what is essentially the outcome of a bargaining game between the firms over the surplus from the auction. This problem obviously disappears when the bidder-optimal core point is unique. Ausubel & Milgrom (2002) have shown

literature on matching before. For an overview see Roth & Sotomayor (1990).

that the bidder-optimal core payoff is unique if and only if the vector of Vickrey-Clarke-Groves payoffs is contained in the core. In that case the two coincide.[11]

Definition. The Vickrey-Clarke-Groves (VCG) payoff, π_n^{VCG}, for firm n is given by

$$\begin{aligned}
\pi_n^{VCG} &= \max_{k\in\mathfrak{K}} \sum_N v_m(k) - \max_{k\in\mathfrak{K}} \sum_{N\backslash n} v_m(k) \\
&= \max_{k\in\mathfrak{K}} \Bigg[P\Bigg(\sum_{N\backslash 0} \min\left\{q^{N\backslash 0}, k\right\} \Bigg) \sum_{N\backslash 0} \min\left\{q^{N\backslash 0}, k\right\} \\
&- \sum_{N\backslash 0} C_m\left(\min\left\{q^{N\backslash 0}, k\right\}\right) + v_0(k) \Bigg] \\
&- \max_{k\in\mathfrak{K}} \Bigg[P\Bigg(\sum_{N\backslash 0} \min\left\{q^{N\backslash 0,n}, k\right\} \Bigg) \sum_{N\backslash 0,n} \min\left\{q^{N\backslash 0,n}, k\right\} \\
&- \sum_{N\backslash 0,n} C_m\left(\min\left\{q^{N\backslash 0,n}, k\right\}\right) + v_0(k) \Bigg]
\end{aligned}$$

where $q^{N\backslash O}$ is the N-optimal output level in the Cournot game defined by Lemma 4.

Although the presence of VCG-payoffs in the core ensures the existence of a unique Nash equilibrium in semi-sincere strategies in the generalized ascending proxy auction it is not sufficient to support a Nash equilibrium with truthful revelation of valuations. Only if the auction terminates at the Vickrey-Clarke-Groves payoffs when bidders submit their valuations truthfully to the proxies do they not have an incentive to shade their bids. Ausubel & Milgrom (2002) identify a technical condition on bidder valuations, buyer-submodularity, for this to be the case. Buyer

[11] Vickrey-Clarke-Groves payoffs are the result of an incentive compatible mechanisms developed by Clarke (1971) and Groves (1973) as a generalization of Vickrey (1961)'s second price auction. Bikhchandani & Ostroy (2002), amongst others, have identified conditions under which the VCG-payoffs are contained in the core and the bidder-optimal core point is unique as a consequence.

submodularity requires the increase in combined coalitional profits from adding another firm to the coalition to decline in the size of the original coalition.

Definition (Buyer submodularity). Valuations are buyer submodular if for all $n \in N$,

$$\max_{k \in \mathfrak{K}} \sum_{L \cup n} v_m(k) - \max_{k \in \mathfrak{K}} \sum_{L} v_m(k) \leq \max_{k \in \mathfrak{K}} \sum_{L' \cup n} v_m(k) - \max_{k \in \mathfrak{K}} \sum_{L'} v_m(k), \, \forall 0 \in L' \subset L$$

Proposition 15. *If the valuations are buyer submodular, truthful reporting is a Nash equilibrium strategy in the generalized ascending proxy auction leading to Vickrey payoffs for every bidder.*

Proof. See Ausubel & Milgrom (2002). □

If truthful reporting of valuations guarantees the firms their VCG payoffs and therefore the largest payoffs they can hope to achieve in the core of the auction game, both the strategic and informational requirement of the mechanism are much simplified. It should be noted that – although the auctioneer is not a buyer in the auction – its valuations affect the final allocation and the prices paid by the firms and therefore have to be included in the buyer-submodularity argument.

3.3 Auctioneer Preferences and Vickrey Payoffs

While the valuations for the firms are derived from the second-period constrained Cournot game, no restrictions have so far been placed on the auctioneer's preferences over capacity allocations. In the following paragraphs some plausible assumptions about the objective of the auctioneer will be made and their effect on the outcome of the auction will examined.

3.3.1 Pure Revenue Maximization

A purely revenue maximizing auctioneer is indifferent among the final capacity allocations and selects the final allocation that maximizes the combined bids by all firms. In this case, $v_0(k) = 0$ for all $k \in \mathfrak{K}$ and the provisionally winning capacity allocation after round t is found by

$$k^{*t} = \arg\max_{k \in \mathfrak{K}} \sum_{N \setminus 0} b_n^t(k)$$

The following results characterize the outcome of the generalized ascending proxy auction.

Proposition 16. *If the auctioneer is indifferent between allocations, $v_0(k) = 0$ for all $k \in \mathfrak{K}$, all coalitionally stable pure strategy Nash equilibria of the generalized ascending proxy auction result in the N-efficient capacity level, $k^{*T} = q^N$.*

Proof. Suppose the auction ends at a capacity vector $\tilde{k} \neq q^N$. Then, by the definition of N-efficiency,

$$P\left(\sum_N q_n^*(\tilde{k})\right) \sum_N q_n^*(\tilde{k}) - \sum_N C_n\left(q_n^*(\tilde{k})\right) < P\left(\sum_N q_n^N\right) \sum_N q_n^N - \sum_N C_n\left(q_n^N\right)$$

Therefore there exists a coalition W of firms for which $P\left(\sum_N q_n^N\right) q_w^N - C_w\left(q_w^N\right) -$

$P\left(\sum_N q_n^*(\tilde{k})\right) q^*w(\tilde{k}) - C_w\left(q_w^*(\tilde{k})\right) > 0$ and

$$\begin{aligned}
P\Bigg(\sum_N q_n^N\Bigg) \sum_W q_w^N - \sum_W C_w\left(q_w^N\right) - P\Bigg(\sum_N q_n^*(\tilde{k})\Bigg) \sum_W q_w^*(\tilde{k}) \\
- \sum_W C_w\left(q_w^*(\tilde{k})\right) \\
> P\Bigg(\sum_N q_n^*(\tilde{k})\Bigg) \sum_{N\backslash W} q_n^*(\tilde{k}) - \sum_{N\backslash W} C_n\left(q_n^*(\tilde{k})\right) - P\Bigg(\sum_{N\backslash O} q_n^N\Bigg) \sum_{N\backslash W} q_n^N \\
- \sum_{N\backslash W} C_n\left(q_n^N\right).
\end{aligned}$$

The coalition W has a profitable deviation that would lead the auction to terminate at $k^{*T} = q^N$ which furthermore does not require transfers between the members of the coalition. □

More importantly, the following proposition establishes the link between truthful reporting and Vickrey-Clarke-Groves payoffs when the auctioneer is indifferent between allocations.

