ABSTRACT

Title of dissertation: ESSAYS IN MONETARY ECONOMICS AND BUSINESS CYCLES

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This dissertation investigates non-linear macroeconomic dynamics within the New Keynesian model during periods with zero short-term nominal interest rates. I implement modern quantitative tools to solve and analyze Dynamic Stochastic General Equilibrium (DSGE) models where the feedback rule that defines monetary policy is subject to the Zero Lower Bound (ZLB) constraint. The revived attention about the importance of the ZLB constraint followed the extreme events that took place in the United States after the financial crisis of 2008.

The first chapter studies aggregate dynamics near the ZLB of nominal interest rates in a medium-scale New Keynesian model with capital. I use Sequential Monte Carlo methods to uncover the shocks that pushed the U.S. economy to the ZLB during the Great Recession and investigate the interaction between shocks and frictions in generating the contraction of output, consumption and investment during 2008:Q3-2013:Q4. I find that a combination of shocks to the marginal efficiency of investment and to households' discount factor generated the prolonged liquidity

trap observed in this period. A comparison between these two sources suggests that investment shocks played a more important role in accounting for the contraction of economic activity. Fiscal and monetary policy stimulus helped the U.S. economy avoid deflation and accelerated the recovery.

The second chapter studies a New-Keynesian model with Markov sunspot shocks that move the economy between a targeted-inflation regime and a deflation regime and fit it to data from the U.S. and Japan. For the U.S. we find that adverse demand shocks have moved the economy to the zero lower bound (ZLB) in 2009 and an expansive monetary policy has kept it there subsequently. In contrast, Japan has experienced a switch to the deflation regime in 1999 and remained there since then, except for a short period. The two scenarios have drastically different implications for macroeconomic policies. Fiscal multipliers are about 20% smaller in the deflationary regime, despite the economy remaining at the ZLB. While a commitment by the central bank to keep rates near the ZLB doubles the fiscal multipliers in the targeted-inflation regime (U.S.), it has no effect in the deflation regime (Japan).

ESSAYS IN MONETARY ECONOMICS AND BUSINESS CYCLES

by

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Dissertation submitted to the Faculty of the Graduate School of the
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Dedication

To my parents

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Chapter 1 What Explains the Great Recession and the Slow Recovery?

1.1 Introduction

From 2007 to 2009 the U.S. economy was caught up in the throes of a severe recession. The Great Recession was the worst episode of economic contraction since the Great Depression, with consumption and investment plunging after the collapse of large financial institutions in September 2008. It has taken over half a decade for the economy to climb back to pre-recession levels. This episode is noteworthy not only because of its depth and subsequent slow recovery, but also because monetary policy quickly became constrained by the Zero Lower Bound (ZLB) on nominal interest rates. The Federal Reserve responded swiftly, lowering the Federal Funds Rate to nearly zero by the first quarter of 2009. The policy rate has remained at the ZLB for over five years to this date. Two natural questions arise: what made the Great Recession so severe and why was the recovery slow?

Five years after the end of the recession, modern macroeconomic models continue to struggle to generate a coherent story about the events that caused such a severe contraction in economic activity. On the one hand, existing medium scale

DSGE models that are able to account for the dynamics of aggregate quantities and prices abstract from the ZLB because of the computational complexity of solving rational expectation models with nonlinearities. On the other hand, models that explicitly account for the ZLB adopt highly stylized frameworks that abstract from investment or assume very simple structures that limit their quantitative capability to match the dynamics during the Great Recession. This paper focuses on bridging this gap. I use the structure of a modern macro model commonly used for quantitative analysis and a set of computational techniques that help me paint a full picture of the causes of the Great Recession and the slow recovery. A key contribution is to uncover the shocks that pushed the nominal interest rate to the ZLB and understand their role in shaping the evolution of aggregate demand, in particular investment, which is often ignored in models with a ZLB constraint.

Given the length of the ZLB spell, the ability to solve quantitative models that incorporate this fundamental constraint on monetary policy is essential for our understanding of the Great Recession. This is precisely the goal of this paper. I look at U.S. data from the perspective of a Dynamic Stochastic General Equilibrium (DSGE) model that incorporates the ZLB constraint. My model builds on the work of Christiano, Eichenbaum and Evans (2005) and Smets and Wouters (2007), which has been widely adopted for quantitative analyses by researchers and policy institutions. I use Sequential Monte Carlo methods (SMC) to structurally estimate the shocks that explain the recession and the slow recovery. Moreover, I use the estimated shocks to investigate how fiscal and monetary policies contributed to the recovery.

Compared to similar work that investigates the causes of the Great Recession in models that allow for the ZLB, this paper provides a broader answer for two reasons. First, I use a medium scale DSGE model that incorporates investment. Adding investment is crucial in order to understand the importance of shocks related to financial frictions, which have been argued to be at the root of the recession. Second, I take a novel approach to the structural estimation of shocks, combining nonlinear solution methods with SMC techniques. To the best of my knowledge, this paper is the first to apply both computational techniques simultaneously in a medium scale New Keynesian DSGE model of the type that is commonly used for policy analysis, and use them to unveil the underlying drivers of the Great Recession. In addition, I use the estimated structural shocks to conduct counterfactual exercises. Among the five disturbances included in the model economy, I consider a shock to the marginal efficiency of investment (investment shock) and a shock to households' subjective discount factor (preference shock). Both shocks represent deeper frictions in the financial sector of the economy in a reduced form way.[1]

I find that the Great Recession originated in a decline in the marginal efficiency of investment. This decline started in the second half of 2007 and worsened in the third quarter of 2008, after the bankruptcy of important financial institutions. The U.S. economy encountered the ZLB as monetary policy responded to large negative

[1]This modeling approach is commonly used in DSGE models. For example, Smets and Wouters (2007) introduce a 'risk-premium' shock that affects the relative price of the nominal bond. In contrast, Justiniano, Primiceri and Tambalotti (2010) use a shock to households' subjective discount factor instead of the 'risk-premium' shock. Up to a first order, both frictions enter in the consumption Euler equation in the same way, but the risk-premium shock also affects the spread between the return on capital and bonds directly.

shock to households' ability to borrow and to the ability of the financial system to channel resources to investment. In the absence of these latter two shocks the recession would have been milder, with output falling 20% less with respect to its pre-recession trend and consumption and investment recovering fully to pre-crisis levels by the end of 2010. My results indicate that the U.S. economy remained at the ZLB in the aftermath of the recession because of stimulative monetary policy that kept the nominal interest rate pegged at zero. During the liquidity trap, fiscal policy provided substantial stimulus, in particular during 2009:Q2-2011:Q2, and its stimulative effect on output helped the economy stave off deflation. Without the fiscal stimulus, inflation would have been close to -3% when the economy hit rock bottom in 2009:Q1 and would have remained negative for another two quarters. The unwinding of the fiscal stimulus program in 2011 and political struggles that resulted in a reversion in the stance of fiscal policy held back the recovery.

A general consensus has emerged among economists that the recession originated in the financial system. However, the source and relative importance of frictions that caused the financial system to fail remain open to debate. For example, Mian, Rao and Sufi (2013) show that households' deleveraging reduced consumption in early 2007 and 2008, leading the collapse of the financial system. On the other hand, Gilchrist and Zakrajšek (2012) point towards depressed investment due to sharp increases in borrowing costs for firms as the leading cause of the recession. Whether shocks affecting households consumption were more important than frictions disrupting investment, is of particular interest for policy evaluation. Should policy have focused on alleviating households' mortgage debt and rehabilitating the housing market? Or should it have concentrated on providing resources to replenish bank capital and avoiding the collapse of financial intermediaries and investment banks? My results indicate that shocks and frictions affecting investment played a prominent role in explaining the Great Recession. I also find that these frictions remain elevated today compared to their pre-recession level, which explains the sluggishness in the economic recovery.

The rest of the chapter is organized as follows. Section 1.2 discusses related literature. Section 1.3 shows some important features of the data and a potential interpretation of the shocks that explain the dynamics around the Great Recession. The DSGE model used for the quantitative analysis is spelled out in detail in Section 1.4. In Section 1.5, I discuss the parameterization and the solution strategy of the nonlinear model that explicitly incorporates the ZLB. Section 1.6 describes how to uncover the structural shocks that pushed the economy to the ZLB. Section 1.7 presents a series of counterfactual exercises to understand the economy's dynamics during and after the Great Recession. Section 1.8 concludes.

1.2 Related Literature

This paper fits within the literature that investigates macroeconomic dynamics in the presence of the Zero Lower Bound constraint. Eggertsson and Woodford (2003*b*) were the first to study the behavior of the economy at the ZLB in a New Keynesian DSGE model. However, to maintain analytical tractability and to characterize optimal policy, their setup abstracts from capital accumulation. To take the economy to the ZLB they study the effect of a temporary, unanticipated rise in households' discount factor that increases the real interest rate and lowers consumption. Their setup delivers sharp insights on the mechanics of ZLB events but it is not suited for quantitative analysis. In this regard, my paper is different because I incorporate capital accumulation and investment, and allow for five different shocks that drive the dynamics of the economy, bringing models that study the ZLB and business cycle dynamics closer together.

Much of the subsequent work on the ZLB adopted the Eggertsson and Woodford (2003*b*) modeling environment. For example, Eggertsson (2009*b*) investigates the effects of alternative fiscal policies at the ZLB, while Christiano, Eichenbaum and Rebelo (2011*a*) studies the size of the fiscal multiplier. These papers assume that a shock to households' discount factors is what causes the ZLB to bind, and hence the narrative around liquidity trap episodes has been centered on frictions that mostly affect consumption. Compared to this line of work, in my model there are two shocks that can push the economy to the ZLB: one that affects households' discount factors and works exactly in the same way as in related literature, and another that disturbs aggregate investment dynamics directly. Because I want to quantify the forces that took the U.S. economy to the ZLB during the Great Recession, I let the data uncover the role of each shock, and investigate their relative contributions to the depth of the recession and the slow economic recovery. Shocks to the marginal efficiency of investment have a long tradition in business cycle analysis since Greenwood, Hercowitz and Huffman (1988), and have recently been rekindled as a dominant source of business cycles fluctuations by Justiniano, Primiceri and Tambalotti (2011). However, my paper is the first to quantify the relative importance of these alternative shocks in generating a liquidity trap.

This paper uses a medium-scale DSGE model along the lines of Christiano, Eichenbaum and Evans (2005) and Smets and Wouters (2007). Variants of such models have been used to study business cycle dynamics, as in Justiniano, Primiceri and Tambalotti (2010), or to investigate trade-offs in monetary policy stabilization as in Justiniano, Primiceri and Tambalotti (2013). Moreover, variants of such models have been widely adopted in policy making institutions in the U.S. and around the world. Compared to this literature, my paper is among the few that solves the full nonlinear dynamics of a medium-scale model subject to a ZLB constraint. Christiano, Eichenbaum and Rebelo (2011*a*) was an early attempt to bring the ZLB into a medium-scale DSGE model but only to study the size of the fiscal multiplier. In related work, Christiano, Eichenbaum and Trabandt (2014) solve a DSGE model that accounts for labor market variables as well as aggregate demand and prices.

I differ from Christiano, Eichenbaum and Trabandt (2014) in a one key aspect. I use a particle filter to perform a formal estimation of the shocks that explain the data as seen through the structure of the model. In contrast, they take a less formal approach, exploiting the first order conditions of their model to map certain observables in the data into unobserved wedges in the model. In doing so they need to take a stand on the observables that best correspond to their proposed wedges, and impose additional restrictions in order to map the observables to the model equilibrium conditions. I do not impose such restrictions, instead letting the data speak freely about the driving forces that caused the recession. As an example of such restrictions, to facilitate a direct measurement of the consumption and financial wedges, Christiano, Eichenbaum and Trabandt (2014) assume a zero covariance between the stochastic discount factor and the ex-ante real interest rate. In my application, the estimation of the shocks respects the nonlinear equilibrium conditions of the model at all times, and hence the estimation and interpretation of the shocks is more transparent.

This paper also sheds light on the importance of financial frictions during the Great Recession. Although, I do not incorporate an explicit mechanism like the financial accelerator of Bernanke, Gertler and Gilchrist (1999), or model financial amplification through bank balance sheets as in Gertler and Karadi (2011), I take a reduced form approach that is useful for measuring the strength and persistence of financial frictions. An additional advantage of my approach is that the frictions that I recover from the data can be rationalized with different microeconomic mechanisms for financial frictions. I consider two shocks that can be interpreted as disruptions in financial markets. In this regard, my paper also echoes the main result in Christiano, Eichenbaum and Trabandt (2014), which attributes most of the fluctuations during the Great Recession to a *financial wedge*. However, I do not need to assume that such a wedge can be recovered directly from data on credit spreads only.[2] In my filtering exercises, I back out the equivalent to a *financial wedge* directly from observed data on consumption, investment and output growth. It turns out that my reduced form measure of financial frictions is closely related to fluctuations in the observed cost of

[2]Their results are sensitive to the particular measure of spreads. When they use Gilchrist and Zakrajšek (2012) measure of credit spreads, their financial wedge is not persistent enough to produces a long lasting recession as the one observed in the U.S.

borrowing for nonfinancial firms during the worst part of the crisis, but my measure remains persistently high even after observed credit spreads return to pre-recession levels.

In terms of methodology this paper builds on the solution methods developed in Judd, Maliar and Maliar (2012), Aruoba, Cuba-Borda and Schorfheide (2013), Gust, Lopez-Salido and Smith (2012*b*), and Fernández-Villaverde et al. (2012*b*) to characterize the full nonlinear equilibrium dynamics of DSGE models subject to occasionally binding constraints.[3] Compared to the discrete state-space solution method based on policy function iteration reviewed in Richter, Throckmorton and Walker (2011), my paper uses a combination of projection and simulation techniques to find the global approximation to the model decision rules. The advantage is that my solution strategy is more suitable for medium-scale models with many state variables. To extract the unobserved shocks that drive the dynamics during the Great Recession, I implement a particle filter adapted from the work in Aruoba, Cuba-Borda and Schorfheide (2013).

1.3 The Great Recession

Before discussing the model, I briefly review the evolution of key macroeconomic aggregates during and after the Great Recession. Figure 1.1 shows the comovement of key macroeconomic variables before and after the Great Recession.

[3] Gust, Lopez-Salido and Smith (2012*b*) are the first to estimate model parameters in a small New Keynesian with a ZLB, but their setup abstracts from capital, which prevents them from studying the evolution of investment in the data.

I look at the cyclical components of quarterly data on Gross Domestic Product (GDP), consumption and investment.[4] All series are expressed in annualized real per-capita terms. I extract the cyclical component using the Hodrick-Prescott filter with a standard smoothing parameter for quarterly observations. I normalize the data to 2007:Q3, which is the quarter prior to the official start of the recession according to the NBER. The y-axis in the figure is expressed in terms of the percent change relative to the peak of the NBER cycle. Output, investment and consumption all experienced a severe and prolonged contraction. Investment fell below trend together with GDP at the start of the recession.[5] Consumption remained above its pre-recession level for another three quarters, then fell rapidly along with investment as the financial crisis intensified. By the trough in the first quarter of 2009, detrended investment had fallen 25% from its peak, while detrended output fell 5% and detrended consumption about 3%.[6]

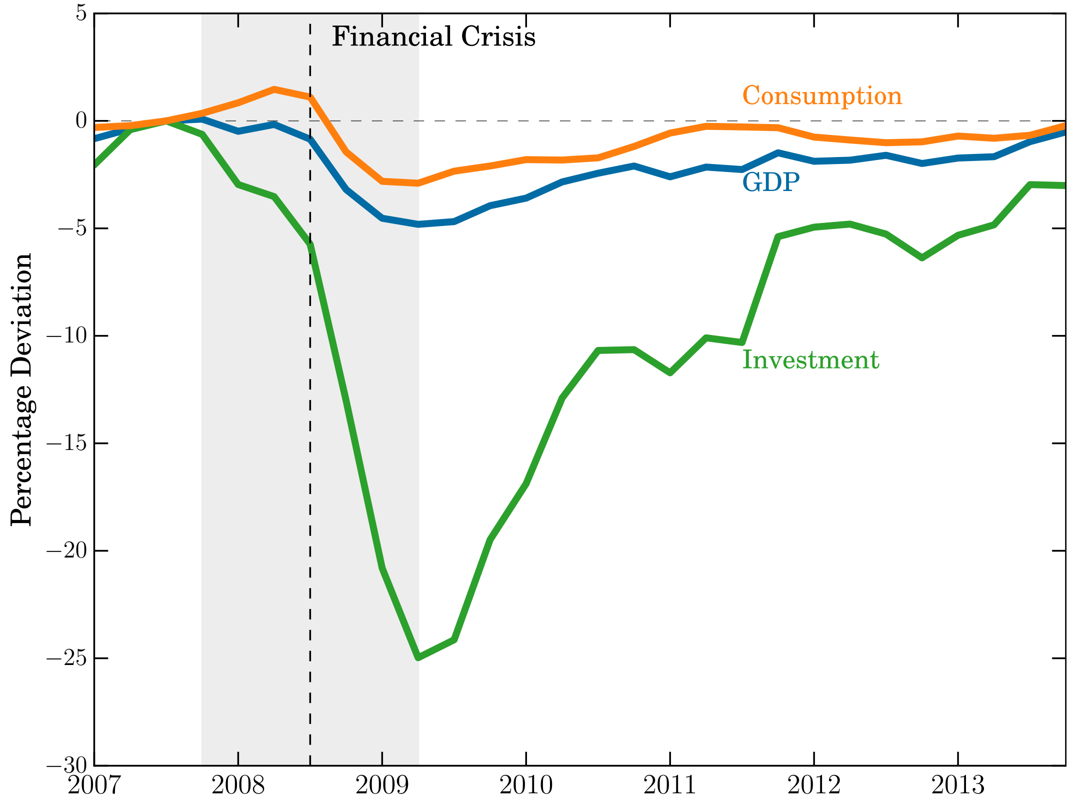
Figure 1.2 shows the evolution of prices and interest rates during the same time period. The figure shows inflation of the GDP deflator and the annualized effective federal funds rate, both expressed in percentage terms. At the onset of the recession and before 2008:Q3, inflation remained roughly around 2% while the

[4]My measure of consumption includes private personal consumption expenditure om non-durable goods and services, whereas my measure of investment combines fixed private investment and the private consumption of durable goods. Additional details on the data series are provided in Section 1.5.2.

[5]Net exports did not contract until 2008:Q3. In fact, the value of total exported goods and services increased 4.5% between 2007:Q3 and 2008:Q2, while imports declined 0.1%. During the same period, government consumption increased 0.2% with respect to its pre-recession trend.

[6]In terms of levels, output fell by 7.3% from peak to through, consumption fell 5%, and investment fell 29%. Additional measures of economic activity also deteriorated sharply. Average weekly hours fell by almost 2% and the civilian unemployment rate rose from 4.4% to 9.8%

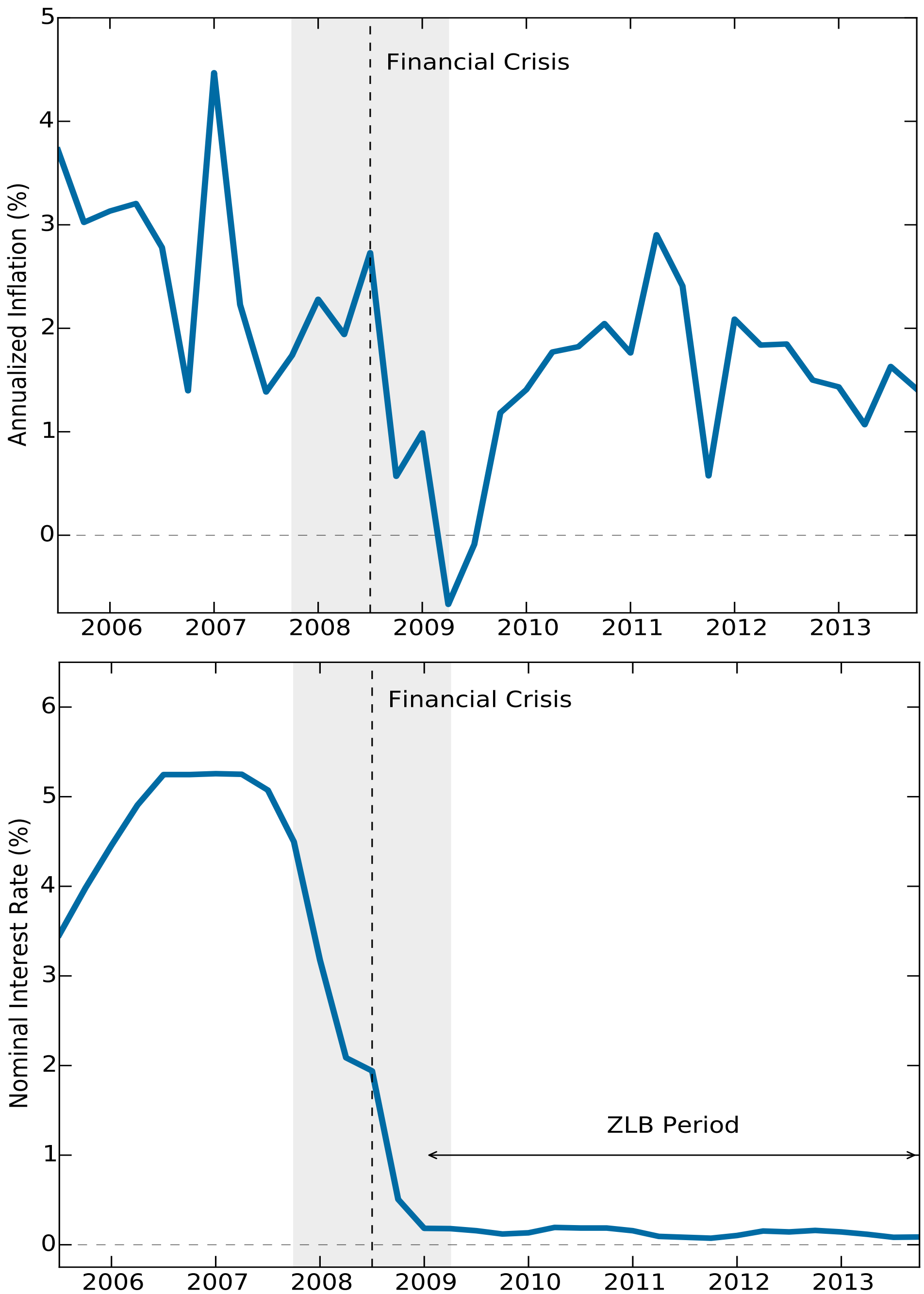
Figure 1.1: U.S. Great Recession: Macroeconomic Comovement



Notes: Output, consumption and investment are expressed in annualized real per-capita terms. All series are detrended using the HP filter ($\lambda = 1600$) and normalized to 2007:Q3. The shaded region indicates the NBER recession.

nominal interest rate fell from 4.75% to 2%. As economic conditions worsened inflation fell rapidly and became negative in the first quarter of 2009. At the same time the nominal interest fell below 0.20%, effectively reaching its lower bound. There is no doubt that from 2007:Q3-2008:Q2 the U.S. economy was in a recession. However up, to that point the evolution of macroeconomic aggregates can be dubbed a plain vanilla recession. As is evident from both figures, from 2008:Q3 onward the story changed substantially, with consumption turning around and quickly falling below pre-recession levels, and investment contracting even at a faster rate. Trying to uncover the forces behind this latter period is challenging precisely because the zero lower bound became binding.

Figure 1.2: Inflation and Interest Rates



Notes: The shaded region indicates the NBER recession.

1.3.1 Potential causes of the Great Recession

I present some informal discussion of the micro foundations for the investment and preference shocks that play a central role in my results. I focus on these shocks because the timing of the dramatic decline in consumption and investment shown in Figure 1.1 points to the prominence of disruptions originating in the financial sector of the economy. In addition, there is a long tradition in the ZLB literature, in particular in small scale New Keynesian models without capital, that relies on shocks to preferences as a simple mechanism to cause contractions in aggregate demand and push the economy to the ZLB.

Preference shock. This type of shock affects the growth rate of aggregate consumption through movements in the real interest rate that tilt the consumption Euler equation.[7] Where do these movements in the real interest rate come from? Guerrieri and Lorenzoni (2011) provide a possible explanation based on tightening of borrowing constraints. A sudden reduction in the debt limit forces household near the constraint to reduce consumption and repay debt; the increased desire for savings induced by precautionary motives puts downward pressure on the nominal interest rate. From this perspective, a tightening of borrowing constraints in a heterogeneous agent economy provides a rationale for an increase in the desire to save and a reduction in the nominal interest rate that can be captured by shocks to households' preferences in the representative agent economy.

[7] A typical linearized Euler equation expressed in percentage deviations and assuming logarithmic utility is: $\hat{c}_{t+1} - \hat{c}_t = \mathbb{E}_t\{\hat{R}_t - \hat{\pi}_{t+1} + \hat{\epsilon}_{t+1}\}$. Here ϵ_t are the shocks to households' subjective discount factor.

Investment shock. With respect to shocks that distort the intertemporal margin of capital accumulation there are various reduced form interpretations. For example, Justiniano, Primiceri and Tambalotti (2010) and Justiniano, Primiceri and Tambalotti (2011) document that a type of investment shock represented as a wedge in the transformation of current investment into installed capital (marginal efficiency of investment) is the main source of business cycle fluctuations, and they provide evidence that such a shock played a significant role in the run-up to the Great Recession. However, they cannot provide estimates of the investment shock after 2008:Q3 because their solution methods are unable to capture the ZLB constraint. A shock to the marginal efficiency of investment can be interpreted as a disruption in financial intermediation that affects the supply of capital and generates fluctuations in its rate of return. Justiniano, Primiceri and Tambalotti (2010) point out that a costly monitoring friction in the spirit of Carlstrom and Fuerst (1997) also gives rise to a wedge that affects the transformation of investment into new capital. This wedge looks similar to a shock that shifts the cost of adjusting investment, and the authors interpret it as a disturbance that raises the cost of monitoring investment projects. This observation is consistent with the large increase in corporate credit spreads observed between 2008:Q1-2010:Q2, as documented by Gilchrist, Yankov and Zakrajšek (2009) and Gilchirst and Zakrajšek (2012).

1.4 The model

This section describes the model used for evaluating the macroeconomic dynamics observed during the Great Recession period. The model economy contains several frictions that introduce nominal and real rigidities that have been shown to be successful in capturing the dynamics of macroeconomic aggregates (Christiano, Eichenbaum and Evans (2005), Smets and Wouters (2007) and Justiniano, Primiceri and Tambalotti (2010)). The frictions in the model include price rigidity in the form of convex cost of price adjustment, habit formation in consumption, variable capital utilization, and investment adjustment costs. The dynamics are driven by exogenous shocks to the growth rate of technological progress, shocks to preferences, shocks to the marginal efficiency of investment, a shock to aggregate demand in the form of government purchases, and a shock to the monetary policy rule.

1.4.1 Households

Preferences. There is a representative household that consumes and supplies labor L_t. Preferences are separable over consumption and labor (hours worked) and take the following functional form:

$$\max_{\{C_t,I_t,u_t,\bar{K}_t,B_t,L_t\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t d_t \left[\ln\left(C_t - hC_{t-1}\right) - \psi_L \frac{L_t^{1+\nu}}{1+\nu} \right] \tag{1.1}$$

The representative household maximizes expected discounted utility, where $\mathbb{E}_t$ denotes the expectation operator conditional on information available in period t and

β is the discount factor. The utility specification allows for external habits in consumption, where the parameter h controls the strength of the habit. The utility cost of labor is controlled by the term ψ_L, and ν represents the inverse of the Frisch elasticity of labor supply. The term d_t is an intertemporal shock that follows a stationary first order autoregressive process

$$\ln d_t = \rho_d \ln d_{t-1} + \varepsilon_t^d, \quad \text{with} \quad \varepsilon_t^d \sim N(0, \sigma_d^2) \tag{1.2}$$

The d_t shock captures exogenous changes in the desire to increase or decrease consumption in the present compared to the future, and in what follows I refer to it simply as the *preference shock*.

Budget constraint. Households receive a nominal wage W_t as compensation for the labor they supply to intermediate firms. The capital stock of the economy $\bar{K}_{t-1}$ is owned by the households, who rent it to intermediate firms every period in exchange for a nominal return R_t^k. In addition to the quantity of capital rented to firms, the household also chooses the intensity of capital utilization in the production process, denoted by u_t, such that the amount of capital that firms use to produce is equal to $K_t = u_t \bar{K}_{t-1}$. A higher intensity of operation of the capital stock entails a real cost for the household denoted by $\mathcal{A}(u_t)$, expressed in terms of the final consumption good. In addition to factor income, the representative household collects interest from holding a one period risk-free nominal bond B_{t-1} issued by the government. This asset pays R_{t-1} dollars in period t. In addition, the household pays a lump sum tax T_t, and receives the profits generated by firms Π_t. Income is allocated to consumption (C_t), investment (I_t) and the purchase of government bonds issued in the current period (B_t). The aggregate price level in this economy is P_t and the period budget constraint in nominal terms is therefore given by

$$P_t C_t + P_t I_t + B_t \leq W_t L_t + R_t^k u_t \bar{K}_{t-1} - P_t \mathcal{A}(u_t) \bar{K}_{t-1} + R_{t-1} B_{t-1} - T_t + \Pi_t \tag{1.3}$$

Investment frictions. Investment decisions are subject to an adjustment cost function $S(.)$ and a shock to the marginal efficiency of investment μ_t. As in Justiniano, Primiceri and Tambalotti (2011), I interpret this shock as a reduced form representation of a friction that disturbs the process of financial intermediation and affects the efficiency with which investment goods are transformed into capital:

$$\bar{K}_t = (1-\delta)\,\bar{K}_{t-1} + \mu_t \left[1 - S\left(\frac{I_t}{I_{t-1}}\right)\right] I_t \tag{1.4}$$

The marginal efficiency of investment evolves according to the process

$$\ln \mu_t = \rho_z \ln \mu_{t-1} + \varepsilon_t^\mu, \quad \text{with} \quad \varepsilon_t^\mu \sim N(0, \sigma_\mu^2) \tag{1.5}$$

Let Ξ_t denote the multiplier associated with the capital accumulation equation and Λ_t the multiplier associated with the budget constraint of the household. The optimal investment allocation implies that the relative price of installed capital in terms of consumption goods is a function of current and future realizations of the investment efficiency shock:

$$\frac{\Xi_t}{P_t \Lambda_t} = \beta \mathbb{E}_t \frac{P_{t+1}\Lambda_{t+1}}{P_t \Lambda_t} \left\{ \left[u_{t+1} \frac{R^k_{t+1}}{P_{t+1}} - \mathcal{A}(u_{t+1}) \right] + (1-\delta) \frac{\Xi_{t+1}}{P_{t+1}\Lambda_{t+1}} \right\}$$

The marginal efficiency of investment, μ_t, affects the transformation of current investment into installed capital directly through the capital accumulation equation and also indirectly through Tobin's Q, which affects the rate on return of capital.

Labor Supply. Labor services produced by the representative household are sold to a perfectly competitive labor market at the aggregate nominal wage rate W_t. The optimality condition for the household's intratemporal allocation is:[8]

$$W_t = d_t \psi_L \frac{L_t^{\nu}}{\Lambda_t}$$

1.4.2 Firms

The production side of the economy consists of perfectly competitive final good producers that buy intermediate goods $Y_{i,t}$ from a continuum of firms that operate in a monopolistically competitive market. The intermediate firms are indexed by $i \in [0, 1]$.

Final good producers. The final good firms buy intermediate inputs from producers and aggregate the intermediate goods using a technology with constant

[8] The shock d_t enters the intratemporal condition because it also affects the marginal disutility of labor. This specification of the preference shock helps the model generate a positive correlation between consumption and hours, to match business cycle facts without the need of a separate shock to preferences for leisure. However, the equilibrium response of hours to the preference shock depends on the wealth effects that the preference shocks generate through changes in consumption.

elasticity of substitution to produce the consumption good Y_t. Taking the prices of inputs $P_{i,t}$ and the price at which they sell the final good P_t as given, the final-good firm chooses its demand for each intermediate input $Y_{i,t}$ to maximize profits:

$$\max_{Y_{i,t}} P_t Y_t - \int_0^1 P_{i,t} Y_{i,t} di, \quad s.t. \quad Y_t \leq \left[\int_0^1 {Y_{i,t}}^{1-\lambda_p} di \right]^{\frac{1}{1-\lambda_p}} \tag{1.6}$$

where λ_p is the inverse of the elasticity of substitution across intermediate inputs, which controls the steady state markup of price over marginal cost. The optimal demand for intermediate goods satisfies $Y_{i,t} = \left(\frac{P_{i,t}}{P_t}\right)^{-\frac{1}{\lambda_p}} Y_t$.

Intermediate-goods firms. Intermediate firms operate a technology that combines labor and capital to produce the intermediate good:

$$Y_{i,t} = \begin{cases} K_{i,t}^{\alpha} \left(A_t L_{i,t}\right)^{1-\alpha} - A_t F & \text{if } K_{i,t}^{\alpha} \left(A_t L_{i,t}\right)^{1-\alpha} > A_t F \\ 0 & \text{otherwise} \end{cases} \tag{1.7}$$

Here $K_{i,t}$ and $L_{i,t}$ denote the firm's demand for effective units of capital and composite labor services respectively. A_t is an aggregate technology shock with growth rate $z_t \equiv A_t/A_{t-1}$. The growth rate z_t follows an exogenous autoregressive process:

$$\ln(z_t/z) = \rho_z \ln(z_{t-1}/z) + \varepsilon_t^z, \quad \text{with} \quad \varepsilon_t^z \sim N(0, \sigma_z^2) \tag{1.8}$$

The term F represents a fixed cost that is calibrated to ensure zero profits in steady state. The growth rate of technology along the balanced growth path is given by $z = \gamma$.

Marginal costs. Intermediate firms rent labor and capital in perfectly competitive markets taking factor prices W_t and R_t^k as given. Each firm solves the following program:

$$\min_{K_{i,t},L_{i,t}} R_t^k K_{i,t} + W_t L_{i,t} \tag{1.9}$$

subject to the production technology (1.7). Cost minimization entails the following equilibrium condition

$$\frac{K_{i,t}}{L_{i,t}} = \frac{\alpha}{1-\alpha}\frac{W_t}{R_t^k} \tag{1.10}$$

Because capital is traded in an economy-wide market, all intermediate producers take as given the aggregate rental rate of capital R_t^k. As a consequence the optimal factor allocation depends only on aggregate prices, so that aggregation is straightforward. Using the definition of aggregate demand for inputs, $K_t = \int_0^1 K_{i,t} di$ and $L_t = \int_0^1 L_{i,t}$, it is easy to show that the following expression for the marginal cost of production holds:[9]

$$MC_t = \alpha^{-\alpha}(1-\alpha)^{\alpha-1}\frac{W_t^{1-\alpha} R_t^{k,\alpha}}{A_t^{1-\alpha}} \tag{1.11}$$

Price setting. Intermediate firms face a cost of adjusting prices in every period. The cost is expressed as a fraction of firms' revenue and is controlled by the convex

[9] Appendix A.1 shows that because optimal factor allocation is identical across firms, so are marginal costs.

function $\Phi_p(P_t/P_{t-1})$. Taking the marginal cost MC_t as given, each intermediate firm solves the following price setting problem:

$$\max_{\{P_{i,t}\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \Lambda_t \left\{ \left[1 - \Phi_p \left(\frac{P_{i,t}}{P_{i,t-1}} \right) \right] P_{i,t} Y_{i,t} - MC_t Y_{i,t} \right\} \tag{1.12}$$

$$s.t. \quad Y_{i,t} = \left(\frac{P_{i,t}}{P_t} \right)^{-\frac{1}{\lambda_p}} Y_t \tag{1.13}$$

1.4.3 Government

Monetary Policy. The monetary authority controls the short term interest rate following an operational rule that responds to deviations of inflation with respect to the central bank's desired level of inflation and the gap of observed output with respect to the non-stochastic level of output along the balanced growth path of the economy, Y^*. Based on the results in Aruoba, Cuba-Borda and Schorfheide (2013), I do not consider equilibria with deflationary dynamics, and I let the central bank's desired level of long-run inflation coincide with the steady-state level of inflation $\pi^* > 0$.[10] The main difference with respect to the standard analysis is that I impose the zero lower bound constraint on the nominal interest rate:

$$R_t = \max \left\{ 1, \left[(r^* \pi^*) \left(\frac{\pi_t}{\bar{\pi}} \right)^{\psi_1} \left(\frac{Y_t}{Y^*} \right)^{\psi_2} \right]^{\rho_R} R_{t-1}^{1-\rho_R} \exp(\varepsilon_t^r) \right\}, \tag{1.14}$$

I assume that the response of the interest rate is smoothed with respect to the previously observed nominal interest rate. The parameter ρ_R controls the speed of

[10] Aruoba, Cuba-Borda and Schorfheide (2013) find that during the Great Recession the U.S. economy remained in a targeted-inflation regime where the steady-state level of inflation is positive.

the adjustment, while ε_t^R is a monetary policy shock, that is normally distributed with mean zero and standard deviation σ_r^2.

Fiscal Policy. The government issues bonds B_t every period to satisfy its flow budget constraint $P_t T_t - P_t G_t = R_{t-1} B_{t-1} - B_t$. The term G_t is government expenditure, and evolves exogenously according to $G_t = \zeta_t Y_t = \left(1 - \frac{1}{g_t}\right) Y_t$, where g_t is an exogenous autoregressive process with mean $\bar{g} = 1/(1-\zeta)$:

$$\ln(g_t) = (1-\rho_g)\ln \bar{g} + \rho_g \ln(g_{t-1}) + \varepsilon_t^g, \quad \text{with} \quad \varepsilon_t^g \sim N(0, \sigma_g^2) \tag{1.15}$$

Here, ζ is the government expenditure to GDP ratio in the steady state.

1.4.4 Market Clearing

The market clearing conditions for this economy are as follows. I consider a symmetric price equilibrium, $P_{i,t} = P_{j,t} = P_t \quad \forall i, j \in [0,1]$, that satisfies: *(i)* the market for capital clears: $\int_0^1 K_{i,t} di = K_t$, *(ii)* the market for labor services clear: $\int_0^1 L_{i,t} di = L_t$ *(iii)* Installed capital $\bar{K}_t$ evolves according to (1.4), such that the market for final goods clears:

$$\left[\frac{1}{g_t} - \Phi\left(\pi_t\right)\right] Y_t = C_t + I_t + \mathcal{A}(u_t)\bar{K}_{t-1} \tag{1.16}$$

By Walras' Law the market for government bonds clears if all other markets clear.

1.4.5 Functional Forms

For estimation and the subsequent quantitative analysis I specify specific functional forms for the price adjustment cost function $\Phi(.)$, the investment adjustment cost function $S(.)$ and the capacity utilization function $\mathcal{A}(.)$, given by:

$$\Phi\left(\frac{P_{i,t}}{P_{i,t-1}}\right) = \frac{\phi_p}{2}\left(\frac{P_{i,t}}{P_{i,t-1}} - \pi^*\right)^2 \tag{1.17}$$

$$S\left(\frac{I_t}{I_{t-1}}\right) = \frac{\xi}{2}\left(\frac{I_t}{I_{t-1}} - \gamma\right)^2 \tag{1.18}$$

$$\mathcal{A}(u_t) = \rho^* \frac{u_t^{1+\chi} - 1}{1+\chi} \tag{1.19}$$

where ϕ_p and ξ are parameters that control the magnitude of the adjustment costs of prices and investment. The parameter χ controls the curvature of the capacity utilization function. The parameters ρ^* and π^* denote the steady state values of the rental rate and the inflation rate respectively.

1.4.6 Equilibrium Conditions and Solution Strategy

The characterization of the equilibrium conditions of the model is relatively standard and is relegated to Appendix A.1.1. The stochastic process for aggregate technology A_t introduces a source of long-run growth in the model. The equilibrium conditions are transformed into a stationary representation by dividing all real variables by the technology factor A_t and all nominal variables by the factor P_tA_t. Further details are presented in Appendix A.1. In what follows small case letters refer to detrended variables, e.g. $x_t \equiv \frac{X_t}{A_t}$.

The computational strategy adopted in this paper relies on the concept of *Functional Rational Expectations Equilibrium (FREE)* employed in Krueger and Kubler (2004*b*) and Malin, Krueger and Kubler (2007). The idea consists of finding a suitable set of functions defined over a compact set that satisfy the first order equilibrium conditions of the model. More precisely, the equilibrium can be characterized in terms of the five policy functions $\mathcal{C} = \{L(\mathbb{S}), q(\mathbb{S}), \lambda(\mathbb{S}), i(\mathbb{S}), \pi(\mathbb{S})\}$,which correspond to hours worked, Tobin's q, marginal utility of wealth, investment and inflation, respectively.

The solution is assumed to be a time-invariant function of a minimum set of state variables $\mathbb{S}$. The state vector is formed by $\mathbb{S} = \left[R_{-1}, c_{-1}, \bar{k}_{-1}, i_{-1}, \mu, d, z, g, \varepsilon^r\right]$, where x_{-1} corresponds to the lagged value of the variable, x denotes its current realization, and x' denotes future realizations. In total the model has $n = 9$ state variables. The choice of $\mathbb{S}$ is fundamental for the characterization of the equilibrium. Because the ZLB creates a kink in the monetary policy rule, the model has two steady states, opening the possibility of multiple equilibrium dynamics (i.e. the control functions that satisfy the equilibrium conditions may not be unique). In fact this is the case in Aruoba, Cuba-Borda and Schorfheide (2013), who show that it is possible to construct a deflationary equilibrium and many non-fundamental equilibria in which the state vector is augmented by an extraneous stochastic process (a sunspot) that moves the equilibrium dynamics from the equilibrium with positive inflation to the equilibrium with deflationary dynamics. They find no evidence that the U.S. economy switched away from the target-inflation equilibrium during the Great Recession. For this reason I focus solely on the targeted-inflation equilibrium in the quantitative analysis of section 1.5.

