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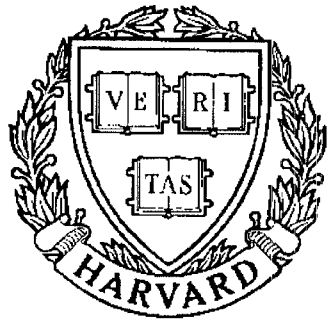

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Frames Generated By Subspace Addition

by Y.C. Pati

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Frames Generated By Subspace Addition *

Y. C. Pati †

Abstract

Given two subspaces M and N of a Hilbert space, and frames associated with each of the subspaces, the question addressed in this report is that of determining when the union of the two frames is a frame for the direct sum space $M \oplus N$. We provide sufficient conditions for the union of the two frames to be a frame for $M \oplus N$ and also estimates for the frame bounds. The results discussed here are given in terms of the relative geometry of subspaces. Some simple examples in which the frame bounds can be expicitly computed are provided to demonstrate accuracy of the frame bound estimates.

1 Introduction

Any vector in a seperable Hilbert space can be expanded via an orthonormal basis, i.e. written as a linear combination of the basis elements. Thus orthonormal bases provide orthogonal decompositions of Hilbert spaces. As a natural generalization of orthonormal bases, Duffin and Schaeffer [3] introduced the concept of *frames* for Hilbert spaces. Frames define non-orthogonal decompositions of a Hilbert space. Any vector in a Hilbert space can be written as a linear combination of the frame elements where the expansion coefficients can be computed via the frame operator (see Section 2). In most applications of frame decompositions (e.g. wavelet transforms) the approach has been to start with a frame for the space of interest and then use the frame to decompose vectors in the space. Here we consider the situation where we start with frames for subspaces of a larger Hilbert space and construct a frame for the sum of two subspaces by taking unions of the subspace frames. Succesive approximation schemes (c.f. [6]) in which approximations to functions are successively refined provide examples of applications in which this approach is useful.

We show in Section 3 that given subspaces M and N of a Hilbert space, and frames associated with each of the subspaces, the union of the two frames is a frame for $M \oplus N$ whenever the minimum angle, θ_m between the two subspaces is bounded away from zero. We also show that the lower frame bound for the union of the two frames can be bounded below in terms of the quantity $(1 - \cos\theta_m)$, and bounded above by the minimum of the lower frame bounds associated with the frames for M and N individually.

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†Electrical Engineering Department and Systems Research Center, University of Maryland, College Park, MD 20742 and Nanoelectronics Processing Facility, Code 6804, Naval Research Laboratories, Washington D.C.

2 Hilbert Space Frames

Definition 2.1 *Given a Hilbert space $\mathcal{H}$ and a sequence of vectors $\{h_n\}_{n\in\mathbb{Z}} \subset \mathcal{H}$, $\{h_n\}$ is called a frame if there exists constants $A > 0$ and $B < \infty$ such that*

$$A\|f\|^2 \leq \sum_n | < f, h_n > |^2 \leq B\|f\|^2, \tag{1}$$

for every $f \in \mathcal{H}$. A and B are respectively the lower and upper frame bounds.

Remarks:

- A frame $\{h_n\}$ with frame bounds $A = B$ is called a *tight frame.*
- Every orthonormal basis is a tight frame with $A = B = 1$.
- A tight frame of normalized vectors for which $A = B = 1$ is an orthonormal basis.

Given a frame $\{h_n\}$ in the Hilbert space $\mathcal{H}$, with frame bounds A and B, the frame operator $S : \mathcal{H} \to \mathcal{H}$ can be defined as follows. For any $f \in \mathcal{H}$,

$$Sf = \sum_n < f, h_n > h_n. \tag{2}$$

We list in the form of a theorem some properties of the frame operator which we shall find useful. Proof of these and other related properties of frames can be found in [5] and/or [2].

Theorem 2.1

(i) S is a bounded linear operator with $AI \leq S \leq BI$, where I is the identity operator in $\mathcal{H}$.

(ii) S is an invertible operator with $B^{-1}I \leq S^{-1} \leq A^{-1}I$.

