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Guardian Maps and the Generalized Stability of Parametrized Families of Matrices and Polynomials

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Guardian Maps and the Generalized Stability of Parametrized Families of Matrices and Polynomials

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Abstract. The generalized stability of families of real matrices and polynomials is considered. (Generalized stability is meant in the usual sense of confinement of matrix eigenvalues or polynomial zeros to a prespecified domain in the complex plane, and includes Hurwitz and Schur stability as special cases.) "Guardian maps" and "semiguardian maps" are introduced as a unifying tool for the study of this problem. Basically these are scalar maps which vanish when their matrix or polynomial argument loses stability. Such maps are exhibited for a wide variety of cases of interest corresponding to generalized stability with respect to domains of the complex plane. In the case of one- and two-parameter families of matrices or polynomials, concise necessary and sufficient conditions for generalized stability are derived. For the general multiparameter case, the problem is transformed into one of checking that a given map is nonzero for the allowed parameter values.

Key words. Stability, Robust stability, Linear systems, Matrices, Polynomials.

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1. Introduction

This paper develops a new approach for the study of generalized stability of parametrized families of matrices or polynomials. The matrix case is emphasized in the exposition.

Generalized stability [V] of a matrix (resp. polynomial) entails that its eigenvalues (resp. zeros) lie in a prespecified domain of the complex plane. The classical stability requirements result upon defining the domain of interest as the open left half complex plane (continuous-time case) or the open unit disk (discrete-time case). Practical considerations relating to damping ratio, bandwidth, vehicle handling qualities, etc., are commonly expressed in terms of the generalized stability formulation, with respect to a suitable domain in the complex plane.

The problem of obtaining necessary and sufficient conditions for the *Hurwitz* stability of polytopes of matrices or polynomials has recently been considered by several authors (see for instance [BD]). This research direction was inspired largely by Kharitonov [K], who studied Hurwitz stability of families of real *polynomials* with uncertain coefficients. Specifically, he studied "interval polynomial" families, i.e., families of polynomials with coefficients belonging to fixed compact intervals of the real line. He showed that the Hurwitz stability of such a family of polynomials is equivalent to that of four distinguished "corner polynomials." Bartlett, Hollot and Lin [BHL] showed that, in the case of an arbitrary polytope of polynomials, it suffices to check the edges (The Edge Theorem). These strong results do not hold however for polytopes of *matrices*. For the case of Hurwitz stability of the convex hull of two real matrices or polynomials, necessary and sufficient conditions have been obtained by Bialas [B1] and, in subsequent independent work, by Fu and Barmish [FB1]. Very recently, the case of one- and two-parameter families of matrices was resolved for several specific domains of the complex plane [GT], [VT], [STA].

In this paper, we introduce and apply to the generalized stability of parametrized families problem the concepts of "guardian map" and "semiguardian map" for sets of matrices or polynomials. These notions allow one to replace the question at hand with that of whether or not the guardian map is nonzero for all members of the family (this key fact is proved in Section 3). These concepts are closely related to work of Gutman [G2] and Gutman and Jury [GJ] on root clustering in domains of the complex plane. The notion of "critical constraints," used in examples in [G2], is similar in spirit to the idea of guardian maps.

Necessary and sufficient conditions are given for stability of one-parameter families relative to domains for which the set of generalized stable matrices is endowed with either a guardian or semiguardian polynomial map. In particular, the technique yields conditions for Hurwitz and Schur (i.e., discrete-time) stability of the convex hull of two matrices or polynomials. Combined with the Edge Theorem, it solves the problem of *generalized* stability of arbitrary polytopes of polynomials.

For the two-parameter case, we consider stability of families of matrices relative to domains to which we can associate a polynomial guardian map. The first step replaces the two-parameter problem by a one-parameter stability problem relative to a new domain. The second step employs a polynomial semiguardian map associated with the new domain to obtain necessary and sufficient conditions for stability.

The paper is organized as follows. Section 2 establishes notation and provides requisite background material. The concepts of guardian and semiguardian maps are introduced in Section 3. Two basic results and several examples are also presented there. These results are applied to one- and two-parameter families of matrices in Section 4 and Section 5 , respectively. In Section 6, a systematic procedure for constructing guardian and semiguardian maps is presented for problems of stability relative to domains with polynomial boundaries. In Section 7, some applications of the techniques are presented. Finally, concluding remarks are given in Section 8.

2. Preliminaries

This section begins by establishing notation, and proceeds to a brief discussion of relevant background material from algebra.

2.1. Notation

$\mathrm{Arg}(s)$: Argument of the complex number s

$\mathring{\mathbb{C}}_-$ ($\mathring{\mathbb{C}}_+$): Open left-half (right-half) complex plane

$\mathcal{D}^c$: Complement of set $\mathcal{D}$

$\overline{\mathcal{D}}$: Closure of set $\mathcal{D}$

$\partial\mathcal{D}$: Boundary of set $\mathcal{D}$

$\mathrm{int}(\mathcal{D})$: Interior of set $\mathcal{D}$

I_n: Identity matrix of dimension n (also denoted I when n is clear from the context)

$\lambda_i(A)$: Eigenvalue of matrix A

$\sigma(A)$: Eigenvalues of matrix A (counting multiplicities)

$\mathcal{Z}(p)$: Zeros of polynomial p (counting multiplicities)

p': Derivative of polynomial p

$\det(A)$: Determinant of matrix A

Ω: Generic open subset of $\mathbb{C}$, symmetric about the real line

Ξ: $\mathbb{C} \setminus [0,1]$

Θ: $\mathbb{C} \setminus [1,\infty)$

$\mathcal{P}_n$: Set of all real polynomials of degree at most n.

$\mathcal{S}_n(\mathcal{D})$: Set of all $n \times n$ real matrices with spectrum inside $\mathcal{D} \subset \mathbb{C}$. $\mathcal{S}_n(\Omega)$ is also used to denote the set of all real polynomials of degree n with zeros inside $\mathcal{D}$. (Sometimes denoted $\mathcal{S}(\mathcal{D})$ when n is clear from the context.)

$\otimes, \oplus$: Kronecker product, Kronecker sum

$A \ominus B$: $A \oplus (-B)$

$A \cdot B$: Bialternate product of A and B (see Section 2.2)

$A^{[2]}$: Schläflian form of order 2 of matrix A ("Upper Schläflian"; see Section 2.2)

$A_{[2]}$: Infinitesimal version of $A^{[2]}$ ("Lower Schläflian"; see Section 2.2)

2.2. Some Algebraic Tools

Kronecker product and sum

Given square matrices A and B having dimension n_1 and n_2, respectively, the *Kronecker product* (e.g. [B3]) of A and B, denoted $A \otimes B$, is the square $n_1 n_2$-dimensional matrix whose ij^{th} $n_2 \times n_2$ block-entry is given by $a_{ij}B$. The *Kronecker sum* $A \oplus B$ of A and B is the $n_1 n_2$-dimensional matrix $A \otimes I_{n_2} + I_{n_1} \otimes B$. Note that $A \oplus A$ is linear in A.

The eigenvalues of $A \otimes B$ and $A \oplus B$ consist of the $n_1 n_2$ products $\lambda_i(A)\lambda_j(B)$ and $n_1 n_2$ sums $\lambda_i(A) + \lambda_j(B)$, respectively, over all ordered pairs (i,j), $i = 1, \ldots, n_1$, $j = 1, \ldots, n_2$.

In fact, this is simply a special case of the following more general result. Let p be a complex polynomial in the variables x_1 and x_2, given by

$$p(x_1, x_2) = \sum_{i,j=0}^{i+j=N} p_{ij} x_1^i x_2^j, \tag{1}$$

and consider the associated function of two complex square matrices A and B

$$P(A, B) := \sum_{i,j=0}^{i+j=N} p_{ij} A^i \otimes B^j. \tag{2}$$

Lemma 1. (Stéphanos[S3]). With the notation above, the eigenvalues of $P(A, B)$ consist of the $n_1 n_2$ values $p(\lambda_i(A), \lambda_j(B))$ over all possible (ordered) pairs $(i, j),\ \ i = 1, \ldots, n_1,\ \ j = 1, \ldots, n_2$.

Schläflian forms

The Schläflian forms,[2] discussed next, have spectral properties akin to those of the Kronecker product and sum with the advantage of reduced dimensionality. Let $x = (x_1, \ldots, x_n)^T$ and $p \geq 2$ be an integer. Denote by $x^{[p]}$ the N_p^n-dimensional vector ($N_p^n := \binom{n+p-1}{p}$) formed by the lexicographic listing of all linearly independent terms of the form

$$x_1^{p_1} x_2^{p_2} \ldots x_n^{p_n}, \quad \sum_{i=1}^{n} p_i = p, \quad p_i \geq 0,\ i = 1, \ldots, n. \tag{3}$$

For a given $n \times n$ matrix A, the associated (*upper*) *Schläflian matrix* of order p (e.g., [A2], [B4], [BZ]), denoted $A^{[p]}$, is the N_p^n-dimensional square matrix defined by the implicit relationship

$$(Ax)^{[p]} = A^{[p]} x^{[p]}, \quad \forall x \in I\!R^n. \tag{4}$$

The related form $A_{[p]}$ (the "lower Schläflian matrix") is defined as follows. Consider the equation $\dot{x} = Ax$ for $x \in I\!R^n$. Then $A_{[p]}$ is defined as the coefficient matrix in the equation

$$\frac{\mathrm{d}x^{[p]}}{\mathrm{dt}} = A_{[p]} x^{[p]}. \tag{5}$$

[2] These are also referred to as power transformations [BZ].

It can be shown that $A_{[2]}$ is the directional derivative of $A^{[2]}$ along the direction I_n [B4]. As such, $A_{[2]}$ is linear in A.

The next result is essentially the same as results in [B4] and [BZ].

Lemma 2. The eigenvalues of $A_{[p]}$ (resp. $A^{[p]}$) consist of the N_p^n sums (products) over *distinct* unordered multiindices of the form

$$\lambda_{i_1}(A) + \cdots + \lambda_{i_p}(A) \quad \left(\text{resp. } \lambda_{i_1}(A) \times \cdots \times \lambda_{i_p}(A)\right). \tag{6}$$

In contrast, recall that the eigenvalues of the Kronecker sum $A \oplus A$ consist of the n^2 sums $\lambda_i(A) + \lambda_j(A)$ over ordered pairs (i,j). In the light of Lemma 2, it is clear that $\sigma(A \oplus A) = \sigma(A_{[2]})$ (not counting multiplicities). Hence the $\frac{n(n+1)}{2} \times \frac{n(n+1)}{2}$ lower Schläflian matrix $A_{[2]}$ may be viewed as a redundancy-free version of the $n^2 \times n^2$ matrix $A \oplus A$, as far as the eigenvalues are concerned. Because of this, $A_{[2]}$ may be used to advantage, instead of $A \oplus A$, in the application of some of the results presented in the sequel. A similar statement clearly holds for $A^{[2]}$ vs. $A \otimes A$.

