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Almost Sure Stability Of Linear Stochastic Systems With Poisson Process Coefficients

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Abstract: We consider the problem of determining the sample path stability of a class of linear stochastic differential equations with point process coefficients. Necessary and sufficient conditions are obtained which are similar in spirit to those derived by Khas'minskii and Pinsky for diffusion processes. The conditions are based on the deep theorems of Furstenburg on the asymptotic behavior of products of random matrices. Estimates on the probabilities of large deviations for stable processes are also given; together with a result on the stabilization of unstable systems by feedback controls.

Key Words: Almost sure stability, invariant measures, large deviations, products of random matrices, stabilization.

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1. The Problem and Main Results.

Consider the linear stochastic system

$$dx(t) = Ax(t)dt + \sum_{i=0}^{m} B_i x(t) dN_i(t), \tag{1.1}$$

$$x(0) = x_0 \in \mathbb{R}^n \setminus \{0\},\ t \geq 0,$$

on the underlying probability space (Ω, F, P) with A and B_i constant $n \times n$ real matrices, and $\{ N_i(t),\ t \geq 0 \}$, $i=1,\ldots,m$, independent Poisson processes - specifically, one dimensional counting process with intensity $\lambda_i > 0$ and right-continuous paths. $N_i(t) \in \{0,1,2,\ldots\}$ counts the number of occurrences in $[0,t]$. We are interested in the almost sure stability properties of the solutions of (1.1). That is, if $|\cdot|$ is any norm on $\mathbb{R}^n$ ($||\cdot||$ is the induced matrix norm), we would like to characterize the asymptotic exponential growth rate

$$\lim_{t \uparrow \infty} \frac{1}{t} \log \left(\frac{|x(t)|}{|x_0|} \right) \tag{1.2}$$

if it exists.

This problem is the analog of the one considered by Khas'minshii [1] and Pinsky [2] for diffusion processes, and by Loparo and Blankenship [3] for systems with jump process coefficients. Like previous results, the expression given here for the growth rate is not an explicit, readily computable one, except in simple cases. The stability properties of the moments of the solution of (1.1) were considered by Marcus [4] [5] (see also [6]). Explicit stability criteria are possible for the moments. Related results on the optimal control and scheduling of systems with Poisson noises are given in [7][8]. See also [9].

The system (1.1) is interpreted in terms of the integral equation

$$x(t) = x_0 + \int_0^t Ax(s)ds + \sum_{i=1}^{m} \int_0^t B_i x(s) dN_i(s) \tag{1.3}$$

with the stochastic integral defined by the calculus explained in [5] [10][1]. Let $\{\tau_j^i,\ j \geq 1\}$ be the interarrival times and $t_j^i = \tau_1^i + \cdots + \tau_j^i$ be the occurrence time for

[1]We could also treat some of the more complicated point process models in [5] [10], but the main ideas are best conveyed by the simple case considered here

the Poisson process $N_i(t)$. Then

$$\int_0^t B_i\, x(s)\, dN_i(s) \triangleq \begin{cases} 0 & , \; N_i(t)=0 \\ \sum_{j=1}^{N_i(t)} B_i\, x(t_j^{i-}) & , \; N_i(t) \geq 1 \, . \end{cases} \tag{1.4}$$

Now, let $\{\tau_j ,\, j \geq 1\}$ be the interarrival times of the sum process $N(t) = N_1(t) + \cdots + N_m(t)$ with intensity $\lambda = \lambda_1 + \cdots + \lambda_m$, and μ_j be the process indicating which N_i under went an increment at the occurrence time $t_j = \tau_1 + \cdots + \tau_j$. We assume the probability of multiple, simultaneous jumps is zero. The process $\{x(t),\, t \geq 0\}$ exists, has right continuous paths, and jumps at t_j, j=1,2, ... If we set $D_i = I + B_i$, then

$$x(t) = exp\,(A\,(t - t_{N(t)}))D_{\tau_{N(t)}} \cdots D_{\tau_1} exp\,(A\,\tau_1) x_0 . \tag{1.5}$$

This expression is the basis of our treatment of the almost sure stability problem. Its composition as a product of random matrices directed our attention to the work of Furstenberg and Kesten [12], Grenander [13] and Furstenberg [15]-[18] on the limits of products of random matrices.

Our main result is based on the following observations. First, for each $i = 1,\ldots,m$, the $\{\tau_j^i,\, j \geq 1\}$ are independent and exponentially distributed with parameter λ_i. The random processes $\{\tau_j ,\, \mu_j ,\, j \geq 1\}$ depend in a complex way on $\{\tau_j^i,\, i = 1, \ldots, m,\, j \geq 1\}$. However, $\{\tau_i ,\, i \geq 1\}$ and $\{\mu_j ,\, j \geq 1\}$ are independent and form independent, identically distributed sequences. This follows from the presumed independence of the $\{N_i(t),\, i = 1, \ldots, m\}$, and will be shown in section 3. As a consequence, we have the following:

Theorem (Stability). *Consider the system* (1.1) *with the stated assumptions on the processes* $N_i(t)$, $i=1,\ldots,m$. *Then*

$$r = \lim_{k\to\infty} \frac{1}{k} E \log \left|\left| D_{\mu_k} e^{A\tau_k} \ldots D_{\mu_1} e^{A\tau_1} \right|\right| < \infty \tag{1.6}$$

exists and

$$r = \lim_{k\to\infty} \frac{1}{k} \log \left|\left| D_{\mu_k} e^{A\tau_k} \ldots D_{\mu_1} e^{A\tau_1} \right|\right| \quad a.s. \tag{1.7}$$

The quantity r is the *asymptotic exponential growth rate* of the process $x(t)$; that is,

$$\frac{|x(t)|}{|x(0)|} \sim e^{rt} \qquad t \text{ large}$$

Hence, $r > 0$ implies almost sure instability and $r < 0$ corresponds to almost sure asymptotic stability. This result is proved in section 3 (Theorem 3.5).

It is possible to obtain a more detailed description of the long term behavior of $\{x(t),\ t \geq 0\}$ by examining the behavior of products of random matrices acting on specific initial states $x(0) = 0$. The key questions are: Does the limit of

$$\frac{1}{k} \log ||D_{\mu_k} e^{A\tau_k} \cdots D_{\mu_1} e^{A\tau_1} x_0||$$

exist? If it does, how is it related to the rate r in (1.7)? To treat these questions, we generalize some results of Furstenburg, Kesten, Grenander and others on random walks on semi-simple Lie groups to general semi-groups (not necessarily groups since the terms D_k may be singular). This analysis is given in section 4. The main result is as follows (Theorem 4.14):

Suppose μ is the measure on the Borel sets $B(\mathbb{R}^{n\times n})$ defined by

$$\mu(\Gamma) \triangleq P\{D_{\mu_1} e^{A\tau_1} \in \Gamma\}, \ \ \Gamma \in B(\mathbb{R}^{n \times n}).$$

Let SG be the closed semi-group generated by the support of μ, i.e.

$$SG \triangleq \textit{smallest closed semi-group containing } \{D_i e^{At}, 0 \leq t < \infty, i = 1, \ldots, m\}.$$

Let ν be an invariant measure for μ; i.e., a solution of the integral equation

$$\mu * \nu = \nu \tag{1.8}$$

Let Q_0 be the collection of extremal invariant probability measures of μ on $M \triangleq S^{n-1} \cup \{0\}$.

Theorem *For all* $\nu \in Q_0$,

$$r_\nu \triangleq \sum_{i=1}^{m} \lambda_i \int_M \int_0^\infty log \mid D_i \, exp(At)u \mid e^{-\lambda t} \, dt \, d\nu(u) < \infty \tag{1.9}$$

and

$$\lim_{t \to \infty} \frac{1}{t} \log \left(\frac{|x(t)|}{|x_0|} \right) = \lambda r_\nu \ \ a.s. \tag{1.10}$$

for all $x_0 \in E_\nu^0$, *an ergodic component corresponding to* $\nu \in Q_0$. *Indeed, there are only finite different values, say,* $r_1 < r_2 < \cdots < r_l = r$, $l \leq n$. *Furthermore, if* $\bigcup_{\nu \in Q_0} E_\nu^0$ *contains a basis of* $\mathbb{R}^n$, *then the system* (1.1) *is asymptotically stable almost surely if* $r_l < 0$, *while the system* (1.1) *is asymptotically unstable if* $r_1 > 0$. *In case* $r_1 < 0$ and $r_l > 0$, *then the stability of the system depends on the initial state* x_0.

To apply these theorems to a specific problem, one must determine r or at least its sign; or, more generally, the collection Q_0 must be constructed and r_ν computed. If the semi-group SG is *transient* or *irreducible,* then r_ν will be independent of ν (even though there may be many ergodic components). (See Theorem 4.10 and Corollary 4.11.) In this case a theorem of Furstenburg ([15], Theorem 8.6) may be used to determine the sign of $r_\nu = r$. Application of this result to specific systems requires a close analysis of the geometric structure of the semi-group associated with those systems. Several examples are given in the next section to illustrate the techniques.

Two final results of interest in engineering practice concern the occurence of "large deviations" in the paths of $\{ x(t),\ t \geq 0 \}$ of a stable system (1.1) and the ability to stabilize a system like (1.1) with feedback controls.

The following result is proved in section 5.

Theorem (Large deviations). *If the system* (1.1) *is asymptotically stable with* $r_\nu < 0$, *then there exist constants* $M(x_0,R)$ and $r_\nu \lambda < \gamma < 0$ *such that*

$$\mathbf{P}\{ \sup_{s \geq t} |x(s)| \geq R \} \leq M(x_0,R)e^{\gamma t},\ t \geq 0. \tag{1.11}$$

The constants may be determined rather precisely, see equation (5.6) for details.

The following result is proved in section 6.

Theorem (Stabilization). *The control system with state and control dependent Poisson noises*

$$dx(t) = Ax(t)dt + Bu(t)dt + Cx(t)dN_1(t) + Du(t)dN_2(t) \tag{1.12}$$

is stabilized by the linear feedback control $u(t) = -Kx(t)$ *almost surely where* K *is any matrix such that*

$$\lambda_1 \int_0^\infty \log \|(I+C)e^{(A-BK)t}\| e^{-\lambda t}\, dt \tag{1.13}$$

$$+ \lambda_2 \int_0^\infty \log \|(I-DK)e^{(A-BK)t}\| e^{-\lambda t}\, dt < 0$$

where λ_i *is the intensity of* $N_i(t)$ *and* λ *is the intensity of* $N(t) = N_1(t) + N_2(t)$. *If*

$D = 0$ *(no control dependent noise) and* (A, B) *is controllable, i.e.,*

$$\text{rank}\,[B, AB, \ldots, A^{n-1}B] = n$$

then (1.12) *is stabilized by any matrix* K *for which the eigenvalues of* $A - BK$ *lie to the left of* $\mathrm{Re}(s) = -\lambda\,|\log||I+C|||$ *in the complex plane.*

2. Examples and Applications.

We would like to use some examples to show how to apply our theorems to determine stability properties of specific systems. As we shall see, in many cases, it is hard to find the necessary invariant measure because it is associated with an integral equation with shift arguments. It is difficult to evaluate a solution from this, although it exists.

Example 2.1. Consider the simple system

$$dx(t) = \begin{pmatrix} k & \omega \\ -\omega & k \end{pmatrix} x(t)dt + \begin{pmatrix} -1 & \alpha \\ \alpha & -1 \end{pmatrix} x(t)dN(t) \tag{2.1}$$

where $N(t)$ is a Poisson process with intensity $\lambda > 0$. Then

$$expAt = e^{kt} \begin{pmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{pmatrix}, \quad \omega > 0$$

$$D = I + B = \alpha \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \alpha \neq 0.$$

In this case, $De^{At} \neq e^{At}D$ and

SG = smallest semi-group containing $\{De^{At}, 0 \leq \mathrm{t} \leq \infty\}$

where μ is the probability measure on SG with density function $\lambda e^{-\lambda t}$, $t \geq 0$ at each element De^{At}. Since D is non-singular, we can take $M = S^0$, the unit circle. In order to solve $\nu = \mu * \nu$, we let $\Gamma \in$ Borel set $B(S^0)$,

$$\nu(\Gamma) = \int_{SG\times S^0} \chi_\Gamma(g \circ x)\, d\mu(g)\, d\nu(x)$$

$$= \int_0^\infty \nu(exp\,(-At)D^{-1}\circ\Gamma)\,\lambda e^{-\lambda t}\, dt. \tag{2.2}$$

For x $\in \Gamma$, x $= (cos\,\theta, \sin\theta)^T$ for some $\theta \geq 0$ and let

$$y = exp\,(-At)D^{-1}x = e^{-kt}\begin{pmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{pmatrix}\frac{1}{\alpha}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$= \frac{1}{\alpha}\, e^{-kt}\begin{pmatrix} -\sin(\omega t-\theta) \\ \cos(\omega t-\theta) \end{pmatrix}.$$

Let ϕ be an angle between the y and x_1-axis. Then

$$\tan\phi = \frac{\cos(\omega t-\theta)}{-\sin(\omega t-\theta)} = -\cot(\omega t-\theta). \tag{2.3}$$

Differentiating (2.3), we get

$$\sec^2\phi\, d\phi = -\csc^2(\omega t-\theta)\, d\theta,$$

so that from (2.3)

$$\frac{d\phi}{d\theta} = \frac{-\csc^2(\omega t-\theta)}{\sec^2\phi} = \frac{-\csc^2(\omega t-\theta)}{1+\cot^2(\omega t-\theta)} = -1.$$

Suppose ν has density function $f\,(\theta)$, $0 \leq \theta \leq 2\pi$. Thus from (2.2),

$$f\,(\theta) = \int_0^\infty f\,(\phi)\left|\frac{d\phi}{d\theta}\right|\lambda e^{-\lambda t}\, dt = \int_0^\infty f\,(\phi)\lambda e^{-\lambda t}\, dt \tag{2.4}$$

and so

$$f\,(\theta) = \frac{1}{2\pi},\ 0\leq\theta\leq 2\pi$$

satisfies (2.4). Since SG is transitive on S, then the Haar measure $\nu(\theta)$ with density f(θ) is a unique invariant measure of μ. Thus,

$$
\begin{aligned}
r_\nu &= \int_{SG\times S} \log \mid g\circ x \mid d\mu(g)d\nu(x) \\
&= \int_0^\infty\int_0^{2\pi} \log \left| De^{At}\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}\right| \lambda e^{-\lambda t}\frac{1}{2\pi}\, d\theta dt \\
&= \int_0^\infty\int_0^{2\pi} \log \left| \alpha e^{kt}\begin{pmatrix}\sin(\theta-\omega t)\\ \cos(\theta-\omega t)\end{pmatrix}\right| \frac{\lambda}{2\pi} e^{-\lambda t}\, d\theta dt \\
&= \int_0^\infty \log \mid \alpha e^{kt} \mid \lambda e^{-\lambda t}\, dt \\
&= \log \mid\alpha\mid + \frac{k}{\lambda}\,.
\end{aligned}
$$

Consequently, if $k < -\lambda \log|\alpha|$, the system (2.1) is asymptotically stable, while for $k > -\lambda\log|\alpha|$, the system (2.1) is asymptotically unstable.