(iii) The sequence $\{S^{-1}h_n\}$ is also a frame, called the dual frame, *with frame bounds B^{-1} and A^{-1}.*

(iv) Given any $f \in \mathcal{H}$, f can be written in terms of the frame elements as

$$f = \sum < f, S^{-1}h_n > h_n = \sum < f, h_n > S^{-1}h_n. \tag{3}$$

(v) Given $f \in \mathcal{H}$, if there exists another sequence of coefficients $\{a_n\}$ (other than the sequence $\{< f, S^{-1}h_n >\}$) such that $f = \sum a_n h_n$, then the a_n's are related to the coefficients given in (3) by the formula,

$$\sum |a_n|^2 = \sum | < f, S^{-1}h_n > |^2 + \sum | < f, S^{-1}h_n > - a_n|^2. \tag{4}$$

3 Frames Generated by Subspace Addition

Definition 3.1 *Given two subspaces M and N of a Hilbert space $\mathcal{H}$, the smallest angle θ_m between M and N is defined as,*

$$\cos\theta_m = \sup_{\substack{x\in M \\ \|x\|=1}} \sup_{\substack{y\in N \\ \|y\|=1}} |< x, y >|,$$

where $\theta_m \in [0, \pi/2]$.

We shall make use of the following Theorem in proving the main result (Theorem 3.2) of this section.

Theorem 3.1 *Let M and N be subspaces of a Hilbert space $\mathcal{H}$ with $M \cap N = \{0\}$. And let P_M and P_N be orthogonal projectors on M and N respectively. Then*

$$\inf_{x\in M\oplus N,\ \|x\|\neq 0} \frac{\|(P_M - P_N)x\|}{\|x\|} = \sin\theta_m.$$

Proof: See [7] ∎

Theorem 3.2 *Let M and N be nontrivial, closed subspaces of a Hilbert space $\mathcal{H}$. Let $\{x_j\}$ be a frame for M with frame bounds A_M and B_M, and $\{y_j\}$ be a frame for N with frame bounds A_N and B_N. Define $Q = \mathrm{Span}(\{x_j\} \bigcup \{y_j\})$.*

Let θ_m denote the minimum angle between the subspaces M and N.

Then if, $\theta_m > 0$ $\{x_j, y_j\}$ is a frame for Q with frame bounds

$$\begin{aligned} A_Q &\geq \tilde{A}_Q = \min(A_M, A_N)(1 - \cos\theta_m) \\ B_Q &\leq \tilde{B}_Q = \max(B_M, B_N)\min(2, \frac{1}{1-\cos\theta_m}) \end{aligned} \tag{5}$$

Proof:

Note that $M \cap N = 0$ by the hypothesis that $\theta_m > 0$. So $Q = M \oplus N$ and thus if $g \in Q$ there is a unique decomposition of g as $g = x + y$ where $x \in M$ and $y \in N$.

Lower Frame Bound:

Take $g \in Q$ and let P_M and P_N be orthogonal projection operators onto M and N respectively.

$$\begin{aligned} \sum_j |< g, x_j >|^2 + |< g, y_j >|^2 &= \sum_j |< P_M g, x_j >|^2 + |< P_N g, y_j >|^2 \\ &\geq A_M\|P_M g\|^2 + A_N\|P_N g\|^2 \\ &\geq \min(A_M, A_N)\left\{\|P_M g\|^2 + \|P_N g\|^2\right\} \end{aligned} \tag{6}$$

Now,

$$\begin{aligned} \|P_M g\|^2 + \|P_N g\|^2 &= \|P_M g - P_N g\|^2 + 2\mathrm{Re}\,\langle P_M g, P_N g\rangle \\ &\geq \left[\inf_g \frac{\|(P_M - P_N)g\|}{\|g\|}\right]^2 \|g\|^2 - 2\,|\langle P_M g, P_N g\rangle| \\ &= \sin^2\theta_m\|g\|^2 - 2\,|\langle P_M g, P_N g\rangle| \\ &\geq \sin^2\theta_m\|g\|^2 - 2\|P_M g\|\|P_N g\|\cos\theta_m \\ &\geq \sin^2\theta_m\|g\|^2 - \left(\|P_M g\|^2 + \|P_N g\|^2\right)\cos\theta_m \end{aligned} \tag{7}$$

