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Topological Via Minimization Problem**

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Abstract

Suppose that we are given a two-layer routing area bounded by a closed continuous curve B , a set of terminals placed on B which are available on both layers, and a set of two terminal-nets. The topological via minimization problem is the problem of routing the nets by zero-width wires such that no two wires corresponding to different nets intersect on the same layer and the number of vias used is minimized. Very recently, it was reported that this problem is NP-hard but the proof contains a critical flaw. In this paper, we present a correct NP-hardness proof of the problem.

1. Introduction

Since the introduction of *channel routing* by Hashimoto and Stevens [9] in 1971, the problem of minimizing the number of vias between conductors on different layers has extensively been studied [1–3,5,10–12,14–19]. In the traditional via minimization problem, the pattern of wire segments are already determined, and the problem is to reassign those wire segments to different layers so as to minimize the number of vias. For this case, complete complexity results have been obtained. Namely, the problem is NP-hard [8] when the maximum junction degree is four or more [5] and is polynomially solvable when it is less than four [3,12,15], where the *junction degree* is the number of wire segments which meet at a single point and which are to be electrically connected.

For the case of the so-called *topological via minimization* (TVM) problem, the situation has been different. In this problem, we are given a two-layer routing area bounded by a closed continuous curve B , a set T of terminals placed on B , and a set N of two-terminal nets. All terminals are assumed to be available on both layers. The problem is to route the nets in N by zero-width wires such that no two wires realizing different nets intersect on the same layer and the number of vias used is minimized.

Hsu [10] introduced this problem in 1983, suspected its NP-hardness, and proposed a heuristic algorithm. Later, Marek-Sadowska [14] claimed that the problem is NP-hard. Based on her claim Chang and Du [1] concluded that the “minimum vertex deletion bipartite subgraph problem” is NP-hard even for circle graphs. The authors noticed the incorrectness of her proof as pointed out in [16]. Since then the complexity issue of the TVM problem has been sought after and very recently Sarrafzadeh and

Lee [18] reported that the problem is NP-hard. Unfortunately, their proof contains a critical flaw. In this paper, we present a correct NP-hardness proof of the TVM problem.

2. Preliminaries

Let $G = (V, E)$ be an undirected graph where V and E are the sets of vertices and edges, respectively. Two distinct vertices v and w are said to be *adjacent* to (resp., *independent* from) each other if $(v, w) \in E$ (resp., $(v, w) \notin E$). If $(v, w) \in E$, the edge is said to be *incident upon* v and w . A subset V' of V is called an *independent set* of G if any two vertices in V' are independent. A sequence of distinct vertices $[v_{i_1}, v_{i_2}, \dots, v_{i_r}]$ is called a *cycle* of length r if $(v_{i_p}, v_{i_{p+1}}) \in E$ for $p = 1, 2, \dots, r - 1$ and $(v_{i_r}, v_{i_1}) \in E$. If r is odd (resp., even), it is called an *odd* (resp., *even*) *cycle*.

A graph $G = (V, E)$ is called a *planar graph* if it can be drawn in the plane in such a way that (i) each vertex in V is represented by a point, (ii) each edge $(u, v) \in E$ is represented by a continuous line connecting the two points which represent u and v , and (iii) no two lines, which represent edges, share any point, except in their ends. Such a drawing is called a *planar embedding* of G and is denoted by \hat{G} . A graph $G = (V, E)$ is called a *bipartite graph* if V can be partitioned into two nonempty independent subsets V_1 and V_2 such that $V_1 \cap V_2 = \phi$. It is well known [6] that a graph is bipartite if and only if there is no odd cycle in it.

Let S be a set of chords on a circle. A graph $G = (V, E)$ is called a *circle graph* for S if there is a one-to-one correspondence between V and S such that two vertices in V are adjacent if and only if their corresponding chords in S intersect. We denote

by $G(S)$ the circle graph for S . Given an instance of the TVM problem, by mapping the terminals on B to points on the circumference of a circle C and regarding the two-terminal nets in N as chords on C , one can obtain the circle graph $G(N) = (N, E)$. Let N_1 and N_2 be two mutually disjoint independent sets of $G(N)$. The nets corresponding to the vertices or chords in N_1 and those in N_2 each can be routed without any vias on their respective layers. Marek-Sadowska [14] showed the following result.

Lemma [14]. Each of the remaining nets in $N - N_1 - N_2$ can be routed with only one via. \square

This lemma implies that the TVM problem is equivalent to that of finding two mutually disjoint independent sets N_1 and N_2 of a circle graph such that $|N_1| + |N_2|$ is a maximum. In general the problem of finding such independent sets of a graph is called the *minimum vertex deletion bipartite subgraph* (MVDB) problem. This problem has been shown to be NP-hard even for cubic graphs and for planar graphs whose maximum vertex degree is four [5], where the degree of a vertex is the number of its incident edges. In the following, we formally define the MVDB problem for circle graphs as a decision problem.

VDB-CIRCLE

Instance: A circle graph $G = (V, E)$ given in the form of a set M of chords on a circle R and a positive integer $K \leq |M|$.

Question: Is there a subset M' of M such that $|M'| \geq K$ and the circle graph $G(M')$ is bipartite?

Two chords d and d' are said to be *independent* if they do not intersect. An *independent chord set* is a set of chords which are pairwise independent. Let $S = [d_1, d_2, \dots, d_r]$ be a sequence of distinct chords. If each pair of consecutive chords d_i and d_{i+1} in S intersect for $i = 1, 2, \dots, r - 1$ but no other intersections occur, S is called a *chord chain* (see Fig. 1 (a)). If in addition to the intersection of every consecutive chord pair, d_r and d_1 intersect, S is called a *chord cycle* (see Fig. 1 (b)). In this case r is called the length of the chord cycle. Furthermore, if r is odd (resp., even), S is called an *odd* (resp., *even*) *chord cycle*. It is clear that the existence of an odd chord cycle in a set M of chords results in the existence of an odd cycle in the circle graph $G(M)$ and vice versa. Thus, $G(M)$ is bipartite if and only if M has no odd chord cycle. Let $M_1, M_2, \dots, M_r \subset M$ be independent chord sets. If any sequence of chords $[d_1, d_2, \dots, d_r]$ such that $d_i \in M_i$ for $i = 1, 2, \dots, r$ forms a chord chain, we call the sequence of independent chord sets $[M_1, M_2, \dots, M_r]$ an *independent chord set chain* (see Fig. 1 (c)). An *independent chord set cycle* is similarly defined (see Fig. 1 (d)).