Proposition 17. *If the auctioneer is indifferent between allocations, $v_0(k) = 0$ for all $k \in \mathfrak{K}$, truthful reporting of valuations $v_n(k)$ by all bidders, $n \in N\backslash 0$ is a Nash equilibrium.*

Proof. The proof proceeds by showing that the valuations derived from the Cournot game are buyer-submodular. First, define an arbitrary coalition $M \subset N$ such that $0 \in M$ and a firm $n \in N\backslash M$. Then the maximum profit the coalition of M and n can achieve, given firm n's output q_n is

$$\max_{q_m, m\in N\backslash n} P\left(\sum_N q_m\right)\left(\sum_M q_m + q_n\right) - \sum_M C_m\left(q_m\right) - C_n\left(q_n\right) \tag{3.7}$$

and let $q^M(q_n)$ be the maximizing argument. Clearly $q^M(q_n)$ is a function of q_n. Furthermore, $q^M_m(q_n) = 0$ for all $m \notin M$. Using the definition of buyer submodularity and $q^M(q_m)$ from equation (3.7), the valuations from the Cournot game are buyer submodular if, for all $0 \in L' \subset L$,

$$
\begin{aligned}
&\int_0^{q^{L\cup n}} \frac{\partial}{\partial Q} P\Big(\sum_L q^L(q_n) + q_n\Big)\left[\sum_L q^L(q_n) + q_n\right] \\
&\qquad\qquad + P\Big(\sum_L q^L(q_n) + q_n\Big) - \frac{\partial}{\partial q_n} C_n(q_n) dq_n \\
&\leq \int_0^{q^{L'\cup n}} \frac{\partial}{\partial Q} P\Big(\sum_{L'} q^{L'}(q_n) + q_n\Big)\left[\sum_L q^{L'}(q_n) + q_n\right] \\
&\qquad\qquad + P\Big(\sum_{L'} q^{L'}(q_n) + q_n\Big) - \frac{\partial}{\partial q_n} C_n(q_n) dq_n \quad (3.8)
\end{aligned}
$$

where $q^{L\cup n}$ and $q^{L'\cup n}$ is the coalition efficient output for firm n as a member of coalition L and L', respectively.

The concavity of the demand function and the fact that $\sum_L q^L(q_n) \leq \sum_{L'} q^{L'}(q_n)$ for each q_n guarantee that the integrand in the second line of equation (3.8) exceeds the one in the first. Furthermore, by Proposition 9, q$^{L\cup n} \leq q^{L'\cup n}$. Since both the definition of submodularity and equation (3.7) involve maximization of combined profits the capacity constraint must be binding and the above argument translates to valuation expressed in capacity vectors. The Cournot market game induces valuations that are bidder submodular. □

The buyer submodularity property of firms' valuations is a direct result of the properties of the demand and cost functions. Adding a firm to an existing coalition raises total coalitional output while reducing optimal production levels for each individual firm. This, in turn, reduces both the market price obtained by the firms as well as their cost of production, resulting in higher combined profits. If the firm is added to a larger coalition, however, the fall in the price caused by the increase in production is larger due to the concavity of the demand function. Furthermore, since firm's production levels decline in coalition size, the convexity of the cost function diminishes the reduction in costs from lower firm output for larger coalitions, as firms are operating in the flatter portion of their cost function. As a consequence the benefit of incorporating a firm into a coalition declines in its original size, making valuations buyer submodular.

The generalized ascending proxy auction therefore possesses a Nash equilibrium in which bidders submit their valuations truthfully when the auctioneer is purely interested in maximizing its profits. While pure profit maximization is likely to be a reasonable assumption about the auctioneer's motivation in many cases, a government auctioneer might have additional aims. This is particularly the case when the increase in firm payoffs from the auction come at the expense of the consumers in the final goods market as in the Cournot setup of the second period market. If consumers are unable to participate directly in the auction – as is likely to be true in general – the auctioneer can counterbalance firms' interests through the choice of its preferences over allocations. In particular, the auctioneer might attempt to maximize total surplus from the auction.

3.3.2 Consumer Surplus Maximization

In contrast to a social planner's perspective on maximizing total surplus achievable given demand $P(Q)$ and the cost functions $C_n(q_n)$, the auctioneer has to consider that firms cannot be forced to produce output above their profit-maximizing level in the second-period market game. A sophisticated auctioneer should be assumed to include this limitation in the calculation of its valuations. Hence, in the Cournot auction game, the definition of total surplus for a given capacity vector has to be constructed by including each firm's reaction function to a given capacity vector into the standard definition. Moreover, since firm surplus is already accounted through their bidding behavior, the auctioneer's valuations should only represent consumer surplus in order to avoid a double-counting of profits. The valuations for the auctioneer that attempts to represent the interests of the consumers in the second-period market are defined as follows.

Definition (Consumer Surplus Valuations). Feasible consumer surplus valuations, $v_0(k)$, are

$$v_0(k) = \int_0^{\sum_N q_n^*(k)} P\left(Q\right) dQ - P\left(\sum_N q_n^*(k)\right) \sum_N q_n^*(k) \tag{3.9}$$

where $q_n^*(k)$ is the constrained Cournot Nash Equilibrium output level defined in Lemma 3.

Definition (Total Feasible Surplus). Total feasible surplus in the Cournot auction game as a function of the capacity allocation k is given by

$$\int_0^{\sum_N q_n^*(k)} P\left(Q\right) dQ - \sum_N C_n\left(q_n^*(k)\right) \tag{3.10}$$

where $q_n^*(k)$ is the constrained Cournot Nash Equilibrium output level defined in Lemma 3.

Then, the sum of the auctioneer's and all firms' valuations are equal total feasible surplus,

$$\begin{aligned}\sum_N v_n(k) &= \int_0^{\sum_N q_n^*(k)} P(Q)\,dQ - P\Big(\sum_N q_n^*(k)\Big)\sum_N q_n^*(k) \\ &\quad + \sum_{N\backslash 0}\left[P\Big(\sum_N q_n^*(k)\Big)q_n^*(k) - C_n\left(q_n^*(k)\right)\right] \\ &= \int_0^{\sum_N q_n^*(k)} P(Q)\,dQ - \sum_{N\backslash 0} C_n\left(q_n^*(k)\right)\end{aligned} \tag{3.11}$$

Proposition 18. *If the auctioneer has consumer surplus valuations, any Nash equilibrium in semi-sincere strategies of the generalized ascending proxy auction maximizes total feasible surplus.*

Proof. The proof follows from Lemma 3, equation 3.11 and the fact that the auctioneer cannot force firms to produce more than their optimal level $q_n^*(k)$. □

It is possible to describe the final capacity allocation of the auction more precisely.

Proposition 19. *The minimal capacity level at which the generalized ascending proxy auction terminates when the auctioneer maximizes consumer surplus is the unconstrained Cournot output level q_n^* for all firms $n \in N$.*

Proof. The auction terminates at the capacity allocation k^* which maximizes total feasible surplus,

$$\begin{aligned}k^* &= \arg\max_{k\in\mathfrak{K}} \sum_N v_n(k) \\ &= \arg\max_{k\in\mathfrak{K}} \int_0^{\sum_N q_n^*(k)} P(Q)\,dQ - \sum_N C_n\left(q_n^*(k)\right)\end{aligned} \tag{3.12}$$

By the first order conditions this is true for k^* such that, for all $n \in N$,

$$P\left(\sum_N q_m^*(k^*)\right) \sum_{m\in N} \frac{\partial q_m}{\partial q_n}\frac{\partial q_n}{\partial k_n}\bigg|_{k^*} - \sum_{m\in N} \frac{\partial}{\partial q_m} C_m\left(q_m^*(k^*)\right) \frac{\partial q_m}{\partial q_n}\frac{\partial q_n}{\partial k_n}\bigg|_{k^*} = 0 \tag{3.13}$$

Equation 3.13 is satisfied in two cases.

1. $\frac{\partial q_n}{\partial k_n}|_{k^*} = 0$: This holds for all $k^* \geq q^*$.

2. $\sum_{m\in N} P\left(\sum_N q_n^*(k^*)\right) \sum_N -\frac{\partial}{\partial q_m} C_m\left(q_m^*(k^*)\right) = 0$: By the definition of q^* this cannot be true for any $k^* < q^*$.

$\square$

The intuition behind this result is as follows. At the $(N\backslash 0)$-optimal capacity vector no firm can outbid all of its competitors for additional capacity and output in the second period is constrained. By subsidizing each firm's bids for higher capacity levels, the auctioneer can profitably relax the second period constraints and raise overall output in the constrained Cournot game. Maximizing the sum of all submitted bids and the auctioneer's valuations in each round of the auction acts as such a subsidy. Once k reaches the unconstrained Cournot equilibrium output levels, however, the firms' capacity constraint is no longer binding and additional units of capital will not raise equilibrium production in the market game, even though total surplus would increase. The inability of the auctioneer to achieve total surplus maximization is then due to its inability to subsidize production, instead of capacity in the set-up of this game.