Example 2.2 (Harmonic oscillator with damping).

Let $y(t)$ be a point process, regarded as the formal derivative of a Poisson process $N(t)$ with intensity λ. Consider the second order system

$$
\ddot{z}(t) + y(t)\dot{z}(t) + [\omega^2+ky(t)]z(t) = 0 \tag{2.5}
$$

$$
z(0),\ \dot{z}(0) \text{ given},\ t\geq 0,\ \omega>0,\ k>0.
$$

Let $x_1(t) = \omega z(t)$, $x_2(t) = \dot{z}(t)$ and $x(t) = [x_1(t), x_2(t)]^T$. Then

$$
dx(t) = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} x(t)dt + \begin{pmatrix} 0 & 0 \\ -\dfrac{k}{\omega} & -1 \end{pmatrix} x(t)dN(t) \tag{2.6}
$$

$$
x(0) = \begin{pmatrix} \omega z(0) \\ \dot{z}(0) \end{pmatrix} \text{ given.}
$$

Set

$$A = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ -\dfrac{k}{\omega} & -1 \end{pmatrix}$$

and

$$D = I + B = \begin{pmatrix} 1 & 0 \\ -\dfrac{k}{\omega} & 0 \end{pmatrix}, \qquad expAt = \begin{pmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{pmatrix}.$$

Let SG be the smallest closed semi-group containing $\{De^{At},\ t \geq 0\}$. The probability measure μ on SG has density $\lambda e^{-\lambda t}$, $t \geq 0$ at each element De^{At}. Since D is singular, we take $M = S^0 \cup \{0\}$. It is easy to see that the only invariant set is

$$E = \left\{ P_1 = \left(\frac{\omega}{\sqrt{\omega^2+k^2}}, \frac{-k}{\sqrt{\omega^2+k^2}} \right), P_2 = \left(\frac{-\omega}{\sqrt{\omega^2+k^2}}, \frac{k}{\sqrt{\omega^2+k^2}} \right), (0,0) \right\}$$

with invariant measure ν of μ being defined by

$$\nu(P_i) = \frac{1}{2}, \quad i=1,\ 2 \text{ and } \nu(0) = 0.$$

Note that $SG \circ S^0 = E$ is invariant, so that the stability of the transient set $F = S^0 \setminus E$ also depends on r_ν though E does not span $\mathbb{R}^2$. (See section 4.) Now, we calculate $r_\nu = r$ as follows.

$$\begin{aligned} r_\nu &= \int_{SG \times M} log \mid gx \mid d\mu(g) d\nu(x) \\ &= \frac{1}{2} \sum_{i=1}^{2} \int_0^\infty log \mid De^{At} P_i \mid \lambda e^{-\lambda t} dt \\ &= \int_0^\infty \log \left| \cos\omega t - \frac{k}{\omega} \sin\omega t \right| \lambda e^{-\lambda t} dt \\ &= \frac{1}{2} \int_0^\infty \log \left[\cos^2\omega t - \frac{2k}{\omega} \cos\omega t \sin\omega t + \frac{k^2}{\omega^2} \sin^2\omega t \right] \lambda e^{-\lambda t} dt \end{aligned}$$

$$= \frac{1}{2}\int_0^\infty \log\left[\frac{1}{2}(1+\frac{k^2}{\omega^2}) + \frac{1}{2}(1-\frac{k^2}{\omega^2})\cos 2\omega t - \frac{k}{\omega}\sin 2\omega t\right]\lambda e^{-\lambda t}\,dt$$

$$= \frac{1}{2}\int_0^\infty \log\left[\frac{1}{2}(1+\frac{k^2}{\omega^2}) + \frac{1}{2}(1+\frac{k^2}{\omega^2})\cos(2\omega t+\alpha)\right]\lambda e^{-\lambda t}\,dt$$

$$= \frac{1}{2}\log\frac{1}{2}\left(1+\frac{k^2}{\omega^2}\right) + \frac{1}{2}\int_0 \log[1+\cos(2\omega t+\alpha)]\lambda e^{-\lambda t}\,dt \tag{2.7}$$

where

$$\tan\alpha = \frac{\omega k}{\frac{1}{2}(\omega^2-k^2)}, \quad -\frac{\pi}{2}\leq\alpha\leq\frac{\pi}{2}.$$

Let

$$I_1 \triangleq \int_0^\infty \log[1+cos\,(2\omega t+\alpha)]\,\lambda e^{-\lambda t}\,dt$$

$$= \int_\alpha^\infty \log[1+cos\;t\,]\frac{\lambda}{2\omega}\,e^{-\lambda(t-\alpha)/2\omega}\,dt. \tag{2.8}$$

Using the fact

$$\int_0^\pi \log(1+cos\;t\,)dt = -\pi\log 2,$$

we have

$$\int_\beta^{\beta+2\pi} \log(1+cos\;t\,)dt = -2\pi\log 2,\ \forall\,\beta.$$

Thus, let p $= \dfrac{\lambda}{2\omega}$,

$$I_1 > -2\pi p\;\log 2\sum_{j=0}^\infty e^{-pj2\pi} = -\frac{2\pi p\;\log 2}{1-e^{-2\pi p}} \tag{2.9}$$

and

$$I_1 < -2\pi p \ \log 2 \ \sum_{j=1}^{\infty} e^{-pj2\pi} = -2\pi p \ \log 2 \frac{e^{-2\pi p}}{1-e^{-2\pi p}} \tag{2.10}$$

$$= -\frac{2\pi p \ \log 2}{e^{2\pi p} - 1}.$$

Thus, from (2.7), (2.8), (2.9) and (2.10), we have

$$-\frac{\pi p \ \log 2}{1 - e^{-2\pi p}} < r_\nu - \frac{1}{2}\log\frac{1}{2}(1 + \frac{k^2}{\omega^2}) < -\frac{\pi p \ \log 2}{e^{2\pi p} - 1}. \tag{2.11}$$

Hence, if $k \leq \omega$, $r_\nu < 0$. What happens for $k > \omega$? We have to calculate k from (2.11) to determine the sign of r_ν. From (2.11), if

$$\frac{1}{2}\log\frac{1}{2}\left(1+\frac{k^2}{\omega^2}\right) \geq \frac{\pi p \ \log 2}{1 - e^{-2\pi p}}$$

or

$$k \geq \omega \left[2 \exp\left(\frac{2\pi p \ \log 2}{1 - e^{-2\pi p}}\right) - 1\right]^{\frac{1}{2}} \tag{2.12}$$

then $r_\nu > 0$ and the system (2.6) is asymptotically unstable; while for

$$\frac{1}{2}\log\frac{1}{2}\left(1+\frac{k^2}{\omega^2}\right) \leq \frac{\pi p \ \log 2}{e^{2\pi p} - 1}$$

or

$$k \leq \omega \left[2 \exp\left(\frac{2\pi p \ \log 2}{e^{2\pi p} - 1}\right) - 1\right]^{\frac{1}{2}}, \tag{2.13}$$

we have $r_\nu < 0$ and the system (2.6) becomes asymptotically stable.

Example 2.3 (Randomly coupled harmonic oscillators) (cf. [25] for m=1). Let $y_{ij}(t)$, i, j=1,...,m, be independent processes which are regarded as formal derivatives of independent Poisson processes $N_{ij}(t)$ with intensities λ_{ij}, respectively.

Consider the following stochastic system of m coupled harmonic oscillators.

$$\ddot{z}_i(t) + \omega_i^2 z_i(t) = \sum_{j=1}^{m} b_{ij}\, y_{ij}(t) z_j(t) \tag{2.14}$$

$$z_i(0),\ \dot{z}_i(0) \text{ given},\ t \geq 0,\ \omega_i > 0,\ i = 1, \ldots, m.$$

Let $x_{2i-1}(t) = \omega z_i(t)$, $x_{2i}(t) = \dot{z}_i(t)$ and $x = [x_1, \ldots, x_{2m}]^T$. Then in standard notation

$$dx(t) = Ax(t)dt + \sum_{i,j=1}^{m} B_{ij}\, x(t) dN_{ij}(t) \tag{2.15}$$

where

$$A = \operatorname{diag}\{A_1, \ldots, A_m\}, \qquad A_i = \begin{pmatrix} 0 & \omega_i \\ -\omega_i & 0 \end{pmatrix},$$

and all the entries of B_{ij} are zero except the entry $e_{2i,2j-1} = \dfrac{b_{ij}}{\omega_i}$. Set

$$D_{ij} = I + B_{ij}.$$

Note that $\operatorname{tr}(A) = 0$ and $\det(D_{ij}) = 1$, so we have $D_{ij}\, e^{At} \in SL(2m)$. We can define a measure μ on $SL(2m)$ with density $\lambda_{ij}\, e^{-\lambda t}$, $t \geq 0$, $\lambda = \sum_{i,j=1}^{m} \lambda_{ij}$ at each element $D_{ij}\, e^{At}$. In this case, it is difficult to determine an invariant measure because the corresponding integral equation is hard to solve. However, we can use Furstenberg's theorem (Theorem 4.12) to show the rate $r > 0$. Let

$$G = \text{smallest subgroup containing } \{D_{ij}\, e^{At},\ 0 \leq t < \infty,\ i,j = 1,\ldots,m\}$$

$$= \text{smallest subgroup containing } \{D_{ij},\ i,j = 1,\ldots,m;\ e^{At},\ 0 \leq t < \infty\}.$$

Then G may not be transitive on S^{2m-1}. If we assume no two ω_i are equal, then the commutant Σ of the smallest subgroup G_1 containing $\{e^{At},\ t \geq 0\}$ is isomorphic to $\mathbb{C}^m$, i.e.

$T \in \Sigma$ if

$$T = diag\ \{T_1, \ldots, T_m\}$$

with

$$T_i = \begin{pmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{pmatrix}, \quad \alpha_i, \beta_i \in \mathbb{R}.$$

Since $Te^{At} = e^{At}T$, and T and e^{At} are normal, they preserve their eigenspace. Thus, the invariant subspaces V of G_1 are of the form $\mathbb{R}^2_{j_1} \times \cdots \times \mathbb{R}^2_{j_l}$, $l < m$.

Before verifying the hypotheses of Furstenberg's theorem, we need a non-degeneracy assumption:

(A) For any index set $J = \{j_1, \ldots, j_l\}$, $l < m$, there exists an $i \notin J$ such that $b_{ik} \neq 0$ for some $k \in J$.

By assumption (A), $\exists\ b_{ik} \neq 0$ so that the entry $e_{2i,2k-1}(D^j_{ik}) = j\dfrac{b_{ik}}{\omega_i}$ tends to infinity as $j \to \infty$. Thus, G is not compact.

Let an index set $J = \{j_1, \ldots, j_l\}$. By assumption (A), $\exists\ i \notin J$ such that $b_{ik} \neq 0$ for some $k \in J$. Then $D_{ik}\ V \not\subset V$. Hence, G is irreducible.

Note that G_1 is connected. There is no finite index subgroup of G_1. Thus, any finite index subgroup H of G must contain G_1 and some mixed powers of $\{D_{ij}\}$. Moreover, the irreducibility of G is due to sufficiently more non-zero entries of D_{ij}, not the exact value b_{ij}, so H is also irreducible.

In the cases where some ω_i are equal. The commutant Σ properly contains $\mathbb{C}^m$ and the invariant subspaces of G_1 are much more complicated.

Consequently, by Theorem 4.12, $r_\nu = r > 0$ and $x(t)$ grows exponentially a.s. This implies that all the states of all subsystems grow exponentially.

Remark. If assumption (A) does not hold, the system can be subdivided into proper subsystems $\{\Sigma_i\}$, which have property (A), and $\overline{\Sigma}$. States of Σ_i grow exponentially a.s. by the above arguments. The remaining subsystem $\overline{\Sigma}$ depends on Σ_i and its state thus grows exponentially a.s. Hence, the system of n coupled harmonic oscillators is asymptotically unstable.

