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The Strongly H_∞ Performance of Constrained Model Predictive Control

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Abstract

An off-line performance index for the Constrained Model Predictive Control (CMPC) is defined by the strongly H_∞ performance criterion in this paper. From the CMPC algorithm, each term of the closed-form of CMPC control law corresponding to an active constraint situation can be decomposed to have an uncertainty block, which is time varying over the control period. To analyze the strongly H_∞ performance and quantify the minimum upper bound of L_2-induced gain of CMPC system with this type of control law, a numerical method, the Linear Matrix Inequality (LMI) technique, was found useful. Several examples are given to show the results on quantification and analysis of the control system performance.

1 Introduction

The robust performance tuning of the unconstrained Model Predictive Control has been developed by Lee and Yu (1994). The off-line performance criterion of Constrained Model Predictive Control (CMPC) was not defined since CMPC has been applied in the industries. Because the control action of CMPC is nonlinear even though the process model is assumed to be linear around the operating point, the robust performance technique of linear time invariant (LTI) systems is helpless to analyze the CMPC performance.

The closed-form control law of CMPC is described as a time varying function. It can be formulated as a linear function containing uncertainty blocks which correspond to the active constraint situation at the optimum of the control system. For details, please see the technical report of " The Closed-Form Control Law of CMPC" [Chiou and Zafiriou, 1993]. To consider the performance of CMPC with this type of time varying control law, a strongly H_∞ performance criterion is introduced in this paper.

The strongly H_∞ performance was defined in Zhou's paper (1992) for uncertain systems in state space. The performance criterion is the H_∞ norm of control system transfer function between

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external input and the system desired output bounded by 1 (or the system L_2-induced gain between desired output and external input is less than 1). If the control system performance satisfies this criterion, then it also implies that the system is stable. The minimum upper bound of L_2-induced gain is also can be obtained by solving an Eigen Value optimization problem.

2 Preliminaries

Although Model Predictive Control (MPC) algorithms have been applied to systems with nonlinear dynamic models [Garcia, 1984; Eaton *et al*, 1989], it is usually assumed that the dynamics are linear, the nonlinearity of the problem arising from the hard constraints. The properties of the controller are independent of the type of model description used for the plant [see, e.g., Morari *et al.*, 1989] . Consider a discrete state space model with disturbance directly added to the output for a process given as:

$$\begin{aligned} x(k+1) &= \phi x(k) + \Theta u(k) \\ y(k) &= Cx(k) + D_d w(k) \end{aligned} \tag{1}$$

where $x(\cdot)$: the state variable; $u(k)$: the manipulated variable; $y(k)$: the output measurement; $w(k)$: the external input; k : the time index; ϕ, Θ, C, D_d are the coefficient matrices of the model. For more details about the prediction and on-line control algorithm of CMPC, please see the preliminaries of the technical report of "The Quadratic Stability of CMPC" [Chiou and Zafiriou, 1994].

3 Strongly H_∞ Performance: Numerical Method

Follow the technical report of "The Closed-Form Control Law of CMPC" and "The Quadratic Stability of CMPC" on formulating the CMPC control law and closed-loop system. The general CMPC closed-loop system can be described as:

$$\bar{x}(k+1) = (\Psi_1 + E_1 F(k) E_2)\bar{x}(k) + (\Psi_2 + E_1 F(k) E_3) w(k) \tag{2}$$

$$z(k) = \mathcal{C}\bar{x}(k) + D_z w(k) \tag{3}$$

$$y(k) = \mathcal{C}\bar{x}(k) + D_d w(k) \tag{4}$$

where $\mathcal{C} = [\ C \ \ 0\ ]$; $z(k)$: the output to be minimized (desired output); D_z : the coefficient matrix for $w(k)$ (in this paper, $D_d = D_z$); The matrices Ψ_1, E_1, E_2, Ψ_2, E_3 are functions of tuning parameters and coefficients of the state space model. The control block diagram of the CMPC system for strongly H_∞ performance analysis is shown in Figure 1.

The definition of strongly H_∞ performance was given for (2) (3) [Zhou, 1992] as:

Definition 1 *The time varying uncertain dynamical system described by equation (2) and (3) is said to satisfy* **strongly H_∞ performance criterion** *if there exists a constant symmetric $\bar{\mathcal{P}} > 0$ such that*

$$\begin{bmatrix} \Psi_1 + E_1 F(k) E_2 & \Psi_2 + E_1 F(k) E_3 \\ \mathcal{C} & D_z \end{bmatrix}^T \begin{bmatrix} \bar{\mathcal{P}} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \Psi_1 + E_1 F(k) E_2 & \Psi_2 + E_1 F(k) E_3 \\ \mathcal{C} & D_z \end{bmatrix} - \begin{bmatrix} \bar{\mathcal{P}} & 0 \\ 0 & I \end{bmatrix} < 0 \tag{5}$$

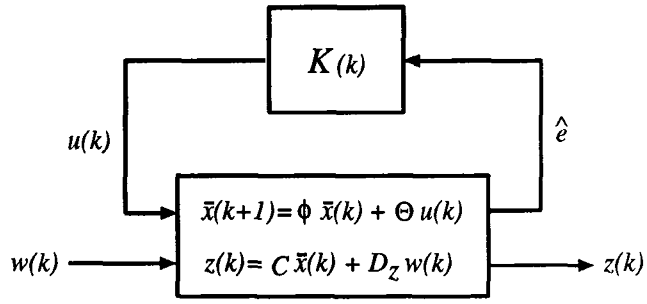

Figure 1: The control block diagram for strongly H_∞ performance analysis; where $\hat{e}$ is the predicted error (the difference between predicted output and reference input); $K(k)$ is the time varying feedback gain of CMPC.

Remark I.
The inequality (5) *also implies:*

$$I - D_z^T D_z - (\Psi_2 + E_1 F(k) E_3)^T \mathcal{P} (\Psi_2 + E_1 F(k) E_3) > 0$$

which means that we need $I - D_z^T D_z > 0$ *at least.*
The definition 1 for the uncertain state space closed-loop system is also implies the following theorem:

Theorem 1 *If a CMPC system can satisfy the strongly H_∞ performance criterion of definition* 1, *then it implies:*

$$\frac{\| z \|_2}{\| w \|_2} < 1$$

with $\bar{x}(0) = 0$. *where*

$$\| z \|_2 = \sum_{k=0}^{nf} z^T(k) z(k), \quad \| w \|_2 = \sum_{k=0}^{nf} w^T(k) w(k)$$

and nf is the final sampling point.

Proof is in Appendix A.

Based on the definition, the strongly H_∞ performance criterion of CMPC is given as following for numerical analysis by Linear Matrix Inequality (LMI) method [Boyd, Ghaoui, Feron, and Balakrishnan, 1993].

Theorem 2 *If there exists a $\mathcal{P}$ such that*

$$\begin{bmatrix} \Psi^T \mathcal{P} \Psi - \mathcal{P} + \mathcal{E}_2^T \mathcal{E}_2 & \Psi^T \mathcal{P} \mathcal{E}_1 \\ \mathcal{E}_1^T \mathcal{P} \Psi & \mathcal{E}_1^T \mathcal{P} \mathcal{E}_1 - I \end{bmatrix} \leq 0$$

where

$$\Psi = \begin{bmatrix} \Psi_1 & \Psi_2 \\ C & D_z \end{bmatrix}, \quad \mathcal{E}_1 = \begin{bmatrix} E_1 \\ 0 \end{bmatrix}, \quad \mathcal{E}_2 = \begin{bmatrix} E_2 & E_3 \end{bmatrix}$$

then the designed CMPC satisfies the strongly H_∞ performance criterion.

Proof is in Appendix A.

4 Quantification of the Minimum Upper Bound of L_2–induced gain

Let the minimum upper bound of L_2–induced gain of CMPC between z and w be γ, and

$$\frac{\| z \|}{\| w \|} < \gamma$$

The minimum upper bound of L_2–induced gain of CMPC between z and w can be quantified by the following optimization problem:

$$\min \gamma^2$$

subject to

$$\begin{bmatrix} \Psi_1 + E_1 F(k) E_2 & \Psi_2 + E_1 F(k) E_3 \\ C & D_z \end{bmatrix}^T \begin{bmatrix} \bar{\mathcal{P}} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \Psi_1 + E_1 F(k) E_2 & \Psi_2 + E_1 F(k) E_3 \\ C & D_z \end{bmatrix} - \begin{bmatrix} \bar{\mathcal{P}} & 0 \\ 0 & \gamma^2 I \end{bmatrix} < 0$$

and

$$\bar{\mathcal{P}} > 0, \quad F^T(k) F(k) \leq I$$

The above optimization problem can be reformulated as:

$$\min \gamma^2 \tag{6}$$

subject to

$$\begin{bmatrix} \Psi_1{}^T \bar{\mathcal{P}} \Psi_1 - \bar{\mathcal{P}} + C^T C + \bar{\lambda} E_2{}^T E_2 & \Psi_1{}^T \bar{\mathcal{P}} \Psi_2 + C^T D_z + \bar{\lambda} E_2{}^T E_3 & \Psi_1{}^T \bar{\mathcal{P}} E_1 \\ \Psi_2{}^T \bar{\mathcal{P}} \Psi_1 + D_z{}^T C + \bar{\lambda} E_3{}^T E_2 & \Psi_2{}^T \bar{\mathcal{P}} \Psi_2 + D_z{}^T D_z - \gamma^2 I + \bar{\lambda} E_3{}^T E_3 & \Psi_2{}^T \bar{\mathcal{P}} E_1 \\ E_1{}^T \bar{\mathcal{P}} \Psi_1 & E_1{}^T \bar{\mathcal{P}} \Psi_2 & E_1{}^T \bar{\mathcal{P}} E_1 - \bar{\lambda} I \end{bmatrix} \leq 0$$

and

$$\bar{\lambda} > 0$$

The above optimization problem is an Eigen Value problem and LMI-Lab [Gahinet and Nemirovskii, 1993] has an useful tool to solve this problem. Several examples are given to show the feasibility of quantifying the minimum upper bound of L_2–induced gain.