The strategic properties of the generalized ascending proxy auction are again related to the valuations of the participants. In particular, buyer-submodularity will lead to the existence of a Nash equilibrium with truthful reporting of valuations. The intuition is similar to the case of the revenue-maximizing auctioneer.[12]

Proposition 20. *If the auctioneer has consumer surplus valuations, truthful reporting of valuations $v_n(k)$ by all bidders, $n \in N \backslash 0$ is a Nash equilibrium.*

Proof. The proof proceeds as in the case where the auctioneer is indifferent (Proposition 17). For a given coalition M, define the following levels of output. Let $q_m^{*M}(q_n)$ be the unconstrained Cournot Nash equilibrium output for members in M, given firm n's output q_n. Furthermore, define q_m^{*M} as the unconstrained Nash equilibrium output of a firm m given by Lemma 3 when only firms in M are producing. Then, buyer submodularity holds if, for all $0 \in L' \subset L$,

$$\int_0^{q_n^{*L\cup n}} \frac{\partial}{\partial Q} P\left(\sum_L q_l^{*L}(q_n) + q_n\right) - \frac{\partial}{\partial q_n} C_n(q_n) dq_n \\ \leq \int_0^{q_n^{*L'\cup n}} \frac{\partial}{\partial Q} P\left(\sum_{L'} q_l^{*L'}(q_n) + q_n\right) - \frac{\partial}{\partial q_n} C_n(q_n) dq_n$$

Since $\sum_L q_l^{*L}(q_n) \geq \sum_{L'} q_l^{*L'}(q_n)$ and $q_n^{*L} \leq q_n^{*L'}$ the inequality holds. □

[12] It should be noted that the strategic considerations only apply to the bidders. The auctioneer is assumed to report its valuations truthfully, even though this is unlikely to be revenue maxmizing.

The reporting of auctioneer preferences that seek to maximize consumer surplus do thus not affect the strategic properties of the generalized ascending proxy auction. Truthful reporting leads to the Vickrey-Clarke-Groves payoff vector, and bidders have no strategy that would provide them with a higher core payoff. This result, together with the buyer-submodularity of the values for a pure profit-maximizing auctioneer, allows a further generalization. For any valuations reported by the auctioneer that are proportional to consumer surplus, there exists a Nash equilibrium with truthful reporting of bidder valuations.

Proposition 21. *For any $0 \leq \alpha \leq 1$, auctioneer valuations*

$$v_0(k) = \alpha \left[\int_0^{\sum_N q_n^*(k)} P\left(Q\right) dQ - P\left(\sum_N q_n^*(k) \right) \sum_N q_n^*(k) \right],$$

there exists a Nash equilibrium in pure strategies with truthful reporting by all bidders $n \in N \backslash 0$.

Proof. The proof follows from the fact that the sum of two submodular functions is itself submodular, and from Propositions 20 and 17. □

In other words, as long as the auctioneer maximizes the weighted sum of revenue and consumer surplus, its preferences do not affect bidding strategies for the firms. Governments are therefore able to choose an optimal trade-off between consumer surplus maximization and raising of revenue without having to consider the strategic implications for bidding behavior.[13]

[13] A relatively higher weight on revenue maximization is utility maximizing whenever other methods of raising funds for governments, such as taxes or fees, generate distortions and an associated deadweight loss.

It can be argued that since the generalized ascending proxy auction leads to the unconstrained Cournot equilibrium in the second period market game, it is an unnecessary complication over a traditional auction that would generate exactly the same output levels, prices and consumer surplus. While this is true, there is a crucial difference between the two mechanisms. If there is enough capacity being auctioned to allow firms to produce Cournot output levels in the second period there would not be excess demand for capacity in the auction and revenue for the auctioneer would be minimal. With the generalized ascending proxy auction, the auctioneer subsidizes aggregate capacity levels, but firms still have to compete over the division of capacity among each other. As a consequence, revenue for the auctioneer is higher and part of the firm's gains from restricting output is in effect appropriated by the (government) auctioneer.

3.4 Summary and Conclusion

Externalities in valuations can lead to inefficient allocations with many traditional auctions for two main reasons. First, if bidding is restricted to naming prices for items, bidders are unable to express their valuations fully, that is they lack the means to communicate to the auctioneer the externalities other bidders may impose on them. As a consequence, the auctioneer does not have sufficient information to compute the efficient allocation. Moreover, bidders cannot subsidize each other's bids in order to prevent an undesirable outcome or influence the allocation of units they do not win themselves. Secondly, even if the auctioneer possesses information about the externalities, as long as prices are defined over goods, the final allocation-price pair might not be stable in the sense that at the given prices bidders might prefer a different bundle.

This paper combines a constrained Cournot market game with an auction for capacity to illustrate the failure of standard auctions to generate a profit-maximizing capacity allocation for the firms. Rather than relying on *ad hoc* assumptions about externalities, valuations are constructed from the profits firms can earn in the market game following the auction. Firms' profits are linked to each other's output decisions through the price of a homogeneous product, and capacity constraints can be used to limit output levels. Since individual reductions in profit from the expansion of output by another firm are smaller than the increase in profit for the expanding firm, it is not profitable to buy capacity solely in order to prevent increases in production. That is firms bidding for units of capacity cannot prevent output to reach the unconstrained Cournot equilibrium level even though this is clearly inefficient from their perspective.

By defining bids over entire allocations instead of units of capacity, the generalized ascending proxy auction allows coalitions of firms to prevent each other from obtaining capacity in excess of the combined profit maximizing level. Furthermore, at the final price-allocation pair no coalition of firms can suggest a different price-allocation combination that would be acceptable to the auctioneer.[14] The auction also allows the auctioneer to include its own preferences explicitly in determining the final allocation. Consumer surplus considerations can thus be expressed. Since

[14]The generalized ascending proxy auction terminates in the core.

firms cannot be forced to produce in excess of the unconstrained Cournot equilibrium levels, the total surplus maximizing output levels cannot be attained, however. The auctioneer can merely relax the capacity constraints for firms to increase production to their Cournot output. In contrast to standard auctions which may lead to the same outcome, the auctioneer can appropriate some of the second period profits from the firms.

Optimal bidding strategies in the context of the Cournot game are straightforward. If the auctioneer is purely revenue maximizing or if its valuations reflect consumer surplus considerations, the generalized ascending proxy auction has a Nash equilibrium in pure strategies where bidders submit their valuations truthfully to the proxy bidding on their behalf. Even when truthful reporting is not a Nash equilibrium, there are equilibria where bidders shade their respective valuations uniformly.

Chapter 4

Risk aversion, information and the unraveling of labor markets

4.1 Introduction

Most markets are open more or less continuously, and unless a market is particularly thin, agents can participate in transactions whenever they wish. The timing of transactions is thus not an issue in itself. Some markets, however, are characterized by their periodic nature, that is transactions are to take place within specified periods of time which are separated by possibly lengthy periods of market inactivity. This constraint may make the timing of transactions a strategic variable, in particular when timing decisions affect the quality or quantity of goods or services available. In some cases, there is a strong tendency for transactions to take place ever earlier, even predating the official start of the trading period.

Roth & Xing (1994) provide a detailed survey of a large number of markets where unraveling, that is the move of transactions to increasingly earlier dates, has occurred. Many of those markets are for entry level professional positions, such as medical doctors, lawyers and junior university appointments. Admission into top-level colleges also seems to have been moving towards earlier deadlines as admission officials attempt to secure the most promising students. Moreover, they report a rising number of basketball players that are recruited straight from high-school without the once customary delay of college training and experience. While these examples are concentrated in the human resource arena, Roth & Xing (1994) also show that early contracting is a problem in the planning of post-season football games. As an indication that unraveling is not a new concern, they quote medieval legislation outlawing the buying and selling before the official start of periodic goods markets.