In Brockett and Blankenship's paper [23], it was noted that any finite state, continuous time Markov processes (FSCT Markov processes) with infinitesimal transition probabilities matrix $A = \{a_{ij}\}_{i,j=1}^{m}$ defined by

$$\dot{p} = A\ p \tag{2.16}$$

where p_i is the probability that $z = z_i$, the i^{th} state, can be modeled by a stochastic differential equation of the form

$$dz(t) = \sum_{i=1}^{\delta} \phi_i(z) dN_i(t) \tag{2.17}$$

where z(t) $\in$ Z=$\{z_1, \ldots, z_m\} \subset \mathbb{C}$, $\delta = m(m-1)$. $\phi_i(z)$ are polynomials with degree $m-1$ interpolating exactly at points of Z such that

$$\phi_i(z) = \begin{cases} 0 & , z \neq z_1 \\ z_{i+1} - z_1 & , z = z_1 \end{cases} \qquad 1 \leq i \leq m-1$$

$$\phi_i(z) = \begin{cases} 0 & , z \neq z_2 \\ z_1 - z_2 & , z = z_2 \end{cases} \qquad i = m$$

$$\phi_i(z) = \begin{cases} 0 & , z \neq z_2 \\ z_{i-(m-1)+1} - z_2 & , z = z_2 \end{cases} \qquad m+1 \leq i \leq 2(m-1) \tag{2.18}$$

$$, \ldots ,$$

$$\phi_i(z) = \begin{cases} 0 & , z \neq z_m \\ z_{i-(m-1)^2} - z_m & , z = z_m \end{cases} \quad (m-1)^2+1 \leq i \leq m(m-1)$$

and N_i are independent Poisson processes with intensity λ_i, respectively, defined by

$$\lambda_i = \begin{cases} a_{i+1,1} & , 1 \leq i \leq m-1 \\ a_{12} & , i = m \\ a_{i-(m-1)+1,2} & , m+1 \leq i \leq 2(m-1) \\ \cdots & \cdots \\ a_{i-(m-1)^2,m} & , (m-1)^2+1 \leq i \leq m(m-1). \end{cases} \tag{2.19}$$

Let

$$\tilde{z} = \begin{pmatrix} z^1 \\ z^2 \\ \cdot \\ \cdot \\ \cdot \\ z^{m-1} \end{pmatrix}.$$

and observe that $d(z^k)$ can be expressed as

$$dz^k(t) = \sum_{i=1}^{\delta} \phi_i^k(z) dN_i(t)$$

where $\phi_i^k(z)$ is also a polynomial of degree $m-1$ such that replacing every z_j on the right hand side of (2.18) by z_j^k. Then there are constant matrices M_i such that

$$d\tilde{z}(t) = \sum_{i=1}^{\delta} M_i \tilde{z}\, dN_i(t). \tag{2.20}$$

If we consider a particular class of stochastic system as defined by

$$dx = A(z)xdt + C(z)dt \tag{2.21}$$

where $A(z)$ and $C(z)$ are polynomials of z, then

$$d\begin{pmatrix} 1 \\ x \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ C(z) & A(z) \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix} dt.$$

We introduce the tensor product as usual

$$\begin{pmatrix} 1 \\ x \end{pmatrix} \otimes \tilde{z} = \tilde{x} \triangleq \begin{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix} \\ z^1 \begin{pmatrix} 1 \\ x \end{pmatrix} \\ \cdot \\ \cdot \\ \cdot \\ z^{m-1} \begin{pmatrix} 1 \\ x \end{pmatrix} \end{pmatrix} .$$

Using the stochastic calculus for point processes, we can obtain a differential equation of the form

$$d\tilde{x} = \tilde{A}\,\tilde{x}\,dt + \sum_{i=1}^{\delta} \tilde{B}_i\,\tilde{x}\,dN_i \tag{2.22}$$

where $\tilde{A}$, $\tilde{B}_i$ are constant matrices. If we let

$$R = \begin{pmatrix} 0 & & 0 \ldots 0 \\ 0 & & 0 \ldots 0 \\ \cdot & I_n & \cdot \quad \cdot \\ \cdot & & \cdot \ldots \cdot \\ \cdot & & \cdot \quad \cdot \\ 0 & & 0 \ldots 0 \end{pmatrix} ,$$

then $x = R\tilde{x}$. Since $z(t)$ stays in the finite set Z, the stability of (2.21) can be obtained from that of (2.22), since our theory may be used to compute the rate r_ν for (2.22)

Example 2.4 (Random telegraph wave).

Let $z(t)$ be random telegraph wave which takes on the value set $Z = \{-1, 1\}$ with transition probability satisfying

$$\frac{d}{dt} \begin{pmatrix} p_1 \\ p_{-1} \end{pmatrix} = \begin{pmatrix} -\lambda & \lambda \\ \lambda & -\lambda \end{pmatrix} \begin{pmatrix} p_1 \\ p_{-1} \end{pmatrix} .$$

Then the differential equation for $z(t)$ becomes

$$dz(t) = -2z(t)dN(t) \tag{2.23}$$
$$z(0) = \pm 1$$

where $N(t)$ is a Poisson process with intensity λ. If we consider the state process

$$dx(t) = [k + \omega z(t)]\, x(t)dt \tag{2.24}$$
$$x(0) = x_0, \quad \omega > 0, \quad t \geq 0,$$

then using (2.23), (2.24) and the fact $z^2(t) = 1$, we get

$$\begin{aligned} d(zx) &= dz\, x + zdx \\ &= -2zxdN + z(k + \omega z)xdt \\ &= \omega xdt + kzxdt - 2zxdN. \end{aligned} \tag{2.25}$$

Combining (2.24) and (2.25), we have

$$d\begin{pmatrix} x \\ zx \end{pmatrix} = \begin{pmatrix} k & \omega \\ \omega & k \end{pmatrix} \begin{pmatrix} x \\ zx \end{pmatrix} dt + \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ zx \end{pmatrix} dN(t)\,. \tag{2.26}$$

Then,

$$expAt = e^{kt} \begin{pmatrix} \cosh \omega t & \sinh \omega t \\ \sinh \omega t & \cosh \omega t \end{pmatrix},$$

$$D = I + B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let SG be the smallest closed semi-group containing $\{De^{At},\, 0 \leq t < \infty\}$ and the measure μ is defined on SG with density $\lambda e^{-\lambda t}$, $t \geq 0$ at each element De^{At}. The corresponding invariant measure ν is difficult to calculate exactly and may not be unique since SG is not transitive on the circle S^0. However, SG is irreducible. By Theorem 4.12, the rate r is independent of ν.

Let

$$X(t) = De^{At} = e^{kt} \begin{pmatrix} \cosh \omega t & \sinh \omega t \\ -\sinh \omega t & -\cosh \omega t \end{pmatrix},$$

then

$$||X(t)||_2 = e^{kt} \ (\cosh 2\omega t \ + \ \sinh 2\omega t)^{1/2} = e^{(k+\omega)t},$$

and

$$r_1 = \int_0^\infty \log||X(t)||_2 \, \lambda e^{-\lambda t} \, dt$$

$$= \int_0^\infty (k+\omega)t \ \lambda e^{-\lambda t} \, dt$$

$$= \frac{k+\omega}{\lambda}.$$

Again, we calculate

$$X(t_2)X(t_1) = e^{k(t_1+t_2)} \begin{pmatrix} \cosh \omega(t_1-t_2) & \sinh \omega(t_1-t_2) \\ \sinh \omega(t_1-t_2) & \cosh \omega(t_1-t_2) \end{pmatrix}$$

with

$$||X(t_2)X(t_1)||_2 = e^{k(t_1+t_2)} [\cosh \omega(t_1-t_2) \ + \ \sinh \omega(t_1-t_2)]$$

$$= e^{k(t_1+t_2)} \ e^{\omega(t_1-t_2)},$$

so that

$$r_2 = \int_0^\infty\int_0^\infty \log||X(t_2)X(t_1)||_2 \, \lambda e^{-\lambda t_1} dt_1 \, \lambda e^{-\lambda t_2} dt_2$$

$$= \int_0^\infty\int_0^\infty [k(t_1+t_2) \ + \ \omega(t_1-t_2)] \lambda e^{-\lambda t_1} dt_1 \, \lambda e^{-\lambda t_2} dt_2$$

$$= \frac{2k}{\lambda} .$$

In general,

$$r_l = \int_0^\infty \cdots \int_0^\infty \log ||X(t_l) \cdots X(t_1)||_2 \, \lambda e^{-\lambda t_1} dt_1 \cdots \lambda e^{-\lambda t_l} dt_l$$

$$= \begin{cases} l\dfrac{k}{\lambda} + \dfrac{\omega}{\lambda}, & l \text{ is odd} \\ l\dfrac{k}{\lambda}, & l \text{ is even.} \end{cases}$$

Thus

$$r = \lim_{l \to \infty} \frac{r_l}{l} = \frac{k}{\lambda} .$$

From (2.26), we know that stability of (2.24) is equivalent to that of (2.26). Hence, the system (2.24) is asymptotically stable for $k < 0$ while it is asymptotically unstable for $k > 0$. This result shows that the random telegraph process $z(t)$ does not affect the stability of the corresponding deterministic system.

3. Products of Random Matrices and Almost Sure Stability

In this section we shall derive the expression for the asymptotic exponential growth rate for paths of the solutions of system (1.1). We begin with a result on a general stochastic Banach algebra with $||\cdot||$ as a norm.

Lemma 3.1 (Polya & Szego [11]). *Let* $\{a_k\} \subset \mathbb{R}$ *and* $\alpha \triangleq \inf_k \frac{a_k}{k}$. *If* $a_{k+l} \leq a_k + a_l$, *then* $\lim_{k \to \infty} \frac{a_k}{k} = \alpha$ *and* $\alpha \neq +\infty$.

Proof. Let $a_0 = 0$. Then

$$a_k \leq a_1 + a_{k-1} \leq \cdots \leq k a_1$$

or

$$\frac{1}{k} a_k \leq a_1$$

which implies that $\alpha < +\infty$. If $\alpha > -\infty$, let $\epsilon > 0$. Choose l so that $\frac{1}{l} a_l < \alpha + \epsilon$. Each integer $k \geq l$ can be written as $k = ql + r$, $0 \leq r \leq l-1$, $q \geq 0$, q, r integers. Let

$$c \triangleq \max\{ |a_0|, |a_1|, \ldots, |a_{l-1}| \} < \infty.$$

Then

$$\alpha \leq \frac{a_k}{k} = \frac{a_{ql+r}}{ql+r} \leq \frac{q a_l + a_r}{ql+r} \leq \frac{a_l}{l} + \frac{a_r}{k} \leq \frac{a_l}{l} + \frac{c}{k}.$$

Now, choose $k \geq [\frac{c}{\epsilon}] + 1$, we have $\alpha \leq \frac{a_k}{k} \leq \alpha + 2\epsilon$. Since ϵ is arbitrary, we know that $\frac{a_k}{k}$ converges and the limit is α. If $\alpha = -\infty$, a similar argument shows that $\frac{a_k}{k} \to -\infty$ as $k \to \infty$.

QED

The following theorem is adapted from Grenander [12 pp. 161].

Theorem 3.2. *If $\{X_i\}_{i=1}^{\infty}$ are independent and identically distributed stochastic elements in a Banach algebra with* E $\log^+||X_1|| < \infty$, *where* $\log^+||X_1|| \triangleq \max\{0, \log||X_1||\}$, *then the limit*

$$r \triangleq \lim_{k \to \infty} \frac{1}{k} E \log||X_k \cdots X_1|| < \infty \tag{3.1}$$

exists and

$$r = \overline{\lim_{k \to \infty}} \frac{1}{k} \log||X_k \cdots X_1|| < \infty \quad a.s. \tag{3.2}$$

Proof. Let

$$\alpha_k \triangleq E \log||X_k \cdots X_1|| \quad a.s.$$

Note that α_k are either finite or they are $-\infty$ after some k_0. In the latter case, r $= -\infty$. In case $r > -\infty$, then α_k are finite for every k and we use the stationarity of $\{X_i\}$. Evidently,

$$\begin{aligned}
\alpha_{k+l} &= E \log||X_{k+l} \cdots X_1|| \\
&\leq E \log(||X_{k+l} \cdots X_{k+1}|| \; ||X_k \cdots X_1||) \\
&= E \log ||X_{k+l} \cdots X_{k+1}|| + E \log ||X_k \cdots X_1|| \\
&= \alpha_l + \alpha_k .
\end{aligned}$$

By Lemma 3.1, the limit $\frac{1}{k} \alpha_k$ exists and is equal to r, where

$$r = \inf_k \frac{1}{k} E \log||X_k \cdots X_1||$$

with $-\infty \leq r < \infty$. Next, let

$$\xi_k \triangleq \frac{1}{k} \log||X_k \cdots X_1||.$$

If $r > -\infty$, let $\epsilon > 0$ be given. Choose l so that $\frac{1}{l} \alpha_l < r + \epsilon$. Once again, any integer $k \geq l$ can be expressed as $k = ql + s$, $0 \leq s \leq l-1$.

$$\xi_k = \frac{1}{k} \log||X_k \cdots X_1||.$$

$$\leq \frac{1}{k} \Big[\log||X_k \cdots X_{ql+1}|| + \log||X_{ql} \cdots X_{(q-1)l+1}|| + \cdots$$

$$+ \log||X_l \cdots X_1|| \Bigg]$$

$$\leq \frac{1}{k}\log||X_k \cdots X_{ql+1}|| + \frac{1}{q}\left[\frac{1}{l}\log||X_{ql} \cdots X_{(q-1)l+1}|| + \cdots \right.$$

$$\left. + \frac{1}{l}\log||X_l \cdots X_1|| \right]. \tag{3.3}$$

By the strong law of large numbers, the quantity in brackets tends to

$$\frac{1}{l} E \log||X_l \cdots X_1|| \quad as \quad q \to \infty \quad a.s.,$$

i.e. $\exists$ q_0 such that $q \geq q_0$ implies

$$\frac{1}{q}\left[\frac{1}{l}\log||X_{ql} \cdots X_{(q-1)l+1}|| + \frac{1}{l}\log||X_l \cdots X_1||\right]$$

$$\leq \frac{1}{l} E \log||X_l \cdots X_1|| + \epsilon$$

$$< r + 2\epsilon. \tag{3.4}$$

If $P\{\log ||X_k \cdots X_{ql+1}|| = -\infty\} > 0$, then $\alpha_k = -\infty$, and so, $r = -\infty$ which is a contradiction. Since E $\log^+||X_1||<\infty$ and $\{X_i\}$ are i.i.d., we have

$$\log||X_k \cdots X_{ql+1}|| \leq \log||X_k|| + \cdots + \log||X_{ql+1}|| < \infty \quad a.s. \tag{3.5}$$

Thus,

$$|\log ||X_k \cdots X_{ql+1}|| \,| < \infty \quad a.s.,$$

so that for k large enough, we have

$$\frac{1}{k}\log ||X_k \cdots X_{ql+1}|| < \epsilon \quad a.s. \tag{3.6}$$

From (3.3), (3.4) and (3.6), we have

$$\xi_k \leq r + 3\epsilon$$

for k large enough. Since ϵ is arbitrary, we have

$$\overline{\lim_{k \to \infty}} \; \xi_k \leq r \quad a.s. \tag{3.7}$$

Now let

$$\Delta_k \triangleq \frac{1}{k} \sum_{i=1}^{k} \log ||X_i|| - \frac{1}{k} \log ||X_k \cdots X_1|| \geq 0.$$

Then for $\{X_i\}$ i.i.d., we have

$$E \; \Delta_k = \frac{1}{k} \sum_{i=1}^{k} E \; \log ||X_i|| - \frac{1}{k} E \; \log ||X_k \cdots X_i||$$

$$= E \; \log ||X_1|| - \frac{1}{k} \alpha_k$$

$$\to E \; \log ||X_1|| - r \quad \text{as } k \to \infty.$$

Let

$$\Delta \triangleq \underline{\lim_{k \to \infty}} \; \Delta_k .$$

Then by the strong law of large numbers, we have

$$\Delta = E \; \log ||X_1|| - \overline{\lim_{k \to \infty}} \; \xi_k .$$

Applying Fatou's lemma, we obtain

$$E \; \Delta = E \; \log ||X_1|| - E \; \overline{\lim_{k \to \infty}} \; \xi_k$$

$$\leq \underline{\lim_{k \to \infty}} \; E \; \Delta_k$$

$$= E \; \log ||X_1|| - r.$$

Therefore, $E \; \overline{\lim_{k \to \infty}} \; \xi_k \geq r$. Combining with (3.7), we have the result $\overline{\lim_{k \to \infty}} \; \xi_k = r$ a.s.