5 Illustrations

Example 1

A SISO multieffect evaporator process is given [Ricker *et al.*, 1989] :

$$\tilde{p}(s) = \frac{2.69(-6s+1)e^{-1.5s}}{100s^2 + 25s + 1}$$

(I) Set constraints on Δu, u over the control horizon (M), and select tuning parameters:

$$P = 7,\ M = 5,\ B = 0,\ D = 0.5,\ \Gamma = 1,\ T_s = 3$$

where T_s is the sampling time. Suppose $D_d = 0.1$, $D_z = 0.1$. By theorem 2 and LMI-Lab tool, we find there exists a $\mathcal{P} > 0$ such the CMPC of this process satisfies the H_∞ performance criterion (definition 1).

$$\mathcal{P} = \begin{bmatrix} 6.2833e+02 & -8.8468e+02 & 2.9612e+02 & -6.2705e+02 & 0.0000e+00 \\ -8.8468e+02 & 1.2469e+03 & -4.1793e+02 & 8.8329e+02 & 0.0000e+00 \\ 2.9612e+02 & -4.1793e+02 & 1.4043e+02 & -2.9572e+02 & 0.0000e+00 \\ -6.2705e+02 & 8.8329e+02 & -2.9572e+02 & 6.2825e+02 & 0.0000e+00 \\ 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 1.9472e-01 \end{bmatrix} > 0$$

Quantifying the minimum upper bound of the L_2-induced gain by using LMI-Lab tool to solve the optimization problem (6), we obtain:

$$\gamma = 0.3867, \ \bar{\lambda} = 4.0342,$$

$$\bar{\mathcal{P}} = \begin{bmatrix} 1.0050e+01 & -1.1119e+01 & 3.0379e+00 & -6.3116e+00 \\ -1.1119e+01 & 1.4636e+01 & -4.7552e+00 & 8.2477e+00 \\ 3.0379e+00 & -4.7552e+00 & 1.9089e+00 & -2.3944e+00 \\ -6.3116e+00 & 8.2477e+00 & -2.3944e+00 & 9.2758e+00 \end{bmatrix} > 0$$

(II) Set constraint on predicted output $\hat{y}(k+2)$, and select the tuning parameters as:

$$P = 7, \ M = 5, \ B = 0, \ D = 0, \ \Gamma = 1, \ T_s = 3$$

Suppose $D_d = 0.1$, $D_z = 0.1$. By theorem 2 and LMI-Lab tool, we find there exists a $\mathcal{P} > 0$ such the CMPC of this process satisfies the H_∞ performance criterion (definition 1).

$$\mathcal{P} = \begin{bmatrix} 4.6112e-01 & -1.9548e-02 & -1.0342e-01 & 0.0000e+00 \\ -1.9548e-02 & 3.2433e-01 & 2.3154e-03 & 0.0000e+00 \\ -1.0342e-01 & 2.3154e-03 & 2.1639e-01 & 0.0000e+00 \\ 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 5.2180e-01 \end{bmatrix} > 0$$

Quantifying the minimum upper bound of the L_2-induced gain by using LMI-Lab tool to solve the optimization problem (6), we obtain:

$$\gamma = 0.2514, \ \bar{\lambda} = 0.7278,$$

$$\bar{\mathcal{P}} = \begin{bmatrix} 1.7879e-01 & -1.5667e-02 & -2.5088e-02 \\ -1.5667e-02 & 1.2416e-01 & 2.5342e-02 \\ -2.5088e-02 & 2.5342e-02 & 6.0682e-02 \end{bmatrix} > 0$$

(III) Set constraints on $\Delta u(k)$, $u(k)$, $\hat{y}(k+7)$, and select tuning parameters:

$$P = 10, \ M = 5, \ B = 0, \ D = 1.0, \ \Gamma = 1, \ T_s = 3$$

Suppose $D_d = 0.05$, $D_z = 0.05$. By theorem 2 and LMI-Lab tool, we find there exists a $\mathcal{P} > 0$ such the CMPC of this process satisfies the H_∞ performance criterion (definition 1).

$$\mathcal{P} = \begin{bmatrix} 8.5776e+02 & -1.2067e+03 & 4.0291e+02 & -8.5579e+02 & 0.0000e+00 \\ -1.2067e+03 & 1.7003e+03 & -5.6905e+02 & 1.2046e+03 & 0.0000e+00 \\ 4.0291e+02 & -5.6905e+02 & 1.9127e+02 & -4.0232e+02 & 0.0000e+00 \\ -8.5579e+02 & 1.2046e+03 & -4.0232e+02 & 8.6005e+02 & 0.0000e+00 \\ 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 5.8630e-01 \end{bmatrix} > 0$$

Quantifying the minimum upper bound of the L_2-induced gain by using LMI-Lab tool to solve the optimization problem (6), we obtain:

$$\gamma = 0.1458, \quad \bar{\lambda} = 1.3031,$$

$$\bar{\mathcal{P}} = \begin{bmatrix} 2.4643e+00 & -1.6440e+00 & 5.8365e-02 & 2.8754e-02 \\ -1.6440e+00 & 1.7296e+00 & -3.9352e-01 & -2.3719e-01 \\ 5.8365e-02 & -3.9352e-01 & 3.8287e-01 & 1.5226e-01 \\ 2.8754e-02 & -2.3719e-01 & 1.5226e-01 & 2.4420e+00 \end{bmatrix} > 0$$

(IV) Set constraint on predicted output $\hat{y}(k+1)$ with softening, and select the tuning parameters as:

$$P = 30, \ M = 1, \ B = 0, \ D = 0, \ \Gamma = 1, \ W = 11.396, \ T_s = 3$$

Suppose $D_d = 0.1$, $D_z = 0.1$. By theorem 2 and LMI-Lab tool, we find there exists a $\mathcal{P} > 0$ such the CMPC of this process satisfies the H_∞ performance criterion (definition 1).

$$\mathcal{P} = \begin{bmatrix} 2.0322e+02 & -1.0745e+02 & -5.1728e+00 & 0.0000e+00 \\ -1.0745e+02 & 9.6219e+01 & -1.1518e+01 & 0.0000e+00 \\ -5.1728e+00 & -1.1518e+01 & 2.0433e+01 & 0.0000e+00 \\ 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 4.2554e+01 \end{bmatrix} > 0$$

Quantifying the minimum upper bound of the L_2-induced gain by using LMI-Lab tool to solve the optimization problem (6), we obtain:

$$\gamma = 0.1633, \quad \bar{\lambda} = 0.0025,$$

$$\bar{\mathcal{P}} = \begin{bmatrix} 2.6088e+00 & -1.8690e+00 & -6.6395e-02 \\ -1.8690e+00 & 1.8782e+00 & -1.4632e-01 \\ -6.6395e-02 & -1.4632e-01 & 2.5346e-01 \end{bmatrix} > 0$$

We can conclude that the CMPC of this process satisfies the strongly H_∞ performance for specified D_d, D_z to control this process under the constraints set on u, Δu over the control horizon M or $u(k)$, $\Delta u(k)$, $\hat{y}(k+7)$ or $\hat{y}(k+2)$ or $\hat{y}(k+1)$ with softening.

Example 2

A SISO process is given [Prett and Garcia, 1988] :

$$\tilde{p}(s) = \frac{4.05e^{-27s}}{50s+1}$$

Set constraint on $\hat{y}(k+7)$, and select tuning parameters:

$$P = 60, \ M = 1, \ B = 0, \ D = 0, \ \Gamma = 1, \ W = 290, \ T_s = 4$$

Suppose $D_d = 0.01$, $D_z = 0.01$. By theorem 2 and LMI-Lab tool, we find there exists a $\mathcal{P} > 0$ such the CMPC of this process satisfies the H_∞ performance criterion (definition 1).

$$
\mathcal{P} = \begin{bmatrix}
4.5067e+03 & 3.2735e+01 & -1.1813e-03 & -1.2707e-03 & 6.2316e-06 & -2.1261e-04 & 4.2326e-02 & 2.2284e-01 & 0.0000e+00 \\
3.2735e+01 & 4.5856e+02 & 3.0884e+01 & 1.4028e-02 & -1.6246e-03 & -3.0949e-06 & 3.4791e-04 & -3.5835e-02 & 0.0000e+00 \\
-1.1813e-03 & 3.0884e+01 & 4.3567e+02 & 2.9104e+01 & 3.1261e-02 & -2.2192e-03 & 2.1854e-05 & -2.7483e-04 & 0.0000e+00 \\
-1.2707e-03 & 1.4028e-02 & 2.9104e+01 & 4.0411e+02 & 2.7423e+01 & 5.0341e-02 & -2.8940e-03 & 2.0468e-05 & 0.0000e+00 \\
6.2316e-06 & -1.6246e-03 & 3.1261e-02 & 2.7423e+01 & 3.6380e+02 & 2.5869e+01 & 6.8078e-02 & -3.5610e-03 & 0.0000e+00 \\
-2.1261e-04 & -3.0949e-06 & -2.2192e-03 & 5.0341e-02 & 2.5869e+01 & 3.1532e+02 & 2.4671e+01 & 7.5841e-02 & 0.0000e+00 \\
4.2326e-02 & 3.4791e-04 & 2.1854e-05 & -2.8940e-03 & 6.8078e-02 & 2.4671e+01 & 2.5433e+02 & 2.3854e+01 & 0.0000e+00 \\
2.2284e-01 & -3.5835e-02 & -2.7483e-04 & 2.0468e-05 & -3.5610e-03 & 7.5841e-02 & 2.3854e+01 & 1.7805e+02 & 0.0000e+00 \\
0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 1.3280e+03
\end{bmatrix} > 0
$$