Therefore,

$$\left(\|P_M g\|^2 + \|P_N g\|^2\right)(1 + \cos\theta_m) \geq \sin^2\theta_m \|g\|^2. \tag{8}$$

Or equivalently,

$$\|P_M g\|^2 + \|P_N g\|^2 \geq \frac{\sin^2\theta_m}{1+\cos\theta_m}\|g\|^2 = \frac{1-\cos^2\theta_m}{1+\cos\theta_m}\|g\|^2 = (1-\cos\theta_m)\|g\|^2 \tag{9}$$

Thus from (6) and (9) we get

$$\sum_j |< g, x_j >|^2 + |< g, y_j >|^2 \geq \min(A_M, A_N)(1-\cos\theta_m)\|g\|^2$$

Upper Frame Bound:

$$\begin{aligned}\sum_j |< g, x_j >|^2 + |< g, y_j >|^2 &\leq B_M\|P_M g\|^2 + B_N\|P_N g\|^2 \\ &\leq B_M\|g\|^2 + B_N\|g\|^2 \leq 2\max(B_M, B_N)\|g\|^2 \end{aligned} \tag{10}$$

Also,

$$\begin{aligned}\|P_M g\|^2 + \|P_N g\|^2 &= \|P_M g - P_N g\|^2 + 2\mathrm{Re}\,\langle P_M g, P_N g\rangle \\ &\leq \|g\|^2 + 2\mathrm{Re}\,\langle P_M g, P_N g\rangle \\ &\leq \|g\|^2 + 2\|P_M g\|\|P_N g\|\cos\theta_m \\ &\leq \|g\|^2 + \left(\|P_M g\|^2 + \|P_N g\|^2\right)\cos\theta_m \end{aligned} \tag{11}$$

Thus,

$$\left(\|P_M g\|^2 + \|P_N g\|^2\right) \leq \frac{1}{1-\cos\theta_m}\|g\|^2. \tag{12}$$

So we also have,

$$\sum_j |< g, x_j >|^2 + |< g, y_j >|^2 \leq \max(B_M, B_N)\frac{1}{1-\cos\theta_m}\|g\|^2 \tag{13}$$

Therefore (by (10) and (13)),

$$B_Q \leq \widetilde{B}_Q = \max(B_M, B_N)\min(2, \frac{1}{1-\cos\theta_m})$$

■

It should be noted that the conditions under which Theorem 3.2 guarantees the union of two frames to be a frame for their combined span, are only sufficient conditions. In fact in all finite-dimensional cases, these conditions are not necessary. In these cases, estimates of the frame bounds can be made using knowledge of the correlations ($\langle x_i, x_j\rangle$) among the frame elements themselves. However as we show by example in the next section, in an infinite-dimensional setting, the union of two frames can fail to form a frame if $\theta_m = 0$.

As can be seen from Equation (5), the estimate $\widetilde{A}_Q$ of Theorem 3.2 for the lower frame bound A_Q is always less than or equal to $\min(A_M, A_N)$. We now show that the actual lower frame bound A_Q must indeed be less than or equal to $\min(A_M, A_N)$. To show this we first prove the following lemma.

Lemma 3.1 *Let M and N be nontrivial, closed subspaces of a Hilbert space $\mathcal{H}$, $M \cap N = \{0\}$. Define $Q = M \oplus N$. Then $\forall x^* \in M$, $\exists\ g^* \in Q$ such that*

$$P_M g^* = x^* \quad \text{and} \quad P_N g^* = 0. \tag{14}$$

In particular,

$$g^* = (I - P_N)(I - P_M P_N)^{-1} x^*,$$

satisfies (14).

