THE INSTITUTE FOR SYSTEMS RESEARCH

ISR TECHNICAL REPORT 2008-32

A Convex Parameterization of all Stabilizing Controllers for Non-Strongly Stabilizable Plants, Under Quadratically Invariant Sparsity Constraints

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Abstract

This paper addresses the design of controllers, subject to sparsity constraints, for linear and time-invariant plants. Prior results have shown that a class of stabilizing controllers, satisfying a given sparsity constraint, admits a convex representation of the Youla–type, provided that the sparsity constraints imposed on the controller are quadratically invariant with respect to the plant and that the plant is strongly stabilizable. Another important aspect of the aforementioned results is that the sparsity constraints on the controller can be recast as convex constraints on the Youla parameter, which makes this approach suitable for optimization using norm-based costs. In this paper, we extend these previous results to non-strongly stabilizable plants. Our extension also leads to a Youla-type representation for the class of controllers, under quadratically invariant sparsity constraints. In our extension, the controller class also admits a representation of the Youla–type, where the Youla parameter is subject to only convex constraints.

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I. Introduction

The design of decentralized control systems is in general a hard problem, partly due to the lack of convexity induced by restrictions on the structure of the controller. Typically, these constraints arise from pre–specified information patterns, such as when the controller consists of interconnected blocks that have access to different measurements.

The theoretical machinery developed in [6] unifies and consolidates many previous results, pinpoints certain tractable decentralized control structures, and outlines the most general known class of convex problems in decentralized control. Also in [6], a numerical computational procedure is proposed for decentralized $\mathcal{H}^2$ optimal synthesis of quadratic invariant, decentralized structures with strongly stabilizable plants. This paper is an extension of the method from [6] to the general case of possibly non–strongly stabilizable plants.

Necessary and sufficient conditions for strong stabilizability, of general, multi–input multi–output, linear and time–invariant plants, are not yet known in the literature. Neither are general methods for designing stable controllers, for the cases in which they do exist. This makes designing stable controllers for strongly stabilizable plants a difficult task even in the centralized setting. More importantly, for most practical situations in control engineering, the working hypothesis is stabilizability only, rather than strong–stabilizability.

For the design method in [6], the optimal controller (in the $\mathcal{H}^2$ sense, for instance) can be synthesized via convex programming, starting from a stable, stabilizing controller. While inheriting this feature, our approach has the increased handiness of relying just on *any* stabilizable controller, not necessarily stable, which in general is far easier to find. This bridges the gap between stability constraints and the main optimization paradigm, hence it has the merit of not over–complicating the final convex program with additional tough constraints related to stabilization.

It followed quite naturally to develop our results over any ring of stable, linear systems, within the general framework established in the seminal paper [8] of Vidyasagar et al. Complying with [8], the notions of *proper* and *strictly proper* are introduced in an abstract setting, and any transfer function is viewed as the ratio of two stable, causal transfer functions. The advantage of using this setup is that it encompasses within a single framework, continuous or discrete-time systems, lumped as well as distributed systems, n-D systems, etc The important special case of linear, time–invariant, 1-D systems, is immediately retrieved by considering the instance of the ring of proper, stable, rational functions.

The core of our approach resides in the so called coordinate–free method proposed in [4], where coprime factorizability of the plant is not needed, to provide a Youla–type parametrization of all stabilizing controllers. Using this parametrization and much in the spirit of [6], our main result shows how to deal with the decentralized problem for quadratic invariant structures, provided the availability of tools to solve the centralized problem.

II. Preliminaries

With all the notation borrowed from [4], $\mathcal{A}$ is the set of stable, causal transfer functions and is assumed to have a commutative ring structure. The set of all transfer functions, which we denote as $\mathcal{F}$, is therefore the field of fractions of $\mathcal{A}$, defined as follows:

$$\mathcal{F} \stackrel{def}{=} \left\{ n/d \;\middle|\; n, d \in \mathcal{A};\ d \text{ not a divisor of zero} \right\} \tag{1}$$