Quantifying the minimum upper bound of the L_2-induced gain by using LMI-Lab tool to solve the optimization problem (6), we obtain:

$$
\gamma = 0.1591, \;\; \bar{\lambda} = 1.6818e-04,
$$

$$
\bar{\mathcal{P}} = \begin{bmatrix}
7.7819e-01 & 2.5576e-02 & 3.7063e-05 & -9.4018e-06 & 6.1118e-06 & -2.9477e-05 & 1.7969e-04 & 1.4027e-03 \\
2.5576e-02 & 8.0463e-02 & 2.4554e-02 & 1.3644e-05 & -6.4086e-06 & 1.5841e-06 & -8.2026e-06 & -1.5626e-04 \\
3.7063e-05 & 2.4554e-02 & 7.7704e-02 & 2.3531e-02 & -5.5106e-06 & -3.0478e-06 & -3.7346e-06 & 2.1245e-05 \\
-9.4018e-06 & 1.3644e-05 & 2.3531e-02 & 7.4946e-02 & 2.2509e-02 & -2.3923e-05 & 1.2805e-06 & -6.3453e-06 \\
6.1118e-06 & -6.4086e-06 & -5.5106e-06 & 2.2509e-02 & 7.2183e-02 & 2.1484e-02 & -3.9650e-05 & 6.0236e-06 \\
-2.9477e-05 & 1.5841e-06 & -3.0478e-06 & -2.3923e-05 & 2.1484e-02 & 6.9414e-02 & 2.0451e-02 & -5.7032e-05 \\
1.7969e-04 & -8.2026e-06 & -3.7346e-06 & 1.2805e-06 & -3.9650e-05 & 2.0451e-02 & 6.6621e-02 & 1.9399e-02 \\
1.4027e-03 & -1.5626e-04 & 2.1245e-05 & -6.3453e-06 & 6.0236e-06 & -5.7032e-05 & 1.9399e-02 & 5.7345e-02
\end{bmatrix} > 0
$$

We can conclude that the CMPC of this process satisfies the strongly H_∞ performance for specified D_d, D_z to control this process under the constraints set on $\hat{y}(k+7)$ with softening.

Example 3

A 2×2 process model of a subsystem of the Shell Control Problem is given [Prett and Garcia, 1988]:

$$
\tilde{p}(s) = \begin{bmatrix} \frac{4.05e^{-27s}}{50s+1} & \frac{1.77e^{-28s}}{60s+1} \\ \frac{5.39e^{-18s}}{50s+1} & \frac{5.72e^{-14s}}{60s+1} \end{bmatrix}
$$

(I) Set constraints on Δu, u over the control horizon (M), and select tuning parameters:

$$P = 6,\ M = 5,\ B = 0,\ D = 1.5I,\ \Gamma = I,\ T_s = 6$$

Suppose $D_d = 0.01I$, $D_z = 0.01I$. By theorem 2 and LMI-Lab tool, we find there exists a $\mathcal{P} > 0$ such the CMPC of this process satisfies the H_∞ performance criterion (definition 1).

$$\mathcal{P} = \left[\begin{matrix}
4.7996e+04 & -2.4854e+03 & 4.1704e+04 & 9.8195e+03 & -8.1497e+01 & 7.4133e+03 & -3.3464e+01 \\
-2.4854e+03 & 1.3303e+02 & -2.2874e+03 & -4.2507e+02 & 3.2890e+00 & -3.9257e+02 & -1.4113e-01 \\
4.1704e+04 & -2.2874e+03 & 4.0082e+04 & 7.1849e+03 & -2.0448e+02 & 6.8824e+03 & 4.5449e+01 \\
9.8195e+03 & -4.2507e+02 & 7.1849e+03 & 9.6200e+04 & -1.0775e+00 & 1.2980e+04 & -5.4015e+00 \\
-8.1497e+01 & 3.2890e+00 & -2.0448e+02 & -1.0775e+00 & 5.2668e+02 & -1.4460e+02 & -7.8238e+01 \\
7.4133e+03 & -3.9257e+02 & 6.8824e+03 & 1.2980e+04 & -1.4460e+02 & 2.6605e+03 & 2.1796e+01 \\
-3.3464e+01 & -1.4113e-01 & 4.5449e+01 & -5.4015e+00 & -7.8238e+01 & 2.1796e+01 & 2.2627e+01 \\
-1.5537e+04 & 8.2260e+02 & -1.4315e+04 & -2.7293e+04 & -3.4267e+01 & -5.4965e+03 & 2.1939e+00 \\
-2.8480e+00 & 1.1069e-02 & -2.9933e+00 & -7.8610e-02 & 2.0714e+01 & -4.5420e+00 & -7.2238e+00 \\
3.2522e+01 & 9.0190e-01 & -6.7324e+01 & 6.9360e+00 & 6.0804e+01 & -2.2533e+01 & -9.3302e+00 \\
-9.8023e+03 & 4.2436e+02 & -7.1765e+03 & -9.6189e+04 & 3.6102e+00 & -1.2977e+04 & 5.4034e+00 \\
-1.7180e+04 & 9.0794e+02 & -1.5758e+04 & -3.0159e+04 & -8.9186e+01 & -6.0518e+03 & 2.7041e+00 \\
0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 \\
0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00
\end{matrix}\right.$$

$$\left.\begin{matrix}
-1.5537e+04 & -2.8480e+00 & 3.2522e+01 & -9.8023e+03 & -1.7180e+04 & 0.0000e+00 & 0.0000e+00 \\
8.2260e+02 & 1.1069e-02 & 9.0190e-01 & 4.2436e+02 & 9.0794e+02 & 0.0000e+00 & 0.0000e+00 \\
-1.4315e+04 & -2.9933e+00 & -6.7324e+01 & -7.1765e+03 & -1.5758e+04 & 0.0000e+00 & 0.0000e+00 \\
-2.7293e+04 & -7.8610e-02 & 6.9360e+00 & -9.6189e+04 & -3.0159e+04 & 0.0000e+00 & 0.0000e+00 \\
-3.4267e+01 & 2.0714e+01 & 6.0804e+01 & 3.6102e+00 & -8.9186e+01 & 0.0000e+00 & 0.0000e+00 \\
-5.4965e+03 & -4.5420e+00 & -2.2533e+01 & -1.2977e+04 & -6.0518e+03 & 0.0000e+00 & 0.0000e+00 \\
2.1939e+00 & -7.2238e+00 & -9.3302e+00 & 5.4034e+00 & 2.7041e+00 & 0.0000e+00 & 0.0000e+00 \\
1.1577e+04 & -2.7769e+00 & 3.3660e+00 & 2.7289e+04 & 1.2780e+04 & 0.0000e+00 & 0.0000e+00 \\
-2.7769e+00 & 3.0512e+00 & 8.6406e-01 & 7.8654e-02 & -3.0631e+00 & 0.0000e+00 & 0.0000e+00 \\
3.3660e+00 & 8.6406e-01 & 1.7028e+01 & -6.8629e+00 & 2.7286e+00 & 0.0000e+00 & 0.0000e+00 \\
2.7289e+04 & 7.8654e-02 & -6.8629e+00 & 9.6192e+04 & 3.0159e+04 & 0.0000e+00 & 0.0000e+00 \\
1.2780e+04 & -3.0631e+00 & 2.7286e+00 & 3.0159e+04 & 1.4140e+04 & 0.0000e+00 & 0.0000e+00 \\
0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 3.1203e-02 & 0.0000e+00 \\
0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 3.1203e-02
\end{matrix}\right] > 0$$

Quantifying the minimum upper bound of the L_2-induced gain by using LMI-Lab tool to solve the optimization problem (6), we obtain:

$$\gamma = 0.5,\quad \bar{\lambda} = 35.2842,$$

$$\bar{\mathcal{P}} = \left[\begin{matrix}
1.9878e+02 & -8.9262e+00 & 1.4357e+02 & 1.1910e+01 & -1.6510e+01 & 3.2109e+01 \\
-8.9262e+00 & 9.9520e-01 & -2.4172e+01 & -1.4336e-01 & 1.6051e+00 & -4.4430e+00 \\
1.4357e+02 & -2.4172e+01 & 7.4176e+02 & -7.7082e+00 & -2.7570e+02 & 1.8752e+02 \\
1.1910e+01 & -1.4336e-01 & -7.7082e+00 & 3.1711e+02 & 1.4424e+01 & 2.9026e+01 \\
-1.6510e+01 & 1.6051e+00 & -2.7570e+02 & 1.4424e+01 & 8.2771e+02 & -2.1191e+02 \\
3.2109e+01 & -4.4430e+00 & 1.8752e+02 & 2.9026e+01 & -2.1191e+02 & 8.4607e+01 \\
-4.9944e+00 & -1.5432e-01 & -3.3337e+00 & 3.6953e+00 & -1.1071e+02 & 1.3319e+01 \\
-3.7614e+01 & 5.4804e+00 & -1.2950e+02 & -6.1519e+01 & -9.0052e+01 & -1.0305e+01 \\
-6.0343e-01 & 2.7835e-01 & -3.4017e-02 & -1.5198e-01 & 8.9095e+00 & -2.2396e-01 \\
4.3244e+00 & 4.1675e-02 & -5.5255e+01 & 2.3389e+00 & 1.6565e+02 & -4.2665e+01 \\
1.3749e+01 & -2.8570e+00 & 4.8968e+01 & -8.3951e+01 & 2.7706e+01 & -1.0559e+01 \\
-4.2903e+00 & 1.3273e+00 & 3.3887e+00 & 4.9291e+01 & -4.9619e+01 & 2.5500e+01
\end{matrix}\right.$$