Proof: First note that $\forall x \in Q$, $(I - P_N)x \in N^\perp \cap Q$ Secondly since $\|P_M P_N\| < 1$, $(I - P_M P_N)^{-1}$ exists and is given by

$$(I - P_M P_N)^{-1} x = \sum_{k=0}^{\infty} (P_M P_N)^k x. \tag{15}$$

For any $x \in M$, $(I - P_M P_N)^{-1} x \in M$ since every term of the series in the right hand side of Equation (15) is in M and M is closed. Now let $x = (I - P_M P_N)^{-1} x^*$ and let $g^* = (I - P_N)x$. Clearly $P_N g^* = 0$. Also,

$$\begin{aligned} P_M g^* &= P_M (I - P_N)(I - P_M P_N)^{-1} x^* \\ &= P_M (I - P_M P_N)^{-1} x^* - P_M P_N (I - P_M P_N)^{-1} x^* \\ &= (I - P_M P_N)^{-1} x^* - P_M P_N (I - P_M P_N)^{-1} x^* \quad (\text{since } (I - P_M P_N)^{-1} x^* \in M) \\ &= (I - P_M P_N)(I - P_M P_N)^{-1} x^* = x^* \end{aligned} \tag{16}$$

■

Theorem 3.3 *Let M and N be subspaces of a Hilbert space $\mathcal{H}$. Let $\{x_j\}$ be a frame for M with frame bounds A_M and B_M , and $\{y_j\}$ be a frame for N with frame bounds A_N and B_N. Let $\theta_m > 0$ denote the minimum angle between the subspaces M and N and define $Q = M \oplus N$. Then if A_Q is the lower frame bound for the frame $\{x_j\} \bigcup \{y_j\}$ of Q,*

$$A_Q \leq \min(A_M, A_N).$$

Proof:

Without loss of generality, assume $A_M \leq A_N$. Thus since A_M is the lower frame bound for the frame $\{x_j\}$ of M, for any $\epsilon > 0$, $\exists\ x^* \in M$ such that

$$\sum |\langle x^*, x_j \rangle|^2 \leq (A_M + \epsilon)\|x^*\|^2.$$

By Lemma 3.1 $\exists\ g^* \in Q$ such that $P_M g^* = x^*$ and $P_N g^* = 0$. Therefore,

$$\begin{aligned} &\sum |\langle g^*, x_j \rangle|^2 + \sum |\langle g^*, y_j \rangle|^2 \\ &\qquad = \sum |\langle P_M g^*, x_j \rangle|^2 + \sum |\langle P_N g^*, y_j \rangle|^2 \\ &\qquad = \sum |\langle x^*, x_j \rangle|^2 + 0 \\ &\qquad \leq (A_M + \epsilon)\|x^*\|^2 \leq (A_M + \epsilon)\|g^*\|^2 \end{aligned}$$

Thus $A_Q \leq A_M + \epsilon$ and since $\epsilon > 0$ is arbritary we have that $A_Q \leq A_M = \min(A_M, A_N)$. ■

4 Examples

In this section we consider a few finite-dimensional examples in which the frame bounds can be explicitly computed. The general methodology in these examples is as follows.

Let $T : \mathcal{H} \to l^2$ be defined such that $T : f \to \{< f, x_j >\}$ where $\{x_j\}$ is a frame for $\mathcal{H}$. Therefore the frame operator $S = T^*T$. If we let $\{e_j\}$ be an orthonormal basis for $\mathcal{H}$ then the matrix representation of T with repect to this basis is given by

$$W = [w_{ij}] = [\langle x_i, e_j \rangle] .$$

Hence the upper and lower frame bounds are given by the upper and lower spectral limits of W^*W. In the finite dimensional case, the frame bounds can be computed as the maximum and minimum eigenvalues of W^*W or equivalently, the squares of the maximum and minimum singular values of T.

Example I: A frame for $\mathbb{R}^2$ from frames for 1-D subspaces

Let, $x = (1,0)^T$ and $y = (\sin\theta, \cos\theta)^T$. Define $M = \text{Span}\{x\}$, $N = \text{Span}\{y\}$; so x is a frame for M with frame bounds $A_M = B_M = 1$ and y is a frame for N with frame bounds $A_N = B_N = 1$. In this case $\theta_m = \theta$. Clearly for any angle $\theta > 0$, $\text{Span}\{x, y\}=\mathbb{R}^2 = Q$. Using the standard orthonormal basis for $\mathbb{R}^2$, we get

$$W = \begin{bmatrix} 1 & 0 \\ \sin\theta & \cos\theta \end{bmatrix}$$

Hence

$$\begin{aligned} A_Q &= \lambda_{\min}(W^*W) &&= 1 - \cos\theta \\ B_Q &= \lambda_{\max}(W^*W) &&= 1 + \cos\theta \end{aligned}$$

Since here the lower frame bound is equal to the lower frame bound estimate of Theorem 3.2, in this case Theorem 3.2 provides both necessary and sufficient conditions. Figure 1 shows the actual upper and lower frame bounds for this example.