If $r = -\infty$, by the strong law of large numbers, we have

$$\frac{1}{q}\left[\frac{1}{l}\log||X_{ql}\cdots X_{(q-1)l+1}|| + \frac{1}{l}\log||X_l\cdots X_1||\right]$$

$$\to \frac{1}{l}\log||X_l\cdots X_1|| \text{ as } q\to\infty.$$

From (3.3) and (3.5), we know that

$$\xi_k \to -\infty \quad a.s.$$

QED

Remark. A similar theorem for products of random matrices was proved by Furstenberg and Kesten [12] who obtained the result $\lim_{k\to\infty}\xi_k = r$ under the weaker assumptions that $||X_i||$ are stationary and metrically transitive by slightly different arguments. We will develop a more general theory for products of random matrices acting on initial vectors in next section.

Before proving the main theorem of this section, we need some elementary results for independent Poisson processes.

Lemma 3.3. *If $\{N_i(t), i=1,\ldots,m\}$ are independent Poisson processes with intensity λ_i, $i=1,\ldots,m$, respectively, then $N(t)=N_1(t)+\cdots+N_m(t)$ is also a Poisson process with intensity $\lambda=\lambda_1+\cdots+\lambda_m$. Let μ_j denote the index i for which N_i increases at the time $\tau_1+\cdots+\tau_j$. Then $\{\mu_j, j\geq 1\}$ are independent and identically distributed as*

$$P(\mu_j=i)=\frac{\lambda_i}{\lambda},\quad j\geq 1. \tag{3.8}$$

Furthermore, $\{\tau_i, \mu_j, i, j\geq 1\}$ are independent.

Proof. Let $\{\tau_j^i, j \geq 1\}$ be the interarrival times for $N_i(t)$, i=1,...,m and $\{\tau_j, j \geq 1\}$ be the interarrival times for the sum process N(t). Then for each i, $\{\tau_j^i, j \geq 1\}$ are independent and exponentially distributed with parameter λ_i. The assumption that $N_1(t), \ldots, N_m(t)$ are independent implies that $\tau_{j_1}^1, \ldots, \tau_{j_m}^m$ are independent. Now, we prove (3.8) by induction. First,

$$\begin{aligned}
&P[\mu_1 = i, \tau_1 < t] \\
&= P[\tau_1^i < t, \tau_1^i < \tau_1^j, j \in \{1,\ldots,m\} \setminus \{i\}] \\
&= \int_0^t dx_i \; \lambda_i e^{-\lambda_i x_i} \prod_{j \neq i} \int_{x_i}^{\infty} dx_j \; \lambda_j \; e^{-\lambda_j x_j} \\
&= \int_0^t dx_i \; \lambda_i e^{-\lambda_i x_i} \prod_{j \neq i} e^{-\lambda_j x_i} \\
&= \int_0^t \lambda_i e^{-\lambda x_i} \, dx_i \\
&= \frac{\lambda_i}{\lambda}(1 - e^{-\lambda t}).
\end{aligned}$$

Now, we assume the kth step is true,

$$P[\mu_1 = i_1, \tau_1 < t_1; \cdots; \mu_k = i_k, \tau_k < t_k] = \prod_{j=1}^{k} \frac{\lambda_{i_j}}{\lambda} (1 - e^{-\lambda t_j}). \tag{3.9}$$

For the collection $\{i_1, \ldots, i_k\} \subset \{1, \ldots, m\}$, let $j_1, \ldots, j_p$ be distinct integers such that $\{j_1, \ldots, j_p\} = \{i_1, \ldots, i_k\}$. Without loss of generality, we can assume $i_1 = j_1$ and $i_k = j_p$. Let l_{j_q} be the number of $i_u \in \{i_1, \ldots, i_k\}$ such that $i_u = j_q$. If $j_q \in \{1, \ldots, m\} \setminus \{j_1, \ldots, j_p\}$, set $l_{j_q} = 0$. Now, we define r_i as follows. Let r_1 be the first index such that $i_1 = i_2 = \cdots = i_{r_1} = j_1$ and $i_{r_1+1} \neq j_1$, and let r_2 be the index such that $i_{r_1+1} = \cdots = i_{r_1+r_2} \neq i_{r_1+r_2+1}$, etc.

Now we prove the (k+1)th step.

$$P\,[\mu_1 = i_1,\ \tau_1 < t_1\ ;\ \cdots\ ;\ \mu_k = i_k\ ,\ \tau_k < t_k\ \ ;\ \mu_{k+1} = i_{k+1},\ \tau_{k+1} < t_{k+1}]$$

$$= P\Big[\tau_1^{j_1} + \ldots + \tau_{r_1}^{j_1} < \tau_1^{j_2},\ \tau_1^{j_1} < t_1\ ,\ \ldots\ ,\ \tau_{r_1}^{j_1} < t_{r_1},\ \tau_1^{j_2} - (\tau_1^{j_1} + \ldots + \tau_{r_1}^{j_1}) < t_{r_1+1}\ ;\ \cdots\ ;$$

$$\tau_1^{j_p} + \ldots + \tau_{l_{j_p}}^{j_p} < \tau_1^{i_{k+1}} + \ldots + \tau_{l_{i_{k+1}}+1}^{i_{k+1}},\ \tau_1^{i_{k+1}} + \ldots + \tau_{l_{i_{k+1}}+1}^{i_{k+1}} - (\tau_1^{j_p} + \ldots + \tau_{l_{j_p}}^{j_p}) < t_{k+1};$$

$$\tau_1^{i_{k+1}} + \ldots + \tau_{l_{i_{k+1}}+1}^{i_{k+1}} < \tau_1^{q} + \ldots + \tau_{l_q+1}^{q},\ \forall\ \ q \in \{1, \ldots, m\} \setminus \{i_{k+1}\}\Big]$$

$$= P\Big[0 < \tau_1^{j_1} < t_1,\ \ldots\ ,\ 0 < \tau_{r_1}^{j_1} < t_{r_1};\ \tau_1^{j_1} + \ldots + \tau_{r_1}^{j_1} < \tau_1^{j_2} < t_{r_1+1} + \tau_1^{j_1} + \ldots + \tau_{r_1}^{j_1}\ ;\ \cdots\ ;$$

$$\tau_1^{j_p} + \ldots + \tau_{l_{j_p}}^{j_p} - (\tau_1^{i_{k+1}} + \ldots + \tau_{l_{i_{k+1}}}^{i_{k+1}}) < \tau_{l_{i_{k+1}}+1}^{i_{k+1}} < t_{k+1} + (\tau_1^{j_p} + \ldots + \tau_{l_{j_p}}^{j_p}) - (\tau_1^{i_{k+1}} + \ldots + \tau_{l_{i_{k+1}}}^{i_{k+1}});$$

$$\tau_1^{i_{k+1}} + \ldots + \tau_{l_{i_{k+1}}+1}^{i_{k+1}} - (\tau_1^{q} + \ldots + \tau_{l_q}^{q}) < \tau_{l_q+1}^{q},\ \forall\ \ q \in \{1, \ldots, m\} \setminus \{i_{k+1}\}\Big]$$

$$= \prod_{u=1}^{r_1} \int_0^{t_u} dx_u^{j_1}\, \lambda_{j_1} exp\,(-\lambda_{j_1} x_u^{j_1}) \int_{x_1^{j_1} + \ldots + x_{r_1}^{j_1}}^{t_{r_1+1} + (x_1^{j_1} + \ldots + x_{r_1}^{j_1})} dx_1^{j_2}\, \lambda_{j_2}\, exp\,(-\lambda_{j_2} x_1^{j_2})\ \cdots$$

$$\int_{x_1^{j_p} + \ldots + x_{l_{j_p}}^{j_p} - (x_1^{i_{k+1}} + \ldots + x_{l_{i_{k+1}}}^{i_{k+1}})}^{t_{k+1} + (x_1^{j_p} + \ldots + x_{l_{j_p}}^{j_p}) - (x_1^{i_{k+1}} + \ldots + x_{l_{i_{k+1}}}^{i_{k+1}})} dx_{l_{i_{k+1}}+1}^{i_{k+1}}\ \ \lambda_{i_{k+1}}\ exp\,(-\lambda_{i_{k+1}} x_{l_{i_{k+1}}+1}^{i_{k+1}})$$

$$\prod_{q \neq i_{k+1}} \int_{x_1^{i_{k+1}} + \ldots + x_{l_{i_{k+1}}+1}^{i_{k+1}} - (x_1^q + \ldots + x_{l_q}^q)}^{\infty} dx_{l_q+1}^{q}\ \ \lambda_q\ exp\,(-\lambda_q\, x_{l_q+1}^{q}). \tag{3.10}$$

Set

$$y_l^i = x_1^i\ +\ \cdots + x_l^i.$$

We calculate the tail of the above integration as follows:

$$I(x)\ \triangleq \int_{y_{l_{j_p}}^{j_p} - y_{l_{i_{k+1}}}^{i_{k+1}}}^{t_{k+1} + y_{l_{j_p}}^{j_p} - y_{l_{i_{k+1}}}^{i_{k+1}}} dx_{l_{i_{k+1}}+1}^{i_{k+1}}\ \ \lambda_{i_{k+1}}\ exp\,(-\lambda_{i_{k+1}} x_{l_{i_{k+1}}+1}^{i_{k+1}})$$

$$\cdot \prod_{q \neq i_{k+1}} \int_{y_{l_{i_{k+1}}}^{i_{k+1}}+1 - y_{l_q}^{q}}^{\infty} dx_{l_q+1}^{q} \; \lambda_q \, exp\,(-\lambda_q \, x_{l_q+1}^{q})$$

$$= \int_{y_{l_{j_p}}^{j_p} - y_{l_{i_{k+1}}}^{i_{k+1}}}^{t_{k+1} + y_{l_{j_p}}^{j_p} - y_{l_{i_{k+1}}}^{i_{k+1}}} dx_{l_{i_{k+1}}+1}^{i_{k+1}} \; \lambda_{i_{k+1}} \, exp\,(-\lambda_{i_{k+1}} x_{l_{i_{k+1}}+1}^{i_{k+1}}) \prod_{q \neq i_{k+1}} exp\,[-\lambda_q \,(y_{l_{i_{k+1}}+1}^{i_{k+1}} - y_{l_q}^{q})]$$

$$= \int_{y_{l_{j_p}}^{j_p} - y_{l_{i_{k+1}}}^{i_{k+1}}}^{t_{k+1} + y_{l_{j_p}}^{j_p} - y_{l_{i_{k+1}}}^{i_{k+1}}} dx_{l_{i_{k+1}}+1}^{i_{k+1}} \; \lambda_{i_{k+1}} \, exp\,(-\lambda x_{l_{i_{k+1}}+1}^{i_{k+1}}) \prod_{q \neq i_{k+1}} exp\,[-\lambda_q \,(y_{l_{i_{k+1}}}^{i_{k+1}} - y_{l_q}^{q})]$$

$$= \frac{\lambda_{i_{k+1}}}{\lambda} \,(1-e^{-\lambda t_{k+1}})\, exp\,[-\lambda(y_{l_{j_p}}^{j_p} - y_{l_{i_{k+1}}}^{i_{k+1}})] \prod_{q \neq i_{k+1}} exp\,[-\lambda_q \,(y_{l_{i_{k+1}}}^{i_{k+1}} - y_{l_q}^{q})]$$

$$= \frac{\lambda_{i_{k+1}}}{\lambda} \,(1-e^{-\lambda t_{k+1}}) \prod_{q \neq j_p} exp\,[-\lambda_q \,(y_{l_{j_p}}^{j_p} - y_{l_q}^{q})]$$

$$= \frac{\lambda_{i_{k+1}}}{\lambda} \,(1-e^{-\lambda t_{k+1}}) \prod_{q \neq j_p} \int_{y_{l_{j_p}}^{j_p} - y_{l_q}^{q}}^{\infty} \lambda_q \, exp\,(-\lambda_q \, x_{l_q+1}^{q}) \; dx_{l_q+1}^{q}. \qquad (3.11)$$

From the final result of (3.10) and (3.11), we get

$$P\,[\mu_1 = i_1,\; \tau_1 < t_1\,;\; \cdots\,;\; \mu_k = i_k\,,\; \tau_k < t_k\;;\; \mu_{k+1} = i_{k+1},\; \tau_{k+1} < t_{k+1}]$$

$$= \prod_{u=1}^{r_1} \int_0^{t_u} dx_u^{j_1} \, \lambda_{j_1} exp\,(-\lambda_{j_1} x_u^{j_1}) \int_{y_{r_1}^{j_1}}^{t_{r_1+1} + y_{r_1}^{j_1}} dx_1^{j_2} \; \lambda_{j_2} \, exp\,(-\lambda_{j_2} x_1^{j_2}) \;\cdots$$

$$\prod_{q \neq j_p} \int_{y_{l_{j_p}}^{j_p} - y_{l_q}^{q}}^{\infty} dx_{l_{q+1}+1}^{q} \; \lambda_q \, exp\,(-\lambda_q \, x_{l_q+1}^{q}) \frac{\lambda_{i_{k+1}}}{\lambda} \,(1-e^{-\lambda t_{k+1}})$$

$$= P\,[0 < \tau_1^{j_1} < t_1,\; \ldots,\; 0 < \tau_{r_1}^{j_1} < t_{r_1};\; \tau_1^{j_1} + \ldots + \tau_{r_1}^{j_1} < \tau_1^{j_2} < t_{r_1+1} + \tau_1^{j_1} + \ldots + \tau_{r_1}^{j_1}\,;\; \cdots\,;$$

$$\tau_1^{j_p}+\ldots+\tau_{l_{j_p}}^{j_p}-(\tau_1^q+\ldots+\tau_{l_q}^q)<\tau_{l_q+1}^q, \forall\, q\neq j_p] \cdot \frac{\lambda_{i_{k+1}}}{\lambda}(1-e^{-\lambda t_{k+1}})$$

$$= P[\mu_1=i_1,\ \tau_1<t_1\ ; \cdots ;\ \mu_k=i_k\ ,\ \tau_k<t_k] \cdot \frac{\lambda_{i_{k+1}}}{\lambda}(1-e^{-\lambda t_{k+1}})$$

$$= \prod_{j=1}^{k+1}\frac{\lambda_{i_j}}{\lambda}(1-e^{-\lambda t_j})$$

by the induction hypothesis. Thus, we have proved (3.9) for any integer k.