Bialternate product

Let A and B be $n \times n$ matrices. To introduce the bialternate product of A and B, we first establish some notation. Let V^n be the $\frac{1}{2}n(n-1)$-tuple consisting of pairs of integers (p,q), $p = 2, 3, \ldots, n$, $q = 1, \ldots, p-1$, listed lexicographically. That is,

$$V^n = [(2,1), (3,1), (3,2), (4,1), (4,2), (4,3), \ldots\ldots, (n, n-1)]. \tag{7}$$

Denote by V_i^n the i^{th} entry of V^n. Denote

$$f\left((p,q);(r,s)\right) = \frac{1}{2}\left(\det \begin{bmatrix} a_{pr} & a_{ps} \\ b_{qr} & b_{qs} \end{bmatrix} + \det \begin{bmatrix} b_{pr} & b_{ps} \\ a_{qr} & a_{qs} \end{bmatrix}\right) \tag{8}$$

where the dependence of f on A and B is kept implicit for simplicity. The *bialternate product* of A and B (e.g., [F], [GJ], [M]), denoted $A \cdot B$, is a $\frac{1}{2}n(n-1)$-dimensional square matrix whose ij^{th} entry is given by[1]

$$(A \cdot B)_{ij} = f\left(V_i^n; V_j^n\right). \tag{9}$$

[1] As far as the properties discussed below are concerned, the particular ordering of V^n is immaterial. In the literature, it is typically left unspecified ([F], [G2] and [GJ]).

Define

$$\Psi(A,A) := \sum_{p,q} \psi_{pq} A^p \cdot A^q, \tag{10}$$

and denote the eigenvalues of the $n \times n$ matrix A by $\lambda_1, \ldots, \lambda_n$.

Lemma 3. (Stéphanos [S3]). With the notation above, the eigenvalues of $\Psi(A,A)$ are the $\frac{1}{2}n(n-1)$ values

$$\psi(\lambda_i, \lambda_j) := \frac{1}{2}\sum_{p,q} \psi_{pq}(\lambda_i^p \lambda_j^q + \lambda_i^q \lambda_j^p), \quad i = 2, \ldots, n \; ; \; j = 1, \ldots, i-1. \tag{11}$$

For example, if A is 3×3 then $\sigma(A \cdot A) = \{\lambda_1\lambda_2, \lambda_1\lambda_3, \lambda_2\lambda_3\}$. In contrast, note that in this case $\sigma(A \otimes A) = \{\lambda_1^2, \lambda_1\lambda_2, \lambda_1\lambda_3, \lambda_2\lambda_1, \lambda_2^2, \lambda_2\lambda_3, \lambda_3\lambda_1, \lambda_3\lambda_2, \lambda_3^2\}$. As another example, it is easily checked (e.g. [G2], [GJ]) that if

$$\mathcal{Q}(A) = (A^2 \cdot I - A \cdot A) \tag{12}$$

for an $n \times n$ matrix A with eigenvalues $\lambda_1, \ldots, \lambda_n$, then

$$\sigma\left(\mathcal{Q}(A)\right) = \left\{ \frac{(\lambda_1-\lambda_2)^2}{2}, \ldots, \frac{(\lambda_1-\lambda_n)^2}{2}, \frac{(\lambda_2-\lambda_3)^2}{2}, \ldots, \frac{(\lambda_2-\lambda_n)^2}{2}, \ldots, \frac{(\lambda_{n-1}-\lambda_n)^2}{2} \right\}. \tag{13}$$

The Bezoutian

Given any polynomial $a(s) = a_n s^n + \cdots + a_1 s + a_0$, $a_n \neq 0$, define the polynomial $\hat{a}(s) := s^n a(s^{-1}) = a_0 s^n + \cdots + a_{n-1}s + a_n$ and the matrix

$$S(a) := \begin{bmatrix} a_1 & a_2 & \cdot & \cdot & \cdot & a_n \\ a_2 & \cdot & \cdot & \cdot & a_n & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & a_n & \cdot & \cdot & \cdot & \cdot \\ a_n & 0 & . & . & . & 0 \end{bmatrix}. \tag{14}$$

The *Bezoutian* $B(a,b)$ of two polynomials a and b may then be expressed as the $n \times n$ matrix, n being the largest of the degrees of a and b, given by

$$B(a,b) := S(a)S(\hat{b})P - S(b)S(\hat{a})P, \tag{15}$$

where P is a certain permutation matrix [LT].

Our interest in the Bezoutian stems from the following result.

Lemma 4. The polynomials $a(s)$ and $b(s)$ have no common zeros if and only if the associated Bezoutian $B(a,b)$ (or, equivalently, the matrix $S(a)S(\hat{b}) - S(b)S(\hat{a})$) is nonsingular.

The same result holds if instead of the Bezoutian, the resultant (or Sylvester) matrix, which has dimension $2n$, is used [LT] (see also [AS]).

3. Guardian and Semiguardian Maps

In this section, the concepts of guardian and semiguardian maps are introduced. These concepts play a key role in subsequent developments. Several examples are presented.

3.1. Guardian Maps

The definition of guardian maps given next has been formulated with a view toward problems more general than the generalized stability issues considered in this paper. (See e.g. Example 3.6 below.) For the purposes of this paper, the set $\mathcal{S}$ of Definition 1 will usually be of the form $\mathcal{S}(\Omega)$, where

$$\mathcal{S}(\Omega) = \{A \in I\!R^{n\times n} : \sigma(A) \subset \Omega\} \tag{16}$$

for matrix stability problems, and

$$\mathcal{S}(\Omega) = \{p \in \mathcal{P}_n : \mathcal{Z}(p) \subset \Omega\} \tag{17}$$

for polynomial stability problems. Here, Ω is an open subset of the complex plane. Such sets $\mathcal{S}(\Omega)$ will be referred to as *(generalized) stability sets.*

Definition 1. Let $\mathcal{X}$ be the set of all $n \times n$ square real (complex) matrices, or the set of all polynomials of degree at most n with real (complex) coefficients, and let $\mathcal{S}$ be an open subset of $\mathcal{X}$. Let ν map $\mathcal{X}$ into $\mathbb{C}$. We say that ν *guards* $\mathcal{S}$ if for all $x \in \overline{\mathcal{S}}$, the equivalence

$$x \in \partial\mathcal{S} \quad \Longleftrightarrow \quad \nu(x) = 0 \tag{18}$$

holds. In this case, we also say that ν is a *guardian map* for $\mathcal{S}$.

Proposition 1 below will be used in subsequent sections to tackle the stability problem for parametrized families of matrices or polynomials relative to domains of the complex plane corresponding to guarded stability sets. Let $r = (r_1, \ldots, r_k) \in U$, where U is a pathwise connected subset of $I\!R^k$, and let $x(r)$ be a matrix or polynomial in $\mathcal{X}$ which depends continuously on the parameter vector r. Given an open subset $\mathcal{S}$ of $\mathcal{X}$, we seek basic conditions for $x(r)$ to lie within $\mathcal{S}$ for all values of r in U.

Proposition 1. Let $\mathcal{S}$ be guarded by the map ν and assume that $x(r^0) \in \mathcal{S}$ for some $r^0 \in U$. Then

$$x(r) \in \mathcal{S} \quad \text{for all } r \in U \qquad \Longleftrightarrow \qquad \nu\big(x(r)\big) \neq 0 \quad \text{for all } r \in U. \tag{19}$$

Proof. Suppose that $x(r^1) \notin \mathcal{S}$ for some $r^1 \in U$. By virtue of the pathwise connectedness of U, there exists a curve $\{r(t) : t \in [t_0, t_1]\}$ within U, such that $r(t_0) = r^0$ and $r(t_1) = r^1$. Now consider $x(r(t))$ as t increases from t_0. Since $x(r^0) \in \mathcal{S}$, it follows that there is a $t^* \in (t_0, t_1]$ such that

$$x(r(t^*)) \in \partial\mathcal{S}.$$

This implies that there is an $r^* \in U$ (namely $r^* = r(t^*)$) such that

$$x(r^*) \in \partial\mathcal{S}.$$

Since ν guards $\mathcal{S}$, we conclude that

$$\nu\big(x(r^*)\big) = 0.$$

This proves sufficiency. Necessity follows from the openness and guardedness of $\mathcal{S}$.

□

3.2. Semiguardian Maps

The following generalization of the concept of guardian maps will prove useful in the development to follow.

Definition 2. Let $\mathcal{S}$ and ν be as in Definition 1. The map ν is said to be *semiguarding* for $\mathcal{S}$ if, for all $x \in \overline{\mathcal{S}}$, the implication

$$x \in \partial\mathcal{S} \quad \Rightarrow \quad \nu(x) = 0 \tag{20}$$

holds. In this case, we also say that ν is a *semiguardian map* for $\mathcal{S}$. An element $x \in \mathcal{S}$ for which $\nu(x) = 0$ is said to be a *blind spot* for $(\nu, \mathcal{S})$.

In the light of Definition 1 and 2, a guardian map for a given set $\mathcal{S}$ is simply a semiguardian map for which the corresponding set of blind spots is empty.

The next proposition is the analogue of Proposition 1 for semiguarded sets of matrices or polynomials, the main difference being that, in this case, the blind spots must be taken into account. The result will prove useful in the study of two-parameter problems in Section 5, and is intended also for sets for which only a semiguardian map is known.

Proposition 2. Let $\mathcal{S}$ be semiguarded by ν and assume that $x(r^0) \in \mathcal{S}$ for some $r^0 \in U$. Then the equivalence

$$x(r) \in \mathcal{S} \quad \text{for all } r \in U \qquad \Longleftrightarrow \qquad x(r) \in \mathcal{S} \quad \text{for all } r \in U_{\mathrm{cr}} \tag{21}$$

holds, where

$$U_{\mathrm{cr}} := \{r \in U : \ \nu\big(x(r)\big) = 0\}. \tag{22}$$

Proof. Similar to that of Proposition 1.

Proposition 2 implies that for the infinite family of real matrices or polynomials $\{x(r) : \ r \in U\}$ to be stable relative to Ω, it suffices to check that the family $\{x(r) : \ r \in U_{\mathrm{cr}}\}$ is stable. In other words, to establish that the family $\{x(r) : \ r \in U\}$ is stable relative to Ω, one has to ensure that $\nu(A(r)) = 0$ corresponds to the family "hitting" the blind spots and not $\partial\mathcal{S}(\Omega)$. In cases where U_{cr} is a finite set, Proposition 2 therefore provides a tool for asserting the stability of the family $\{x(r) : \ r \in U\}$.

Remark 3.1. The assumption $x(r^0) \in \mathcal{S}$ for some $r^0 \in U$ appearing in Proposition 2 is, strictly speaking, required only in the case $U_{\mathrm{cr}} = \emptyset$.

Remark 3.2. For some sets $\mathcal{S}$, the semiguardian map ν factors as $\nu = \nu_1\nu_2$ where $\nu_2(x) = 0$ implies $x \notin \mathcal{S}$ (see Example 3.9 and Proposition 7). In this situation, the set U_{cr} in the Proposition above may be replaced by $U^1_{\mathrm{cr}} := \{r \in U : \ \nu_1(x(r)) = 0\}$, with the requirement that the set $U^2_{\mathrm{cr}} := \{r \in U : \ \nu_2(x(r)) = 0\}$ be empty.

3.3. Examples

We now give some examples of guardian and semiguardian maps and the associated guarded and semiguarded sets of matrices or polynomials. The number of examples given is justified by the important role each plays in the development to follow. Before proceeding with the examples, we point out that for every guardian or semiguardian map for the matrix case there are corresponding maps for the polynomial case, and vice versa. For instance, note that a family of polynomials may be viewed as characteristic polynomials of an associated family of companion matrices. Similarly, stability of a family of matrices may be reformulated in terms of the associated family of characteristic polynomials.

Example 3.1. The map $\nu : A \mapsto \det(A)$ guards the set of nonsingular matrices; similarly, $\nu : p \mapsto p(0)$ guards the set of all polynomials that do not vanish at zero.