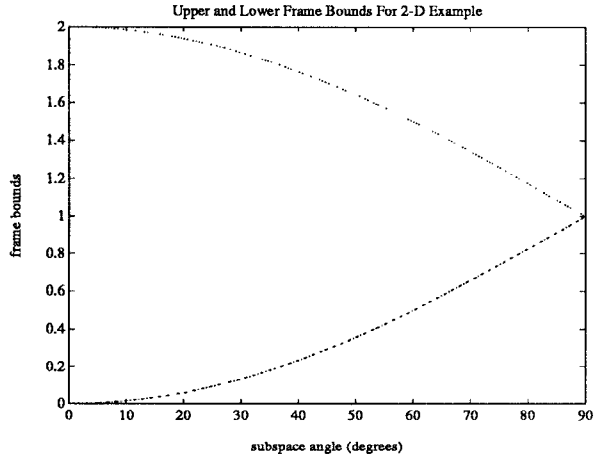

Figure 1: Actual upper and lower frame bounds for two-dimensional example

Example II: A Frame for $\mathbb{R}^3$ from frames for $\mathbb{R}^2$ and $\mathbb{R}$

Let $x_1 = (1,0,0)^T$, $x_2 = (0,1,0)^T$ and $y = (\cos\omega\cos\theta, \sin\omega\cos\theta, \sin\theta)^T$. Let $M = \text{Span}\{x_1, x_2\}$ and $N = \text{Span}\{y\}$. Here $\theta_m = \theta$. So for any $\theta > 0$ $M \oplus N = \mathbb{R}^3$. For this example,

$$W = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \cos\omega\cos\theta & \sin\omega\cos\theta & \sin\theta \end{bmatrix}$$

In this example as well, we have,

$$\begin{aligned} A_Q &= \lambda_{\min}(W^*W) = 1 - \cos\theta \\ B_Q &= \lambda_{\max}(W^*W) = 1 + \cos\theta \end{aligned}$$

Example III: Other Frames for $\mathbb{R}^3$ from frames for $\mathbb{R}^2$ nd $\mathbb{R}$.

Let $x_1 = (1,0,0)^T$, $x_2 = (\cos\gamma, \sin\gamma, 0)^T$ and $y = (\cos\omega\cos\theta, \sin\omega\cos\theta, \sin\theta)^T$. Let $M = \text{Span}\{x_1, x_2\}$ and $N = \text{Span}\{y\}$. Here $A_N = 1$, $A_M = 1 - \cos\gamma$ and $\theta_m = \theta$. So for any $\theta > 0$ $M \oplus N = \mathbb{R}^3$. So,

$$W = \begin{bmatrix} 1 & 0 & 0 \\ \cos\gamma & \sin\gamma & 0 \\ \cos\omega\cos\theta & \sin\omega\cos\theta & \sin\theta \end{bmatrix}$$

In this case, analytical expressions for the eigenvalues of W^*W are quite complicated. Therefore we shall demonstrate the lower frame bounds numerically for a few values of γ and ω. Figures 2–6 each show for a particular value of γ, the actual lower frame bounds and the estimate provided by Theorem 3.2 for different values of ω. It can be seen that the lower frame