Let x and y be two points on a circle such that the Euclidean distance between them is not equal to the diameter of the circle. These two points define two arcs, the shorter and the longer arcs. Assume that x and y appear in this order during the clockwise traversal of the shorter arc. In this situation, x is said to be *to the right of* y , and any point z on this arc, except x and y , is said to be *located between* x and y on the circle (see Fig. 2 (a)). Furthermore, if h is the chord with endpoints x and y , we call x (resp., y) the *right* (resp., *left*) *endpoint* of h (see Fig. 2 (b)). Finally, for two sets, P and Q , of points or chords, the relation that P is *to the right of* Q is similarly defined as long as no ambiguity arises.

3. NP-hardness Proof of the TVM Problem

In this section, we show that the TVM problem is NP-hard by proving the NP-completeness of the VDB-CIRCLE problem.

Theorem 1. The VDB-CIRCLE problem is NP-complete.

Proof. It is clear that the VDB-CIRCLE problem belongs to the class NP. Therefore, it is sufficient to show that a known NP-complete problem is polynomially transformable to this problem. We use the following problem which was shown to be NP-complete by Lichtenstein [13]. We denote by (l_1, l_2, \dots, l_r) a conjunctive clause with literals l_1, l_2, \dots, l_r .

Planar 3-Satisfiability (P3SAT)

Instance: A set $U = \{v_i \mid 1 \leq i \leq n\}$ of n Boolean variables and a set $C = \{c_j \mid 1 \leq j \leq m\}$ of m clauses over U such that each clause c_j contains exactly three literals. Furthermore, the following graph is planar:

$$G_C = (V_C, E_C), \text{ where}$$

$$V_C = \{c_j \mid 1 \leq j \leq m\} \cup \{v_i \mid 1 \leq i \leq n\} \quad \text{and}$$

$$E_C = \{(c_j, v_i) \mid v_i \text{ or } \bar{v}_i \text{ is contained in } c_j\} \cup \{(v_i, v_{i+1}) \mid 1 \leq i < n\} \cup \{(v_n, v_1)\}.$$

Question: Is C satisfiable? Namely, is there a truth assignment for U such that each clause in C is true?

Let $U = \{v_i \mid 1 \leq i \leq n\}$ and $C = \{c_j \mid 1 \leq j \leq m\}$ be a given instance of the P3SAT problem such that $G_C = (V_C, E_C)$ is planar. Note that V_C consists of two types of vertices *clause vertices* c_1, c_2, \dots, c_m and *variable vertices* v_1, v_2, \dots, v_n , and that

G_C contains the cycle $[v_1, v_2, \dots, v_n]$, which we call the *variable cycle* of G_C . Without loss of generality, we assume that (i) $m \geq 2$, (ii) every variable or its complement is contained in some clause in C , (iii) both a variable and its complement are not contained in a single clause, and (iv) no duplicate of a literal is contained in a clause. In any planar embedding of G_C , each vertex corresponding to a clause in C is located either inside or outside the variable cycle X of G_C . For example, in Fig. 3, we show a planar embedding of G_C for the instance $U = \{v_1, v_2, v_3, v_4, v_5\}$ and $C = \{(v_1, \bar{v}_4, \bar{v}_5), (v_1, v_3, v_4), (v_1, v_2, \bar{v}_4)\}$. Given a planar embedding \hat{G}_C of G_C , we construct an instance of the VDB-CIRCLE problem in the form of a set of chords M . We first draw a sufficiently large circle R which corresponds to the variable cycle X of G_C . We then create *variable gadgets* and *clause gadgets* for the variables in U and the clauses in C , respectively.

We first show the construction of variable gadgets. Let α_i be the number of clauses which contain v_i or \bar{v}_i , and let $\beta_i = 4\alpha_i + 1$. We create an independent chord set chain $[N_1^i, N_2^i, \dots, N_{\beta_i}^i]$ such that $|N_j^i| = m(\beta_i - 1)/2$ (resp., $m(\beta_i + 1)/2$) if j is odd (resp., even). We also create independent chord sets $S_1^i, S_2^i, \dots, S_{\beta_i-1}^i$ such that $|S_j^i| = m\beta_i$ and that N_j^i, S_j^i and N_{j+1}^i form an independent chord set cycle for $j = 1, 2, \dots, \beta_i - 1$. For example, Fig. 4 illustrates the variable gadget for variable v_i with $\alpha_i = 1$ for the case of $m = 3$. In the figure, dotted (resp., solid) lines denote odd (resp., even) numbered chord sets N_1^i, N_3^i and N_5^i (resp., N_2^i and N_4^i), and heavy solid lines denote S_1^i, S_2^i, S_3^i and S_4^i . The number attached to each line indicates the cardinality of the corresponding chord set. The even numbered chord sets $N_2^i, N_4^i, \dots, N_{\beta_i-1}^i$ are intended to correspond to the variable v_i , and the odd numbered chord sets $N_1^i, N_3^i, \dots, N_{\beta_i}^i$ to

its complement \bar{v}_i . We place the variable gadgets on the circle R in the same order as the corresponding vertices are located on the variable cycle X in \hat{G}_C . If necessary, we make the lengths of chords in each gadget sufficiently small compared with the diameter of R so that the gadgets are apart from each other.

We now describe how to construct clause gadgets. We first assign integers called *indices* to edges in G_C other than those on the cycle X in the following way. For each variable vertex v_i , $i = 1, 2, \dots, n$, we first scan counter-clockwise those incident edges of v_i that lie inside X in \hat{G}_C , assigning numbers one to the first edge, two to the second edge, \dots , γ_i to the last edge, and then scan clockwise those incident edges of v_i that lie outside X , assigning numbers $\gamma_i + 1$ to the first edge, $\gamma_i + 2$ to the second edge, \dots , α_i to the last edge, where γ_i is the number of incident edges of v_i that lie inside X . Obviously, every edge other than those on the cycle X is assigned one and only one index. Note that such an edge corresponds to a literal in a clause. For example, see Fig. 5.