The next two examples provide the simplest useful illustrations of the concept of guardian map: both the set of Hurwitz stable matrices (or polynomials) and the set of Schur stable matrices (or polynomials) are guarded.

Example 3.2. The map $\nu : A \mapsto \det(A \oplus A)$ guards the set of $n \times n$ Hurwitz stable matrices $\mathcal{S}(\mathring{\mathbb{C}}_-)$. This follows from the property that the spectrum of $A \oplus A$ consists of all pairwise sums of eigenvalues of A (see Section 2). Another such guardian map ν is given by $\nu(A) = \det(A_{[2]})$. Note that each of these maps guards $\mathcal{S}(\mathring{\mathbb{C}}_+)$ as well. Similarly, the set of Hurwitz stable real *polynomials* of the form

$$p(s) := a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 \tag{23}$$

is guarded by the map $\nu : p \mapsto \det H(p)$ where $H(p)$ is the Hurwitz matrix associated with p and is given by

$$H(p) = \begin{bmatrix} a_{n-1} & a_{n-3} & a_{n-5} & . & . & . & . & 0 \\ a_n & a_{n-2} & a_{n-4} & . & . & . & . & 0 \\ 0 & a_{n-1} & a_{n-3} & . & . & . & . & 0 \\ 0 & a_n & a_{n-2} & . & . & . & . & 0 \\ . & . & . & . & . & . & . & . \\ 0 & . & . & . & . & a_3 & a_1 & 0 \\ 0 & . & . & . & . & . & a_2 & a_0 \end{bmatrix}. \tag{24}$$

This follows from *Orlando's formula* [G1], [B1]:

$$\det H(p) = (-1)^{\frac{n(n-1)}{2}} \left(\frac{1}{2}\right)^n a_n^{n-1} \prod_{1\le i\le k\le n} (x_i + x_k), \tag{25}$$

where $x_1, \ldots, x_n$ are the zeros of $p(s)$.

Example 3.3. The map $\nu : A \mapsto \det\,(A \otimes A - I \otimes I)$ guards the set of Schur stable matrices, i.e., of matrices with eigenvalues in in the open unit disk. Another guardian map for this set is given by $\nu(A) = \det\bigl(A^{[2]} - I^{[2]}\bigr)$. Here too, both maps guard the generalized stability set corresponding to the outside of the unit disk. Similarly, a guardian map for the set of Schur stable real polynomials of the form (23) can be readily obtained from results in [JP], [AB]: Define the $(n-1)\times(n-1)$ matrix $D(p)$ by

$$D(p) = \begin{bmatrix} a_n & a_{n-1} & a_{n-2} & \cdot & \cdot & a_3 & a_2 - a_0 \\ 0 & a_n & a_{n-1} & \cdot & \cdot & a_4 - a_0 & a_3 - a_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & -a_0 & -a_1 & \cdot & \cdot & a_n - a_{n-4} & a_{n-1} - a_{n-3} \\ -a_0 & -a_1 & -a_2 & \cdot & \cdot & -a_{n-3} & a_n - a_{n-2} \end{bmatrix}. \tag{26}$$

Then we have the following expression for $\det D(p)$:

$$\det D(p) = \prod_{\substack{k=1\\ i<k}}^{n} (1 - x_i x_k). \tag{27}$$

Clearly, $\det D(p)$ vanishes whenever p has a pair of conjuguate eigenvalues on the unit circle. Note however that $\det D(p)$ does not necessarily vanish if p only has 1 (or -1) as eigenvalue. Taking this into account, we conclude that the set of Schur stable real polynomials is guarded by $\nu : p \mapsto p(1)p(-1)\det D(p)$.

In the next two examples we exhibit guardian maps for sets of matrices having all their eigenvalues in the domains depicted in Fig. 1 and Fig. 2.

Example 3.4. Given $\beta > 0$, the map $\nu : A \mapsto \det\,[(A + i\beta I) \ominus (A - i\beta I)]$ guards $\mathcal{S}(\Omega^\beta)$ where $\Omega^\beta := \{s : |\Im m(s)| < \beta\}$. Indeed, $\nu(A) = 0$ if and only if A has two eigenvalues

$\lambda_1 = x + iy_1$ and $\lambda_2 = x + iy_2$ such that $y_2 - y_1 = 2\beta$. Therefore, if $A \in \overline{\mathcal{S}(\Omega^\beta)}$ then $\nu(A) = 0$ if and only if A has some eigenvalues on $\partial\Omega^\beta$, i.e., $A \in \partial\mathcal{S}(\Omega^\beta)$.

Example 3.5. Given $\theta_0 \in [+\frac{\pi}{2}, \pi)$, $\nu: A \mapsto \det\,(e^{i\theta_0}A \ominus e^{-i\theta_0}A)$ guards $\mathcal{S}(\Omega_{\theta_0})$ where Ω_{θ_0} is the sector given by $\Omega_{\theta_0} := \{s: |\mathrm{Arg}(s)| > \theta_0\}$. Again, $\nu(A) = 0$ if and only if A has eigenvalues $\lambda_k = r_k e^{i\theta_k}, k = 1, 2$ such that $r_1 e^{i\theta_1} e^{i\theta_0} - r_2 e^{i\theta_2} e^{-i\theta_0} = 0$, i.e., if and only if $r_1 = r_2$ and $\theta_2 - \theta_1 = 2\theta_0 \pmod{2\pi}$. Since $\frac{\pi}{2} \le \theta_0 < \pi$, this says that for all $A \in \overline{\mathcal{S}(\Omega_{\theta_0})}$, $\nu(A) = 0$ if and only if A has at least one pair of eigenvalues on $\partial\Omega_{\theta_0}$ or a single eigenvalue at 0. Note that $\Omega_{\theta_0} = \mathring{\mathbb{C}}_-$ for $\theta_0 = \frac{\pi}{2}$.

As a final example of guardian maps, Example 3.6 below is of interest when assessing "strict aperiodicity" of linear systems wherein the eigenvalues (poles) are real, negative and distinct [J2], [J3]. It is also a building block for a semiguardian map to be considered below.

Example 3.6. The map $\nu: A \mapsto \det(A^2 \cdot I - A \cdot A)$ guards the set of all real matrices having only algebraically simple eigenvalues; this follows directly from (11). Similarly, it follows from Lemma 4 that the map $\nu: p \mapsto \det B(p, p')$ guards the set of all real polynomials with only algebraically simple zeros. Using this, it can be seen that the map $\nu: A \mapsto \det(A^2 \cdot I - A \cdot A)\,\det(A)$ (resp. $\nu: p \mapsto \det B(p, p')\; p(0)$) guards the set of strictly aperiodic matrices (resp. polynomials).

Our first examples of semiguardian maps are related to Examples 3.5 and 3.6. In Section 6, we shall provide a systematic procedure for constructing "polynomic" guardian and semiguardian maps for stability sets corresponding to a large class of domains in the complex plane (boundaries of which are defined by polynomials).

Example 3.7. Given $\beta > 0$ and $\Omega = \{s: |\Im m(s)| > \beta\}$, the set $\mathcal{S}_n(\Omega)$ is not guarded by $\nu: A \mapsto \det\,[(A + i\beta I) \ominus (A - i\beta I)]$ if $n > 3$. Indeed, for $n > 3$, one can easily construct a matrix $A \in \mathcal{S}(\Omega)$ with a pair of eigenvalues $\lambda_1 = x + iy_1$, $\lambda_2 = x + iy_2$ such that $y_2 - y_1 = 2\beta$. Clearly, this cannot be done if $n \le 3$, and $\mathcal{S}_n(\Omega)$ is guarded by ν for $n = 2, 3$. However, for any n, ν is a semiguardian map for $\mathcal{S}_n(\Omega)$ and any matrix having the eigenvalues λ_1, λ_2 above is a blind spot for $(\nu, \mathcal{S}_n(\Omega))$.

Example 3.8. For $\theta_0 \in (\frac{\pi}{2}, \pi)$, let the interior of the complement of the sector Ω_{θ_0}

be denoted by $\Omega = \{s : \mid \text{Arg } (s)| < \theta_0\}$. Again the set $\mathcal{S}_n(\Omega)$ is not guarded by $\nu : A \mapsto \det\ (e^{i\theta_0}A \ominus e^{-i\theta_0}A)$ for any $n \geq 2$. Indeed, if $A \in \mathcal{S}(\Omega)$ has eigenvalues $e^{i\theta}$ and $e^{-i\theta}$ where $\theta = \pi - \theta_0 < \theta_0$, then one eigenvalue of $e^{i\theta_0}A \ominus e^{-i\theta_0}A$ is given by $e^{i\theta_0}e^{i\theta} - e^{-i\theta_0}e^{-i\theta} = e^{i\pi} - e^{-i\pi} = 0$, implying $\nu(A) = 0$ although A is a stable matrix. Here as in the previous example, ν is a semiguardian map for $\mathcal{S}_n(\Omega)$, for any n. The set of blind spots includes any matrix with at least one eigenvalue in the mirror image (w.r.t. the imaginary axis) of $\partial\Omega \setminus \{0\}$ (dashed lines in Fig. 3).

Our final example will be used in the study of two-parameter families of matrices or polynomials.

Example 3.9. Let α, β be finite real numbers, with $\alpha \neq \beta$, and let $\Omega := \mathbb{C} \setminus [\alpha, \beta]$. Then the set of stable matrices $\mathcal{S}(\Omega)$ is semiguarded by the map

$$\nu : \ A \mapsto \det\left(A^2 \cdot I - A \cdot A\right) \det\left((A - \alpha I)(A - \beta I)\right). \tag{28}$$

For $\Omega = \mathbb{C} \setminus (-\infty, \beta]$, $\mathcal{S}(\Omega)$ is semiguarded by the map

$$\nu : \ A \mapsto \det\left(A^2 \cdot I - A \cdot A\right) \det(A - \beta I). \tag{28}$$

This can be seen by referring back to Example 3.6 and noting that $\mathcal{S}(\mathbb{C} \setminus \mathbb{R})$ is semiguarded by $\nu : \ A \mapsto \det\left(A^2 \cdot I - A \cdot A\right)$. In the case of polynomials, analogous semiguardian maps are given by

$$\nu : \ p \mapsto \det B(p, p')\ p(\alpha)p(\beta). \tag{30}$$

and

$$\nu : \ p \mapsto \det B(p, p')\ p(\beta), \tag{31}$$

respectively. Similar expressions based on the polynomial resultant [LT] may also be used as semiguardian maps. In all these cases, the blind spots include any matrix (polynomial) having at least one multiple eigenvalue (zero) off the real axis.

In the sequel, to highlight the stability aspect of this work we choose to focus on the case of stability sets $\mathcal{S}(\Omega)$, with the understanding that the results hold in general. In addition, for ease of exposition we discuss only the matrix case. The development can be carried out, *mutatis mutandis*, for the polynomial case.

4. One-Parameter Families

In this section, we derive necessary and sufficient conditions for stability of a one-parameter family of matrices

$$A(r) = A_0 + rA_1 + \cdots + r^m A_m \ , \tag{32}$$

$r \in [0,1]$,[3] relative to a given domain $\Omega \subset \mathbb{C}$. In (32), A_k, $k = 1, \ldots, m$, are given $n \times n$ real matrices. In the remainder of this section, the family $\{A(r) : r \in [0,1]\}$ is denoted by $\mathcal{A}$.