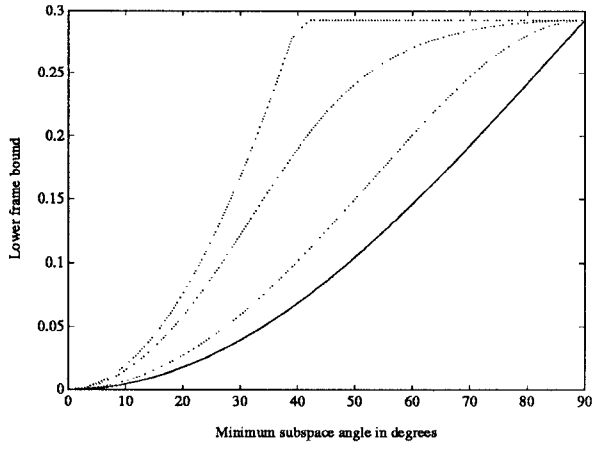

Figure 2: Example III with $\gamma = \pi/4$. Solid Line: Lower frame bound estimate; Dashed lines: Actual lower frame bound for different values of ω

bound estimate becomes increasingly accurate as γ approaches $\pi/2$. For small γ the estimate is quite conservative for certain values of ω, however in these cases, there also exist values of ω for which the lower frame bound is close to the estimate of Theorem 3.2. In this loose sense, the estimate of Theorem 3.2 is as good an estimate as can be derived using knowledge of the minimum subspace angle alone.

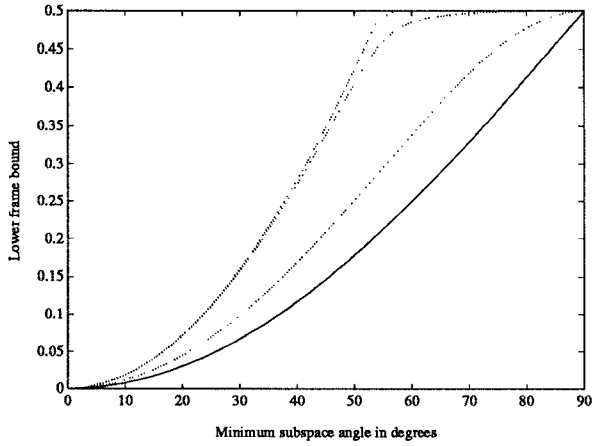

Figure 3: Example III with $\gamma = \pi/3$. Solid Line: Lower frame bound estimate; Dashed lines: Actual lower frame bound for different values of ω

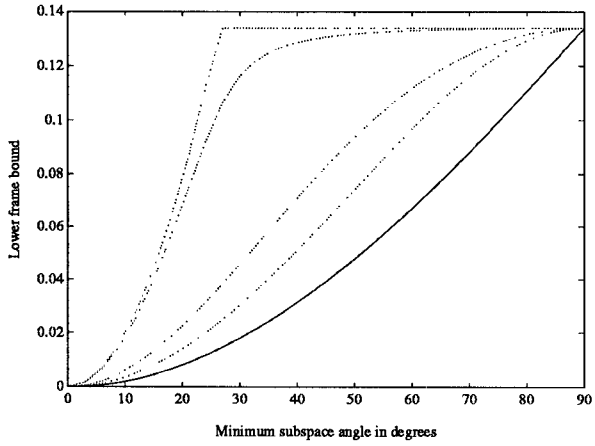

Figure 4: Example III with $\gamma = \pi/6$. Solid Line: Lower frame bound estimate; Dashed lines: Actual lower frame bound for different values of ω

Example IV: Violation of Lower Frame Bound when $\theta_m = 0$

By this infinite-dimensional example (which can be found in [4]) we show that the lower frame bound can indeed be zero in the case where $\theta_m = 0$.

Let $\{e_j\}$ be the standard orthonormal basis for l^2 and let $\psi_j = e_{3j}$, $\phi_j = \sqrt{1 - 1/j}\, e_{3j} + \sqrt{1/j}\, e_{3j+1}$. Thus the sequences $\{\psi_j\}$, and $\{\phi_j\}$ are orthonormal sequences and thereby frames for their respective closed spans. However if we consider the union of the two frames and take $e_{3k+1} \in \text{Span}\{\psi_j,\ \phi_j\}$ as a test vector, it is easily seen that

$$\sum_j |\langle e_{3k+1}, \psi_j \rangle|^2 + \sum_j |\langle e_{3k+1}, \phi_j \rangle|^2 = \frac{1}{k} = \frac{1}{k}\|e_{3k+1}\|^2.$$

Hence since $\frac{1}{k} \to 0$ as $k \to \infty$, the sequence $\{\psi_j,\ \phi_j\}$ is not a frame for its span.