Accordingly, $\mathcal{F}^{p\times m}$ will stand for the set of transfer function matrices (matrices with all entries in $\mathcal{F}$) with p rows and m columns. Let $\mathcal{Z}$ be any be any prime ideal of $\mathcal{A}$ with $\mathcal{A} \neq \mathcal{Z}$ and such that $\mathcal{Z}$ includes all the divisors of zero of $\mathcal{A}$. Define the subsets $\mathcal{P}$ and $\mathcal{P}_s$ of $\mathcal{F}$ as

$$\mathcal{P} \stackrel{def}{=} \left\{ \frac{a}{b} \in \mathcal{F} | a \in \mathcal{A},\ b \in \mathcal{A} - \mathcal{Z} \right\},$$

$$\mathcal{P}_s \stackrel{def}{=} \left\{ \frac{a}{b} \in \mathcal{F} | a \in \mathcal{Z},\ b \in \mathcal{A} - \mathcal{Z} \right\}.$$

We shall call every transfer function in $\mathcal{P}$ ($\mathcal{P}_s$) causal (strictly causal). Similarly, if every entry of some transfer function matrix is in $\mathcal{P}$ ($\mathcal{P}_s$) then the transfer matrix will be called causal (strictly causal).

III. Parametrization of Stabilizing Controllers via the Coordinate–Free Approach

In Fig.1 we depict the standard feedback interconnection between a generalized plant and controller. Here, w is the vector of reference signals, while ν_1 and ν_2 are the disturbance signals and sensor noise respectively. In addition, u are the controls, y are the measurements and z the

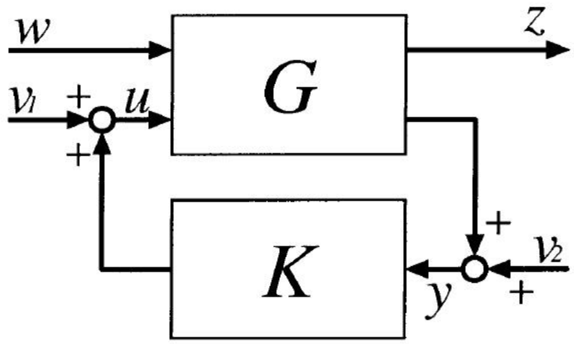

Fig. 1. Feedback interconnection between the generalized plant and the controller

regulated outputs (in general some error signals). For convenience of notation, G is partitioned accordingly with $G_{zw} \in \mathcal{F}^{n_z \times n_w}$, $G_{zu} \in \mathcal{F}^{n_z \times n_u}$, $G_{yw} \in \mathcal{F}^{n_y \times n_w}$ and $G_{yu} \in \mathcal{F}^{n_y \times n_u}$. Here, the integers n_w, n_u, n_y and n_z denote the dimensions of w, u, y and z respectively. The generalized plant G lies in $\mathcal{F}^{(n_y+n_z)\times(n_u+n_w)}$ and the controller in $\mathcal{F}^{n_u \times n_y}$. We adopt the superscript T as the notation for matrix transposition. Assuming that the loop is *well posed* – that is $(I - KG_{yu})$ is invertible over $\mathcal{F}^{n_u \times n_u}$ – then the transfer matrix from $[w^T \ \ \nu_1^T \ \ \nu_2^T \ \ ]^T$ to $[z^T \ \ u^T \ \ y^T \ \ ]^T$ is given by

$$\Theta(G,K) = \begin{bmatrix} G_{zw} + G_{zu}K(I - G_{yu}K)^{-1}G_{yw} & G_{zu}(I - KG_{yu})^{-1} & G_{zu}K(I - G_{yu}K)^{-1} \\ K(I - G_{yu}K)^{-1}G_{yw} & (I - KG_{yu})^{-1} & K(I - G_{yu}K)^{-1} \\ (I - G_{yu}K)^{-1}G_{yw} & G_{yu}(I - KG_{yu})^{-1} & (I - G_{yu}K)^{-1} \end{bmatrix}. \tag{2}$$

If the transfer matrix $\Theta(G,K)$ is over $\mathcal{A}$ then we call it *stable*, or we say that K is a *a stabilizing controller* of G or equivalently that K *stabilizes* G. If a stabilizable controller of G exists, we say that G is *stabilizable*.