Definition 1 *A FRE Equilibrium is defined by the compact set* $\mathbb{S} \in \mathbb{R}^n$ *and the set of control functions* $\mathcal{C} = \{L(\mathbb{S}), q(\mathbb{S}), \lambda(\mathbb{S}), i(\mathbb{S}), \pi(\mathbb{S})\}$ *such that:*

$$\lambda(\mathbb{S}) = \beta R_t \mathbb{E} \frac{\lambda(\mathbb{S}')}{\pi(\mathbb{S}')} \frac{1}{\gamma e^{z'}} \tag{1.20}$$

$$\lambda(\mathbb{S}) = \frac{\gamma e^{d+z}}{\gamma c e^{z} - h c_{-1}} + h\beta \mathbb{E} \frac{e^{d'}}{\gamma c' e^{z'} - hc} \tag{1.21}$$

$$q(\mathbb{S}) = \beta \mathbb{E} \frac{\lambda(\mathbb{S}')}{\gamma e^{z'} \lambda(\mathbb{S})} \{\rho(\mathbb{S}')u' - \mathcal{A}(u') + (1-\delta) q(\mathbb{S}')\} \tag{1.22}$$

$$1 - e^{\mu} q(\mathbb{S}) \left[1 - S(\Delta i) - dS(\Delta i)\Delta i\right] = \beta \mathbb{E} q(\mathbb{S}') \frac{\lambda(\mathbb{S}')}{\lambda(\mathbb{S})} \frac{1}{\gamma e^{z'}} e^{\mu'} dS(\Delta i')\Delta i'^2 \tag{1.23}$$

$$(\frac{1}{\lambda_p} - 1) \left[1 - \Phi_p(\pi(\mathbb{S}))\right] - \frac{mc}{\lambda_p} + d\Phi_p(\pi(\mathbb{S}))\pi(\mathbb{S}) = \beta \mathbb{E} \frac{\lambda(\mathbb{S}')}{\lambda(\mathbb{S})} \Phi_p(\pi(\mathbb{S}'))\pi(\mathbb{S}') \frac{y'}{y} \tag{1.24}$$

$$w = \psi_L \frac{e^d L(\mathbb{S})^{\nu}}{\lambda(\mathbb{S})} \tag{1.25}$$

$$\rho = \frac{\alpha}{1-\alpha} \frac{L(\mathbb{S})}{\bar{k}_{-1} u} \gamma e^{z} \tilde{w} \tag{1.26}$$

$$u = d\mathcal{A}^{-1}(\rho) \tag{1.27}$$

$$mc = \alpha^{-\alpha}(1-\alpha)^{\alpha-1} w^{1-\alpha} \rho^{\alpha} \tag{1.28}$$

$$c = \left[\frac{1}{\bar{g} e^{g}} - d\Phi_p(\pi(\mathbb{S}))\right] y - \mathcal{A}(u) \frac{\bar{k}_{-1}}{\gamma e^{z}} - i(\mathbb{S}) \tag{1.29}$$

$$\bar{k} = (1-\delta) \frac{\bar{k}_{-1}}{\gamma e^{z}} + \mu \left[1 - S(\Delta i)\right] i(\mathbb{S}) \tag{1.30}$$

$$y = k^{\alpha} L(\mathbb{S})^{1-\alpha} - \mathcal{F} \tag{1.31}$$

$$R = \max\left\{1, \left[(r^* \pi^*) \left(\frac{\pi(\mathbb{S})}{\pi^*}\right)^{\psi_1} \left(\frac{y}{y^*}\right)^{\psi_2}\right]^{\rho_R} R_{-1}^{1-\rho_R} e^{\varepsilon^r}\right\} \tag{1.32}$$

To simplify notation I use: $\Delta i = \frac{i(\mathbb{S})}{i_{-1}} e^{z} \gamma$, $d\Phi_p = \partial \Phi_p(x)/\partial x$,$dS = \partial S(x)/\partial x$

and $d\mathcal{A} = \partial\mathcal{A}(x)/\partial x$, where x stands in for the argument of each function as defined in Section 1.4.5.

1.5 Quantitative Results

1.5.1 Computational Strategy

The FRE Equilibrium definition requires the solution of an infinite dimensional nonlinear system of equilibrium conditions in order to characterize the functions in $\mathcal{C}(\mathbb{S})$. The solution strategy adopted here uses two approximations. First, the compact state space $\mathbb{S}$ is represented using ergodic set methods as in Judd, Maliar and Maliar (2012) and Aruoba, Cuba-Borda and Schorfheide (2013). This reduces the problem to finding the best approximation to the true functions in $\mathcal{C}(\mathbb{S})$ using a grid $m_i \in \mathcal{M} \subset \mathbb{S}, \quad i = 1, \ldots, M$, that represents the region of the state space that is relevant to characterize the solution. Section 1.5.3 explains how to construct this grid such that it contains enough nodes where the ZLB binds. Second, the unknown equilibrium functions in $\mathcal{C}^{(j)} \in \mathcal{C}(\mathbb{S})$ are approximated by piece-wise continuous functions characterized by a set of coefficients $\theta \in \mathbb{R}^{2\times N}$. These coefficients are used to construct linear combinations of basis functions $\mathcal{T}_j : \mathbb{S} \to \mathbb{R}$ that are evaluated in each solution node. In particular I use Chebyshev polynomials, defined as $T_j(x) = \cos(j \times \arccos(x))$, where $x \in [-1, 1]$, which are combined using a *complete polynomial* rule in order to form the multidimensional basis function $\mathcal{T}_j$.[11]

Because I will look for an approximate solution to the functional equations

[11] Additional details of the construction of the basis functions is provided in section A.2.

that satisfy the equilibrium conditions, a criterion that informs about how close my approximation is to the "true" solution is needed.[12] The metric for the approximation is given by a set of residual functions $\mathcal{R}(\mathbb{S})$ that are obtained from the equilibrium conditions (1.20)-(1.24). For example, the residual for equation (1.20) is:

$$\mathcal{R}_1(\mathbb{S}) = \lambda(\mathbb{S}) - \beta R \mathbb{E} \frac{\lambda(\mathbb{S}')}{\pi(\mathbb{S}')} \frac{1}{\gamma e^{z'}}. \tag{1.33}$$

Appendix A.2 explains how to construct all the residual functions used to solve the nonlinear model. To evaluate the expectations that appear in the residual functions, I use a sparse-grid approximation based on the integration rules discussed in Heiss and Winschel (2006).

1.5.2 Parameter Estimation with Pre-ZLB data

Estimation of the model subject to the ZLB constraint is a computationally intensive enterprise because it requires the nonlinear solution to be computed for a large number of parameter vectors. Instead I follow a two-step procedure. First I estimate a log-linearized version of the model using data prior to the ZLB episode from *1984:Q1 - 2008:Q3*. I estimate the model parameters using a first-order approximation of the DSGE model equilibrium conditions, and characterize the posterior distribution of the parameters using the Random Walk Metropolis-Hastings algorithm

[12]Note here that because of the multiplicity of equilibria, constructing the approximate function numerically requires an initial guess that converges to the desired equilibrium. It turns out that using the decision rules of the linearized model as the initial guess, the nonlinear decision rule converge to the dynamics of the equilibrium with positive steady state inflation.

described in An and Schorfheide (2007). Conditional on the parameters obtained from the pre-ZLB period, I solve the model enforcing the ZLB and use Sequential Monte Carlo methods to extract the underlying states and shocks corresponding to the period *2008:Q2 - 2013:Q4*.

Some parameters are fixed before the estimation because the likelihood is not informative with respect to them. The parameter ζ is set to 0.22 in order to match the long-run average ratio of government consumption expenditure to Gross Domestic Product observed in NIPA data from 1960-2013. The parameter λ_p is fixed at 0.1667, implying a steady state price markup of 20%. This value is slightly lower than that estimated in medium-scale DSGE models, which find a steady price-markup of 28%. Since I do not use data on hours worked for estimation I set the parameter $\nu = 1$, implying a Frisch elasticity of labor supply equal to one. This value is large with respect to the microeconomic evidence for this elasticity along the intensive margin reported in Chetty et al. (2013). Nevertheless it is within the range of estimated values obtained by Rios-Rull et al. (2012*b*). I normalize the steady state level of hours worked to 1/3 using the parameter ψ_L.

Data. I use quarterly data on five macroeconomic variables covering the period 1984:Q1 to 2008:Q3.[13] I map the model variables to data on output growth, consumption growth, investment growth, inflation and the nominal interest rate. I use data on Gross Domestic Product (GDP) to measure output growth. Consumption is the sum of personal consumption of non-durable goods (PCND) and services

[13] All the series were extracted from the FRB St. Louis FRED Database, with the original name of the data series shown in parenthesis. Additional details are provided in section A.3.

(PCND). Investment includes the personal consumption of durable goods (PCDG), fixed private investment (FPI) and the change in inventories (CBI). All these series are scaled by the civilian non-institutionalized population aged over sixteen years (CNP16OV) and deflated using the implicit GDP price deflator (GDPDEF). Growth rates are computed as one period log differences expressed in percentages. Inflation is computed as the percentage log-difference of the implicit price deflator. Finally the nominal interest rate is measured using the quarterly average of the Federal Funds Rate (FEDFUNDS).

Table 1.1 presents parameter estimates based on the results of the MH simulator. I obtain 100,000 draws of parameters from the posterior distribution and construct summary statistics of the posterior distribution based on the last 50,000 draws of the sequence. A few results from the estimation are discussed next. The estimated value of the share of capital in the intermediate firms' production function is 0.18. In estimated DSGE models this parameter is usually below the commonly used value of 0.33 obtained from long-run averages of the capital share in aggregate output. With respect to the parameter h that controls the consumption habit and the persistence of consumption with respect to nominal shocks, I obtain a value of 0.55. This degree of habit persistence is relatively modest compared to the most recent estimates of Christiano, Eichenbaum and Trabandt (2014).

The parameter χ, which controls the elasticity of capacity utilization with respect to the rental rate of capital, is an important parameter in determining the persistence of inflation in response to demand shocks. There is wide range of variation in previous estimates of this parameter. For example, Christiano, Eichenbaum and Evans (2005) assume a very small value of 0.01, whereas Justiniano, Primiceri and Tambalotti (2010) obtain an estimate in the range of 3 to 7. My estimate for χ is closer to the latter, implying that the rental rate is very sensitive to changes in capacity utilization. This implies that, all else equal, marginal costs will respond strongly to movements in capital utilization, making inflation less persistent and also more volatile.

The parameter controlling nominal rigidities is an important one for the transmission of shocks in the model and deserves additional attention. I estimate the price adjustment cost parameter ϕ_p indirectly using the implied slope of the Phillips curve $\kappa(\phi_p)$. The estimate of the slope of the Phillips curve is steep, with a value of $\kappa = 0.21$ at the posterior mean. To give a sense of the degree of price stickiness implied by the model, I compute the associated frequency of price adjustment in a first order approximation of the Phillips curve derived under Calvo pricing. The estimated value of κ implies that firms would adjust prices roughly every three quarters, compared to the four to six quarters commonly obtained in the DSGE literature. Nonetheless, the posterior credible set of the estimated Phillips curve parameter is consistent with the wide range of values reported in Schorfheide (2008*a*).

I use informative priors to estimate the parameters of the monetary policy rule. There is a significant amount of persistence in the determination of the interest rate, reflected in the estimate of ρ_R. The response of the nominal rate to inflation is within the range of estimated parameters in the literature. The output gap response parameter seems low because the policy rule is expressed in terms of quarterly percentage deviations of output with respect to its balanced growth path. The estimated parameters that control the growth rate of technology and the inflation rate along the balanced growth path imply a long run rate of output growth of 2% and a long run inflation rate of 2.3% in annualized terms.

Fit of the Estimated Model. The parameters were estimated using a Bayesian framework, and to check the empirical fit of the model I report posterior predictive checks. For this, I use simulated trajectories from the model and draws from the posterior distribution of parameters to construct a set of statistics $\mathcal{S}(\tilde{\mathbb{Y}}^T) \in \mathbb{R}^n$. The same sample statistic can be constructed using observed data $\mathcal{S}(\mathbb{Y}^T)$. If the observed sample statistic lies far in the tail of the predictive distribution, this indicates that the model has trouble capturing the data along that dimension.

Figure 1.3 shows the posterior predictive checks for the empirical distribution of four statistics $\mathcal{S}(\tilde{\mathbb{Y}}^T)$: the mean, standard deviation, first order autocorrelation and the correlation with GDP growth.[14] The posterior predictive checks reveal that the estimated model captures well the mean of all the observed series except that of investment growth. The reason is that neutral technology shocks are the only source of long-run growth in the model economy. Hence all variables grow at the rate (γ) along the balanced growth path. However, in the data investment has a different long-run growth rate. A simple way to resolve this discrepancy is to introduce an additional source long-run technological progress that only affects investment and that can disentangle the long-run growth rate of both variables. With respect to standard deviations and first order autocorrelations the DSGE model does well in matching the empirical counterparts, although on average it tends to over predict the

[14] The algorithm used to construct the predictive distribution is discussed in Section A.2.5.

Table 1.1: DSGE Model Parameters

	Parameter Description	Prior Mean	Posterior Mean	90% Credible Sets		Prior	Prior SD
h	Habit persistence	0.5	0.5467	0.4582	0.6328	B	0.10
α	Capital share	0.3	0.1806	0.1607	0.1991	N	0.05
ξ	Inv. Adj. Cost	4.0	4.0491	2.5816	5.5861	G	1.00
χ	Cap. Utilization Cost	5.0	5.2988	3.7511	6.9456	G	1.00
$\kappa(\phi_p)$	Phillips Curve	0.3	0.2127	0.0994	0.3262	G	0.20
Taylor Rule:							
ρ_r	Smoothing	0.5	0.7310	0.6756	0.7833	B	0.20
ψ_1	Inflation Gap	1.5	1.6758	1.5299	1.8157	N	0.10
ψ_2	Output Gap	0.005	0.0748	0.0297	0.1154	N	0.05
Shock Process:							
ρ_z	Persistence z shock	0.4	0.0933	0.0134	0.1669	B	0.20
ρ_g	Persistence g shock	0.6	0.9886	0.9794	0.9983	B	0.20
ρ_μ	Persistence μ shock	0.6	0.7000	0.5982	0.8117	B	0.20
ρ_d	Persistence d shock	0.6	0.9475	0.9170	0.9829	B	0.20
$100\sigma_z$	Std. Dev. z shock	0.2	0.9171	0.7333	1.1097	IG	1.00
$100\sigma_g$	Std. Dev. g shock	0.5	0.2746	0.2428	0.3052	IG	1.00
$100\sigma_\mu$	Std. Dev. μ shock	0.5	3.8763	2.4248	5.3372	IG	1.00
$100\sigma_d$	Std. Dev. d shock	0.2	1.2436	0.8029	1.6882	IG	1.00
$100\sigma_r$	Std. Dev. ϵ^r shock	0.2	0.1746	0.1476	0.2004	IG	1.00
Balanced Growth Path:							
$\gamma^{(q)}$	Long Run Growth	0.5	0.4978	0.4604	0.5343	N	0.025
$\pi^{(q)}$	Inflation Rate	0.5	0.5734	0.4302	0.7169	N	0.10
$r^{(q)}$	Discount rate	0.3	0.1848	0.1010	0.2646	G	0.10
Implied Parameters:							
β	Discount Factor		0.9982				
ϕ_p	Price Adj. Cost		15.1062				
$rr^{(q)}$	Real rate		2.7255				

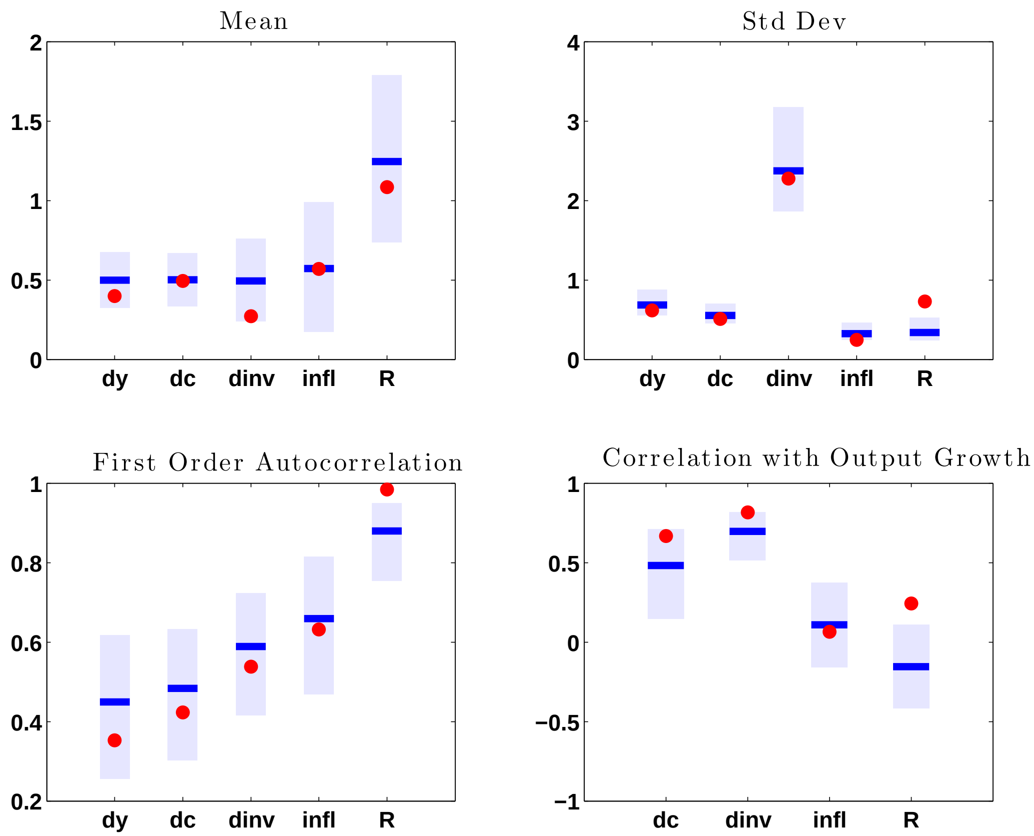
Notes: The parameters were estimated using 1984:Q1-2008:Q3 data. The credible sets are obtained from the 5th and 95th percentiles of the posterior distribution. $r^{(q)} = 100(\beta^{-1} - 1)$

autocorrelations. A similar picture emerges for the correlation with output growth.

Lastly the model tends to under predict the second moments of the nominal interest rate. This could be explained if in practice the monetary authority responds more strongly to periods of low economic activity compared to periods of high economic activity. For example, in the face a negative output gap, the Fed may decide to lower the nominal interest rate quickly. This would generate greater volatility in the observed nominal interest rate but it would reduce its autocorrelation. Similarly, when the economy is in a recovery, the Fed may decide not to increase the nominal rate too quickly, to avoid halting the expansion. This would introduce a higher autocorrelation in observed interest rates but it would lower its volatility. In fact, Aruoba, Bocola and Schorfheide (2013) find evidence that supports the view that the Fed adjusts the nominal interest rate asymmetrically. My specification of the policy rule does not allow for such asymmetry, and hence the posterior predictive distribution fails to account for these dynamics. Nonetheless, the estimated model with pre-ZLB data captures important features of the dynamics of output, consumption, investment, inflation and the nominal rate.

Taking as given the estimated parameters at their posterior mean, I now explain how to incorporate the ZLB into the solution of the model, which is the backbone of the quantitative exercise of section 1.7.

Figure 1.3: Posterior Predictive Checks



Notes: The red dot corresponds to the observed statistic. The dark-blue horizontal bar is the mean of the simulated statistic. The light-blue bands correspond to the 5th and 95th percentiles of the posterior predictive density.

1.5.3 Incorporating the Zero Bound

Before moving to the quantitative analysis, I solve the nonlinear model that incorporates the zero bound. Using the computational strategy described in Section 1.4.6 and the posterior mean of the parameters in Table 1.1, I approximate the FRE equilibrium on a grid constructed using simulation based methods. As in Judd, Maliar and Maliar (2012) I construct a representation of the ergodic set of the model using a clustered grid algorithm. However, as emphasized in Aruoba, Cuba-Borda and Schorfheide (2013), the essential ergodic set does not capture events where the ZLB is binding. Following their computational strategy, I augment the essential ergodic set with grid points that capture binding ZLB periods in the U.S. experience from 2009:Q1-2013:Q4. The additional grid points are obtained from the filtered distribution of states during this period.[15]

Decision rules. The functional equations that characterize the equilibrium dynamics during normal times when the ZLB is not binding (nb) and when the ZLB is binding (b) are parameterized by the vector of unknown coefficients $\Theta = \{\theta^{L,r}, \theta^{q,r}, \theta^{\pi,r}, \theta^{\lambda,r}, \theta^{i,r}\}$, where $r = \{b, nb\}$ denotes one of the two possible regimes of the nominal interest rate.

I use piece-wise smooth functions as in Aruoba, Cuba-Borda and Schorfheide (2013) because they provide a flexible approximation that allows each of the control functions to inherit the kink induced by the zero bound constraint. For example, consider the approximation of the marginal utility of wealth:

$$\lambda(\mathbb{S}) \approx \begin{cases} \sum \theta_j^{\lambda,nb} \mathcal{T}_j(\mathbb{S}) & \text{if } R(\mathbb{S}) > 1, \quad j = 1, \ldots, N \\ \sum \theta_j^{\lambda,b} \mathcal{T}_j(\mathbb{S}) & \text{if } R(\mathbb{S}) = 1, \quad j = 1, \ldots, N \end{cases} \tag{1.34}$$

A total of $2 \times N$ coefficients and basis functions are used to approximate the decision rules over the grid of points $\mathcal{M}$. The piece-wise smooth approximations consist of using one set of coefficients $\theta_j^{\lambda,nb} \in \mathbb{R}^N$ to approximate the functional equation in the regions of the state space where the zero bound is not binding, while a second

[15] The construction of the filtering distributions is explained in Section 1.6.1, and additional details are provided in section A.4.

set of coefficients $\theta_j^{\lambda,b} \in \mathbb{R}^N$ approximates the decision rule when the constraint is active.

Using a third order approximation with a complete basis of Chebyshev polynomials, each element of Θ contains $N = 220$ unknown coefficients; hence a total of $2,200$ unknowns need to be solved numerically. The objective function that pins down the unknown coefficients is given by the sum of squared residuals of the equilibrium conditions $\mathcal{R}(\mathbb{S}, \Theta)$ evaluated at the $M = 600$ grid points, out of which 440 correspond to the essential ergodic set obtained from simulating the model, and the remaining 160 points are obtained from the filtered states described previously. I use a Newton-based solver to find the coefficients (Θ) that minimize $\sum_{i=1}^{M} \mathcal{R}(\mathbb{S}_i, \Theta)^2$.

Accuracy of nonlinear solution After obtaining the vector of coefficients Θ, I check the accuracy of the solution using the bounded rationality measure from Judd (1998), also known as Euler Equation errors. This approach scales the approximation errors $\mathcal{R}(\mathbb{S}, \Theta)$ as a fraction of current consumption and expresses them in terms of unit-free quantities. Table 1.2 shows a summary of the Euler error accuracy measure computed for all residual functions. The average approximation error is on the order of 10^{-3}, which means that the representative agent's loss from following the approximate decision rules is 0.1 cents for every dollar spent.[16]

[16] Figure A.1 shows the full distribution of the Euler equation errors.

Table 1.2: Euler Equation Errors

	Euler Equation	Marginal Utility of Wealth	Capital Euler Equation	Investment Equation	Pricing Equation
Mean	-3.00	-2.63	-3.29	-2.82	-2.35
Min	-6.14	-5.74	-5.33	-5.88	-5.52
Max	-2.14	-1.70	-2.48	-1.80	-1.54

1.6 What Caused the Great Recession?

This section investigates the forces that explain the dynamics of quantities and prices during the Great Recession. I use the implied equilibrium dynamics of the model to match U.S. data from 2000:Q1 to 2013:Q4. A key challenge in understanding the forces driving the events in the aftermath of the financial crisis in 2008:Q3 is backing out the structural shocks that rationalize the data. This is a complicated task because the presence of the ZLB renders the model highly nonlinear.

Prior to the Great Recession, the vast majority of DSGE models ignored the ZLB, because it seemed unlikely that the U.S. economy would ever face shocks large enough to make this constraint binding.[17] Without the ZLB, equilibrium dynamics are well approximated with a linear state-space system, and the estimation of structural innovations can be performed using the Kalman filter, for example as in Bauer, Nicholas and Rubio-Ramirez (2003). In my application the structural innovations are Gaussian but the ZLB causes the dynamics of the economy to be highly

[17]Assuming the ZLB away has the added benefit that linear approximations are enough to characterize equilibrium dynamics, which allows the use of fast solution methods that open the door for estimating medium scale DSGE models.

nonlinear, which complicates inference about the underlying unobserved variables.[18]

I tackle the inference problem using Sequential Monte Carlo methods (SMC) to numerically approximate the distribution of states that rationalizes the sequence of observed output, consumption, and investment growth as well as inflation and the nominal interest rate during the period 2008:Q3 - 2013:Q4.[19] The exercise is similar to Aruoba, Cuba-Borda and Schorfheide (2013) in terms of extracting the filtered states and shocks, but in addition in this paper I use the estimated structural shocks to perform counterfactual exercises about the evolution of observed macroeconomic variables.

1.6.1 Inference of unobserved states

I briefly discuss the inference problem and sketch the mechanics of the filtering algorithm used to estimate unobserved states in nonlinear models. The solution of the system of equilibrium conditions of the model has the following nonlinear state-space representation of the dynamics of the endogenous variables:

$$\mathbb{S}_t = g\left(\mathbb{S}_{t-1}, u_t\right), \quad u_t \sim N(0, \Sigma_u) \tag{1.35}$$

$$\mathbb{Y}_t = m(\mathbb{S}_t) + \varepsilon_t, \quad \varepsilon_t \sim N(0, \Sigma_\varepsilon), \tag{1.36}$$

[18] An example of a New Keynesian model with a linear structure but with non-Gaussian innovations is studied in Curdia, Del Negro and Greenwald (2013).

[19] Herbst and Schorfheide (2014) provide a detailed description of SMC methods, as well as practical guidelines for implementing different algorithms.

Where (1.35) is the transition equation that describes the evolution of the state variables $\mathbb{S}_t$ as a function of the previous position of the system $\mathbb{S}_{t-1}$, and the realization of structural shocks u_t. The relationship of observable variables to the model state variables is given by the measurement equation (1.36), which is augmented by disturbances ε_t, which represent measurement errors that may create a discrepancy between the model implied series and their observed counterparts. I set the variance of the measurement error in (1.36) to 10% of the sample variance in the observables. The introduction of measurement error is not an arbitrary device to increase the fit of the model; rather it is essential for the evaluation of the observation density described below.

Consider s_t and y_t as realizations generated by the nonlinear system. Let Y^t denote a time series of observations from $1, \ldots, t$. The state-space system described above induces an *observation density* $p(y_t|s_t, Y^{t-1})$ and a *transition density* denoted $p(s_t|s_{t-1}, Y^t)$. The filtering problem consists of learning about the realizations of the unobserved states and shocks $\{s_t, u_t\}$ given the sequence of observations Y^t. In other words, the objective is to characterize the shape of the *filtering density* $p(s_t|Y^t)$. In a nutshell, SMC methods start with a discrete approximation of the filtering density $p(s_{t-1}|Y^{t-1})$ characterized by a collection of particles (a swarm) $\left\{\pi_{t-1}^{(i)}, W_{t-1}^{(i)}\right\}_{i=1}^{N_p}$, where N_p is the number of particles, and use the information contained in the current observation y_t together with the state transition equation of the model to update the particles, creating a new particle swarm $\left\{\pi_t^{(i)}, W_t^{(i)}\right\}_{i=1}^{N_p}$ such that the following approximation holds for any function $h(s_t)$: $\frac{1}{N_p}\sum_{i=1}^{Np} h(s_t^{(i)})W_t^{(i)} \approx \int h(s_t)p(s_t|Y^t)ds_t$. For this particular application I set $N_p = 100,000$ particles and start the the approximation to $p(s_0|Y^0)$ using simulations from the solution of the model.

1.6.2 What shocks explain the Great Recession?

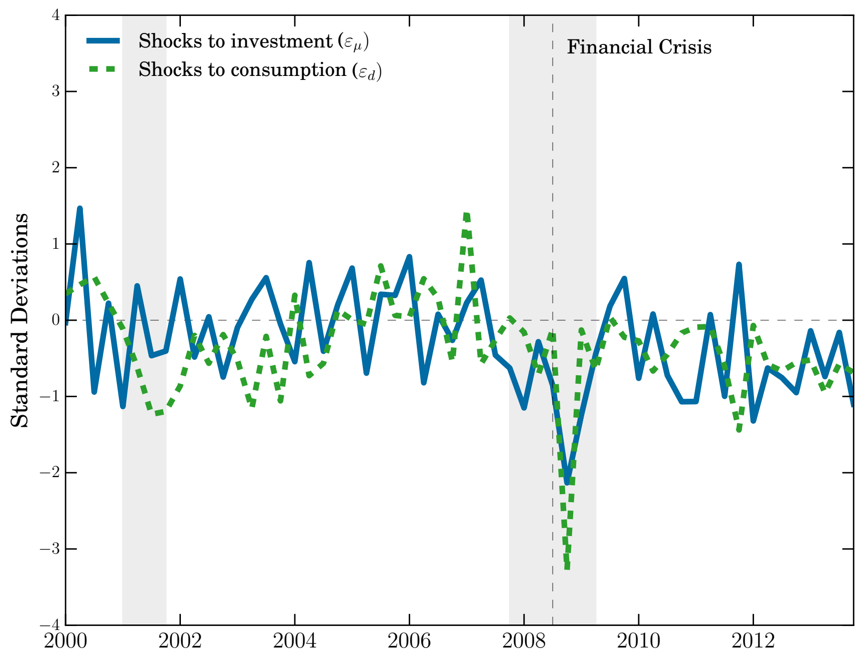
I recover the structural shocks from the approximation of the filtering density $p(s_t|Y^t)$. To illustrate, Figure 1.4 presents the mean of the filtered innovations $\mathbb{E}(s_t|Y^t)$ for *t=2000:Q1* to *t=2013:Q4*, for the marginal efficiency of investment (ε_μ) and the discount factor (ε_d).[20] Feeding these innovations, together with the filtered endogenous states $R_{-1}, c_{-1}, \bar{k}_{-1}, i_{-1}$, through the system (1.35) - (1.36) recovers the observed evolution of consumption, investment and output, as well as the dynamics of inflation and the nominal interest rate. With the exception of 2008:Q3, when financial distress was at its highest, all the innovations are within two standard deviations in both the pre and post recession periods.

Preference and Investment Shocks. From Figure 1.4 can be seen that the economic downturn during the Great Recession is associated with a combination of negative shocks to consumption and investment. The sequence of negative shocks started in the second half of 2007 and continued through the second half of 2008, up to this point, these shocks were no different in magnitude to those observed in previous periods. This suggests that early on the Great Recession started as regular downturn episode accompanied by a build-up of frictions affecting financial intermediation prior to the financial crisis. Prior to 2008:Q3, the negative investment shocks capture the decline in consumption of durable goods and investment in residential

[20]For technology and government expenditure I show the path of their time series process later in the section.

structures, which were the components of aggregate demand that were affected by the deterioration in the housing market at the onset of the recession.

Figure 1.4: Structural Innovations



Notes: The solid blue line is constructed as $\mathbb{E}(\mu_t|Y^t) - \rho_\mu \mathbb{E}(\mu_{t-1}|Y^t)$. Similarly the dashed green line is computed as $\mathbb{E}(d_t|Y^t) - \rho_d \mathbb{E}(d_{t-1}|Y^t)$. The gray areas indicate NBER recession dates.

In the third quarter of 2008, a combination of negative shocks to the marginal efficiency of investment and households' discount factors are necessary to explain the contraction in macroeconomic aggregates. Both shocks are larger than two standard deviations, with the shock to preferences larger than then shock to the marginal efficiency of investment. The size of the shocks after 2008:Q3 imply that the forces that pushed the economy into the trough and triggered a binding zero lower bound were unusual given the structure of the model. What is more interesting is the timing of both shocks, because the model needs to account for both the contraction in investment and consumption to generate the sharp decline in demand leading to the -10% contraction of output growth in the last quarter of 2008. Given that the estimated preference shock is larger, one might conclude that this shock is the leading explanation of the Great Recession. However, as will be discussed in section 1.7, shocks to investment have a larger role in accounting for the contraction of output.

In general the New Keynesian model studied in this paper generates ZLB episodes infrequently, about 0.1% of the time in the ergodic distribution. Because the ZLB happens so rarely, the model requires large initial impulses to generate a large enough decline in economic activity to explain the observed low interest rates in the data. In models without capital, explaining such an event requires even larger innovations of the structural shocks. For example, Fernández-Villaverde et al. (2012*b*) an eight standard deviation shock to the discount factor generate a ZLB episode of four quarters. Similarly Gust, Lopez-Salido and Smith (2012*b*), using a small New Keynesian model without capital, explain the decline in output and the prolonged ZLB episode using a negative shock to households' discount factor of roughly five standard deviations. From an ex-ante perspective, the combination of shocks that generates the ZLB event in my model has a probability of (0.0001%).[21] Compared to a single shock of five standard deviations, my results are not very different in terms of the extreme nature of the initial impulses. However, as I discuss next, trying to match the decline in consumption and investment using only shocks to preferences would be impossible.

Why are shocks to both preferences and investment necessary to explain the

[21] This corresponds to the probability of drawing a negative $2.1\sigma_{\epsilon_\mu}$ and a negative $2.6\sigma_{\epsilon_d}$, which are the shocks that deliver the ZLB episode of 2009:Q1.

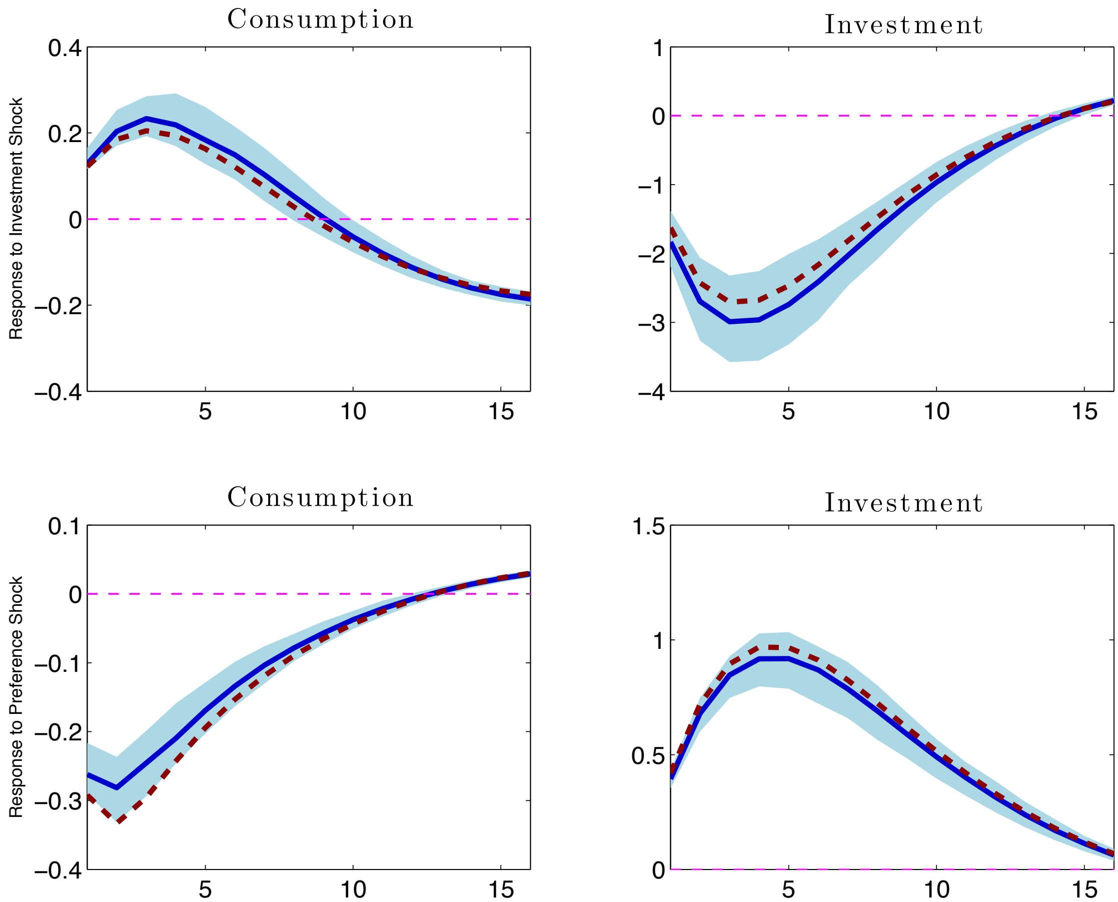
data starting in 2008:Q3? The reason is that in models with capital, preference shocks alone trigger wealth effects that generate negative comovement between consumption and investment. This is illustrated in Figure 1.5, where I plot impulse responses to preference and investment shocks away from the ZLB. In 2008:Q2, before the financial crisis unfolded, the Federal Funds rate was 2.1%. Because negative preference shocks shift output away from consumption towards investment, while negative shocks to the marginal efficiency of investment produce the opposite effect, the model cannot match simultaneously the decline in both components of aggregate demand with a single large shock.

Figure 1.5 highlights the counterfactual response in investment following negative preference shocks.[22] To further illustrate this point, I use the particle filter and U.S. data to calculate the marginal contribution to the log-likelihood of each of the observations in the period 2007:Q4-2009:Q1 for two different specifications of the model. First, I shut down the investment shock while keeping the other parameters at their estimated values. I then repeat this exercise shutting down the preference shock.[23]. Table 1.3 presents the results of this exercise. A more negative number indicates a worse fit of the model for a given observation. Note that in all periods trying to explain the data with investment shocks or preference shocks alone worsens the likelihood. Early in the recession, from 2007:Q4-2008:Q2, the model without the preference shock performs better than the model without the investment shock.

[22] Resolving the comovement problem remains a challenge for DSGE models. This issue was originally pointed out by Barro and King (1984).

[23] Specifically in the first experiment $\sigma_\mu = 0.0001$, while for the second experiment $\sigma_d = 0.0001$.

Figure 1.5: Comovement Problem Away from the ZLB



Notes: The dark blue lines correspond to the impulse responses of the nonlinear model. The light blue shade denote the 20%-80% confidence interval. The red dashed lines are obtained from the linear solution of the model ignoring the ZLB. All impulse responses correspond to a one standard deviation shock.

The reason is that early in the recession there was a slight increase in consumption but a decline in investment (see Figure 1.1), so one shock is enough to match both dynamics. In 2008:Q3, however, both investment and consumption plunged, so trying to match the decline in both variables with a single shock is impossible, as illustrated by the sharp deterioration in the log-likelihood.

Technology shocks Figure 1.6 shows the comparison between the filtered innovations to technology (z_t) and their directly measured counterpart using quar-

Table 1.3: Marginal Contribution to the Log-Likelihood

	Baseline	**Without Preference Shock**	**Without Investment Shock**
2007:Q4	-1.6	-1.4	-9.6
2008:Q1	-4.8	-3.6	-43.4
2008:Q2	-3.2	-12.3	-24.3
2008:Q3	-6.0	-8.9	-34.8
2008:Q4	-33.4	-207.2	-149.6
2009:Q1	-10.6	-70.7	-121.7

terly data on total factor productivity (TFP) for the U.S. business sector.[24] The correlation between the filtered and observed measure of technology shocks is 0.52, and there is also a clear negative TFP shock in 2008:Q4 observed in the data.

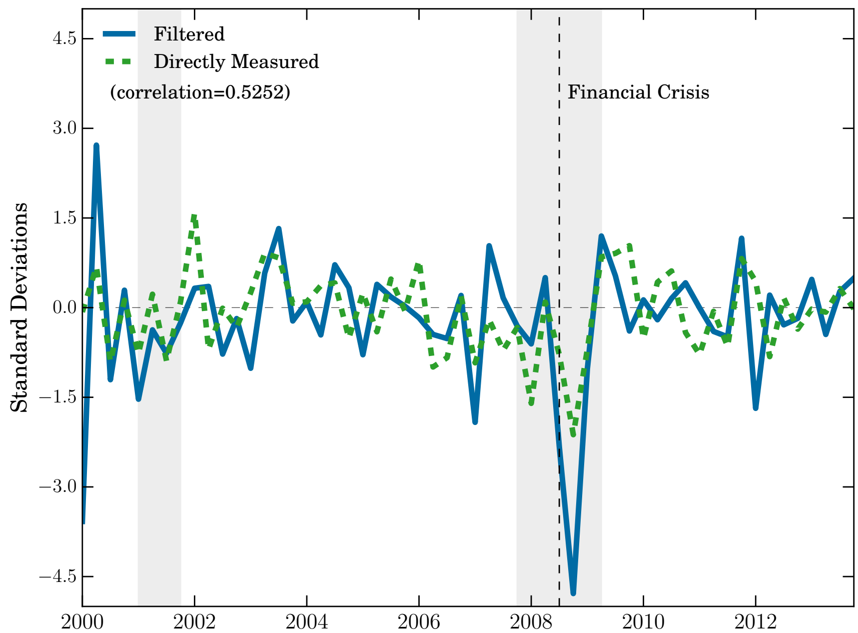
Why is a large negative technology shock necessary to explain the data starting in 2008:Q4? The reason is the absence of persistent deflation in the data. When the ZLB is binding, prices fall due to a self-reinforcing loop of high real interest rates, declining marginal costs and low aggregate demand, so explaining the absence of a more severe deflation in the U.S. remains a puzzle. After 2008:Q3, inflation turned negative only in the first two quarters of 2009, bouncing back to positive territory thereafter.

Although inflation has remained below the Federal Reserve's target of 2% in the aftermath of the recession, the U.S. economy has avoided a deflationary spiral.

[24] This measure is produced by the Federal Reserve Bank of San Francisco, and methodological details are presented in Fernald (2012). I fit an AR(1) process with drift to the quarterly growth rate of observed TFP, exactly as in the description of the model, and report the fitted residuals.

The negative TFP shocks observed in the data explain in part why prices did not fall more dramatically and persistently as the economy approached the ZLB. Negative technology shocks increase marginal costs, counteracting the deflationary pressures at the ZLB, and if the shocks are negative enough it is possible to reproduce inflation and align the model prediction with the data.

Figure 1.6: Filtered and Directly Measured Innovations to TFP



Notes: The dashed green line is the direct measure of technology shocks are obtained from the TFP series discussed in Fernald (2012). The solid blue line is the mean filtered state z_t obtained from $\mathbb{E}(s_t|Y^t)$. The gray areas indicate NBER recession dates.

Using a different time series representation for technology growth, Christiano, Eichenbaum and Trabandt (2014) also rationalize the absence of deflation assuming a one-time negative productivity shock in 2008:Q3. My estimates of the unobserved technology shocks (z_t) square well with those obtained from direct measurement of TFP. The advantage of my filtering procedure is that I let the data speak through the model to recover the shocks more transparently. For instance I do not need to assume the size of the negative technology shock in 2008:Q3, or that it has to be a one-time innovation in that particular period. The fact that this is actually the case is a result I obtain from the estimation.

Fiscal and monetary policy shocks. Figure 1.7 shows estimated shocks to government spending. Two positive shocks in 2009:Q1 and 2009:Q2 and a large negative shock in 2011:Q2 stand out. These events correspond to well-identified events related to changes in discretionary fiscal expenditures. The positive shocks correspond to the *American Reinvestment and Recovery Act*, which was enacted quickly after the collapse in the financial sector.[25] The negative shocks observed from 2010:Q4 and continuing until 2012:Q3 correspond to the prolonged struggle about the stance of fiscal policy between the White House and Congress. Examples of this struggle include the 2010 year-end debate about extending tax cuts and federal unemployment benefits, the fiscal entrapment surrounding the Federal debt limit in the first half of 2011, the downgrade in the credit rating of U.S. federal government debt in August 2011, and concerns about the fiscal cliff in early 2013.