Let $t_i \to\infty$, $i=1,\ldots,k$, we obtain

$$P[\mu_1=i_1, \ldots, \mu_k=i_k] = \prod_{j=1}^{k}\frac{\lambda_{i_j}}{\lambda}\ . \tag{3.12}$$

Thus,

$$P[\mu_k=i_k] = \sum_{i_{k-1}=1}^{m}\cdots\sum_{i_1=1}^{m} P[\mu_1=i_1, \ldots, \mu_k=i_k]$$

$$= \sum_{i_{k-1}=1}^{m}\cdots\sum_{i_1=1}^{m}\prod_{j=1}^{k}\frac{\lambda_{i_j}}{\lambda}$$

$$= \sum_{i_{k-1}=1}^{m}\cdots\sum_{i_2=1}^{m}\prod_{j=2}^{k}\frac{\lambda_{i_j}}{\lambda}$$

$$= \frac{\lambda_{i_k}}{\lambda}\ . \tag{3.13}$$

Since k is arbitrary, we have

$$P[\mu_1=i_1, \ldots, \mu_k=i_k] = \prod_{j=1}^{k}\frac{\lambda_{i_j}}{\lambda} = \prod_{j=1}^{k}P[\mu_j=i_j],$$

so that $\{\mu_j\}$ are independent and identically distributed as (3.13). Also from (3.9), we get

$$P[\tau_1<t_1, \ldots, \tau_k<t_k] = \sum_{i_k=1}^{m}\cdots\sum_{i_1=1}^{m} P[\mu_1=i_1,\ \tau_1<t_1\ ; \cdots ;\ \mu_k=i_k\ ,\ \tau_k<t_k]$$

$$= \sum_{i_k=1}^{m} \cdots \sum_{i_1=1}^{m} \prod_{j=1}^{k} \frac{\lambda_{i_j}}{\lambda}(1-e^{-\lambda t_j})$$

$$= \prod_{j=1}^{k} (1-e^{-\lambda t_j}). \tag{3.14}$$

Thus,

$$P[\tau_k < t_k] = \lim_{\substack{t_j \to \infty \\ j=1,\ldots,k-1}} P[\tau_1 < t_1, \ldots, \tau_{k-1} < t_{k-1}, \tau_k < t_k] \tag{3.15}$$

$$= 1 - e^{-\lambda t_k}.$$

From (3.14) and (3.15), we have

$$P[\tau_1 < t_1, \ldots, \tau_k < t_k] = \prod_{j=1}^{k} (1-e^{-\lambda t_j}) = \prod_{j=1}^{k} P[\tau_j < t_j].$$

This shows that $\{\tau_j\}$ are independent and exponentially distributed as in (3.15) with intensity λ. Consequently, $N(t)$ is a Poisson process with parameter λ. Furthermore, any collections of $\{\tau_i\}$, $\{\mu_j\}$ are independent by (3.9), (3.13) and (3.15).

QED

Lemma 3.4. $\lim_{t \to \infty} \frac{1}{t} N(t) = \lambda$ *a.s.*

Proof. Let $t_k = \sum_{i=1}^{k} \tau_i$ be the waiting time for k renewals. $N(t)$ is the number of occurrences in $[0,t]$. Then $t_{N(t)} \leq t < t_{N(t)+1}$, so that

$$\frac{t_{N(t)}}{N(t)} \leq \frac{t}{N(t)} < \frac{t_{N(t)+1}}{N(t)} \tag{3.16}$$

Since

$$P(N(t) \le k) = \sum_{i=0}^{k} \frac{(\lambda t)^k}{k!} e^{-\lambda t} \to 0 \quad as \quad t \to \infty,$$

it follows that $N(t) \to +\infty$ as $t \to \infty$ a.s. By the strong law of large numbers, we have

$$\lim_{t \to \infty} \frac{t_{N(t)}}{N(t)} = E\tau_1 = \frac{1}{\lambda} \quad a.s.$$

and

$$\lim_{t \to \infty} \frac{t_{N(t)+1}}{N(t)} = \lim_{t \to \infty} \frac{t_{N(t)+1}}{N(t)+1} \cdot \frac{N(t)+1}{N(t)} = \frac{1}{\lambda} \quad a.s.$$

Thus, by (3.16), the result follows

$$\lim_{t \to \infty} \frac{t}{N(t)} = \frac{1}{\lambda} \quad a.s.$$

QED

We are now in a position to prove the main result.

Theorem 3.5. *Consider the stochastic system* (1.1) *with the stated assumptions on the processes* $N_i(t)$, $i=1, \ldots, m$. *Then*

$$r \triangleq \lim_{k \to \infty} \frac{1}{k} E \log \left|\left| D_{\mu_k} e^{A\tau_k} \ldots D_{\mu_1} e^{A\tau_1} \right|\right| < \infty \tag{3.17}$$

exists and

$$r = \overline{\lim_{k \to \infty}} \frac{1}{k} \log \left|\left| D_{\mu_k} e^{A\tau_k} \ldots D_{\mu_1} e^{A\tau_1} \right|\right| \quad a.s. \tag{3.18}$$

Consequently, if $r < 0$, *then the system* (1.1) *is asymptotically stable almost surely. If* $r > 0$, *it is unstable.*

Proof. By Lemma 3.3, we know that $\{\tau_i\}$, $\{\mu_j\}$ are i.i.d. and any collections of them are independent, so that $\{D_{\mu_i} e^{A\tau_i}\}$ forms an i.i.d. sequence. Evidently,

$$
\begin{aligned}
E\ \log^+ \| D_{\mu_1} e^{A\tau_1} \| &\leq E\ \log^+ (\| D_{\mu_1} \| e^{\|A\|\tau_1}) \\
&\leq E\ \log^+ \| D_{\mu_1} \| + \|A\| E(\tau_1) \\
&\leq \frac{1}{\lambda} (\sum_{i=1}^{m} \lambda_i \log^+ \|D_i\| + \|A\|) < \infty .
\end{aligned}
$$

By Theorem 3.2, r exists and (3.18) holds. Also,

$$
\begin{aligned}
\log\ |x(t)| &= \log\ | exp\,[A\,(t - t_{N(t)})] D_{\mu_{N(t)}} exp\,(A\,\tau_{N(t)}) \cdots D_{\mu_1} exp\,(A\,\tau_1) x_0 | \\
&\leq \log\ |x_0| + \log\ \| exp\,[A\,(t - t_{N(t)})] \| \\
&+ \log\ \| D_{\mu_{N(t)}} exp\,(A\,\tau_{N(t)}) \cdots D_{\mu_1} exp\,(A\,\tau_1) \| \qquad (3.19)
\end{aligned}
$$

Since $t_{N(t)} \leq t < t_{N(t)+1}$, then $0 \leq t - t_{N(t)} \leq \tau_{N(t)+1}$ and $P(\tau_{N(t)} < \infty) = 1$, so that

$$
\begin{aligned}
\frac{1}{t} \log\ \| exp\,[A\,(t - t_{N(t)})] \| &\leq \frac{1}{t} \log\ \| exp\,(A\,\tau_{N(t)+1}) \| \\
&\leq \frac{1}{t} \|A\|\, \tau_{N(t)+1} \\
&\to 0 \text{ as } t \to +\infty \quad a.s.
\end{aligned}
$$

and for $x_0 \neq 0$,

$$
\lim_{t \to \infty} \frac{1}{t} \log |x_0| = 0.
$$

By (3.18), we have

$$
\begin{aligned}
&\overline{\lim_{t \to \infty}} \frac{1}{t} \log\ \| D_{\mu_{N(t)}} exp\,(A\,\tau_{N(t)}) \cdots D_{\mu_1} exp\,(A\,\tau_1) \| \\
&= \overline{\lim_{t \to \infty}} \frac{1}{N(t)} \log\ \| D_{\mu_{N(t)}} exp\,(A\,\tau_{N(t)}) \cdots D_{\mu_1} exp\,(A\,\tau_1) \| \cdot \frac{N(t)}{t} \\
&= r\lambda \qquad a.s.
\end{aligned}
$$

Thus from (3.19), we obtain

$$\overline{\lim_{t \to \infty}} \frac{1}{t} \log |x(t)| \leq r\lambda \quad a.s. \tag{3.20}$$

Since we can take a time sequence $t_k = \tau_1 + \cdots + \tau_k$ and for any ball $B(R) = \{x \in \mathbb{R}^n, |x| \leq R\}$, we have

$$\sup_{x_0 \in B(R)} \overline{\lim_{t \to \infty}} \frac{1}{t} \log |x(t)| = r\lambda \quad a.s.$$

Hence, if $r < 0$, then $|x(t)| \to 0$ a.s. for any initial state. If $r > 0$, then $\exists x_0$ such that

$$\lim_{t \to \infty} |x(t)| = \infty \quad a.s.$$

and then the system (1.1) is unstable.

QED

Remark. In the critical case $r = 0$, we have to investigate the order of $\log |x(t)|$. We conjecture that it is unstable.

4. Random Walks and Almost Sure Stability.

In the previous section, we derived an almost sure stability theorem based on the asymptotic growth rate of products of random matrices. In this section we are interested in the behavior of random products of matrices acting on initial states x_0, i.e., does the limit of

$$\frac{1}{k} \log ||D_{\mu_k} e^{A\tau_k} \cdots D_{\mu_1} e^{A\tau_1} x_0||$$

exist? If it does, is it equal to the rate r computed in the last section? In order to obtain a more precise result, we would like to treat random products of matrices in some way as random walks on the sphere. Results for random walks on semi-simple Lie groups are known in H. Furstenberg's papers [15], [16] and [18], see also [14] and [17] for related results. Multiplicative ergodic results of products of non-singular matrices can also be found in [19] and [20], see also [21]. In this section, we generalize these to general semi-groups, since the terms D_i arising in our model may not be non-singular.

Consider SG as a topological semi-group of $n \times n$ real matrices such that $(g_1, g_2) \to g_1 g_2$, $g_1, g_2 \in SG$, is continuous in the matrix norm sense. Let μ be a regular probability measure on SG. Without loss of generality, we will assume SG is the closed semi-group generated by the support of μ. Define a SG action on $M \triangleq S^{n-1} \cup \{0\}$, where S^{n-1} is a unit sphere in $\mathbb{R}^n$. If $g \in SG$, $x \in M$, then

$$g \circ x = \begin{cases} \dfrac{gx}{||gx||} & \text{if } ||gx|| > 0 \\ 0 & \text{else} \end{cases} \tag{4.1}$$

where gx is the product of a matrix g and a vector $x \in \mathbb{R}^n$.

Definition. A regular, Borel, probability measure ν on M is said to be an *invariant measure* of μ if $\mu * \nu = \nu$, i.e.

$$\int_{SG \times M} f(g \circ x) d\mu(g) d\nu(x) = \int_M f(x) d\nu(x)$$

for any continuous function f on M. We denote $\nu(f) \triangleq \int_M f(x) d\nu(x)$.

Remark. The measure $\overline{\nu}$ which has support $\{0\}$ is always an invariant measure of μ.

Lemma 4.1. *If there exists an $x_0 \in S^{n-1}$ such that $g_i \cdots g_1 x_0 \neq 0$, $\forall$ $g_j \in supp\, \mu$, i.e. $0 \notin SG$, then there exists an invariant probability measure $\nu \neq \overline{\nu}$ of μ on M with $supp\, \nu \subseteq S^{n-1}$.*

Proof. Let ν_0 be any probability measure on M such that $\nu_0(\{x_0\}) = 1$ and $\nu_0(M \backslash \{x_0\}) = 0$. Consider the sequence $\nu_k = \frac{1}{k}\sum_{i=0}^{k-1} \mu^i{}_* \nu_0$, $k \geq 1$ where μ^i is i-fold convolution of μ. For each continuous function f on M, we have

$$\nu_k(f) = \int_M f(x)\,d\nu_k(x)$$

$$= \frac{1}{k}\sum_{i=0}^{k-1} \int_{SG \times \underset{i\ times}{\cdots} \times SG \times M} f(g_i \ldots g_1 o x)\,d\mu(g_i) \cdots d\mu(g_1)\,d\nu_0(x).$$

$$= \frac{1}{k}\sum_{i=0}^{k-1} \int_{SG \times \underset{i\ times}{\cdots} \times SG} f(g_i \ldots g_1 o x_0)\,d\mu(g_i) \cdots d\mu(g_1). \tag{4.2}$$

Since M is compact, f is bounded by its sup norm $\|f\| < \infty$, so that $|\nu_k(f)| \leq \|f\|$, $\forall k$. By the Banach-Alaoglu Theorem, there exists a subsequence $\nu_l \to \nu$, a probability measure, in the sense $\nu_l(f) \to \nu(f)$ for each continuous f on M. Since

$$\mu_* \nu_l - \nu_l = \frac{l+1}{l}\,\nu_{l+1} - \frac{1}{l}\,\nu_0 - \nu_l$$

$$= (\nu_{l+1} - \nu_l) + \frac{1}{l}(\nu_{l+1} - \nu_0) \to 0,$$

as $l \to \infty$, we have $\mu_* \nu = \nu$. Finally, if $0 \notin SG$, then from (4.2), we see that $\nu_l(\{0\}) = 0$, so that $\nu(\{0\}) = 0$.

QED

Remark. If every element in *supp* μ is non-singular, then the conclusion of Lemma 4.1 holds.

Let Q be a collection of all the invariant probability measures of μ. Then Q is non-empty convex and compact in the weak$*$ topology. By the Krein-Milman Theorem, Q is equal to the closed convex hull of Q_0, the set of its extreme elements. Let $\{X_i\}$ be independent SG-valued random variables with common distribution μ. We define

$$W_k(u) \triangleq X_k \cdots X_1 u, \quad u \in M \tag{4.3}$$

and a random walk when $W_k(u)$ is projected on M. Let Z_0 be a random variable which is independent of $\{X_i\}$ and has distribution $\nu \in Q_0$. We define a random process as follows

$$Z_k \triangleq X_k \circ Z_{k-1}. \tag{4.4}$$

By induction Z_k has distribution $\mu_* \nu = \nu$, $k \geq 1$. $\{Z_k\}$ is stationary since $\{Z_k, Z_{k+1}, \ldots, Z_{k+m}\}$ is determined by the distribution ν of Z_k and the transition probability

$$\mu_x \triangleq \mu_* \delta_x, \quad x \in M$$

which is independent of k. Since X_{k+1} is independent of $\{X_k, Z_k, \ldots, X_1, Z_1, Z_0\}$, it is easy to check $\{X_{k+1}, Z_k\}$ is a stationary Markov process. We have the following lemma.

Lemma 4.2. *If ν is an extremal invariant probability measure of μ, then the corresponding process $\{Z_k\}$ is ergodic in the sense that invariant random variables of the shift operator w.r.t. the above process are constant. Thus, the ergodic theorem holds, i.e.,*

$$\lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} f(Z_i) = \nu(f) \quad a.s. \tag{4.5}$$

for any continuous function f on M. In addition, if ν', $\nu'' \in Q_0$. then either $\nu' = \nu''$ or they are mutually singular.