Of particular interest is the feedback system displayed in Figure 2, where the transfer function matrices $K \in \mathcal{F}^{n_u \times n_y}$ and $P \in \mathcal{F}^{n_y \times n_u}$ represent the controller and the plant respectivelly. Denote by $H(P,K)$ the transfer function matrix from $[\nu_2^T \ \ \nu_1^T \ \ ]^T$ to $[y^T \ \ u^T \ \ ]^T$ (provided that

$(I + KP)$ is nonsingular):

$$H(P, K) = \begin{bmatrix} (I + PK)^{-1} & -P(I + KP)^{-1} \\ K(I + PK)^{-1} & (I + KP)^{-1} \end{bmatrix} \tag{3}$$

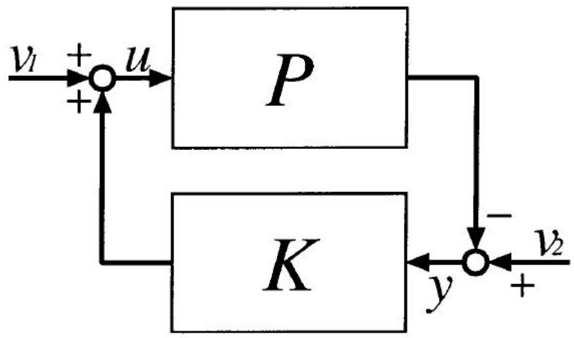

Fig. 2. Feedback System

Analogously with the generalized–feedback system in (2), if the transfer matrix $H(P, K)$ is over $\mathcal{A}$ we call it *stable* or we say that K is a *stabilizing controller* of P or equivalently that K *stabilizes* P. If a stabilizable controller of P exists, we say that P is *stabilizable*. It is important to note here that $H(P, K)$ can be envisioned as part of the transfer function (2) (the two by two block in the bottom right corner). This is further related to the following Lemma from [4], (which is in fact a generalization of the well-known Theorem 4.3.2 in [1].)

Lemma III.1. *[4, Lemma 1] Let G and K be a generalized plant and its controller over $\mathcal{F}$, with G stabilizable and agreeingly partitioned as in (2). Then $\Theta(G, K)$ is stable if and only if $H(-G_{yu}, K)$ is stable.*

We define the set $\mathcal{C}$ of stabilizing controllers of P by

$$\mathcal{C} \stackrel{def}{=} \Big\{ K \Big| K \in \mathcal{F}^{n_y \times n_u} \text{ and } K \text{ stablilizes } P \Big\}. \tag{4}$$

Clearly, from Lemma III.1, the set $\mathcal{C}$ is the same set as

$$\mathcal{C} = \Big\{ K \Big| K \in \mathcal{F}^{n_y \times n_u} \text{ and } K \text{ stabilizes } G \Big\},$$

for any generalized plant G for which $G_{yu} = -P$.

Of central importance in the sequel is the following result (a summary of Theorem 4.2 and Theorem 4.3 in [3]) as it provides a useful Youla–like parametrization of the stabilizing controllers of $H(P, K)$.

Theorem III.2. *[3] **i)** Given the integers n_u and n_y, and a plant P in the set $\mathcal{F}^{n_y \times n_u}$, consider the following set:*

$$\mathcal{H}_P \stackrel{def}{=} \left\{ H(P,K) \in \mathcal{A}^{(n_u+n_y)\times(n_u+n_y)} \middle| \; K \text{ is a stabilizing controller for } P \right\}$$

Given an arbitrary K_0 that stabilizes P, and therefore for which $H(P, K_0) \in \mathcal{A}^{(n_u+n_y)\times(n_u+n_y)}$, the following holds:

$$\mathcal{H}_P = \left\{ \Omega(Q) \;\middle|\; Q \in \mathcal{A}^{(n_u+n_y)\times(n_u+n_y)},\ Q \text{ causal and } \Omega(Q) \ \text{ nonsingular} \right\}$$

where for any causal Q in the set $\mathcal{A}^{(n_u+n_y)\times(n_u+n_y)}$, $\Omega(Q)$ is defined as follows:

$$\Omega(Q) \stackrel{def}{=} \left(H(P,K_0) - \begin{bmatrix} I_{n_y} & O \\ O & O \end{bmatrix} \right) Q \left(H(P,K_0) - \begin{bmatrix} O & O \\ O & I_{n_u} \end{bmatrix} \right) + H(P,K_0). \tag{5}$$