To emphasize that the recovered filtered shocks capture the developments in actual U.S. fiscal policy, Figure 1.7 compares the filtered model estimate of g_t to a direct measure of the government spending shock based on the observed ratio of government consumption and investment to GDP. The empirical measure of g_t is recovered using the same equation that describes the evolution of government

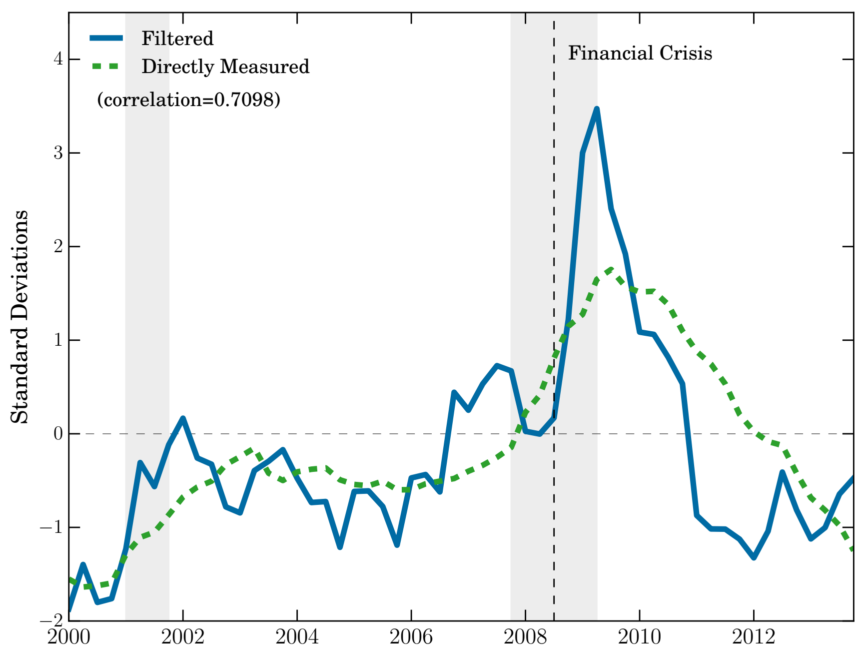
[25] The "stimulus bill" was signed into law in February 17, 2009, less than a month after the change in U.S. presidential administration.

expenditure in the model:

$$\frac{G_t}{Y_t} = \left(1 - \frac{1}{g_t}\right) \tag{1.37}$$

The correlation between the empirical measure of g_t and the filtered measure is roughly 0.7.[26] Fiscal policy was clearly expansionary from 2009:Q1 until 2011:Q2, and tightened afterwards. As shown in section 1.7, this fiscal swing had a negative effect on the economic recovery.

Figure 1.7: Fiscal Policy Shocks: Filtered vs Data

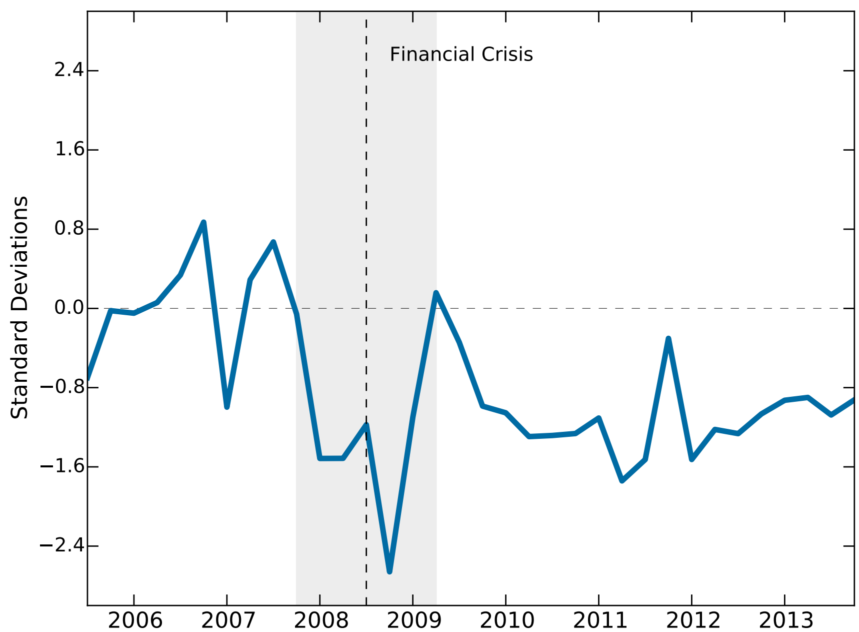


Notes: The dashed green line corresponds to the empirical measure of the g_t process. The solid blue line is the mean filtered state g_t obtained from $\mathbb{E}(s_t|Y^t)$. Data on government consumption and gross investment used to construct g_t comes from Table 1.1.5 of the National Income and Product Accounts. The gray areas indicate NBER recession dates.

[26]Note that I do not use the share of government consumption G_t/Y_t when estimating the filtered shocks. I use only the same data on consumption, investment and GDP growth. Nevertheless, the sequence of filtered g_t shocks tracks well its data counterpart. The latter is constructed using quarterly information on government consumption and expenditure including gross investment.

Figure 1.8 show the sequence of estimated monetary policy shocks. From 2007:Q3 onward, the estimated monetary policy shocks ε^r are negative. From 2007:Q3 onward these shocks reflect an aggressive response by the Federal Reserve to counteract contractions in aggregate demand. However, once the economy hit the ZLB in 2009:Q1, the continued sequence of negative monetary policy shocks has a more subtle interpretation.

Figure 1.8: Estimated Monetary Policy Shocks



Notes: The solid blue line is the mean filtered state ε_t^r obtained from $\mathbb{E}(s_t|Y^t)$.

In December 2008, the Federal Open Market Committee adopted a language that suggested an active policy decision of maintaining the nominal interest near zero for a substantial period of time.[27] The FOMC adopted more explicit language in 2011:Q3, when it announced that the federal funds rate would likely remain at

[27]For instance, the first FOMC statement of 2009 indicates that *"economic conditions are likely to warrant exceptionally low levels of the federal funds rate for some time."*

zero until mid-2013. In 2012:Q1 the reference date for the *forward guidance* policy was extended until late 2014. My model does not take into account the shift towards an explicit forward guidance policy. However, the negative monetary policy shocks obtained after 2009:Q1, and in particular during the period 2011:Q3-2013:Q4, reflect the commitment of the Federal Reserve to maintain a zero interest rate policy even if the monetary policy rule would otherwise have called for an earlier *lift-off* in the nominal rate.

The path for the estimated monetary policy shocks is consistent with Woodford (2011)'s notion of monetary policy accommodation which means that the central bank does not tighten policy in response to increased government purchases. In Section 1.7.2 I show that monetary policy was accommodative enough to generate a substantial output increase in response to the fiscal stimulus, in particular during the period 2009:Q1-2010:Q1.

1.6.3 Interpreting the structural shocks

To complement the evidence presented in the previous section I compare the filtered shocks with some direct measures of the underlying financial frictions that caused the recession. Instead of focusing on the filtered innovations, ϵ_t^μ and ϵ_t^d, these sequences are transformed into the implied paths of the marginal efficiency of investment μ_t and the subjective discount factor adjusted by the preference shock, $\tilde{\beta} = \beta \frac{d_t}{d_{t-1}}$. The latter reflects the effective level of patience or impatience of individuals ex-post and is related to households' desire to save.[28] To aid interpretation, I compare the path of the marginal efficiency of investment to the observed evolution of credit spreads, and the path of the adjusted discount factor to the observed evolution of the U.S. personal savings rate.

Figure 1.9 compares the paths of the marginal efficiency of investment (inverted scale) and credit spreads. The measure of credit spreads comes from the corporate bond spread measure constructed by Gilchrist and Zakrajšek (2012) (GZ).[29] Both the marginal efficiency of investment and the credit spreads data are scaled in terms of standard deviations to plot them on the same scale.

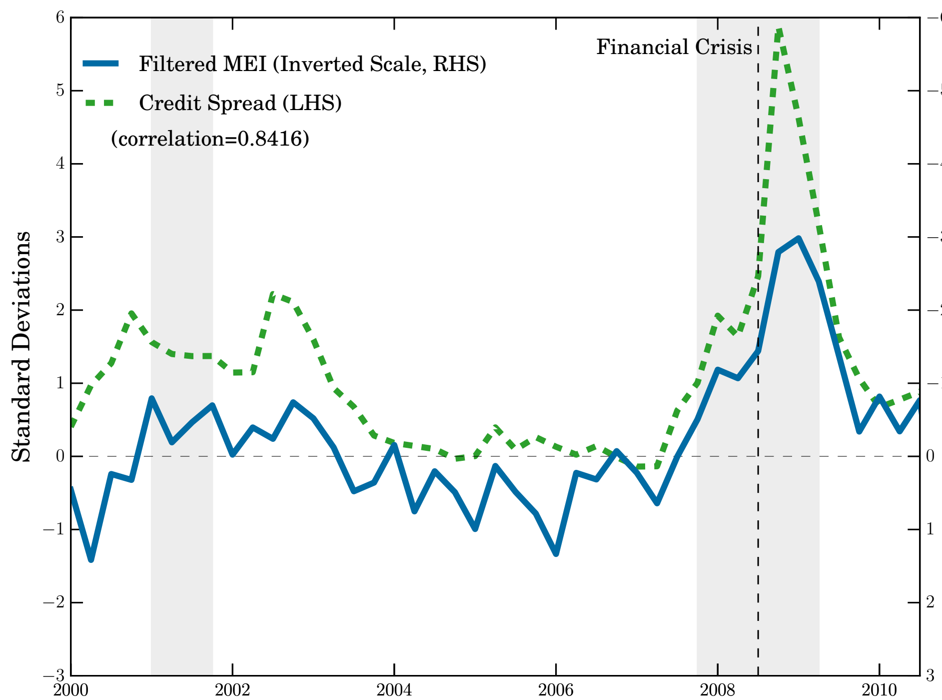
As credit spreads rose with the onset of the recession, the marginal efficiency of investment declined. In fact, in 2007:Q3, μ_t was near its steady state of zero. During the first three quarters of the recession, from 2007:Q4 to 2008:Q2, the marginal efficiency of investment declined about 6% below its steady state, and eventually fell 12% below its steady state by 2009:Q1. The spike in the observed cost of borrowing for nonfinancial firms in this period coincides with the deterioration of the estimated marginal efficiency of investment. The tight correlation (-0.84) between the filtered marginal efficiency of investment and credit spreads is a strong signal that the investment shock I recover from the model captures disruptions in credit markets. Following the end of the recession, credit spreads declined and the marginal

[28] To derive this object, use equation (1.20) and assume that the habit persistence parameter is zero, $h = 0$. This implies a simple consumption Euler equation of the form: $c_t = \beta \frac{d_t}{d_{t-1}} \frac{R_{t-1}}{\gamma e^{z_t}} c_{t-1}$.

[29] The GZ spread measure is constructed using secondary market prices of senior unsecured fixed coupon corporate bonds of U.S. nonfinancial firms. Compared to the simple measure of Baa-Aaa corporate bond spreads, it has the advantage that it adjusts for the duration mismatch between the cash flow of corporate bonds and the risk-free security.

efficiency of investment recovered. However, the financial frictions remained at levels that continued to depress aggregate demand. After 2009:Q1, the marginal efficiency of investment remained around 5.0% below its steady state.

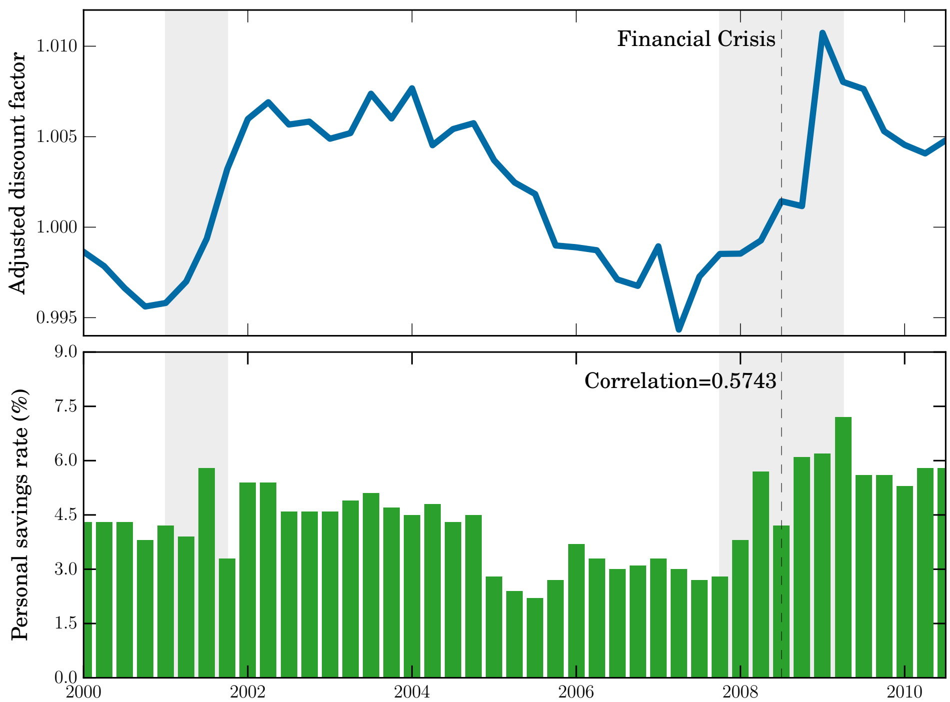
Figure 1.9: Marginal Efficiency of Investment and Credit Spreads



Notes: The dashed green line corresponds to the credit spread measure of Gilchrist and Zakrajšek (2012). The solid blue line is the mean filtered state, μ_t, obtained from $\mathbb{E}(s_t|Y^t)$. All series are standardized. The gray areas indicate NBER recession dates.

As discussed above, another possible cause of the Great Recession that has received considerable attention is the tightening of borrowing constraints for households. For instance, Mian and Sufi (2012) favor this view as the leading cause of the economic collapse. In my model it is not possible to capture household borrowing and lending because of the representative agent structure. However, the shock to preferences serves as a stand in for deeper frictions that cause agents to reduce leverage, causing a contraction in aggregate consumption.

Figure 1.10: Subjective Discount Factor and Saving Rates



Notes: The personal savings rate is defined as the ratio of personal savings to disposable personal income obtained from FRED, Federal Reserve Bank of St. Louis. The solid blue line is the mean filtered state, $\tilde{\beta}_t$, constructed from $\mathbb{E}(s_t|Y^t)$. The gray areas indicate NBER recession dates.

To illustrate this point, Figure 1.10 plots the adjusted discount factor against the U.S. personal savings rate. A value of the adjusted discount factor above $\beta = 0.9981$ indicates an increased desire to save by households, driven by negative shocks to d_t. I interpret an increase in the adjusted discount factor above β as a tightening of borrowing constraints. The measure of the savings rate I use is the ratio of personal saving to disposable personal income. The adjusted discount factor tracks the increase in the personal savings rate, which jumped from 2.3% prior to the start of the recession to 7% by the end of the episode.

A possible objection to interpreting the preference shock as a reduced form measure of tightening borrowing constraints is that the increase in the personal savings rate could have been driven by a faster decline in disposable income that did not affect borrowing limits directly. This was unlikely to be the case. Table 1.4 shows the evolution of households' stock of debt using the Federal Reserve Bank of New York Consumer Credit Panel data. Total debt peaked in 2008:Q3 at 12.7 trillion dollars, and declined continuously from that point on. The first row shows the year-over-year change in the stock of debt. The second row shows the cumulative growth with respect to 2008:Q3. The reduction in debt in the data coincides with the increase in households' estimate of the adjusted discount factor. This provides further support to the interpretation given to this shock.

Table 1.4: Evolution of Households' Stock of Debt

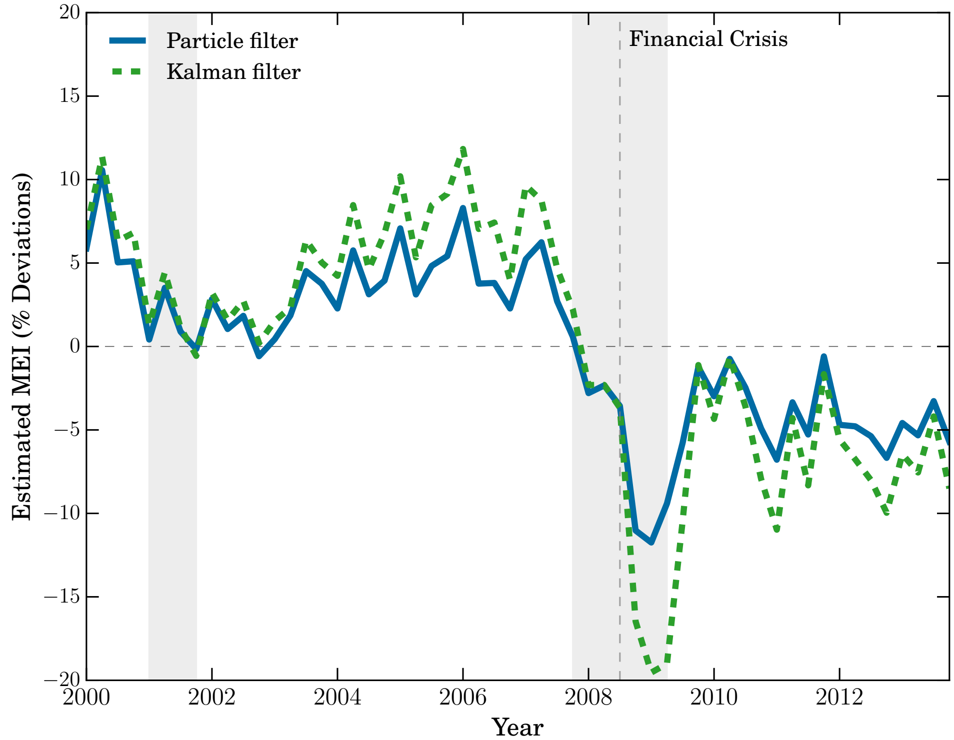
	2008:Q3	2008:Q4	2009:Q4	2010:Q4	2011:Q4
Annual change (%)	4.47	2.41	-3.98	-3.72	-1.51
% Change from 2008:Q3	-	-0.04	-4.02	-7.59	-8.98

Notes: Data comes from the Federal Reserve Bank of New York Consumer Credit Panel, Brown et al. (2010).

1.6.4 How much amplification does the ZLB generate?

Because the feedback response of the nominal interest rate is hampered when the economy reaches the ZLB, the equilibrium dynamics of all other endogenous variables change substantially. The max operator in equation (1.14) generates a non-differentiability in all decision rules around the region where the ZLB binds. How much do these non-linearities affect the inference about the causes of the Great Recession? To illustrate this point I extract an alternative set of shocks using a linearized version of the model that ignores the ZLB, and compare them with those obtained from the non-linear model.

Figure 1.11: Amplification due to the ZLB



Notes: The solid blue line is the mean filtered state, μ_t, constructed from $\mathbb{E}(\mu_t|Y^t)$. The dashed green line is the analog from the Kalman filter. The gray areas indicate NBER recession dates.

Figure 1.11 shows the time path of the marginal efficiency of investment estimated using the Kalman filter in the linearized version of the model, compared with the baseline results obtained with the particle filter. Prior to the ZLB period that starts in 2009:Q1 the estimated shocks from the linear and non-linear model are similar. However once the economy hits the ZLB, the estimated shocks in the nonlinear model are consistently smaller in absolute value. Because of the kinks in the decision rules, shocks of similar size have larger effects in the regions where the ZLB is binding.

1.7 Counterfactual Experiments

This section analyzes the effect of structural shocks on macroeconomic dynamics, with a focus on the Great Recession and its aftermath (2008:Q3 to 2013:Q4). I use the model solution and the filtered states to construct counterfactual responses of output, consumption, investment, the nominal rate and inflation, and compare them to their realized values. The counterfactual exercises that follow answer three questions: What caused the ZLB to bind? What was the contribution of each shock to aggregate demand? and How much did fiscal and monetary policy helped during the economic recovery?

Constructing counterfactual paths. I briefly explain the algorithm used to construct the counterfactual paths. The first step is to recover the mean of the filtering density $\bar{s}_t = \mathbb{E}p(s_t|Y^t)$. Feeding the path $\bar{s}_t$ into (1.36), then I can reconstruct the observed U.S. time series. The counterfactual paths are constructed according to the following algorithm:

Algorithm 1 ***Counterfactual Paths***

1. *Construct the posterior mean ($\bar{s}_t$) of the state vector $\bar{s}_t = \mathbb{E}(s_t|Y^t)_{2008:Q2}^{2013:Q4}$, using the particle approximation of $p(s^t|Y^t)$.*

2. *The actual data path $\mathbb{Y}_t$ can be reconstructed by feeding $\{\bar{s}_t\}_{2008:Q2}^{2013:Q4}$ into equation (1.35) and equation (1.36).*

3. *The counterfactual path $\tilde{\mathbb{Y}}_t^i$ is constructed by setting $\{\varepsilon_t^i = 0\}_{2008:Q2}^{2013:Q4} \in \bar{s}_t$, for a subset of shocks $i \subset \{\mu, d, z, g, R\}$*

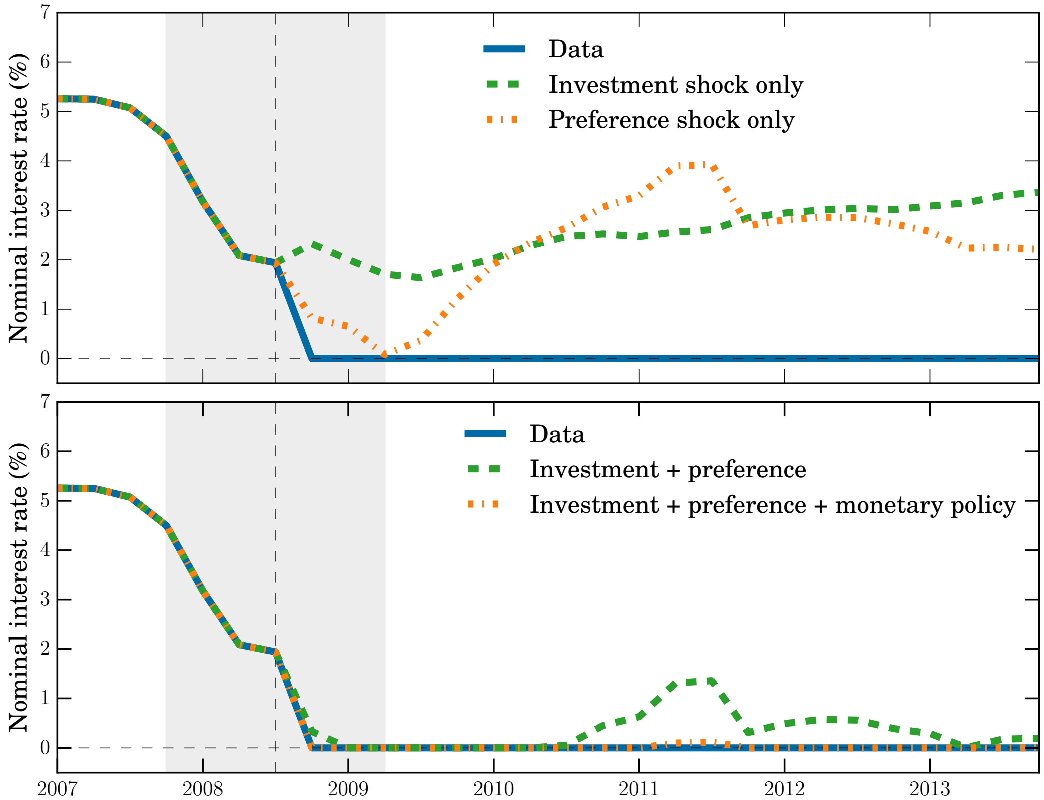
1.7.1 What drove and kept the economy at the ZLB?

As already discussed in Section 1.6.2, the initial force that pushed the economy to the ZLB and a deep recession was a combination of large negative investment and preference shocks. To assess the importance of each of these impulses, Figure 1.12 feeds specific combinations of the filtered shocks through the model measurement equation from 2008:Q3 onward.

The top panel shows that neither shocks to preferences nor shocks to the marginal efficiency of investment alone can independently push the U.S. economy into a liquidity trap. This result follows from the comovement problem illustrated earlier. For example, in response to a negative shock to the discount factor, households reduce consumption. But because the capital accumulation margin is not directly distorted by the preference shock, investment increases. The higher investment pushes aggregate demand upward and hence the feedback rule calls for an increase in the nominal rate. The bottom panel shows that the combination of investment and preference shocks pushed the economy to the ZLB in the model, following closely the observed path of the nominal interest rate in the data. If only these two shocks are present, the nominal rate turns effectively zero in 2009:Q2, and remains at the ZLB for a year until 2010:Q2. Afterwards, the nominal rate rises consistent with the mean reversion of the processes for μ_t and d_t and the associated economic recovery. However, the economic slump caused by the shocks that took place around the financial crisis was large and persistent.

The figure also shows an experiment in which monetary policy shocks are also

Figure 1.12: Counterfactual Paths - Nominal Interest Rate



Notes: The shaded region indicates the NBER recession. The vertical dashed line corresponds to the start of the counterfactual exercise in 2008:Q3.

fed into the simulation. The simulated economy is at the ZLB starting in 2009:Q2 and monetary shocks initially have no effect. As aggregate demand picks up during the recovery, the Federal Reserve uses the unanticipated shocks to the feedback rule to keep the nominal interest rate pegged at zero from 2010:Q3 onward. The model cannot account for unconventional policies that were implemented in response to the financial meltdown, in particular forward guidance that used the language of FOMC announcements to signal future monetary policy actions. However the fact the nominal rate stayed at the ZLB after the economy started to recover indicates monetary policy actions were indeed expansionary.

Another important element that is not considered in the above experiments is fiscal policy. An increase in government expenditure puts upward pressure on inflation and would ordinarily trigger an hike in the nominal rate. When the nominal interest rate adjusts to changes in government expenditure, fiscal policy is less effective in stimulating the economy (e.g. Christiano, Eichenbaum and Rebelo (2011a)). By committing to keep the nominal interest rate close to the ZLB, the Federal Reserve increased the effectiveness of fiscal stimulus. This explains why the estimated sequence of monetary policy shocks is negative during and after the Great Recession.

1.7.2 Response of Aggregate Demand

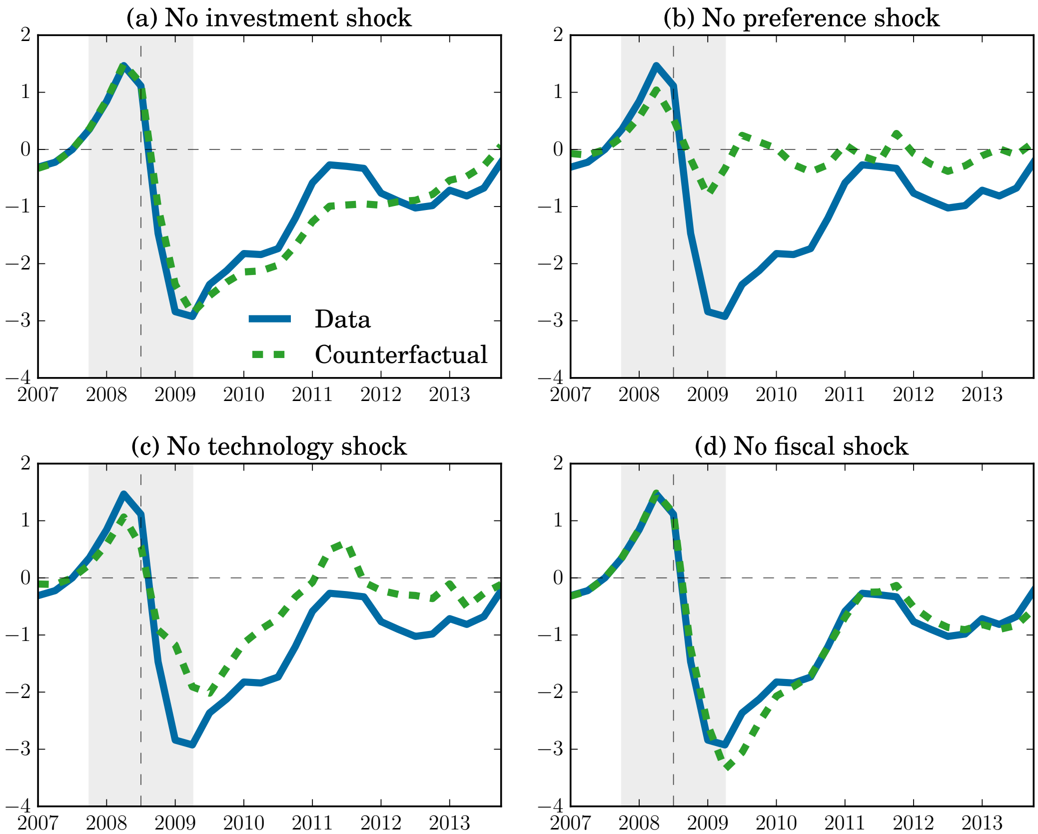
What explain the dynamics of consumption and investment during and after the Great Recession? To answer these questions I construct the counterfactual paths implied by shutting only the $i - th$ shock and then compare the simulated paths from the model with the data. In terms of the notation in algorithm 1, the difference between $\mathbb{Y}_t$ and $\tilde{\mathbb{Y}}_t^i$ measures the contribution of the i^{th} structural innovation to the evolution of the observed time series. In both the model and the data I use the level of the series instead of the growth rates and use the Hodrick-Prescott filter to extract their cyclical component. Figure 1.13 shows consumption data and its counterfactual paths, all expressed in percentage deviations with respect to their pre-recession values. Each panel corresponds to an experiment in which only the indicated shock is shut down.

Panel (b) and (c) show that the decline in consumption can be attributed mostly to technology and preference shocks, with the latter playing a more important role and highlighting the importance of shocks affecting households borrowing ability during the recession. Nonetheless, technology shocks have a significant role in consumption dynamics. The estimated negative technology shock in 2008:Q4 tends to reduce consumption through wealth effects on labor supply but also through households' intertemporal margin because it increases current and future inflation. The latter translates into a reduction in the real interest rate, stimulating consumption through intertemporal substitution. Panel (c) reveals that the contractionary effect dominates making the observed contraction in consumption more pronounced in the aftermath of the financial crisis.

Panel (a) shows that investment shocks played almost no role in accounting for the decline in consumption. Once again this disconnect is a reflection of the comovement problem. Meanwhile, panel (d) shows the fiscal stimulus during the period 2009-2011 had small crowding out effects on private consumption. At the ZLB the additional inflation caused by increased government expenditure lowers the real interest rate, thus potentially increasing private consumption. The counterfactual exercise shows that consumption would have decreased an additional 0.5% on average during 2009, had it not been for the stimulative effect of government expenditure on inflation.

Figure 1.14 shows a similar counterfactual exercise for investment. The differences are starker. Panel (a) shows that the dominant force explaining the decline in investment is the deterioration in the marginal efficiency of investment. The protracted recovery in the aftermath of the recession is due to the persistence of shocks

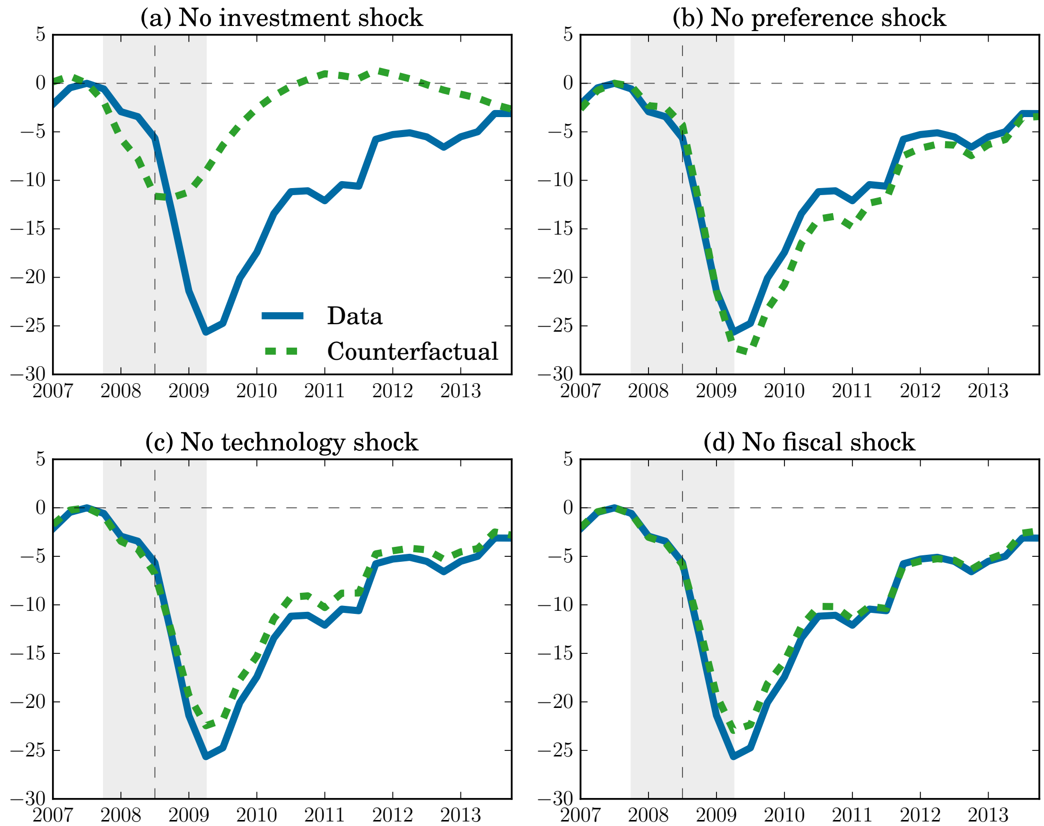
Figure 1.13: Counterfactual Paths - Consumption



Notes: All series are detrended using the HP filter ($\lambda = 1600$) and normalized to 2007:Q3. The shaded region indicates the NBER recession. The vertical dashed line corresponds to the start of the counterfactual exercise in 2008:Q3.

affecting investment allocations, as captured by the filtered estimates of μ_t, which remained below trend throughout. Contrary to the effects on consumption, government expenditure did have a mild crowding out effect on aggregate investment. This presents a trade-off for the effectiveness of fiscal policy during a liquidity trap that cannot be appreciated in New Keynesian models without investment. The economic reason for this effect is that higher government expenditure does not reduce the excess return between risky capital and risk-free bonds, which is driven by the μ_t process. By lowering the real rate, increased government spending shifts resources

Figure 1.14: Counterfactual Paths - Investment

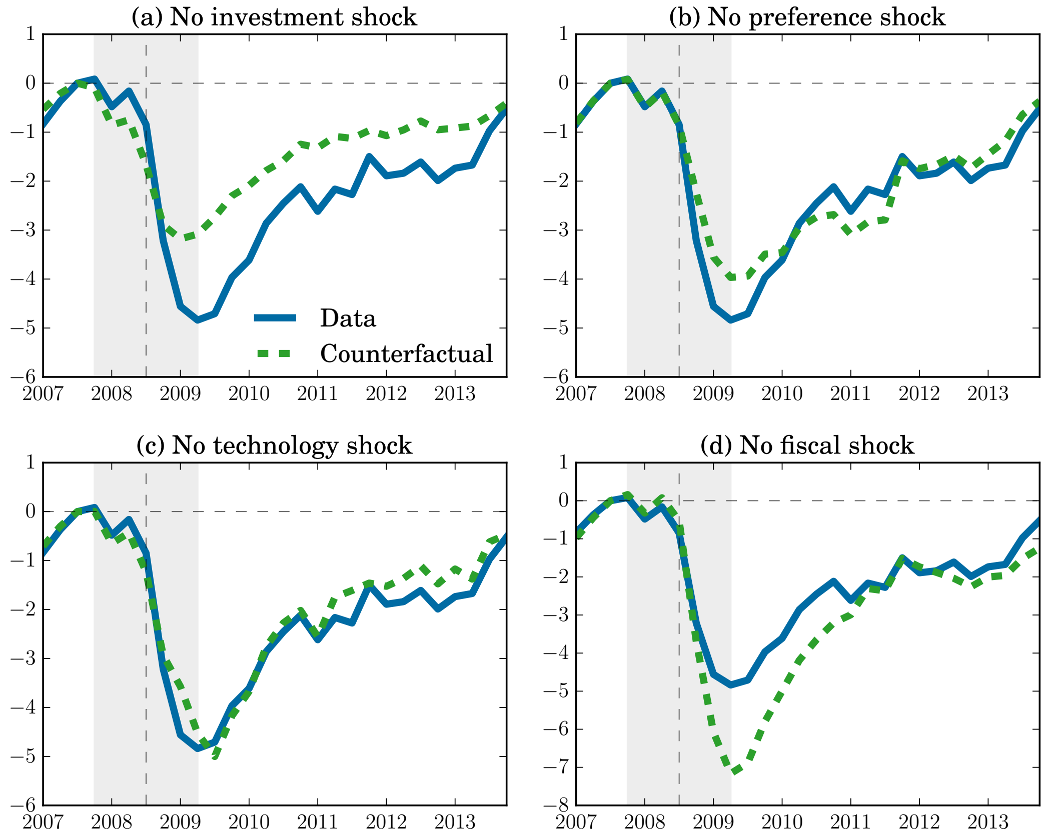


Notes: All series are detrended using the HP filter ($\lambda = 1600$) and normalized to 2007:Q3. The shaded region indicates the NBER recession. The vertical dashed line corresponds to the start of the counterfactual exercise in 2008:Q3.

away from already low levels of investment towards consumption, worsening the economic downturn.

Figure 1.15 is the key counterfactual exercise. It shows the overall effect of the different shocks on the dynamics of output. I focus on two results. Panel (a) shows that the contribution of investment shocks to output was substantial. The financial frictions captured by the investment shock caused a significant contraction of output. Absent the sharp decline in the marginal efficiency of investment, output would have been almost 2% higher in the aftermath of the financial crisis. The

Figure 1.15: Counterfactual Paths of Output



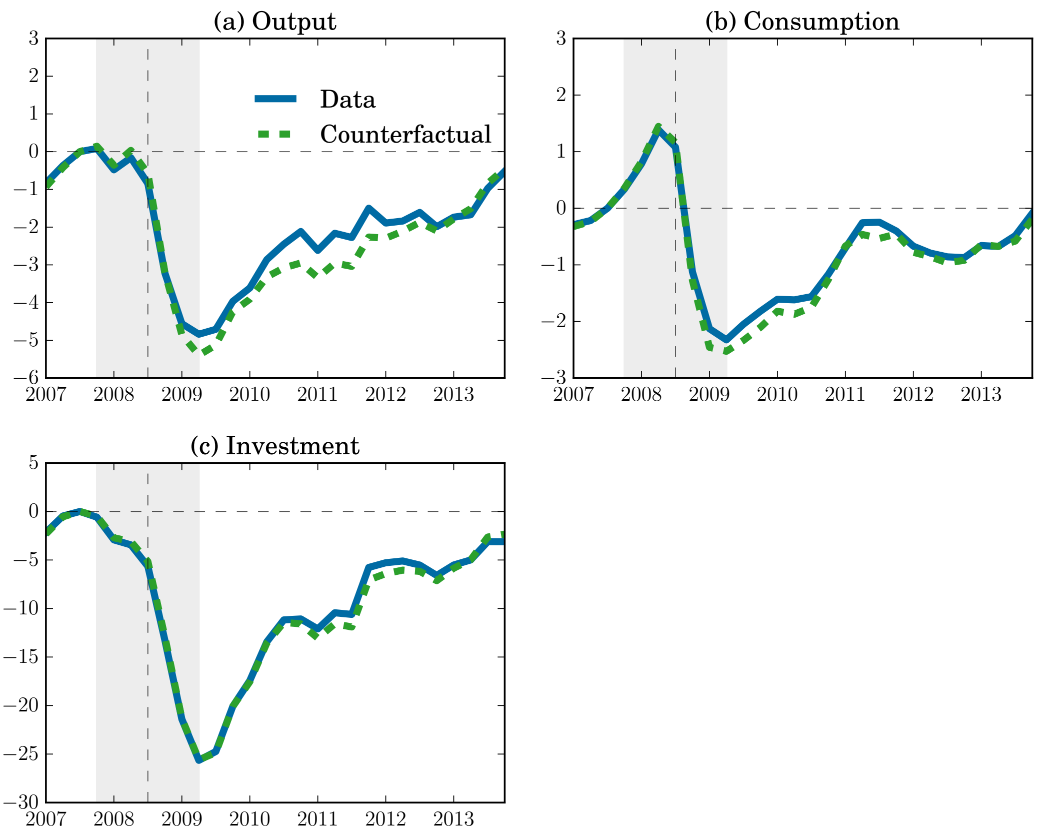
Notes: All series are detrended using the HP filter ($\lambda = 1600$) and normalized to 2007:Q3. The shaded region indicates the NBER recession. The vertical dashed line corresponds to the start of the counterfactual exercise in 2008:Q3

second result is illustrated in panel (d). Expansionary fiscal policy had a positive effect on output. For example, absent fiscal policy in 2009:Q2 output would have contracted an additional 2.5%. From 2009-2011 the effects of fiscal policy helped reduce the magnitude of the economic contraction by between a third and a half.

The contribution of preference shocks is milder, as evidenced by panel (b). The reason is that the bulk of the contraction in output is accounted for by the decline in investment, which can be explained mostly by the negative marginal efficiency of investment shock. This is despite the fact that in 2009:Q2 the innovation to households' preferences (ϵ_d) was twice as large as the innovation to the marginal efficiency of investment (ϵ_μ).

Figure 1.16 shows the counterfactual paths of aggregate demand without monetary policy shocks. The estimated monetary policy shocks during this period were negative and kept the nominal interest rate at the ZLB. Panel (a) shows that absent such monetary stimulus, output would have contracted almost an additional 0.5% in 2009:Q2 followed by a slower recovery. Panels (b) and (c) shows that the absence of monetary accommodation reverses the effect of the fiscal expansion mostly through a crowding out of private consumption while investment remains almost unaltered.

Figure 1.16: Effect of Monetary Policy on Aggregate Demand

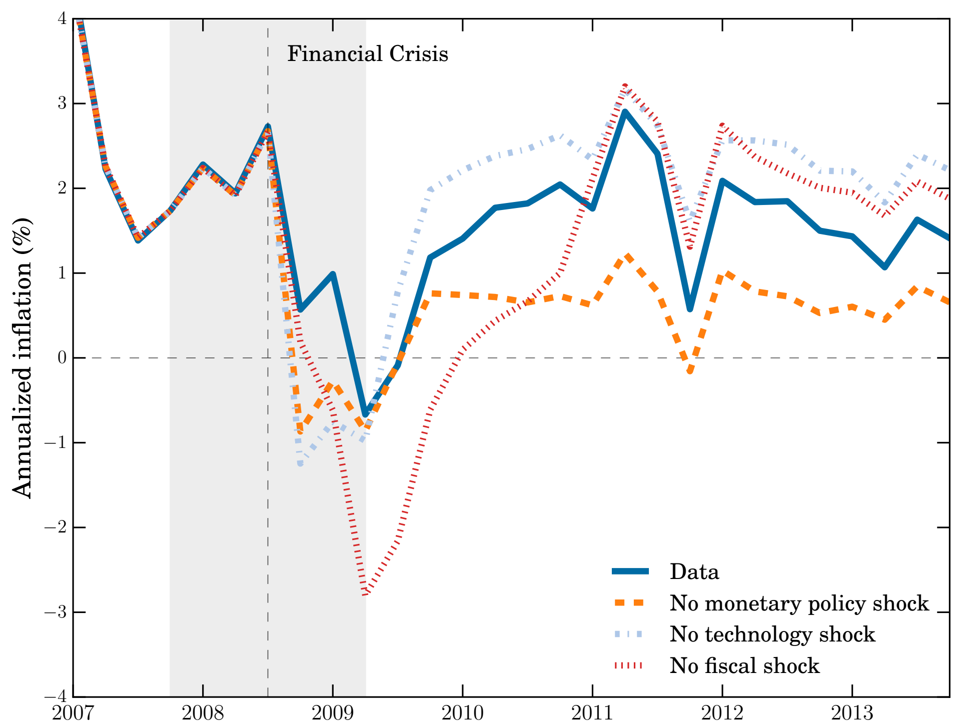


Notes: All series are detrended using the HP filter ($\lambda = 1600$) and normalized to 2007:Q3. The shaded region indicates the NBER recession. The vertical dashed line corresponds to the start of the counterfactual exercise in 2008:Q3

1.7.3 Where did the deflation go?

A puzzle that continues to generate debate among macroeconomists is the absence of persistent deflation during and after the Great Recession. As noted by Hall (2011), any model that delivers a Phillips curve equation relating prices to some measure of economic activity (e.g. unemployment or marginal costs) would predict persistent deflation when there is substantial slack in the economy. However, despite the severity of the Great Recession, inflation has for the most part remained positive. Figure 1.17 shows different counterfactual paths for inflation constructed using algorithm 1. I focus only on the shocks that would have predicted deflation.