Proof. If $\{Z_k\}$ is not ergodic, then from [22, Chapter X, Theorem 1.1, pp. 460], there exists a non-constant bounded invariant random variable which is a measurable function of Z_0, say $\phi(Z_0)$. By stationarity, $\phi(Z_k) = \phi(Z_0)$, $k \geq 1$. Since we can translate and multiply ϕ by constants without violating the measurability of Z_0, we can assume $\epsilon \leq \phi \leq 1-\epsilon$ for some $\epsilon > 0$ and ϕ is not a constant function. Let ν_1 be such that $d\nu_1 \triangleq \phi d\nu$. Then

$$\begin{aligned}
\mu_* \nu_1(f) &= \int_{SG \times M} f(g \circ x)\phi(x) d\mu(g) d\nu(x) \\
&= E(f(Z_1)\phi(Z_0)) \\
&= E(f(Z_1)\phi(Z_1)) \\
&= \int_M f(x)\phi(x) d\nu(x) \\
&= \nu_1(f).
\end{aligned}$$

Let $c \triangleq \int_M \phi(x) d\nu(x)$. Since $0 < \phi < 1$. Then $0 < c < 1$ and $\frac{1}{c}\nu_1 \in Q$. In the same manner, let ν_2 be such that $d\nu_2 \triangleq (1-\phi)d\nu$. Then $\frac{1}{1-c}\nu_2 \in Q$. Since ϕ is not constant, $\frac{1}{c}\nu_1 \neq \nu$. But,

$$\nu = \nu_1 + \nu_2 = c\left(\frac{1}{c}\nu_1\right) + (1-c)\left(\frac{1}{1-c}\nu_2\right)$$

which contradicts the extremality of ν. By the strong law of large numbers [22, Chapter V, Theorem 6.1, pp. 219], we have the desired result (4.5). Now, if $\nu', \nu'' \in Q_0$, $\nu' \neq \nu$ and B $= supp\, \nu' \bigcap supp\, \nu'' \neq \emptyset$, then there exists $B_1 \subset B$ such that $\nu'(B_1) \neq \nu''(B_1)$. From (4.5), we have for u $\in$ B,

$$\nu'(B_1) = \lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} \chi_{B_1}(X_i \cdots X_1 o u) = \nu''(B_1)$$

which is a contradiction. Thus, $supp\, \nu' \bigcap supp\, \nu''$ is of measure zero w.r.t. both ν' and ν''. Hence ν' and ν'' are mutually singular.

QED

Remark. Since $\{X_i\}$ is i.i.d., then $\{X_i\}$ is ergodic by [22, Chapter X, Theorem 1.2, pp. 460]. If $\{Z_i\}$ is the process corresponding to $\nu \in Q_0$, then $\{Z_i\}$ is also ergodic by Lemma 4.2, so that $\{(X_{k+1}, Z_k)\}$ is a stationary Markov ergodic process. Hence, we can apply the ergodic theorem [22, Chapter V, Theorem 6.1, pp. 219] to conclude that

$$\lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} f(X_i, Z_{i-1}) = \mu \times \nu(f) \quad a.s. \tag{4.6}$$

for all f defined on $SG \times M$ such that $\mu \times \nu(f^+) < \infty$, where $f^+ = \frac{1}{2}(\,|f| + f\,)$.

Lemma 4.3. *If $\nu \in Q$, then $E =$ supp ν is a closed invariant set, i.e., $SG \circ E \subset E$. Conversely, if E is a closed invariant set, then $\exists\ \nu \in Q$ such that supp $\nu \subset E$.*

Proof. If $\nu = \mu * \nu$, i.e.

$$\int_M f(x)\,d\nu(x) = \int_{SG \times M} f(g \circ x)\,d\mu(g)\,d\nu(x) \tag{4.7}$$

for all continuous function f on M. Let $E = supp\ \nu$ and

$$H \triangleq \{g \in SG \mid g \circ E \subset E\}. \tag{4.8}$$

Consider $f = \chi_E$. Then (4.7) becomes

$$1 = \int_{SG \times E} \chi_E(g \circ x) d\mu(g) d\nu(x).$$

Thus, $\mu(\mathrm{H}) = 1$. Since E is closed, H is a closed sub semi-group in SG. Since SG is the smallest closed sub semi-group that can support μ, we have $H = SG$ and E is an invariant set. Conversely, if E is a closed invariant set in M, let ν_0 be any probability measure on M with support contained in E. Then

$$\mu^k {}_*\nu_0(E) = \int_{SG \times \underset{k \text{ times}}{\cdots} \times SG \times E} \chi_E(g_k \cdots g_1 \circ x) d\mu(g_k) \cdots d\mu(g_1) d\nu_0(x) = 1$$

for $SG \circ E \subset E$. Thus supp $(\mu^k {}_*\nu_0) \subset E$. By the argument used to prove Lemma 4.1, $\nu_k \triangleq \frac{1}{k}\sum_{i=0}^{k-1} \mu^i {}_*\nu_0 \to \nu$, say, and $\mu_*\nu = \nu$. Since supp $\nu_k \subset E$, $\forall k$, and E is closed, we have supp $\nu \subseteq$ E.

QED

Lemma 4.4. *If SG is transitive on S^{n-1}, i.e., $g \circ x = y$ always has a solution $g \in$ SG for all x, $y \in \mathrm{S}^{n-1}$, then $Q_0 \setminus \{\overline{\nu}\}$ has at most one invariant measure ν_0 of μ such that supp $\nu_0 = \mathrm{S}^{n-1}$.*

Proof. If $\exists \nu_0 \in Q_0 \setminus \{\overline{\nu}\}$, let $E = supp\ \nu_0$. Then E is an invariant set. Since SG is transitive on S^{n-1}, we have $E \supseteq \mathrm{S}^{n-1}$. Since $\overline{\nu} \in Q_0$ and different invariant probability measures of μ in Q_0 are mutually singular, we have $supp\ \nu_0 = S^{n-1}$.

QED

Remark. We have established the one to one correspondence between an invariant probability measure ν and an invariant set $E_\nu \triangleq supp\ \nu$. If $\nu \in Q_0$, then the interior of E_ν are disjoint. Let $F = \mathrm{S}^{n-1} \setminus \bigcup_{\nu \in Q_0} E_\nu$. We call F a *transient set*. Thus, we have a partition of S^{n-1} into invariant sets $\{E_\nu\}$ and a transient set F. Since F cannot contain an invariant set, every states in F will eventually go to $\bigcup_{\nu \in Q_0} E_\nu$ by actions of SG.

Remark. From Lemma 4.3, we know that μ has a unique invariant probability measure on S^{n-1} iff there is only one distinct invariant set on S^{n-1}.

Combining the above lemmas, we obtain a key result.

Theorem 4.5. *Let SG be a closed sub semi-group of $n \times n$ matrices and $\nu \in Q_0$ be an extremal invariant probability measure of μ on $M = \mathrm{S}^{n-1} \cup \{0\}$. Assume that*

$$\int_{SG \times M} \log^+ |\, gu \,|\, d\,\mu(g)\,d\,\nu(u) < \infty$$

and let

$$r_\nu = \int_{SG \times M} \log |\, gu \,|\, d\,\mu(g)\,d\,\nu(u) < \infty. \tag{4.9}$$

If $\{X_i\}$ is i.i.d. with common distribution μ, then we have almost all $u \in \operatorname{supp} \nu$,

$$\lim_{k \to \infty} \frac{1}{k} \log |\, W_k(u) \,| = r_\nu \quad a.s. \tag{4.10}$$

Proof. Consider

$$f(X_{k+1}, Z_k) = \begin{cases} \log |\, X_{k+1} Z_k \,| &, \ |\, Z_k(\omega) \,| = 1 \\ -\infty &, \ Z_k(\omega) = 0 \end{cases}$$

on $SG \times M$. Then $f^+(X_{k+1}, Z_k)$ is integrable by assumption. Since $\{X_{k+1}, Z_k\}$ is ergodic, the law of large numbers tells us

$$r_\nu = \mu \times \nu(f) = \lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} f(X_i, Z_{i-1}) \quad a.s.$$

$$= \lim_{k \to \infty} \frac{1}{k} \log | X_k \cdots X_1 Z_0 | \quad a.s. \tag{4.11}$$

The last equality is easy to check if $| Z_i(\omega) | = 1$ for all i and ω. Note that once $Z_i(\omega) = 0$, we have $Z_l(\omega) = 0$ for all $l \geq i$, so that both sides of last equality of (4.11) are $-\infty$. Hence (4.10) holds for almost all $u \in \text{supp}\ \nu$.

QED

Remark. If $u \in \text{supp}\ \nu$ such that (4.10) holds, then for any $\alpha \neq 0$, we have

$$\lim_{k \to \infty} \frac{1}{k} \log | W_k(\alpha u) | = \lim_{k \to \infty} \frac{1}{k} [\log | W_k(u) | + \log | \alpha |] \tag{4.12}$$

$$= \lim_{k \to \infty} \frac{1}{k} \log | W_k(u) | + 0$$

$$= r_\nu \quad a.s.$$

Definition. Let $\nu \in Q_0 \setminus \{\overline{\nu}\}$ be an extremal invariant probability measure of μ. We call

$$E_\nu^0 \triangleq \{ u \in \mathit{supp}\ \nu \mid u \text{ satisfies (4.10) } a.s. \},$$

an *ergodic component* of the process $\{X_{k+1}, Z_k\}$ and

$$F^0 \triangleq S^{n-1} \setminus (\cup E_\nu^0),$$

a *transient component*.

Lemma 4.6. *Let $\nu_1, \nu_2 \in Q_0 \setminus \{\overline{\nu}\}$ corresponding r_1, r_2, respectively. If $r_1 \leq r_2$, then*

$$\overline{\lim_{k\to\infty}} \frac{1}{k} \log | W_k(\alpha_1 u_1 + \alpha_2 u_2) | \leq r_2 \tag{4.13}$$

for $u_1 \in E_1^0$, $u_2 \in E_2^0$ *and* $|\alpha_1| + |\alpha_2| > 0$.

Proof. Case (i) $r_1 > -\infty$. From (4.10), we know that for each $\epsilon > 0$, there exists a $T(\epsilon) > 0$ such that $| W_k(\alpha_i u_i) | \leq |\alpha_i| e^{k(r_i + \epsilon)}$, $i = 1,2$, for all $k > T(\epsilon)$. Thus,

$$\begin{aligned} | W_k(\alpha_1 u_1 + \alpha_2 u_2) | &\leq | \alpha_1 W_k(u_1) | + | \alpha_2 W_k(u_2) | \\ &\leq |\alpha_1| e^{k(r_1+\epsilon)} + |\alpha_2| e^{k(r_2+\epsilon)} \\ &\leq (|\alpha_1| + |\alpha_2|) e^{k(r_2+\epsilon)}. \end{aligned}$$

Letting $\epsilon \downarrow 0$. we have the result.

Case (ii) $-\infty = r_1 < r_2$. Then for each $\epsilon > 0$, $\exists$ $T(\epsilon) > 0$ such that

$$| W_k(\alpha_1 u_1) | = |\alpha_1| \, | W_k(u_1) | < |\alpha_1| \epsilon$$

and

$$| W_k(\alpha_2 u_2) | = |\alpha_2| \, | W_k(u_2) | \leq |\alpha_2| e^{k(r_2+\epsilon)}$$

for $k \geq T(\epsilon)$. Hence,

$$\begin{aligned} | W_k(\alpha_1 u_1 + \alpha_2 u_2) | &\leq | \alpha_1 W_k(u_1) | + | \alpha_2 W_k(u_2) | \\ &\leq |\alpha_1| \epsilon + |\alpha_2| e^{k(r_2+\epsilon)} \\ &\leq (|\alpha_1| + |\alpha_2|) e^{k(r_2+\epsilon)} \end{aligned}$$

for k sufficiently large. As $\epsilon \downarrow 0$, we have the same result.

Case (iii) $r_2 = -\infty$. For each $N > 0$, $\exists$ $T(N) > 0$ such that

$$| W_k(\alpha_i u_i) | = |\alpha_i| \, | W_k(u_i) | \leq |\alpha_i| e^{-kN}, \; i = 1,2,$$

for $k > T(N)$. Thus,

$$| W_k(\alpha_1 u_1 + \alpha_2 u_2) | \le | \alpha_1 | e^{-kN} + | \alpha_2 | e^{-kN}$$
$$= (| \alpha_1 | + | \alpha_2 |) e^{-kN}.$$

As $N \to \infty$, we have the desired result (4.13).

QED

Lemma 4.7. *There are at most n ergodic components E_i^0 corresponding to different values of r_i with $\nu_i \in Q_0 \setminus \{\bar{\nu}\}$.*

Proof. Choose arbitrary l ergodic components E_i^0 corresponding to different r_i, i=1, ..., l. We claim the set $\{u_i\}$, where $u_i \in E_i^0$, are independent. Without loss of generality, we can assume $r_1 < r_2 < \cdots < r_l$. Suppose $\exists i \le l$, $u_i = \sum_{j=1}^{i-1} \alpha_j u_j$. Then Lemma 4.6 implies

$$r_i = \lim_{k \to \infty} \frac{1}{k} \log | W_k(u_i) | \le \max \{r_1, \ldots, r_{i-1}\} = r_{i-1}$$

which is a contradiction. We complete the proof by noting that there are only n independent vectors on S^{n-1}.

QED

Lemma 4.8. *Let $\nu_1, \nu_2 \in Q_0 \setminus \{\bar{\nu}\}$ with $r_1 < r_2$. Then*

$$\varliminf_{k \to \infty} \frac{1}{k} \log | W_k(\alpha_1 u_1 + \alpha_2 u_2) | \ge r_2 \tag{4.14}$$

for $u_1 \in E_1^0$, $u_2 \in E_2^0$ and $\alpha_2 \neq 0$. Thus,

$$\lim_{k \to \infty} \frac{1}{k} \log | W_k(\alpha_1 u_1 + \alpha_2 u_2) | = r_2. \tag{4.15}$$

Proof. If $\alpha_1 = 0$, (4.14) is trivially satisfied, so we assume $\alpha_1 \neq 0$.