$$
\left.\begin{matrix}
-4.9944e+00 & -3.7614e+01 & -6.0343e-01 & 4.3244e+00 & 1.3749e+01 & -4.2903e+00 \\
-1.5432e-01 & 5.4804e+00 & 2.7835e-01 & 4.1675e-02 & -2.8570e+00 & 1.3273e+00 \\
-3.3337e+00 & -1.2950e+02 & -3.4017e-02 & -5.5255e+01 & 4.8968e+01 & 3.3887e+00 \\
3.6953e+00 & -6.1519e+01 & -1.5198e-01 & 2.3389e+00 & -8.3951e+01 & 4.9291e+01 \\
-1.1071e+02 & -9.0052e+01 & 8.9095e+00 & 1.6565e+02 & 2.7706e+01 & -4.9619e+01 \\
1.3319e+01 & -1.0305e+01 & -2.2396e-01 & -4.2665e+01 & -1.0559e+01 & 2.5500e+01 \\
9.9336e+01 & 2.7095e+01 & -3.7965e+01 & -8.9081e+00 & 5.8990e+00 & -1.8623e+01 \\
2.7095e+01 & 1.0436e+02 & -4.4906e+00 & -5.2631e+01 & 6.8097e+00 & -4.1501e+00 \\
-3.7965e+01 & -4.4906e+00 & 2.3475e+01 & -1.0900e+01 & 1.9721e-02 & 2.5736e+00 \\
-8.9081e+00 & -5.2631e+01 & -1.0900e+01 & 1.0810e+02 & 8.2570e+00 & -3.4248e-01 \\
5.8990e+00 & 6.8097e+00 & 1.9721e-02 & 8.2570e+00 & 2.1622e+02 & -5.2946e+01 \\
-1.8623e+01 & -4.1501e+00 & 2.5736e+00 & -3.4248e-01 & -5.2946e+01 & 1.9511e+02
\end{matrix}\right] > 0
$$

(II) Set constraints on predicted output $\hat{y}_1(k+6), \hat{y}_2(k+4)$, and select the tuning parameters as:

$$
P = 7,\ M = 2,\ B = 0,\ D = 0,\ \Gamma = I,\ T_s = 6
$$

Suppose $D_d = 0.05I$, $D_z = 0.05I$. By theorem 2 and LMI-Lab tool, we find there exists a $\mathcal{P} > 0$ such the CMPC of this process satisfies the H_∞ performance criterion (definition 1).

$$
\mathcal{P} = \left[\begin{matrix}
1.0019e+03 & -2.2305e+01 & -2.0255e+01 & 9.4871e-02 & 3.7338e+01 & -1.0452e+01 \\
-2.2305e+01 & 5.0865e-01 & 2.8318e-02 & 3.1213e-02 & -5.7876e-01 & 1.1841e-01 \\
-2.0255e+01 & 2.8318e-02 & 1.6946e+01 & -5.5930e-01 & -1.2768e+01 & 5.3731e+00 \\
9.4871e-02 & 3.1213e-02 & -5.5930e-01 & 2.4500e+00 & 1.0831e-02 & -2.5580e-01 \\
3.7338e+01 & -5.7876e-01 & -1.2768e+01 & 1.0831e-02 & 1.7061e+01 & -5.4842e+00 \\
-1.0452e+01 & 1.1841e-01 & 5.3731e+00 & -2.5580e-01 & -5.4842e+00 & 2.4223e+00 \\
-1.9720e+01 & 4.0805e-01 & 1.8412e+00 & -1.1011e-03 & -3.0785e+00 & 8.8465e-01 \\
-2.1161e+00 & 1.8717e-01 & -5.0799e+00 & 5.2963e-01 & 7.5081e-01 & -2.0461e+00 \\
-6.2569e+00 & 1.4746e-01 & -2.8048e-01 & 6.8899e-04 & 4.6931e-01 & -1.3470e-01 \\
-7.2538e+00 & 1.1596e-01 & 2.3288e+00 & -2.8573e-03 & -1.7634e+00 & 8.0173e-01 \\
0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 \\
0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00
\end{matrix}\right.
$$

$$
\left.\begin{matrix}
-1.9720e+01 & -2.1161e+00 & -6.2569e+00 & -7.2538e+00 & 0.0000e+00 & 0.0000e+00 \\
4.0805e-01 & 1.8717e-01 & 1.4746e-01 & 1.1596e-01 & 0.0000e+00 & 0.0000e+00 \\
1.8412e+00 & -5.0799e+00 & -2.8048e-01 & 2.3288e+00 & 0.0000e+00 & 0.0000e+00 \\
-1.1011e-03 & 5.2963e-01 & 6.8899e-04 & -2.8573e-03 & 0.0000e+00 & 0.0000e+00 \\
-3.0785e+00 & 7.5081e-01 & 4.6931e-01 & -1.7634e+00 & 0.0000e+00 & 0.0000e+00 \\
8.8465e-01 & -2.0461e+00 & -1.3470e-01 & 8.0173e-01 & 0.0000e+00 & 0.0000e+00 \\
1.4091e+00 & -4.2110e-02 & -1.9241e-01 & 1.5296e-02 & 0.0000e+00 & 0.0000e+00 \\
-4.2110e-02 & 1.0845e+01 & -9.2938e-03 & -1.3207e-02 & 0.0000e+00 & 0.0000e+00 \\
-1.9241e-01 & -9.2938e-03 & 4.7080e-01 & -3.1790e-05 & 0.0000e+00 & 0.0000e+00 \\
1.5296e-02 & -1.3207e-02 & -3.1790e-05 & 1.1765e+01 & 0.0000e+00 & 0.0000e+00 \\
0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 4.1458e+00 & 0.0000e+00 \\
0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 4.1458e+00
\end{matrix}\right] > 0
$$

Quantifying the minimum upper bound of the L_2-induced gain by using LMI-Lab tool to solve the optimization problem (6), we obtain:

$$
\gamma = 0.1270,\ \bar{\lambda} = 8.9453,
$$

$$
\bar{\mathcal{P}} = \left[\begin{array}{cccccccccc}
6.5691e+01 & -1.4965e+00 & -2.4766e-01 & -9.3984e-02 & 2.3611e+00 & -3.1030e-01 & -1.5539e+00 & -7.7927e-01 & -2.2741e-01 & -7.7550e-01 \\
-1.4965e+00 & 3.9973e-02 & -1.8874e-01 & 3.9030e-02 & 6.4789e-02 & -5.0164e-02 & 1.7017e-02 & 4.7487e-02 & 1.1884e-02 & 3.5865e-02 \\
-2.4766e-01 & -1.8874e-01 & 7.4326e+00 & -6.9435e-01 & -5.6435e+00 & 2.4465e+00 & 8.3638e-01 & -8.1823e-01 & -2.7703e-01 & -8.9760e-01 \\
-9.3984e-02 & 3.9030e-02 & -6.9435e-01 & 1.6378e+00 & 1.2142e-01 & -4.2331e-02 & -5.3319e-02 & 6.9380e-03 & 7.7087e-02 & 3.6602e-03 \\
2.3611e+00 & 6.4789e-02 & -5.6435e+00 & 1.2142e-01 & 7.9352e+00 & -2.4687e+00 & -1.3871e+00 & 5.4244e-02 & 4.1339e-01 & 1.5469e+00 \\
-3.1030e-01 & -5.0164e-02 & 2.4465e+00 & -4.2331e-02 & -2.4687e+00 & 9.4623e-01 & 3.9595e-01 & -2.4010e-01 & -1.2236e-01 & -4.3544e-01 \\
-1.5539e+00 & 1.7017e-02 & 8.3638e-01 & -5.3319e-02 & -1.3871e+00 & 3.9595e-01 & 6.3348e-01 & -6.3303e-03 & -1.9324e-01 & 3.8540e-02 \\
-7.7927e-01 & 4.7487e-02 & -8.1823e-01 & 6.9380e-03 & 5.4244e-02 & -2.4010e-01 & -6.3303e-03 & 8.5136e-01 & 4.4299e-03 & -5.5785e-03 \\
-2.2741e-01 & 1.1884e-02 & -2.7703e-01 & 7.7087e-02 & 4.1339e-01 & -1.2236e-01 & -1.9324e-01 & 4.4299e-03 & 2.8080e-01 & -1.6988e-02 \\
-7.7550e-01 & 3.5865e-02 & -8.9760e-01 & 3.6602e-03 & 1.5469e+00 & -4.3544e-01 & 3.8540e-02 & -5.5785e-03 & -1.6988e-02 & 1.5300e+00
\end{array}\right] > 0
$$

(III) Set constraints on $u(k)$, $\Delta u(k)$ and predicted output $\hat{y}_1(k+8), \hat{y}_2(k+7)$, and select the tuning parameters as:

$$P = 10,\ M = 2,\ B = 0,\ D = 0,\ \Gamma = I,\ T_s = 6$$

Suppose $D_d = 0.01I$, $D_z = 0.01I$. By theorem 2 and LMI-Lab tool, we find there exists a $\mathcal{P} > 0$ such the CMPC of this process satisfies the H_∞ performance criterion (definition 1).