Figure 1.17: Counterfactual Paths of Inflation



Notes: Inflation is measured by the annualized quarterly percentage change in the GDP deflator. The shaded region indicates the NBER recession. The vertical dashed line corresponds to the start of the counterfactual exercise in 2008:Q3.

The counterfactuals show that fiscal and monetary policy helped prevent a sustained decline in prices. The negative technology shock helped reduce the deflationary pressures in the immediate aftermath of the financial crisis, but thereafter it was mostly the stimulative effect of fiscal and monetary policy on aggregate demand that kept inflation on positive terrain. Del Negro, Giannoni and Schorfheide (2015) offer an alternative explanation of the deflation puzzle based on estimates of nominal rigidities that imply a very slow frequency of price adjustment.[30] In comparison, my results complement the findings of Aruoba, Cuba-Borda and Schorfheide (2013) calling attention to the role of policies that helped avoid the perils of a switch to an equilibrium with deflationary dynamics.

1.7.4 Role of nominal rigidities

I resolve the model increasing the value of the parameter ϕ_p that controls the cost of price adjustment. The new value of ϕ_p implies a slope of the Phillips curve of $\kappa(\phi_p) = 0.041$, which is close to the degree of nominal price rigidity estimated in Christiano, Eichenbaum and Trabandt (2014) and Justiniano, Primiceri and Tambalotti (2010).

The main difference is the balance between the contribution of technology and preference shocks. With prices that adjust less frequently, one can mechanically account for the mild decline in inflation observed in the data without relying on large negative technology shocks. If anything, increasing the degree of nominal rigidity

[30] Del Negro, Giannoni and Schorfheide (2015) augment the Smets and Wouters (2007) model with financial frictions and find that the degree of nominal rigidities is stronger when they use credit spread data to identify the structural parameters of the model.

helps the model fit better the dynamics of inflation around the Great Recession. With higher nominal rigidities, consumption becomes even more responsive to shocks to preferences. But the degree of nominal rigidity does not substantially affect the contribution of investment shocks in explaining the contraction and slow recovery of output.

1.8 Conclusions

In this paper I examine the potential causes of the U.S. Great Recession and the subsequent slow recovery through the lens of a medium scale New Keynesian model subject to the Zero Lower Bound constraint. Using Sequential Monte Carlo Methods, I recover the shocks and unobservable states that caused the Great Recession. I find that the recession started with a decline in the marginal efficiency of investment, reflecting frictions in the process of financial intermediation. When the financial crisis unfolded in 2008:Q3, these frictions were exacerbated, pushing the economy to the ZLB.

To explain the economy's dynamics prior and during the recession, the model relies on disturbances to the marginal efficiency of investment and households' subjective discount factor. Both shocks are necessary to explain why the economy reached the liquidity trap, although shocks to the marginal efficiency of investment are more important overall. In particular, the persistent deterioration in the marginal efficiency of investment played the dominant role in explaining the slow recovery. Absent the negative shocks to investment from 2008:Q3 onward, investment would have contracted about 45% less than it actually did. Moreover, investment would have recovered faster, returning to its pre-recession levels by mid-2010.

Discretionary government expenditure provided substantial stimulus to reduce the severity of the recession. Government consumption helped sustain aggregate demand, and with the economy at the ZLB the fiscal stimulus helped avoid a deflationary spiral. Monetary policy also helped stimulate the economy by keeping the nominal interest rate at the ZLB even after the frictions affecting consumption and investment started to subdue. However, the impasses between the White House and Congress with regard to the long-run outlook of fiscal policy generated a slowdown and held back the economic recovery from 2011 onward.

In this paper I attempt to bridge the gap between the solution of quantitative models with a ZLB constraint and medium-scale DSGE models commonly used for the study of business cycles and policy analysis. Solving this class of models is important to understanding the joint dynamics of aggregate demand, inflation and a nominal interest rate that is subject to the ZLB constraint. I leave extensions, such as the study of different price setting mechanisms to better understand the dynamics of inflation, and the importance of explicit financial frictions when the economy is at the ZLB, for future research.

Chapter 2 Macroeconomic Dynamics Near the ZLB: A Tale of Two Countries

(coauthored with Boragăn Aruoba and Frank Schorfheide)

2.1 Introduction

Japan has experienced near-zero interest rates since 1995 and in the U.S. the federal funds rate dropped below 20 basis points in December 2008 and has stayed near zero in the aftermath of the Great Recession. Simultaneously, Japan experienced a deflation of about 1% per year. Investors' access to money, which yields a zero nominal return, prevents interest rates from falling below zero and thereby creates a zero lower bound (ZLB) for nominal interest rates. The ZLB is of great concern to policy makers because if an economy is at the ZLB, the central bank is unable to stimulate the economy or react to deflation using a conventional monetary policy that reduces interest rates.

One prominent explanation for the prolonged spell of zero interest rates and deflation in Japan since the late 1990s is that the economy moved toward an undesirable or unintended steady state. Once the ZLB is explicitly included in a standard New Keynesian dynamic stochastic general equilibrium (DSGE) model with an interest-rate feedback rule, there are typically two steady states. In the targeted-inflation steady state inflation equals the value targeted by the central bank and nominal interest rates are strictly positive. In the second steady state, the deflation steady state, nominal interest rates are zero and inflation rates are negative. Benhabib, Schmitt-Grohé and Uribe (2001*a*) were the first to study equilibria in which an economy transitions from the neighborhood of the targeted-inflation steady state to the undesirable deflation steady state.

While *ex post* the U.S. did not experience an extended period of deflation, a potential switch to a deflation regime that resembles the economic experience of Japan was a real concern to U.S. policy makers. For instance, James Bullard, the President of the Federal Reserve Bank of St. Louis, discussed the possibility of various shocks, including actions or announcements by the Federal Reserve, leading the U.S. economy to settle near the deflation steady state Bullard (2010):

> During this recovery, the U.S. economy is susceptible to negative shocks that may dampen inflation expectations. This could push the economy into an unintended, low nominal interest rate steady state. Escape from such an outcome is problematic. [...] The United States is closer to a Japanese-style outcome today than at any time in recent history. [...] Promising to remain at zero for a long time is a double-edged sword. The policy is consistent with the idea that inflation and inflation expectations should rise in response to the promise and that this will eventually lead the economy back toward the targeted equilibrium. But the policy is

> also consistent with the idea that inflation and inflation expectations will instead fall and that the economy will settle in the neighborhood of the unintended steady state, as Japan has in recent years.

The key contribution of our paper is to provide a formal econometric analysis of the likelihood that Japan and the U.S. shifted to a regime that can be described by fluctuations around a deflation steady state in a standard New Keynesian DSGE model. While many authors have suggested that Japan's experience resembles the outcomes predicted by the deflation steady state, to the best of our knowledge, this paper is the first to provide a full-fledged econometric assessment of this hypothesis. We construct a sunspot equilibrium for an estimated small-scale New Keynesian DSGE model with an explicit ZLB constraint, in which a sunspot shock can move the economy from a targeted-inflation regime to a deflation regime. While this sunspot shock is formally exogenous in our model, we offer an informal interpretation according to which agents coordinate their expectations and actions based on the central bank's statements about the stance of monetary policy. Our paper also makes an important technical contribution: it is the first paper to use global projection methods to compute a sunspot equilibrium for a DSGE model with a full set of stochastic shocks that can be used to track macroeconomic time series.

We estimate our model based on U.S. and Japanese data on output growth, inflation, and interest rates, using observations that pre-date the episodes of zero nominal interest rates. Conditioning on these parameter estimates, we use a nonlinear filter to extract the sequence of shocks that can explain the data. Most importantly, we obtain estimates of the probability that the economies were in either the targeted-inflation or the deflation regime. We find that the U.S. and Japanese ZLB experiences were markedly different: Japan shifted from the targeted-inflation regime into the deflation regime in the second quarter of 1999. From an econometric perspective, our sunspot model fits the Japanese data remarkably well. Despite the simplicity of our DSGE model's structure the filtered shock innovations are by and large consistent with the probabilistic assumptions of independence and normality underlying the model specification. The U.S. on the other hand, remained in the targeted-inflation regime throughout our sample period. It experienced a sequence of bad shocks during the Great Recession that pushed interest rates toward zero, followed by an expansionary monetary policy that has kept interest rates at zero since then. The large shocks necessary to capture the Great Recession are highly unlikely under the probabilistic structure of the model, which is a common problem for DSGE models with Gaussian innovations.

To illustrate the consequences of being in either regime, we conduct a sequence of expansionary fiscal policy experiments, conditioning on states that are associated with the ZLB episodes in the U.S. and Japan, and compare the outcomes of these policies in the two countries. The two regimes have drastically different implications for macroeconomic policies. Fiscal multipliers are about 20% smaller in the deflationary regime, despite the economy remaining at the ZLB. While a commitment by the central bank to keep rates near the ZLB doubles the fiscal multipliers in the targeted-inflation regime (U.S.), it has no effect in the deflation regime (Japan).

Our paper is related to four strands of the literature: sunspots and multiplicity of equilibria in New Keynesian DSGE models; global projection methods for the solution of DSGE models; the use of particle filters to extract hidden states based on nonlinear state-space models; and the size of government spending multipliers at the ZLB.

The relevance of sunspots in economic models was first discussed in Cass and Shell (1983), who define sunspots as "extrinsic uncertainty, that is, random phenomena that do not affect tastes, endowments, or production possibilities." Sunspot shocks can affect economic outcomes in environments in which there does not exist a unique equilibrium. Multiplicity of equilibria in New Keynesian DSGE models arises for two reasons. First, a passive monetary policy – meaning that in response to a one percent deviation of inflation from its target the central bank raises nominal interest rates by less than one percent – can generate local indeterminacy in the neighborhood of a steady state. An econometric analysis of this type of multiplicity is provided by Lubik and Schorfheide (2004). Second, the kink in the monetary policy rule induced by the ZLB generates a second steady-state in which nominal interest rates are zero and inflation rates are negative. Because in the neighborhood of this second steady state the central bank is unable to lower interest rates in response to a drop in inflation, the local dynamics are indeterminate. As a result it is generally possible to construct a large number of equilibria in New Keynesian DSGE models. Benhabib, Schmitt-Grohé and Uribe (2001*a*,*b*) were the first to construct equilibria in which the economy transitions from the targeted-inflation steady state toward the deflation steady state. More recently, Schmitt-Grohé and Uribe (2012) study an equilibrium in which confidence shocks combined with downward nominal wage rigidity can deliver jobless recoveries near the ZLB in a mostly analytical analysis. Cochrane (2013) abstracts from the existence of the deflationary steady state and constructs multiple liquidity trap equilibria by assuming that after exiting the ZLB monetary policy remains passive and exploiting the resulting local indeterminacy. Armenter (2014) considers the multiplicity of Markov equilibria in a model in which monetary policy is not represented by a Taylor rule but it is optimally chosen to maximize social welfare.

Our paper focuses on an equilibrium in which a Markov-switching sunspot shock moves the economy from the vicinity of one steady state to the vicinity of the other steady state. This equilibrium allows us to provide a formal econometric assessment of whether Japan or the U.S. have shifted toward the deflation steady state during their respective ZLB episodes. Such a sunspot equilibrium has been recently analyzed by Mertens and Ravn (2014), but in a model with a much more restrictive exogenous shock structure. Our paper is the first to compute a sunspot equilibrium in a New Keynesian DSGE model that is rich enough to track macroeconomic time series and to use a filter to extract the evolution of the hidden sunspot shock.

In terms of solution method, our work is most closely related to the papers by Judd, Maliar and Maliar (2010), Maliar and Maliar (2014), Fernández-Villaverde et al. (2012*a*), and Gust, Lopez-Salido and Smith (2012*a*).[1] All of these papers use

[1]Most of the other papers that study DSGE models with a ZLB constraint take various shortcuts to solve the model. In particular, following Eggertsson and Woodford (2003*a*), many authors assume that an exogenous Markov-switching process pushes the economy to the ZLB. The subsequent exit from the ZLB is exogenous and occurs with a prespecified probability. The absence of other shocks makes it impossible to use the model to track actual data. Unfortunately, model

global projection methods to approximate agents' decision rules in a New Keynesian DSGE model with a ZLB constraint. However, these papers solely consider an equilibrium in which the economy is always in the targeted-inflation regime – what we could call a targeted-inflation equilibrium – and some details of the implementation of the solution algorithm are different.

To improve the accuracy of the model solution, we introduce two novel features. First, we use a piece-wise smooth approximation with two separate functions characterizing the decisions when the ZLB is binding and when it is not. This means all our decision rules allow for kinks at points in the state space where the ZLB becomes binding. Second, when constructing a grid of points in the models' state space for which the equilibrium conditions are explicitly evaluated by the projection approach, we combine draws from the ergodic distribution of the DSGE model with values of the state variables obtained by applying our filtering procedure. Our modification of the ergodic-set method proposed by Judd, Maliar and Maliar (2010) ensures that the model solution is accurate in a region of the state space that is unlikely ex ante under the ergodic distribution of the model, but very important ex post to explain the observed data. This modification turns out to be very important when solving a model tailored to fit U.S. data.

With respect to the empirical analysis, the only other paper that combines a projection solution with a nonlinear filter to track U.S. data throughout the Great Recession period to extract estimates of the fundamental shocks is Gust, Lopez-

properties tend to be very sensitive to the approximation technique and to implicit or explicit assumptions about the probability of leaving the ZLB, see Braun, Körber and Waki (2012) and Fernández-Villaverde et al. (2012*a*).

Salido and Smith (2012*a*). However, their empirical analysis is restricted to the targeted-inflation equilibrium and focuses on the extent to which the ZLB constrained the ability of monetary policy to stabilize the economy. Moreover, ours is the first paper to use a nonlinear DSGE model with an explicit ZLB constraint to study the ZLB experience of Japan.

The effect of an increase in government spending when the economy is at the ZLB has been studied by Braun, Körber and Waki (2012), Christiano, Eichenbaum and Rebelo (2011*b*), Fernández-Villaverde et al. (2012*a*), Eggertsson (2009*a*), and Mertens and Ravn (2014). Christiano, Eichenbaum and Rebelo (2011*b*) argue that the fiscal multiplier at the ZLB can be substantially larger than one. In general, the government spending multiplier crucially depends on whether the expansionary fiscal policy triggers an exit from the ZLB. The longer the exit from the ZLB is delayed, the larger the government spending multiplier. Mertens and Ravn (2014) emphasize that in what we would call a deflation equilibrium, the effects of expansionary government spending can be substantially smaller from the effects in the standard targeted-inflation equilibrium.

The remainder of the paper is organized as follows. Section 2.2 presents a simple two-equation model that we use to illustrate the multiplicity of equilibria in monetary models with ZLB constraints. We also highlight the types of equilibria studied in this paper. The New Keynesian model that is used for the quantitative analysis is presented in Section 2.3, and the solution of the model is discussed in Section 2.4. Section 2.5 contains the quantitative analysis, and Section 2.6 concludes. Detailed derivations, descriptions of algorithms, and additional quantitative results are summarized in an Online Appendix.

2.2 A Two-Equation Example

We begin with a simple two-equation example to characterize the sunspot equilibrium that we will study in the remainder of this paper in the context of a New Keynesian DSGE model with interest-rate feedback rule and ZLB constraint. The example is adapted from Benhabib, Schmitt-Grohé and Uribe (2001*a*) and Hursey and Wolman (2010). Suppose that the economy can be described by the Fisher relationship

$$R_t = r\mathbb{E}_t[\pi_{t+1}] \tag{2.1}$$

and the monetary policy rule

$$R_t = \max\left\{1,\ r\pi_*\left(\frac{\pi_t}{\pi_*}\right)^{\psi}\exp[\sigma\epsilon_t]\right\}, \quad \epsilon_t \sim iidN(0,1), \quad \psi > 1. \tag{2.2}$$

Here R_t denotes the gross nominal interest rate, π_t is the gross inflation rate, and ϵ_t is a monetary policy shock. The gross nominal interest rate is bounded from below by one. Throughout this paper we refer to this bound as the ZLB because it bounds the net interest rate from below by zero. Combining (2.1) and (2.2) yields a nonlinear expectational difference equation for inflation

$$\mathbb{E}_t[\pi_{t+1}] = \max\left\{\frac{1}{r},\ \pi_*\left(\frac{\pi_t}{\pi_*}\right)^{\psi}\exp[\sigma\epsilon_t]\right\}. \tag{2.3}$$

This model has two steady states ($\sigma = 0$), which we call the targeted-inflation steady state and the deflation steady state, respectively. In the targeted-inflation steady state, inflation equals π_*, and the nominal interest rate is $R = r\pi_*$. In the deflation steady state, inflation equals $\pi_D = 1/r$, and the nominal interest is $R_D = 1$.

The presence of two steady states suggests that the nonlinear rational expectation difference equation (2.3) has multiple stable stochastic solutions. We find solutions to this equation using a guess-and-verify approach. A solution that fluctuates around the targeted-inflation steady state is given by

$$\pi_t^{(*)} = \pi_* \gamma_* \exp\left[-\frac{1}{\psi}\sigma\epsilon_t\right], \quad \gamma_* = \exp\left[\frac{\sigma^2}{2(\psi - 1)\psi^2}\right]. \tag{2.4}$$

We can also obtain a solution that fluctuates around the deflation steady state:

$$\pi_t^{(D)} = \pi_* \gamma_D \exp\left[-\frac{1}{\psi}\sigma\epsilon_t\right], \quad \gamma_D = \frac{1}{\pi_* r}\exp\left[-\frac{\sigma^2}{2\psi^2}\right]. \tag{2.5}$$

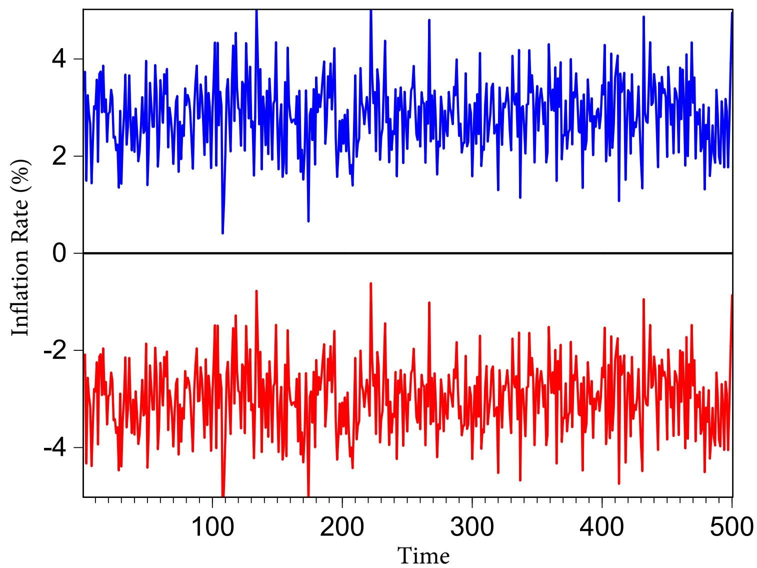
This second solution differs from (2.4) only with respect to the constant γ_D, and has the same dynamics. We refer to $\pi_t^{(*)}$ as the targeted-inflation equilibrium and $\pi_t^{(D)}$ as the deflation equilibrium associated with (2.3).[2]

In the remainder of the paper we will focus on an equilibrium in which a two-state Markov-switching sunspot shock $s_t \in \{0, 1\}$ triggers movements from a

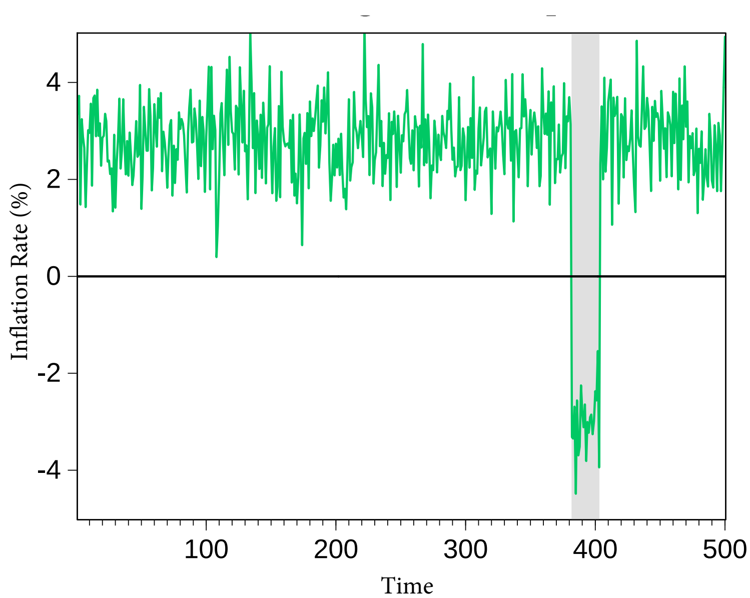
[2]There can be other equilibria similar to (2.5) where the economy spends time around the deflation steady state. Some of these can be simply constructed using (2.5) by changing the dynamics in the region where the ZLB binds. See Appendix B.1 for an example.

Figure 2.1: Inflation Dynamics in the Two-Equation Model

Targeted-Inflation and Deflation Equilibria



Sunspot Equilibrium



Notes: In the top panel, the blue line shows the targeted-inflation equilibrium, and the red line shows the deflation equilibrium. In the bottom panel, the shaded area corresponds to periods in which the system is in the deflation regime.

targeted-inflation regime to a deflation regime and vice versa:

$$\pi_t^{(s)} = \pi_* \gamma(s_t) \exp\left[-\frac{1}{\psi} \sigma \epsilon_t \right]. \tag{2.6}$$

The constants $\gamma(1)$ and $\gamma(0)$ are similar in magnitude (but not identical) to γ_* and γ_D in (2.4) and (2.5), respectively. The precise values depend on the transition probabilities of the Markov switching process and ensure that (2.3) holds in every period t. The fluctuations of $\pi_t^{(s)}$ around $\pi_* \gamma(s_t)$ are identical to the fluctuations in the above targeted-inflation and deflation equilibria. Throughout this paper, we will assume that the sunspot process evolves independently from the fundamental shocks.[3] A numerical illustration is provided in Figure 2.1. The top panel compares the paths of net inflation under the targeted-inflation equilibrium (2.4) and the deflation equilibrium (2.5). The difference between the inflation paths is the level shift due to the constants γ_* versus γ_D. The bottom panel shows the sunspot equilibrium with visible shifts from the targeted-inflation regime to the deflation regime (shaded areas) and back.

There exist many other solutions to (2.3). The local dynamics around the deflation steady state, ignoring the ZLB constraint, are indeterminate, and it is possible to find alternative deflation equilibria. For example, Benhabib, Schmitt-Grohé and Uribe (2001*a*) studies alternative equilibria in which the economy transitions from the targeted-inflation regime to a deflation regime and remains in the deflation

[3] For the simple example in this section we can easily construct equilibria in which the Markov transition is triggered by ϵ_t.

regime permanently in continuous-time perfect foresight monetary models. Such equilibria can also be constructed in our model, and one of them is discussed in more detail in Appendix B.1.

2.3 A Prototypical New Keynesian DSGE Model

Our quantitative analysis will be based on a small-scale New Keynesian DSGE model. Variants of this model have been widely studied in the literature and its properties are discussed in detail in Woodford (2003). The model economy consists of perfectly competitive final-goods-producing firms, a continuum of monopolistically competitive intermediate goods producers, a continuum of identical households, and a government that engages in active monetary and passive fiscal policy. To keep the dimension of the state space manageable, we abstract from capital accumulation and wage rigidities. We describe the preferences and technologies of the agents in Section 2.3.1, and summarize the equilibrium conditions in Section 2.3.2.

2.3.1 Preferences and Technologies

Households. Households derive utility from consumption C_t relative to an exogenous habit stock and disutility from hours worked H_t. We assume that the habit stock is given by the level of technology A_t, which ensures that the economy evolves along a balanced growth path. We also assume that the households value transaction services from real money balances, detrended by A_t, and include them in the utility function. The households maximize

$$\mathbb{E}_t\left[\sum_{s=0}^{\infty}\beta^s\left(\frac{(C_{t+s}/A_{t+s})^{1-\tau}-1}{1-\tau}-\chi_H\frac{H_{t+s}^{1+1/\eta}}{1+1/\eta}+\chi_M V\left(\frac{M_{t+s}}{P_{t+s}A_{t+s}}\right)\right)\right], \tag{2.7}$$

subject to budget constraint

$$P_tC_t+T_t+M_t+B_t=P_tW_tH_t+M_{t-1}+R_{t-1}B_{t-1}+P_tD_t+P_tSC_t. \tag{2.8}$$

Here β is the discount factor, $1/\tau$ is the intertemporal elasticity of substitution, η is the Frisch labor supply elasticity, and P_t is the price of the final good. The households supply labor services to the firms, taking the real wage W_t as given. At the end of period t, households hold money in the amount of M_t. They have access to a bond market where nominal government bonds B_t that pay gross interest R_t are traded. Furthermore, the households receive profits D_t from the firms and pay lump-sum taxes T_t. SC_t is the net cash inflow from trading a full set of state-contingent securities.

Detrended real money balances $M_t/(P_tA_t)$ enter the utility function in an additively separable fashion. An empirical justification of this assumption is provided by Ireland (2004). As a consequence, the equilibrium has a block diagonal structure under the interest-rate feedback rule that we will specify below: the level of output, inflation, and interest rates can be determined independently of the money stock. We assume that the marginal utility $V'(m)$ is decreasing in real money balances m and reaches zero for $m=\bar{m}$, which is the amount of money held in steady state by households if the net nominal interest rate is zero. Since the return on holding money is zero, it provides the rationale for the ZLB on nominal rates. The usual transversality condition on asset accumulation applies.

Firms. The final-goods producers aggregate intermediate goods, indexed by $j \in [0, 1]$, using the technology:

$$Y_t = \left(\int_0^1 Y_t(j)^{1-\nu} dj \right)^{\frac{1}{1-\nu}}. \tag{2.9}$$

The firms take input prices $P_t(j)$ and output prices P_t as given. Profit maximization implies that the demand for inputs is given by

$$Y_t(j) = \left(\frac{P_t(j)}{P_t} \right)^{-1/\nu} Y_t. \tag{2.10}$$

Under the assumption of free entry into the final-goods market, profits are zero in equilibrium, and the price of the aggregate good is given by

$$P_t = \left(\int_0^1 P_t(j)^{\frac{\nu-1}{\nu}} dj \right)^{\frac{\nu}{\nu-1}}. \tag{2.11}$$

We define inflation as $\pi_t = P_t / P_{t-1}$.

Intermediate good j is produced by a monopolist who has access to the following production technology:

$$Y_t(j) = A_t H_t(j), \tag{2.12}$$

where A_t is an exogenous productivity process that is common to all firms and $H_t(j)$ is the firm-specific labor input. Labor is hired in a perfectly competitive factor market at the real wage W_t. Intermediate-goods-producing firms face quadratic price adjustment costs of the form

$$AC_t(j) = \frac{\phi}{2}\left(\frac{P_t(j)}{P_{t-1}(j)} - \bar{\pi}\right)^2 Y_t(j), \tag{2.13}$$

where ϕ governs the price stickiness in the economy and $\bar{\pi}$ is a baseline rate of price change that does not require the payment of any adjustment costs. In our quantitative analysis, we set $\bar{\pi} = 1$, that is, it is costless to keep prices constant. Firm j chooses its labor input $H_t(j)$ and the price $P_t(j)$ to maximize the present value of future profits

$$\mathbb{E}_t\left[\sum_{s=0}^{\infty}\beta^s Q_{t+s|t}\left(\frac{P_{t+s}(j)}{P_{t+s}}Y_{t+s}(j) - W_{t+s}H_{t+s}(j) - AC_{t+s}\right)\right]. \tag{2.14}$$

Here, $Q_{t+s|t}$ is the time t value to the household of a unit of the consumption good in period $t+s$, which is treated as exogenous by the firm.

Government Policies. Monetary policy is described by an interest rate feedback rule of the form

$$R_t = \max\left\{1,\ \left[r\pi_*\left(\frac{\pi_t}{\pi_*}\right)^{\psi_1}\left(\frac{Y_t}{\gamma Y_{t-1}}\right)^{\psi_2}\right]^{1-\rho_R} R_{t-1}^{\rho_R} e^{\sigma_R \epsilon_{R,t}}\right\}. \tag{2.15}$$

Here r is the steady-state real interest rate, π_* is the target-inflation rate, and $\epsilon_{R,t}$ is a monetary policy shock. The key departure from much of the New Keynesian

DSGE literature is the use of the max operator to enforce the ZLB. Provided that the ZLB is not binding, the central bank reacts to deviations of inflation from the target rate π_* and deviations of output growth from its long-run value γ.

The government consumes a stochastic fraction of aggregate output and government spending evolves according to

$$G_t = \left(1 - \frac{1}{g_t}\right) Y_t. \tag{2.16}$$

The government levies a lump-sum tax T_t (or provides a subsidy if T_t is negative) to finance any shortfalls in government revenues (or to rebate any surplus). Its budget constraint is given by

$$P_t G_t + M_{t-1} + R_{t-1} B_{t-1} = T_t + M_t + B_t. \tag{2.17}$$

Exogenous shocks. The model economy is perturbed by three (fundamental) exogenous processes. Aggregate productivity evolves according to

$$\ln A_t = \ln \gamma + \ln A_{t-1} + \ln z_t, \text{ where } \ln z_t = \rho_z \ln z_{t-1} + \sigma_z \epsilon_{z,t}. \tag{2.18}$$

Thus, on average, the economy grows at the rate γ, and z_t generates exogenous fluctuations of the technology growth rate. We assume that the government spending shock follows the AR(1) law of motion

$$\ln g_t = (1 - \rho_g) \ln g_* + \rho_g \ln g_{t-1} + \sigma_g \epsilon_{g,t}. \tag{2.19}$$

While we formally introduce the exogenous process g_t as a government spending shock, we interpret it more broadly as an exogenous demand shock that contributes to fluctuations in output. The monetary policy shock $\epsilon_{R,t}$ is assumed to be serially uncorrelated. We stack the three innovations into the vector $\epsilon_t = [\epsilon_{z,t}, \epsilon_{g,t}, \epsilon_{r,t}]'$ and assume that $\epsilon_t \sim iidN(0, I)$.[4] In addition to the fundamental shock processes, agents in the model economy observe an exogenous sunspot shock s_t, which follows a two-state Markov-switching process

$$\mathbb{P}\{s_t = 1\} = \begin{cases} (1 - p_{00}) & \text{if } s_{t-1} = 0 \\ p_{11} & \text{if } s_{t-1} = 1 \end{cases} . \tag{2.20}$$

2.3.2 Equilibrium Conditions

Since the exogenous productivity process has a stochastic trend, it is convenient to characterize the equilibrium conditions of the model economy in terms of detrended consumption $c_t \equiv C_t/A_t$ and detrended output $y_t \equiv Y_t/A_t$. Also, we define

$$\mathcal{E}_t \equiv I\!\!E_t \left[\frac{c_{t+1}^{-\tau}}{\gamma z_{t+1} \pi_{t+1}} \right] \tag{2.21}$$

$$\xi(c, \pi, y) \equiv c^{-\tau} y \left\{ \frac{1}{\nu} \left(1 - \chi_h c^\tau y^{1/\eta}\right) + \phi(\pi - \bar{\pi}) \left[\left(1 - \frac{1}{2\nu}\right) \pi + \frac{\bar{\pi}}{2\nu} \right] - 1 \right\}, \tag{2.22}$$

[4] Unlike some of the other papers in the ZLB literature, e.g. Christiano, Eichenbaum and Rebelo (2011*b*) and Fernández-Villaverde et al. (2012*a*), we do not include a discount factor shock in the model. We follow the strand of the literature that has estimated three-equation DSGE models that are driven by a technology shock, a demand (government spending), and a monetary policy shock and has documented that such models fit U.S. data for output growth, inflation, and interest rates reasonably well before the Great Recession.

which will be useful in the computational algorithm. A detailed derivation of the equilibrium conditions is provided in Appendix B.2. The consumption Euler equation is given by

$$c_t^{-\tau} = \beta R_t \mathcal{E}_t. \tag{2.23}$$

In a symmetric equilibrium, in which all firms set the same price $P_t(j)$, the price-setting decision of the firms leads to the condition

$$\xi\left(c_t, \pi_t, y_t\right) = \phi\beta I\!E_t\left[c_{t+1}^{-\tau} y_{t+1}(\pi_{t+1} - \bar{\pi})\pi_{t+1}\right] \tag{2.24}$$

The aggregate resource constraint can be expressed as

$$c_t = \left[\frac{1}{g_t} - \frac{\phi}{2}\left(\pi_t - \bar{\pi}\right)^2\right] y_t. \tag{2.25}$$

This constant reflects both government spending as well as the resource cost (in terms of output) caused by price changes. Finally, we reproduce the monetary policy rule:

$$R_t = \max\left\{1, \left[r\pi_*\left(\frac{\pi_t}{\pi_*}\right)^{\psi_1}\left(\frac{y_t}{y_{t-1}} z_t\right)^{\psi_2}\right]^{1-\rho_R} R_{t-1}^{\rho_R} e^{\sigma_R \epsilon_{R,t}}\right\}. \tag{2.26}$$

We do not use a measure of money in our empirical analysis and therefore drop the equilibrium condition that determines money demand.

As in the two-equation model in Section 2.2, the New Keynesian model with the ZLB constraint has two steady states, which we refer to as the targeted-inflation and the deflation steady state. In the targeted-inflation steady state, inflation equals π_* and the gross interest rate equals $r\pi_*$, while in the deflation steady state, inflation equals $1/r$ and the interest rate equals one.

2.4 Solution Algorithm

We now discuss some key features of the algorithm that is used to solve the nonlinear DSGE model presented in the previous section. Additional details can be found in Appendix B.4.1. We utilize a global approximation following Judd (1992) where the decision rules are assumed to be combinations of Chebyshev polynomials. The minimum set of state variables associated with our DSGE model is

$$\mathcal{S}_t = (R_{t-1}, y_{t-1}, g_t, z_t, \epsilon_{R,t}, s_t). \tag{2.27}$$

An (approximate) solution of the DSGE model is a set of decision rules $\pi_t = \pi(\mathcal{S}_t;\Theta)$, $\mathcal{E}_t = \mathcal{E}(\mathcal{S}_t;\Theta)$, $c_t = c(\mathcal{S}_t;\Theta)$, $y_t = y(\mathcal{S}_t;\Theta)$, and $R_t = R(\mathcal{S}_t;\Theta)$ that solve the nonlinear rational expectations system (2.21), (2.23), (2.24), (2.25), and (2.26), where $\Theta \equiv \{\Theta_i\}$ for $i = 1, ..., N$ parameterize the decision rules. Note that conditional on $\pi(\mathcal{S}_t;\Theta)$ and $\mathcal{E}(\mathcal{S}_t;\Theta)$, equations (2.23), (2.25) and (2.26) determine $c(\mathcal{S}_t;\Theta)$, $y(\mathcal{S}_t;\Theta)$, and $R(\mathcal{S}_t;\Theta)$, and therefore these equations hold exactly.

The solution algorithm amounts to specifying a grid of points $\mathcal{G} = \{\mathcal{S}_1, \ldots, \mathcal{S}_M\}$ in the model's state space and solving for Θ such that the sum of squared residuals associated with (2.21) and (2.24) are minimized for $\mathcal{S}_t \in \mathcal{G}$. There are three non-standard aspects of our solution method that we will now discuss in more detail: first, the piecewise smooth representation of the functions $\pi(\cdot;\Theta)$ and $\mathcal{E}(\cdot;\Theta)$; second, our iterative procedure of choosing grid points $\mathcal{G}$; and third, our method of initializing Θ when constructing the decision rules for the sunspot equilibrium.

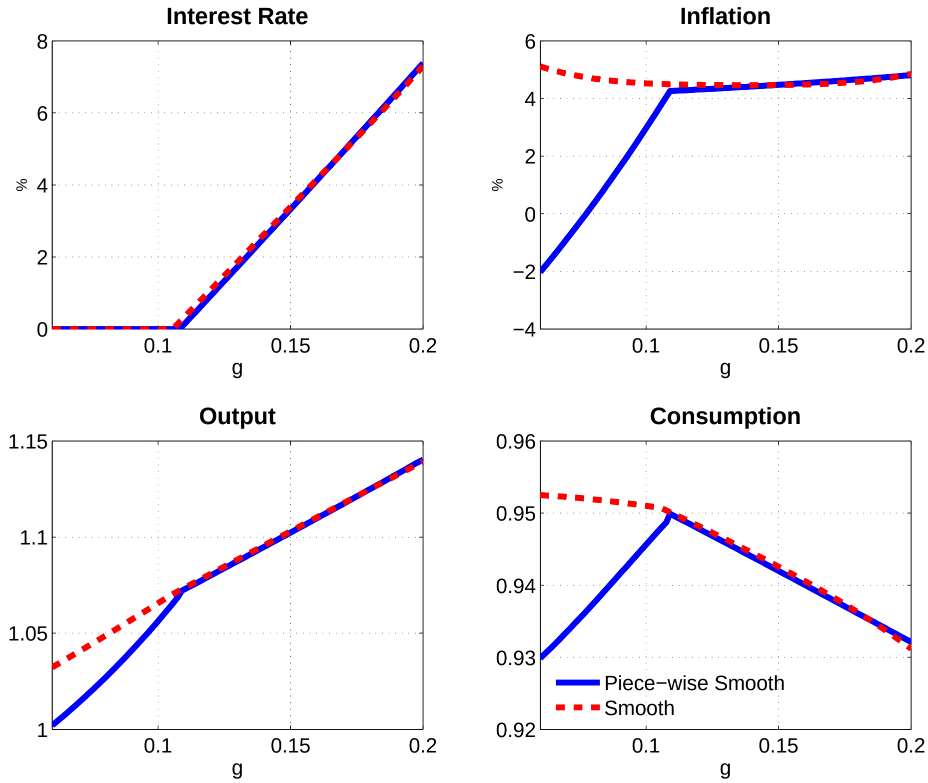
Piece-wise Smooth Decision Rules. We show in Appendix B.3 that the solution to a simplified linearized version of our DSGE model entails piece-wise linear decision rules. While Chebyshev polynomials, which are smooth functions of the states, can in principle approximate functions with a kink, such approximations are quite inaccurate for low-order polynomials. Thus, unlike Judd, Maliar and Maliar (2010), Fernández-Villaverde et al. (2012*a*) and Gust, Lopez-Salido and Smith (2012*a*), we use a piece-wise smooth approximation of the functions $\pi(\mathcal{S}_t)$ and $\mathcal{E}(\mathcal{S}_t)$ by postulating

$$\pi(\mathcal{S}_t;\Theta) = \begin{cases} f_\pi^1(\mathcal{S}_t;\Theta) & \text{if } s_t = 1 \text{ and } R(\mathcal{S}_t) > 1 \\ f_\pi^2(\mathcal{S}_t;\Theta) & \text{if } s_t = 1 \text{ and } R(\mathcal{S}_t) = 1 \\ f_\pi^3(\mathcal{S}_t;\Theta) & \text{if } s_t = 0 \text{ and } R(\mathcal{S}_t) > 1 \\ f_\pi^4(\mathcal{S}_t;\Theta) & \text{if } s_t = 0 \text{ and } R(\mathcal{S}_t) = 1 \end{cases} \tag{2.28}$$

and similarly for $\mathcal{E}(\mathcal{S}_t,\Theta)$, where the functions f_j^i are linear combinations of a complete set of Chebyshev polynomials up to fourth order. Our method is flexible enough to allow for a kink in all decision rules and not just R_t, which has a kink by its construction.

In our experience, the flexibility of the piece-wise smooth approximation yields more accurate decision rules, especially for inflation. Figure 2.2 shows a slice of the decision rules where we set $s_t = 1$, $R_{t-1} = 1$, $y_{t-1} = y_*$, $z = 0$, and $\epsilon_{R,t} = 0$ and vary

Figure 2.2: Sample Decision Rules



Note: This figure shows the decision rules assuming parameter values $p_{11} = 1$ and $\eta = \infty$ (linear disutility of labor). The x-axis shows the state variable g, while the other state variables are fixed at $s_t = 1$, $R_{t-1} = 1$, $y_{t-1} = y_*$, $z = 0$, and $\epsilon_{R,t} = 0$.

g_t in a wide range. The solid blue decision rules are based on the piece-wise smooth approximation in (2.28), whereas the dashed red decision rules are obtained using a single set of Chebyshev polynomials, which impose smoothness on all decision rules except for $R(\mathcal{S}_t, \Theta)$. When approximated smoothly, the decision rules fail to capture the kinks that are apparent in the piece-wise smooth approximation. For instance, the decision rule for output illustrates that the (marginal) government-spending multiplier is sensitive to the ZLB – it is noticeably larger in the area of the state space where the ZLB binds – and the decision rule for inflation shows a very significant change in slope, neither of which is captured by the smooth approximation.

Choice of Grid Points. With regard to the choice of grid points, projection methods that require the solution to be accurate on a fixed grid, e.g., a tensor product grid, become exceedingly difficult to implement as the number of state variables increases above three. While the Smolyak grid proposed by Krueger and Kubler (2004*a*) can alleviate the curse of dimensionality to some extent, we build on recent work by Judd, Maliar and Maliar (2010)[5], with a significant modification: we combine simulated grid points (obtained using a time-separated-grid algorithm) with states obtained from the data using a nonlinear filter. While Japanese data between 1981 and 2013 can be comfortably explained by the ergodic distribution associated with the sunspot solution of the DSGE model, U.S. data since 2008 are much more difficult to reconcile with the DSGE model. For the U.S., one needs shocks that are several standard deviations away from the center of the ergodic distribution to reach the ZLB in 2009. Thus, it is crucial to combine draws from the ergodic distribution with states that are extracted from data on output growth, inflation, and interest rates to generate the grid $\mathcal{G}$. This ensures that our approximation remains accurate in the area of the state space that is relevant for the empirical analysis. This is an iterative process. For a given solution given by Θ, we simulate the model and get a set of points that characterize the ergodic distribution. Then we run a particle filter, details of which are provided in Section 2.5.4, to obtain the grid points which

[5]The work by Judd, Maliar, and Maliar evolved considerably over time. We initially built on the working paper version, Judd, Maliar and Maliar (2010), which proposed to simulate the model to be solved, to distinguish clusters on the simulated series, and to use the clusters' centers as a grid for projections. In the published version of the paper, Maliar and Maliar (2014) also consider ϵ-distinguishable (EDS) grids and locally-adaptive EDS grids. Their locally-adaptive grids are similar in spirit to our approach, which tries to control accuracy in a region of the state space that is important for the substantive analysis, even if it is far in the tails of the ergodic distribution.

are consistent with the U.S. data since 2008.