For each clause $c_j = (l_{j1}, l_{j2}, l_{j3})$, $j = 1, 2, \dots, m$, let p_{jr} be the index of the edge that corresponds to l_{jr} , and q_{jr} be the integer such that $l_{jr} = v_{q_{jr}}$ or $\bar{v}_{q_{jr}}$, $r = 1, 2, 3$. We construct the clause gadget for c_j as follows. We first create an independent chord set E_j of m chords whose endpoints are both located between the leftmost left endpoint of a chord in $S_{4p_{j1}-2}^{q_{j1}}$ (resp., $S_{4p_{j1}-1}^{q_{j1}}$) and the rightmost right endpoint of a chord in $S_{4p_{j1}-1}^{q_{j1}}$ (resp., $S_{4p_{j1}}^{q_{j1}}$) if $l_{j1} = v_{q_{j1}}$ (resp., $\bar{v}_{q_{j1}}$). We define the *clause arc* for c_j as the short arc on R defined by the endpoints of the innermost chord in E_j , that is, the leftmost right and the rightmost left endpoints among those of its chords. Note that no endpoint of the chords in $\bigcup_{s=1}^{\beta_{q_{j1}}} N_s^{q_{j1}}$ is located on this arc. We then create three chords h_{j1} , h_{j2} ,

and h_{j3} such that one endpoint of h_{jr} , $r = 1, 2, 3$, is located between the rightmost right endpoint of a chord in $N_{4p_{jr}-2}^{q_{jr}}$ (resp., $N_{4p_{jr}-1}^{q_{jr}}$) and the leftmost right endpoint of a chord in $S_{4p_{jr}-3}^{q_{jr}}$ (resp., $S_{4p_{jr}-2}^{q_{jr}}$), if $l_{jr} = v_{q_{jr}}$ (resp., $\bar{v}_{q_{jr}}$), and the other on the clause arc for c_j . We adjust the endpoints of h_{j1} , h_{j2} and h_{j3} on the clause arc for c_j , if necessary, so that they form an odd chord cycle. For example, Figs. 6 (a) and (b) show h_{j1} for the cases of $l_{j1} = v_{q_{j1}}$ and $l_{j1} = \bar{v}_{q_{j1}}$, respectively, where we assume that $\alpha_{q_{j1}} = 1$, and Fig. 6 (c) depicts the clause gadget for clause $c_j = (l_{j1}, l_{j2}, l_{j3}) = (v_a, \bar{v}_b, \bar{v}_c)$ with $p_{j1} = 1$, $p_{j2} = 2$ and $p_{j3} = 1$. In the figures, striped lines are used to denote the chords of E_j .

We now have defined the set M of chords. Note that chord h_{j1} intersects exactly all chords in $N_{4p_{j1}-3}^{q_{j1}}$, $N_{4p_{j1}-1}^{q_{j1}}$ and $S_{4p_{j1}-3}^{q_{j1}}$ (resp., $N_{4p_{j1}-2}^{q_{j1}}$, $N_{4p_{j1}}^{q_{j1}}$ and $S_{4p_{j1}-2}^{q_{j1}}$) if $l_{j1} = v_{q_{j1}}$ (resp., $\bar{v}_{q_{j1}}$), and those in E_j , and the chords h_{j2} and h_{j3} . Note also that if $l_{jr} = v_{q_{jr}}$ (resp., $\bar{v}_{q_{jr}}$), h_{jr} intersects all chords in $N_{4p_{jr}-3}^{q_{jr}}$ (resp., $N_{4p_{jr}-2}^{q_{jr}}$) and no chord in $N_s^{q_{jr}}$ for any even (resp., odd) number s , $1 \leq s \leq \beta_{q_{jr}}$, and that any three chords consisting of h_{jr} , a chord in $N_{4p_{jr}-3}^{q_{jr}}$ (resp., $N_{4p_{jr}-2}^{q_{jr}}$) and a chord in $S_{4p_{jr}-3}^{q_{jr}}$ (resp., $S_{4p_{jr}-2}^{q_{jr}}$) form an odd chord cycle.

As for the value of K , we set

$$\begin{aligned} K &= \sum_{i=1}^n \left(\sum_{j=1}^{(\beta_i+1)/2} |N_{2j-1}^i| + \sum_{j=1}^{\beta_i-1} |S_j^i| \right) + \sum_{j=1}^m |E_j| + m \\ &= \sum_{i=1}^n (m(\beta_i^2 - 1)/4 + \beta_i(\beta_i - 1)m) + m^2 + m. \end{aligned}$$

It is well known that given a planar graph G , we can obtain a planar embedding of G in polynomial time [4,7]. The edge indices can also be set in polynomial time. In the above transformation, we create $O(m^3)$ and $O(m)$ chords for each variable in U and

each clause in C , respectively. Therefore, the total number of chords is $O(nm^3)$, and hence the transformation can be done in polynomial time. As an example, we depict in Fig. 7 a complete instance of our problem constructed from the instance given in Fig. 3.

We now show that C is satisfiable if and only if there is a subset M' of M such that $|M'| \geq K$ and $G(M')$ is bipartite. Suppose that C is satisfiable with a given truth assignment for U . Let M' be the chord set consisting of

1. All chords in $(\bigcup_{i=1}^n (\bigcup_{j=1}^{\beta_i-1} S_j^i)) \cup (\bigcup_{i=1}^m E_i)$,
2. Either all chords in $\bigcup_{j=1}^{(\beta_i-1)/2} N_{2j}^i$ if $v_i = \text{true}$, or all chords in $\bigcup_{j=1}^{(\beta_i+1)/2} N_{2j-1}^i$ if $v_i = \text{false}$, for each variable v_i , $i = 1, 2, \dots, n$, and
3. Exactly one of the three chords h_{j1}, h_{j2} and h_{j3} whose corresponding literal is *true* for each clause c_j , $j = 1, 2, \dots, m$.

Clearly $|M'| = K$. We show below that $G(M')$ is bipartite.

Let $H = M' \cap \{h_{j1}, h_{j2}, h_{j3} \mid j = 1, 2, \dots, m\}$. $M' - H$ consists of mutually disjoint independent chord set chains $[S_{2j-1}^i, N_{2j}^i, S_{2j}^i]$, $j = 1, 2, \dots, (\beta_i - 1)/2$, (resp., $[N_1^i, S_1^i]$, $[S_{2j}^i, N_{2j}^i, S_{2j}^i]$, $j = 1, 2, \dots, (\beta_i - 3)/2$, and $[S_{\beta_i-1}^i, N_{\beta_i}^i]$) if $v_i = \text{true}$ (resp., *false*) for $i = 1, 2, \dots, n$. Furthermore, it is easy to see that the chords in each such chord set chain intersect at most one chord h in H and that h and the chords in $M' - H$ do not form an odd chord cycle. For example, see Fig. 8, where we assume that (i) $\alpha_i = 2$, (ii) $c_a = (\cdot, v_i, \cdot)$ and $c_b = (v_i, \cdot, \cdot)$, (iii) $v_i = \text{true}$, and (iv) h_{a2} and h_{b1} are in H . Therefore, no chord cycle formed by chords in M' has a chord contained in a variable gadget, and hence a chord cycle formed by chords in M' , if any, consists of chords in H only. Let H_1 (resp., H_2) = $M' \cap \{h_{j1}, h_{j2}, h_{j3} \mid c_j \text{ lies inside (resp., outside) } X \text{ in}$

$\hat{G}_C, j = 1, 2, \dots, m\}$. Note that $H = H_1 \cup H_2$. Since \hat{G}_C is a planar embedding of G_C , it is easy to see that H_1 and H_2 are independent chord sets, which implies that no odd cycle can be formed by chords in H only. Therefore, M' has no odd cycle and hence $G(M')$ is bipartite.