Here I_{n_y} and I_{n_u} denote the identity matrices of dimension n_y and n_u respectively.
***ii)** For $\Omega(Q)$ defined in (5), with $\Omega(Q)$ in the set $\mathcal{A}^{(n_u+n_y)\times(n_u+n_y)}$, consider the following partition:*

$$\Omega(Q) = \begin{array}{c} \begin{array}{cc} \overbrace{\qquad\qquad}^{n_y} & \overbrace{\qquad\qquad}^{n_u} \end{array} \\ \begin{bmatrix} \Omega_{11}(Q) & \Omega_{12}(Q) \\ \Omega_{21}(Q) & \Omega_{22}(Q) \end{bmatrix} \end{array} \begin{array}{l} \\ \}\, n_y \\ \}\, n_u \end{array} \tag{6}$$

If $\Omega_{11}(Q)$ is nonsingular then any *stabilizing controller of P can be written as*

$$K(Q) = \Omega_{21}(Q)\Omega_{11}^{-1}(Q) \tag{7}$$

for some causal matrix Q in the set $\mathcal{A}^{(n_u+n_y)\times(n_u+n_y)}$, where Ω_{21} and Ω_{11} are the $(2,1)$- and $(1,1)$- blocks of $\Omega(Q)$ respectively.

Remark III.3. *We would like to point out here, that reference [4] contains a typo which is repeated for several times throughout the paper. Specifically, in Section III of [4], the expression of K is given as $\Omega_{21}\Omega_{22}^{-1}$. It can be easily seen from (3) that under the assumptions of Theorem*

III.2, the actual expression of K is the one given in (7). References to the results in [4] are made in the sequel, taking into account the above mentioned typo.

Remark III.4. *Using Lemma III.1, we conclude that Theorem III.2 ii) provides a parametrization of all stabilizing controllers of $\Theta(G, K)$ for any generalized plant G for which $G_{yu} = -P$.*

From this point onward we shall make the hypothesis on G_{yu} to be strictly causal, that is

$$G_{yu} \in \mathcal{P}_s. \tag{8}$$

This is necessary as to guarantee several conditions in a way made precise by the following remark (see [4]).

Remark III.5. *[4] The assumption of strict causality of G_{yu} implies that every stabilizing controller is causal [2, Prop. 6.2] and that the closed loop is well–posed [9, pp.119] for every stabilizing controller [3, Prop. 5].*

IV. Problem Formulation

A. The Standard Control Problem

Assume that a consistent norm has been adopted for transfer matrices over $\mathcal{F}$. A standard problem in control is the following: in the generalized feedback system from Figure 1, with the given causal and stabilizable plant matrix G, design a stabilizing controller K that minimizes the norm of the top left corner entry of $\Theta(G, K)$ which is the transfer function from w to z, namely

$$\min_{K \text{stabilizes } G} \left\| G_{zw} + G_{zu}\, K\big(I - G_{yu}K\big)^{-1} G_{yw} \right\|. \tag{9}$$

The following result, [4, Theorem 1] will be instrumental in our proposed approach, as it makes clear the equivalence between the standard control problem (9) and the model–matching problem of minimizing the norm of some affine (and therefore convex) function in the argument Q – the Youla parameter from Theorem III.2.

Theorem IV.1. *[4, Theorem 1] Let G be a stabilizable, generalized plant such that the block $G_{yu} \in \mathcal{F}^{n_y \times n_u}$ is strictly causal. Given* any *stabilizable controller $K_0 \in \mathcal{F}^{n_u \times n_y}$ of G_{yu}, the standard control problem (9) is equivalent to*

$$\min_{Q \in \mathcal{A}^{(n_u+n_y)\times(n_u+n_y)}} \left\| T_1 - T_2 \, Q \, T_3 \right\| \tag{10}$$

subject to Q causal and stable, where an optimal solution K^* to (9) can always be obtained from the optimal Q in (10), denoted with Q^*, via $K^* = \Omega_{21}(Q^*)\Omega_{11}^{-1}(Q^*)$. Here T_1, T_2 and T_3 are the transfer function matrices defined below:

$$\begin{aligned} T_1 &\stackrel{def}{=} G_{zw} + G_{zu} \, K_0(I - G_{yu}K_0)^{-1}G_{yw} \, , \\ T_2 &\stackrel{def}{=} \begin{bmatrix} G_{zu}K_0(I - G_{yu}K_0)^{-1} & G_{zu}(I - K_0G_{yu})^{-1} \end{bmatrix} , \\ T_3 &\stackrel{def}{=} \begin{bmatrix} (I - G_{yu}K_0)^{-1}G_{yw} \\ K_0(I - G_{yu}K_0)^{-1}G_{yw} \end{bmatrix} . \end{aligned} \tag{11}$$