Initialization of Θ. Recall that the sunspot equilibrium decision rules are obtained by solving for Θ that minimizes the sum of squared residuals associated with (2.21) and (2.24) for $\mathcal{S}_t \in \mathcal{G}$. We start the solution process by solving the model assuming $p_{11} = p_{00} = 1$, that is, both sunspot regimes are absorbing states. This means, essentially, the decision rules evaluated at $s_t = 1$ ($s_t = 0$) resemble those that would be obtained in the targeted-inflation equilibrium (a minimal-state-variable deflation equilibrium). Once these decision rules are accurately obtained, after some iterations of the simulate-filter-solve algorithm, we use them as initial guesses of the decision rules of the full model with $p_{11} < 1$ and $p_{00} < 1$. When the transition probabilities are nonzero, the agents anticipate regime changes to occur in the future and this changes their decision rules. Still the initial guesses prove to be reasonably accurate.[6] We parameterize each f_j^i for $i = 1, ..., 4$ and $j = \pi, \mathcal{E}$ with 126 parameters for a total of 1,008 elements in Θ and use $M = 624$ including the grid points from the ergodic distribution and the filtered states. For a given set of filtered states and simulated grid, the solution takes about two minutes on a single-core Windows-based computer using MATLAB. The approximation errors are on the order of 10^{-4} or smaller, expressed in consumption units.

[6] We do the filtering iteration three times and within each iteration we do the simulation-solve iteration five times. Further iterations do not change the results in any appreciable way.

2.5 Quantitative Analysis

The data sets used in the empirical analysis are described in Section 2.5.1. In Section 2.5.2, we estimate the parameters of the DSGE model for the U.S. and Japan using data from before the economies reached the ZLB. These parameter estimates are the starting point for the subsequent analysis. In Section 2.5.3, we compare the ergodic distributions of interest rates and inflation under the parameter estimates obtained for the two countries. In Section 2.5.4 we show that the Japanese economy shifted to the deflation regime at the end of the 1990s, which triggered a long spell of zero nominal interest rates. The U.S., on the other hand, stayed in the targeted-inflation regime after 2009 when interest rates reached the ZLB. Adverse demand shocks contributed to the low interest rates initially, and subsequently an expansionary monetary policy kept interest rates at zero. We offer an interpretation of the evolution of the estimated sunspot shocks in Section 2.5.5. Finally, Section 2.5.6 compares the effects of an expansionary fiscal policy in the U.S. and Japan.

2.5.1 Data

The subsequent empirical analysis is based on real per-capita GDP growth, GDP deflator inflation, and interest rate data for the U.S. and Japan. The U.S. interest rate is the federal funds rate and for Japan we use the Bank of Japan's uncollateralized call rate. Further details about the data are provided in Appendix B.5. The time series are plotted in Figure 2.3. The U.S. sample starts in 1984:Q1, after the start of the Great Moderation, whereas the time series for Japan start in 1981:Q1.

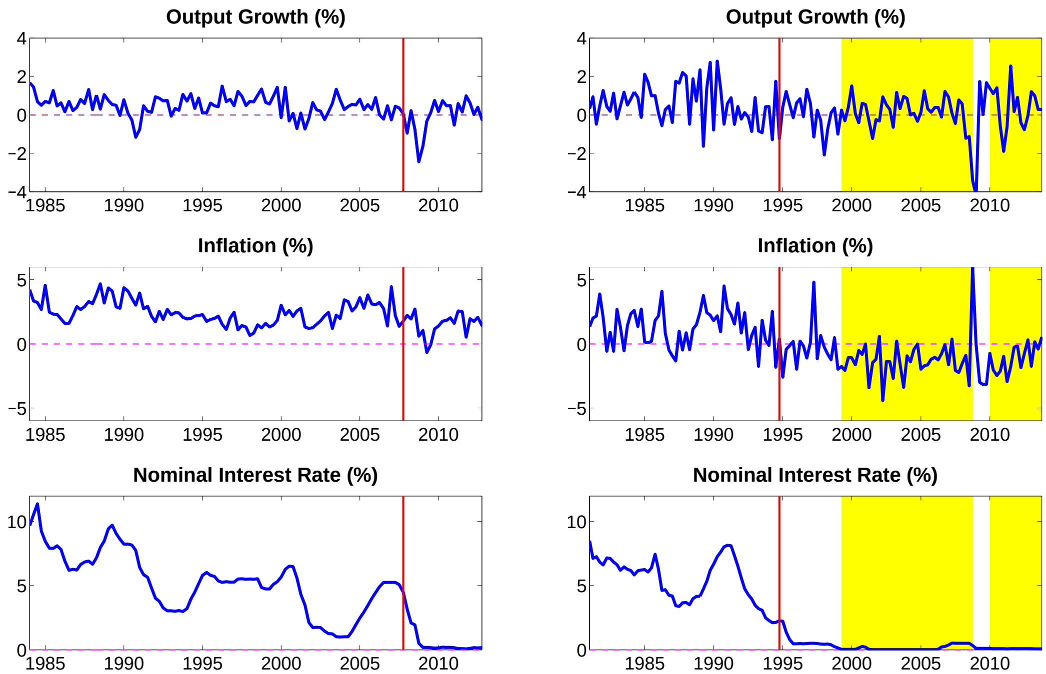
The vertical lines denote the end of the estimation sample, which is 2007:Q4 for the U.S. and 1994:Q4 for Japan. We chose the endpoints for the estimation sample such that the economies are unambiguously in the targeted-inflation regime and away from the ZLB during the estimation period. For the U.S. the fourth quarter of 2007 marks the beginning of the Great Recession, which was followed by a long-lasting spell of zero interest rates starting in 2009. In Japan, short-term interest rates dropped below 50 basis points in 1995:Q4 and have stayed at or near zero ever since. A key feature of the deflation regime in our model is that inflation rates are negative. While the U.S. experienced only two quarters of negative inflation (2009:Q2 and Q3) and two quarters of inflation around 0.5% (2011:Q4 and 2013:Q2), inflation in Japan has been negative (or near zero) for most quarters since 1995. These features of the data are important for the identification of the sunspot regimes.

2.5.2 Model Estimation

We verified that the decision rules for the targeted-inflation regime, in the region of the state space for which the ZLB is far from being binding, are well approximated by the decision rules obtained from a second-order perturbation solution of the DSGE model that ignores the ZLB. Because the perturbation solution is much easier to compute and numerically more stable than the global approximation to the sunspot equilibrium discussed in Section 2.4, we end the estimation samples for the U.S. and Japan in 2007:Q4 and 1994:Q4, respectively. To obtain posterior esti-

Figure 2.3: Data

U.S. 1984-2013 Japan 1981-2013



Note: See Section 2.5.1 for the data definitions. The vertical red line in each figure show the end of the estimation sample. The yellow shading is explained in Section 2.5.4 and it shows the periods during which $\mathbb{P}[\{s_t = 1\}|Y_{1:t}] < 0.1$ as assessed by the nonlinear filter.

mates of the DSGE model parameters we use a particle Markov chain Monte Carlo approach along the lines of Fernández-Villaverde and Rubio-Ramírez (2007) and Andrieu, Doucet and Holenstein (2010), which approximates the likelihood function with a particle filter and embeds that approximation into a Metropolis-Hastings sampler.

The parameter estimates are in Table 2.1.[7] A subset of the parameters were fixed prior to estimation. We choose values for γ, β, and π^* such that the steady state of the model matches the average output growth, inflation, and interest rates over

[7]The prior distribution as well as the implementation of the posterior sampler are described in Appendix B.6.

Table 2.1: DSGE Model Parameters

Parameter	Description	U.S.	Japan
τ	Inverse IES	2.23	1.14
		(1.85, 2.66)	(0.72, 1.70)
κ	Slope (linearized) Phillips curve	0.26	0.55
		(0.16, 0.39)	(0.36, 0.77)
ψ_1	Taylor rule: weight on inflation	1.52	1.49
		(1.45, 1.60)	(1.41, 1.58)
ρ_R	Interest rate smoothing	0.59	0.6
		(0.51, 0.68)	(0.47, 0.71)
ρ_g	Persistence: demand shock	0.92	0.88
		(0.88, 0.94)	(0.82, 0.94)
ρ_z	Persistence: technology shock	0.16	0.04
		(0.05, 0.30)	(0.01, 0.09)
$100\sigma_R$	Std dev: monetary policy shock	0.23	0.23
		(0.18, 0.30)	(0.17, 0.30)
$100\sigma_g$	Std dev: demand shock	0.54	1.02
		(0.41, 0.70)	(0.71, 1.51)
$100\sigma_z$	Std dev: technology shock	0.54	1.02
		(0.44, 0.66)	(0.82, 1.26)
The Following Parameters Were Fixed During Estimation			
$100\ln\gamma$	Quarterly growth rate of technology	0.48	0.56
$400(1/\beta-1)$	Annualized discount rate	0.87	1.88
$400\ln\pi^*$	Annualized inflation rate	2.52	1.28
$(G/Y)_*$	SS consumption/output ratio	0.15	0.16
η	Frisch elasticity	0.85	0.72
ψ_2	Taylor rule: weight on output growth	0.80	0.30
ν	EOS intermediate inputs	0.10	0.10
p_{00}	Prob of staying in deflation regime	0.95	0.95
p_{11}	Prob of staying in targeted-inflation regime	0.99	0.99

Notes: We report posterior means and 90% credible intervals (5th and 95th percentile of the posterior distribution) in parentheses. EOS is elasticity of substitution; SS is steady state. Note that $g_* = 1/(1-(G/Y)_*)$. For the U.S. the estimation sample covers 1984:Q1-2007:Q4. For Japan the estimation sample is 1981:Q1-1994:Q4.

the estimation sample period.[8] The steady state government expenditure-to-output ratio is determined from national accounts data. Since our sample does not include observations on labor market variables, we fix the Frisch labor supply elasticity. Based on Ríos-Rull et al. (2012*a*), who provide a detailed discussion of parameter values that are appropriate for DSGE models of U.S. data, we set $\eta = 0.85$ for the U.S. Our value for Japan is based on Kuroda and Yamamoto (2008), who use micro-level data to estimate labor supply elasticities along the intensive and extensive for males and females. The authors report a range of values which we aggregated into $\eta = 0.72$.

We fix the value of ψ_2 based on estimates of linearized DSGE models with an output-growth rule.[9] The parameter ν, which captures the elasticity of substitution between intermediate goods, is set to 0.1. It is essentially not separately identifiable from the price adjustment cost parameter ϕ. Finally, we need to specify values for the transition probabilities p_{00} and p_{11}. These parameters determine the expected durations of staying in each regime. Since there is no clear empirical observation to identify the transition probabilities, we informally chose $p_{00} = 0.95$ and $p_{11} = 0.99$. These values make the deflation regime ($s_t = 0$) less persistent than the targeted-inflation regime ($s_t = 1$) and imply unconditional regime probabilities of 0.17 ($s_t = 0$) and 0.83 ($s_t = 1$), respectively.

[8] In a nonlinear model, the average of the ergodic distribution is generally different from the steady state. However, over the estimation period, the nonlinearities are not very strong and the discrepancy is small.

[9] For Japan we use the average of the estimates from Ichiue, Kurozumi and Sunakawa (2013) and Fujiwara, Hirose and Shintani (2011), which are 0.50 and 0.17 respectively. For the US we use the estimate of Aruoba and Schorfheide (2013).

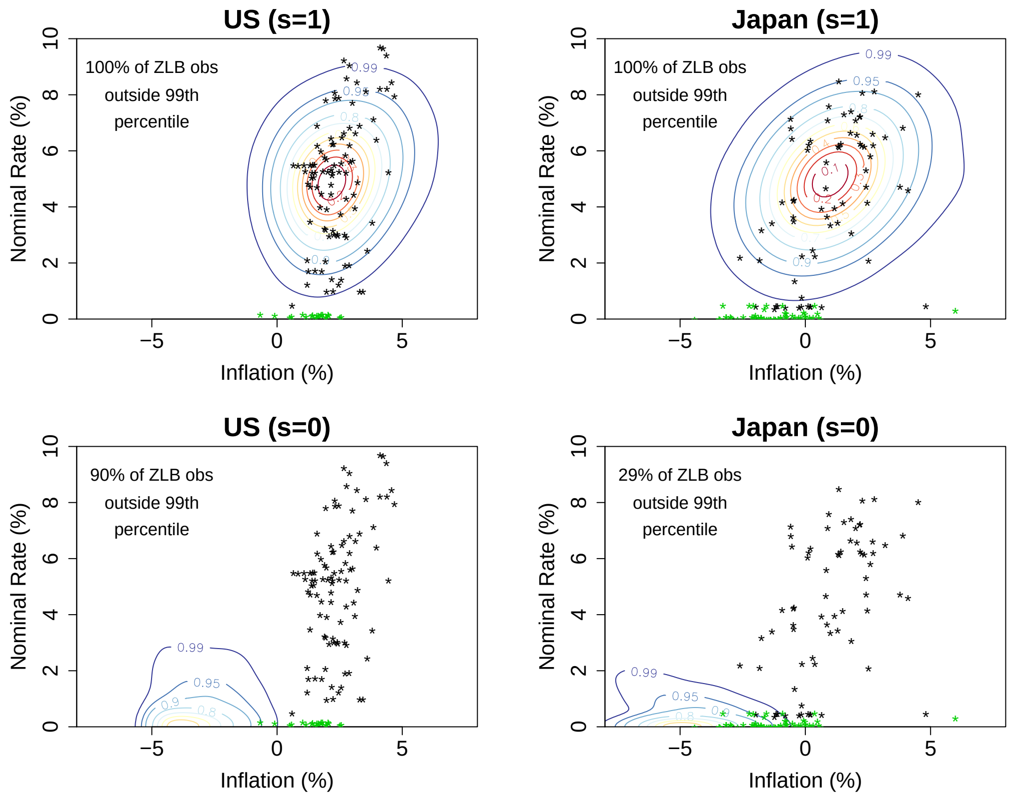
For the remaining parameters, we report posterior mean estimates and 90% credible intervals in Table 2.1. Overall, the estimates reported in the table are in line with the estimates reported elsewhere in the literature. Most notable are the estimates of the degree of price rigidity. Rather than reporting estimates for the adjustment cost parameter ϕ, we report estimates for the implied slope of the New Keynesian Phillips curve in a linearized version of the DSGE model (without ZLB constraint). This transformation takes the form $\kappa = \tau(1-\nu)/(\nu\pi_*^2\phi)$. The slope estimate is 0.26 for the U.S. and 0.55 for Japan, implying fairly flexible prices and relatively small real effects of unanticipated interest rate changes.[10]

2.5.3 Equilibrium Dynamics

To understand how our model behaves at its ergodic distribution, we simulate a long sequence of draws using the estimates for both countries. Figure 2.4 depicts contour plots of the ergodic distributions of inflation and interest rates for the two countries in columns and for the two regimes in rows. In the contour plots each line represents one percentile with the outermost line showing the 99^{th} percentile. In each panel we show the data used to estimate the model using black stars and the post-estimation data using green stars. There are a number of noteworthy results. First, the ergodic distributions are centered near the respective steady state values, with the mean inflation when $s = 1$ slightly below π_* and mean inflation when $s = 0$ below $1/r$. Second, focusing on the top row, the estimation data falls squarely inside

[10] A survey of DSGE model-based New Keynesian Phillips curve is provided in Schorfheide (2008*b*). Our estimates fall within the range of the estimates obtained in the literature.

Figure 2.4: Ergodic Distribution and Data



Notes: In each panel we report the joint probability density function (kernel density estimate) of the annualized net interest rate and inflation, represented by the contours. Black stars show the data used in estimation. Green stars show the rest of the data.

the ergodic distributions for $s = 1$ with only a few observations with high interest rates for the U.S. Third, the ZLB is not observed in the ergodic distribution for $s = 1$ for either country, while about 85% of observations feature the ZLB when $s = 0$ for both countries. This is not surprising since the estimation samples of both countries cover a period of above-zero interest rates and low macroeconomic volatility. Finally, deflation is very unlikely in the U.S. when $s = 1$, with only a 1.1% probability, while in Japan this probability is much higher at 22.9%. When $s = 0$, on the other hand, inflation is never positive.

To provide more details about the ergodic distribution, annualized output growth is virtually identical across the two regimes for both countries. An important difference between the two regimes is the correlation of (detrended) output and inflation. When $s = 1$, this correlation is strongly positive – 0.83 for the U.S. and 0.73 for Japan – which is naturally consistent with the data, albeit somewhat stronger. When $s = 0$, on the other hand, the correlation becomes strongly negative, around -0.95 for both countries. This result is linked to the findings of Eggertsson (2009*a*) and Mertens and Ravn (2014), who show that positive demand shocks may lead to a negative comovement of prices and output in the deflation regime. Since the majority of fluctuations in our model is explained by the demand shock, this delivers the negative correlation.[11]

To understand what drives the correlation between inflation and output in the model it is useful to consider a special case with $\psi_2 = 0$, $\rho_R = 0$ and $\tau = 1$. Combing the Euler equation (2.23) with the policy rule (2.15) and the aggregate resource constraint (2.25) yields the following expression:

$$y_t^{-1} = \beta \max \left\{ 1, \left[r\pi_* \left(\frac{\pi_t}{\pi_*} \right)^{\psi_1} \right] \right\} \left[\frac{1}{g_t} - \frac{\phi}{2} \left(\pi_t - \bar{\pi} \right)^2 \right] \mathbb{E}_t \left[\frac{\left[\frac{1}{g_{t+1}} - \frac{\phi}{2} \left(\pi_{t+1} - \bar{\pi} \right)^2 \right]^{-1}}{\gamma z_{t+1} \pi_{t+1} y_{t+1}} \right] \tag{2.29}$$

The relation given by (2.29) can be interpreted as an Aggregate Demand curve (AD), because it relates output demand and inflation in the $(y - \pi)$ plane. When the ZLB does not bind, the Taylor principle implies that the nominal interest rate

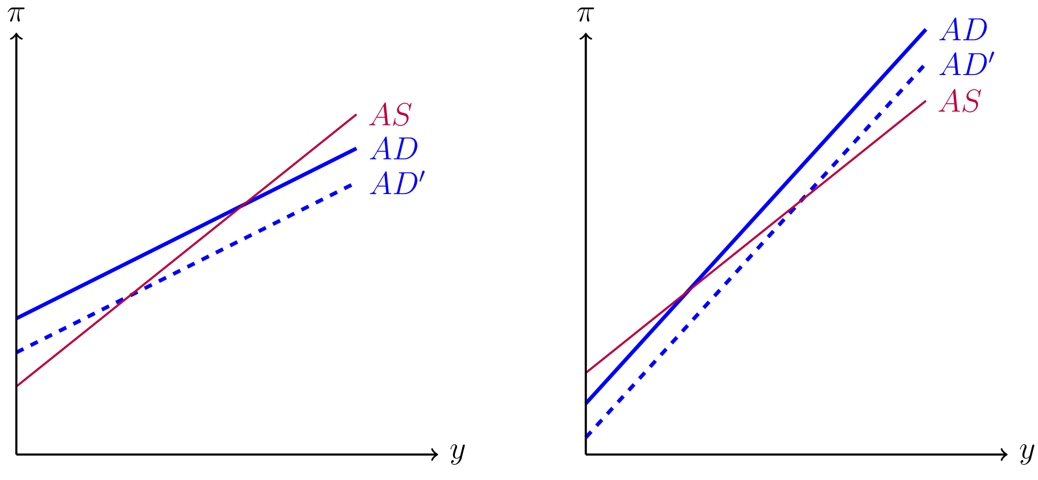
[11] We show impulse responses for the U.S. economy in Appendix B.7.

responds more than one-to-one to changes in inflation, hence the AD schedule is downward sloping. On the other hand, when the ZLB binds, the nominal interest rate no longer responds to changes in inflation, thus an increase in inflation lowers the real interest rate and stimulates consumption. The associated increase in output implies that the AD schedule bends and becomes upward sloping. In addition to the AD curve, Equation 2.24 defines an upward sloping schedule in the $(y - \pi)$ plane that has the interpretation of an Aggregate Supply (AS) curve. The equilibrium values of output and inflation are obtained solving the system of equations given by the AS and AD curves.

Figure 2.5 shows the effect of a fiscal expansion in both the deflationary and targeted inflation regime when the economy is at the ZLB. As discussed above, in both cases the AD schedule is upward slopping. A fiscal expansion shifts the AD curve out because for any level of inflation higher government expenditure increases the demand for the final good. In the $s_t = 0$ regime deflationary spells are more frequent and persistent compared to the $s_t = 1$ regime, hence the real interest rate is higher and consumption is lower along the the AD schedule. This implies that the AD curve is flatter in the $s_t = 0$ regime.

Panel (a) shows that in the $s_t = 0$ regime an increase in government expenditure has a deflationary effect, whereas in panel (b) expansionary fiscal policy increases inflation in the $s_t = 1$ regime. Ultimately the final effect on output and inflation depends on the response of the AS curve arising from wealth effects on labor supply. Mertens and Ravn (2014) show that the deflationary effect limits the output expansion from a rightward shift in the AS curve. Our quantitative findings

Figure 2.5: Response of Inflation to Fiscal Stimulus at the ZLB



(a) Deflationary Regime ($s_t = 0$) (b) Target Inflation Regime ($s_t = 1$)

Notes: The dashed blue line corresponds to the AD schedule with higher government expenditure.

in Section 2.5.6 further illustrate this point for the particular case of Japan.

The focus of this paper is not normative, but it is worth mentioning that the deflationary regime is not necessarily "bad" in terms of welfare. Average consumption across the two regimes are identical and the volatility of consumption is 24% higher in the deflationary regime. The distance between actual and desired inflation (0%) is larger in the deflationary regime relative to the targeted-inflation regime, which means the adjustment costs will be larger. These observations would imply a lower welfare for the deflationary regime. However, the interest rate is much closer, in fact most of the time exactly equal to zero (the Friedman rule) and thus the welfare cost due to holding money is much smaller. We leave a full-blown normative analysis along the lines of Aruoba and Schorfheide (2011) to future work.

2.5.4 Evidence For the Deflation Regime in the U.S. and Japan

The DSGE model has a nonlinear state-space representation of the form

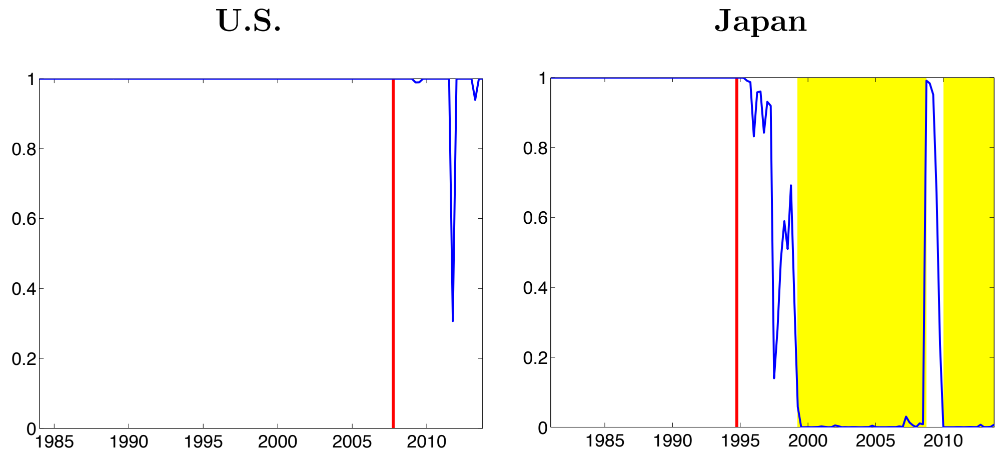
$$\begin{aligned} d_t &= \Psi(x_t) + \nu_t \\ x_t &= F_{s_t}(x_{t-1}, \epsilon_t) \\ \mathbb{P}\{s_t = 1\} &= \begin{cases} (1 - p_{00}) & \text{if } s_{t-1} = 0 \\ p_{11} & \text{if } s_{t-1} = 1 \end{cases} \end{aligned} \tag{2.30}$$

Here d_t is the 3×1 vector of observables consisting of output growth, inflation, and nominal interest rates and $D_{1:t}$ is the sequence $\{d_1, \ldots, d_t\}$. The vector x_t stacks the continuous state variables, which are given by $x_t = [R_t, y_t, y_{t-1}, z_t, g_t, A_t]'$, and $s_t \in \{0, 1\}$ is the Markov-switching process. The first equation in (2.30) is the measurement equation, where $\nu_t \sim N(0, \Sigma_\nu)$ is a vector of measurement errors. The second equation corresponds to the law of motion of the continuous state variables. The vector $\epsilon_t \sim N(0, I)$ stacks the innovations $\epsilon_{z,t}$, $\epsilon_{g,t}$, and $\epsilon_{R,t}$. The functions $F_0(\cdot)$ and $F_1(\cdot)$ are generated by the model solution procedure described in Section 2.4. The third equation represents the law of motion of the Markov-switching process. Conditioning on the posterior mean estimates obtained in Section 2.5.2, we now use a sequential Monte Carlo filter (also known as the particle filter)[12] to extract estimates of the sunspot shock process s_t, and the latent state x_t.

The main result is presented in Figure 2.6, which depicts the filtered prob-

[12]This filter is a more elaborate version of the filter that underlies the estimation in Section 2.5.2. It is described in detail in Appendix B.8. A recent survey of sequential Monte Carlo methods is provided by Creal (2012).

Figure 2.6: Filtered Probability of Targeted-Inflation Regime



Notes: The solid red vertical bar indicates the end of the estimation sample. The shaded area indicates time periods for which the filtered probability for the targeted-inflation regime falls below 10%.

abilities $\mathbb{P}[s_t = 1|D_{1:t}]$ of being in the targeted-inflation regime. According to our estimates, the experience of the U.S. and Japan was markedly different. With the exception of 2011:Q4, when the probability of the deflation regime increased to about 70%, the U.S. has been in the targeted-inflation regime. In 2009:Q2, the probability of the deflation regime is small, but non-zero, vindicating Bullard's (2010) concern of a shift to the deflationary regime. Japan, on the other hand, experienced a switch to the deflation regime in 1999:Q2, and, except for the period from 2008:Q4 to 2009:Q3, has stayed in the deflation regime.[13] Recall from Figure 2.3 that the U.S. interest rates have been essentially zero since 2009:Q1, whereas in Japan interest rates have been below 50 basis points since 1995:Q4, and essentially zero since 1999:Q1. While in the case of the U.S. the ZLB spell is interpreted as evidence in

[13]A large decline in oil prices led to a decrease in the import deflator which in turn generated a large jump in the GDP deflator to about 6% in 2008:Q4. If we remove this observation, then the temporary switch to the targeted-inflation regime vanishes. If we use CPI instead of the GDP deflator as our price measure, we find a long spell of the deflation regime from 2000 to 2008 as well as a subsequent shorter spell.

favor of the targeted-inflation regime, for Japan it is attributed toward a shift into the deflation equilibrium. The key reason for this difference is the behavior of inflation. The U.S. experienced only three quarters of low or negative inflation rates, whereas prices have been on average falling for many years in Japan. The ergodic distributions depicted in Figure 2.4 highlight that the deflation regime not only implies that interest rates are close to zero, it also implies that inflation is negative with very high probability. Accordingly, it shows that none of the ZLB observations fall inside 99% of the ergodic distribution for the targeted-inflation refime for either country, while about 70% of ZLB observations for Japan are well inside the ergodic distribution for the deflation regime.

In the absence of a switch to the deflation regime, the U.S. reaches the ZLB in response to very large negative innovations (greater than 2 standard deviations) to the latent demand shock process g_t. Since the DSGE model has a fairly strong mean reversion, a sequence of expansionary monetary policy shocks are necessary to prevent the nominal interest rate from rising. In the absence of these monetary policy shocks, U.S. nominal interest rates would have averaged 1.3% whereas average inflation would have been 0.4% instead of 1.6% after 2009. In Japan, the switch to the deflation regime pushed the economy toward the ZLB. While interest rates are close to zero in the deflation regime, Figure 2.4 shows that inflation rates should be less than -2.5% with very high probability. The average inflation rate between 1999 and 2008 is about -1.3%. The model rationalizes the relatively low observed deflation with a sequence of demand shock innovations that is on average slightly negative. Recall that in the deflation regime a negative $\epsilon_{g,t}$ tends to raise inflation.

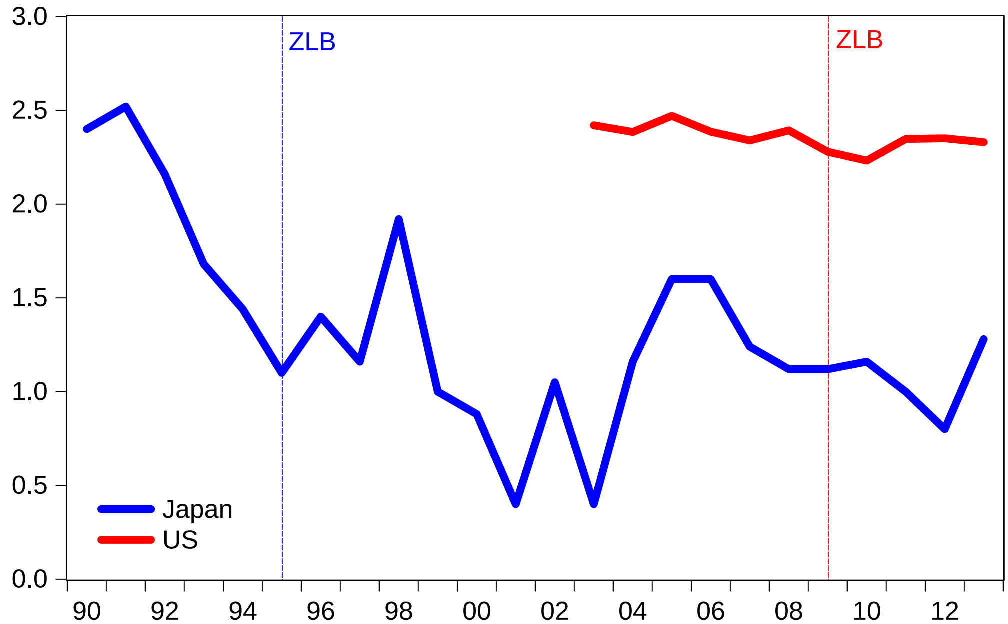
2.5.5 Interpretation of Results

From the perspective of our model both the U.S. and the Japanese economy experienced a sequence of adverse demand shocks that led to a fall in interest rates.[14] In the U.S. it was the financial crisis that unfolded during 2008 and peaked in the fourth quarter. For Japan some of the obvious culprits are the burst of the housing bubble (1992:Q1), the East-Asian / Korean crises (1997) and the Russian Financial Crisis (1998Q3). Following these events, short-term interest rates have been zero both in the U.S. and Japan. The key finding of our empirical analysis is that the two countries stayed at the ZLB for very different reasons. Japan experienced a switch of the sunspot variable s_t from the targeted-inflation regime to the deflation regime in 1999:Q2. The Japanese economy essentially stayed in the deflation regime until the end of our sample in 2013. For the U.S., on the other hand, there is no strong evidence of a switch to the deflation regime. A change in the sunspot regime means that the agents in the economy coordinated their expectations and actions based on some extraneous information. While this information is not directly observed by us, we will compare aspects of monetary policy in Japan and the U.S. that may have contributed to agents' expectation formation and, through the lens of our model, determined whether a regime switch occurred.

Mechanically, s_t is an exogenous process in our model and agents' decision rules and expectations about the future are indexed by s_t. Since a switch in s_t triggers changes in expectations, we can interpret the sunspot shock also as an exogenous

[14] The filtered $\epsilon_{g,t}$ shocks are plotted in Figure B-4 and Figure B-5 in the Appendix.

Figure 2.7: 10-Year Inflation Expectations



Notes: Units are annualized percentages. Vertical lines show the quarter where interest rates fall to the ZLB in each country.

shock to expectations. In Figure 2.7 we plot 10-year inflation expectations for Japan and the U.S. starting five years prior to each country's respective ZLB episode. For Japan we use the Consensus Forecasts and for the U.S. we use the results from Aruoba (2014), which are based on surveys. The vertical lines in the figure depict the start of the ZLB episode of the two countries. For the U.S., long-run inflation expectations simply do not move during or after the financial crisis and they show small fluctuations around 2.3%. For Japan, the expectations are around 2.5% prior to the burst of the housing price bubble and they gradually decline to 0.5% by 2003. Of course the realized quarterly or annual inflation is consistently negative throughout this period as well. Thus, the evidence in Figure 2.7 is consistent with the interpretation that Japan experienced a shock to inflation expectations whereas the U.S. did not.

Inflation expectations are closely tied to expectations about future monetary policy. In Japan the policy rate was pushed to the ZLB in 1999, but any further action such as committing to a particular target or quantitative easing (QE) was expressly ruled out. A speech by the then-governor Hayami (1999) explains that this policy is in effect "until deflationary concerns subside" (Page 1). He then goes on to imply that rates may go up before inflation becomes positive, if the Bank of Japan decides that price stability may be at jeopardy at some future point in time. In fact, the Bank of Japan increased its policy rate in August 2000 based on inflation concerns, even though prices had been continuously falling for many quarters. He also dismisses the need for QE, arguing that a cut in the interest rate achieves what QE can achieve, no more, no less. When QE was finally implemented in 2001, the policy wasn't explained clearly and previous claims by bank officials about the perceived ineffectiveness of QE were not refuted. To sum up, as Ito and Mishkin (2006), who provide and excellent (and critical) summary of the actions taken by the Bank of Japan and the Japanese government, put it: "The Bank of Japan had a credibility problem, particularly under the Hayami Regime [1998-2003], in which the markets and the public did not expect the Bank of Japan to pursue expansionary monetary policy in the future, which would ensure that deflation would end. These mistakes in the management of expectations are a key reason why Japan found itself in a deflation that it is finding very difficult to get out of" (Page 165).

The actions of U.S. policymakers following the financial crisis of 2008 contrast greatly with the actions of the Bank of Japan. The Federal Reserve and in general policy makers in the U.S. reacted to the financial crisis very forcefully, using unconventional tools early on. By the end of 2008 as the federal funds rate target was brought to near-zero levels, several rounds of large-scale asset purchase policies were implemented to provide liquidity to the banking system and lower long-term interest rates. Moreover, the Federal Reserve implemented a policy of "forward guidance." Starting from the December 2008 policy announcement, the Federal Reserve made its intention of keeping the federal funds rate near zero for an extended period of time very clear. The December 2008 press release includes the following statement: "The committee anticipates that weak economic conditions are likely to warrant exceptionally low levels of the federal funds rate for some time." The statement was strengthened by changing "some time" to "an extended period" three months later. Starting in August 2011, the Federal Reserve was even more specific, providing explicit time frames for the low rates.

Thus, a plausible interpretation of our empirical findings is the following. In the U.S. the expansionary unconventional monetary policies of the Federal Reserve kept inflation expectations anchored and prevented a switch to the deflation regime. The Bank of Japan, on the other hand, did not convince the public that it would pursue an aggressive expansionary monetary policy, which triggered an adverse shock to inflation expectations and moved the economy into the deflation regime. Ueda (2012) provides a very thorough review of the policies used in the U.S. and Japan and he concludes that "the entrenched nature of deflationary expectations, however, seems to have prevented [the zero interest rate and QE policies to increase inflation expectations on a significant scale for a sustained period]. Unfortunately, the

Japanese economy seems to be trapped in an 'equilibrium' whereby only exogenous forces generate movements to a better equilibrium with a higher rate of inflation" (Page 20). This, of course, is precisely the point we show formally in this paper.

2.5.6 Policy Experiments

During their respective ZLB episodes, both Japan and the U.S. engaged in unprecedented fiscal and monetary interventions. The U.S. enacted the American Recovery and Reinvestment Act (ARRA) in February 2009, which consisted of various fiscal interventions, a significant part of which was government spending. Similarly, there have been numerous fiscal programs in Japan starting in 1998, some of which were explicitly aimed at dealing with various local shocks (e.g., the 2011 earthquake) or global shocks (e.g., the global financial crisis), and starting in 2010 with deflation. We provide a summary of these programs in Table B-1. All of these policies were aimed at increasing real economic activity, increasing inflation from deflationary levels (or preventing it from going there), or both. In this section, our main goal is to demonstrate how these fiscal policies may have drastically different effects on the economy, depending on whether a shift to the deflation regime or an adverse sequence of shocks in the targeted-inflation regime is what is keeping the economy near the ZLB.

The recent literature has emphasized that the effects of expansionary fiscal policies on output may be larger if the economy is at or near the ZLB. In the absence of the ZLB, a typical interest rate feedback rule implies that the central bank raises nominal interest rates in response to rising inflation and output caused by an increase in government spending. This monetary contraction raises the real interest rate, reduces private consumption, and overall dampens the stimulating effect of the fiscal expansion. If the economy is at the ZLB, the expansionary fiscal policy is less likely to be accompanied by a rise in interest rates because the feedback portion of the policy rule tends to predict negative interest rates. Without a rising nominal interest rate, the increase in inflation that results from the fiscal expansion reduces the real rate. In turn, current-period demand is stimulated, amplifying the positive effect on output. In fact, the decision rules depicted in Figure 2.2 show that when the ZLB starts to bind, the response of output to an increase in government spending is larger, and consumption goes up.[15]

2.5.6.1 Details of the Policy Experiments

Due to the nonlinearity of our DSGE model, the effect of policy interventions captured by impulse response functions depends on the initial state of the economy. Rather than conditioning on one particular time period, we average results for several periods. We distinguish between ZLB periods (2009:Q1 to 2011:Q1 for the U.S. and 1999:Q2 to 2005:Q2 for Japan) and non-ZLB periods (1984:Q1 to 2005:Q2 for the U.S. and 1981:Q2 to 1991:Q2 for Japan).

[15]To be clear, the typical exercise in the literature is not a standard impulse response analysis. The analysis assumes the existence of a very large impulse other than the policy impulse being considered that affects the economy and causes the ZLB to bind. This shock is assumed to be large enough so that even after the policy impulse, which would have increased the nominal interest rate, the ZLB continues to bind. As an example, Fernández-Villaverde et al. (2012*a*) uses an eight-standard-deviation shock to the discount factor to keep the economy at the ZLB.

The policy effect for a particular quarter is computed as follows. Suppose that we condition on the state of the economy in period $t-1$ and track the economy for H periods. First we compute $\mathbb{P}[s_{t+h}=1|D_{1:t+h}]$, where $D_{1:t+h}$ denotes the sequence of observations $d_1,\dots,d_{t+h}$ for $h=0,1,\dots,H$. If this probability exceeds 10% we set $\tilde{s}_{t+h}=1$; otherwise we set $\tilde{s}_{t+h}=0$. Second, we compute an estimate of the remaining states: $\tilde{x}_{t-1}=\mathbb{E}[x_{t-1}|\tilde{s}_t,D_{1:t}]$ as well as estimates of the shocks $\tilde{\epsilon}_{i,t+h}=\mathbb{E}[\epsilon_{i,t+h}|\tilde{s}_{t:t+h},D_{1:t+h}]$ for $i=g,r,z$. Third, we compute the non-intervention path by iterating the state-transition equations forward based on the filtered shocks $\tilde{\epsilon}_{i,t+h}$. By construction, the non-intervention path reproduces the actual data. Fourth, we generate the intervention paths for consumption, output, inflation and interest rates (signified by an I superscript) by setting $\epsilon^I_{g,t}=\tilde{\epsilon}_{g,t}+f$ (f represents the size of the fiscal intervention), $\epsilon^I_{r,t}=\tilde{\epsilon}_{r,t}$, $\epsilon^I_{z,t}=\tilde{\epsilon}_{z,t}$, and $\epsilon^I_{i,t+h}=\tilde{\epsilon}_{i,t+h}$ for $i=g,r,z$ and $h>0$ and iterating the state-transition equation forward based on the $\epsilon^I_{i,t+h}$'s. We also compute cumulative government spending multipliers for the first H periods following the intervention:

$$\mu_H=\frac{\sum_{h=0}^{H}(Y^I_{t+h}-Y_{t+h})}{\sum_{h=0}^{H}(G^I_{t+h}-G_{t+h})}. \tag{2.31}$$

Note that according to our timing convention $H=0$ corresponds to the multiplier upon impact of the shock.

After conducting the same policy intervention for every period t in the ZLB (non-ZLB) period, we record the median and various percentiles of the government spending multiplier and the difference between the paths with and without the intervention. For the ZLB period this methodology conditions on the economy being at the ZLB, integrating out the conditions that cause the economy to stay at the ZLB.

We consider two policy experiments, beginning with a pure fiscal expansion where g increases by $1.5\sigma_g$. This is a reasonably large intervention, which is also in line with the actual policy interventions in these countries.[16] The second experiment couples the same fiscal intervention with a commitment by the central bank to keep interest rates at or near the ZLB. This central bank intervention is implemented using a sequence of unanticipated monetary policy shocks $\epsilon_{R,t}$.[17] To avoid implausibly large interventions, we choose these shocks such that they are no larger than two standard deviations in absolute value, and the interest-rate intervention is no larger than one percentage point in annualized terms in any quarter. Thus, we implicitly assume that the central bank would renege on a policy to keep interest rates near zero for an extended period of time in states of the world in which output growth and/or inflation turn out to be high. For each experiment, we report the paths of key variables following the policy interventions, as well as the cumulative government spending multiplier. Appendix B.4.2 provides some more details.

[16]For example, when we looked at the funding for federal contracts, grants, and loans portion of ARRA as disbursed in the first two quarters of the program, which amounts to just over 1% of GDP, this is equivalent to a g shock of size $1.4\sigma_g$. Table B-1 also shows that there were sizable fiscal programs in Japan, some of which were upwards of 3% of GDP.

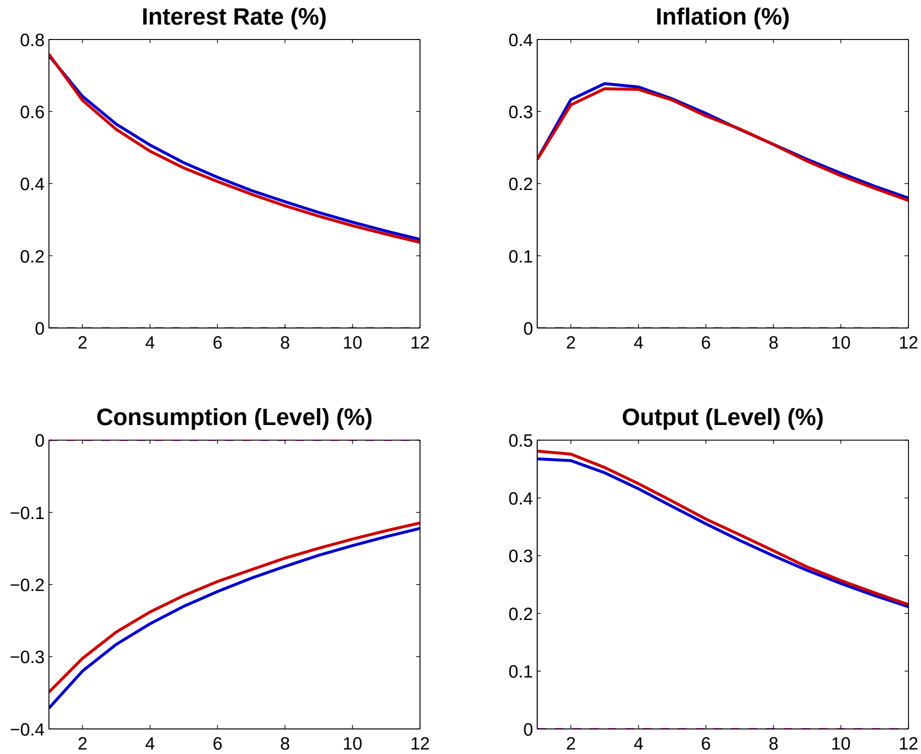
[17]A detailed discussion about the advantages and disadvantages of using unanticipated versus anticipated monetary policy shocks to generate predictions conditional on an interest rate path is provided in Del Negro and Schorfheide (2012*b*).