$$
\mathcal{P} = \left[\begin{array}{cccccccccccc}
3.2813e+02 & -7.5736e+00 & 5.4797e+00 & 1.6292e+00 & 5.1625e+00 & 3.3665e-01 & -8.7435e+00 & 3.5144e-01 & 6.1233e-01 & 1.4220e+00 & 0.0000e+00 & 0.0000e+00 \\
-7.5736e+00 & 1.9816e-01 & -1.0523e+00 & -6.5316e-02 & 5.9958e-01 & -3.2093e-01 & 1.0242e-01 & 1.1789e-01 & 1.0628e-02 & 7.9282e-02 & 0.0000e+00 & 0.0000e+00 \\
5.4797e+00 & -1.0523e+00 & 3.8256e+01 & 1.1602e+00 & -3.2891e+01 & 1.3519e+01 & 4.6095e+00 & -4.5420e+00 & -1.1798e+00 & -5.3432e+00 & 0.0000e+00 & 0.0000e+00 \\
1.6292e+00 & -6.5316e-02 & 1.1602e+00 & 3.6027e+00 & -1.1981e-02 & 4.2334e-01 & 8.6759e-04 & -8.7307e-01 & -3.1075e-04 & 2.7321e-03 & 0.0000e+00 & 0.0000e+00 \\
5.1625e+00 & 5.9958e-01 & -3.2891e+01 & -1.1981e-02 & 4.0155e+01 & -1.3556e+01 & -7.6241e+00 & 1.6730e+00 & 1.9760e+00 & 6.6000e+00 & 0.0000e+00 & 0.0000e+00 \\
3.3665e-01 & -3.2093e-01 & 1.3519e+01 & 4.2334e-01 & -1.3556e+01 & 5.1258e+00 & 2.2047e+00 & -1.2903e+00 & -5.6799e-01 & -2.2140e+00 & 0.0000e+00 & 0.0000e+00 \\
-8.7435e+00 & 1.0242e-01 & 4.6095e+00 & 8.6759e-04 & -7.6241e+00 & 2.2047e+00 & 2.4526e+00 & 4.6000e-04 & -7.9820e-01 & -9.1386e-01 & 0.0000e+00 & 0.0000e+00 \\
3.5144e-01 & 1.1789e-01 & -4.5420e+00 & -8.7307e-01 & 1.6730e+00 & -1.2903e+00 & 4.6000e-04 & 1.3985e+00 & 3.0617e-05 & 4.6211e-02 & 0.0000e+00 & 0.0000e+00 \\
6.1233e-01 & 1.0628e-02 & -1.1798e+00 & -3.1075e-04 & 1.9760e+00 & -5.6799e-01 & -7.9820e-01 & 3.0617e-05 & 4.0740e-01 & 9.3841e-02 & 0.0000e+00 & 0.0000e+00 \\
1.4220e+00 & 7.9282e-02 & -5.3432e+00 & 2.7321e-03 & 6.6000e+00 & -2.2140e+00 & -9.1386e-01 & 4.6211e-02 & 9.3841e-02 & 1.8941e+00 & 0.0000e+00 & 0.0000e+00 \\
0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 2.2103e-01 & 0.0000e+00 \\
0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 2.2103e-01
\end{array}\right] > 0
$$

Quantifying the minimum upper bound of the L_2-induced gain by using LMI-Lab tool to solve the optimization problem (6), we obtain:

$$\gamma = 0.0982, \;\; \bar{\lambda} = 5.4204,$$

$$\bar{\mathcal{P}} = \begin{bmatrix}
3.3427e+00 & -1.5269e-01 & 2.5306e+00 & 3.7439e-02 & -5.2026e-01 & 5.4478e-01 & 2.5787e-02 & -5.1530e-01 & -3.5678e-02 & 1.0707e-02 \\
-1.5269e-01 & 1.9201e-02 & -4.5286e-01 & 1.2629e-01 & 6.4096e-02 & -8.2410e-02 & 9.6419e-04 & 6.0984e-02 & 1.3759e-03 & -1.0136e-03 \\
2.5306e+00 & -4.5286e-01 & 1.3381e+01 & -2.9770e+00 & -5.4604e+00 & 3.2177e+00 & -3.7088e-02 & -1.3937e+00 & 1.5604e-02 & -1.1139e+00 \\
3.7439e-02 & 1.2629e-01 & -2.9770e+00 & 1.5868e+01 & 1.2082e+00 & 3.1506e-02 & 1.1766e-01 & -1.1565e+00 & 2.3127e-02 & 2.5487e-01 \\
-5.2026e-01 & 6.4096e-02 & -5.4604e+00 & 1.2082e+00 & 1.4646e+01 & -3.4703e+00 & -2.2250e+00 & -9.3982e-01 & 1.9120e-01 & 2.7271e+00 \\
5.4478e-01 & -8.2410e-02 & 3.2177e+00 & 3.1506e-02 & -3.4703e+00 & 1.3087e+00 & 3.1487e-01 & -2.3217e-01 & -2.4079e-02 & -6.9609e-01 \\
2.5787e-02 & 9.6419e-04 & -3.7088e-02 & 1.1766e-01 & -2.2250e+00 & 3.1487e-01 & 1.8014e+00 & 2.7807e-01 & -7.5289e-01 & 4.1844e-01 \\
-5.1530e-01 & 6.0984e-02 & -1.3937e+00 & -1.1565e+00 & -9.3982e-01 & -2.3217e-01 & 2.7807e-01 & 1.4558e+00 & -1.3565e-02 & -4.9641e-01 \\
-3.5678e-02 & 1.3759e-03 & 1.5604e-02 & 2.3127e-02 & 1.9120e-01 & -2.4079e-02 & -7.5289e-01 & -1.3565e-02 & 6.9419e-01 & -3.3420e-01 \\
1.0707e-02 & -1.0136e-03 & -1.1139e+00 & 2.5487e-01 & 2.7271e+00 & -6.9609e-01 & 4.1844e-01 & -4.9641e-01 & -3.3420e-01 & 2.2892e+00
\end{bmatrix} > 0$$

(IV) Set constraints on $u_1(k)$, $\Delta u_1(k)$ and $\hat{y}_1(k+5)$ with softening, and select the tuning parameters as:

$$P = 73, \; M = 1, \; B = 0, \; D = 0, \; \Gamma = I, \; W = 10.0835, \; T_s = 6$$

Suppose $D_d = 0.1I$, $D_z = 0.1I$. By theorem 2 and LMI-Lab tool, we find there exists a $\mathcal{P} > 0$ such the CMPC of this process satisfies the H_∞ performance criterion (definition 1).

$$\mathcal{P} = \left[\begin{array}{cccccc}
3.2798e+04 & -9.2980e+02 & 5.8437e+03 & 3.2330e+03 & -2.2906e+02 & 1.3074e+03 \\
-9.2980e+02 & 6.7103e+01 & -1.4417e+03 & -4.8814e+02 & 4.8356e+01 & -3.2055e+02 \\
5.8437e+03 & -1.4417e+03 & 4.6164e+04 & 6.2547e+03 & -1.3194e+04 & 1.2239e+04 \\
3.2330e+03 & -4.8814e+02 & 6.2547e+03 & 2.6614e+04 & 1.1247e+04 & -5.8885e+02 \\
-2.2906e+02 & 4.8356e+01 & -1.3194e+04 & 1.1247e+04 & 3.7362e+04 & -8.9180e+03 \\
1.3074e+03 & -3.2055e+02 & 1.2239e+04 & -5.8885e+02 & -8.9180e+03 & 4.1596e+03 \\
-9.3120e+02 & 3.9210e+01 & -1.1828e+03 & 2.7321e+03 & -3.9368e+03 & 2.7018e+02 \\
-2.0123e+03 & 5.5038e+02 & -1.2523e+04 & -9.5986e+03 & -7.7673e+03 & -1.4876e+03 \\
1.9273e+02 & -7.1725e-01 & 2.4651e+02 & -5.8499e+02 & 3.9241e+02 & 9.5503e+00 \\
-5.6625e+02 & -3.2264e+01 & -5.7935e+02 & 1.2694e+03 & 7.5042e+03 & -1.3041e+03 \\
0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 \\
0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00
\end{array} \right.$$

$$
\left.\begin{matrix}
-9.3120e+02 & -2.0123e+03 & 1.9273e+02 & -5.6625e+02 & 0.0000e+00 & 0.0000e+00 \\
3.9210e+01 & 5.5038e+02 & -7.1725e-01 & -3.2264e+01 & 0.0000e+00 & 0.0000e+00 \\
-1.1828e+03 & -1.2523e+04 & 2.4651e+02 & -5.7935e+02 & 0.0000e+00 & 0.0000e+00 \\
2.7321e+03 & -9.5986e+03 & -5.8499e+02 & 1.2694e+03 & 0.0000e+00 & 0.0000e+00 \\
-3.9368e+03 & -7.7673e+03 & 3.9241e+02 & 7.5042e+03 & 0.0000e+00 & 0.0000e+00 \\
2.7018e+02 & -1.4876e+03 & 9.5503e+00 & -1.3041e+03 & 0.0000e+00 & 0.0000e+00 \\
3.5274e+03 & 1.2577e+02 & -1.1690e+03 & -9.5431e+02 & 0.0000e+00 & 0.0000e+00 \\
1.2577e+02 & 8.9670e+03 & 2.7292e+02 & -2.0900e+03 & 0.0000e+00 & 0.0000e+00 \\
-1.1690e+03 & 2.7292e+02 & 6.6701e+02 & -1.2912e+02 & 0.0000e+00 & 0.0000e+00 \\
-9.5431e+02 & -2.0900e+03 & -1.2912e+02 & 3.1473e+03 & 0.0000e+00 & 0.0000e+00 \\
0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 1.2744e+02 & 0.0000e+00 \\
0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 1.2744e+02
\end{matrix}\right] > 0
$$