Case (i) $r_1 > \infty$. From (4.12) and $\epsilon > 0$, $\exists$ $T(\epsilon) > 0$ such that $k > T(\epsilon)$, we have for $\alpha_i \neq 0$, $i = 1,2$,

$$
\begin{aligned}
| W_k(\alpha_1 u_1) | &\leq e^{k(r_1+\epsilon)} \\
| W_k(\alpha_2 u_2) | &\geq e^{k(r_2-\epsilon)}.
\end{aligned}
\tag{4.16}
$$

Without loss of generality, we can assume $\epsilon < \frac{1}{2}(r_2 - r_1)$. Let $\delta = r_2 - r_1 - 2\epsilon > 0$. Thus

$$
\begin{aligned}
| W_k(\alpha_1 u_1 + \alpha_2 u_2) | &\geq | W_k(\alpha_2 u_2) | - | W_k(\alpha_1 u_1) | \\
&\geq e^{k(r_2-\epsilon)} - e^{k(r_1+\epsilon)} \\
&\geq (1 - e^{-\delta})\, e^{k(r_2-\epsilon)}
\end{aligned}
$$

for $k \geq 1$. Letting $\epsilon \downarrow 0$, we have (4.14).

Case (ii) $r_1 = -\infty$. For each $\epsilon > 0$, $\exists$ $T(\epsilon) > 0$ such that whenever $k > T(\epsilon)$, we have $| W_k(\alpha_1 u_1) | < \epsilon$ and (4.16) holds. Hence,

$$
| W_k(\alpha_1 u_1 + \alpha_2 u_2) | \geq | e^{k(r_2-\epsilon)} - \epsilon | \geq c e^{k(r_2-\epsilon)}
$$

for some $c > 0$. By letting $\epsilon \downarrow 0$, we have (4.14). Consequently, we know (4.15) with Lemma 4.6 and (4.14).

QED

Lemma 4.9. *Let* $\nu_1 \neq \nu_2 \in Q_0 \setminus \{\overline{\nu}\}$ with $r_1 = r_2$. *If* $|\alpha_1| + |\alpha_2| > 0$ *and* $u_1 \in E_1^0$, $u_2 \in E_2^0$, *then*

$$
\lim_{k\to\infty} \frac{1}{k} \log | W_k(\alpha_1 u_1 + \alpha_2 u_2) | = r_2 = r_1. \tag{4.17}
$$

Proof. First note that if $r_2 = -\infty$, then (4.17) is true by Lemma 4.6. Now assume $r_2 > -\infty$. Without loss of generality, we can assume $\alpha_1 \neq 0$, $\alpha_2 \neq 0$. Since $|W_k(u_i)| = e^{kr_i + o(k)}$, i=1, 2, then,

$$|W_k(\alpha_1 u_1 + \alpha_2 u_2)| \geq \Big| |\alpha_2| \, |W_k(u_2)| - |\alpha_1| \, |W_k(u_1)| \Big|$$

$$= (|\alpha_2| \; e^{o(k)} - |\alpha_1|) e^{kr_2 + o(k)}.$$

If $o(k)$ has no finite limit as $k \to \infty$, then (4.14) holds for all α_i, $i = 1, 2$ and Lemma 4.6 implies (4.17). Since $\nu_1 \neq \nu_2$, $E_1^0 \bigcap E_2^0 = \emptyset$, so that $\{u_1, u_2\}$ spans a two dimensional subspace D in $\mathbb{R}^n$. Suppose $\lim_{k \to \infty} o(k) = a < \infty$. We finish our proof by noting that for those $u = \alpha_1 u_1 + \alpha_2 u_2$ with $\dfrac{|\alpha_1|}{|\alpha_2|} = e^a$, u can also expressed in terms of other two vectors in D such that (4.17) holds.

QED

Theorem 4.10. *If SG is irreducible in the sense that SG cannot have a non-trivial invariant subspace in $\mathbb{R}^n$, then r_ν is independent of $\nu \in Q_0 \setminus \{\overline{\nu}\}$ and the limit in* (4.10) *holds for all $u \neq 0$.*

Proof. Let $\nu \in Q_0 \setminus \{\overline{\nu}\}$ and so $\mu_* \nu = \nu$. By the assumptions on SG, ν cannot be supported on a linear subvariety on S^{n-1}, i.e. a proper subspace of $\mathbb{R}^n$ projected on S^{n-1}. Thus, $\exists \; \{u_1, \ldots, u_n\}$, an independent set in S^{n-1}, such that

$$\lim_{k \to \infty} \frac{1}{k} \log |W_k(u_i)| = r_\nu \;\; a.s. \qquad i = 1, \ldots, n.$$

By Theorem 3.5 and Lemma 4.6, we have

$$\sup_{u \neq 0} \lim_{k \to \infty} \frac{1}{k} \log |\, W_k(u) \,| = \lim_{k \to \infty} \frac{1}{k} \log ||X_k \cdots X_1|| = r \leq r_{\nu}. \qquad \textbf{(4.18)}$$

Let $\nu' \in Q_0$. If $u \in E^0_{\nu'}$, then (4.18) implies $r_{\nu'} \leq r_{\nu}$. We can reverse the order of ν and ν' to get the equality $r_{\nu'} = r_{\nu} = r$. Thus, the rate is independent of the choice of an extremal measure. By Lemma 4.9, (4.10) holds for all $u \neq 0$.

QED

Corollary 4.11. *If SG is transitive on* S^{n-1} *and $\nu \in Q_0 \setminus \{\overline{\nu}\}$ exists, then* (4.10) *holds for all $u \neq 0$.*

Proof. The result follows from Theorem 4.10 by noting that transitivity of SG on S^{n-1} implies irreducibility of SG.

QED

In general, it is hard to determine an invariant measure and calculate the exact value r_{ν} by integration. But in many cases, we can determine stability of a given system if we know the sign of r_{ν}. At this stage, we state a known result of Furstenberg in [15, Theorem 8.6, pp. 426] without proof.

Theorem 4.12. *Let G, generated by the support of μ, be a non-compact subgroup of $SL(n)$. If either condition*

(i) *all subgroups of G of finite index are irreducible*

or

(ii) *G is connected and irreducible*

is satisfied, then $r_\nu = r > 0, \forall \ \nu \in Q_0 \setminus \{\overline{\nu}\}$.

Corollary 4.13. *Let the group* G*, generated by the support of* μ *be semi-simple in* $GL(n)$*. If* G *is non-compact and irreducible, then* $r_\nu = r > 0 \ \forall \ \nu \in Q_0$.

Remark. If SG is in $GL(n)$, then let G be the group generated by the support of μ and $SG \subseteq G$. Then

$$\frac{1}{k}\log|X_k \cdots X_1 u| = \frac{1}{k}\log|Y_k \cdots Y_1 u| + \frac{1}{nk}\log|\det(X_k \cdots X_1)| \tag{4.19}$$

where

$$Y_i = \frac{X_i}{\operatorname{sgn}(\det X_i)\,|\det X_i|^{1/n}} \tag{4.20}$$

belongs to $SL(n)$ if either $\det X_i > 0, \forall \ i$ or n is odd. Moreover, if the corresponding $\tilde{G}$ in $SL(n)$ of G satisfies conditions of Theorem 4.12, then $\lim_{k\to\infty}\frac{1}{k}\log|Y_k \cdots Y_1 u| > 0$ for all $u \neq 0$. In addition, if $|\det(X_i)| \geq 1$, then we know the limit (4.19) is greater than zero.

Finally, we can use Lemma 3.3 to obtain a more precise result than Theorem 3.5 for system (1.1).

Theorem 4.14. *Consider the system* (1.1) *with the assumptions stated in section 3 on the processes* $N_i(t)$, $i=1,\ldots,m$*. Let* μ *be a measure on* $\mathbb{R}^{n\times n}$ *defined by*

$$\mu(\Gamma) \triangleq P\{D_{\mu_1} e^{A\tau_1} \in \Gamma\},\ \Gamma \in B(\mathbb{R}^{n\times n}) \tag{4.21}$$

and SG *be the closed semi-group generated by the support of* μ*, i.e.,*

$$SG = \textit{smallest semi-group containing } \{D_i e^{At},\ 0 \leq t < \infty,\ i=1,\ldots,m\}.$$

(4.22)

Consider Q_0, *a collection of extremal invariant probability measures of* μ *on* M. *Then*

$$r_\nu = \sum_{i=1}^{m} \lambda_i \int_M \int_0^\infty \log | D_i e^{At} u | e^{-\lambda t} \, dt \, d\nu(u) < \infty, \quad \nu \in Q_0 \setminus \{\overline{\nu}\} \tag{4.23}$$

and

$$\lim_{t \to \infty} \frac{1}{t} \log \frac{| x(t) |}{| x_0 |} = \lambda r_\nu \ \ a.s., \tag{4.24}$$

for all $x_0 \in E_\nu^0$. *There are only finite different values, say* $r_1 < r_2 < \cdots < r_l$, $l \leq n$. *In addition, if* $\cup E_\nu^0$ *contains a basis of* $\mathbb{R}^n$, *then the system* (1.1) *is asymptotically stable almost surely if* $r_l < 0$ *while the system* (1.1) *is asymptotically unstable almost surely if* $r_1 > 0$. *In case* $r_1 < 0$ *and* $r_l > 0$, *then the stability of the system depends on the initial state* x_0.

Proof. Let

$$W_k(x_0) = D_{\mu_k} e^{A \tau_k} \cdots D_{\mu_1} e^{A \tau_1} x_0. \tag{4.25}$$

If $x_0 \in E_\nu^0$, $\nu \in Q_0$,

$$\begin{aligned} \log | x(t) | &= \log | e^{[A(t - t_{N(t)})]} W_{N(t)}(x_0) | \\ &\leq ||A|| \tau_{N(t)+1} + \log | W_{N(t)}(x_0) |, \end{aligned}$$

so that using Lemma 3.4 and Theorem 4.5,

$$\begin{aligned} \overline{\lim_{t \to \infty}} \frac{1}{t} \log | x(t) | &\leq \lim_{t \to \infty} \frac{1}{t} ||A|| \tau_{N(t)+1} + \lim_{t \to \infty} \frac{1}{N(t)} \log | W_{N(t)}(x_0) | \cdot \frac{N(t)}{t} \\ &= 0 + r_\nu \lambda \ \ a.s. \end{aligned} \tag{4.26}$$

On the other hand, using $| e^{At} z | \geq e^{(-||A|| t)} | z |$, we have

$$\log | x(t) | \geq - ||A|| \tau_{N(t)+1} + \log | W_{N(t)}(x_0) | .$$

Thus,

$$\underline{\lim}_{t\to\infty} \frac{1}{t} \log | x(t) | \geq - \lim_{t\to\infty} \frac{1}{t} ||A|| \tau_{N(t)+1} + \lim_{t\to\infty} \frac{1}{N(t)} \log | W_{N(t)}(x_0) | \cdot \frac{N(t)}{t}$$

$$= 0 + r_\nu \lambda \quad a.s. \tag{4.27}$$

From (4.26) and (4.27), we prove (4.24). By Lemma 4.7, there are at most n ergodic components corresponding to different values $r_1 < r_2 < \cdots < r_l$, $l \leq n$. If $\cup E_\nu^0$ contains a basis of $\mathbb{R}^n$, then the asymptotic growth rate associated with any initial state is one of the r_i by an argument similar to the proof of Theorem 4.10. Thus, the last result just follows from (4.24).

QED

Remark. If SG is transitive, then there is at most one ergodic component. If SG is only irreducible, there may be many ergodic components, but r_ν is independent of the choice of $\nu \in Q_0 \setminus \{\overline{\nu}\}$. Stability of the system (1.1) depends on the sign of the rate r. If $\{E_\nu^0\}$ doesnot contain a basis of $\mathbb{R}^n$, there is no result corresponding to (4.10) for $u \in F^0$, the transient component. The behavior of transient states must be investigated individually. An example will illustrate the difficulty.

Consider

$$A = \begin{pmatrix} 0 & 0 \\ -e^t & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}.$$

Then

$$expAt = \begin{pmatrix} 1 & 0 \\ -e^{t} & 1 \end{pmatrix}, \qquad D = I + B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It can be shown that the unique invariant probability measure $\nu \neq \overline{\nu}$ of μ concentrates on two points $P_1 = (0,1)$ and $P_2 = (0, -1)$ on the circle with probability $\nu(P_1) = \nu(P_2) = \frac{1}{2}$. The corresponding rate $r_\nu = 0$ because trajectories starting at P_1 or P_2 are fixed. But trajectories starting at transient states in $F^0 = S \setminus \{P_1, P_2\}$ go to infinity with rate $= 1$.

5. Large Deviations of Asymptotically Stable Systems.

In this section, we assume the system (1.1) is asymptotically stable with $r_\nu < 0$ and the same assumptions as in the previous sections. Now for $x_0 \in E_\nu^0$, we have

$$\lim_{t \to \infty} \frac{1}{t} \log \frac{|x(t)|}{|x_0|} = r_\nu \lambda < 0.$$

Then for each $\epsilon > 0$ with $r_\nu \lambda + \epsilon < 0$, $\exists$ $\mathrm{T}(\epsilon) > 0$ such that for $t \geq \mathrm{T}(\epsilon)$, we have

$$\frac{1}{t} \log |x(t)| < r_\nu \lambda + \epsilon < 0. \tag{5.1}$$

Since the sample path of $x(t)$ is piecewise right continuous with finite jumps during any finite interval, $\exists$ $M_1(\epsilon) > 0$ such that

$$|x(t)| \leq M_1(\epsilon) \quad a.s. \qquad \forall\ t \in [0, \mathrm{T}(\epsilon)].$$

Let

$$M(\epsilon) = \max \{1, M_1(\epsilon)\, e^{-(r_\nu \lambda + \epsilon)\mathrm{T}(\epsilon)}\}.$$

Then

$$| x(t) | \le M(\epsilon)\, e^{(r_\nu \lambda + \epsilon)T(\epsilon)} \le M(\epsilon)\, e^{(r_\nu \lambda + \epsilon)t} \quad \forall \; t \in [0, T(\epsilon)].$$

so that with (5.1)

$$| x(t) | \le M(\epsilon)\, e^{(r_\nu \lambda + \epsilon)t} \quad a.s. \text{ for all } t \ge 0. \tag{5.2}$$

Thus, from the Markov inequality, we get

$$P\,(| x(t) | \ge R) \le \frac{E\, | x(t) |}{R} \le \frac{M(\epsilon)}{R}\, e^{(r_\nu \lambda + \epsilon)t}.$$

We would like to obtain a similar result for large deviations,

$$P\,(\sup_{s \ge t} | x(s) | \ge R) \le M(\epsilon, x_0, R)\, e^{\gamma t}, \quad t \ge 0,$$

where $0 > \gamma > r_\nu \lambda$. Problems of this kind with wide band noise were considered in [24].