2.5.6.2 Pure Fiscal Policy Intervention

The impulse responses for the fiscal-only policy intervention for the U.S. is presented in Figure 2.8 and the multipliers for all policy experiments are summarized in Table 2.2. In each panel of Figure 2.8 the blue line indicates the response of the economy during non-ZLB periods and the red line shows the response of the economy during the ZLB periods. Recall that in the U.S. the ZLB is reached within the targeted-inflation regime by large adverse demand shocks. Even though these shocks lie far in the tails of the ergodic distribution, the response of the economy in the ZLB period closely resembles the response during non-ZLB periods, which in turn is the "standard" response to a government spending shock in a New Keynesian DSGE model: on impact output goes up by slightly less than 0.5% and inflation increases by about 25 basis points. As a result, the central bank raises the nominal interest rate by over 75 basis points, which means roughly a 50 basis point increase in the real interest rate. This reduces consumption by over 0.35%, which is the standard crowding-out effect of government spending. All of these changes yield a fiscal multiplier of 0.62 on impact, which goes up to 0.70 at the end of three years.

In light of the results reported in the literature on government spending multipliers during ZLB episodes it may be surprising that our impulse responses during non-ZLB and ZLB periods depicted in Figure 2.8 are so similar. The reason for the similarity is that despite being at the ZLB prior to the impact of the shock, the economy leaves the ZLB as soon as the shock hits, because we do not keep the economy

Figure 2.8: Fiscal-Only Intervention - U.S.



Notes: The blue line shows the pointwise median response of the economy in "normal" times and the red line shows the pointwise median response of the economy in the ZLB period. See Section 2.5.6.1 for the definitions. The interest and inflation rates are annualized. The bottom panels show the percentage change in the level of consumption and output.

Table 2.2: Cumulative Fiscal Multipliers

	U.S.				**Japan**			
H	0	3	7	11	0	3	7	11
	Ergodic Distribution							
Fiscal	0.62	0.67	0.68	0.70	0.58	0.56	0.56	0.56
	ZLB Episode							
Fiscal	0.62	0.67	0.69	0.70	0.47	0.46	0.46	0.46
Fiscal and Monetary	1.16	1.23	1.25	1.24	0.47	0.46	0.46	0.46

Note: The multiplier is defined in (2.31).

at the ZLB through another concurrent shock.[18] As soon as the economy exits the

[18]When Fernández-Villaverde et al. (2012*a*) conduct a similar exercise without forcing the ZLB,

ZLB, the additional channel that boosts the output response through the reduction of the real interest rate is absent. A second reason for the relatively small multiplier is the Frisch labor supply elasticity of $\eta = 0.85$. The multiplier is increasing in η and almost reaches one if $\eta = \infty$, i.e., preferences are quasi linear.[19]

For Japan a very different picture emerges. Results are presented in Figure 2.9. As we discussed in Section 2.5.4, Japan remains at the deflationary regime ($s_t = 0$) throughout the ZLB period and thus behaves very differently relative to the non-ZLB period. In particular, as a result of the fiscal intervention, the inflation rate falls sharply by 100 basis points in the ZLB period, while it increases, as the conventional wisdom would suggest, during non-ZLB periods. This decline in inflation is large enough to wipe out any desire for the central bank to increase the interest rate and thus the economy stays at the ZLB. A constant interest rate along with a decline in inflation increases the real interest rate and this depresses consumption further.

Note that this is the channel emphasized in the literature as being responsible for increasing the multiplier at the ZLB but working in exactly the opposite direction since inflation falls. At the end, output still goes up as a result of this intervention but the increase is reduced by about 0.15%, which is almost a fifth of the response during the non-ZLB periods. In terms of multipliers, the impact multiplier in normal times is 0.58 and it is 0.47 in the ZLB period.

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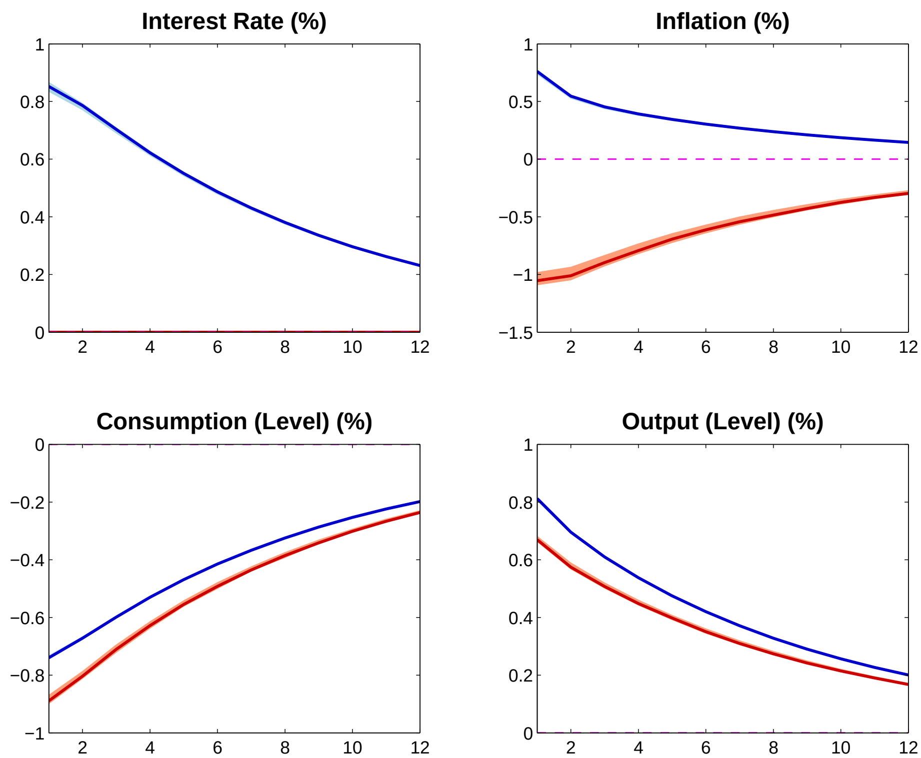
they get a multiplier around 0.5. See footnote 15 for further details.

[19] Christiano, Eichenbaum and Rebelo (2011*b*) obtain multipliers larger than one even away from the ZLB by using a utility function where consumption and leisure are close complements so that when employment increases in response to a government spending shock, so does consumption.

Figure 2.9: Fiscal-Only Intervention - Japan

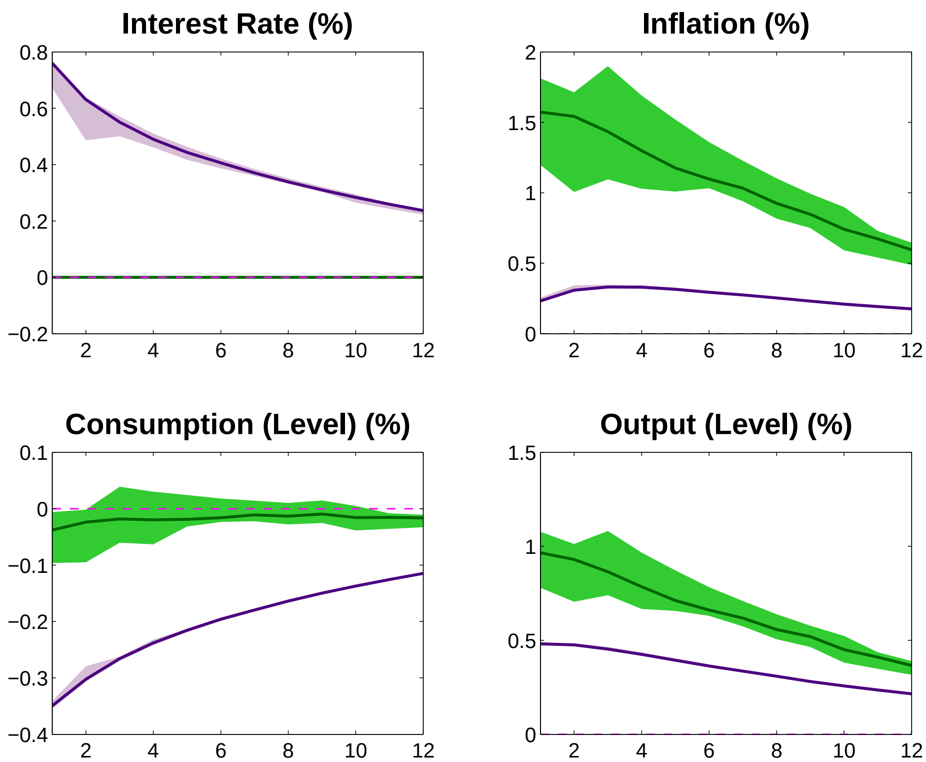


Notes: The blue line shows the pointwise median response of the economy in "normal" times and the red line shows the pointwise median response of the economy in the ZLB period. The shaded areas are the upper and lower 20% percentiles of the distribution of responses. See Section 2.5.6.1 for the definitions. The interest and inflation rates are annualized. The bottom panels show the percentage change in the level of consumption and output.

2.5.6.3 Combined Fiscal and Monetary Policy Intervention

We now combine the fiscal intervention with the promise of the central bank to keep rates at or near the ZLB. We only consider the ZLB period for this exercise since in "normal" times the interest rate is far from the ZLB and an expansionary monetary policy that pushes the interest rate all the way to zero would be unrealistic. In Japan the interest rate remains zero after the fiscal-only intervention, thus we consider the combined fiscal and monetary policy only for the U.S. The results are

Figure 2.10: Combined Fiscal and Monetary Intervention at the ZLB - U.S.



Notes: The purple line shows the pointwise median response of the economy to the fiscal-only intervention (the red line in Figure 2.8) and the green line shows the pointwise median response of the economy to the combined intervention. The shaded areas are the upper and lower 20% percentiles of the distribution of responses. The interest and inflation rates are annualized. The bottom panels show the percentage change in the level of consumption and output.

reported in Figure 2.10.

In all of the twelve quarters under consideration, the central bank manages to pull the interest rate all the way to the ZLB, despite the increasing urge not to do so due to higher inflation and output responses. As a result, the output response exactly doubles to almost 1%, a large fraction of which comes from the smaller decline in consumption, since the channel through the real interest rate is in effect. The impact multiplier increases by 87% and after three years the multiplier is still 77% larger. All of this shows that, unlike Japan which is in the deflationary regime, when the economy is at the ZLB because of adverse demand shocks within the targeted-inflation regime, the monetary stimulus we consider provides a very large additional boost to the fiscal intervention. In this regard, our empirical findings are consistent with earlier results reported in the literature. However, our interpretation is different. The reason that the fiscal intervention has a large effect is because an expansionary monetary policy keeps interest rates at zero. This interpretation is consistent with the filtered monetary policy shocks shown in Appendix B.8.5. On average, these shocks have been negative after 2009, meaning that from an ex-post perspective, monetary policy, through the lens of our model, has been expansionary in the aftermath of the Great Recession.

2.6 Conclusion

We solve a small-scale New Keynesian DSGE model with the ZLB constraint and Markov sunspot shocks that can move the economy between a targeted-inflation regime and a deflation regime. An economy may stay at or near the ZLB either by successive exogenous shocks (e.g. adverse demand or technology shocks or expansionary monetary policy shocks) in the former regime or by a regime switch to the latter. We develop a framework that can distinguish these two possibilities and apply it to the ZLB episode of the U.S. since 2008 and Japan since late 1990s. According to our estimation results, the U.S. and Japanese experiences were markedly different. Adverse demand shocks have moved the U.S. economy to the ZLB in 2009 and, subsequently, an expansive monetary policy has kept interest rates close to zero. In contrast, the Japanese economy stayed at the ZLB by a switch to the deflation regime in 1999. While both economies were affected by adverse demand shocks that pushed them to the ZLB, we argue that the U.S. economy did not experience a regime switch due to the strong and committed response of the Federal Reserve that coordinated private inflation expectations near its target. The Bank of Japan, on the other hand, was unable to coordinate expectations, perhaps due to its weak reaction to the adverse shocks early on, and the regime switch took place.

The U.S. and Japan's experiences of moving to the ZLB have drastically different policy implications. Fiscal multipliers are about 20% smaller in the deflationary regime, despite the economy remaining at the ZLB. While a commitment by the central bank to keep rates near the ZLB doubles the fiscal multipliers in the targeted-inflation regime (U.S.), it has no effect in the deflation regime (Japan). Moreover, our results show that Japan experienced persistent deflation because of the switch to the deflationary regime and this may explain why numerous fiscal policies enacted in Japan in the last 15 years were not able generate positive inflation.

Solving for the sunspot equilibrium is computationally challenging. We leave extensions to larger DSGE models and equilibria in which the regime shifts are triggered by fundamental shocks to future research. The latter will be important in formalizing the idea we explored in this paper where central bank's actions other than their interest rate decisions may help coordinate private expectations and induce a switch in regimes. Finally, in future work we plan to conduct a normative analysis in the sunspot equilibrium.

Appendix A: Appendix for Chapter 1

A.1 DSGE model

A.1.1 Model description

The economy is composed by households, firms and the government. Next I describe the optimization problem of each agent in the economy:

Households. The representative agent in this economy solves the following problem:

$$\max_{\{C_t, I_t, L_t, \bar{K}_t, B_t, u_t\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t d_t \left[\ln\left(C_t - bC_{t-1}\right) - \psi_L \frac{L_t^{1+\nu}}{1+\nu} \right]$$

$$s.t.$$

$$P_t C_t + P_t I_t + B_t \leq W_t L_t + R_t^k u_t \bar{K}_{t-1} - P_t \mathcal{A}(u_t) \bar{K}_{t-1} + R_{t-1} B_{t-1} - P_t T_t + \Pi_t$$

$$\bar{K}_{t+1} = (1-\delta)\bar{K}_t + \mu_t \left(1 - S\left(\frac{I_t}{I_{t-1}}\right)\right) I_t$$

Let Λ_t be the multiplier associated with the nominal budget constraint and Ξ_t the multiplier associated with the law of motion of installed capital. The optimality condition for consumption is:

$$\Lambda_t P_t = \frac{d_t}{C_t - hC_{t-1}} - h\beta\mathbb{E}_t\frac{d_{t+1}}{C_{t+1} - hC_t}$$

And the associated Euler equation is:

$$\Lambda_t = \beta R_t \mathbb{E}_t \Lambda_{t+1}$$

The investment decision is governed by:

$$\Lambda_t P_t = \mu_t \Xi_t \left[1 - S\left(\frac{I_t}{I_{t-1}}\right) - S'\left(\frac{I_t}{I_{t-1}}\right)\frac{I_t}{I_{t-1}}\right] + \beta\mathbb{E}_t\Xi_{t+1}\mu_{t+1}S'\left(\frac{I_{t+1}}{I_{t-1}}\right)\left(\frac{I_{t+1}}{I_t}\right)^2$$

For capital accumulation the optimality condition is:

$$\Xi_t = \beta\mathbb{E}_t\left\{\Lambda_{t+1}\left[u_{t+1}R^k_{t+1} - P_{t+1}\mathcal{A}(u_{t+1})\right] + (1-\delta)\Xi_{t+1}\right\}$$

The optimal level of capacity utilization satisfies the condition:

$$R^k_t = P_t\mathcal{A}'(u_t)$$

The first order condition for labor supply is trivial:

$$\Lambda_t W_t = \psi_L L^\nu_t$$

Intermediate-goods firms Firms operate a technology that combines labor and capital to produce the intermediate good. Taking the demand for their products as given, intermediate-goods firms have to choose their demand for labor and capital input and set the price at which they sell their product. The problem can be broken in these two stages.

Optimal factor demand First firms takes the price of its output as given and rent capital $K_{i,t}$ and labor $H_{i,t}$ from households to minimize costs subject to its production technology. To hire labor firms pay the real wage W_t and a rental rate of capital R_t^k that are determined at the aggregate level.

$$\begin{aligned} \min W_t H_{i,t} + R_t^k K_{i,t} \\ s.t. \quad Y_{i,t} \leq K_{i,t}^{\alpha} \left(A_t H_{i,t}\right)^{1-\alpha} - A_t F \end{aligned}$$

This problem yields the following first order conditions:

$$\begin{aligned} W_t H_t(i) = \Psi_t (1-\alpha) K_t^{\alpha}(i) \left(H_t(i)\right)^{1-\alpha} \\ R_t^k K_t(i) = \Psi_t \alpha K_t^{\alpha}(i) \left(H_t(i)\right)^{1-\alpha} \end{aligned}$$

In the first stage optimal factor demand yields the following condition:

$$\frac{R_t^k}{W_t} = \frac{\alpha}{1-\alpha} \frac{H_t(i)}{K_t(i)}$$

This implies that all firms choose the same demand for factors of production, and one can write an expression for marginal costs as:

$$
\begin{aligned}
MC_t &= \alpha^{-\alpha}(1-\alpha)^{\alpha-1} R_t^{k\alpha}(W_t)^{1-\alpha} \\
MC_t &= \alpha^{-\alpha}(1-\alpha)^{\alpha-1}\left(\frac{\alpha}{1-\alpha}\frac{H_t}{K_t}W_t\right)^{\alpha}(W_t)^{1-\alpha} \\
MC_t &= (1-\alpha)^{-1}\left(\frac{H_t}{K_t}\right)^{\alpha} W_t \\
MC_t &= \frac{W_t}{(1-\alpha)(K_t/H_t)^{\alpha}}
\end{aligned}
$$

Pricing decision . Taking the marginal cost as given, In the second stage firms set prices to maximize (nominal) profits: $D_t = \left[1-\Phi_p\left(P_{i,t}/P_{it-1}\right)\right]P_{i,t}Y_t(i) - MC_tY_t(i)$ To the solve the following program:

$$
\begin{aligned}
&\max_{\{P_{i,t}\}_{t=0}^{\infty}} \quad \mathbb{E}_t\sum_{t=0}^{\infty}\beta^s\Lambda_t\left\{\left[1-\Phi_p\left(\tfrac{P_{i,t}}{P_{i,t-1}}\right)\right]P_{i,t}Y_{i,t} - MC_tY_{i,t}\right\} \\
&s.t. \qquad\qquad Y_t(i) = \left(\tfrac{P_{i,t}}{P_t}\right)^{-\theta}Y_t
\end{aligned}
$$

Then we have:

$$
\begin{aligned}
&\Lambda_t\left\{\left[1-\Phi_t\left(\frac{P_{i,t}}{P_{it-1}}\right)\right]\frac{P_{i,t}^{1-\theta}}{P_t^{-\theta}}Y_t - MC_t\left(\frac{P_{i,t}}{P_t}\right)^{-\theta}Y_t\right\} \\
&+\beta\Lambda_{t+1}\left\{\left[1-\Phi_t\left(\frac{P_{it+1}}{P_{i,t}}\right)\right]P_{it+1}Y_{it+1} - MC_{t+1}Y_{it+1}\right\} \quad + \quad \ldots
\end{aligned}
$$

To ease the notation I use $\theta = \frac{1}{\lambda_p}$. The first order conditions of this problem is:

$$
\begin{aligned}
&\Lambda_t\left\{(1-\theta)\left[1-\Phi_p\left(\tfrac{P_{i,t}}{P_{it-1}}\right)\right]\left(\tfrac{P_{i,t}}{P_t}\right)^{-\theta}Y_t - \Phi_p'\left(\tfrac{P_{i,t}}{P_{it-1}}\right)\left(\tfrac{P_{i,t}}{P_t}\right)^{-\theta}Y_t\tfrac{P_{i,t}}{P_{it-1}}\right. \\
&\left.+\theta MC_t\left(\tfrac{P_{i,t}}{P_t}\right)^{-\theta}\tfrac{Y_t}{P_t}\right\} + \beta\mathbb{E}_t\Lambda_{t+1}\left\{\Phi_p'\left(\tfrac{P_{it+1}}{P_{i,t}}\right)\left(\tfrac{P_{it+1}}{P_{i,t}}\right)^2 Y_{it+1}\right\} = 0
\end{aligned}
$$

Symmetric price equilibrium. In a symmetric equilibrium this reduces to:

$$\Lambda_t \left\{ (1-\theta)\left[1-\Phi_p\left(\pi_t\right)\right] Y_t + \theta \left(\frac{MC_t}{P_t}\right) Y_t - \Phi_p'(\pi_t)\pi_t Y_t \right\} \quad + \quad \beta \Lambda_{t+1} \Phi_p'(\pi_{t+1}) \pi_{t+1}^2 Y_{t+1} = 0$$

In a symmetric equilibrium the optimal pricing decision yields the following equilibrium condition:

$$\Phi_p'(\pi_t)\pi_t + (\theta-1)\left[1-\Phi_p\left(\pi_t\right)\right] - \theta\left(\frac{MC_t}{P_t}\right) = \beta \mathbb{E}_t \frac{\Lambda_{t+1}}{\Lambda_t} \Phi_p'(\pi_{t+1}) \pi_{t+1}^2 \frac{Y_{t+1}}{Y_t}$$

Marginal costs. Before setting prices the firm decides the optimal factor demand after minimizing production costs given by the term $W_t L_{i,t} + R_t^k K_{i,t}$ subject to the production technology described earlier. Here I assume that labor and capital are traded in an economy wide factor market which results in simple solution in which all intermediate firms choose the same capital labor ratio: $K_{i,t}/L_{i,t} = \frac{\alpha}{1-\alpha}\frac{W_t}{R_t^k}$. As a consequence all intermediate firms face identical marginal costs given by:

$$MC_t = \alpha^{-\alpha}(1-\alpha)^{-(1-\alpha)} W_t^{1-\alpha} R_t^{k\alpha} A_t^{-(1-\alpha)}.$$

A.1.2 Steady state

To determine the steady state first note that:

$$\rho = \frac{\gamma}{\beta} - 1 + \delta$$

From the pricing equation we have:

$$mc = \frac{(\theta - 1)}{\theta} = \frac{1}{\lambda_p}$$

From this point on it will be easier to characterize the steady state as a function of the capital-labor ratio:

$$\begin{aligned} w &= mc(1-\alpha)(k/L)^{\alpha} \\ \rho &= \alpha mc(k/L)^{\alpha-1} \end{aligned}$$

Which implies that,

$$(k/L) = \left(\frac{\rho}{\alpha mc}\right)^{\frac{1}{\alpha-1}}$$

And then we ca replace in the expression for w to obtain,

$$w = mc(1-\alpha)\left(\frac{\rho}{\alpha mc}\right)^{\frac{\alpha}{\alpha-1}} = mc^{1/1-\alpha}(1-\alpha)\left(\frac{\rho}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}$$

From the production function we can obtain,

$$\frac{y}{L} = (k/L)^{\alpha} - \frac{\mathcal{F}}{L}$$

And since $\mathcal{F}$ is set such that profits are zero in steady state and using the result that $\Phi_p(\pi^*) = 0$, we have:

$$\frac{\mathcal{F}}{L} = (k/L)^{\alpha} - \rho(k/L) - w$$

Which implies that,

$$\frac{y}{L} = \rho(k/L) - w$$

Now using the law of motion of effective capital and the definition of effective capital we have:

$$i/k = \gamma - (1 - \delta)$$

And the investment output ratio is:

$$\frac{i}{y} = \frac{i}{k}\frac{k}{L}\frac{L}{y}$$

Finally from the resource constraint:

$$\frac{c}{y} = \frac{1}{\bar{g}} - \frac{i}{y}$$

Now we need to solve for the steady state value of hours worked L in order to recover the rest of the objects of the steady state. From the consumption-leisure optimality condition we obtain:

$$w = \frac{\psi_L L^\nu}{\lambda}$$

And from the definition of the marginal utility of wealth we obtain:

$$\lambda = \frac{1}{c} \frac{\gamma - h\beta}{\gamma - b}$$

Hence we can write:

$$\lambda L = \left(\frac{c}{L}\right)^{-1} \frac{\gamma - h\beta}{\gamma - b}$$

And note that $\frac{c}{L} = \frac{1}{\bar{g}} \frac{y}{L} - \frac{i}{L} \frac{L}{y} \frac{y}{L}$. Combining with the previous equation,

$$w = \frac{\psi_L L^{1+\nu}}{\lambda L}$$

Hence we can recover the steady state level of hours as:

$$L = \left(\frac{w \lambda L}{\psi_L}\right)^{\frac{1}{1+\nu}}$$

With the steady state level of hours we can directly recover the steady state of $k, c, i, y, \mathcal{F}$.

A.2 Computing the Nonlinear Solution

I explain how to solve the nonlinear decision rules. I use the notation $\mathbb{S} = \left[R_{-1}, c_{-1}, \bar{k}_{-1}, i_{-1}, \mu, d, z, g, \varepsilon^r\right]$ to summarize the state variables, and approximate the decision rules $\mathcal{C} = \{L(\mathbb{S}), q(\mathbb{S}), \lambda(\mathbb{S}), i(\mathbb{S}), \pi(\mathbb{S})\}$

A.2.1 Residuals

To find the policy functions that solves the above system of equilibrium conditions I minimize the sum of squared residuals with respect to the unknown coefficients Θ. To that end, I first define the residual functions that will serve as metric for the solution procedure described later.

$$\mathcal{R}_1(\mathbb{S}) = \lambda(\mathbb{S}) - \beta R_t \mathbb{E}\frac{\lambda(\mathbb{S}')}{\pi(\mathbb{S}')}\frac{1}{\gamma e^{z'}} \tag{A.1}$$

$$\mathcal{R}_2(\mathbb{S}) = \lambda(\mathbb{S}) - \frac{\gamma e^{d+z}}{\gamma c e^{z} - h c_{-1}} - h\beta\mathbb{E}\frac{e^{d'}}{\gamma c' e^{z'} - hc} \tag{A.2}$$

$$\mathcal{R}_3(\mathbb{S}) = q(\mathbb{S}) - \beta\mathbb{E}\frac{\lambda(\mathbb{S}')}{\gamma e^{z'}\lambda(\mathbb{S})}\left\{\rho(\mathbb{S}')u' - \mathcal{A}(u') + (1-\delta)q(\mathbb{S}')\right\} \tag{A.3}$$

$$\begin{aligned}\mathcal{R}_4(\mathbb{S}) &= 1 - e^{\mu} q(\mathbb{S})\left[1 - S(\Delta i) - dS(\Delta i)x\right] \\ &- \beta\mathbb{E}q(\mathbb{S}')\frac{\lambda(\mathbb{S}')}{\lambda(\mathbb{S})}\frac{1}{\gamma e^{z'}}e^{\mu'}dS(\Delta i')x'^2\end{aligned} \tag{A.4}$$

$$\begin{aligned}\mathcal{R}_5(\mathbb{S}) &= (\frac{1}{\lambda_p} - 1)\left[1 - \Phi_p(\pi(\mathbb{S}))\right] - \frac{mc}{\lambda_p} + d\Phi_p(\pi(\mathbb{S}))\pi(\mathbb{S}) \\ &- \beta\mathbb{E}\frac{\lambda(\mathbb{S}')}{\lambda(\mathbb{S})}\Phi_p(\pi(\mathbb{S}'))\pi(\mathbb{S}')\frac{y'}{y}\end{aligned} \tag{A.5}$$

A.2.2 Expectations

To evaluate the expectations that form part of the residual equations (A.1) - (A.5), I use deterministic integration methods based on a Gauss-Hermite quadrature rule. The exogenous components of the state vector, $\mathbb{S}'$, is constructed using a non-product using the sparse grid algorithm of Heiss and Winschel (2006). For example, suppose that we want to compute $\mathbb{E}[f(\mathbf{x})]$ for $\mathbf{x} \in \mathbb{R}^D$ where, $\mathbf{x}$ is a vector of random variables distributed according to $N(0, I_D)$.

Define first the Q^{th} order discrete approximation to any univariate function $\mathbb{E}g(x)$ to be:

$$V_Q \equiv \mathbb{E}[g(x)] \approx \sum_{i=1}^{Q} g(x_i)w_i$$

Where x_i and w_i are the Gauss-Hermite nodes and weights as in Judd (1998). Usually to evaluate $\mathbb{E}f(\mathbf{x})$ one would construct a tensor product approximation using the x_i, and w_i for each element in $\mathbf{x}$. However this approach becomes computationally costly as the dimensionality of the function of interest and the accuracy of the approximation Q increases. To simplify, take the case of a bivariate function $D = 2$, and define the set of indexes $\mathcal{I}_Q^{D=2} = \left\{\mathbf{i} \in \mathbb{N}^D : \sum_{d=1}^{D} i_d = D + Q\right\}$, where $\mathbb{N}$ is the set of all positive integers. The *level-k* sparse grid approximation with $D = 2$ dimensions is given by:

$$\mathbb{E}[f(\mathbf{x})] \approx \sum_{q=k-D}^{k-1} (-1)^{k-1-q} \binom{D-1}{k-1-q} \sum_{\mathbf{i}\in\mathcal{I}_Q} V_{i1} \otimes \cdots \otimes V_{iD}$$

A.2.3 Computational Algorithm

Algorithm 2 *The solution algorithm proceeds as follows.*

1. *Without loss of generality begin in step j with a guess for the unknown coefficients $\Theta^{(j)}$.*

2. *Construct the approximated decision rules:*

$$\mathcal{C}^{(j)} = \{L(\mathbb{S};\Theta^{(j)}), q(\mathbb{S};\Theta^{(j)}), \lambda(\mathbb{S};\Theta^{(j)}), i(\mathbb{S};\Theta^{(j)}), \pi(\mathbb{S};\Theta^{(j)})\}$$

3. *Construct the following objects using the decision rules that correspond to a non-binding ZLB (Θ^{nb}): $w, \rho, u, mc, c, \bar{k}, y$, using equations* (1.25) - (1.31).

4. *Compute the notional interest rate using* (1.32). *If the the notional rate violates the ZLB go back to step 3, set $R = 1$ and recompute all objects using Θ^b.*

5. *Repeat steps 1 - 4, to construct and evaluate the objects inside the expectations in* (A.1) - (A.5).

6. *Updated the vector of unknown coefficients to $\Theta^{(j+1)}$ using any minimization routine until a solution for $\min_\Theta \sum_{i=1}^{M} \mathcal{R}(\mathbb{S}_i,\Theta)^2$ is found.*

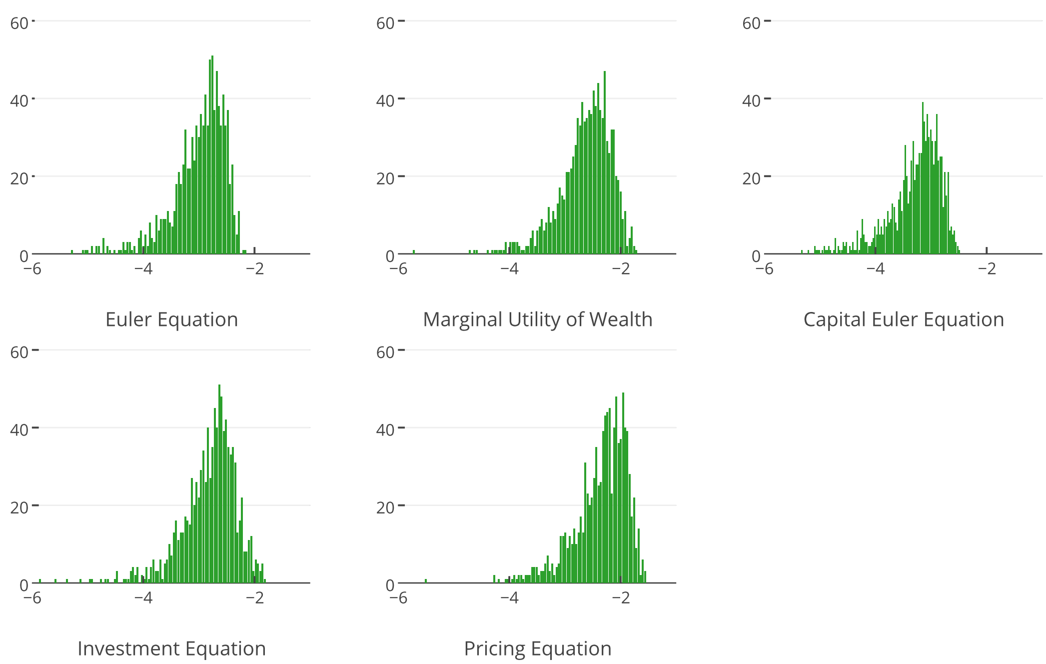
A.2.4 Accuracy of the Nonlinear Solution

The accuracy of the numerical solution is evaluated with respect to the residual functions defined in Section A.2.1. If one could obtain the actual policy functions instead of the approximated ones, then Equation A.1-Equation A.5 would be satisfied exactly. A measure of the "exactness" of the approximated policy rules can be measured by how much does the approximated decision rules fail to satisfy the residual equations exactly. To take an specific example, take Equation A.1:

$$\mathcal{E}\mathcal{E}_1 = \log 10 \left| 1 - \frac{\beta R_t \mathbb{E} \frac{\lambda(\mathbb{S}')}{\pi(\mathbb{S}')} \frac{1}{\gamma e^{z'}}}{\lambda(\mathbb{S})} \right|$$

The above expression measured in terms of consumption units is expressed in $\log 10$ for ease of interpretation. Figure A.1 shows the distribution of the Euler Errors for all the residual functions.

Figure A.1: Distribution of Euler Equation Errors in $\log 10$ units



A.2.5 Posterior Predictive Checks

An brief introduction to the use of predictive checks can be found in del Negro and Schorfheide (2012*a*). The predictive checks rests upon the construction of the predictive distribution of a sequence of simulated data $p(\tilde{\mathbb{Y}}^T|\Omega_T)$ where Ω_T is the information set available up to period T and includes the realization of the observed data $\mathbb{Y}^T$ and the draws from the posterior distribution of the parameters of the model $p(\theta|\Omega_T)$. 3 describes how to obtain draws from the predictive distribution. Once these draws are obtained they can be transformed into empirical moments of interest $\mathcal{S}(\tilde{\mathbb{Y}}^T) \in \mathbb{R}^n$, for example sample means, covariances, autocovariances, correlations, etc. I compute the posterior predictive checks for the five observable series used for estimation. The predictive distribution is constructed using $T = 100$ periods initialized at the deterministic steady state of the model and sampling $N = 1,000$ realizations of the posterior distribution of estimated parameters.

Algorithm 3 *Drawing from* $p(\tilde{\mathbb{Y}}^T|\Omega_T)$

1. *Fix a draw of the parameter vector from the posterior distribution,* θ_j*,* $j = 1, \dots, N$.
2. *Use the model solution to simulate a sequence of observables* $\tilde{\mathbb{Y}}_t(\theta_j)$*,* $t = 1, \dots, T$.
3. *Construct the vector of moments of interest* $\mathcal{S} : \tilde{\mathbb{Y}}^T(\theta_j) \rightarrow \mathcal{S}(\tilde{\mathbb{Y}}^T) \in \mathbb{R}^{n\times 1}$.
4. *The posterior predictive distribution* $p(\mathcal{S}(\tilde{\mathbb{Y}}^T)|\Omega_T)$ *can be characterized with*

the empirical distribution of $\mathcal{S}(\tilde{\mathbb{Y}}^T)$.

5. *Compare the distribution of* $\mathcal{S}(\tilde{\mathbb{Y}}^T)$ *with the corresponding statistic based on actual data* $\mathcal{S}(\mathbb{Y}^T)$.

A.3 Data for Estimation and Filtering

All information comes from the Federal Reserve Bank of St. Louis FRED data service (mnemonics in parenthesis), for the period 1984:Q1 to 2013:Q4

1. **Price Level.** Is the implicit GDP price deflator (GDPDEF) index.

2. **Population.** Is the civilian non-institutionalized population over sixteen years (CNP16OV).

3. **Per Capita Output Growth** Real per-capita output growth is constructed using data on Gross Domestic Product (GDP) expressed in billions of dollars:

$$\Delta y_t = \left[\ln\left(\frac{GDP_t}{CNP16OV_t \times GDPDEF_t}\right) - \ln\left(\frac{GDP_{t-1}}{CNP16OV_{t-1} \times GDPDEF_{t-1}}\right)\right] \times 100$$

4. **Per Capita Consumption Growth.** Real per-capita consumption growth is constructed first summing Personal Consumption Services (PCESV) + Personal Consumption Durables Divided (PCND). Let $CONS_t = PCESV_t + PCND_t$, expressed in billions of dollars.

$$\Delta c_t = \left[\ln\left(\frac{CONS_t}{CNP16OV_t \times GDPDEF_t}\right) - \ln\left(\frac{CONS_{t-1}}{CNP16OV_{t-1} \times GDPDEF_{t-1}}\right)\right] \times 100$$

5. **Per Capita Investment Growth.** Real per-capita investment growth is constructed first summing Personal Consumption Durable Goods(PCEDG) + Fixed Private Investment(FPI) + Change in Private Inventories(CBI). Let $INV_t = PCEDG_t + FPI_t + CBI_t$, expressed in billions of dollars.

$$\Delta i_t = \left[\ln\left(\frac{INV_t}{CNP16OV_t \times GDPDEF_t}\right) - \ln\left(\frac{INV_{t-1}}{CNP16OV_{t-1} \times GDPDEF_{t-1}}\right)\right] \times 100$$

6. **Inflation.** The inflation rate is measured as the quarterly change of GDPDEF:

$$\pi_t = [\ln GDPDEF_t - \ln GDPDEF_{t-1}] \times 100$$

7. **Interest Rate.** The nominal interest rate is measured using the quarterly rate of the federal funds rate (FEDFUNDS):

$$R_t = FEDFUNDS_t/4$$

A.4 Filtering

Here I describe the algorithm used to approximate the filtering density $p(s_t|\mathcal{Y}^t)$ used to recover the unobserved states of the economy. The exposition follows Creal (2009) and Andrieu, Doucet and Holenstein (2010). An in depth treatment with applications to the New Keynesian models can be found in Herbst and Schorfheide (2014).

To simplify the discussion I consider a single variable dynamic model, whose dynamics can be represented in the form of a nonlinear state-space system:

$$s_t = g\left(s_{t-1}, u_t\right), \quad u_t \sim N(0, \sigma_u) \tag{A.4.1}$$

$$y_t = m(s_t) + \varepsilon_t, \quad \varepsilon_t \sim N(0, \sigma_\varepsilon) \tag{A.4.2}$$

This systems gives rise to the measurement density $p(y_t|s_t)$ and the transition density $p(x_t|x_{t-1})$, which inherit their Markov structure from the transition equation of the model. Given a sequence of observations $y^t = \{y_1, \ldots, y_t\}$ we are interested in recovering the sequence of states that generated them $x^t = \{x_1, \ldots, x_t\}$. But because the system is stochastic we can only characterize the joint distribution where the states comes from, $p(x^t|y^t)$, known as the joint filtering distribution.

A.4.1 Filtering Distribution Decomposition.

A key step to apply particle filter methods, is to decompose the the joint filtering distribution:

$$
\begin{aligned}
p(s^t|y^t) &= \frac{p(y_t|s^t, y^{t-1})p(s^t|y^{t-1})}{p(y_t|y^{t-1})} \\
&= \frac{p(y_t|s^t, y^{t-1})}{p(y_t|y^{t-1})} p(s_t|s^{t-1}, y^{t-1})p(s^{t-1}|y^{t-1}) \\
&= \frac{p(y_t|s_t)p(s_t|s_{t-1})}{p(y_t|y^{t-1})} p(s^{t-1}|y^{t-1}) \propto p(y_t|s_t)p(s_t|s_{t-1})p(s^{t-1}|y^{t-1})
\end{aligned}
$$

The last steps follows from the Markov property of the transition and measurement densities. This makes clear that it is possible to characterize the filtering density sequentially starting from some initial joint density of states $p(s^{t-1}|y^{t-1})$. The scaling factor in the last equation is the marginal contribution of observation y_t to the likelihood.

A.4.2 Marginal Distribution Decomposition.

An alternative to trying to uncover the joint distributions of states, the filtering problem can be cast in terms of the marginal distribution $p(s_t|y^t)$. This is the key approach taken in the implementation of Sequential Monte Carlo methods. One can thing of this decomposition as a form of sequential learning, yielding greater flexibility to the filtering approach.

Suppose we have access to an initial distribution of states $p(s_0)$. Given the evolution of the system to a known position, y^{t-1}, s_{t-1}, the marginal predictive density $p(s_t|y^{t-1})$ can be obtained integrating out the transition equation of the model according to:

$$p(s_t|y^{t-1}) = \int p(s_t|s_{t-1})p(s_{t-1}|y^{t-1})ds_{t-1}$$

Following the same steps as before, the marginal filtering density is given by:

$$p(s_t|y^t) = \frac{p(y_t|s_t)p(s_t|y^{t-1})}{p(y_t|y^{t-1})} = \frac{p(y_t|s_t)}{p(y_t|y^{t-1})} \int p(s_t|s_{t-1})p(s_{t-1}|y^{t-1})ds_{t-1} \quad \text{(A.4.3)}$$

Given the additive nature of the measurement errors in the observation equation the *observation density:* $p(y_t|s_t)$ can be readily evaluated. However, there is no closed form expression for the objects that form the marginal predictive density $p(s_t|y^{t-1})$ or the marginal likelihood $p(y_t|y^{t-1})$

A.4.3 Sequential Monte Carlo Approximation

The key challenge in uncovering the marginal distribution of unobserved states is solving for the integrals that shown in the previous section. I will use Monte Carlo methods to approximate these objects. The key idea, is to start with a probability mass function represented by a collection of particles $\{\pi_{t-1}^{(j)}\}_{j=1}^{N}$ with associated weights $\{W_{t-1}^{(j)}\}_{j=1}^{N}$ to approximate the filtering density $p(s_{t-1}|y^{t-1})$ and systematically use the model transition and measurement equations to update this approximation to obtain $p(s_t|y^t)$.

A.4.3.1 Importance Sampler.

If we could simply draw the particles and weights $\{\pi_t^{(j)}, W_t^j\}_{j=1}^N$ from the target distribution, $p(s_t|y^t)$ then for any function $h(.)$ it is possible to construct a Monte Carlo estimator:

$$\frac{1}{M}\sum_{j=1}^{M} h(s_t^j)W_t^j \xrightarrow{a.s.} \mathbb{E}\left[h(s_t)|y^t\right] = \int h(s_t)p(s_t|y^t)ds_t \tag{A.4.4}$$

Because this is not possible in practice because the target distribution is unknown, the approximation has to be constructed in a different way. The solution is to use a known distribution, known as importance density $g_t(s_t|s_{t-1}, y^t)$, such that:

$$\mathbb{E}\left[h(s_t)|y^t\right] = \int h(s_t)\frac{p(s_t|y^t)}{g_t(s_t|s_{t-1}, y^t)}g_t(s_t|s_{t-1}, y^t)ds_t$$

Note that now we can approximate this integral drawing from the known importance density $g_t(s_t|s_{t-1}, y^t)$. The importance density is indexed at time t, meaning that it can be adjusted as new information is incorporated into the approximation. Also note that since we draw from the importance density instead of drawing from the target density, the draws are reweighted using the importance weights, $w_t^j = \frac{p(s_t|y^t)}{g_t(s_t|s_{t-1},y^t)}$.

A.4.3.2 Sequential Importance Sampler.