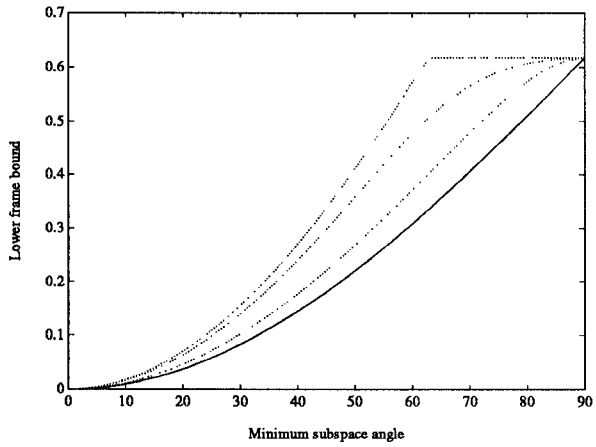

Figure 5: Example III with $\gamma = 3\pi/8$. Solid Line: Lower frame bound estimate; Dashed lines: Actual lower frame bound for different values of ω

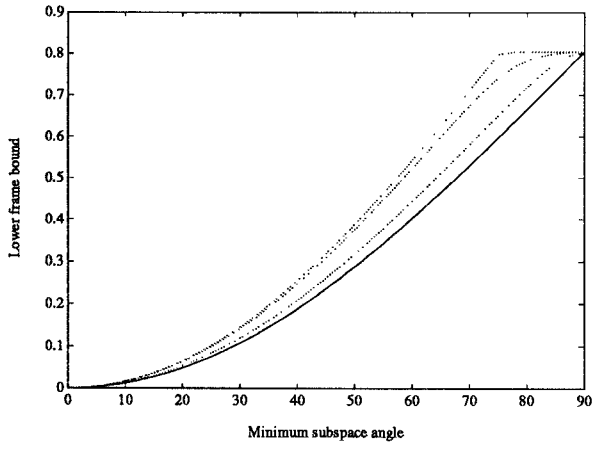

Figure 6: Example III with $\gamma = 7\pi/16$. Solid Line: Lower frame bound estimate; Dashed lines: Actual lower frame bound for different values of ω

5 Conclusions

In this report we have provided a geometric characterization of conditions which guarantee that the union of two frames is a frame for the appropriate direct sum space. The main result of this report is contained in Theorem 3.2 which says that given frames for subspaces M and N, the union of the frames is a frame for the direct sum space $M \oplus N$ provided that the minimum angle between the two subspaces is bounded away from zero. An estimate for the lower frame bound can be made in terms of the quantity $1 - \cos\theta_m$. As mentioned in Section 4, the lower frame bound estimate in Theorem 3.2 is in a sense the best estimate which can be made using the minimum subspace angle alone. Furthermore, we have shown that the lower frame bound is nonincreasing with respect to the lower frame bounds for the original subspaces.

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References

[1] Ake Bjorck and Gene H. Golub. Numerical methods for computing angles betweeen linear subspaces. *Mathematics of Computation*, 27(123), July 1973.

[2] Ingrid Daubechies. The wavelet transform, time-frequency localization and signal analysis. *IEEE Transactions on Information Theory*, 36(5), September 1990.

[3] R. J. Duffin and A. C. Schaeffer. A class of nonharmonic fourier series. *Trans. Amer. Math. Soc.*, 72:341–366, 1952.

[4] J. Gillis. Reconstruction of stochastic processes using frames. Technical Report SRC-TR-91-14, University of Maryland, Systems Research Center, College Park, MD, 1991.

[5] Christopher E. Heil and David F. Walnut. Continuous and discrete wavelet transforms. *SIAM Review*, 31(4):628–666, December 1989.

[6] Y. C. Pati and P. S. Krishnaprasad. In preparation.

[7] G. W. Stewart. Error and perturbation bounds for subspaces associated with certain eigenvalue problems. *SIAM Review*, 15(4), October 1973.