Conversely, suppose that we have a chord set M' such that $|M'| \geq K$ and $G(M')$ is bipartite. If M' contains a chord in an independent chord set such as E_j, S_j^i and N_j^i , we can add all the remaining chords of this independent chord set to M' without violating the bipartiteness of the corresponding graph. Thus, we assume in the following, that each such independent chord set is either contained in or disjoint from M' . If M' is disjoint from a chord set E_j for some $j, 1 \leq j \leq m$, or a chord set S_j^i for some i and $j, 1 \leq i \leq n$ and $1 \leq j \leq \beta_i - 1$, the following operations will always produce a new chord set M'' such that $|M''| \geq |M'|$ and $G(M'')$ is bipartite.

Case 1. M' is disjoint from E_j for some $j, 1 \leq j \leq m$.

- (a) If two chords in $\{h_{j1}, h_{j2}, h_{j3}\}$ are in M' (all three can not be in M'), remove one of them from M' , and
- (b) $M'' \leftarrow M' \cup E_j$.

It is clear that $G(M'')$ is still bipartite, and $|M''| \geq |M'|$ since $|E_j| = m \geq 2$.

Case 2. M' is disjoint from S_j^i for some i and $j, 1 \leq i \leq n$ and $1 \leq j \leq m$.

- (a) If both N_j^i and N_{j+1}^i are subsets of M' , remove N_j^i from M' , or
- (b) If $N_j^i \subset M'$ (hence $M' \cap N_{j+1}^i = \phi$) (resp., $N_{j+1}^i \subset M'$ (hence $M' \cap N_j^i = \phi$)) and there is a chord h_{jr} in M' for some j and $r, 1 \leq j \leq m$ and $1 \leq r \leq 3$, such that h_{jr} intersects the chords in N_j^i (resp., N_{j+1}^i), remove the chord h_{jr} from M' , and
- (c) $M'' \leftarrow M' \cup S_j^i$.

In case (a), S_j^i and N_j^i are interchanged. From the construction of M , it is easy to see that such an interchange dose not introduce new odd cycles. In case (b), the chords in S_j^i intersect only the chords in either N_j^i or N_{j+1}^i , and h_{jr} . Thus, in both cases, the resultant graph is bipartite. Since we remove at most $m(\beta_i + 1)/2 + 1$ chords but add $m\beta_i$ chords and $\beta_i = 4\alpha_i + 1 \geq 5$, $|M''| \geq |M'|$.

We now assume that M' contains all chords in $\bigcup_{j=1}^m E_j$ and $\bigcup_{i=1}^n (\bigcup_{j=1}^{\beta_i-1} S_j^i)$. Since M' contains all chords in $\bigcup_{i=1}^n (\bigcup_{j=1}^{\beta_i-1} S_j^i)$, M' can contain at most $\delta_i = m(\beta_i^2 - 1)/4$ chords from the chord set $\bigcup_{j=1}^{\beta_i} N_j^i$ by including either the odd numbered chord sets $N_1^i, N_3^i, \dots, N_{\beta_i}^i$ or the even numbered chord sets $N_2^i, N_4^i, \dots, N_{\beta_i-1}^i$ for each variable v_i , $i = 1, 2, \dots, n$. Let N'_i be a set of chords selected from $\bigcup_{j=1}^{\beta_i} N_j^i$ such that $N'_i \subset M'$. If $|N'_i| < \delta_i$, it is easy to see that $|N'_i| \leq \delta_i - m$. In this case, the following operations will yield a new chord set M'' such that $|M''| \geq |M'|$ and $G(M'')$ is bipartite:

- (a) Remove from M' all chords in N'_i and those chords in H that intersect a chord in $\bigcup_{j=1}^{(\beta_i+1)/2} N_{2j-1}^i$, and
- (b) $M'' \leftarrow M' \cup (\bigcup_{j=1}^{(\beta_i+1)/2} N_{2j-1}^i)$.

Clearly $G(M'')$ is bipartite. Since $M' \supset E_j$ and $G(M')$ was bipartite, M' contained at most one chord in $\{h_{j1}, h_{j2}, h_{j3}\}$ for each $j = 1, 2, \dots, m$. Thus, operation (a) removed at most $\delta_i - m + m = \delta_i$ chords from M' . Since operation (b) added exactly δ_i chords back to get M'' , we have that $|M''| \geq |M'|$. Therefore, we can assume that M' contains all chords in either $\bigcup_{j=1}^{(\beta_i+1)/2} N_{2j-1}^i$ or $\bigcup_{j=1}^{(\beta_i-1)/2} N_{2j}^i$ for each variable v_i , $i = 1, 2, \dots, n$. We assign to v_i the value *true* if $M' \supset \bigcup_{j=1}^{(\beta_i-1)/2} N_{2j}^i$ and *false* if $M' \supset \bigcup_{j=1}^{(\beta_i+1)/2} N_{2j-1}^i$. Clearly, this gives a consistent truth assignment for U .

We now show that each clause $c_j \in C$, $j = 1, 2, \dots, m$, is true with this assignment.