All guardian and semiguardian maps considered thus far are "polynomic," in the sense of the following definition.

Definition 3. A guardian map ν is said to be *polynomic* if it is a polynomial function of the entries (matrix case) or coefficients (polynomial case) of its argument.

The case in which a polynomic guardian map for $\mathcal{S}(\Omega)$ is available is considered first. We then consider the case in which only a polynomic semiguardian map is available.

4.1. Polynomic Guardian Maps

Let ν be a polynomic guardian map for $\mathcal{S}(\Omega)$. Then $\nu(A(r))$ is a polynomial in r. From Proposition 1, it follows that the family $\mathcal{A}$ is stable relative to Ω if and only if (i) A_0 is stable relative to Ω (i.e., $A_0 \in \mathcal{S}(\Omega)$), and (ii) the univariate polynomial $\nu(A(r))$ has no zeros in $[0,1]$. In other words, *if $\mathcal{S}(\Omega)$ is guarded by ν and $A(0) \in \mathcal{S}(\Omega)$ then a necessary and sufficient condition for the family $\mathcal{A}$ to be stable relative to Ω is that the polynomial $\nu(A(r))$ be stable relative to Ξ.* In such situations, it therefore suffices to merely check that a scalar polynomial has no zeros in a certain interval; this can be done using a finite algorithm based on Sturm sequences (see e.g. [J1]).

The necessary and sufficient condition given above assumes availability of a guardian map $\nu(A)$ explicitly in the form of a polynomial in the entries of A. However, examples considered in Section 3, as well as results on a class of domains considered in Section 6,

[3] In fact, the discussion applies for r constrained to lie in *any* interval (not necessarily compact).

show that guardian and semiguardian maps often occur in the form

$$\nu(A) = \det \mathcal{F}(A) \tag{33}$$

where $\mathcal{F}$ is a polynomial mapping on $I\!R^{n\times n}$. A necessary and sufficient condition analogous to the one above, but not requiring expansion of the determinant (33), is now formulated.

With $A(r)$ as in Eq. (32), we may write

$$\mathcal{F}(A(r)) = \sum_{i=0}^{q} r^i F_i(A_0,\ldots,A_m) \tag{34}$$

Note that

$$F_0(A_0,\ldots,A_m) = \mathcal{F}(A_0). \tag{35}$$

In the sequel, F_i denotes $F_i(A_0,\ldots,A_m)$ for $i = 0,1,\ldots,q$.

Theorem 1. Let $\mathcal{S}(\Omega)$ be guarded by a map ν of the form (33), and let $A_0 \in \mathcal{S}(\Omega)$. Then $A(r) \in \mathcal{S}(\Omega)$ for all $r \in [0,1]$ if and only if $M(A_0,\ldots,A_m) \in \mathcal{S}(\Theta)$ where $\Theta = \mathbb{C} \setminus [1,\infty)$ and

$$M(A_0,\ldots,A_m) = \begin{bmatrix} 0 & I & . & . & 0 \\ \vdots & \vdots & \vdots & & \\ . & . & . & 0 & I \\ -M_0 & . & . & . & -M_{q-1} \end{bmatrix} \tag{36}$$

with

$$M_i = F_0^{-1} F_{q-i}, \qquad i = 0,\ldots,q-1 \tag{37}$$

if $q \geq 2$, and

$$M(A_0,\ldots,A_m) = -F_0^{-1} F_1 \tag{38}$$

if $q = 1$.

Proof. From Proposition 1, we have $A(r) \in \mathcal{S}(\Omega)$ for all $r \in [0,1]$ if and only if

$$\nu(A(r)) \neq 0 \text{ for all } r \in [0,1]. \tag{39}$$

Since $A_0 \in \mathcal{S}(\Omega)$ and ν guards $\mathcal{S}(\Omega)$, it follows that $\nu(A_0) = \det\mathcal{F}(A_0) \neq 0$. Therefore F_0 is invertible. Thus (33) and (34) imply

$$\nu(A(r)) = \det F_0 \, \det(I + rM_{q-1} + \cdots + r^q M_0). \tag{40}$$

This implies that $\nu(A(r))$ is nonvanishing for all $r \in [0,1]$ if and only if

$$\chi(\mu) := \det\left(\mu^q I + \mu^{q-1} M_{q-1} + \ldots + \mu M_1 + M_0 \right) \neq 0 \tag{41}$$

for all $\mu \in [1, \infty)$, where $\mu := \frac{1}{r}$. Since $\chi(\mu)$ is the characteristic polynomial of $M(A_0, \ldots, A_m)$ if $q > 1$ (also of $-M_{q-1}$ if $q = 1$), we have that $A(r) \in \mathcal{S}(\Omega)$ for all $r \in [0,1]$, if and only if $M(A_0, \ldots, A_m)$ has no eigenvalues in $[1, \infty)$.

□

Stability of the matrix $M(A_0, \ldots, A_m)$ relative to Θ is therefore necessary and sufficient for stability of the family $\mathcal{A}$ relative to the guarded set $\mathcal{S}(\Omega)$.

4.2. Polynomic Semiguardian Maps

Let ν be a semiguardian map for $\mathcal{S}(\Omega)$ and assume that $A(0) \in \mathcal{S}(\Omega)$. From Proposition 2, we have that the family $\mathcal{A}$ is stable relative to Ω if and only if $A(r) \in \mathcal{S}(\Omega)$ for all $r \in U_{\text{cr}}$ where U_{cr} is the set of zeros of the polynomial $\nu(A(r))$ belonging to $[0,1]$.

For semiguardian maps of the form (33), if the matrix $M(A_0, \ldots, A_m)$ is well-defined, the condition $M(A_0, \ldots, A_m) \in \mathcal{S}(\Theta)$ of Section 4.1 remains sufficient for stability of the family $\mathcal{A}$, but is no longer necessary. The test is inconclusive if $M(A_0, \ldots, A_m)$ has an eigenvalue in $[1, \infty)$. Suppose that $\nu(A_0) = \det \mathcal{F}(A_0) \neq 0$. Then the matrix $M(A_0, \ldots, A_m)$ given by (36) (or (38)) is well defined. Define the *critical subset* of the spectrum of $M(A_0, \ldots, A_m)$ by

$$\Sigma_{\text{cr}} := \sigma(M(A_0, \ldots, A_m)) \cap [1, \infty). \tag{42}$$

If Σ_{cr} is nonempty, denote it by $\{\mu_1, \ldots, \mu_\ell\}$. Since $\mu = \frac{1}{r}$, the set $U_{\text{cr}} = \{r \in [0,1] : \nu(A(r)) = 0\}$ will then be given by $\{\mu_1^{-1}, \ldots, \mu_\ell^{-1}\}$. Proposition 2 now yields the following.

Theorem 2. *Let $\mathcal{S}(\Omega)$ be semiguarded by a map ν of the form (33), and let $\nu(A_0) \neq 0$. If $\Sigma_{\text{cr}} = \emptyset$, i.e., $M(A_0, \ldots, A_m)$ has no eigenvalues in $[1, \infty)$, then the family $\mathcal{A}$ is stable relative to Ω if and only if $A_0 \in \mathcal{S}(\Omega)$. If, however, $\Sigma_{\text{cr}} = \{\mu_1, \ldots, \mu_\ell\} \neq \emptyset$, then the family $\mathcal{A}$ is stable relative to Ω if and only if*

$$A(\mu_i^{-1}) \in \mathcal{S}(\Omega), \quad i = 1, \ldots, \ell. \tag{43}$$

Remark 4.1. A result analogous to Theorem 2 may be obtained with ν a polynomial map rather than being specifically of the form (33). The companion matrix associated with the scalar polynomial $\nu(A(r))$ then plays the role of $M(A_0, \ldots, A_m)$, and the assumption $\nu(A_0) \neq 0$ is then no longer relevant.

The results above may be applied to the special case of generalized stability of the convex hull of two matrices or polynomials. Combined with Edge Theorem [BHL], these results therefore provide a solution to the generalized stability of a polytope of polynomials. The special case of Hurwitz and Schur stability of the convex hull of two matrices or polynomials is presented next.

4.3. Hurwitz and Schur Stability of the Convex Hull of Two Matrices or Polynomials

As applications of Theorem 1, we consider Hurwitz and Schur stability of the convex hull of two real matrices. For the former problem, a known result is obtained [B1], [FB1].

Given two $n \times n$ real matrices A_0 and A_1, the convex hull $\mathrm{co}(A_0, A_1)$ of A_0 and A_1 consists of the matrices

$$\begin{aligned} A(r) &= (1-r)A_0 + rA_1 \\ &= A_0 + r(A_1 - A_0), \end{aligned}$$

for $r \in [0, 1]$.

Hurwitz stability

Let $\Omega = \overset{\circ}{\mathbb{C}}_-$ and recall that $\mathcal{S}(\Omega)$ is guarded by $\nu: \ A \mapsto \det\mathcal{F}(A)$, where $\mathcal{F}(A)$ may denote either $A_{[2]}$ or $A \oplus A$.

Corollary 1. Let A_0 be Hurwitz stable. Then $\mathrm{co}(A_0, A_1)$ is Hurwitz stable if and only if $\mathcal{F}^{-1}(A_0)\mathcal{F}(A_1)$ has no eigenvalues in $(-\infty, 0]$. Here, $\mathcal{F}(A)$ can denote either $A_{[2]}$ or $A \oplus A$.

Proof. Since $\mathcal{F}$ is linear,

$$\begin{aligned} \nu\big(A(r)\big) &= \det \mathcal{F}\big(A_0 + r(A_1 - A_0)\big) \\ &= \det \big(\mathcal{F}(A_0) + r\mathcal{F}(A_1 - A_0)\big). \end{aligned}$$

Hence $F_0 = \mathcal{F}(A_0)$ and $F_1 = \mathcal{F}(A_1 - A_0)$, in the notation of Theorem 1. Applying Theorem 1 with $q = 1$ yields that $\mathrm{co}(A_0, A_1)$ is Hurwitz if and only if $M(A_0, A_1 - A_0) =$

$-\mathcal{F}^{-1}(A_0)\mathcal{F}(A_1 - A_0) \in \mathcal{S}(\Theta)$, i.e., has no eigenvalue in the interval $[1, \infty)$. Finally, $-\mathcal{F}_0^{-1}\mathcal{F}(A_1 - A_0) = I - \mathcal{F}^{-1}(A_0)\mathcal{F}(A_1)$, and the result follows.

□

The analogue of Corollary 1 for the case of the convex hull of two polynomials p_0 and p_1 ([B1, Theorem 1], [FB1, Theorem 3.1]) follows along the same lines, from Example 3.2.

Corollary 2. Let p_0 be Hurwitz stable. Then $\mathrm{co}(p_0, p_1)$ is Hurwitz stable if and only if the $n \times n$ matrix $H^{-1}(p_0)H(p_1)$ has no eigenvalues in $(-\infty, 0]$. Here, $H(p)$ denotes the Hurwitz matrix associated with the polynomial p.