The unraveling of periodic markets is undesirable for two reasons. The first has to do with the purpose of the market institution itself. By providing a well-specified time and location – not necessarily in terms of geographical space – in which buyers and sellers can interact, a market coordinates supply and demand, reduces search and transaction costs and increases the information available to the to the participants. Since unraveling moves a significant proportion of the transactions outside of the market, it impedes this coordinating function and, in extreme cases, may lead to its dissolution. The second problem arises when unraveling causes trades being executed before all information becomes known. If information is revealed over time, *ex-post* efficiency requires that agents wait as long as possible before they trade. Early trades therefore cannot be *ex-post* efficient.

In general, observers and participants in markets agree on this, and attempts are often made to reverse the trend and implement binding transactions dates. In some cases these are adhered to, in others, however, the incentive to contract early seems to be too large. At the same time, Roth & Xing (1994) observe, unraveling is not a universal phenomenon. Indeed, there are many markets where the contract procedures and dates are stable and market participants seem to have no interest in moving early. It is thus reasonable to ask how market institutions or the (behavioral) characteristics of the participants in a market affect the timing decision.

This paper attempts to cast some light on the role of information and risk attitude in the unraveling process. The importance of risk aversion and insurance in unraveling has been pointed out before and will be discussed in the section 4.2. Section 4.3 will set up a general model of a two period matching market and link information and risk to early contracting. Since most of the unraveling seems to occur in labor markets, this focus on matching seems appropriate. Preferences are common, but there is uncertainty over the quality of agents which is only resolved in the second period. Unlike previous work, preferences are over ranks rather than the quality of agents. This complicates the analysis significantly. Three different assumptions about the information structure of the game will be made and their effect on the matching process be examined. Unraveling is found to be a possibility in all three paradigms. The generality of the model and the very nature of the matching process will preclude very detailed predictions but point to the importance of preferences and information. Section 4.4 will conclude and outline a future research strategy.

4.2 Literature Overview

Early contracting can be used by risk-averse agents to insure themselves against uncertainty in a market. Several authors have considered this possibility. Typically, they examine a labor market where firms and workers match pairwise in order to produce output. Matching is possible in two periods and uncertainty, introduced in several different ways, is resolved just prior to the second trading period. Li & Rosen (1998), for example, assume that workers are unproductive with a certain probability while firms are always productive. This individual uncertainty introduces aggregate risk about the relative supply of productive labor in the second period and, in consequence, about the distribution of output between workers and firms. In the unique equilibrium, workers that are productive with high probability contract early while the others prefer to wait for the second-period spot market.[1]

Li & Suen (2004) augment Li & Rosen (1998)'s model to allow for unproductive firms. Since aggregate uncertainty is necessary to induce early contracting, they introduce a random shock to the number of workers in the second period. Multiple equilibria with early contracting are possible due to the non-monotone relationship between the number of early contracts and the probability of being on the long side of the spot-market. This non-monotonicity is caused by the uncertainty over firms' productivity and workers that are more likely to be productive being matched earlier. The higher the uncertainty regarding the number of productive workers in the second period and the more risk-averse workers and firms are, the larger becomes the number of worker-firm pairs that contract early. Restricting the ability of agents to set the distribution of output negotiated in the first period is found to reduce unraveling.

Abandoning the assumption of binary productivity, Li & Suen (2000) examine

[1]Output is assumed to be claimed by the side of the market that is in short supply in the spot market. By contracting early, workers and firms can agree to a division of output, removing the risk associated with being on the long side of the market.

a model where production is a function of both workers' and firms' quality. Two sided uncertainty is introduced through a continuous distribution of productivity. In equilibrium, matching in the second period is positive assortative in the sense that higher productivity workers are matched with higher productivity firms and output is shared.[2] Since quality is not known with certainty in the first period, early contracting cannot be assortative and reduces the variation of production agents expect albeit at the cost of lower expected output. As in the case of Li & Suen (2004) and Li & Rosen (1998), agents that have a higher expected productivity have a greater incentive to contract early.

Suen (2000) shows that unraveling does not have to start at the top of the type distribution. Productivity is again distributed continuously and final production depends on the quality of both the firm and worker in the final matching. While firm productivity is known by all agents, in the first period the quality of workers is unknown to firms and workers alike. With this set-up and endogenous division of final output, Suen (2000) finds that only mediocre firms have an incentive to contract early. Highly productive firms prefer to wait until workers' productivity is known in order to attain a best possible matching. Low-productivity firms cannot afford the wage demands of workers of *ex-ante* average workers.

Risk aversion and the consequential desire by agents to avoid market uncertainty is the common motivation for unraveling in those studies. But while this may indeed be the reason for early contracting in actual markets, there is evidence that it may not be the sole reason. Roth & Xing (1997), for example, observe that the

[2] See, for example, Becker (1981).

short contracting period for making appointments in the market for clinical psychologists forces employers to make offers as early as possible (and future employees to accept quickly) for fear of being unmatched at the end of the official market period. Although it seems that the market has not unraveled, the time constraint may lead to inefficient and potentially unstable matchings. Unstable matchings in themselves are blamed by Roth (1991) for market unraveling. If the mechanism allocating workers to employers in the official market produces matchings that are not stable, that is matchings where agents have both an incentive and an opportunity to *re*-match after the market, unraveling may be a way to achieve a stable matching outside of the market. In this case, risk aversion may not be necessary for early contracting, and a better designed allocation algorithm may be able to bring transactions back into the official market. Niederle & Roth (2003) study the effect of the contracting process on unraveling.

Interpreting the marriage market as a game of incomplete information about potential marriage partners, Bergstrom & Bagnoli (1993) show that relatively more undesirable men get married younger, while men with higher expected quality wait until their quality is revealed. Women always marry young with the more attractive ones being matched to the older (and more desirable) men. Unlike most of the literature on unraveling as a means to avoid uncertainty, their model does not rely on payments between the two sides of the market.

4.3 The model

In accordance with the empirical evidence, the majority of the theoretical literature has concentrated on labor markets, with firms and workers forming pairwise matchings in order to produce some kind of output. Unraveling is interpreted as a response of risk-averse agents to uncertainty.[3] Uncertainty is with respect to the quality of a potential partner which is measured either as their probability of being productive, or through parameters that enter positively into a production function. This individual uncertainty, however, may not be sufficient to lead to early contracting, and aggregate uncertainty has to be introduced mainly through a stochastic relative supply of labor and the rule for sharing output after the matching. Sharing rules and monetary transfers between agents are thus crucial in the analysis.

The approach taken in this paper is slightly different. The focus is still on the role of risk aversion; the value of a potential partner, however, is not measured through a productivity parameter, but purely as its ranking *vis-a-vis* all other market participants. Furthermore, the role of information is stressed. Since the aim is more to highlight certain issues, the model is kept very general even though the price to be paid for this generality is an absence of clear predictions.

[3] Throughout this paper the terms *risk* and *uncertainty* are used synonymously. While technically the stochastic elements introduced in this section, and indeed the ones used in the early contracting literature, fall into the risk category of Knight (1921)'s terminology, the improvement in readability is hoped to justify the abuse of notation. For a detailed treatment of risk and uncertainty, see Fishburn (1994).

4.3.1 Setup and notation

Firms and workers meet in a two period job market. Firms can hire at most one worker and workers can accept no more than one position. Job offers can be made by firms in both periods. If a worker accepts an offer in the first period, the offer is binding and both the firm and the worker exit the job market prior to the second period. Offers do not carry over from the first to the second period, that is, a worker has to accept or decline an offer in the period it is made.[4] Workers cannot propose job offers.