Having defined the idea of an importance sampler, now I address how to approximate $p(s_t|y^t)$ sequentially. Suppose that a swarm of particles $\{\pi_{t-1}^j, W_{t-1}^j\}_{j=1}^N$ approximates $p(s_{t-1}|y^{t-1})$ according to Equation A.4.4. Using Equation A.4.3 we can write:

$$\mathbb{E}[h(s_t)|y^t] = \frac{\int h(s_t)p(y_t|s_t)p(s_t|s_{t-1})p(s_{t-1}|y^{t-1})ds_{t-1}}{p(y_t|y^{t-1})} \tag{A.4.5}$$

Concentrate the numerator, note that the particle approximation to $p(s_{t-1}|y^{t-1})$ is known. Conditional on this approximation, we draw (s_t^j) from an importance density $g_t(s_t|s_{t-1}, y^t)$ to generate the following approximation:

$$\int h(s_t)p(y_t|s_t)p(s_t|s_{t-1})p(s_{t-1}|y^{t-1})ds_{t-1} \approx \frac{1}{N}\sum_{j=1}^{N} h(s_t^j)w_t^j W_{t-1}^j \tag{A.4.6}$$

Where the *incremental weights*, $w_t = \frac{p(y_t|s_t)p(s_t|s_{t-1})}{g_t(s_t|s_{t-1},y^t)}$, are computed for each of the draws from the importance density. A by-product of this approximation is a expression for the likelihood:

$$p(y_t|y^{t-1}) \approx \frac{1}{N}\sum_{j=1}^{N} w_t^j W_{t-1}^j \tag{A.4.7}$$

Because these weights are not drawn from the target density, we need to re-scale them in order to update the approximation that we are really interested on. Let the unscaled weights be given by, $\tilde{W}_t^i = w_t^i W_{t-1}^i$. The normalized weights are $W_t^i =$

$\frac{\tilde{W}_t^i}{\sum_{j=1}^N \tilde{W}_t^j}$, and then the approximation of the filtering density $p(s_t|y^t)$ is given by:

$$\mathbb{E}\left[h(s_t)|y^t\right] = \frac{\frac{1}{N}\sum_{j=1}^N h(s_t^j)\tilde{W}_t^j}{\sum_{j=1}^N \tilde{W}_t^j} = \frac{1}{N}\sum_{j=1}^N h(s_t^j)\left(\frac{\tilde{W}_t^j}{\sum_{j=1}^N \tilde{W}_t^j}\right) \tag{A.4.8}$$

The approximation $\{\pi_t^i, W_t^i\}_{i=1}^N$ suffers from particle degeneracy, in the sense that some of the draws from the importance density have a negligible weight. To mitigate this problem, the particles are resampled at the end of each step, keeping only those that have positive weights.

Appendix B: Appendix for Chapter 2

B.1 Solving the Two-Equation Model

The model is characterized by the nonlinear difference equation

$$\mathbb{E}_t[\pi_{t+1}] = \max\left\{\frac{1}{r},\ \pi_*\left(\frac{\pi_t}{\pi_*}\right)^{\psi}\exp[\epsilon_t]\right\}. \tag{B.1.1}$$

We assume that $r\pi_* \geq 1$ and $\psi > 1$.

The Targeted-Inflation Equilibrium and Deflation Equilibrium. Consider a solution to (B.1.1) that takes the following form

$$\pi_t = \pi_*\gamma\exp[\lambda\epsilon_t]. \tag{B.1.2}$$

We now determine values of γ and λ such that (B.1.1) is satisfied. We begin by calculating the following expectation

$$\begin{aligned}
\mathbb{E}_t[\pi_{t+1}] &= \pi_*\gamma\frac{1}{\sqrt{2\pi\sigma^2}}\int\exp[\lambda\epsilon]\exp\left[-\frac{1}{2\sigma^2}\epsilon^2\right]d\epsilon \\
&= \pi_*\gamma\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left[\frac{1}{2}\lambda^2\sigma^2\right]\int\exp\left[-\frac{1}{2\sigma^2}(\epsilon-\lambda\sigma^2)^2\right]d\epsilon \\
&= \pi_*\gamma\exp\left[\frac{1}{2}\lambda^2\sigma^2\right].
\end{aligned}$$

Combining this expression with (B.1.1) yields

$$\gamma \exp[\lambda^2\sigma^2/2] = \max \left\{ \frac{1}{r\pi_*},\ \gamma^\psi \exp[(\psi\lambda + 1)\epsilon_t] \right\}. \tag{B.1.3}$$

By choosing $\lambda = -1/\psi$, we ensure that the right-hand side of (B.1.3) is always constant. Thus, (B.1.3) reduces to

$$\gamma \exp[\sigma^2/(2\psi^2)] = \max \left\{ \frac{1}{r\pi_*},\ \gamma^\psi \right\} \tag{B.1.4}$$

Depending on whether the nominal interest rate is at the ZLB ($R_t = 1$) or not, we obtain two solutions for γ by equating the left-hand-side of (B.1.4) with either the first or the second term in the max operator:

$$\gamma_D = \frac{1}{r\pi_*} \exp\left[-\frac{\sigma^2}{2\psi^2} \right] \quad \text{and} \quad \gamma_* = \exp\left[\frac{\sigma^2}{2(\psi-1)\psi^2} \right]. \tag{B.1.5}$$

The derivation is completed by noting that

$$\begin{aligned} \gamma_D^\psi &= \frac{1}{r\pi_*} \exp\left[-\frac{\sigma^2}{2\psi} \right] \leq \frac{1}{r\pi_*} \\ \gamma_*^\psi &= \exp\left[\frac{\sigma^2}{2(\psi-1)\psi} \right] \geq 1 \geq \frac{1}{r\pi_*}. \end{aligned}$$

A Sunspot Equilibrium. Let $s_t \in \{0, 1\}$ denote the Markov-switching sunspot process. Assume the system is in the targeted-inflation regime if $s_t = 1$ and that it is in the deflation regime if $s_t = 0$ (the 0 is used to indicate that the system is near the ZLB). The probabilities of staying in state 0 and 1, respectively, are denoted by

ψ_{00} and ψ_{11}. We conjecture that the inflation dynamics follow the process

$$\pi_t^{(s)} = \pi_* \gamma(s_t) \exp[-\epsilon_t/\psi] \tag{B.1.6}$$

In this case condition (B.1.4) turns into

$$\begin{aligned}
\mathbb{E}_t[\pi_{t+1}|s_t = 0]/\pi_* &= \big(\psi_{00}\gamma(0) + (1-\psi_{00})\gamma(1)\big)\exp[\sigma^2/(2\psi^2)] = \frac{1}{r\pi_*} \\
\mathbb{E}_t[\pi_{t+1}|s_t = 1]/\pi_* &= \big(\psi_{11}\gamma(1) + (1-\psi_{11})\gamma(0)\big)\exp[\sigma^2/(2\psi^2)] = [\gamma(1)]^{\psi}.
\end{aligned}$$

This system of two equations can be solved for $\gamma(0)$ and $\gamma(1)$ as a function of the Markov-transition probabilities ψ_{00} and ψ_{11}. Then (B.1.6) is a stable solution of (B.1.1) provided that

$$[\gamma(0)]^{\psi} \le \frac{1}{r\pi_*} \quad \text{and} \quad [\gamma(1)]^{\psi} \ge \frac{1}{r\pi_*}.$$

Sunspot Shock Correlated with Fundamentals. As before, let $s_t \in \{0, 1\}$ be a Markov-switching sunspot process. However, now assume that a state transition is triggered by certain realizations of the monetary policy shock ϵ_t. In particular, if $s_t = 0$, then suppose $s_{t+1} = 0$ whenever $\epsilon_{t+1} \le \underline{\epsilon}_0$, such that

$$\psi_{00} = \Phi(\underline{\epsilon}_0),$$

where $\Phi(\cdot)$ is the cumulative density function of a $N(0, 1)$. Likewise, if $s_t = 1$, then let $s_{t+1} = 0$ whenever $\epsilon_{t+1} > \underline{\epsilon}_0$, such that

$$\psi_{11} = 1 - \Phi(\underline{\epsilon}_1).$$

To find the constants $\gamma(0)$ and $\gamma(1)$, we need to evaluate

$$\begin{aligned} & \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\underline{\epsilon}} \exp\left[-\frac{1}{2\sigma^2}(\epsilon + \sigma^2/\psi)^2\right] d\epsilon \\ = \;& \mathbb{P}\left\{\frac{\epsilon + \sigma^2/\psi}{\sigma} \leq \frac{\underline{\epsilon} + \sigma^2/\psi}{\sigma}\right\} = \Phi\left(\frac{\underline{\epsilon} + \sigma^2/\psi}{\sigma}\right). \end{aligned}$$

Thus, condition (B.1.4) turns into

$$\begin{aligned} \frac{1}{r\pi_*} &= \left[\gamma(0)\Phi(\underline{\epsilon}_0)\Phi\left(\frac{\underline{\epsilon}_0 + \sigma^2/\psi}{\sigma}\right) + \gamma(1)(1 - \Phi(\underline{\epsilon}_0))\left(1 - \Phi\left(\frac{\underline{\epsilon}_0 + \sigma^2/\psi}{\sigma}\right)\right)\right] \exp[\sigma^2/(2\psi^2)] \\ \gamma^{\psi}(1) &= \left[\gamma(1)(1 - \Phi(\underline{\epsilon}_1))\left(1 - \Phi\left(\frac{\underline{\epsilon}_1 + \sigma^2/\psi}{\sigma}\right)\right) + \gamma(0)\Phi(\underline{\epsilon}_1)\Phi\left(\frac{\underline{\epsilon}_1 + \sigma^2/\psi}{\sigma}\right)\right] \exp[\sigma^2/(2\psi^2)]. \end{aligned}$$

This system of two equations can be solved for $\gamma(0)$ and $\gamma(1)$ as a function of the thresholds $\underline{\epsilon}_0$ and $\underline{\epsilon}_1$. Then (B.1.6) is a stable solution of (B.1.1) provided that

$$[\gamma(0)]^{\psi} \leq \frac{1}{r\pi_*} \quad \text{and} \quad [\gamma(1)]^{\psi} \geq \frac{1}{r\pi_*}.$$

Benhabib, Schmitt-Grohé and Uribe (2001*a*) Dynamics. BSGU constructed equilibria in which the economy transitioned from the targeted-inflation equilibrium to the deflation equilibrium. Consider the following law of motion for inflation

$$\pi_t^{(BGSU)} = \pi_* \gamma_* \exp[-\epsilon_t/\psi] \exp\left[-\psi^{t-t_0}\right]. \tag{B.1.7}$$

Here, γ_* was defined in (B.1.5) and $-t_0$ can be viewed as the initialization period for the inflation process. We need to verify that $\pi_t^{(BGSU)}$ satisfies (B.1.1). From the derivations that lead to (B.1.4) we deduce that

$$\gamma_* \mathbb{E}_{t+1}\big[\exp[-\epsilon_{t+1}/\psi]\big] = \gamma_*^{\psi}.$$

Since

$$\exp\big[-\psi^{t+1-t_0}\big] = \big(\exp\big[-\psi^{t-t_0}\big]\big)^{\psi},$$

we deduce that the law of motion for $\pi_t^{(BGSU)}$ in (B.1.7) satisfies the relationship

$$\mathbb{E}_t[\pi_{t+1}] = \pi_* \left(\frac{\pi_t}{\pi_*}\right)^{\psi} \exp[\epsilon_t].$$

Moreover, since $\psi > 1$, the term $\exp\big[-\psi^{t-t_0}\psi\big] \longrightarrow 0$ as $t \longrightarrow \infty$. Thus, the economy will move away from the targeted-inflation equilibrium and at some suitably defined t_* reach the deflation equilibrium and remain there permanently. Overall the inflation dynamics take the form

$$\pi_t = \pi_* \begin{cases} \gamma_* \exp[-\epsilon_t/\psi] \exp\big[-\psi^{t-t_0}\big] & \text{if } t \le t_* \\ \gamma_D \exp[-\epsilon_t/\psi] & \text{otherwise} \end{cases}, \tag{B.1.8}$$

where γ_* and γ_D were defined in (B.1.5).

Alternative Deflation Equilibria. Around the deflation steady state, the system is locally indeterminate. This suggests that we can construct alternative solutions to (B.1.1). Consider the following conjecture for inflation

$$\pi_t = \pi_* \gamma \min \big\{ \exp[-c/\psi], \exp[-\epsilon/\psi] \big\}, \tag{B.1.9}$$

where c is a cutoff value. The intuition for this solution is the following. Large positive shocks ϵ that could push the nominal interest rate above one, are offset by downward movements in inflation. Negative shocks do not need to be offset because they push the desired gross interest rate below one, and the max operator in the policy rule keeps the interest rate at one. Formally, we can compute the expected value of inflation as follows:

$$\begin{aligned} \mathbb{E}_t[\pi_{t+1}] &= \pi_* \gamma \Bigg[\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{c} \exp[-c/\psi] \exp\left[-\frac{1}{2\sigma^2}\epsilon^2 \right] d\epsilon \\ & \frac{1}{\sqrt{2\pi\sigma^2}} \int_{c}^{\infty} \exp[-\epsilon/\psi] \exp\left[-\frac{1}{2\sigma^2}\epsilon^2 \right] d\epsilon \\ &= \pi_* \gamma \left[\exp[-c/\psi] \Phi(c/\sigma) + \exp\left[\frac{\sigma^2}{2\psi^2} \right] \int_{c}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2\sigma^2} (\epsilon + \sigma^2/\psi)^2 \right] d\epsilon \right] \\ &= \pi_* \gamma \left[\exp[-c/\psi] \Phi(c/\sigma) + \exp\left[\frac{\sigma^2}{2\psi^2} \right] \left(1 - \Phi\left(\frac{c}{\sigma} + \frac{\sigma}{\psi} \right) \right) \right] \end{aligned} \tag{B.1.10}$$

Here $\Phi(\cdot)$ denotes the cdf of a standard Normal random variable. Now define

$$f(c, \psi, \sigma) = \left[\exp[-c/\psi] \Phi(c/\sigma) + \exp\left[\frac{\sigma^2}{2\psi^2} \right] \left(1 - \Phi\left(\frac{c}{\sigma} + \frac{\sigma}{\psi} \right) \right) \right].$$

Then another solution for which interest rates stay at the ZLB is given by

$$\bar{\gamma} = \frac{1}{r_* \pi_* f(c, \psi, \sigma)}$$

It can be verified that for c small enough, the condition

$$\frac{1}{r_* \pi_*} \geq \bar{\gamma}^{\psi} \min \left\{ \exp[-c+\epsilon],\, 1 \right\}$$

is satisfied.

B.2 Equilibrium Conditions for the Model of Section 1.4

In this section we sketch the derivation of the equilibrium conditions presented in Section 1.4.

B.2.1 Households

The representative household solves

$$\max_{C_{t+s}, H_{t+s}, B_{t+s}} \mathbb{E}_t \left[\sum_{s=0}^{\infty} \beta^s \left(\frac{(C_{t+s}/A_{t+s})^{1-\tau} - 1}{1-\tau} - \chi_H \frac{H_{t+s}^{1+1/\eta}}{1+1/\eta} + \chi_M V \left(\frac{M_{t+s}}{P_{t+s} A_{t+s}} \right) \right) \right],$$

subject to:

$$P_t C_t + T_t + B_t + M_t = P_t W_t H_t + M_{t-1} + R_{t-1} B_{t-1} + P_t D_t + P_t SC_t,$$

Consumption and bond holdings. Let $\beta^s \lambda_{t+s}$ be the Lagrange multiplier on the household budget constraint, the first-order condition with respect to consumption and bond holdings are given by:

$$
\begin{aligned}
P_t \lambda_t &= \left(\frac{C_t}{A_t}\right)^{-\tau} \frac{1}{A_t} \\
\lambda_t &= \beta R_t \lambda_{t+1}.
\end{aligned}
$$

Combining the previous definition with the bond holding first order condition we obtain the consumption Euler equation:

$$
1 = \beta \mathbb{E}_t \left[\left(\frac{C_{t+1}/A_{t+1}}{C_t/A_t} \right)^{-\tau} \frac{1}{\gamma z_{t+1}} \frac{R_t}{\pi_{t+1}} \right].
$$

We define the stochastic discount factor as:

$$
Q_{t+1|t} = \left(\frac{C_{t+1}/A_{t+1}}{C_t/A_t} \right)^{-\tau} \frac{A_t}{A_{t+1}} = \left(\frac{C_{t+1}/A_{t+1}}{C_t/A_t} \right)^{-\tau} \frac{1}{\gamma z_{t+1}}.
$$

Labor-Leisure Choice. Taking first-order conditions with respect to H_t yields the standard intratemporal optimality condition for the allocation of labor

$$
\frac{W_t}{A_t} = \chi_H \left(\frac{C_t}{A_t}\right)^{\tau} H_t^{1/\eta}.
$$

B.2.2 Intermediate Goods Firms

Each intermediate good producers buys labor services $H_t(j)$ at the real wage W_t. Firms face nominal rigidities in terms of price adjustment costs and the adjustment costs expressed as a fraction of firms' real output is given by the function

$\Phi_p\left(\frac{P_t(j)}{P_{t-1}(j)}\right)$. We assume that the adjustment cost function twice-continously differentiable and weakly convex $\Phi_p' \geq 0$ and $\Phi_p'' \geq 0$. The firm maximizes real profits with respect to $H_t(j)$ and $P_t(j)$:

$$\mathbb{E}_t \sum_{s=0}^{\infty} \beta^s Q_{t+s|t}\left(\frac{P_{t+s}(j)}{P_{t+s}} A_{t+s} H_{t+s}(j) - \Phi_p\left(\frac{P_{t+s}(j)}{P_{t+s-1}(j)}\right) A_{t+s} H_{t+s}(j) - W_{t+s} H_{t+s}(j)\right) \quad ,$$

subject to

$$A_t H_t(j) = \left(\frac{P_t(j)}{P_t}\right)^{-1/\nu} Y_t.$$

We use $\mu_{t+s}\beta^s Q_{t+s|t}$ to denote the Lagrange multiplier associated with this constraint.

Price setting decision. Setting $Q_{t|t} = 1$, the first-order condition with respect to $P_t(j)$ is given by:

$$\begin{aligned} 0 \quad = \quad & \frac{A_t H_t(j)}{P_t} - \Phi_p'\left(\frac{P_t(j)}{P_{t-1}(j)}\right) \frac{A_t H_t(j)}{P_{t-1}(j)} - \frac{\mu_t}{\nu}\left(\frac{P_t(j)}{P_t}\right)^{-1/\nu-1} \frac{Y_t}{P_t} \\ & + \beta \mathbb{E}_t\left[Q_{t+1|t} \Phi_p'\left(\frac{P_{t+1}(j)}{P_t(j)}\right) A_{t+1} H_{t+1}(j) \frac{P_{t+1}(j)}{P_t^2(j)}\right]. \end{aligned}$$

Firms' labor demand. Taking first-order conditions with respect to $H_t(j)$ yields

$$W_t \quad = \quad \frac{P_t(j)}{P_t} A_t - \Phi_p\left(\frac{P_t(j)}{P_{t-1}(j)}\right) A_t - \mu_t A_t.$$

Symmetric equilibrium. We restrict attention to a symmetric equilibrium where all firms choose the same price $P_t(j) = P_t \, \forall j$. This assumption implies that in equilibrium all firms face identical marginal costs and demand the same amount of labor input. Combining the firms' price setting and labor demand first order conditions and in the presence of quadratic costs of price adjustment, $\Phi_p\left(\frac{P_t(j)}{P_{t-1}(j)}\right) = \frac{\phi}{2}\left(\frac{P_t(j)}{P_{t-1}(j)} - \bar{\pi}\right)^2$, we obtain:

$$(1-\nu) - \chi_H \left(\frac{C_t}{A_t}\right)^{\tau} H_t^{1/\eta} - \frac{\phi}{2}\left(\frac{P_t}{P_{t-1}-\bar{\pi}}\right) \quad + \\ \nu\phi\left(\frac{P_t}{P_{t-1}} - \bar{\pi}\right)\frac{P_t}{P_{t-1}} = \nu\beta\mathbb{E}_t\left[Q_{t+1|t}\frac{P_{t+1}}{P_t}\Phi_p'\left(\frac{P_{t+1}}{P_t}\right)\frac{Y_{t+1}}{Y_t}\right].$$

B.2.3 Equilibrium Conditions

The technology process introduces a long-run trend in the variables of the model. To make the model stationary we use the following transformations: $y_t = Y_t/A_t$, $c_t = C_t/A_t$, and note that $Y_t/Y_{t-1} = \frac{y_t}{y_{t-1}}\gamma z_t$. We also define the gross inflation rate $\pi_t = P_t/P_{t-1}$. The equilibrium conditions shown in the main text follow inmediately:

$$1 = \beta\mathbb{E}_t\left[\left(\frac{c_{t+1}}{c_t}\right)^{-\tau}\frac{1}{\gamma z_{t+1}}\frac{R_t}{\pi_{t+1}}\right] \qquad \text{(B.2.1)}$$

$$1 = \frac{1}{\nu}\left(1-\chi_h c^{\tau} y^{1/\eta}\right) + \phi(\pi_t - \bar{\pi})\left[\left(1-\frac{1}{2\nu}\right)\pi_t + \frac{\bar{\pi}}{2\nu}\right] \qquad \text{(B.2.2)}$$

$$-\phi\beta\mathbb{E}_t\left[\left(\frac{c_{t+1}}{c_t}\right)^{-\tau}\frac{y_{t+1}}{y_t}(\pi_{t+1}-\bar{\pi})\pi_{t+1}\right]$$

$$c_t = \left[\frac{1}{g_t} - \frac{\phi}{2}\left(\pi_t - \bar{\pi}\right)^2\right] y_t \qquad \text{(B.2.3)}$$

$$R_t = \max\left\{1, \left[r\pi_*\left(\frac{\pi_t}{\pi_*}\right)^{\psi_1}\left(\frac{y_t}{y_{t-1}}z_t\right)^{\psi_2}\right]^{1-\rho_R} R_{t-1}^{\rho_R} e^{\sigma_R \epsilon_{R,t}}\right\}. \qquad \text{(B.2.4)}$$

B.3 An Approximate Solution To a Simplified Model

In this section we will derive an approximate piece-wise linear solution for the DSGE model. Rather than constructing a sunspot equilibrium, we will focus on the targeted-inflation equilibrium and a minimal-state-variable deflation equilibrium. The main purpose is to highlight the kink in the decision rules, which motivates the piece-wise smooth numerical approximation used for the full model. We consider the case of quasi-linear preferences with $\chi_h = 1$ and $\eta = \infty$ and will impose further restrictions below to simplify the analytical derivations. The equilibrium conditions (in terms of detrended variables, i.e., $c_t = C_t/A_t$ and $y_t = Y_t/A_t$) take the form

$$
\begin{aligned}
1 &= \beta \mathbb{E}_t \left[\left(\frac{c_{t+1}}{c_t} \right)^{-\tau} \frac{1}{\gamma z_{t+1}} \frac{R_t}{\pi_{t+1}} \right] && \text{(B.3.1)} \\
1 &= \frac{1}{\nu}\left(1 - c^{\tau}\right) + \phi(\pi_t - \bar{\pi}) \left[\left(1 - \frac{1}{2\nu}\right) \pi_t + \frac{\bar{\pi}}{2\nu} \right] && \text{(B.3.2)} \\
&\quad - \phi \beta \mathbb{E}_t \left[\left(\frac{c_{t+1}}{c_t} \right)^{-\tau} \frac{y_{t+1}}{y_t} (\pi_{t+1} - \bar{\pi}) \pi_{t+1} \right] \\
c_t &= \left[\frac{1}{g_t} - \frac{\phi}{2} \left(\pi_t - \bar{\pi}\right)^2 \right] y_t && \text{(B.3.3)} \\
R_t &= \max \left\{ 1, \left[r \pi_* \left(\frac{\pi_t}{\pi_*} \right)^{\psi_1} \left(\frac{y_t}{y_{t-1}} z_t \right)^{\psi_2} \right]^{1-\rho_R} R_{t-1}^{\rho_R} e^{\sigma_R \epsilon_{R,t}} \right\}. && \text{(B.3.4)}
\end{aligned}
$$

B.3.1 Approximation of Targeted-Inflation Equilibrium

Steady State. Steady-state inflation equals π_*. Let $\lambda = \nu(1-\beta)$, then

$$
\begin{aligned}
r &= \gamma/\beta \\
R_* &= r\pi_* \\
c_* &= \left[1 - \upsilon - \frac{\phi}{2}(1-2\lambda)\left(\pi_* - \frac{1-\lambda}{1-2\lambda}\bar{\pi}\right)^2 + \frac{\phi}{2}\frac{\lambda^2}{1-2\lambda}\bar{\pi}^2\right]^{1/\tau} \\
y_* &= \frac{c_*}{\left[\frac{1}{g_*} - \frac{\phi}{2}(\pi_* - \bar{\pi})^2\right]}.
\end{aligned}
$$

Log-Linearization. We omit the hats from variables that capture deviations from the targeted-inflation steady state. The linearized consumption Euler equation (B.3.1) is

$$
c_t = \mathbb{E}_t[c_{t+1}] - \frac{1}{\tau}(R_t - \mathbb{E}_t[\pi_{t+1} + z_{t+1}]).
$$

The price setting equation (B.3.2) takes the form

$$
\begin{aligned}
0 \;=\; & -\frac{\tau c_*^\tau}{\nu} c_t + \phi\pi_*\left[\left(1 - \frac{1}{2\nu}\right)\pi_* + \frac{\bar{\pi}}{2\nu}\right]\pi_t + \phi\pi_*(\pi_* - \bar{\pi})\left(1 - \frac{1}{2\nu}\right)\pi_t \\
& - \phi\beta\pi_*(\pi_* - \bar{\pi})\Big(\tau c_t - y_t - \mathbb{E}_t[\tau c_{t+1} - y_{t+1}] + \mathbb{E}[\pi_{t+1}]\Big) - \phi\beta\pi_*^2\mathbb{E}_t[\pi_{t+1}].
\end{aligned}
$$

Log-linearizing the aggregate resource constraint (B.3.2) yields

$$
c_t = y_t - \frac{1/g_*}{1/g_* - \phi(\pi_* - \bar{\pi})^2} g_t - \frac{\phi\pi_*(\pi_* - \bar{\pi})}{1/g_* - \phi(\pi_* - \bar{\pi})^2}\pi_t
$$

Finally, the monetary policy rule becomes

$$R_t = \max\left\{ -\ln(r\pi_*),\ (1-\rho_R)\psi_1\pi_t + (1-\rho_R)\psi_2(y_t - y_{t-1} + z_t) + \rho R_{t-1} + \sigma_R\epsilon_{R,t} \right\}.$$

Approximate Piecewise-Linear Solution in Special Case. To simplify the exposition, we impose the following restrictions on the DSGE model parameters: $\tau = 1$, $\gamma = 1$, $\bar{\pi} = \pi_*$, $\psi_1 = \psi$, $\psi_2 = 0$, $\rho_R = 0$, $\rho_z = 0$, and $\rho_g = 0$. We obtain the system

$$\begin{aligned} R_t &= \max\left\{ -\ln(r\pi_*),\ \psi\pi_t + \sigma_R\epsilon_{R,t} \right\} \\ c_t &= \mathbb{E}_t[c_{t+1}] - (R_t - \mathbb{E}_t[\pi_{t+1}]) \\ \pi_t &= \beta\mathbb{E}_t[\pi_{t+1}] + \kappa c_t. \end{aligned} \tag{B.3.5}$$

It is well known that if the shocks are small enough such that the ZLB is non-binding, the linearized system has a unique stable solution for $\psi > 1$. Since the exogenous shocks are *iid* and the simplified system has no endogenous propagation mechanism, consumption, output, inflation, and interest rates will also be *iid* and can be expressed as a function of $\epsilon_{R,t}$. In turn, the conditional expectations of inflation and consumption equal their unconditional means, which we denote by μ_π and μ_c, respectively.

The Euler equation in (B.3.5) simplifies to the static relationship

$$c_t = -R_t + \mu_c + \mu_\pi. \tag{B.3.6}$$

Similarly, the Phillips curve in (B.3.5) becomes

$$\pi_t = \kappa c_t + \beta \mu_\pi. \tag{B.3.7}$$

Combining (B.3.6) and (B.3.7) yields

$$\pi_t = -\kappa R_t + (\kappa + \beta)\mu_\pi + \kappa \mu_c. \tag{B.3.8}$$

We now can use (B.3.8) to eliminate inflation from the monetary policy rule:

$$R_t = \max \left\{ -\ln(r\pi_*),\ -\kappa\psi R_t + (\kappa+\beta)\psi\mu_\pi + \kappa\psi\mu_c + \sigma_R \epsilon_{R,t} \right\} \tag{B.3.9}$$

Define

$$R_t^{(1)} = -\ln(r\pi_*) \quad \text{and} \quad R_t^{(2)} = \frac{1}{1+\kappa\psi}\Big[(\kappa+\beta)\psi\mu_\pi + \kappa\psi\mu_c + \sigma_R\epsilon_{R,t}\Big].$$

Let $\bar{\epsilon}_{R,t}$ be the value of the monetary policy shock for which $R_t = -\ln(r\pi_*)$ and the two terms in the max operator of (B.3.9) are equal

$$\sigma_R \bar{\epsilon}_{R,t} = -(1+\kappa\psi)\ln(r\pi_*) - (\kappa+\beta)\psi\mu_\pi - \kappa\psi\mu_c.$$

To complete the derivation of the equilibrium interest rate, it is useful to distinguish the following two cases. Case (i): suppose that $\epsilon_{R,t} < \bar{\epsilon}_{R,t}$. We will verify that $R_t = R_t^{(1)}$ is consistent with (B.3.9). If the monetary policy shock is less than the threshold value, then

$$(\kappa+\beta)\psi\mu_\pi + \kappa\psi\mu_c + \sigma_R\bar{\epsilon}_{R,t} < -(1+\kappa\psi)\ln(r\pi_*).$$

Thus,

$$-\kappa\psi R_t^{(1)} + (\kappa+\beta)\psi\mu_\pi + \kappa\psi\mu_c + \sigma_R\epsilon_{R,t} < -\kappa\psi R_t^{(1)} - (1+\kappa\psi)\ln(r\pi_*) = -\ln(r\pi_*),$$

which confirms that (B.3.9) is satisfied.

Case (ii): suppose that $\epsilon_{R,t} > \bar{\epsilon}_{R,t}$. We will verify that $R_t = R_t^{(2)}$ is consistent with (B.3.9). If the monetary policy shock is greater than the threshold value, then

$$(\kappa+\beta)\psi\mu_\pi + \kappa\psi\mu_c + \sigma_R\bar{\epsilon}_{R,t} > -(1+\kappa\psi)\ln(r\pi_*).$$

In turn,

$$\begin{aligned}
&-\kappa\psi R_t^{(2)} + (\kappa+\beta)\psi\mu_\pi + \kappa\psi\mu_c + \sigma_R\epsilon_{R,t} \\
&\quad = -\frac{\kappa\psi}{1+\kappa\psi}\left[(\kappa+\beta)\psi\mu_\pi + \kappa\psi\mu_c + \sigma_R\epsilon_{R,t}\right] + (\kappa+\beta)\psi\mu_\pi + \kappa\psi\mu_c + \sigma_R\epsilon_{R,t} \\
&\quad = \frac{1}{1+\kappa\psi}\left[(\kappa+\beta)\psi\mu_\pi + \kappa\psi\mu_c + \sigma_R\epsilon_{R,t}\right] \\
&\quad > -\ln(r\pi_*),
\end{aligned}$$

which confirms that (B.3.9) is satisfied.

We can now deduce that

$$R_t(\epsilon_{R,t}) = \max\left\{-\ln(r\pi_*),\ \frac{1}{1+\kappa\psi}\left[\psi(\kappa+\beta)\mu_\pi + \kappa\psi\mu_c + \sigma_R\epsilon_{R,t}\right]\right\}. \quad \text{(B.3.10)}$$

Combining (B.3.6) and (B.3.10) yields equilibrium consumption

$$c_t(\epsilon_{R,t}) = \begin{cases} \frac{1}{1+\kappa\psi}\left[(1-\psi\beta)\mu_\pi + \mu_c - \sigma_R\epsilon_{R,t}\right] & \text{if } R_t \geq -\ln(r\pi_*) \\ \ln(r\pi_*) + \mu_c + \mu_\pi & \text{otherwise} \end{cases}. \quad \text{(B.3.11)}$$

Likewise, combining (B.3.7) and (B.3.10) delivers equilibrium inflation

$$\pi_t(\epsilon_{R,t}) = \begin{cases} \frac{1}{1+\kappa\psi}\left[(\kappa+\beta)\mu_\pi + \kappa\mu_c - \kappa\sigma_R\epsilon_{R,t}\right] & \text{if } R_t \geq -\ln(r\pi_*) \\ \kappa\ln(r\pi_*) + (\kappa+\beta)\mu_\pi + \kappa\mu_c & \text{otherwise} \end{cases}. \quad \text{(B.3.12)}$$

If $X \sim N(\mu, \sigma^2)$ and C is a truncation constant, then

$$\mathbb{E}[X|X \geq C] = \mu + \frac{\sigma\phi_N(\alpha)}{1-\Phi_N(\alpha)},$$

where $\alpha = (C-\mu)/\sigma$, $\phi_N(x)$ and $\Phi_N(\alpha)$ are the probability density function (pdf) and the cumulative density function (cdf) of a $N(0,1)$. Define the cutoff value

$$C = -(1+\kappa\psi)\ln(r\pi_*) - (\kappa+\beta)\psi\mu_\pi - \kappa\psi\mu_c. \quad \text{(B.3.13)}$$

Using the definition of a cdf and the formula for the mean of a truncated normal random variable, we obtain

$$
\begin{aligned}
\mathbb{P}[\epsilon_{R,t} \geq C/\sigma_R] &= 1 - \Phi_N(C_y/\sigma_R) \\
\mathbb{E}[\epsilon_{R,t} \,|\, \epsilon_{R,t} \geq C/\sigma_R] &= \frac{\sigma_R \phi_N(C/\sigma_R)}{1 - \Phi_N(C/\sigma_R)}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\mu_c &= \frac{1 - \Phi_N(C_y/\sigma_R)}{1 + \kappa\psi}\Bigg[(1 - \psi\beta)\mu_\pi + \mu_c\Bigg] - \frac{\sigma_R \phi_N(C_y/\sigma_R)}{(1 + \kappa\psi)(1 - \Phi_N(C_y/\sigma_R))} \quad \text{(B.3.14)} \\
&\quad + \Phi_N(C_y/\sigma_R)\Bigg[\ln(r\pi_*) + \mu_c + \mu_\pi\Bigg] \\
\mu_\pi &= \frac{1 - \Phi_N(C_y/\sigma_R)}{1 + \kappa\psi}\Bigg[(\kappa + \beta)\mu_\pi + \kappa\mu_c\Bigg] - \frac{\kappa\sigma_R \phi_N(C_y/\sigma_R)}{(1 + \kappa\psi)(1 - \Phi_N(C_y/\sigma_R))} \quad \text{(B.3.15)} \\
&\quad + \Phi_N(C_y/\sigma_R)\Bigg[\kappa \ln(r\pi_*) + (\kappa + \beta)\mu_\pi + \kappa\mu_c\Bigg]
\end{aligned}
$$

The constants C, μ_c, and μ_π can be obtained by solving the system of nonlinear equations composed of (B.3.13) to (B.3.15).

B.3.2 Approximation of Deflation Equilibrium

Steady State. As before, let $\lambda = \nu(1-\beta)$. The steady-state nominal interest rate is $R_D = 1$, and provided that $\beta/(\gamma\pi_*) < 1$ and $\psi_1 > 1$:

$$
\begin{aligned}
r &= \gamma/\beta \\
\pi_D &= \beta/\gamma \\
c_D &= \left[1 - \upsilon - \frac{\phi}{2}(1-2\lambda)\left(\pi_D - \frac{1-\lambda}{1-2\lambda}\bar{\pi}\right)^2 + \frac{\phi}{2}\frac{\lambda^2}{1-2\lambda}\bar{\pi}^2\right]^{1/\tau} \\
y_D &= \frac{c_D}{\left[\frac{1}{g_*} - \frac{\phi}{2}(\pi_D - \bar{\pi})^2\right]}.
\end{aligned}
$$

Log-Linearization. We omit the tildes from variables that capture deviations from the deflation steady state. The linearized consumption Euler equation (B.3.1) is

$$
c_t = \mathbb{E}_t[c_{t+1}] - \frac{1}{\tau}(R_t - \mathbb{E}_t[\pi_{t+1} + z_{t+1}]).
$$

The price-setting equation (B.3.2) takes the form

$$
\begin{aligned}
0 \;=\; & -\frac{\tau c_D^\tau}{\nu}c_t + \phi\beta\left[\left(1 - \frac{1}{2\nu}\right)\beta + \frac{\bar{\pi}}{2\nu}\right]\pi_t + \phi\beta(\beta - \bar{\pi})\left(1 - \frac{1}{2\nu}\right)\pi_t \\
& -\phi\beta^2(\beta - \bar{\pi})\Big(\tau c_t - y_t - \mathbb{E}_t[\tau c_{t+1} - y_{t+1}] + \mathbb{E}[\pi_{t+1}]\Big) - \phi\beta^3\mathbb{E}_t[\pi_{t+1}].
\end{aligned}
$$

Log-linearizing the aggregate resource constraint (B.3.2) yields

$$
c_t = y_t - \frac{1/g_*}{1/g_* - \phi(\beta - \bar{\pi})^2}g_t - \frac{\phi\beta(\beta - \bar{\pi})}{1/g_* - \phi(\beta - \bar{\pi})^2}\pi_t
$$

Finally, the monetary policy rule becomes

$$\begin{aligned} R_t &= \max \Big\{ 0,\ -(1-\rho_R)\ln(r\pi_*) - (1-\rho_R)\psi_1 \ln(\pi_*/\beta) \\ &\quad +(1-\rho_R)\psi_1\pi_t + (1-\rho_R)\psi_2(y_t - y_{t-1} + z_t) + \rho R_{t-1} + \sigma_R \epsilon_{R,t} \Big\}. \end{aligned}$$

Approximate Piecewise-Linear Solution in Special Case. As for the approximate analysis of the targeted-inflation equilibrium, we impose the following restrictions on the DSGE model parameters: $\tau = 1$, $\gamma = 1$, $\bar{\pi} = \pi_*$, $\psi_1 = \psi$, $\psi_2 = 0$, $\rho_R = 0$, $\rho_z = 0$, and $\rho_g = 0$. In the deflation equilibrium, the steady-state inflation rate is $\pi_D = \beta$. To ease the expositions, we assume that the terms $|\pi_D - \bar{\pi}|$ that appear in the log-linearized equations above are negligible. Denote percentage deviations of a variable x_t from its deflation steady state by $\tilde{x}_t = \ln(x_t/x_D)$. If we let $\kappa_D = c_D/(\nu\phi\beta^2)$ and using the steady-state relationship $r = 1/\beta$

$$\begin{aligned} \tilde{R}_t &= \max \Big\{ 0,\ -(\psi-1)\ln(r\pi_*) + \psi\tilde{\pi}_t + \sigma_R\epsilon_{R,t} \Big\} \\ \tilde{c}_t &= \mathbb{E}_t[\tilde{c}_{t+1}] - (\tilde{R}_t - \mathbb{E}_t[\tilde{\pi}_{t+1}]) \qquad \text{(B.3.16)} \\ \tilde{\pi}_t &= \beta\mathbb{E}_t[\tilde{\pi}_{t+1}] + \kappa_D\tilde{c}_t. \end{aligned}$$

Provided that $\psi > 1$, the ZLB is binding with high probability if the shock standard deviation σ_R is small. In this case, $\tilde{R}_t = 0$. An equilibrium in which all variables are *iid* can be obtained by adjusting the constants in (B.3.10) to (B.3.12):

$$
\begin{aligned}
\tilde{R}_t(\epsilon_{R,t}) &= \max\left\{0,\ \frac{1}{1+\kappa\psi}\left[\psi(\kappa+\beta)\mu_\pi^D + \kappa\psi\mu_c^D - (\psi-1)\ln(r\pi_*) + \sigma_R\epsilon_{R,t}\right]\right\} \\
\tilde{c}_t(\epsilon_{R,t}) &= \begin{cases} \frac{1}{1+\kappa\psi}\left[(1-\psi\beta)\mu_\pi^D + \mu_c^D + (\psi-1)\ln(r\pi_*) - \sigma_R\epsilon_{R,t}\right] & \text{if } \tilde{R}_t \geq 0 \\ \mu_c^D + \mu_\pi^D & \text{otherwise} \end{cases} \qquad \text{(B.3.17)} \\
\tilde{\pi}_t(\epsilon_{R,t}) &= \begin{cases} \frac{1}{1+\kappa\psi}\left[(\kappa+\beta)\mu_\pi^D + \kappa\mu_c^D + \kappa(\psi-1)\ln(r\pi_*) - \kappa\sigma_R\epsilon_{R,t}\right] & \text{if } \tilde{R}_t \geq 0 \\ (\kappa+\beta)\mu_\pi^D + \kappa\mu_c^D & \text{otherwise} \end{cases} .
\end{aligned}
$$

In this simple model, the decision rules have a kink at the point in the state space where the two terms in the max operator of the interest rate equation are equal to each other. In the targeted-inflation equilibrium, this point in the state space is given by

$$
\bar{\epsilon}_R^* = \frac{1}{\sigma_R}\left[-(1+\kappa\psi)\ln(r\pi_*) - (\kappa+\beta)\psi\mu_\pi^* - \kappa\psi\mu_c^*\right],
$$

whereas in the deflation equilibrium, it is

$$
\bar{\epsilon}_R^D = \frac{1}{\sigma_R}\left[(\psi-1)\ln(r\pi_*) - (\kappa+\beta)\psi\mu_\pi^D - \kappa\psi\mu_c^D\right],
$$

Once $\epsilon_{R,t}$ falls below the threshold value $\bar{\epsilon}_R^*$ or $\bar{\epsilon}_R^D$, its marginal effect on the endogenous variables is zero. To the extent that $\bar{\epsilon}_R^D > 0 > \bar{\epsilon}_R^*$, it takes a positive shock in the deflation equilibrium to move away from the ZLB, whereas it takes a large negative monetary shock in the targeted-inflation equilibrium to hit the ZLB.

B.4 Computational Details

B.4.1 Model Solution Algorithm

Algorithm 1 (Model Solution)

1. Construct solutions for the targeted-inflation equilibrium ($s_t = 1$ with probability one) and the deflation equilibrium ($s_t = 0$ with probability one):

   (a) Start with a guess for Θ. For the targeted-inflation equilibrium, this guess is obtained from a first-order linear approximation around the targeted-inflation steady state. For the deflation equilibrium, it is obtained by assuming constant decision rules for inflation and $\mathcal{E}$ at the deflation steady state.

   (b) Given this guess, simulate the model for a large number of periods. We use 10,000 simulations after a burn-in period of 150 observations.

   (c) Use the cluster-grid algorithm in Judd, Maliar and Maliar (2010) to obtain a collection of grid points for the model solution. For the deflation equilibrium we use the a time-separted grid instead, because this algorithm suits the behavior of this equilibrium better, since there are many periods when the economy is on the "edge" of the ergodic distribution at the ZLB. Label these grid points as $\{\mathcal{S}_1, ..., \mathcal{S}_M\}$. For a fourth-order approximation, we use $M = 130$.

   (d) Solve for the Θ by minimizing the sum of squared residuals using Algorithm 2.

2. Repeat steps (b)-(d) a sufficient number of times so that the ergodic distribution remains unchanged from one iteration to the next.

3. Initialize the sunspot solution decision rules for $s_t = 1$ ($s_t = 0$) with the targeted-inflation equilibrium decision rules decision rules that come from the targeted-inflation (deflation) equilibrium obtained in step 1. Given this guess, simulate the sunspot model for a large number of periods as in (b). For the sunspot model this simulation also includes the simulated path of the sunspot variable s_t.

4. Given the simulated path, obtain the grid for the state variables over which the approximation needs to be accurate. For the sunspot equilibrium, we use the same time-separated grid algorithm to deliver the grid points that represent the ergodic set. For a fourth order approximation of Japan we set M = 624 and obtain 50% of this points conditioning on $s_t = 1$ and the remaining are conidtioned on $s_t = 0$. This oversamples points from the $s_t = 0$ regime to increase the accuracy of the solution.