Discrete-time (Schur) stability

Let Ω be the open unit disk. From Example 3.3, $\mathcal{S}(\Omega)$ is guarded by $\nu:\ A \mapsto \det\mathcal{F}(A)$ where $\mathcal{F}(A)$ may be taken as either $A \otimes A - I \otimes I$ or $A^{[2]} - I^{[2]}$. Although the latter map is preferable from a computational point of view, the former is used here for notational convenience. Denoting $A_1 - A_0$ by $\tilde{A}_1$, we have that

$$\begin{aligned}
\mathcal{F}(A(r)) &= \mathcal{F}(A_0 + r\tilde{A}_1) \\
&= (A_0 \otimes A_0) - I \otimes I + r\left[A_0 \otimes \tilde{A}_1 + \tilde{A}_1 \otimes A_0\right] + r^2 \tilde{A}_1 \otimes \tilde{A}_1 \\
&=: F_0 + rF_1 + r^2 F_2.
\end{aligned} \tag{44}$$

We now apply Theorem 1 with F_0, F_1 and F_2 as in Eq. (44) and $\tilde{A}_1$ identified with A_1.

Corollary 3. Let all the eigenvalues of A_0 have magnitude less than 1. Then the same is true for any matrix in $\mathrm{co}(A_0, A_1)$ if and only if $M(A_0, \tilde{A}_1)$ has no eigenvalues in $[1, \infty)$, where

$$M(A_0, \tilde{A}_1) = \begin{bmatrix} 0 & I \\ -F_0^{-1}F_2 & -F_0^{-1}F_1 \end{bmatrix} \tag{45}$$

and F_0, F_1 and F_2 are as in (44).

A similar result ([AB]) for the case of Schur stability of the convex hull of two polynomials $\mathrm{co}(p_0, p_1)$ may be obtained using the map given in Example 3.3 and is given next. For a proof, see [S1].

Corollary 4. Let p_0 and p_1 be Schur stable. Then $\mathrm{co}(p_0, p_1)$ is Schur stable if and only if the matrix $D^{-1}(p_0)D(p_1)$ has no eigenvalues in $(-\infty, 0)$. Here, the matrix $D(p)$ is the one defined by (26).

5. Two-Parameter Families

In this section, we consider stability of *two-parameter* families of real matrices relative to a domain Ω for which $\mathcal{S}(\Omega)$ is endowed with a polynomic *guardian* map ν_Ω, which we assume to be real valued.[4] The matrices we study, denoted $A(r_1, r_2)$, are polynomic functions of the parameters r_1, r_2 which are taken to lie in $[0,1]$.[5] Since both ν_Ω and $A(r_1, r_2)$ are polynomic in their arguments, we may express $\nu_\Omega(A(r_1, r_2))$ in the form of a bivariate polynomial, $\nu_\Omega(r_1, r_2)$, viz.

$$\nu_\Omega(r_1, r_2) := \nu_\Omega\big(A(r_1, r_2)\big) = \sum_{i_1=0}^{s_1} \sum_{i_2=0}^{s_2} \nu_{i_1, i_2} r_1^{i_1} r_2^{i_2}. \tag{46}$$

Assume that $A(0,0) \in \mathcal{S}(\Omega)$. Proposition 1 then implies that $A(r_1, r_2) \in \mathcal{S}(\Omega)$ for all $r_1,\ r_2 \in [0,1]$ precisely when

$$\nu_\Omega(r_1, r_2) \neq 0 \qquad \text{for all } (r_1, r_2) \in [0,1] \times [0,1]. \tag{47}$$

Note that $\nu_\Omega(r_1, r_2)$ is not identically zero, as $\nu_{0,0} = \nu_\Omega(A(0,0)) \neq 0$ since $A(0,0) \in \mathcal{S}(\Omega)$ and since $\mathcal{S}(\Omega)$ is guarded by ν_Ω.

In view of condition (47), the generalized stability problem being considered is reducible to a *positivity* question (e.g., [B2]) which can be addressed using work of [WZ], [AS]. This approach was recently taken in [GT] and [VT], in addressing the two-parameter generalized stability problem relative to several specific domains.

Notwithstanding the amenability of the problem at hand to solution by positivity methods, we now consider this problem using semiguardian maps. This calculation is given

[4] All of the examples considered thus far are of this type. In the cases of Examples 3.4 and 3.5, real guardian maps can be obtained using results of Section 6.

[5] More general intervals may be considered.

because it leads one to consider the issue of guardedness of *approximations* of $\mathcal{S}(\Xi)$ (only a polynomic *semiguardian* map is known for $\mathcal{S}(\Xi)$), which is anticipated to be important in multiparameter problems.

The bivariate polynomial $\nu_\Omega(r_1, r_2)$ can be rewritten in the form of a univariate polynomial in, say r_2, viz.

$$\nu_\Omega(r_1, r_2) = \alpha_0(r_1) + \alpha_1(r_1)r_2 + \cdots + \alpha_{s-1}(r_1){r_2}^{s_2-1} + \alpha_{s_2}(r_1){r_2}^{s_2} \tag{48}$$

where each coefficient $\alpha_i(r_1), \quad i = 0, \ldots, s_2$ is a polynomial in r_1. Denote by p_{r_1} the polynomial in r_2 resulting upon fixing r_1 in (48). The bivariate polynomial $\nu_\Omega(r_1, r_2)$ does not vanish for any $r_1,\ r_2 \in [0, 1]$ if and only if

$$p_{r_1} \in \mathcal{S}(\Xi) \ \text{ for each } r_1 \in [0, 1]. \tag{49}$$

Consequently, a generalized stability question for a two-parameter family of matrices has been reduced to a similar question for a related one-parameter family of polynomials relative to the specific domain Ξ.

From Example 3.9, the map ν_Ξ given by

$$\nu_\Xi(p) = \det B(p, p')\ p(0)p(1) \tag{50}$$

is semiguarding for $\mathcal{S}(\Xi)$. Letting $\nu_1(p)$, $\nu_2(p)$ denote $\det B(p, p')$ and $p(0)p(1)$, respectively, we obtain by combining Remark 3.2 and Proposition 2 that $p_{r_1} \in \mathcal{S}(\Xi)$ for all $r_1 \in [0, 1]$ if and only if the polynomial $p_0 \in \mathcal{S}(\Xi)$, $U^1_{\text{cr}} = \emptyset$ and

$$p_{r_1} \in \mathcal{S}(\Xi) \ \text{ for all } \ r_1 \in U^2_{\text{cr}}, \tag{51}$$

where

$$U^1_{\text{cr}} := \{r_1 \in [0, 1] : \ \nu_1(p_{r_1}) = 0\}, \tag{52}$$

$$U^2_{\text{cr}} := \{r_1 \in [0, 1] : \ \nu_2(p_{r_1}) = 0\}. \tag{53}$$

From (50), it is clear that $\nu_\Xi(p_{r_1})$ is a polynomial in the parameter r_1, which we assume not to be identically zero. Thus the set $U_{\text{cr}} := U^1_{\text{cr}} \cup U^2_{\text{cr}}$ is finite. For the case in which U_{cr} is empty, we have, by Remark 3.1, that $p_{r_1} \in \mathcal{S}(\Xi)$ for all $r_1 \in [0,1]$ if and only if $p_0 \in \mathcal{S}(\Xi)$. Hence, in the current setting, $\mathcal{A}$ is stable relative to Ω if and only if $p_0 \in \mathcal{S}(\Xi)$. Suppose, on the other hand, that $U_{\mathrm{cr}}^2 = \emptyset$ and $U_{\mathrm{cr}}^1 =: \{\mu_1, \ldots, \mu_\ell\}$ where the μ_i's belong to $[0,1]$. Then the requirement that $p_{\mu_i} \in \mathcal{S}(\Xi), \quad i = 1, \ldots, \ell$ is necessary and sufficient for stability of the family $\mathcal{A}$ relative to Ω.

The following theorem summarizes the foregoing discussion.

Theorem 3. Let Ω be a subset of the complex plane such that $\mathcal{S}(\Omega)$ is guarded by a given real polynomic map ν_Ω. Let $\mathcal{A} := \{A(r_1, r_2) : \ (r_1, r_2) \in [\alpha_1, \beta_1] \times [\alpha_2, \beta_2]\}$ be a nominally stable family of real matrices; e.g., $A(\alpha_1, \alpha_2) \in \mathcal{S}(\Omega)$. Then the family $\mathcal{A}$ is stable relative to Ω if and only if $U_{\mathrm{cr}}^2 = \emptyset$ and the univariate polynomials p_{α_1}, and $p_{r_1}, \ r_1 \in U_{\mathrm{cr}}^1$, have no zeros in $[\alpha_2, \beta_2]$. Here, for each r_1, p_{r_1} denotes the univariate polynomial $\nu_\Omega(A(r_1, .))$,

$$U_{\mathrm{cr}}^1 := \left\{r_1 \in [\alpha_1, \beta_1] : \quad \det B(p_{r_1}, p'_{r_1}) = 0\right\}$$

and

$$U_{\mathrm{cr}}^2 := \{r_1 \in [\alpha_1, \beta_1] : \quad p_{r_1}(\alpha_2) p_{r_1}(\beta_2) = 0\} .$$

6. Techniques for Constructing Guardian and Semiguardian Maps

6.1. Generating New Guarded and Semiguarded Sets from Known Ones

The next proposition states properties which provide means for the construction of new domains from existing ones with the resulting generalized stability sets being guarded or semiguarded.

Proposition 3. Let $\mathcal{S}$, $\mathcal{S}_1$ and $\mathcal{S}_2$ be subsets of $I\!R^{n \times n}$.

(i) Assume that $\mathcal{S}$ is guarded (resp. semiguarded) by ν. Then $-\mathcal{S} := \{-A : \ A \in \mathcal{S}\}$ is guarded (resp. semiguarded) by $\nu_- : \ A \mapsto \nu(-A)$. In particular, if $\mathcal{S} = \mathcal{S}(\Omega)$, $\mathcal{S}(-\Omega)$ is guarded by ν_-, where $-\Omega := \{-s : \ s \in \Omega\}$.

(ii) Let $\mathcal{S}$ be guarded (resp. semiguarded) by ν and let $\alpha \in I\!R$. Then $\mathcal{S} + \alpha I := \{A + \alpha I : \ A \in \mathcal{S}\}$ is guarded (resp. semiguarded) by $\nu^{(\alpha)} : \quad A \mapsto \nu(A - \alpha I)$. In particular, if $\mathcal{S} = \mathcal{S}(\Omega)$, $\mathcal{S}(\Omega^{(\alpha)})$ is guarded (resp. semiguarded) by $\nu^{(\alpha)}$, where $\Omega^{(\alpha)} := \{s + \alpha : \ s \in \Omega\}$.

(iii) Let $\mathcal{S}$ be guarded (resp. semiguarded) by ν and let $\rho \in I\!R \setminus \{0\}$. Then $\rho\mathcal{S} := \{\rho A: \ A \in \mathcal{S}\}$ is guarded (resp. semiguarded) by $\nu_\rho : \quad A \mapsto \nu(\frac{A}{\rho})$. In particular, if $\mathcal{S} = \mathcal{S}(\Omega)$, $\mathcal{S}(\rho\Omega)$ is guarded (resp. semiguarded) by ν_ρ.

(iv) Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be guarded (resp. semiguarded) by ν_1 and ν_2, respectively. Then $\mathcal{S}_1 \cap \mathcal{S}_2$ is guarded (resp. semiguarded) by $\nu : \quad A \mapsto \nu_1(A)\nu_2(A)$. In particular, if $\mathcal{S}_1 = \mathcal{S}(\Omega_1)$ and $\mathcal{S}_2 = \mathcal{S}(\Omega_2)$, then $\mathcal{S}(\Omega_1 \cap \Omega_2)$ is guarded (resp. semiguarded) by ν. The analogous statement holds for polynomials.

(v) Let $\mathcal{S}$ be guarded (resp. semiguarded) by ν. Then any connected component of $\mathcal{S}$ is guarded (resp. semiguarded) by ν.