Since $|M'| \geq K$, it is easy to see that M' contains no less than m chords in H . Since $E_j \subset M'$, at most one of the three chords h_{j1}, h_{j2} , and h_{j3} is in M' , and hence, exactly one of them is in M' . Let h_{jt} be the chord in $M' \cap \{h_{j1}, h_{j2}, h_{j3}\}$. Since $G(M')$ is bipartite, h_{jt} and any chord in $\bigcup_{r=1}^{\beta_{q_{jt}}} N_r^{q_{jt}} \cap M'$ do not intersect. For otherwise there would be an odd chord cycle including h_{jt} , a chord in $S_x^{q_{jt}}$ and a chord in $N_s^{q_{jt}}$ for some x and s . From the construction of the chord set M , the corresponding literal l_{jt} is equal to (i) $v_{q_{jt}}$ if h_{jt} and every chord in $N_{4p_{jt}-3}^{q_{jt}}$ intersect and (ii) $\bar{v}_{q_{jt}}$ if h_{jt} and any chord in $N_{4p_{jt}-3}^{q_{jt}}$ do not intersect. Thus, if $v_{q_{jt}}$ is assigned *true*, that is, if $\bigcup_{r=1}^{(\beta_{q_{jt}}-1)/2} N_{2r}^{q_{jt}} \subset M'$, then h_{jt} and $N_{4p_{jt}-3}^{q_{jt}}$ intersect (for h_{jt} intersects $N_r^{q_{jt}}$ for some r), and hence l_{jt} must be $v_{q_{jt}}$. On the other hand, if $v_{q_{jt}}$ is assigned *false*, then h_{jt} and $N_{4p_{jt}-3}^{q_{jt}}$ do not intersect, and hence l_{jt} must be $\bar{v}_{q_{jt}}$. In both cases literal l_{jt} gets the value *true*. Therefore, clause c_j is true. \square

By Lemma 1 and Theorem 1, we establish the following result.

Theorem 2. The TVM problem is NP-hard. \square

Remark. We now comment on the NP-complete proof of the VDB-CIRCLE problem given by Sarrafzadeh and Lee [18]. The main difference between our proof and theirs lie in the structure of variable gadgets. In their construction, they use a chord set cycle $[N_1^i, N_2^i, \dots, N_{2\alpha_i}^i]$ such that $|N_j^i| = m$, instead of a chord set chain. They also use the chord sets $S_1^i, S_2^i, \dots, S_{2\alpha_i}^i$ such that $|S_j^i| = 2m$ for $j = 1, 2, \dots, 2\alpha_i$ and $N_{2\alpha_i}^i, S_{2\alpha_i}^i$ and N_1^i , and N_j^i, S_j^i and N_{j+1}^i for $j = 1, 2, \dots, 2\alpha_i - 1$, each form a chord set cycle. Although they do not describe exactly where to locate the chords of a clause gadget, which is the same as ours, we may obtain a set of chords as shown in Fig. 9 (a) for

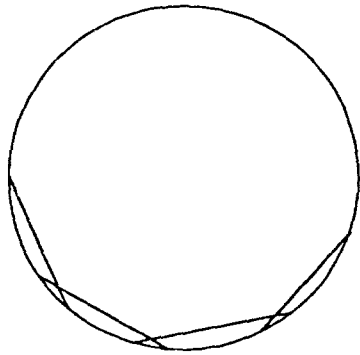
the instance given in Fig. 3. Suppose that we set $v_1 = \text{false}$, $v_2 = \text{false}$, $v_3 = \text{true}$, $v_4 = \text{false}$ and $v_5 = \text{true}$. Clearly, this truth assignment satisfies the instance of Fig. 3. However, according to this truth assignment, we will obtain such a chord set as shown in Fig. 9 (b). Since this chord set obviously contains an odd chord cycle, it can not be a solution to the instance of the VDB-CIRCLE problem. \square

References

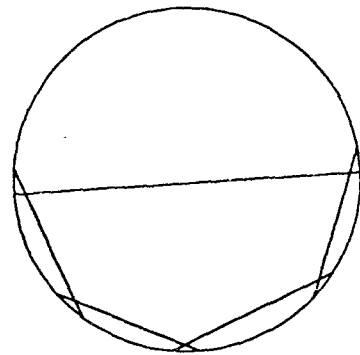
- [1] K. C. Chang and D. H-C. Du, "Efficient algorithms for layer assignment problem," *IEEE Trans. on Computer-Aided Design*, Vol. CAD-6, pp. 67-78, Jan. 1987.
- [2] K. C. Chang and H. C. Du, "Layer assignment problem for three-layer routing," *IEEE Trans. on Computers*, Vol. 37, No. 5, pp 625-632, May 1988.
- [3] R. W. Chen, Y. Kajitani and S. P. Chan, "A graph-theoretic via minimization algorithm for two-layer printed circuit board," *IEEE Trans. on Circuit and Systems*, Vol. CAS30-5, pp. 284-299, May 1983.
- [4] N. Chiba, T. Nishizeki, S. Abe and T. Ozawa, "A linear time algorithm for embedding planar graphs using PQ-trees," *J. Comput. System Sci.*, Vol. 30, pp. 54-76, 1985.
- [5] H-A. Choi, K. Nakajima and C. S. Rim, "Graph bipartization and via minimization," *SIAM Jour. on Discrete Mathematics*, Vol. 2, Feb. 1989; see also *Proc. of the 25th Annual Allerton Conf. on Communication, Control, and Computing*, Montecillo, IL, Sept. 1987 pp. 4-13.
- [6] Even, *Graph algorithms*, Computer Science Press, Rockville, MD, 1979.

- [7] I. Fary, "On straight line representation of planar graphs," *Acta Sci. Math.* (Szeged), Vol. 11, pp. 229–233, 1948.
- [8] M. R. Garey and D. S. Johnson, *Computers and intractability; a guide to the theory of NP-completeness*, W. H. Freeman and Company, San Francisco, CA, 1979.
- [9] A. Hashimoto and J. Stevens, "Wire routing by channel assignment within large apertures," *Proc. 8th Design Automation Workshop*, June 1971, pp. 155–169.
- [10] C. P. Hsu, "Minimum via topological routing," *IEEE Trans. on Computer-Aided Design of ICAS*, Vol. CAD-2, pp. 235–246, Oct. 1983.
- [11] Y. Kajitani, "On via hole minimization of routing on a 2-layer board," *Proc. 1980 Int'l Conference on Circuits and Computers*, June 1980, pp. 295–298.
- [12] Y. S. Kuo, T. C. Chern, and W-K. Shih, "Fast Algorithms for Optimal Layer Assignment," *Proc. 25th Design Automation Conference*, June 1988, pp. 554–559.
- [13] D. Lichtenstein, "Planar formula and their uses," *SIAM J. on Computing*, Vol. 11, No. 2, pp. 329–343, May 1982.
- [14] M. Marek-Sadowska, "An unconstrained topological via minimization problem for two-layer routing," *IEEE Trans. on Computer-Aided Design of ICAS*, Vol. CAD-3, pp. 184–190, July 1984.
- [15] N. J. Naclerio, S. Masuda and K. Nakajima, "Via minimization for gridless layout," *Proc. 24th Design Automation Conference*, June 1987, pp. 159–165.