B. The Decentralized Control Problem

For $p \geq 1$, we denote the set of integers from 1 to p with $\overline{1,p}$. Throughout the sequel we consider that the transfer function matrix $G_{yu} \in \mathcal{F}^{n_y \times n_u}$ is partitioned in p block–rows and m block–columns. The i-th block–row has n_y^i rows, while the j-th block–column has n_u^j columns. Obviously, $\sum_{i=1}^{p} n_y^i = n_y$ and $\sum_{j=1}^{m} n_u^j = n_u$. For $(i,j) \in \overline{1,p} \times \overline{1,m}$, we denote by

$$[G_{yu}]_{ij} \in \mathcal{F}^{n_y^i \times n_u^j}$$

the transfer matrix at the intersection of the i-th block–row and j-th block–column of G_{yu}. Henceforth, we shall use this square bracketed notation for block indexing of transfer function matrices. Analogously, the controller's transfer function matrix $K \in \mathcal{F}^{n_u \times n_y}$ is partitioned in m block–rows and p block–columns, where the j-th block–row has n_u^j rows and the i-th block–column has n_y^i columns. Correspondingly, $[K]_{ji}$ is the notation for the element of $\mathcal{F}^{n_u^j \times n_y^i}$ at the intersection of the j-th block–row and i-th block–column of K.

The decentralized setting will be modeled throughout the paper via the sparsity constraints paradigm, as it has been proved to be a suitable method to formalize many problems in decentralized control. The notation we introduce next is entirely concordant with the one used in [5] and [6] to define the sparsity constraints.

For the $\{0, 1\}$ boolean algebra the operations $(+, \cdot)$ are defined as usual: $0+0 = 0\cdot 1 = 1\cdot 0 = 0 \cdot 0 = 0$ and $1 + 0 = 0 + 1 = 1 + 1 = 1 \cdot 1 = 1$. By a binary matrix we mean a matrix whose entries belong to the set $\{0, 1\}$. Naturally, the addition and multiplication of binary matrices is carried out over the Boolean algebra and under the aforementioned assumptions, the addition and multiplication of binary matrices matrices are defined as in the real case.

Binary matrices will be denoted by capital letters with the "bin" superscript, in order to be distinguished from transfer function matrices over $\mathcal{F}$ which are represented in the sequel by plain capital letters. Furthermore, for binary matrices only, the notation $A^{\text{bin}} \leq B^{\text{bin}}$ means that $a_{ij} \leq b_{ij}$ for all i and j, that is for all the entries of A^{bin} and B^{bin} respectively.

Henceforth, we adopt the convention that the transfer function matrices are indexed by blocks while binary matrices are indexed by each individual entry.

For any binary matrix with m rows and p columns $K^{\text{bin}} \in \{0, 1\}^{m\times p}$, we can define the following linear subspace of $\mathcal{F}^{n_u\times n_y}$:

$$\mathrm{Sparse}(K^{\text{bin}}) \overset{def}{=} \Big\{K \in \mathcal{F}^{n_u\times n_y} \Big|\ \big(K^{\text{bin}}_{ij} = 0\big) \implies \big([K]_{ij} = 0\big); \quad (i, j) \in \overline{1, m} \times \overline{1, p}\Big\} \tag{12}$$

Hence $\mathrm{Sparse}(K^{\text{bin}})$ is the set of all controllers K in $\mathcal{F}^{n_u\times n_y}$ for which $[K]_{ij} = 0$ whenever $K^{\text{bin}}_{ij} = 0$, where by $[K]_{ij} = 0$ we mean that the (i, j)-th block of K is the zero matrix.

Conversely, for any $K \in \mathcal{F}^{n_u\times n_y}$ define $\mathrm{Pattern}(K) \in \{0, 1\}^{m\times p}$ to be the binary matrix given by

$$\mathrm{Pattern}(K)_{ij} = \begin{cases} 0 & \text{if the block } [K]_{ij} = 0; \\ 1 & \text{otherwise .} \end{cases}$$

Accordingly, the binary value of $\mathrm{Pattern}(K)_{kl}$ determines whether controller k may use measurements from the output of the system l, since $[K]_{kl}$ is the map from the outputs of subsystem l to the inputs of subsystem k, while $[G_{yu}]_{ij}$ represents the map from the inputs of subsystem j to the outputs of subsystem i.