The ranking of firms is common knowledge; the ranking of workers, in contrast, becomes common knowledge between the first and second period in which the market operates. This assumption reflects several characteristics of many actual job markets, in particular for entry level positions. In most cases, the amount of public information about potential employers far exceeds the information available about job seekers.[5] The revelation of information about employees is intended to model the process of unraveling: contracting early implies contracting with less information. Three assumptions are made about the information available in the first period. In the first scenario, both workers and firms receive a signal about workers' rankings. In two other cases the information structure is asymmetric and either workers or firms receive the signal. Both assumptions can be justified with respect to actual market situations. On one hand it can be claimed that agents possess more infor-

[4] In the terminology of Niederle & Roth (2003), offers are exploding.

[5] The academic job market, where the ranking of universities is widely published, is a particularly extreme example of this phenomenon.

mation about themselves than outside observers do. On the other hand, applicants may lack knowledge about their competitors' quality and therefore their own ranking, while firms have experience in hiring and may be able to rank candidates with some accuracy.

Firms and workers have identical preferences regarding their respective potential matches. While this is clearly an oversimplification, it has the advantage of allowing the analysis to focus on the effects of early contracting. Since common preferences imply the existence of a unique stable matching in the second period market, any stable matching mechanism will result in the same final matching. The precise nature of the second period market will therefore not affect the results obtained. While this assumption might not hold very well in practice, there does seem to be at least *some* agreement over which employers or workers are more desirable in most actual labor markets.

In order to set up the model properly, some notation is needed. Let $I = \{1, \ldots, I\}$ be the set of firms and $N = \{1, \ldots, N\}$ the set of workers attempting to match in the job market. The quality ranking of firms is a function $q = \{q_1, q_2, \ldots, q_{|I|}\}$ from the set of firms I to the set of all permutations on $\{1, 2, \ldots, |I|\}$, such that q_i is the ranking of firm i.

Denote by s^2 the quality vector of workers and S^2 the set of all possible such vectors. Let $s^1 \in S^1$ be a signal for s^2. Define a cumulative distribution function $G : S^1 \mapsto [0, 1]$ and its associated density function g. The conditional distribution and density functions for worker quality are given by $F(s^2|s^1) : S^1 \times S^2 \mapsto [0, 1]$ and $f(s^2|s^1)$, respectively. The n-th element of s^2 is s^2_n. Moreover, let r be a ranking function from any subset Z of $\mathbb{R}$ of size k to the set of all permutations of $\{1, 2, \ldots, k\}$ such that, for all $p, q \in Z$, $r(p) \leq r(q)$ if an only if $p \leq q$. The n-th element of $r(S^1)$ is r_n. Then $r(s^2)$ produces a ranking of workers based on their realized quality and the conditional density function of worker rankings can then be found as $f(r(s^2)|s^1)$.

Assumption. The ranking of firms $q = \left\{q_1, q_2, \ldots, q_{|I|}\right\}$, as well as the distribution function $G(s^1)$ and the conditional density $f(s^2|s^1)$ are common knowledge.

Worker-specific densities are defined as follows.

Definition. The *individual conditional distribution function* of rankings $f_n(r_n|s)$ for every $n \in N$ is defined as

$$f_n(r_n|s^1) = \int_{\tilde{S^2}} f(r(s)|s^1)ds$$

where $\tilde{S^2} = \{s^2 \in S^2 | r_n(s^2) = r_n\}$. The *individual conditional cumulative distribution function* of rankings $F_n(r_n|s)$ for every $n \in N$ is defined as

$$F_n(r_n|s^1) = \int_{\tilde{S^2}} f(r(s)|s^1)ds$$

where $\tilde{S^2} = \{s^2 \in S^2 | r_n(s^2) \leq r_n\}$.

The purpose of this set-up is to define a ranking of firms that is common knowledge. Workers, in contrast, draw a quality and an associated signal from a distribution. The ranking of workers depends on their quality parameter s^2. Without loss of generality, and for the purpose of consistency between s^2 and r, it is postulated that workers with a higher quality parameter are ranked higher. From the conditional distribution of s^2 it is possible to calculate the probability that a particular worker attains a specific rank once her quality becomes known.[6] Due to the nature of the ranking function, the ranking of an individual worker depends on the quality of *all* workers. The definition of the individual conditional distribution function takes this into account.

A matching of firms and workers assigns a single worker to every firm. Unmatched agents are assumed to be matched to themselves.

Definition. A *matching* μ for firms and workers is a mapping from $I \cup N$ onto itself of order two such that,

1. $|\mu(i)| = |\mu(n)| = 1$
2. $\mu(i) \in N$ or $\mu(i) = i$ for all $i \in I$
3. $\mu(n) \in I$ or $\mu(n) = n$ for all $n \in N$

Preferences are represented by utility functions u and v for workers and firms, respectively. Neither firms nor workers receive utility from the quality of their match; instead they are only interested in obtaining a partner of the highest possible rank. This assumption deviates from the literature where productivity tends to be the crucial characteristic. It is somewhat motivated by job markets, where positions often cannot be held over between years and competition between firms may make them more sensitive to relative quality or ranking of job candidates than overall quality.[7] Since firms and workers are homogeneous except for their quality and

[6] The ranking function translates a continuous density of qualities into a discrete distribution of rankings. It is monotone but not continuous.

[7] This is not to suggest that the overall quality of candidates does not matter, but to highlight

ranking, it is convenient to refer to a particular firm or worker by its quality or ranking, such that r_n refers to both a ranking and the worker for which $r_n(s^2) = r_n$.

Assumption. Firms and workers have identical utility functions, $u_i(r_n) = u(r_n)$ and $v_n(q_i) = v(q_i)$, which are common knowledge.

Assumption. Both workers and firms prefer to be matched to higher ranked agents, $v(q_i) \geq v(q_j)$ whenever $q_j \leq q_i$ and $u(r_m) \geq u(r_n)$ whenever $r_m \leq r_n$. Being unmatched is the least preferred outcome, $v(n) < min_i\{v(q_i)\}$ and $u(i) < min_n\{u(r_n)\}$.

4.3.2 The matching market

With these preliminaries the matching market can be described. The market consists of two periods, $t \in \{1, 2\}$. At the beginning of $t = 1$ all agents are unmatched. A signal s^1 is drawn from the distribution $G(S^1)$ and is revealed either to all firms, all workers or both firms and workers depending on the information scenarios described below. All strategic action takes place in the first period. Firms can make an offer to a single worker or not make any offer at all; workers who receive at least one offer can accept *one* or decline all of them. Analogous to the definition of a matching, firms who do not make an offer to a worker are treated as making an offer to themselves. Thus the set of available actions for a firm i consists of the union of the set of all firms, N, and itself, $\mathcal{A}_i^I = N \cup i$. Similarly, the set of actions of a worker n is the union of the set of firms, I and itself, $\mathcal{A}_n^N = I \cup n$. The set of mixed

the importance of ranking in some markets.

strategy profiles for firms and workers depend on the information structure. If an offer is accepted, the matched pair is removed from the market and a preliminary matching μ_1 can be defined in which $\mu_1(i) = n$ for all firms $i \in I$ whose offer to a worker $n \in N$ has been accepted and $\mu_1(n) = i$ for all workers $n \in N$ who have accepted an offer from a firm $i \in I$. All other agents are matched to themselves: $\mu_1(j) = j$, $\forall j \in I : \mu_1(j) \notin N$ and $\mu_1(m) = m$, $\forall m \in N : \mu_1(m) \notin I$.