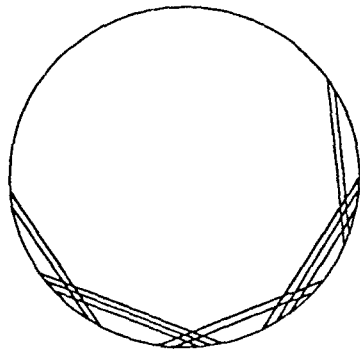
- [16] N. J. Naclerio, S. Masuda and K. Nakajima, "The via minimization problem is NP-complete," *IEEE Trans. on Computers*, to appear; see also *Proc. 1987 Conference on Infor. Sci. and Systems*, Baltimore, MD, March 1987, pp. 611–616.
- [17] C. S. Rim, T. Kashiwabara and K. Nakajima, "Exact algorithms for multilayer topological via minimization," Tech. Report TR88-88, Systems Research Center, Univ. of Maryland, College Park, MD, 1988.
- [18] M. Sarrafzadeh and D. T. Lee, "A new approach to topological via minimization," Tech. Report CIMS-87-01, Northwestern University, Evanston, IL, Nov. 1987.
- [19] X-M. Xiong and E. S. Kuh, "The constrained via minimization problem for PCB and VLSI design," *Proc. 25th Design Automation Conference*, June 1988, pp. 573–578.



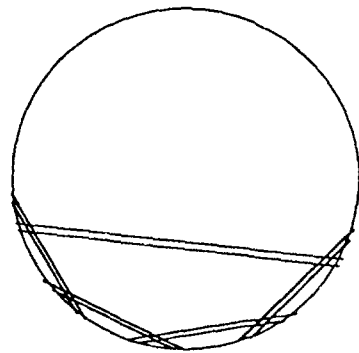
(a) A chord chain.



(b) A chord cycle.

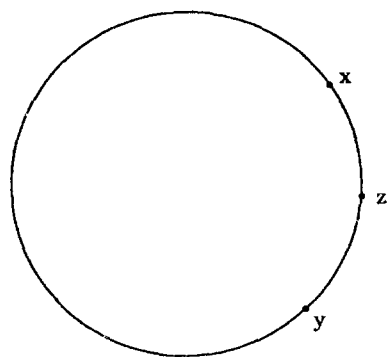


(a) A chord set chain.

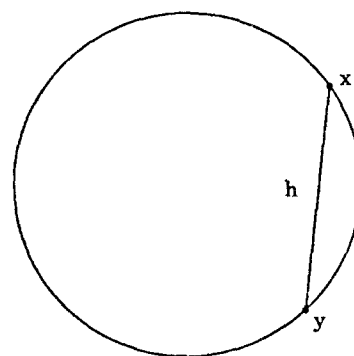


(b) A chord set cycle.

Fig. 1. A chord chain, chord cycle, chord set chain, and chord set cycle.



(a)



(b)

Fig. 2. (a) Point x is to the left of point y . Point z is located between points x and y .
 (b) A chord h with its left endpoint x and right endpoint y .

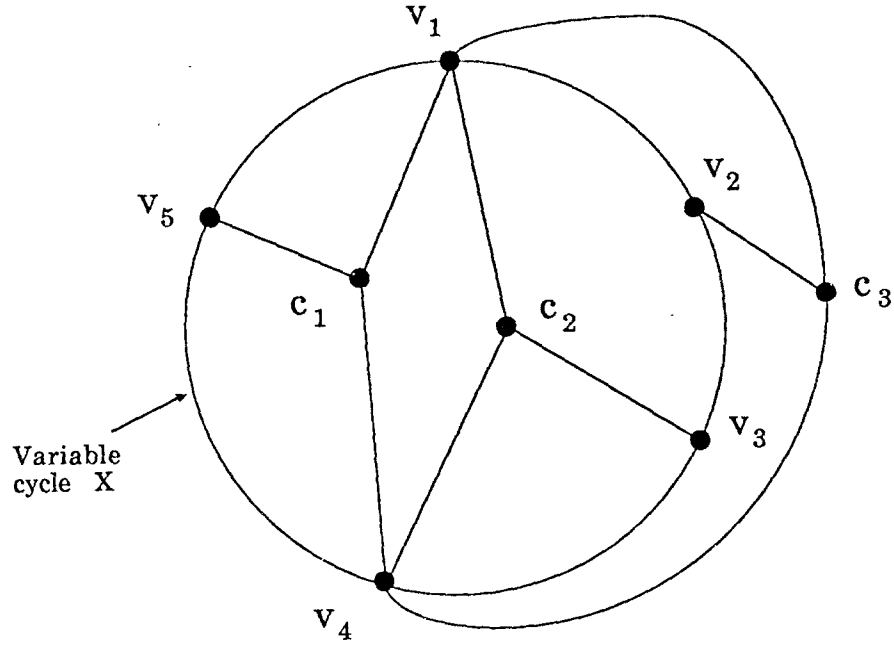


Fig. 3. A planar embedding of G_C for the instance $U = \{v_1, v_2, v_3, v_4, v_5\}$ and $C = \{c_1 = (v_1, \bar{v}_4, \bar{v}_5), c_2 = (v_1, v_3, v_4), c_3 = (v_1, v_2, \bar{v}_4)\}$ of the P3SAT problem.

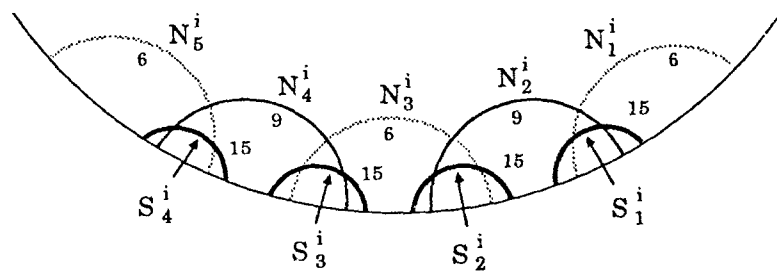


Fig. 4. Variable gadget for variable v_i with $\alpha_i = 1$ and $m = 3$.