Quantifying the minimum upper bound of the L_2-induced gain by using LMI-Lab tool to solve the optimization problem (6), we obtain:

$$\gamma = 0.1896, \;\; \bar{\lambda} = 0.0103,$$

$$
\bar{\mathcal{P}} = \left[\begin{matrix}
4.1409e-01 & -2.5520e-02 & 4.4698e-01 & -4.0323e-02 & -1.5011e-02 & 8.4752e-02 \\
-2.5520e-02 & 6.8584e-03 & -1.3538e-01 & 1.1332e-01 & -6.9858e-03 & -1.2403e-02 \\
4.4698e-01 & -1.3538e-01 & 3.2590e+00 & -2.3602e+00 & -5.0235e-01 & 4.7276e-01 \\
-4.0323e-02 & 1.1332e-01 & -2.3602e+00 & 9.2103e+00 & 2.3630e-01 & 1.1907e-01 \\
-1.5011e-02 & -6.9858e-03 & -5.0235e-01 & 2.3630e-01 & 2.6233e+00 & -4.8114e-01 \\
8.4752e-02 & -1.2403e-02 & 4.7276e-01 & 1.1907e-01 & -4.8114e-01 & 2.3891e-01 \\
4.2934e-03 & -4.6149e-03 & 4.6690e-02 & -6.6261e-03 & -2.1207e-01 & 2.3397e-02 \\
-1.4157e-01 & 1.7047e-02 & -2.2725e-01 & -5.9038e-01 & -2.4168e-01 & -3.7790e-02 \\
-1.3198e-02 & -1.8746e-03 & 2.0432e-02 & 2.0700e-02 & 7.9551e-03 & -1.9656e-03 \\
2.5181e-02 & -9.9932e-03 & 5.2981e-02 & -2.1848e-01 & 7.1412e-01 & -1.1461e-01
\end{matrix}\right.
$$

$$
\left.\begin{matrix}
4.2934e-03 & -1.4157e-01 & -1.3198e-02 & 2.5181e-02 \\
-4.6149e-03 & 1.7047e-02 & -1.8746e-03 & -9.9932e-03 \\
4.6690e-02 & -2.2725e-01 & 2.0432e-02 & 5.2981e-02 \\
-6.6261e-03 & -5.9038e-01 & 2.0700e-02 & -2.1848e-01 \\
-2.1207e-01 & -2.4168e-01 & 7.9551e-03 & 7.1412e-01 \\
2.3397e-02 & -3.7790e-02 & -1.9656e-03 & -1.1461e-01 \\
3.7042e-01 & -1.0256e-02 & -8.0273e-02 & 1.8267e-01 \\
-1.0256e-02 & 6.7150e-01 & -3.0323e-02 & -8.8723e-02 \\
-8.0273e-02 & -3.0323e-02 & 2.1745e-01 & -4.1776e-02 \\
1.8267e-01 & -8.8723e-02 & -4.1776e-02 & 1.0459e+00
\end{matrix}\right] > 0
$$

We can conclude that the CMPC of this process satisfies the strongly H_∞ performance for specified D_d, D_z to control this process under the constraints set on u, Δu over the control horizon M or $\hat{y}_1(k+6), \hat{y}_2(k+4)$ or $u(k)$, $\Delta u(k)$, $\hat{y}_1(k+8)$, $\hat{y}_2(k+7)$ and $u_1(k)$, $\Delta u_1(k), \hat{y}_1(k+5)$ with softening.

5.1 Strongly Robust H_∞ Performance Analysis of CMPC with Scalar Uncertainty in the Model

Suppose we have a process described by state space model with scalar uncertainty in the coefficient matrix of manipulated variable as following:

$$
\begin{aligned}
x(k+1) &= \Phi x(k) + (\Theta_n + W_n\Delta_p)u(k) \\
z(k) &= Cx(k) + D_z w(k) \\
y(k) &= Cx(k) + D_d w(k)
\end{aligned}
$$

where Θ_n is the nominal coefficient matrix of manipulated variable; W_n is the uncertainty weighting matrix; Δ_p is a scalar uncertainty. The the closed-loop of the CMPC system can be written as:

$$\bar{x}(k+1) = (\Psi_1 + E_1\bar{\Delta}E_2)\bar{x}(k) + (\Psi_2 + E_1\bar{\Delta}E_3)w(k) \tag{7}$$

$$z(k) = C\bar{x}(k) + D_z w(k) \tag{8}$$

$$y(z) = C\bar{x}(k) + D_d w(k) \tag{9}$$

where

$$\bar{\Delta} = \begin{bmatrix} F(k) & 0 & 0 \\ 0 & \Delta_p & 0 \\ 0 & 0 & \Delta_p F(k) \end{bmatrix}$$

(For details, please see the technical reports "The Quadratic Stability of CMPC" and "The Closed-Form Control Law of CMPC"). Follow the theorem 2 to analyze the strongly H_∞ performance of CMPC as shown in the following SISO examples.

5.2 Examples for Illustrating Strongly Robust H_∞ Performance Analysis

All examples are for SISO only. The same method can be extended to MIMO, if the coefficients of the state space model are affine functions of uncertainties.

Example 4

A SISO process model of a subsystem of the Shell Control Problem is given [Prett and Garcia, 1988] as a state space model:

$$\begin{aligned} x(k+1) &= \phi x(k) + (4.05 + 2.11\Delta_p)\Theta u(k) \\ y(k) &= Cx(k) + D_d w(k) \end{aligned}$$

where $-1 \leq \Delta_p \leq 1$ and Δ_p is time varying;

$$\phi = \begin{bmatrix} 0.9231 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\Theta = [\,1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\,]^T$$

$$C = [\,0\ 0\ 0\ 0\ 0\ 0\ 0.0198\ 0.0571\,]$$

(I) Set constraints on u, Δu over the control horizon M, and select tuning parameters:

$$P = 150,\ M = 3,\ B = 0,\ D = 4,\ \Gamma = 1,\ T_s = 4$$

Suppose $D_d = 0.01$, $D_z = 0.01$. By theorem 2 and LMI-Lab tool, we find there exists a $\mathcal{P} > 0$ such the CMPC of this process satisfies the H_∞ performance criterion (definition 1).

$$
\mathcal{P} = \left[\begin{array}{cccccc}
7.0973e-01 & -6.6443e-01 & 1.1814e-02 & 4.4295e-03 & 1.7944e-03 & 5.2090e-04 \\
-6.6443e-01 & 1.2066e+00 & -5.7952e-01 & 2.3457e-02 & 8.5268e-03 & 3.3517e-03 \\
1.1814e-02 & -5.7952e-01 & 1.0775e+00 & -5.3634e-01 & 1.7616e-02 & 6.4379e-03 \\
4.4295e-03 & 2.3457e-02 & -5.3634e-01 & 9.6378e-01 & -4.7709e-01 & 1.5481e-02 \\
1.7944e-03 & 8.5268e-03 & 1.7616e-02 & -4.7709e-01 & 8.3981e-01 & -4.0771e-01 \\
5.2090e-04 & 3.3517e-03 & 6.4379e-03 & 1.5481e-02 & -4.0771e-01 & 6.9395e-01 \\
-2.9676e-04 & 1.5965e-03 & 2.5107e-03 & 5.5925e-03 & 1.3346e-02 & -3.2378e-01 \\
2.5129e-04 & -5.9527e-06 & 7.7936e-04 & 1.9779e-03 & 5.2858e-03 & 1.3452e-02 \\
-2.0288e+00 & 1.2063e+00 & 4.5635e-01 & 1.7477e-01 & 6.7393e-02 & 2.6719e-02 \\
0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00
\end{array} \right.
$$

$$
\left. \begin{array}{cccc}
-2.9676e-04 & 2.5129e-04 & -2.0288e+00 & 0.0000e+00 \\
1.5965e-03 & -5.9527e-06 & 1.2063e+00 & 0.0000e+00 \\
2.5107e-03 & 7.7936e-04 & 4.5635e-01 & 0.0000e+00 \\
5.5925e-03 & 1.9779e-03 & 1.7477e-01 & 0.0000e+00 \\
1.3346e-02 & 5.2858e-03 & 6.7393e-02 & 0.0000e+00 \\
-3.2378e-01 & 1.3452e-02 & 2.6719e-02 & 0.0000e+00 \\
5.0774e-01 & -2.0493e-01 & 1.1303e-02 & 0.0000e+00 \\
-2.0493e-01 & 1.8549e-01 & -6.5234e-03 & 0.0000e+00 \\
1.1303e-02 & -6.5234e-03 & 6.5288e+01 & 0.0000e+00 \\
0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 1.9265e-01
\end{array} \right] > 0
$$