Before going further, we note that if $a_i(.,s)$ are F_s-measurable where

F_s = σ-algebra generated by $\{N_i(\tau), 0 \le \tau \le s\,, i = 1, \ldots, m\}$

and $\{N_i(\tau)\}$ are independent Poisson processes with intensity λ_i, respectively. Then

$$\sum_{i=1}^{m} \int_0^t a_i(\omega,s)\, d\tilde{N}_i(\omega,s)$$

is a martingale because N_i has independent increments, where

$$\tilde{N}_i(\omega,s) \triangleq N_i(\omega,s) - \lambda_i s.$$

In addition, we need to construct integrable martingales in exponential form as in the following lemma.

Lemma 5.1. *Let* $\beta > 0$

$$m_\beta(t) = \beta \sum_{i=1}^{m} \int_0^t a_i(\omega,s)\, d\tilde{N}_i(\omega,s)$$

and

$$< m_\beta(t) > = \sum_{i=1}^{m} \lambda_i \int_0^t [exp(\beta a_i(\omega,s)) - 1 - \beta a_i(\omega,s)]\, ds.$$

Then $exp\,[m_\beta(t) - < m_\beta(t) >]$ is an integrable martingale with mean equal to one.

Proof. Let $y_\beta(t) = exp\,(m_\beta(t) - < m_\beta(t) >)$. Then

$$y_\beta(t) = \exp\{ - \sum_{i=1}^{m} \lambda_i \int_0^t (e^{\beta a_i} - 1)\, ds + \sum_{i=1}^{m} \int_0^t \beta a_i\, dN_i \}$$

$$\triangleq exp\,[z_\beta(t)].$$

Thus, using the differential rule for point processes, we get

$$dy_\beta(t) = y_\beta(t)[- \sum_{i=1}^{m} \lambda_i\, (e^{\beta a_i} - 1)]\, dt + \sum_{i=1}^{m} [exp\,(z_\beta(t) + \beta a_i) - exp\,(z_\beta(t)]\, dN_i(t)$$

$$= - \sum_{i=1}^{m} \lambda_i\, (e^{\beta a_i} - 1) y_\beta(t)\, dt + \sum_{i=1}^{m} (e^{\beta a_i} - 1)\, exp\,(z_\beta(t))\, dN_i(t)$$

$$= \sum_{i=1}^{m} (e^{\beta a_i} - 1)\, y_\beta(t)\, d\tilde{N}_i(t)$$

and so

$$y_\beta(t) = 1 + \int_0^t (e^{\beta a_i} - 1)\, y_\beta(s)\, d\tilde{N}_i(s).$$

We have $E\ y_\beta(t) = 1$ and the result follows immediately from the above discussion.

QED

Remark. Note that $e^{\beta a_i} - 1 - \beta a_i \geq 0$. It follows that $<m_\beta(t)>$ is non-decreasing. We call $<m_\beta(t)>$ the *increasing process* associated with $m_\beta(t)$.

Now, we return to the problem of large deviations. Let

$$\rho = \log \; |x(t)|$$

and

$$\theta = \frac{x(t)}{|x(t)|} .$$

Then

$$d\,\rho(t) = \left(\frac{\partial \rho(t)}{\partial x}\right)^T Ax(t)dt + \sum_{i=1}^{m} [\log|x(t)+B_i x(t)| - \log|x(t)|]dN_i(t)$$

$$= \frac{x(t)^T}{|x(t)|^2} Ax(t)dt + \sum_{i=1}^{m} \log\left(\frac{|(I+B_i)x(t)|}{|x(t)|}\right) dN_i(t)$$

$$= \theta(t)^T A\,\theta(t)dt + \sum_{i=1}^{m} \log\;|D_i\,\theta(t)|\;dN_i(t)$$

where

$$D_i = I + B_i .$$

Let

$$\mathbf{L}\rho(t) \triangleq \theta(t)^T A\,\theta(t) + \sum_{i=1}^{m} \lambda_i \log\;|D_i\,\theta(t)| .$$

Then

$$\rho(t) = \rho(0) + \int_0^t \mathbf{L}\rho(s)ds + \sum_{i=1}^{m}\int_0^t \log\;|D_i\,\theta(s)|\;d\tilde{N}_i(s). \tag{5.3}$$

The last term

$$m_\beta(t) \triangleq \sum_{i=1}^{m} \beta \int_0^t \log \ | D_i \theta(s) | \ d\ddot{N}_i(s)$$

is a zero mean, right continuous martingale and

$$E \ m_1^2(t) = \sum_{i=1}^{m} \lambda_i \int_0^t E \ \log^2 \ | D_i \theta(s) | \ ds.$$

$$= \leq (\sum_{i=1}^{m} \lambda_i \log^2 || D_i ||) t.$$

From the lemma in [24, p.459], we know that

$$\frac{1}{t} \ m_1(t) \to 0 \quad as \quad t \to \infty \quad a.s.$$

Thus in (5.3)

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t \mathbf{L}\rho(s) ds \ = \lim_{t \to \infty} \frac{\rho(t)}{t} = r_\nu \cdot \lambda < 0 \quad a.s. \tag{5.4}$$

In the same manner used to derive (5.2), we know $\exists \ C(\epsilon)$ such that

$$\int_0^t \mathbf{L}\rho(s) ds \ \leq (r_\nu \lambda + \epsilon) t \ + \ C(\epsilon) \quad a.s. \quad t \geq 0$$

and $r_\nu \lambda + \epsilon < 0$. The increasing process corresponding to $m_\beta(t)$, $\beta > 0$ is

$$< m_\beta(t) > = \sum_{i=1}^{m} \lambda_i \int_0^t [\ | D_i \theta(s) |^\beta - 1 - \beta \log | D_i \theta(s) | \ ] ds.$$

We will use the integrable martingale $exp \ [m_\beta(t) - < m_\beta(t) >]$ with mean 1 to derive the large deviation result following the technique used in [24]. Let

$[t]$ = integral part of t

(t) = fractional part of t

and

$$\overline{M}_\beta = \sup_{|\theta|=1} \{ \sum_{i=1}^{m} \lambda_i [| D_i \theta |^{\beta} - 1 - \beta \log | D_i \theta |]\} < \infty. \tag{5.5}$$

Then

$$P \{ \sup_{s \geq t} | x(s) | \geq R \} \leq \sum_{j=[t]}^{\infty} P \{ \sup_{j+(t) \leq s < j+(t)+1} | x(s) | \geq R \}$$

$$= \sum_{j=[t]}^{\infty} P \{ \sup_{j \leq s-(t) < j+1} \beta \log | x(s) | \geq \beta \log R \}$$

for any $\beta > 0$, and

$$\sup_{j \leq s-(t) < j+1} \{\beta \log | x(s) | \} \leq \sup_{j \leq s-(t) < j+1} \{\beta [\log | x(s) | - \log | x_0 |$$

$$- \int_0^s \mathbf{L} \log | x(\tau) | \, d\tau]\}$$

$$+ \sup_{j \leq s-(t) < j+1} \{\beta [\log | x_0 | + \int_0^s \mathbf{L} \log | x(\tau) | \, d\tau]\}$$

$$\leq \sup_{j \leq s-(t) < j+1} \{[m_\beta(s) - \langle m_\beta(s) \rangle]$$

$$+ \beta [\log | x_0 | + (r_\nu \lambda + \epsilon)(j+(t))$$

$$+ C(\epsilon) + \overline{M}_\beta (j+(t)+1)]\}.$$

Hence, using the martingale inequality, we get

$$P \left\{ \sup_{j \leq s-(t) < j+1} | x(s) | \geq R \right\}$$

$$\leq P \left\{ \sup_{j \leq s-(t) < j+1} \{ exp\, [m_\beta(s) - \langle m_\beta(s) \rangle] \} \geq \right.$$

$$\left. exp\, (\beta [\log R - \log | x_0 | - (r_\nu \lambda + \epsilon)(j+(t)) - C(\epsilon) - \overline{M}_\beta (j+(t)+1)]) \right\}$$

$$\leq \left(\frac{|x_0|}{R} \right)^{\beta} e^{(r_\nu \lambda + \epsilon)(j+(t)) + C(\epsilon) + \overline{M}_\beta (j+(t)+1)}$$

$$= \left(\frac{|x_0|}{R} \right)^{\beta} e^{C(\epsilon) + \overline{M}_\beta} e^{(r_\nu \lambda + \epsilon + \overline{M}_\beta)(j+(t))}.$$

We can choose $\overline{\beta} > 0$ as small as we like in (5.5) so that

$$\gamma(\overline{\beta}, \epsilon) = r_\nu \lambda + \epsilon + \overline{M}_{\overline{\beta}} < 0.$$

Thus,

$$P\ \{\sup_{s \geq t} |x(s)| \geq R\} \leq \sum_{j=[t]}^{\infty} \left(\frac{|x_0|}{R} \right)^{\overline{\beta}} e^{C(\epsilon) + \overline{M}_{\overline{\beta}}} e^{\gamma(\overline{\beta}, \epsilon)(j+(t))}$$

$$= \left(\frac{|x_0|}{R} \right)^{\overline{\beta}} \frac{e^{C(\epsilon) + \overline{M}_{\overline{\beta}}}}{1 - e^{\gamma(\overline{\beta}, \epsilon)}} e^{\gamma(\overline{\beta}, \epsilon) t}, \quad t \geq 0. \tag{5.6}$$

Consequently, we have proved the following theorem.

Theorem 5.1. *If the system* (1.1) *is asymptotically stable with* $r_\nu < 0$, *then* $\exists$ *constant* $M(x_0, R, \epsilon, \beta)$ *and* $0 > \gamma(\beta, \epsilon) > r_\nu \lambda$ *such that*

$$P\ \{\sup_{s \geq t} |x(s)| \geq R\} \leq M(x_0, R, \epsilon, \beta)\, e^{\gamma(\beta, \epsilon) t}, \quad t \geq 0.$$

6. Stabilization.

In this section, we examine the control problem of stabilizing a linear system with Poisson noise disturbances using feedback controls. Consider the linear system with state- and control-dependent noises.

$$dx(t) = Ax(t)dt + Bu(t)dt + Cx(t)dN_1(t) + Du(t)dN_2(t) \tag{6.1}$$

$$x(0) = x_0$$

where A, C are constant $n \times n$ matrices; B, D are constant $n \times m$ matrices; $N_1(t)$ and $N_2(t)$ are independent Poisson processes with intensities λ_1 and λ_2, respectively. We want to stabilize the above system (6.1) by feedback control

$$u(t) = -Kx(t) \tag{6.2}$$

with K a constant $m \times n$ matrix. Substituting (6.2) into (6.1), we obtain

$$dx(t) = (A-BK)x(t)dt + Cx(t)dN_1(t) - DKx(t)dN_2(t). \tag{6.3}$$

Now, let $C_1 = C$, $C_2 = -DK$ and $\{\mu_i\}$ be a random process with values $\{1,2\}$ such that $\mu_i = j$ means that $N_j(t_i)$ increases at the occurrence times $\{t_i\}$ for the sum process $N(t) = N_1(t) + N_2(t)$ as before. Then the state trajectory is

$$x(t) = exp\,(A-BK)(t-t_{N(t)})(I+C_{\mu_{N(t)}})exp\,(A-BK)\tau_{N(t)} \cdots$$
$$\cdot (I+C_{\mu_1})exp\,(A-BK)\tau_1\, x_0$$

where $\{\tau_i\}$ are the interarrival times of the sum Poisson process N(t) with intensity $\lambda = \lambda_1 + \lambda_2$. Stability depends on the Lyapunov characteristic number

$$r_\nu(K) = \lambda_1 \int_M \int_0^\infty \log|(I+C)exp\,(A-BK)t\;x_0|\; e^{-\lambda t}\, dt\, d\,\nu(x_0)$$
$$+ \lambda_2 \int_M \int_0^\infty \log|(I-DK)exp\,(A-BK)t\;x_0|\; e^{-\lambda t}\, dt\, d\,\nu(x_0) \tag{6.4}$$

where ν is a normalized extremal solution of

$$\nu(\Gamma) = \lambda_1 \int_\Gamma \int_0^\infty \chi_\Gamma((I+C)exp\,(A-BK)t\,ox)\; e^{-\lambda t}\, dt\, d\,\nu(x)$$
$$+ \lambda_2 \int_\Gamma \int_0^\infty \chi_\Gamma((I-DK)exp\,(A-BK)t\,ox)\; e^{-\lambda t}\, dt\, d\,\nu(x) \tag{6.5}$$

for all Γ in the Borel sets of $M = S^{n-1} \cup \{0\}$ and χ is the characteristic function with values $\{0,1\}$. Let

$$r(K) = \lim_{k\to\infty} \frac{1}{k} E \log||(I+C_{\mu_k})exp(A-BK)\tau_k \cdots (I+C_{\mu_1})exp(A-BK)\tau_1||. \quad (6.6)$$

From (6.4) and (6.6), we know that $r_\nu(K) \le r(K)$. We would like to have $r(K) < 0$ for some K. It is sufficient to have

$$\begin{aligned} &E \log||(I+C_{\mu_1})exp(A-BK)\tau_1|| \\ &= \lambda_1 \int_0^\infty \log||(I+C)exp(A-BK)t||\, e^{-\lambda t}\, dt \\ &+ \lambda_2 \int_0^\infty \log||(I-DK)exp(A-BK)t||\, e^{-\lambda t}\, dt \\ &< 0 \end{aligned} \quad (6.7)$$

for some matrix K. Thus, we have proved the following theorem.

Theorem 6.1. *Consider the system* (6.1). *If condition* (6.7) *is satisfied for some constant matrix K, then the feedback control $u(t) = -Kx(t)$ can stabilize the system almost surely.*

Remark. Suppose $D \equiv 0$. If (A, B) is controllable in the sense that

$$\text{rank}\,[B, AB, \ldots, A^{n-1}B] = n$$

then we can locate the modes of the system arbitrarily by suitable K. Thus,

$$\begin{aligned} &E \log||(I+C)exp(A-BK)\tau_1|| \\ &= \lambda \int_0^\infty \log||(I+C)exp(A-BK)t||\, e^{-\lambda t}\, dt \\ &\le \log||I+C|| + \int_0^\infty \log||exp(A-BK)t||\, \lambda e^{-\lambda t}\, dt \end{aligned}$$

$$\leq \log||I+C|| + \int_0^\infty \sigma t \, \lambda e^{-\lambda t} \, dt$$

$$= \log||I+C|| + \frac{\sigma}{\lambda} \tag{6.8}$$

where we can find K so that the eigenvalues of $A - BK$ lie to the left of $\sigma < -\lambda|$ log $||$I+C$|||$ in the complex plane. Actually, if $||I+C|| \leq 1$ we can choose $\sigma < 0$. Thus (6.8) is less than 0 and condition (6.7) is satisfied.

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