Let $K^{\text{bin}} \in \{0, 1\}^{m\times p}$ be the requested sparsity pattern of the controller. Define the subset $\mathcal{S}$

of $\mathcal{F}^{n_u \times n_y}$ as

$$\mathcal{S} \stackrel{def}{=} \left\{ K \in \mathcal{F}^{n_u \times n_y} \middle| \, \text{Pattern}(K) = K^{\,\text{bin}} \right\} \tag{13}$$

as the set of transfer function matrices that satisfy the cotroller's imposed sparsity structure. Similarly, $P^{\,\text{bin}}$ in the set $\{0, 1\}^{m \times p}$, defined as

$$P^{\,\text{bin}} \stackrel{def}{=} \text{Pattern}(G_{yu}) \tag{14}$$

will be the sparsity pattern of the G_{yu} block of the generalized plant.

Remark IV.2.

We are ready now for the main result of this subsection. Suppose that the generalized plant G is stabilizable with a controller $K_0 \in \mathcal{S}$. The decentralized version of the standard problem (9) is formulated by simply adding the extra constraint $K \in \mathcal{S}$, specifically

$$\min_{\substack{K \text{stabilizes } G \\ K \in \mathcal{S}}} \left\| G_{zw} + G_{zu} \, K \Big(I - G_{yu} K \Big)^{-1} G_{yw} \right\| . \tag{15}$$

The following result (in fact a Corollary of Theorem IV.1) is central in our proposed approach:

Corollary IV.3. *Let G be a stabilizable, generalized plant such that the block $G_{yu} \in \mathcal{F}^{n_y \times n_u}$ is strictly causal. Given* any *stabilizable controller $K_0 \in \mathcal{S}$ of G_{yu}, the decentralized control problem (15) is equivalent to*

$$\min_{\substack{Q \in \mathcal{A}^{(n_u+n_y)\times(n_u+n_y)} \\ \Omega_{21}(Q)\Omega_{11}^{-1}(Q) \in \mathcal{S}}} \left\| T_1 \; - \; T_2 \, Q \, T_3 \right\| . \tag{16}$$

where an optimal solution K^* to (15) can always be obtained from the optimal Q in (16), denoted with Q^*, via $K^* = \Omega_{21}(Q^*)\Omega_{11}^{-1}(Q^*)$. Here, T_1, T_2 and T_3 are as in (11).

Proof: As mentioned at the end of the previous section, throughout the paper we are under the assumption that the G_{yu} block is strictly causal. This assumption has the desirable feature of ensuring that (see Remark III.5) every stabilizing controller is causal and that the closed loop is well–posed for any stabilizing controller. Moreover, it implies [4, pp. 232] that the $\Omega_{11}(Q)$ block is nonsingular for any causal Q in the set $\mathcal{A}^{(n_u+n_y)\times(n_u+n_y)}$. Later in the paper, we will point out (Corollary V.2) a straightforward argument on the invertibility of $\Omega_{11}(Q)$.

We start with the conclusion of Theorem IV.1 on the equivalence between (9) and (10). Theorem III.2 ii) implies that *for any* stabilizing controller $K \in \mathcal{S}$, there exists a $Q \in \mathcal{A}^{(n_u+n_y)\times(n_u+n_y)}$ such that $K = \Omega_{21}(Q)\Omega_{11}^{-1}(Q)$ (and therefore $\Omega_{21}(Q)\Omega_{11}^{-1}(Q) \in \mathcal{S}$). Moreover, *for any* $Q \in \mathcal{A}^{(n_u+n_y)\times(n_u+n_y)}$ it follows also by Theorem III.2 that $\Omega_{21}(Q)\Omega_{11}^{-1}(Q)$ is a stabilizing controller of G. Hence the constraint $K \in \mathcal{S}$ in (15) being equivalent to the constraint $\Omega_{21}(Q)\Omega_{11}^{-1}(Q) \in \mathcal{S}$, yields the decentralized model–matching problem (16).

■

Problems (15) and (16) are in fact versions with a smaller feasible set of the equivalent (via Theorem IV.1) problems (9) and (10). Here the additional constraint arises from the required decentraized structure of the controller , i.e. $K \in \mathcal{S}$.