Definition. The set of unmatched firms after period 1, I' is given by

$$I' = \{i \in I | \mu_1(i) = i\}$$

The set of unmatched workers after period 1, N' is given by

$$N' = \{n \in N | \mu_1(n) = n\}$$

In the second period, the quality of all workers s^2 and hence their ranking $r(s^2)$ becomes common knowledge. It is then possible to rank the firms and workers still in the market relative to each other. In particular, let r'_n be the n-th element of $r(s')$ where $s' = s^2 \times \mathbf{J}$ and $\mathbf{J}$ is and $|N| \times |N'|$ matrix that selects the signals for all workers in N' from s^2. Similarly, q'_i can be defined as the relative ranking of the firms in I'. An external mechanism matches the remaining firms and workers by rank.

Assumption. In period 2, for all $i \in I'$ and $n \in N'$,

$$\mu_2(q'_i) = r'_n$$

$$\mu_2(r'_n) = q'_i$$

and $\mu_2(k) = \mu_1(k)\ \forall k \notin I' \cup N'$.

Remark. Assortative matching, that is matching agents by their rank, produces a stable match when preferences on both sides of the market are common. Since this matching is unique and since both rankings and the utility functions are common knowledge at $t = 2$, the external mechanism is outcome equivalent to all stable matching mechanisms. Assumption 4.3.2 is thus less restrictive than initially apparent. It should be pointed out, however, that the final assignment of workers, $\mu = \mu_1 \times \mu_2$, to firms need not be stable. Although, by assumption 4.3.2 there exists no worker-firm pair (i, n) where $i \in I'$ and $n \in N'$ that can block the matching, blocking by a pair that consists of at least one member $k \notin I' \cup N'$ is not ruled out.

The optimal strategy profile of both firms and workers in the first period depends on their respective expectations about the match they can achieve in the second period. These expectations, in turn, are contingent on the information available to the agents. Three information structures are particularly appealing for their simplicity and practical importance. In a symmetric structure, both firms and workers receive the quality signal vector s^1. In two asymmetric set-ups, either the workers or the firms obtain information about the signal vector.

4.3.3 Symmetric Information

At the beginning of the first period workers and firms receive information about the signal vector s^1. With this they are able to calculate the conditional density functions $f_n(r_n|s^1)$.

Assumption. The vector of signals s^1 is common knowledge.

While it is imaginable that in an actual job market firms have information about all applicants, the assumption that workers have information about each others' attributes is somewhat strong. Nonetheless, complete symmetry is a convenient baseline case against which to compare other information structures.

A mixed strategy vector for a firm, σ_i, can then be defined as a probability distribution over the set of feasible actions $a_1 \in \mathcal{A}_i^I$ as a function of the revealed signal s^1, $\sigma_i(a_i|s^1)$. Similarly, the mixed strategy vector $\sigma_n(a_n|s^1)$ for a worker describes a probability distribution over the set of feasible actions for worker n, $a_n \in \mathcal{A}_n^N$, contingent on the realized signal s^1. In other words, firms make an offer to a particular worker with a certain probability which depends on their own quality and the expected quality of the particular workers derived from the signal s^1. Workers accept one of the offers they potentially receive based the realized signal and the resulting expected quality in the second period.

With this, equilibrium behavior can be described.

Proposition 22. *There exists a Nash equilibrium in which firms make and workers accept no offer in period 1,*

$$\sigma_i(i|s^1) = 1$$

$$\sigma_n(n|s^1) = 1$$

for all $i \in I$, $n \in N$ and $s^1 \in S^1$.

Proof. All agents are matched in the second period, such that $q_1 = r_n$. Since any first-period offer by firm will be declined, a firm cannot raise its expected utility by contracting early. A similar argument holds for workers. $\square$

The existence of an equilibrium in which both workers and firms wait until the second period to be matched, seems to indicate that unraveling may not be a problem in the type of job market described above. This result depends on workers not accepting *any* offer in the first period. It is not clear, however, that workers – if faced with an offer in period 1 – should automatically refuse being matched before the second period. The following proposition make this clear.

Proposition 23. *The Nash equilibrium described in proposition 22 is not a trembing-hand perfect equilibrium.*

Proof. Suppose each firm i makes an offer to a particular n worker with probability ϵ_i^n and makes no offer at all with probability $1 - \sum_N \epsilon_i^n$. Then, the optimal strategy for a worker is to play a completely mixed strategy σ_n^ϵ in which an offer from a firm i such that $i = \arg\min_{q_j}$ subject to $u_(q_i) \geq \int u(r_n) dF(r_n|s^1)$ is accepted with the maximum probability allowed. As ϵ_i^n decreases, this strategy does not change, ruling out a limit in which workers decline all offers in the first period. □

Intuitively, it makes no sense for workers to decline a first-period offer from a firm which will generate a higher utility than can be expected from being matched in the second period. This, in turn, implies that there may exist a firm that prefers making an offer in period 1. In other words, the Nash equilibrium in which firms and workers never match in the first period relies on weakly dominated strategies and is therefore not trembling-hand perfect. Without restrictions on preferences, it is nevertheless not possible to predict if there is in fact a worker-firm pair that will contract in the first period or not.

The conditions necessary to find a worker and a firm that can contract early can be related to their respective risk attitude. Risk aversion by workers has been pointed out as a factor in unraveling labor markets by other authors, but the risk attitude of firms can play an equally important part. Furthermore, since rankings are discrete, and the number of firms and workers is limited, risk aversion has to be strong enough for workers to accept an offer from a firm that is ranked lower than the worker's expected ranking. If workers are not risk averse, early contracting can still take place as long as firms possess risk-loving utility functions.

Proposition 24. *A pair of firm and worker, (n, i), can deviate profitably from the equilibrium in proposition 22, if and only if their respective utility functions fulfill the two conditions,*

1. $u(q_i) \geq \int u(r_n) dF_n(r_n|s^1)$
2. $\int v(r_n) dF_n(r_n|s^1) > v(q_i)$

Proof. Suppose firm i with $q(i) = q_i$ makes an offer to worker n with expected quality $E[r_n|s^1]$. By condition 2, the firm prefers being matched to n, by condition 1, worker n will accept the offer. If condition 1 is not met, a deviating firm cannot find a worker who it would prefer to being matched in the second period and who would accept its proposal. $\square$

Corollary 3. *If both firms and workers are risk neutral, unraveling will not occur.*

This proposition is interesting for two reasons. First, it shows that risk aversion by workers is not necessary for unraveling, as long as firms are sufficiently risk-loving.

Moreover, if the conditional density of rankings for workers $f_n(r_n|s^1)$ is sufficiently non-degenerate, unraveling is more likely to involve firms and workers in the middle of the ranking distribution. In this case, the expected ranking of workers are found mainly in the middle of the ranking distribution, making early matching unattractive for firms at the top of the distribution unless they exhibit considerable risk-seeking behavior. By the same argument, workers are unwilling to match with low ranked firms.

An implication of this observation is the role of the conditional density function $f_n(r_n|s^1)$. The more information the signal s^1 provides, that is, the more precise the prediction of a workers' final rankings is, the wider will be the distribution of expected rankings and, in consequence, the more firms can make profitable offers in the first period that will be accepted. This suggests that labor markets in which little information is available to potential early contractors are less likely to exhibit unraveling.

With unraveling possibly being a problem, all agents matching in the first period may be an equilibrium in the labor market with symmetric information. The following proposition describes the equilibrium strategies.