Since any convex domain (symmetric w.r.t. to the real axis, symmetric for short) with polygonal boundary may be generated from the two basic domains Ω^β and Ω_θ, using the basic operations in Proposition 3, we have the following subsidiary result.

Proposition 4. Let Ω be any (symmetric) convex domain with polygonal boundary. Then $\mathcal{S}(\Omega)$ is guarded by a polynomic map. Moreover, Proposition 3 can be used to construct a guardian map.

6.2. Domains with Polynomial Boundary

In this section, we construct guardian and semiguardian maps for generalized stability sets corresponding to a whole class of domains of the complex plane. Specifically, we consider domains whose boundaries are given by a polynomial equations $p(x, y) = 0$ where x and y denote real and imaginary parts, respectively.

Denote

$$\Omega = \{s = x + iy : \quad p(x, y) < 0\}$$

where

$$p(x, y) = \sum_{k,\ell} p_{k\ell} x^k y^{2\ell} \ , \tag{54}$$

is a real polynomial. The fact that we focus on real matrices is accounted for by considering polynomials containing only even powers of y. Thus only domains symmetric w.r.t. the real axis are considered.

Associate with p the real valued polynomial

$$q(\lambda,\bar\lambda) = p\left(\frac{\lambda+\bar\lambda}{2},\frac{\lambda-\bar\lambda}{2i}\right)$$

$$= \sum_{k,\ell} p_{k\ell}(-1)^\ell \left(\frac{1}{2}\right)^{k+2\ell} (\lambda+\bar\lambda)^k(\lambda-\bar\lambda)^{2\ell}. \tag{55}$$

Rewrite (55) as

$$q(\lambda,\bar\lambda) = \sum_{k,\ell} q_{k\ell}\lambda^k\bar\lambda^\ell \tag{56}$$

where the coefficients $q_{k\ell}$ are real.

With this notation, Ω and $\partial\Omega$ have the alternative expressions

$$\Omega = \{\lambda \in \mathbb{C}: \quad q(\lambda,\bar\lambda) < 0\} \tag{57}$$

$$\partial\Omega = \{\lambda \in \mathbb{C}: \quad q(\lambda,\bar\lambda) = 0\} \tag{58}$$

Consider the mapping $\mathcal{F}:\ \mathbb{R}^{n\times n} \to \mathbb{R}^{n^2\times n^2}$ given by

$$\mathcal{F}(A) := \sum_{k,\ell} q_{k\ell} A^k \otimes A^\ell. \tag{59}$$

Lemma 1 implies that with $\sigma(A) = \{\lambda_1,\ldots,\lambda_n\}$,

$$\sigma(\mathcal{F}(A)) = \{q(\lambda_i,\lambda_j): \quad i,j=1,\ldots,n\}. \tag{60}$$

Now suppose that $A \in \partial\mathcal{S}(\Omega)$. Then some eigenvalue of A satisfies $\lambda_i \in \partial\Omega$, i.e., $q(\lambda_i,\bar\lambda_i) = 0$. It then follows from (60) that $\mathcal{F}(A)$ is singular ($\det \mathcal{F}(A) = 0$). We obtain the following propositions.

Proposition 5. Assume that ν is not identically zero. Then the map

$$\nu: \quad A \mapsto \det \sum_{k,\ell} q_{k\ell} A^k \otimes A^\ell \tag{61}$$

is semiguarding for $\mathcal{S}(\Omega)$.

Proposition 6. The map (61) guards $\mathcal{S}(\Omega)$ if and only if q satisfies the condition[7]

$$q(\lambda,\bar{\lambda})<0 \text{ and } q(\mu,\bar{\mu})<0 \quad\Rightarrow\quad q(\lambda,\mu)\neq 0. \qquad \text{(Condition } \mathcal{C}\text{)}$$

Maps such as (61) involve determinants of matrices the size of which increases rapidly as n does. Alternative formulas, based on the bialternate product, exist which involve matrices of dimension $\frac{n(n-1)}{2}$, which is approximately half that of $\mathcal{F}(A)$ for large n.

Consider the mapping $\mathcal{M}$ from $I\!R^{n\times n}$ to $I\!R^{\frac{n(n-1)}{2}\times\frac{n(n-1)}{2}}$ given by

$$\mathcal{M}(A)=\sum_{k,\ell} q_{k\ell}A^k\cdot A^\ell. \tag{62}$$

Lemma 3 implies that

$$\sigma(\mathcal{M}(A))=\left\{\sum_{k,\ell} q_{k\ell}(\lambda_i^k\lambda_j^\ell+\lambda_i^\ell\lambda_j^k)/2:\ i=1,\ldots,n-1;\ j=i+1,\ldots,n\right\}. \tag{63}$$

Note that

$$\sigma(\mathcal{M}(A))=\left\{q(\lambda_i,\lambda_j):\quad i=1,\ldots,\ n-1;\ j=i+1,\ldots,\ n\right\}. \tag{64}$$

(Compare with Eq. (60).) This follows from (63) and the fact (implied by (55)) that

$$q(\lambda,\mu)=q(\mu,\lambda)=\frac{1}{2}\big(q(\lambda,\mu)+q(\mu,\lambda)\big). \tag{65}$$

Proposition 7. Suppose that $\det\mathcal{M}(A)$ is not identically zero.

(a) If $\partial\Omega\cap I\!R=\emptyset$, then the map

$$\nu:\ A\mapsto \det\mathcal{M}(A) \tag{66}$$

is semiguarding for $\mathcal{S}(\Omega)$.

---

7 "Ω-transformability" [GJ].

(b) Let $\partial\Omega \cap I\!R = \cup_{i=1}^{\xi}[\alpha_i, \beta_i]$ with $\alpha_1 \leq \beta_1 < \cdots < \alpha_i \leq \beta_i < \cdots < \alpha_\xi \leq \beta_\xi$. Denote $\mathcal{P}(A) = \prod_{i=1}^{\xi}(A - \alpha_i I)(A - \beta_i I)$ where, by convention, the factor $(A - \alpha_1 I)$ (resp. $(A - \beta_\xi I)$) is omitted when α_1 (resp. β_ξ) is $-\infty$ (resp. $+\infty$). Then the mapping

$$\nu : \quad A \mapsto \det\mathcal{M}(A)\det\mathcal{P}(A) \tag{67}$$

is semiguarding for $\mathcal{S}(\Omega)$.

Proof. Let $A \in I\!R^{n\times n} \cap \partial\mathcal{S}(\Omega)$:

(a) If A has an eigenvalue $\lambda \in \partial\Omega$ then $\bar{\lambda}$, also an eigenvalue of A, is distinct from λ. Therefore $q(\lambda, \bar{\lambda}) = 0 \in \sigma(\mathcal{M}(A))$ by virtue of (62). Hence $\nu(A) = 0$.

(b) Let $\lambda \in \partial\Omega$ be an eigenvalue of A. That is, $q(\lambda, \bar{\lambda}) = 0$. If $\lambda \notin I\!R$, then $\bar{\lambda}$ is also an eigenvalue of A, distinct from λ. Consequently, $q(\lambda, \bar{\lambda}) = 0 \in \sigma(\mathcal{M}(A))$ and $\nu(A) = 0$. If $\lambda \in I\!R \cap \partial\Omega$, then $\lambda \in [\alpha_i, \beta_i]$ for some $i \in \{1, \ldots, \xi\}$. Since by assumption $A \in \partial\Omega$, then it is the limit of a sequence of matrices $\{A_k\}$ with each $A_k \in \mathcal{S}(\Omega)$. It follows that there is a $j \in \{1, \ldots n\}$ such that $\lambda = \lim_{k\to\infty} \lambda_j(A_k)$. If $\lambda \in (\alpha_i, \beta_i)$ ($\beta_i > \alpha_i$ assumed) then there is a positive K such that for all $k > K$

$$\lambda_j(A_k) \in \mathbb{C} \setminus I\!R \quad \text{and} \quad \lambda = \lim_{k\to\infty} \lambda_j(A_k).$$

Since $\{A_k\}$ is a sequence of real matrices, we have that $\lambda = \lim_{k\to\infty} \bar{\lambda}_j(A_k)$ as well. Consequently, λ must be an eigenvalue of A of multiplicity at least 2. By virtue of (62), $q(\lambda, \bar{\lambda}) = q(\lambda, \lambda) \in \sigma(\mathcal{M}(A))$, i.e., $\nu(A) = 0$. If $\alpha_i \neq \beta_i$ and λ is either α_i or β_i, then λ might be a simple eigenvalue of A, in which case $q(\lambda, \lambda) = 0$ is no longer an eigenvalue of $\mathcal{M}(A)$. The case when α_i and β_i are finite and $\lambda = \alpha_i = \beta_i$ is handled similarly. The reason for the introduction of the second factor $\det\mathcal{P}(A)$ in the expression (67) for ν should now be clear.

$\square$

Remark 6.1. Proposition 6 applies for the maps of Proposition 7 as well.

We have exhibited semiguardian maps for generalized stability sets corresponding to domains with polynomial boundary. Determining whether or not these maps are also guardian maps requires further investigation. One needs to check whether or not the polynomial q satisfies Condition $\mathcal{C}$. Sufficient conditions for Condition $\mathcal{C}$ were obtained by Gutman and Jury [GJ] for the cases in which the degree of polynomial p (or q) is $1, 2, 3$ or 4. Another result in this direction is given next.

Proposition 8. Suppose that

$$q_{kk} \geq 0, \quad \forall\, k \geq 1, \tag{68}$$

$$q_{k\ell} = 0, \quad \forall\, k \neq \ell, \;\; k\ell \neq 0. \tag{69}$$

Then q satisfies Condition $\mathcal{C}$.

Proof. Proceeding by contradiction, assume that for some pair (λ, μ), $q(\lambda, \mu) = 0$, $q(\lambda, \bar{\lambda}) < 0$ and $q(\mu, \bar{\mu}) < 0$. Since the coefficients of q are real, we also have $q(\bar{\lambda}, \bar{\mu}) = 0$. Set

$$w := q(\lambda, \bar{\lambda}) + q(\mu, \bar{\mu}).$$

Clearly $w < 0$ and

$$\begin{aligned} w &= q(\lambda, \bar{\lambda}) + q(\mu, \bar{\mu}) - \big(q(\lambda, \mu) + q(\bar{\lambda}, \bar{\mu})\big) \\ &= \sum_{k,\ell} q_{k\ell}\big(\lambda^k \bar{\lambda}^\ell + \mu^k \bar{\mu}^\ell - (\lambda^k \mu^\ell + \bar{\lambda}^k \bar{\mu}^\ell)\big). \end{aligned}$$

From (55) and (56), we have

$$q_{k0} = q_{0k}, \quad \forall k \;\geq 1. \tag{70}$$

It now follows from (69) and (70) that

$$\begin{aligned} w = &\sum_{k=1} q_{k0}\;\left(\lambda^k + \mu^k - (\lambda^k + \bar{\lambda}^k) + \bar{\lambda}^k + \bar{\mu}^k - (\mu^k + \bar{\mu}^k)\right) \\ &+ \sum_{k=1} q_{kk}\;\left(\lambda^k \bar{\lambda}^k + \mu^k \bar{\mu}^k - (\lambda^k \mu^k + \bar{\lambda}^k \bar{\mu}^k)\right). \end{aligned}$$

The first summation yields zero. Under assumption (68), the second summation is non-negative, as can be seen by noting that, with $\alpha = r_1 e^{j\theta_1}$ and $\beta = r_2 e^{j\theta_2}$,

$$\begin{aligned} \alpha\bar{\alpha} + \beta\bar{\beta} - (\alpha\beta + \bar{\alpha}\bar{\beta}) &= r_1^2 + r_2^2 - 2r_1 r_2 \cos\,(\theta_1 + \theta_2) \\ &\geq (r_1 - r_2)^2. \end{aligned}$$

This contradicts the fact that $w < 0$.