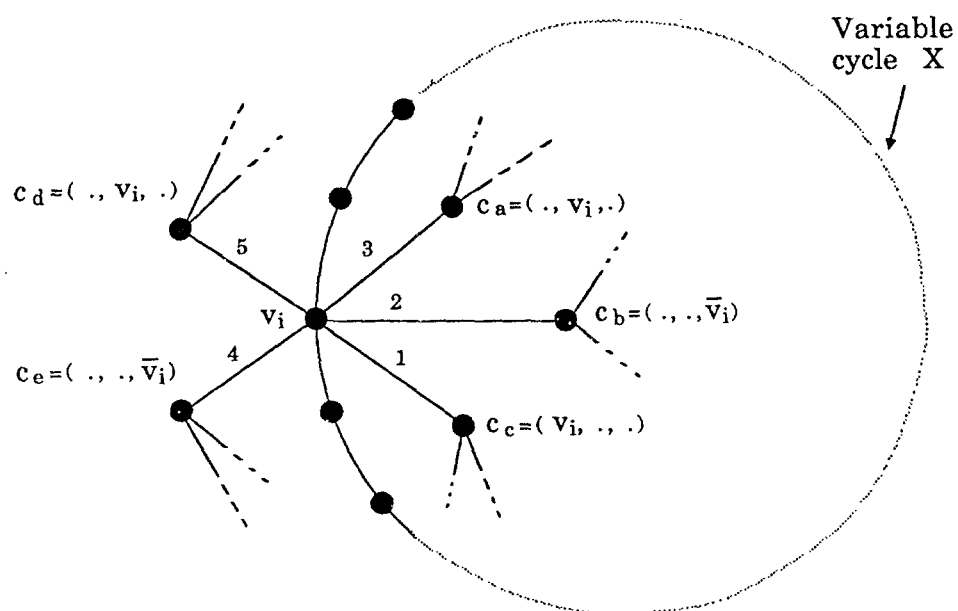
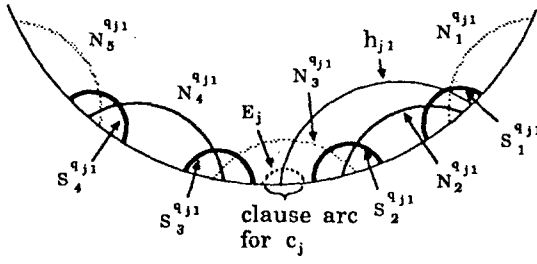
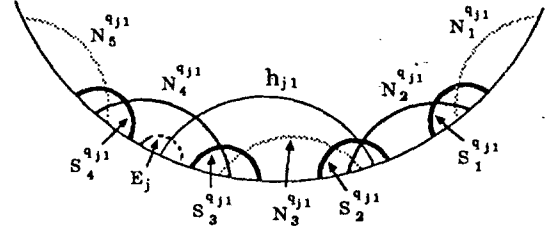


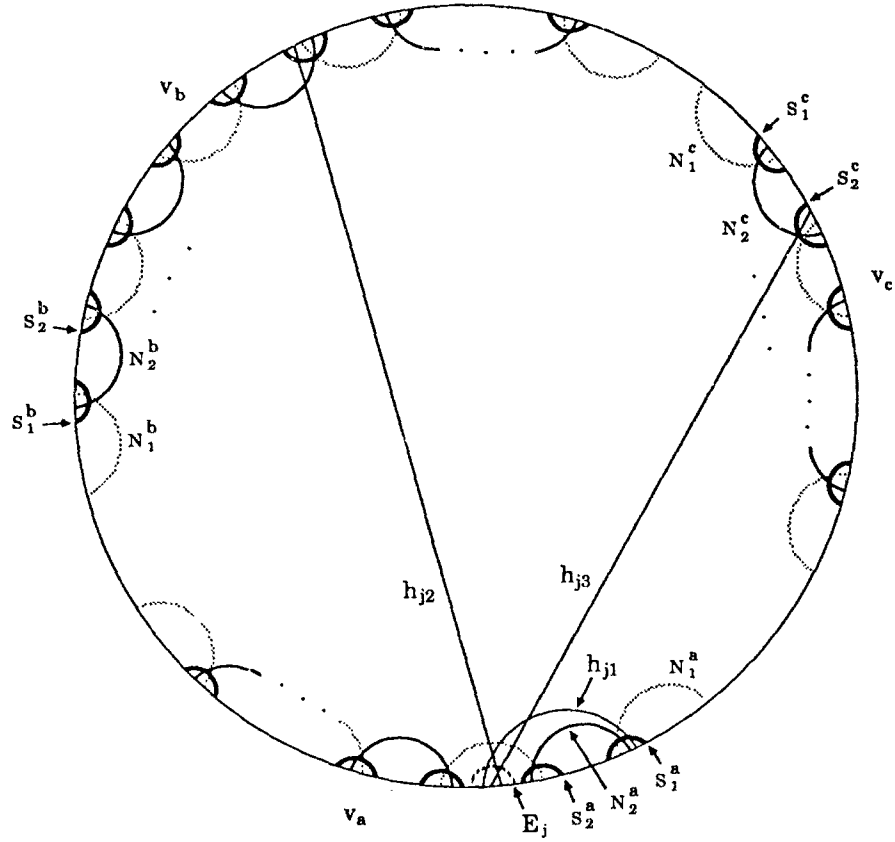
Fig. 5. Assignment of indices to edges not on cycle X .



(a) $l_{j1} = v_{j1}$.



(b) $l_{j1} = \bar{v}_{j1}$.



(c) Clause gadget for $c_j = (l_{j1}, l_{j2}, l_{j3}) = (v_a, \bar{v}_b, \bar{v}_c)$ with $p_{j1} = 1$, $p_{j2} = 2$ and $p_{j3} = 1$.

Fig. 6 Illustrations for the construction of a clause gadget.