Quantifying the minimum upper bound of the L_2-induced gain by using LMI-Lab tool to solve the optimization problem (6), we obtain:

$$
\gamma = 0.7939, \ \ \bar{\lambda} = 5.5404,
$$

$$
\bar{\mathcal{P}} = \left[\begin{array}{cccccc}
2.7140e+00 & -2.6849e+00 & 2.1109e-01 & 8.2817e-02 & 2.1350e-02 & 3.0312e-03 \\
-2.6849e+00 & 5.0278e+00 & -2.5508e+00 & 1.3445e-01 & 5.7979e-02 & 1.6997e-02 \\
2.1109e-01 & -2.5508e+00 & 4.4456e+00 & -2.2780e+00 & 1.1423e-01 & 4.7608e-02 \\
8.2817e-02 & 1.3445e-01 & -2.2780e+00 & 3.8840e+00 & -1.9648e+00 & 9.9287e-02 \\
2.1350e-02 & 5.7979e-02 & 1.1423e-01 & -1.9648e+00 & 3.2831e+00 & -1.6207e+00 \\
3.0312e-03 & 1.6997e-02 & 4.7608e-02 & 9.9287e-02 & -1.6207e+00 & 2.6113e+00 \\
-1.4279e-03 & 4.1648e-03 & 1.4443e-02 & 3.7196e-02 & 8.3489e-02 & -1.2277e+00 \\
3.8020e-03 & -1.5691e-03 & 1.1438e-03 & 1.0927e-02 & 3.1334e-02 & 7.6090e-02 \\
-2.8097e+00 & 1.7549e+00 & 6.0521e-01 & 1.4307e-01 & 2.1691e-02 & 1.3974e-02
\end{array} \right.
$$

$$
\left. \begin{array}{ccc}
-1.4279e-03 & 3.8020e-03 & -2.8097e+00 \\
4.1648e-03 & -1.5691e-03 & 1.7549e+00 \\
1.4443e-02 & 1.1438e-03 & 6.0521e-01 \\
3.7196e-02 & 1.0927e-02 & 1.4307e-01 \\
8.3489e-02 & 3.1334e-02 & 2.1691e-02 \\
-1.2277e+00 & 7.6090e-02 & 1.3974e-02 \\
1.8110e+00 & -7.1523e-01 & 4.0507e-02 \\
-7.1523e-01 & 5.9799e-01 & -3.7860e-02 \\
4.0507e-02 & -3.7860e-02 & 1.7598e+02
\end{array} \right] > 0
$$

(II) Set constraint on $\hat{y}(k+11)$, and select tuning parameters:

$$
P = 60, \ M = 1, \ B = 0, \ D = 0, \ \Gamma = 1, \ T_s = 4
$$

Suppose $D_d = 0.01$, $D_z = 0.01$. By theorem 2 and LMI-Lab tool, we find there exists a $\mathcal{P} > 0$ such the CMPC of this process satisfies the H_∞ performance criterion (definition 1).

$$
\mathcal{P} = \begin{bmatrix}
6.3242e-02 & -1.3911e-02 & 1.4562e-05 & 1.0854e-05 & 8.5045e-06 & 6.9561e-06 & 6.2747e-06 & 8.7309e-06 & 0.0000e+00 \\
-1.3911e-02 & 2.7509e-02 & -1.2505e-02 & 1.7704e-05 & 1.2417e-05 & 9.2364e-06 & 7.1533e-06 & 5.6566e-06 & 0.0000e+00 \\
1.4562e-05 & -1.2505e-02 & 2.4553e-02 & -1.1015e-02 & 2.1631e-05 & 1.4463e-05 & 1.0375e-05 & 8.4132e-06 & 0.0000e+00 \\
1.0854e-05 & 1.7704e-05 & -1.1015e-02 & 2.1409e-02 & -9.4184e-03 & 2.5875e-05 & 1.6730e-05 & 1.2763e-05 & 0.0000e+00 \\
8.5045e-06 & 1.2417e-05 & 2.1631e-05 & -9.4184e-03 & 1.8018e-02 & -7.6745e-03 & 3.0788e-05 & 2.0941e-05 & 0.0000e+00 \\
6.9561e-06 & 9.2364e-06 & 1.4463e-05 & 2.5875e-05 & -7.6745e-03 & 1.4271e-02 & -5.7091e-03 & 3.9318e-05 & 0.0000e+00 \\
6.2747e-06 & 7.1533e-06 & 1.0375e-05 & 1.6730e-05 & 3.0788e-05 & -5.7091e-03 & 9.9342e-03 & -3.3367e-03 & 0.0000e+00 \\
8.7309e-06 & 5.6566e-06 & 8.4132e-06 & 1.2763e-05 & 2.0941e-05 & 3.9318e-05 & -3.3367e-03 & 4.0653e-03 & 0.0000e+00 \\
0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 1.1353e-01
\end{bmatrix} > 0
$$

Quantifying the minimum upper bound of the L_2-induced gain by using LMI-Lab tool to solve the optimization problem (6), we obtain:

$$\gamma = 0.1964, \quad \bar{\lambda} = 3.6506,$$

$$
\bar{\mathcal{P}} = \begin{bmatrix}
1.7913e-01 & -1.2635e-02 & 4.9196e-05 & 3.6532e-05 & 3.0098e-05 & 3.7581e-05 & 8.0657e-05 & 3.2512e-04 \\
-1.2635e-02 & 2.9541e-02 & -1.0913e-02 & 6.6585e-05 & 3.6897e-05 & 2.2114e-05 & 5.3831e-06 & -8.2464e-05 \\
4.9196e-05 & -1.0913e-02 & 2.6019e-02 & -9.1666e-03 & 7.8309e-05 & 3.1978e-05 & 6.1917e-06 & -4.4219e-05 \\
3.6532e-05 & 6.6585e-05 & -9.1666e-03 & 2.2411e-02 & -7.3620e-03 & 8.5880e-05 & 2.2631e-05 & -2.0210e-05 \\
3.0098e-05 & 3.6897e-05 & 7.8309e-05 & -7.3620e-03 & 1.8690e-02 & -5.4809e-03 & 8.6469e-05 & 5.4532e-06 \\
3.7581e-05 & 2.2114e-05 & 3.1978e-05 & 8.5880e-05 & -5.4809e-03 & 1.4817e-02 & -3.4882e-03 & 7.2458e-05 \\
8.0657e-05 & 5.3831e-06 & 6.1917e-06 & 2.2631e-05 & 8.6469e-05 & -3.4882e-03 & 1.0709e-02 & -1.3002e-03 \\
3.2512e-04 & -8.2464e-05 & -4.4219e-05 & -2.0210e-05 & 5.4532e-06 & 7.2458e-05 & -1.3002e-03 & 5.7352e-03
\end{bmatrix} > 0
$$

(III) Set constraint on $\hat{y}(k+7)$ with softening, and select tuning parameters:

$$P = 100, \ M = 1, \ B = 0, \ D = 0, \ W = 74, \ \Gamma = 1, \ T_s = 4$$

Suppose $D_d = 0.01$, $D_z = 0.01$. By theorem 2 and LMI-Lab tool, we find there exists a $\mathcal{P} > 0$ such the CMPC of this process satisfies the H_∞ performance criterion (definition 1).

$$
\mathcal{P} = \begin{bmatrix}
8.0024e-02 & -4.4655e-02 & 1.8905e-05 & 1.4524e-05 & 1.4842e-05 & 1.4417e-05 & 1.4906e-05 & 2.8959e-05 & 0.0000e+00 \\
-4.4655e-02 & 8.6049e-02 & -4.0177e-02 & 2.1153e-05 & 1.3145e-05 & 1.2279e-05 & 1.1988e-05 & 9.9309e-06 & 0.0000e+00 \\
1.8905e-05 & -4.0177e-02 & 7.6736e-02 & -3.5432e-02 & 2.5297e-05 & 1.6399e-05 & 1.4666e-05 & 1.4334e-05 & 0.0000e+00 \\
1.4524e-05 & 2.1153e-05 & -3.5432e-02 & 6.6825e-02 & -3.0347e-02 & 2.9968e-05 & 2.0350e-05 & 1.9637e-05 & 0.0000e+00 \\
1.4842e-05 & 1.3145e-05 & 2.5297e-05 & -3.0347e-02 & 5.6119e-02 & -2.4798e-02 & 3.5912e-05 & 2.8144e-05 & 0.0000e+00 \\
1.4417e-05 & 1.2279e-05 & 1.6399e-05 & 2.9968e-05 & -2.4798e-02 & 4.4268e-02 & -1.8552e-02 & 4.7691e-05 & 0.0000e+00 \\
1.4906e-05 & 1.1988e-05 & 1.4666e-05 & 2.0350e-05 & 3.5912e-05 & -1.8552e-02 & 3.0500e-02 & -1.1040e-02 & 0.0000e+00 \\
2.8959e-05 & 9.9309e-06 & 1.4334e-05 & 1.9637e-05 & 2.8144e-05 & 4.7691e-05 & -1.1040e-02 & 1.1795e-02 & 0.0000e+00 \\
0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 0.0000e+00 & 1.0492e-01
\end{bmatrix} > 0
$$

Quantifying the minimum upper bound of the L_2-induced gain by using LMI-Lab tool to solve the optimization problem (6), we obtain:

$$\gamma = 0.0999, \quad \bar{\lambda} = 4.9202,$$

$$
\bar{\mathcal{P}} = \begin{bmatrix}
2.2030e-01 & -7.3742e-02 & 7.6637e-05 & 1.0502e-04 & 1.2365e-04 & 1.8169e-04 & 3.3359e-04 & 8.1456e-04 \\
-7.3742e-02 & 1.4451e-01 & -6.4872e-02 & 1.0435e-04 & 9.6180e-05 & 7.0296e-05 & 3.1293e-05 & -9.9815e-05 \\
7.6637e-05 & -6.4872e-02 & 1.2641e-01 & -5.5728e-02 & 1.3530e-04 & 1.0079e-04 & 5.5289e-05 & -2.9048e-05 \\
1.0502e-04 & 1.0435e-04 & -5.5728e-02 & 1.0762e-01 & -4.6197e-02 & 1.6588e-04 & 1.0289e-04 & 3.0650e-05 \\
1.2365e-04 & 9.6180e-05 & 1.3530e-04 & -4.6197e-02 & 8.7958e-02 & -3.6150e-02 & 1.9929e-04 & 1.0760e-04 \\
1.8169e-04 & 7.0296e-05 & 1.0079e-04 & 1.6588e-04 & -3.6150e-02 & 6.7091e-02 & -2.5356e-02 & 2.5263e-04 \\
3.3359e-04 & 3.1293e-05 & 5.5289e-05 & 1.0289e-04 & 1.9929e-04 & -2.5356e-02 & 4.4358e-02 & -1.3247e-02 \\
8.1456e-04 & -9.9815e-05 & -2.9048e-05 & 3.0650e-05 & 1.0760e-04 & 2.5263e-04 & -1.3247e-02 & 1.7400e-02
\end{bmatrix} > 0
$$

We can conclude that the CMPC of this process satisfies the strongly H_∞ Performance for specified D_d, D_z to control this process under the constraint set on u, Δu over the M or $\hat{y}(k+11)$ or $\hat{y}(k+7)$ with softening.