Proposition 25. *There exists a Nash equilibrium in which firms make offers in period 1 and all workers accept, that is, for all $i \in I$, $n \in N$ and $s^1 \in S^1$,*

$$\sigma_i(m|s^1) = 1$$

where $m \in N$ such that $q_i = r(E[r_m|s^1])$,

$$\sigma_n(j|s^1) = 1$$

where $j = \arg\min_{j \in I} q_j$ *subject to* $\sigma_j(n|s^1) = 1$*.*

Proof. Three types of deviations are potentially profitable. First, a firm i can make an offer to a worker m such that $q_i > r(E[r_m|s^1])$. Since this worker will also receive an offer from a firm j with $q_j < q_i$, firm i's offer will not be accepted. If all other agents follow their equilibrium strategy, the set of unmatched agents after period 1 resulting from this deviation, $U = N' \cup I'$ consists of firm q_i and worker n such that $r(E[r_n|s^1]) = q_i$, $U = \{n, i\}$. Second, assume worker n employs any strategy σ' such that $\sigma'(j|s^1) = 0$ for all j with $q_j \geq r(E[r_n|s^1])$, the set of unmatched agent after period 1 will also be $U = \{n, i | q_i = r(E[r_n|s^1])\}$. Similarly, if either firm i does not make any offer, or worker n declines all offers, $U = \{n, i | q_i = r(E[r_n|s^1])\}$.

In all of those cases, the mechanism in period 2 will match firm i and worker n, and the final matching is not affected at all by the deviations.

□

The intuition is simple. As long as only a single agent deviates and all others follow their equilibrium strategy, the worker-firm pair that remains unmatched and will be matched to each other in the second period is the same that would have been matched in equilibrium anyway. Thus they cannot gain by holding out for a contract in the second period. This argument, however, holds only if the set of unmatched agents consists only of single worker and firm. Since the expected ranking of a worker is unlikely to be equal to her realized ranking in period 2, if two or more workers remain unmatched after the first period, their ranking relative to each other may change between the periods. In consequence, their final partner may be different from their equilibrium match. This leads a possible deviation by a coalition of a firm and a worker who are not matched in the equilibrium of proposition 25.

Proposition 26. *The equilibrium in proposition 25 is not coalition proof with respect to a coalition consisting of a firm and worker, $\{i, n\}$ such that $q_i < r(E[r_n|s^1])$.*

Proof. Consider strategies $\sigma_i(i|s^1) = 1$ and $\sigma_n(n|s^1) = 1$ for the deviating firm and worker respectively. Then, the set of unmatched agents consists of firms i and j with $q_i < q_j$ and workers n and m with $r(E[r_m|s^1]) < r(E[r_n|s^1])$. In equilibrium, firms i and j would have been matched to worker m and n, respectively, $\mu(i) = m, \mu(j) = n$. The mechanism in the second period matches the two top ranked agents with each other, so that $\mu(i) = \arg\min_{k \in N'} r_k \leq r_m$. Hence firm i cannot lose, but possibly obtain a better match by waiting. Similarly, if $r_n > r_m$, worker n will be matched to the same firm as in equilibrium; if $r_n < r_m$, however, n will be matched to the higher-ranked firm. Hence worker n weakly prefers to wait for the second period. □

Again, the intuition is straightforward. If a high-ranked firm waits and a lower ranked firm remains unmatched, their equilibrium matches are also available in the second period. Since the higher-ranked firm is able to obtain the worker with the higher realized rank it can only gain by waiting. In the worst case it is matched with the worker it would have been matched in equilibrium; in the best case the other firm's equilibrium match turns out to be better and the higher-ranked firm can achieve a better match. The worker with the lower expected rank, similarly, cannot do worse than her equilibrium match, but can be matched to the preferred firm if her realized ranking is higher than the other worker's. By the symmetric argument, the lower-ranked firm and the higher ranked worker can only be made worse off.

These results suggest that there might be equilibria in which some firms match early while others wait until the second period. As long as the set of agents waiting until period 2 consists of only two firms and workers, proposition 26 has shown this to be the case, indeed. It is not possible to generalize this finding to cases where more than two agents on each side of the market contract late. In fact, if worker preferences exhibit universal risk aversion, that is, for any i and all $s^1 \in S^1$, there exists a firm n such that the conditions in proposition 24 hold, there exists no trembling-hand perfect equilibrium with late matching by more than two worker-firm pairs. Moreover, the only equilibrium with some firms matching in the second period that survives trembling hand perfection for all preferences and quality distributions is early contracting by all firms and workers but the top ranked firm and the lowest ranked worker. For any other worker there exist a combination of preferences and type-distributions such that she would accept a proposal by a slightly lower ranked firm in the first period.

To some extent the instability of the second-period matching equilibrium is due to the fact that workers have as much information about their expected ranking as firms do. This allows them to evaluate an offer in the first period without having to make an inference about their expected ranking. In the next section, workers no longer receive the signal.

4.3.4 Asymmetric Information: Firms receive signal

In contrast to the previous information structure, only firms receive information about the signal vector s^1. Workers know the distribution of s^1, $G(s^1)$.

Assumption. The vector of signals s^1 is known by firms only. The distribution of signals, $G(s^1)$, and conditional density functions $f_n(r_n|s^1)$ are common knowledge.[8]

This assumption is likely to reflect the informational structure in many labor markets. Firms – both due to their experience in hiring and since they receive applications from a large number of workers – tend to have more information about the expected quality of possible employees than they might have themselves. In contrast to the symmetric information case, if a firm makes an early offer to a worker, she has to form beliefs about the realization of s^1 before deciding whether to accept or not. Workers' strategies can thus no longer be conditional on the observed signal. Nonetheless, there is still an equilibrium with matching in period 2 identical to the symmetric set-up.

Proposition 27. *There exists a Nash equilibrium in which firms make and workers accept no offer in period 1,*

$$\sigma_i(i|s^1) = 1$$

$$\sigma_n(n) = 1$$

for all $i \in I$, $n \in N$ and $s^1 \in S^1$.

[8]Unless the conditional distribution of ranks are identical for all firms, $f_n(r_n|s^1) = f(r_n|s^1)$, this assumption does not imply that the expected rank is the same for all workers.

Proof. The proof is identical to proposition 22. □

The shortcomings of this equilibrium are the same as in the symmetric information case. Since workers have no information about signal s^1 in the first period, the condition for successful early contracting are different, however. If a worker receives an offer in the first period, she updates her beliefs about the signal s^1 taking into account the offer she has received. Specifically, all realizations of s^1 which would make the early offer non-profitable for the firm will be assigned zero probability. Then, the worker can calculate the expected utility from waiting and compare this to the proposed match. This can be expressed more technically.

Proposition 28. *A pair of firm and worker, (n, i), can deviate profitably from the equilibrium in proposition 27, if and only if their respective utility functions fulfill the two conditions,*

1. $\int v(r_n)dF_n(r_n|s^1) > v(q_i)$
2. $u(q_i) \geq \int_{S^*} \int u(r_n)dF_n(r_n|s^1)dG^*(s^1)$

where

$$S^* = \left\{ s^1 \in S^1 | \int v(r_n)dF_n(r_n|s^1) > v(q_i) \right\}$$

$$\frac{\partial}{\partial s^1}G^*(s^1) = \frac{g(s^1)}{1 - \int_{S^1 \setminus S^*} g(s)ds}$$

Unless the offer originates with the lowest ranked firm, proposition 28 implies that a worker having received an offer revises her expected ranking upwards. Moreover, the higher the rank of the proposing firm, the higher the expected ranking conditional on the offer. This has two important consequences. A worker who would have accepted an out of equilibrium offer from a firm under the symmetric information assumption, might no longer accept as she forms unrealistic but rational beliefs about her expected ranking. In that sense, withholding information from workers may reduce the likelihood of early contracting and the unraveling of the market. At the same time, however, an offer by a lower ranked firm might convince a worker that his expected ranking is lower than it actually is, leading her to accept an offer she would not have considered under symmetric information. The effect of asymmetric information on the incidence of early contracting is thus not obvious.

Since firms possess all relevant information in the first period, the early contracting equilibrium of proposition 25 also exists when workers do not know the signal vector s^1. Furthermore, the same coalition or a high-ranked firm and a lower ranked worker can deviate. The reason for this is that in equilibrium the firms' offers reveal enough information about the realization of s^1 to the workers for them to face incentives similar to the symmetric case. This is no longer true when firms have less information than workers.