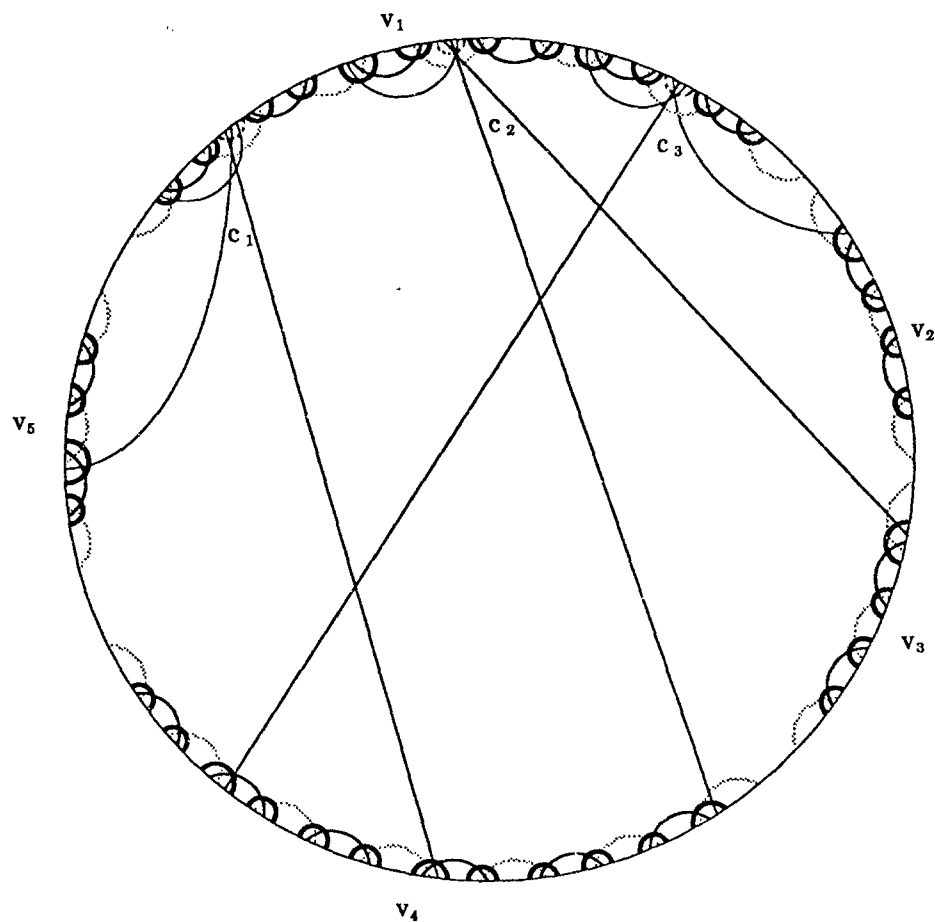


Fig. 7. A complete instance of the VDB-CIRCLE problem constructed from the instance of Fig. 3.

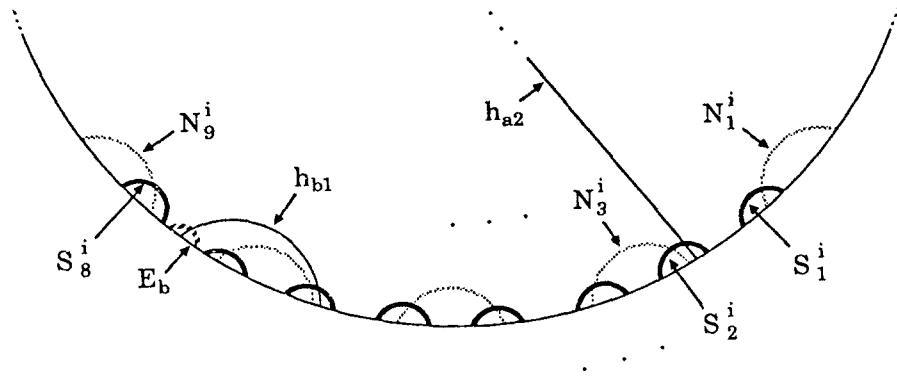
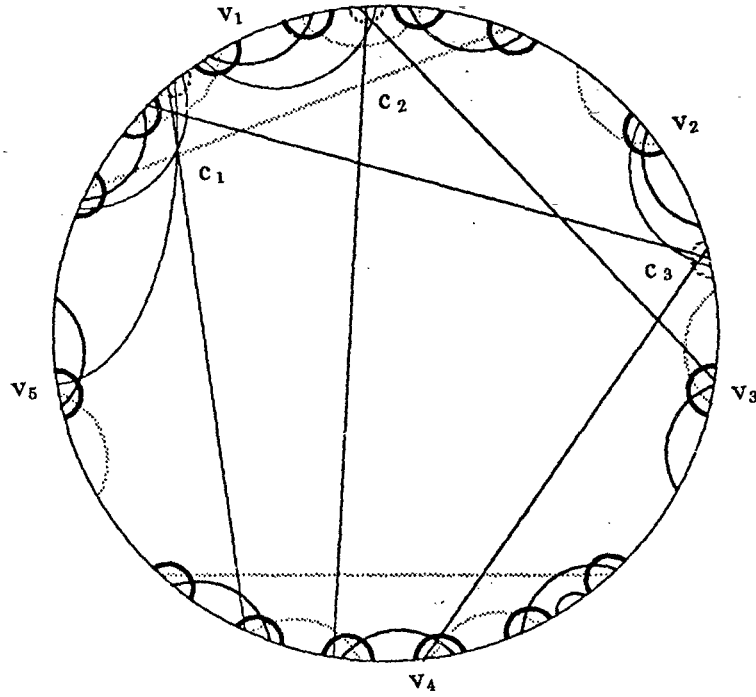
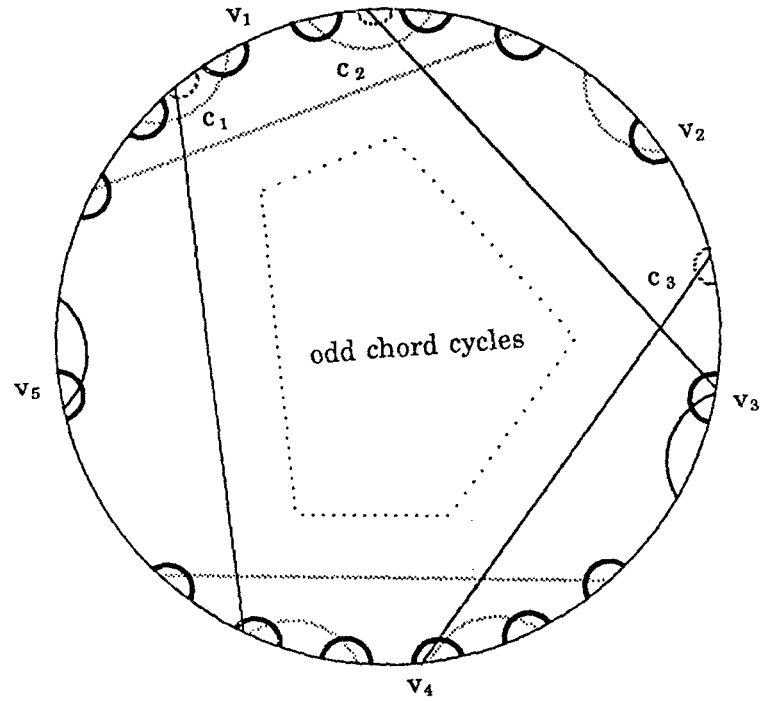


Fig. 8. Chords of the variable component for v_i which are contained in M' when $v_i = \text{true}$.



(a)



(b)

Fig. 9. (a) Sarafzadeh and Lee's [18] transformation for the instance shown in Fig. 3.
 (b) The chord set obtained by the truth assignment $v_1 = \text{false}$, $v_2 = \text{true}$, $v_3 = \text{false}$, $v_4 = \text{true}$ and $v_5 = \text{false}$.