$\square$

Example 6.1. Let $\Omega = \{s = x + iy : \ x + y^2 < 0\}$. Here $p(x,y) = x + y^2$ and

$$q(\lambda,\mu) = \frac{1}{2}\lambda + \frac{1}{2}\mu + \frac{1}{2}\lambda\mu - \frac{1}{4}\lambda^2 - \frac{1}{4}\mu^2$$

It follows from Proposition 8 that q satisfies Condition $\mathcal{C}$, and therefore that $\mathcal{S}(\Omega)$ is guarded by both

$$\nu : \ A \mapsto \det\mathcal{F}(A) \quad \text{and} \quad \nu : \ A \mapsto \det\, \mathcal{M}(A)\det(A).$$

Here,

$$\begin{aligned}\mathcal{F}(A) &= \frac{1}{2}(A \otimes I + I \otimes A) + \frac{1}{2}A \otimes A - \frac{1}{4}(A^2 \otimes I + I \otimes A^2) \\ &= \frac{1}{2}(A \oplus A) - \frac{1}{4}(A \ominus A)^2, \end{aligned} \tag{71}$$

and

$$\begin{aligned}\mathcal{M}(A) &= \frac{1}{2}(A \cdot I + I \cdot A) + \frac{1}{2}A \cdot A - \frac{1}{4}(A^2 \cdot I + I \cdot A^2) \\ &= A \cdot I + \frac{1}{2}A \cdot A - \frac{1}{2}I \cdot A^2. \end{aligned} \tag{72}$$

7. Examples of Application

We apply some of the results obtained in this paper to three examples. The first two examples deal with the stability of one-parameter families of matrices relative to certain domains which arise in control system design. For the second of these examples, we compute the largest interval of parameter variation containing 0 for which stability is preserved. The final example concerns the Hurwitz stability of a two-parameter family of matrices. Programs implementing the techniques in this paper and using the symbolic language MACSYMA,[8] were developed and applied to construct the examples [S2].

[8] MACSYMA is a registered trademark of Symbolics, Inc., Cambridge, MA.

7.1. Robust Stability with Adequate Step Response.

To ensure adequate maximum settling time and minimum damping, it is often required in the design of compensators for linear control systems that the eigenvalues (poles) of the closed loop system be confined within a domain $\Omega := \{s : \Re e(s) < -\sigma \ ; \ |\mathrm{Arg}(s)| > \theta\ \}$, for given $\sigma > 0$ and $\theta \in (\frac{\pi}{2}, \pi)$ (see Fig. 4) [V]. In this example we investigate the stability, with respect to Ω, of the one-parameter family of matrices $\mathcal{A}$ given by

$$A(r) = \begin{bmatrix} r-3 & 1 & 2r+1 \\ r & -1 & -1 \\ 1 & r+1 & -3 \end{bmatrix}, \quad r \in [-\frac{1}{2}, \frac{1}{2}],$$

in the case $\sigma = -1$ and $\theta = \frac{3\pi}{4}$. We first provide a guardian map for $\mathcal{S}(\Omega)$. Letting Ω_θ denote the set of Example 3.5, and using the notation of Proposition 3, we observe that $\Omega = \Omega_\theta \cap \overset{\circ}{\mathbb{C}}{}_{-}^{(-\sigma)}$. Since guardian maps for $\mathcal{S}(\Omega_\theta)$ and $\mathcal{S}(\overset{\circ}{\mathbb{C}}_{-})$ are available, it follows from Proposition 3 that $\mathcal{S}(\Omega)$ is guarded by the map ν specified by

$$\nu(A) = \ \det(e^{i\theta} A \ominus e^{-i\theta} A)\ \det\big((A + \sigma I)_{[2]}\big). \tag{73}$$

For a 3×3 matrix,

$$A_{[2]} = \begin{bmatrix} 2a_{11} & 2a_{12} & 2a_{13} & 0 & 0 & 0 \\ a_{21} & a_{11}+a_{22} & a_{23} & a_{12} & a_{13} & 0 \\ a_{31} & a_{32} & a_{11}+a_{33} & 0 & a_{12} & a_{13} \\ 0 & 2a_{21} & 0 & 2a_{22} & 2a_{23} & 0 \\ 0 & a_{31} & a_{21} & a_{32} & a_{22}+a_{33} & a_{23} \\ 0 & 0 & 2a_{31} & 0 & 2a_{32} & 2a_{33} \end{bmatrix}. \tag{74}$$

Let us for convenience write $\nu(A) = \nu_1(A)\nu_2(A)$ where the ν_i's are the respective factors appearing in (73). We obtain

$$\begin{aligned} \nu_1(A(r)) &= i\sqrt{2}(-32r^9 - 16r^8 - 344r^7 + 336r^6 + 6636r^5 + 13604r^4 + 15408r^3 \\ &\qquad - 48008r^2 + 38298r - 22536), && (75) \\ \nu_2(A(r)) &= -32r^6 - 192r^5 + 288r^3 + 64r^2 - 640r + 312. && (76) \end{aligned}$$

It is easily checked that $A(0) \in \mathcal{S}(\Omega)$, and, using Sturm sequences for example, that ν_1 and ν_2 have no zeros in $[-\frac{1}{2}, \frac{1}{2}]$. We may conclude by virtue of Proposition 1 that the family

$\mathcal{A}$ is stable relative to Ω. Parenthetically, the real zeros of of ν_1 and ν_2 are given by

$$\{-5.7588, 0.5919\} \text{ and } \{-2.9859, 0.95126, 3.737\},$$

respectively. Thus, an eigenvalue of $A(r)$ leaves Ω, for the "first time," at $r = 0.5919$.

7.2. Largest Interval for Generalized Stability

In this example, we investigate the stability of the one-parameter family of matrices

$$A(r) = \begin{bmatrix} r^2 - 1 & r + 1 \\ r^2 - 2r - 1 & -1 \end{bmatrix}$$

relative to the domain (see Fig. 5) $\Omega = \{s : \ \rho_1 < |s| < \rho_2 \ ; \quad |\mathrm{Arg}(s)| > \theta\}$, with

$$\theta = \frac{2\pi}{3}, \quad \rho_1 = \frac{\sqrt{2}}{2}, \quad \rho_2 = 2.$$

Such domains arise in flight controller design [A1, p. 394]. We seek to compute the largest open interval $(\underline{r}, \overline{r})$, containing 0, such that $A(r) \in \mathcal{S}(\Omega)$ for all $r \in (\underline{r}, \overline{r})$ (note that the eigenvalues of $A(0)$ are $-1 \pm i$, hence $A(0) \in \mathcal{S}(\Omega)$). To apply Proposition 3, we write

$$\Omega = \Omega_\theta \cap B(\rho_1) \cap \mathrm{int}(B^c(\rho_2))$$

where $B(\rho)$ denotes the open disk of radius ρ. Thus $\mathcal{S}(\Omega_2)$ is guarded by the map ν_2 given by

$$\begin{aligned} \nu_2(A) &= \det(e^{i\theta} A \ominus e^{-i\theta} A) \det(A^{[2]} - \rho_1^2 I^{[2]}) \det(A^{[2]} - \rho_2^2 I^{[2]}) \\ &=: \nu_1(A)\nu_2(A)\nu_3(A). \end{aligned}$$

Here, $A^{[2]}$ is given by

$$\begin{bmatrix} a_{11}^2 & 2a_{11}a_{12} & a_{12}^2 \\ a_{11}a_{21} & a_{11}a_{22} + a_{12}a_{21} & a_{12}a_{22} \\ a_{21}^2 & 2a_{21}a_{22} & a_{22}^2 \end{bmatrix}. \tag{77}$$

For the example at hand we obtain

$$\nu_1(A(r)) = -3r^7 - 3r^6 + 21r^5 + 24r^4 - 36r^3 - 51r^2 + 12,$$

$$\nu_2(A(r)) = \frac{1}{8}\left(8r^9 - 76r^7 - 52r^6 + 244r^5 + 318r^4 - 170r^3 - 492r^2 - 282r - 51\right),$$

$$\nu_3(A(r)) = -r^9 + 13r^7 + 10r^6 - 55r^5 - 52r^4 + 79r^3 + 58r^2 - 12r - 40.$$

The real zeros of the polynomials above are

$$r = -2, \quad r \approx -1.246, \quad r = -1, \quad r \approx 0.445, \quad r \approx 1.801, \quad r = 2;$$

$$r \approx -1.384, \quad r \approx -1.112, \quad r \approx -0.5578, \quad r \approx -0.4332, \quad r \approx 1.919, \quad r \approx 1.942, \quad r \approx 2.252;$$

$$r = -2, \quad ,r = 1, \quad r = 2, \quad r \approx 3.152;$$

respectively. Clearly, the matrix $A(r)$ is not stable relative to Ω for all values of $r \in I\!R$. It is also seen furthermore that the maximal interval of stability is $[-0.4332,\ 0.445]$.

7.3. Hurwitz Stability of a Two-Parameter Family

In this example, $\Omega = \overset{\circ}{\mathbb{C}}_-$ and

$$A(r_1, r_2) = \begin{bmatrix} -3 - r_2 + 3r_2^2 & -1 + r_2 + 4r_1r_2 \\ -1 + 2r_1 & -2 + 3r_1 + r_2 - r_2^2 \end{bmatrix},$$

$r_1,\ r_2 \in [0, 1]$. Note that $A(0,0)$ is Hurwitz stable. A guardian map for $\mathcal{S}(\Omega)$ is given by

$$\nu_\Omega(A(r_1, r_2)) = \det A_{[2]}(r_1, r_2),$$

where $A_{[2]}$ is given by

$$\begin{bmatrix} 2a_{11} & 2a_{21} & 0 \\ a_{21} & a_{11} + a_{22} & a_{12} \\ 0 & 2a_{12} & 2a_{22} \end{bmatrix}. \tag{78}$$

We obtain

$$\begin{aligned} \nu_\Omega(A(r_1, r_2)) &= -(100 - 200r_1 + 84r_1^2) + (20r_1 + 148r_1^2 - 96r_1^3)r_2 + (120 - 284r_1 + 100r_1^2)r_2^2 \\ &\quad - (80 - 40r_1 + 64r_1^2)r_2^3 + (28 + 36r_1)r_2^4 + 32r_2^5 - 24r_2^6 \\ &=: p_{r_1}(r_2). \end{aligned}$$

In this case, we find that

$$U_{\mathrm{cr}}^1 \approx \{0.5689,\ 0.5725,\ 0.8207\} \quad \text{and} \quad U_{\mathrm{cr}}^2 \approx \{0.5663,\ 0.7142,\ 1\} \neq \emptyset.$$

It follows from Theorem 3 that the family of matrices being considered is not Hurwitz stable. For $r_1 = 0.5663$, p_{r_1} has ($r_2 =$) 1.0 as zero, implying that $A(0.5663, 1.0)$ is Hurwitz unstable (its eigenvalues are $\{-1.3, 0.0\}$).