   For the US we obtained 268 grid points from the ergodic distribution using the cluster-grid algorithm. Again we obtain 50% of the points conditioning on $s_t = 1$ and the rest conditioning on $s_t = 0$. The remaining 356 points come from the filtered states. We use 36 filtered states corresponding to the period 2000:Q1-2008:Q4 and 320 points corresponding to filtered states using multiple particles per period from 2009:Q1-2013:Q4.

5. Solve for the Θ by minimizing the sum of squared residuals using Algorithm 2.

6. Repeat steps 2.-5. a sufficient number of times so that the ergodic distribution of the sunspot model remains unchanged from one iteration to the next. For the US sunspot equilibrium, we also iterate between solution and filtering to make sure the filtered states used in the solution grid remain unchanged.

Algorithm 2 (Determinining the Approximate Decision Rules)

1. For a generic grid point $\mathcal{S}_i$ and the current value for Θ, compute $f_\pi^1(\mathcal{S}_i;\Theta)$, $f_\pi^2(\mathcal{S}_i;\Theta)$, $f_\mathcal{E}^1(\mathcal{S}_i;\Theta)$, and $f_\mathcal{E}^2(\mathcal{S}_i;\Theta)$.

2. Assume $\zeta_i \equiv I\{R(\mathcal{S}_i,\Theta) > 1\} = 1$ and compute π_i, and $\mathcal{E}_i$, as well as y_i and c_i using (2.23) and (2.25).

3. Compute R_i based on (2.15) using π_i and y_i obtained in (2). If R_i is greater than unity, then ζ_i is indeed equal to one. Otherwise, set $\zeta_i = 0$ (and thus $R_i = 1$) and recompute all other objects.

4. The final step is to compute the residual functions. In each regime $s_t = \{0,1\}$ there are four residuals, corresponding to the four functions being approximated. For a given set of state variables $\mathcal{S}_i$, only two of them will be relevant since we either need the constrained decision rules or the unconstrained ones. Taking into account the transition of the sunspot the residual functions will

be given by

$$\mathcal{R}^1(\mathcal{S}_i) = \mathcal{E}_i - \int\int\int\int \frac{c(\mathcal{S}')^{-\tau}}{\gamma z' \pi(\mathcal{S}')} dF(z')\, dF(g')\, dF(\epsilon'_R)\, dF(s') \tag{B.4.1}$$

$$\mathcal{R}^2(\mathcal{S}_i) = \xi(c_i, \pi_i, y_i) \tag{B.4.2}$$

$$- \phi\beta \int\int\int\int c(\mathcal{S}')^{-\tau} y(\mathcal{S}') [\pi(\mathcal{S}') - \bar{\pi}]\, \pi(\mathcal{S}') dF(z')\, dF(g')\, dF(\epsilon'_R)\, dF(s')$$

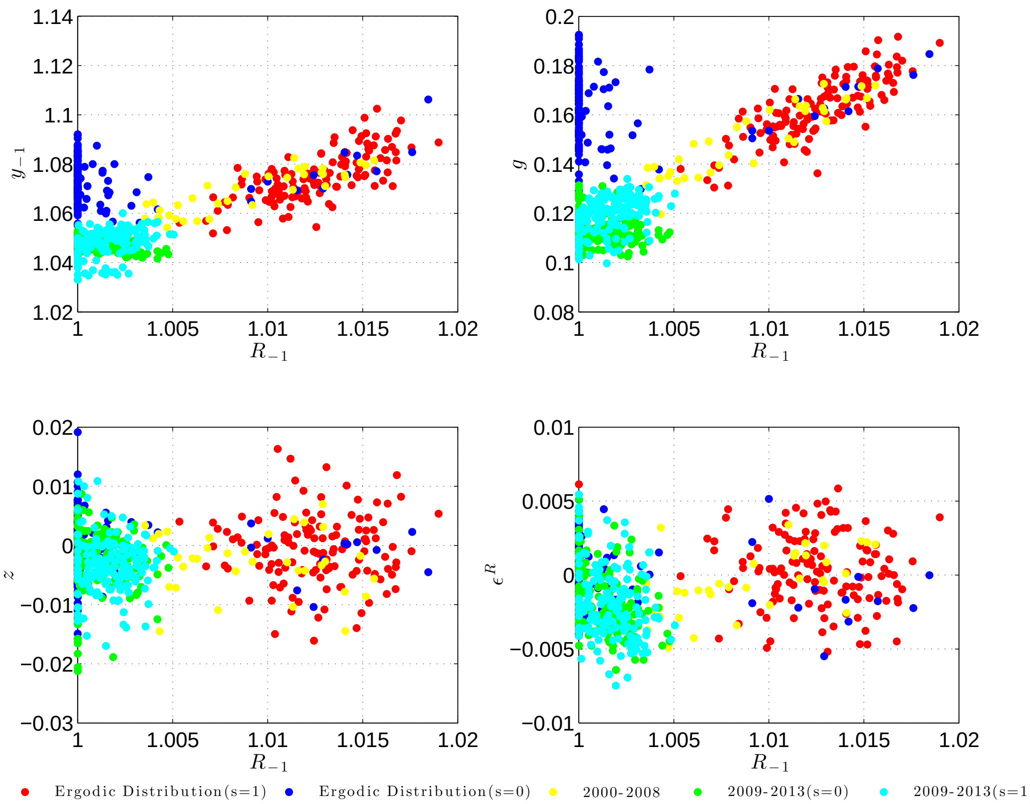
Note that this step involves computing $\pi(\mathcal{S}')$, $y(\mathcal{S}')$, $c(\mathcal{S}')$, and $R(\mathcal{S}')$ which is done following steps (1)-(3) above for each value of $\mathcal{S}'$. We use a non product monomial integration rule to evaluate these integrals.

5. The objective function to be minimized is the sum of squared residuals obtained in (4).

For the target-inflation (deflation) regimes, we first solve for a second-order polynomial approximation of the decision rules and move to a third- and fourth-order polynomial using the previous order solution as initial guess. We use analytical derivatives of the objection function, which speeds up the solution by two orders of magnitude. As a measure of accuracy, we compute the approximation errors from B.4.1 and B.4.3, converted to consumption units. For the sunspot equilibrium the approximation errors are in the order of 10^{-4} or smaller.

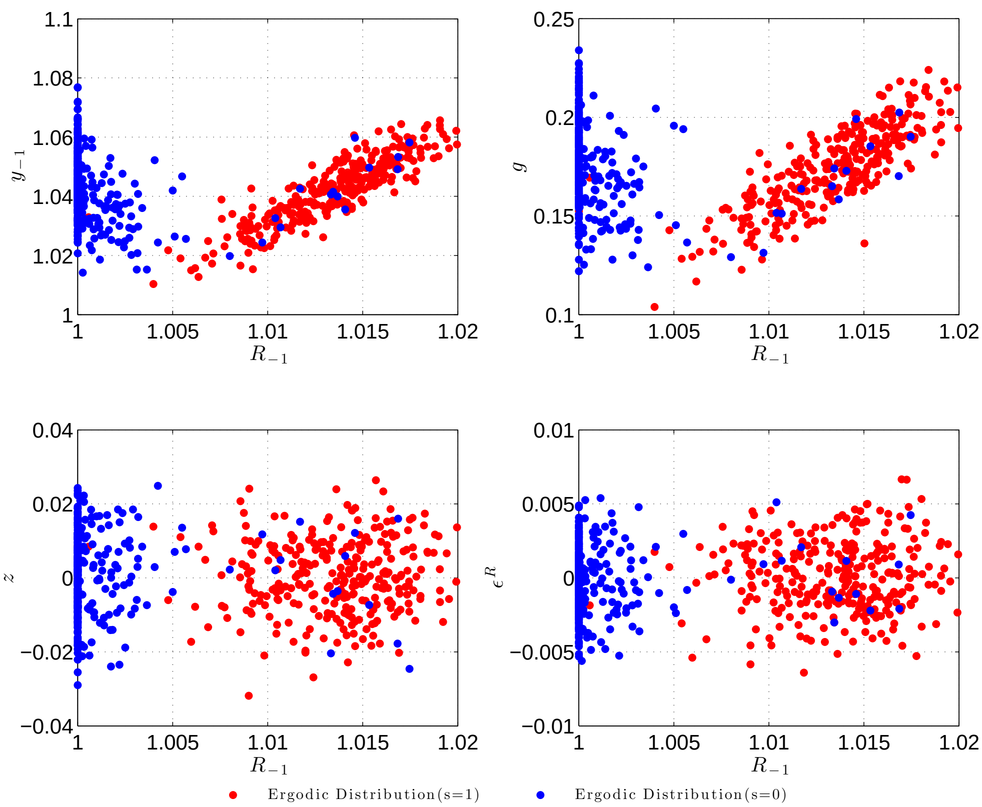
Figures B-1 and B-2 show the solution grid for the sunspot equilibrium. For each panel, we have R_{t-1} on the x axis and one of the other state variables on the y axis. The red (blue) dots are the grid points that represent the ergodic distribution conditional on $s_t = 1$ ($s_t = 0$). For the U.S. we include filtered grid points into the

Figure B-1: Solution Grid for the Targeted-Inflation Equilibrium - US



construction of the grid. The yellow dots denote filtered states between 2000 to 2008; the green (turquoise) dots represent filtered states from 2009 to 2013 conditioning on $s_t = 1$ ($s_t = 0$). It is evident that for the U.S. the filtered states lie in the tails of the ergodic distribution of the sunspot equilibrium. By adding these filtered states to the grid points, we ensure that our approximation will be accurate in these low-probability regions.

Figure B-2: Solution Grid for the Targeted-Inflation Equilibrium - Japan



B.4.2 Details of Policy Experiments

Algorithm 3 (Effect of Combined Fiscal and Monetary Policy Intervention)

Here we describe how we complement the fiscal policy intervention of size f with a commitment of the central bank to keep the policy rate at or near the ZLB. We use $\tilde{x}$ to denote the mean value of x obtained from the particle filter. (See Section 2.5.6.1 for details.) We use $H = 11$, which means the central bank's commitment is in place for three years.

For some t in the ZLB period of U.S. or Japan, we do the following:

1. Initialize the simulation by setting $(R_{t-1}, y_{t-1}, z_{t-1}, g_{t-1}, s_t) = (\tilde{R}_{t-1}, \tilde{y}_{t-1}, \tilde{z}_{t-1}, \tilde{g}_{t-1}, \tilde{s}_t)$

2. Generate baseline trajectories based on the innovation sequence $\{\epsilon_{i,t+h}\}_{h=0}^{H} = \{\tilde{\epsilon}_{i,t+h}\}_{h=0}^{H}$ and $\{s_{t+h}\}_{h=0}^{H} = \{\tilde{s}_{t+h}\}_{h=0}^{H}$, which essentially means in the baseline trajectories all output growth, inflation and the interest rate equals their data counterparts up to a measurement error we use in filtering.

3. Generate the innovation sequence for the counterfactual trajectories according to

$$\epsilon_{g,t}^{I} = f + \tilde{\epsilon}_{g,t}; \quad \epsilon_{g,t+h}^{I} = \tilde{\epsilon}_{g,t+h} \quad \text{for} \quad h = 1, \ldots, H;$$

$$\epsilon_{z,t+h}^{I} = \tilde{\epsilon}_{z,t+h} \quad \text{for} \quad h = 1, \ldots, H;$$

$$s_{t+h}^{I} = \tilde{s}_{t+h}$$

In periods $t+h$ for $h = 0, \ldots, H$, conditional on $\epsilon^I_{g,t+h}, \epsilon^I_{z,t+h}$ and s^I_{t+h}, determine $\epsilon^I_{R,t+h}$ by solving for the smallest $\hat{\epsilon}_{R,t+h}$ such that it is less than $2\sigma_R$ in absolute value, that yields either

$$R^I_{t+h}(\epsilon^I_{R,t+h} = \hat{\epsilon}_{R,t+h}) = 1 \quad \text{or} \quad 400\bigg(R^I_t(\epsilon^I_{R,t+h} = 0) - R^I_{t+h}(\epsilon^I_{R,t+h} = \hat{\epsilon}_{R,t+h})\bigg) = 1.$$

4. Conditional on $(R_{t-1}, y_{t-1}, z_{t-1}, g_{t-1})$, compute $\{R_{t+h}, y_{t+h}, \pi_{t+h}\}^H_{h=0}$ and $\{R^I_{t+h}, y^I_{t+h}, \pi^I_{t+h}\}^H_{h=0}$ based on $\{\epsilon_t, s_t\}$ and $\{\epsilon^I_t, s^I_t\}$, respectively, and let

$$IRF(x_{t+h}|\epsilon_{g,t}, \epsilon_{R,t:t+h}) = \ln x^I_{t+h} - \ln x_{t+h}. \tag{B.4.3}$$

In tables and figures we report the median and the point-wise 20% and 80% response across all possible initial period t. When we consider only a fiscal policy, we set $\epsilon^I_{R,t+h} = 0$ for $h = 0, ..., H$ as well.

B.5 Data

B.5.1 United States

For the US we collected data from the FRB St. Louis FRED database. We obtained real GDP (GDPC96) and converted into per capita terms using the Civilian Noninstitutional Population (CNP16OV). The population series is smoothed applying an eight-quarter backward-looking moving average filter. The measure of the price level is the GDP deflator (GDPDEF) and the inflation rate is computed as its log difference annualized and in percents. The interest rate is the average effective federal funds rate (FEDFUNDS) averaged over each quarter.

B.5.2 Japan

For Japan we collected real GDP (RGDP) from the Cabinet Office's National Accounts. We used the statistical release of benchmark year 2005 that covers the period 1994.Q1 - 2013.Q4. To extend the sample we collected RGDP figures from the benchmark year 2000 and constructed a series spanning the period 1981.Q1-2013.Q1 using the quarterly growth rate of the RGDP benchmark year 2000. Our measure of per-capita output is RGDP divided by the total population of 15 years and over. We smoothed the quarterly growth of the population series using an eight quarter backward-looking moving average filter. We obtained population data from the Statistics Bureau of the Ministry of Foreign Affairs Historical data Table b-1. For the price level we use the implicit GDP deflator index from the Cabinet Office. We also extend the benchmark year 2005 release using the growth rate of the index from the benchmark year 2000 figures. For the nominal interest rate we use the Bank of Japan's uncollateralized call rate (STSTRACLUCON) from 1986:M7-2013:M12. To complete the series from 1981.M1 - 1985.M6 we use the monthly average of the collateralized overnight call rate (STSTRACLCOON). Finally the monthly figures are transformed using quarterly averages over the sample period.

B.5.3 Fiscal Programs in Japan

Table B-1 shows a list of fiscal programs that were in effect in Japan from 1998 to 2013. For each program we show the size of the program, and the amount paid directly by the central (national) government as a percentage of GDP and a short description. In the last three columns we show the major concerns of the government in passing each measure, focusing on concerns about real activity, exchange rate and deflation. We also provide a link to the official statement of each program.

Table B-1: Fiscal Programs in Japan

Program	Total program	National expenditure	Description	Real Economy	Exchange rate	Deflation
Apr-1998	3.12	0.90	Comprehensive economic measures to accelerate recovery http://www5.cao.go.jp/98/b/19980424b-taisaku-e.html	x		
Nov-1998	4.68	1.48	Emergency economic recovery package http://www5.cao.go.jp/98/b/19981116b-taisaku-e.html	x		
Nov-1999	3.57	1.29	Economic stimulus and revitalize economy http://www5.cao.go.jp/99/b/19991111b-taisaku-e2.html	x		
Oct-2000	2.16	0.76	Additional stimulus to consolidate economic recovery http://www5.cao.go.jp/2000/b/1019b-taisaku1-e.html	x		
Dec-2001	0.81	0.51	Fear of global recession and commitment to avoid deflationary spiral http://www5.cao.go.jp/keizai1/2001/1227daijin-speech-e.pdf	x		
Dec-2002	2.97	0.60	Program to accelerate economic reform http://www5.cao.go.jp/keizai1/2002/1212program-e.pdf	x		
Aug-2008	2.33	0.36	Economic package to reduce uncertainty with respecto slowdown in global economy http://www5.cao.go.jp/keizai1/2008/0918summary-english.pdf	x		
Oct-2008	5.37	0.96	Economic measures in response of global recession http://www5.cao.go.jp/keizai1/2008/081201outline-english.pdf	x		
Apr-2009	12.06	3.27	Response to rapid deterioration of external sector and risk in financial system http://www5.cao.go.jp/keizai1/2009/0420summary-english.pdf	x		
Dec-2009	5.18	1.53	Emergency economic measures to secure economic recovery and fight deflation http://www5.cao.go.jp/keizai1/2009/091228_emergency_economic.pdf	x	x	x
Sep-2010	2.03	0.19	Economic measures to overcome deflation http://www5.cao.go.jp/keizai1/2010/2010esp2.pdf	x	x	x
Oct-2010	4.37	1.06	Additional economic measures to control currency appreciation and deflation http://www5.cao.go.jp/keizai1/2010/101008.pdf	x	x	x
Jan-2013	4.17	2.70	Post earthquake stimulus to fight currency appreciation and deflation http://www5.cao.go.jp/keizai1/2013/130111_emergency_economic_measures.pdf	x	x	x
Dec-2013	3.84	1.13	Revitalization of economy and end deflation http://www5.cao.go.jp/keizai1/2013/20131205_economic_measures_all.pdf	x	x	x

Notes: The table shows the major fiscal programs in Japan from 1998 to 2013. The second and third column shows the size of the total program and the portion spent by the central government as a percentage of GDP. The last three columns show what major concerns the policy makers had for each program. We provide the URL of the description of each program.

B.6 DSGE Model Estimation

B.6.1 Estimation Sample

For Japan we restrict the estimation sample to the period 1981:Q1 to 1994:Q4. Effectively the nominal interest reached the zero lower bound in the third quarter of 1998, however we stop estimation earlier to avoid non-linearities caused by the zero lower bound. Our choice for the estimation sample is consistent with other studies that used perturbation-based techniques to estimate structural parameters for the Japanese economy, e.g. Sugo and Ueda (2008) and Fujiwara, Hirose and Shintani (2011). For the US we estimate the model from 1984:Q1 to 2007:Q4. For a similar reason we truncate the estimation before the nominal interest rate reached the zero lower bound.

B.6.2 Priors for estimation

We use similar priors for both countries. For instance the prior mean for τ implies a risk-aversion coefficient of 2. We specify the prior for the price-adjustment-cost parameter ϕ indirectly through a prior for the slope κ of the New-Keynesian Phillips curve in a linearized version of the model. For both countries this prior encompasses values that imply an essentially flat as well as a fairly steep Phillips curve, with a prior mean of 0.3. The prior for the inflation response coefficient in the monetary policy rule is centered at 1.5 with a tighter prior because it was difficult to identify this parameter from the data. Finally, we use diffuse priors for the parameters associated with the exogenous shock processes. Marginal prior distributions

Table B-2: Priors for Estimation

		US		Japan	
Parameter	Density	Param (1)	Param (2)	Param (1)	Param (2)
τ	Gamma	2	0.25	2	0.5
κ	Gamma	0.3	0.1	0.3	0.1
ψ_1	Gamma	1.5	0.05	1.5	0.05
ρ_r	Beta	0.5	0.2	0.6	0.2
ρ_g	Beta	0.8	0.1	0.6	0.2
ρ_z	Beta	0.2	0.1	0.4	0.2
$100\sigma_r$	Inv Gamma	0.3	4.0	0.3	4.0
$100\sigma_g$	Inv Gamma	0.4	4.0	0.4	4.0
$100\sigma_z$	Inv Gamma	0.4	4.0	0.4	4.0

Notes: Para (1) and Para(2) are the mean and the standard deviations for Beta and Gamma distributions; s and ν for the Inverse Gamma distribution, where $p_{\mathcal{IG}}(\sigma|\nu, s) \propto \sigma^{-\nu-1}e^{-\nu s^2/2\sigma^2}$.

for all DSGE model parameters are summarized in Table B-2. We assume that the parameters are *a priori* independent. Thus, the joint prior distribution is given by the product of the marginals.

B.6.3 Posterior Simulator

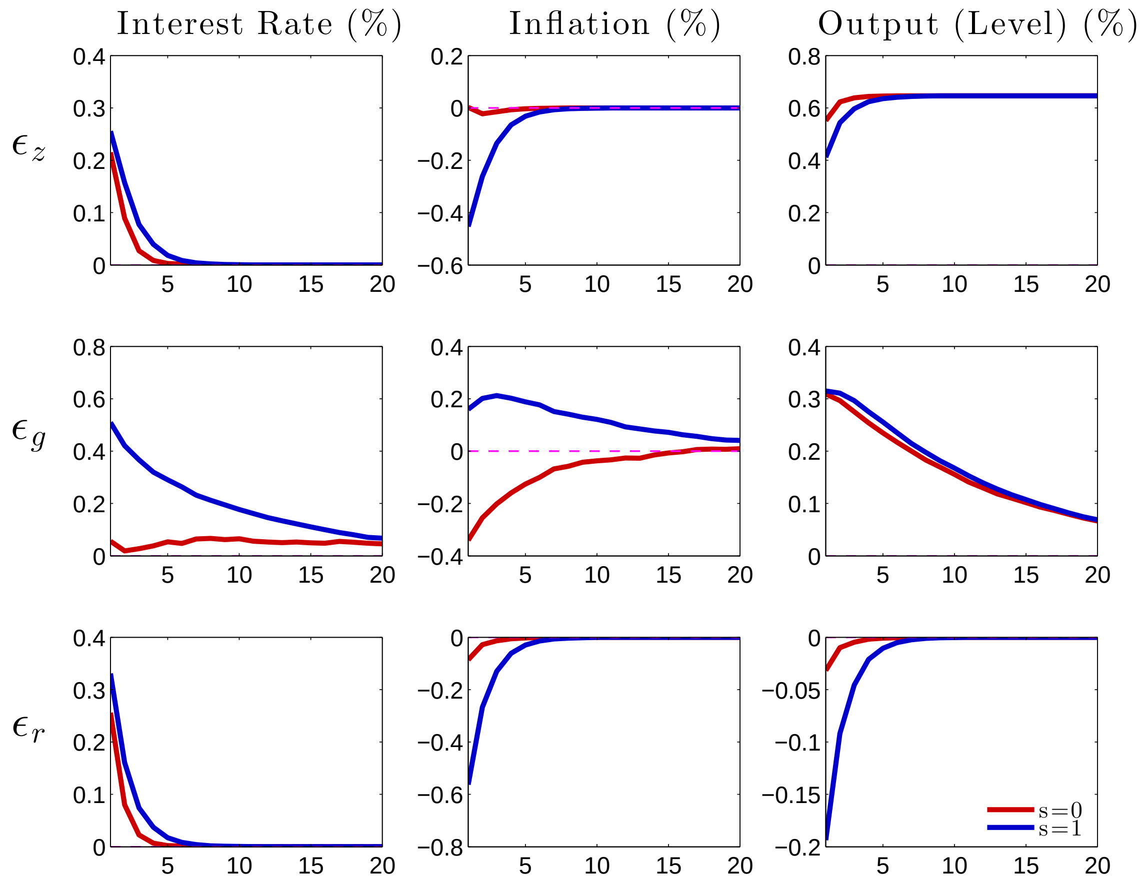
We estimate a second-order approximation of the DSGE model using the random walk Metropolis algorithm (RWM) described in An and Schorfheide (2007). To initialize the RWM chain we first estimate a log-linearized version of the DSGE model to obtain a covariance matrix for the proposal distribution. Using the posterior mode and the covariance matrix of the log-linearized model we then run the RWM algorithm using the particle filter to evaluate the likelihood of the non-linear model. The covariance matrix of the proposal distribution is scaled such that the RWM algorithm has an acceptance rate of approximately 50%. We use 50,000 particles to approximate the likelihood and set the variance of the measurement errors to 10% of the sample variance of the observables to help estimation. We obtain 100,000 draws of parameters from the posterior distribution. Summary statistics of the posterior distribution are based on the last 50,000 draws of the sequence.

B.7 Impulse Responses

We show the impulse response to one standard deviation shocks in Figure B-3. The variables are in columns and the three shocks, ϵ_z, ϵ_g and ϵ_r are in rows. Since these are responses from a nonlinear model, we need to explain how we run the experiments. We start with 100 draws from the ergodic distribution for the U.S., conditioning on $s = 0$ and $s = 1$ separately. Then using these 100 draws as initial states along with their associated s value, we compute the response of the economy to each shock relative to a baseline with no impulse. In both economies all exogenous variables evolve according to their stochastic processes and throughout the duration of the exercise the value of s remains the same. In the figure we report the point-wise median response in percentage units.

The responses when $s = 1$ is entirely standard: a technology shock increases output and interest rates and reduces inflation, a demand shock increases all there variables and a monetary policy shock increases interest rates and reduces inflation and output. With $s = 0$ a few significant differences emerge. First, the effect of shocks die out quicker. Second, monetary policy shocks have a very muted effect on inflation and output. Third, and most importantly, a positive government spending (demand) shock reduces inflation, as opposed to increase it as in when $s = 1$. This is because the aggregate demand curve becomes upward sloped when $s = 0$, similar

Figure B-3: Impulse Responses - U.S. Ergodic Distribution



to what Mertens and Ravn (2014) discuss.

B.8 Particle Filter For Sunspot Equilibrium

The particle filter is used to extract information about the state variables of the model from data on output growth, inflation, and nominal interest rates over the periods 1984:Q1 to 2013:Q4 (U.S.) and 1981:Q1 to 2013:Q4 (Japan).

B.8.1 State-Space Representation

Let d_t be the 3×1 vector of observables consisting of output growth, inflation, and nominal interest rates. The vector x_t stacks the continuous state variables, which are given by $x_t = [R_t, y_t, y_{t-1}, z_t, g_t, A_t]'$ and $s_t \in \{0, 1\}$, is the Markov-switching process.

$$d_t = \Psi(x_t) + \nu_t \tag{B.8.1}$$

$$\mathbb{P}\{s_t = 1\} = \begin{cases} (1 - p_{00}) & \text{if } s_{t-1} = 0 \\ p_{11} & \text{if } s_{t-1} = 1 \end{cases} \tag{B.8.2}$$

$$x_t = F_{s_t}(x_{t-1}, \epsilon_t) \tag{B.8.3}$$

The first equation is the measurement equation, where $\nu_t \sim N(0, \Sigma_\nu)$ is a vector of measurement errors. The second equation represents the law of motion of the Markov-switching process. The third equation corresponds to the law of motion of the continuous state variables. The vector $\epsilon_t \sim N(0, I)$ stacks the innovations $\epsilon_{z,t}$, $\epsilon_{g,t}$, and $\epsilon_{R,t}$. The functions $F_0(\cdot)$ and $F_1(\cdot)$ are generated by the model solution procedure. We subsequently use the densities $p(d_t|x_t)$, $p(s_t|s_{t-1})$, and $p(x_t|x_{t-1}, s_t)$ to summarize the measurement and the state transition equations.

B.8.2 Sequential Importance Sampling Approximation

Let $w_t = [x_t', s_t]'$ and $D_{t_0:t_1} = \{d_{t_0}, \ldots, d_{t_1}\}$. Particle filtering relies on sequential importance sampling approximations. The distribution $p(w_{t-1}|D_{1:t-1})$ is approximated by a set of pairs $\{(z_{t-1}^{(i)}, \pi_{t-1}^{(i)})\}_{i=1}^N$ in the sense that

$$\frac{1}{N}\sum_{i=1}^{N} f(w_{t-1}^{(i)})\pi_{t-1}^{(i)} \xrightarrow{a.s.} \mathbb{E}[f(w_{t-1})|D_{1:t-1}], \tag{B.8.4}$$

where $w_{t-1}^{(i)}$ is the i'th particle, $\pi_{t-1}^{(i)}$ is its weight, and N is the number of particles. An important step in the filtering algorithm is to draw a new set of particles for period t. In general, these particles are drawn from a distribution with a density that is proportional to $g(w_t|D_{1:t}, w_{t-1}^{(i)})$, which may depend on the particle value in period $t-1$ as well as the observation d_t in period t. This procedure leads to an importance sampling approximation of the form:

$$\begin{aligned}
\mathbb{E}[f(w_t)|D_{1:t}] &= \int_{w_t} f(w_t)\frac{p(d_t|w_t)p(w_t|D_{1:t-1})}{p(d_t|D_{1:t-1})}dw_t \qquad \text{(B.8.5)}\\
&= \int_{w_{t-1:t}} f(w_t)\frac{p(d_t|w_t)p(w_t|w_{t-1})p(w_{t-1}|D_{1:t-1})}{p(d_t|D_{1:t-1})}dw_{t-1:t}\\
&\approx \frac{\frac{1}{N}\sum_{i=1}^{N} f(w_t^{(i)})\frac{p(d_t|w_t^{(i)})p(w_t^{(i)}|w_{t-1}^{(i)})}{g(w_t^{(i)}|D_{1:t},w_{t-1}^{(i)})}\pi_{t-1}^{(i)}}{\frac{1}{N}\sum_{j=1}^{N} \frac{p(d_t|w_t^{(j)})p(w_t^{(j)}|w_{t-1}^{(j)})}{g(w_t^{(j)}|D_{1:T},w_{t-1}^{(i)})}\pi_{t-1}^{(i)}}\\
&= \frac{1}{N}\sum_{i=1}^{N} f(w_t^{(i)})\left(\frac{\tilde{\pi}_t^{(i)}}{\frac{1}{N}\sum_{j=1}^{N}\tilde{\pi}_t^{(j)}}\right) = \frac{1}{N}\sum_{i=1}^{N} f(w_t^{(i)})\pi_t^{(i)},
\end{aligned}$$

where the unnormalized and normalized probability weights are given by

$$\tilde{\pi}_t^{(i)} = \frac{p(d_t|w_t^{(i)})p(w_t^{(i)}|w_{t-1}^{(i)})}{g(w_t^{(i)}|D_{1:T}, w_{t-1}^{(i)})}\pi_{t-1}^{(i)} \quad \text{and} \quad \pi_t^{(i)} = \frac{\tilde{\pi}_t^{(i)}}{\sum_{j=1}^{N}\tilde{\pi}_t^{(j)}}, \tag{B.8.6}$$

respectively. In simple versions of the particle filter, $w_t^{(i)}$ is often generated by simulating the model forward, which means that $g(w_t^{(i)}|D_{1:T}, w_{t-1}^{(i)}) \propto p(w_t^{(i)}|w_{t-1}^{(i)})$, and the formula for the particle weights simplifies considerably. Unfortunately, this approach is quite inefficient in our application, and we require a more elaborate density $g(\cdot|\cdot)$ described below that accounts for information in d_t. The resulting extension of the particle filter is known as auxiliary particle filter, e.g. Pitt and Shephard (1999).

B.8.3 Filtering

Initialization. To generate the initial set of particles $\{(w_0^{(i)}, \pi_0^{(i)}\}_{i=1}^N$, for each i, simulate the DSGE model for T_0 periods, starting from the targeted-inflation steady state, and set $\pi_0^{(i)} = 1$.

Sequential Importance Sampling. For $t = 1$ to T:

1. $\{w_{t-1}^{(i)}, \pi_{t-1}^{(i)}\}_{i=1}^N$ is the particle approximation of $p(w_{t-1}|D_{1:t-1})$. For $i = 1$ to N:

    (a) Draw $w_t^{(i)}$ conditional on $w_{t-1}^{(i)}$ from $g(w_t|D_{1:t}, w_{t-1}^{(i)})$.

    (b) Compute the unnormalized particle weights $\tilde{\pi}_t^{(i)}$ according to (B.8.6).

2. Compute the normalized particle weights $\pi_t^{(i)}$ and the effective sample size $ESS_t = N^2/\sum_{i=1}^N (\pi_t^{(i)})^2$.

3. Resample the particles via deterministic resampling (see Kitagawa (1996)). Reset weights to be $\pi_t^{(i)} = 1$ and approximate $p(w_t|D_{1:t})$ by $\{(w_t^{(i)}, \pi_t^{(i)})\}_{i=1}^n$.

B.8.4 Tuning of the Filter

In the empirical analysis, we set $T_0 = 50$ and $N = 1,000,000$. We also fix the measurement error variance for output growth, inflation, and interest rates to be equal to 10% of the sample variance of these series. We assume that the economies are in the targeted-inflation regime during the initialization period. Since our model has discrete and continuous state variables, we write

$$p(w_t|w_{t-1}) = \begin{cases} p_0(x_t|x_{t-1}, s_t = 0)\mathbb{P}\{s_t = 0|s_{t-1}\} & \text{if } s_t = 0 \\ \\ p_1(x_t|x_{t-1}, s_t = 1)\mathbb{P}\{s_t = 1|s_{t-1}\} & \text{if } s_t = 1 \end{cases}$$

and consider proposal densities of the form

$$q(w_t|w_{t-1}, d_t) = \begin{cases} q_0(x_t|x_{t-1}, d_t, s_t = 0)\lambda(w_{t-1}, d_t) & \text{if } s_t = 0 \\ \\ q_1(x_t|x_{t-1}, d_t, s_t = 1)(1 - \lambda(w_{t-1}, d_t)) & \text{if } s_t = 1 \end{cases},$$

where $\lambda(x_{t-1}, d_t)$ is the probability that $s_t = 0$ under the proposal distribution. We use $q(\cdot)$ instead of $g(\cdot)$ to indicate that the densities are normalized to integrate to one.

We effectively generate draws from the proposal density through forward iteration of the state transition equation. To adapt the proposal density to the observation d_t, we draw $\epsilon_t^{(i)} \sim N(\mu^{(i)}, \Sigma^{(i)})$ instead of the model-implied $\epsilon_t \sim N(0, I)$. In slight abuse of notation (ignoring that the dimension of x_t is larger than the dimension of ϵ_t and that its distribution is singular), we can apply the change of variable formula to obtain a representation of the proposal density

$$q(x_t^{(i)}|x_{t-1}^{(i)}) = q_\epsilon(F^{-1}(x_t^{(i)}|x_{t-1}^{(i)})) \left| \frac{\partial F^{-1}(x_t^{(i)}|x_{t-1}^{(i)})}{\partial x_t} \right|$$

Using the same change-of-variable formula, we can represent

$$p(x_t^{(i)}|x_{t-1}^{(i)}) = p_\epsilon(F^{-1}(x_t^{(i)}|x_{t-1}^{(i)})) \left| \frac{\partial F^{-1}(x_t^{(i)}|x_{t-1}^{(i)})}{\partial x_t} \right|$$

By construction, the Jacobian terms cancel and the ratio that is needed to calculate the unnormalized particle weights for period t in (B.8.6) simplifies to

$$\tilde{\pi}_t^{(i)} = p(d_t|w_t^{(i)}) \frac{\exp\left\{-\frac{1}{2}\epsilon_t^{(i)\prime}\epsilon_t^{(i)}\right\}}{|\Sigma_\epsilon^{(i)}|^{-1/2}\exp\left\{-\frac{1}{2}(\epsilon_t^{(i)} - \mu^{(i)})'[\Sigma^{(i)}]^{-1}(\epsilon_t^{(i)} - \mu^{(i)})\right\}} \pi_{t-1}^{(i)}.$$

The choice of μ and Σ is described below.

Let $w_{t-1|t-1}$ be a particle filter approximation of $\mathbb{E}[w_{t-1}|D_{1:t-1}]$ and define

$$\bar{\pi}_t(\epsilon_t) = p(d_t|F(w_{t-1|t-1}, \epsilon_t)) \exp\left\{-\frac{1}{2}\epsilon_t'\epsilon_t\right\} |\Sigma_\epsilon|^{1/2}\pi_{t-1}^{(i)}.$$

We use a grid search over ϵ_t to determine a value $\bar{\epsilon}$ that maximizes this objective function and then set $\mu^{(i)} = \bar{\epsilon}$. Moreover, we let $\Sigma^{(i)} = I$. (Executing the grid search conditional on each $w_{t-1}^{(i)}$, $i = 1, \ldots, N$ turned out to be too time consuming.)

B.8.5 Filtered Shocks

The filtered innovations are summarized in Figure B-4 and Figure B-5. The shaded area indicates time periods in which the economy is in the deflation regime. The vertical red line indicates the end of the estimation sample.

Figure B-4: Filtered Shocks: U.S. 1984:Q1 - 2013:Q4

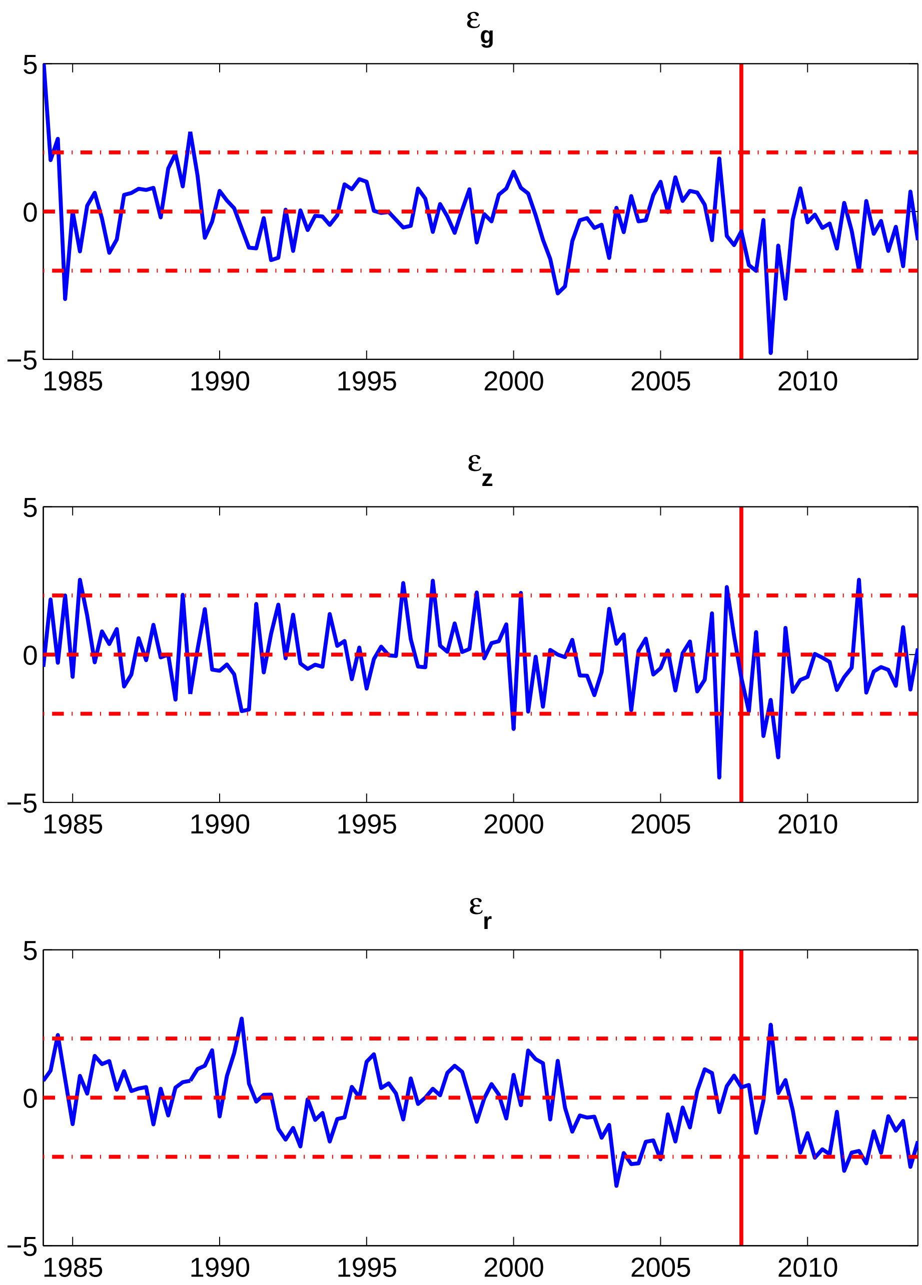
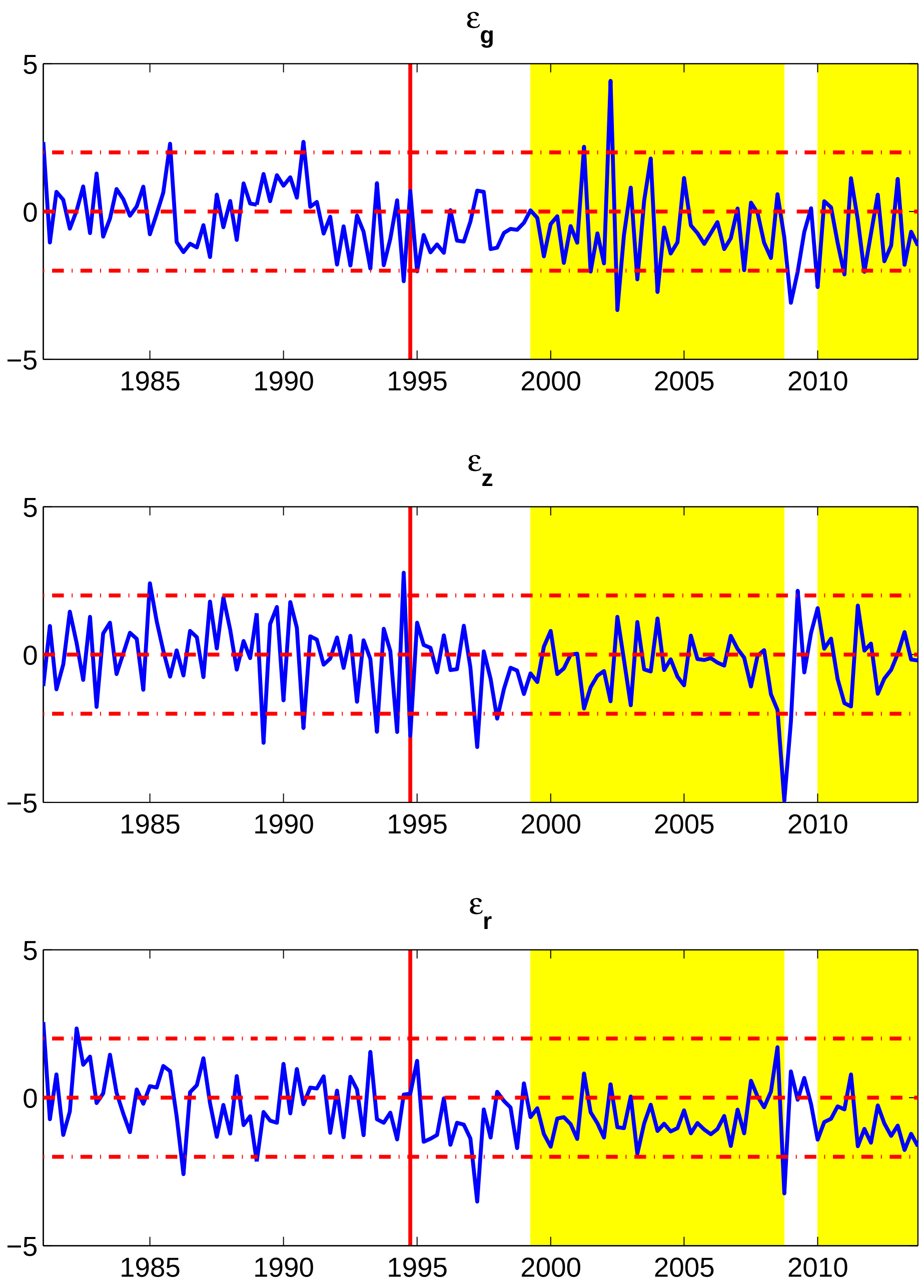


Figure B-5: Filtered Shocks: Japan 1981:Q1 - 2013:Q4



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