6 Conclusions

The strongly H_∞ performance of CMPC is analyzed and the minimum upper bound of L_2-induced gain is also quantified for specified coefficient matrix of external input in this paper. The control law of CMPC is a predicted error (the difference between predicted output and reference input) feedback control law. Each term in the control law corresponding to the active constraint situation can be decomposed to have a time varying uncertainty block. The closed-loop system is affine function of uncertainties. Based on this type of closed-loop system, the analyzing and quantifying work is accomplished. Several examples show the analyzing and quantifying results.

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Appendix A: The Proofs of Theorems 1, 2

(1) **The proof of theorem** 1

The inequality (5) can be rewritten as:

$$\begin{bmatrix} (\Psi_1{}^T\bar{\mathcal{P}} + E_2{}^T F(k)E_1{}^T\bar{\mathcal{P}})(\Psi_1 + E_1F(k)E_2) + \mathcal{C}^T\mathcal{C} - \bar{\mathcal{P}} & (\Psi_1{}^T\bar{\mathcal{P}} + E_2{}^T F(k)E_1{}^T\bar{\mathcal{P}})(\Psi_2 + E_1F(k)E_3) + \mathcal{C}^T D_z \\ (\Psi_2{}^T\bar{\mathcal{P}} + E_3{}^T F(k)E_1{}^T\bar{\mathcal{P}})(\Psi_1 + E_1F(k)E_2) + D_z{}^T\mathcal{C} & (\Psi_2{}^T\bar{\mathcal{P}} + E_3{}^T F(k)E_1{}^T\bar{\mathcal{P}})(\Psi_2 + E_1F(k)E_3) + D_z{}^T D_z - I \end{bmatrix} < 0$$

Multiply the above inquality from left hand side by $[\ \bar{x}^T(k)\ \ w^T(k)\ ]$ and right hand side by $[\ \bar{x}^T(k)\ \ w^T(k)\ ]^T$, and let $\bar{p}(k) = F(k)(E_2\bar{x}(k) + E_3w(k))$ then we have:

$$\begin{aligned} &\bar{x}^T(k)\Psi_1{}^T\bar{\mathcal{P}}\Psi_1\bar{x}(k) - \bar{x}^T(k)\bar{\mathcal{P}}\bar{x}(k) + \bar{x}^T(k)\mathcal{C}^T\mathcal{C}\bar{x}(k) + \bar{x}^T(k)\Psi_1{}^T\bar{\mathcal{P}}\Psi_2w(k) + \bar{x}^T(k)\mathcal{C}^T D_zw(k) + \\ &\bar{x}^T(k)\Psi_1{}^T\bar{\mathcal{P}}E_1\bar{p}(k) + w^T(k)\Psi_2{}^T\bar{\mathcal{P}}\Psi_2w(k) + w^T(k)\Psi_2{}^T\bar{\mathcal{P}}\Psi_1\bar{x}(k) + w^T(k)\Psi_2{}^T\bar{\mathcal{P}}E_1\bar{p}(k) + w^T(k)D_z{}^T\mathcal{C}\bar{x}(k) + \\ &\bar{p}^T(k)E_1{}^T\bar{\mathcal{P}}E_1\bar{p}(k) + \bar{p}^T(k)E_1{}^T\bar{\mathcal{P}}\Psi_1\bar{x}(k) + \bar{p}^T(k)E_1{}^T\bar{\mathcal{P}}\Psi_2w(k) + w^T(k)D_z{}^T D_zw(k) - w^T(k)w(k) < 0 \end{aligned} \tag{10}$$

$\Rightarrow$

$$(\Psi_1\bar{x}(k)+\Psi_2w(k)+E_1\bar{p}(k))^T\bar{\mathcal{P}}(\Psi_1\bar{x}(k)+\Psi_2w(k)+E_1\bar{p}(k)) - \bar{x}^T(k)\bar{\mathcal{P}}\bar{x}(k) + (\mathcal{C}\bar{x}(k)+D_zw(k))^T(\mathcal{C}\bar{x}(k)+D_zw(k)) - w^T(k)w(k) < 0$$

$\Rightarrow$

$$\Delta V + z^T(k)z(k) - w^T(k)w(k) < 0 \tag{11}$$

where $\Delta V = V(k+1) - V(k)$ and $V(k) = \bar{x}^T(k)\bar{\mathcal{P}}\bar{x}(k)$ which is a Lyapunov function of system (2). From inquality (11) and $\bar{x}(0) = 0$, we can have:

$$V(nf) + \sum_{k=0}^{nf} z^T(k)z(k) - \sum_{k=0}^{nf} w^T(k)w(k) < 0$$

but $V(nf) > 0$. Hence,

$$\frac{\| z \|_2}{\| w \|_2} < 1$$

□

(2) **The proof of theorem** 2

Follow the procedures in expanding inequality (5) in the proof of theorem 1, then we have the inquality (10). Rewrite the inquality as:

$$\begin{bmatrix} \bar{x}(k) \\ w(k) \\ \bar{p}(k) \end{bmatrix}^T \begin{bmatrix} \Psi_1{}^T\bar{\mathcal{P}}\Psi_1 - \bar{\mathcal{P}} + \mathcal{C}^T\mathcal{C} & \Psi_1{}^T\bar{\mathcal{P}}\Psi_2 + \mathcal{C}^T D_z & \Psi_1{}^T\bar{\mathcal{P}}E_1 \\ \Psi_2{}^T\bar{\mathcal{P}}\Psi_1 + D_z{}^T\mathcal{C} & \Psi_2{}^T\bar{\mathcal{P}}\Psi_2 + D_z{}^T D_z - I & \Psi_2{}^T\bar{\mathcal{P}}E_1 \\ E_1{}^T\bar{\mathcal{P}}\Psi_1 & E_1{}^T\bar{\mathcal{P}}\Psi_2 & E_1{}^T\bar{\mathcal{P}}E_1 \end{bmatrix} \begin{bmatrix} \bar{x}(k) \\ w(k) \\ \bar{p}(k) \end{bmatrix} < 0 \tag{12}$$

whenever

$$\bar{p}^T\bar{p} = [F(k)(E_2\bar{x}(k) + E_3w(k))]^T[F(k)(E_2\bar{x}(k) + E_3w(k))] \le \begin{bmatrix} \bar{x}(k) \\ w(k) \end{bmatrix}^T \begin{bmatrix} E_2{}^TE_2 & E_2{}^TE_3 \\ E_3{}^TE_2 & E_3{}^TE_3 \end{bmatrix} \begin{bmatrix} \bar{x}(k) \\ w(k) \end{bmatrix}$$

$\Rightarrow$

$$\begin{bmatrix} \bar{x}(k) \\ w(k) \\ \bar{p}(k) \end{bmatrix}^T \begin{bmatrix} -E_2{}^TE_2 & -E_2{}^TE_3 & 0 \\ -E_3{}^TE_2 & -E_3{}^TE_3 & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \bar{x}(k) \\ w(k) \\ \bar{p}(k) \end{bmatrix} \le 0 \tag{13}$$

By $\mathcal{S}$ procedure, there exits a $\bar{\lambda} > 0$ to combine these two inequalities (12) (13) as:

$$\begin{bmatrix} {\Psi_1}^T\bar{\mathcal{P}}\Psi_1 - \bar{\mathcal{P}} + \mathcal{C}^T\mathcal{C} + \bar{\lambda}{E_2}^T E_2 & {\Psi_1}^T\bar{\mathcal{P}}\Psi_2 + \mathcal{C}^T D_z + \bar{\lambda}{E_2}^T E_3 & {\Psi_1}^T\bar{\mathcal{P}}E_1 \\ {\Psi_2}^T\bar{\mathcal{P}}\Psi_1 + {D_z}^T\mathcal{C} + \bar{\lambda}{E_3}^T E_2 & {\Psi_2}^T\bar{\mathcal{P}}\Psi_2 + {D_z}^T D_z - I + \bar{\lambda}{E_3}^T E_3 & {\Psi_2}^T\bar{\mathcal{P}}E_1 \\ {E_1}^T\bar{\mathcal{P}}\Psi_1 & {E_1}^T\bar{\mathcal{P}}\Psi_2 & {E_1}^T\bar{\mathcal{P}}E_1 - \bar{\lambda}I \end{bmatrix} \leq 0 \qquad (14)$$

Let

$$\mathcal{P} = \begin{bmatrix} \bar{\mathcal{P}}/\bar{\lambda} & 0 \\ 0 & I/\bar{\lambda} \end{bmatrix}$$

and reformulate inequality (14), then we have:

$$\begin{bmatrix} \Psi^T\mathcal{P}\Psi - \mathcal{P} + \mathcal{E}_2^T\mathcal{E}_2 & \Psi^T\mathcal{P}\mathcal{E}_1 \\ \mathcal{E}_1^T\mathcal{P}\Psi & \mathcal{E}_1^T\mathcal{P}\mathcal{E}_1 - I \end{bmatrix} \leq 0$$

□

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