8. Concluding Remarks

This paper has introduced a new framework for the study of generalized stability of parametrized families of matrices and polynomials. Stability relative to a wide variety of domains in the complex plane can be addressed within this framework.

Proposition 1 and 2 give necessary and sufficient conditions for generalized stability of multiparameter problems. Where Proposition 1 applies and the family of interest is polynomic, the problem is reducible to a positivity test.

Although the presentation has emphasized the case of generalized stability sets $\mathcal{S}(\Omega)$, the results apply as well to other open sets of matrices or polynomials. A case in point is the set of strictly aperiodic matrices (see Example 3.6).

Finally, the results presented in this paper can be extended in a straightforward manner to the case of matrices with complex entries or polynomials with complex coefficients. In particular, guardian and semiguardian maps can be readily constructed for sets of complex matrices corresponding to domains of the complex plane with polynomial boundaries.

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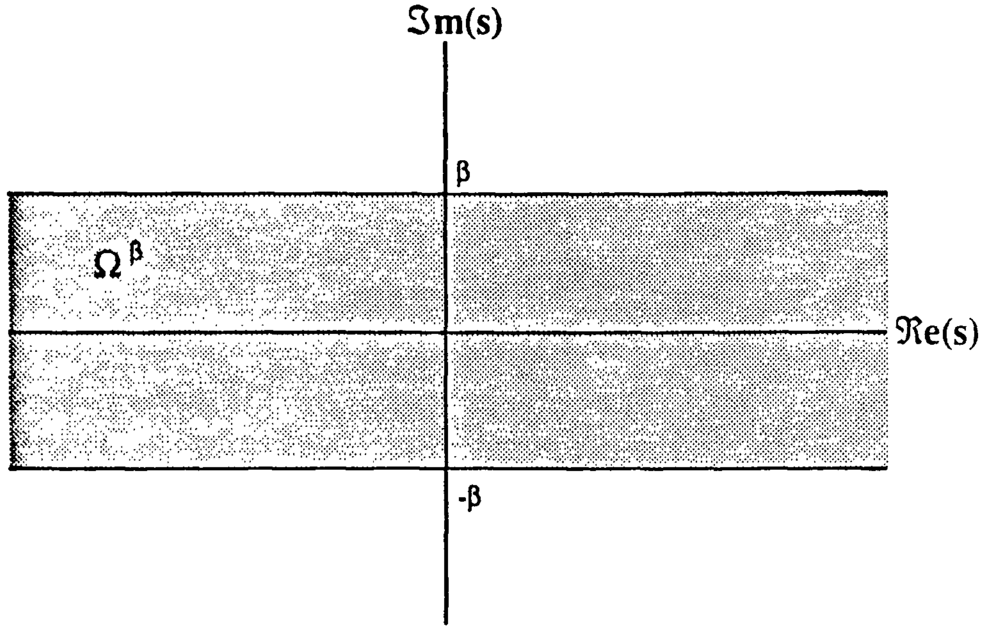

Fig. 1. Stability domain for Example 3.4

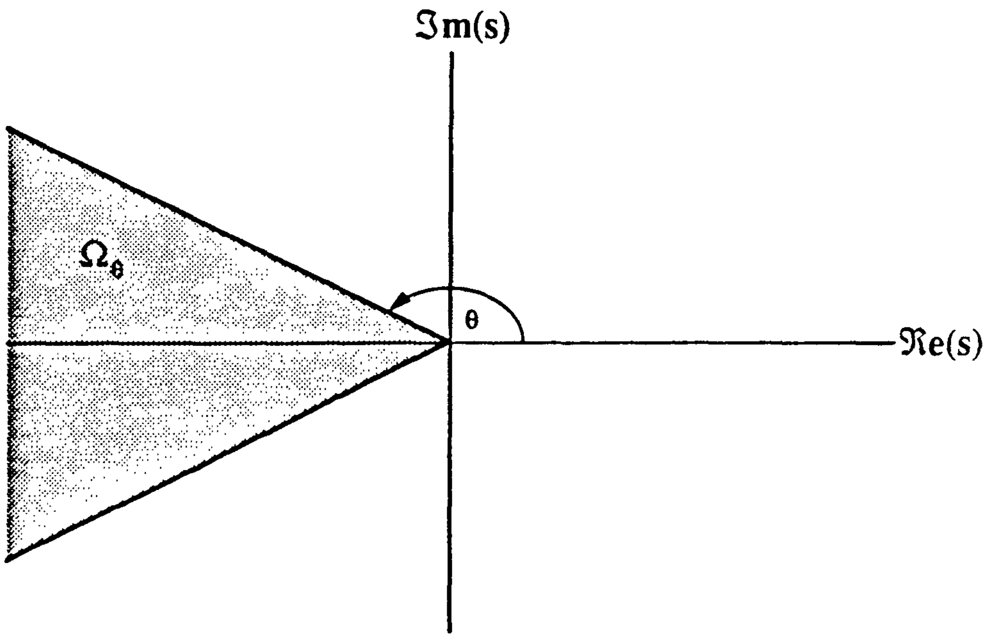

Fig. 2. Stability domain for Example 3.5

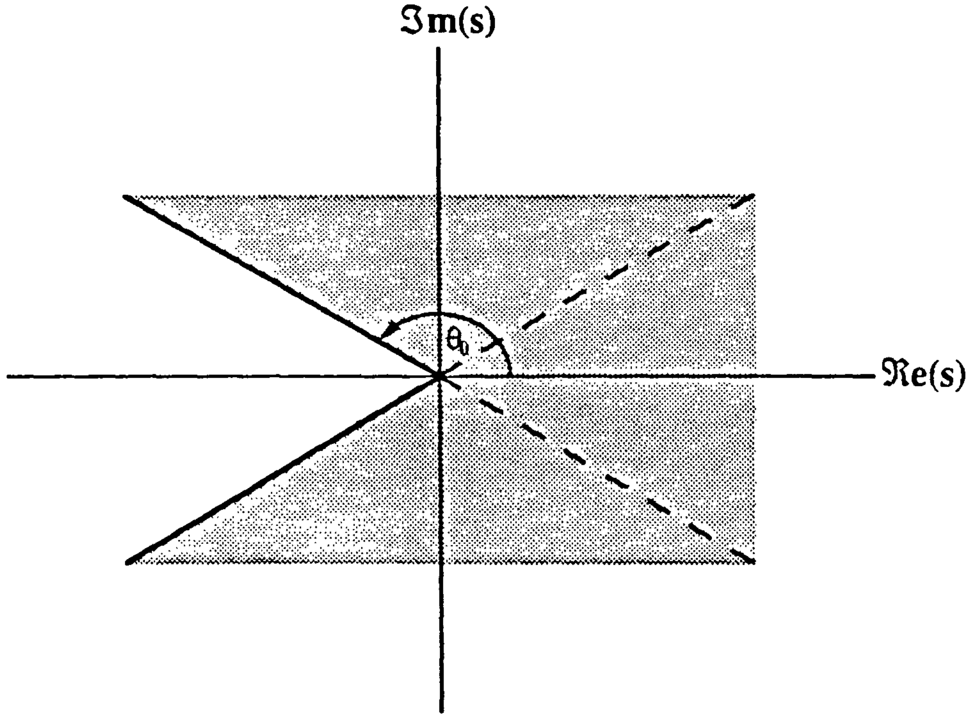

Fig. 3. Eigenvalue location of typical blind spots
for Example 3.8

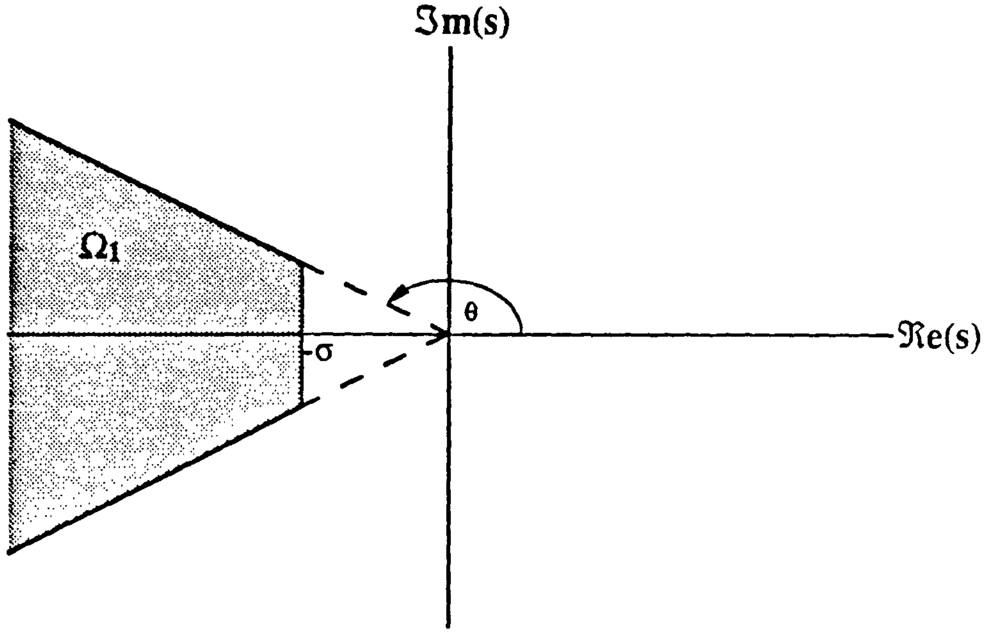

Fig. 4. Stability domain for Example 7.1

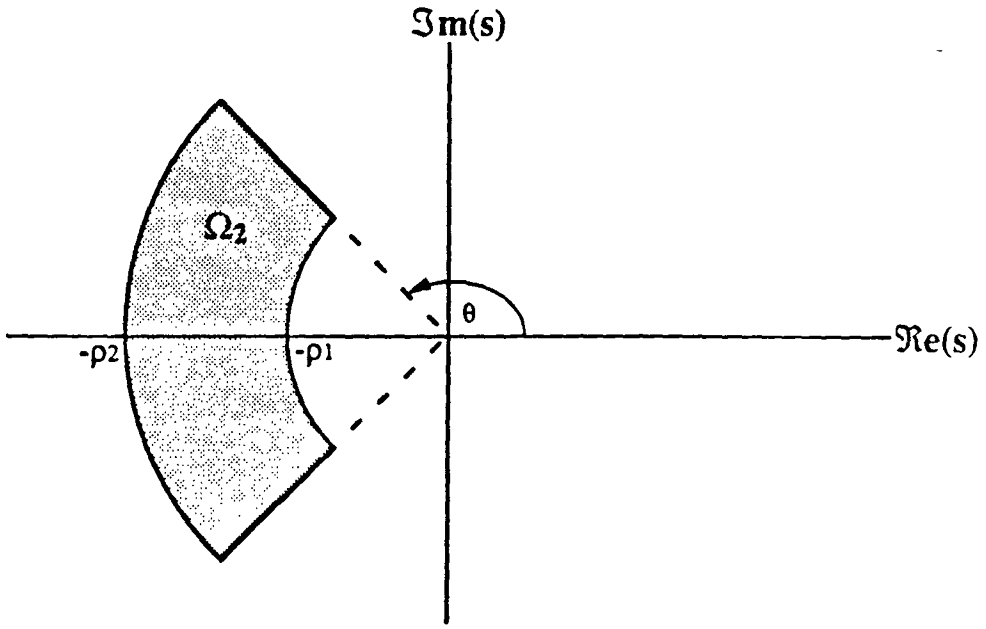

Fig. 5. Stability domain for Example 7.2