4.3.5 Asymmetric Information: Workers receive signal

This case is the mirror image of the information structure discussed in the previous section. Instead of firms, it is the workers who obtain information about s^1, while firms only know the distribution function $G(s^1)$. Thus workers cannot make their strategies contingent on the realized signal.

Assumption. The vector of signals s^1 is known by workers only. The distribution of signals, $G(s^1)$, and conditional density functions $f_n(r_n|s^1)$ are common knowledge.

Since the equilibrium with matching in the second period (introduced in proposition 22) is independent of the information available in the first period, it survives the change of information structure.

Proposition 29. *There exists a Nash equilibrium in which firms make and workers accept no offer in period 1,*

$$\sigma_i(i) = 1$$

$$\sigma_n(n|s^1) = 1$$

for all $i \in I$, $n \in N$ and $s^1 \in S^1$.

Proof. The proof is identical to proposition 22. □

Matching with full information in the second period is thus an equilibrium under all three informational assumptions. Unfortunately, however, this equilibrium relies on weakly dominated strategies and may not be coalitionally stable. If workers possess information about the signal vector s^1 and firms only have expectations the conditions for unraveling are described in the following proposition.

Proposition 30. *A pair of firm and worker, (n, i), can deviate profitably from the equilibrium in proposition 27, if and only if their respective utility functions fulfill the two conditions,*

1. $\int_{S^*} \int v(r_n) dF_n(r_n|s^1) dG^*(s^1) > v(q_i)$
2. $u(q_i) \geq \int u(r_n) dF_n(r_n|s^1)$

where

$$S^* = \left\{ s^1 \in S^1 | u_(q_i) \geq \int u(r_n) dF_n(r_n|s^1) \right\}$$
$$\frac{\partial}{\partial s^1} G^*(s^1) = \frac{g(s^1)}{1 - \int_{S^1 \setminus S^*} g(s) ds}$$

Not surprisingly these conditions are in a sense the opposite from the case where firms possess more information than workers. In order for a firm to make an offer to a worker, it has to evaluate the information it obtains from the decision to accept – or not – by the worker. An acceptance is then almost bad news since it implies that the ranking the worker expects is low enough for her to accept, and the firm updates its belief about the probabilities a particular signal s^1 was received by the workers.

Despite the similarities between all three information structures, there does seem to be one important difference. When the signal vector s^1 and the preferences of workers are common knowledge, firms can identify workers in the first period who would accept an out of equilibrium offer. Similarly, when only firms have information about s^1, they can find workers for who the conditions necessary for early contracting are met. Their proposal then signals to the worker that accepting might be in their own best interest. In contrast, if firms have no information, they might not be able to identify a potential early-matching partner, even though they know that she exists. The crucial difference between the two asymmetric cases is thus the information available to the proposing party. Indeed, if workers rather than firms took the initiative in early contracting the result would be reversed.

Although this observation may suggest that there might not be an equilibrium with early matching, proposition 31 shows that this is not the case. Unless the conditional densities $f_n(r_n|s^1)$ are identical for all workers, firms can arrive at an expected ranking in period 1 even though they lack information about s^1.

Proposition 31. *If $f_n(r_n|s^1) \neq f_m(r_n|s^1)$ $\forall n, m \in N$, there exists a Nash equilibrium in which firms make offers in period 1 and all workers accept. The equilibrium strategies are as follows. For all $i \in I$, $n \in N$ and $s^1 \in S^1$,*

$$\sigma_i(m) = 1$$

where $m \in N$ such that $q_i = r(E[r_m])$,

$$\sigma_n(j|s^1) = 1$$

where $j = \arg\min_{j \in I} q_j$ subject to $\sigma_j(n|s^1) = 1$.

Proof. The proof follows proposition 25. □

If firms cannot distinguish the expected quality of worker, that is if $f_n(r_n|s^1) = f(r_n|s^1)$, this equilibrium no longer exists. There *does* exist an equilibrium with complete contracting in the first period, however. Firms make random offers and workers accept the highest offer they receive. Yet such an outcome is not attractive since it does not use all information available, even if it is only to one of the parties. Given that workers receive (and accept) an offer from a firm below their expected ranking with non-zero probability, they have an incentive to communicate information about s^1 to the firms.[9] Moreover, random contracting implies that firms

[9]Communication between workers and firms, other than related to contracts, are ruled out in this model.

and workers with expected rankings above the median would prefer matching to be restricted to the second period *ex ante*. This raises the question about possible deviations from an equilibrium with contracting in period 1.

Proposition 32. *The equilibrium in proposition 31 is not coalition proof with respect to a coalition consisting of a firm and worker, $\{i, n\}$ such that $q_i < r(E[r_n|s^1])$.*

Proof. Since the deviating coalition and its strategy is independent of the information structure, its existence is not affected by firms' inability to observe s^1 and the proof follows proposition 26. □

As with the other information structures, the existence of equilibria with partial early contracting depends on the utility functions of workers and firms and without further parameterizing the model little can be said.

4.4 Conclusion

Unraveling, that is contracting before the official opening of a market, has been a major problem in many real world labor markets. Although this phenomenon may have different reasons, it is generally considered undesirable by the participants in the affected markets (Roth & Xing 1994). Several authors have provided theoretical justification for early contracting, focusing mainly on early contracting as a mechanism to insure against uncertainties in the distribution of surplus from the matching of workers to firms. Risk aversion is thus a crucial aspect of unraveling labor markets.

This paper provides a general framework to examine the effect of risk aversion on early contracting under three different assumptions about the distribution of information between firms and workers. A two period labor market is considered in which workers' quality is revealed in the second period. A signal – revealed to either workers, firms or both at the beginning of the first period – is correlated with the final quality. Preferences over firms are common knowledge and identical for all workers. Although very formalized, this setup reflects the characteristics of many labor markets, where there is general agreement over the desirability of a match and where information about workers is revealed over time through exams, perhaps, or the production of job relevant material such as dissertations or portfolios.

Under all assumptions about the distribution of information in the first period there exists an equilibrium in which firms only make offers in the second and workers accept *no* offer in the first period. As a result all matching takes place under full information. Though desirable from the perspective of efficiency, this equilibrium is not realistic. Since the rejection of all offers in the first period as well as the refusal to make any offers are weakly dominated strategies the resulting equilibrium is not trembling hand perfect. Alternative equilibrium strategies are suggested which include offers in the first period. Nonetheless, early contracting is only an equilibrium outcome if certain conditions on preferences of firms and workers are met. Loosely speaking, workers have to be risk averse or firms risk loving with respect to expectations appropriate to the relevant information structure. That is, the conditions for unraveling depend on the information available to the two sides of the market.

In a second equilibrium firms and workers match in the first period. Since the final rankings of workers are not known at the time the offers are made and accepted, firms use the information implicit in the signal to make an offer. Then the worker with the nth highest expected ranking will be matched to the nth highest ranked firm. As long as firms are able to obtain an expected ranking of workers this equilibrium survives even if they do not possess information about the signal. As the matching is based on expected rankings it is generally not efficient and not stable *ex post*. Moreover, the equilibrium is not coalitionally stable, as a high ranked firm and a low ranked worker can profitably deviate by not matching in the first period. The existence of larger deviating coalitions depends on preferences and the densities of the stochastic process, as does the existence of equilibria with partial matching in both periods.

The generality of the analysis, while making precise conclusions elusive, allows the framework to be adapted for further research. First, parameterizing the density functions of the signal and the final quality may allow clearer predictions in the context of the two period matching model. More interestingly perhaps, the effect of the protocol for making and reacting to offers under different information assumptions can be examined. If firms have no information about worker quality, for example, allowing workers to apply rather than wait for offers might lead to a different set of equilibria.

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