C. Sparsity Constraints on the Youla Parameter

The binary matrix K^{bin} in the set $\{0,1\}^{m\times p}$ will denote from now on the sparsity pattern of the feasible decentralized controllers. Similarly, P^{bin} in the set $\{0,1\}^{p\times m}$ is the sparsity structure of the G_{yu} block of G.

Consider now the following natural partition of the Youla parameter Q (Q is in the set $\mathcal{A}^{(n_u+n_y)\times(n_u+n_y)}$) :

$$Q = \begin{array}{cc} \overbrace{\phantom{Q_{11}}}^{n_y} \; \overbrace{\phantom{Q_{12}}}^{n_u} & \\ \left[\begin{array}{cc} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{array}\right] & \begin{array}{l} \}\, n_y \\ \}\, n_u \end{array} \end{array} \tag{17}$$

Assume for $Q_{12} \in \mathcal{A}^{n_y\times n_u}$ the same partition by blocks as for G_{yu}, from the beginning of the previous subsection. That is, Q_{12} is partitioned in p block–rows and m block–columns and the i-th block–row has n_y^i rows, while the j-th block–column has n_u^j columns. Hence for any $(i,j) \in \overline{1,p} \times \overline{1,m}$ we get that $[Q_{12}]_{ij} \in \mathcal{A}^{n_y^i\times n_u^j}$. Similarly, assume for $Q_{21} \in \mathcal{A}^{n_u\times n_y}$ the same partition by blocks as for K: m block–rows and p block–columns and for any $(j,i) \in \overline{1,m}\times\overline{1,p}$,

$[Q_{21}]_{ji} \in \mathcal{A}^{n_u^j \times n_y^i}$. It follows that Q_{11} is naturally partitioned in p block–rows times p block–columns and the i-th block–row has n_y^i rows, while the i-th block–column has n_y^i columns. Consequently, for any $(i,j) \in \overline{1,p} \times \overline{1,p}$ we get that $[Q_{11}]_{ij} \in \mathcal{A}^{n_y^i \times n_y^j}$. Similarly, Q_{22} has m block–rows and m block–columns and the j-th block–row has n_u^j rows, while the j-th block–column has n_u^j columns.

For the transfer function matrices $Q_{11},\ Q_{12},\ Q_{21}$, and Q_{22} we define next their corresponding sparsity patterns. As usually, in (18) below, Pattern $(\cdot)$ is referred blockwise.

$$\begin{aligned}
\mathbb{Q}_{11} &\overset{def}{=} \Big\{ Q_{11} \in \mathcal{A}^{n_y \times n_y} \Big| \text{Pattern}\ (Q_{11}) = P^{\text{bin}}\, K^{\text{bin}} + I_m \Big\}, \\
\mathbb{Q}_{12} &\overset{def}{=} \Big\{ Q_{12} \in \mathcal{A}^{n_y \times n_u} \Big| \text{Pattern}\ (Q_{12}) = P^{\text{bin}}\, K^{\text{bin}}\, P^{\text{bin}} + P^{\text{bin}} \Big\}, \\
\mathbb{Q}_{21} &\overset{def}{=} \Big\{ Q_{21} \in \mathcal{A}^{n_u \times n_y} \Big| \text{Pattern}\ (Q_{21}) = K^{\text{bin}} \Big\}, \\
\mathbb{Q}_{22} &\overset{def}{=} \Big\{ Q_{22} \in \mathcal{A}^{n_u \times n_u} \Big| \text{Pattern}\ (Q_{22}) = K^{\text{bin}}\, P^{\text{bin}} \Big\}.
\end{aligned} \tag{18}$$

Next, define the subset of $\mathcal{A}^{(n_u+n_y)\times(n_u+n_y)}$

$$\mathbb{Q} \overset{def}{=} \left\{ \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \Big| Q_{11} \in \mathbb{Q}_{11},\ Q_{12} \in \mathbb{Q}_{12},\ Q_{21} \in \mathbb{Q}_{21},\ Q_{22} \in \mathbb{Q}_{22} \right\}. \tag{19}$$

Define the following binary matrix in the set $\{0,1\}^{(p+m)\times(p+m)}$

$$Q^{\text{bin}} \overset{def}{=} \begin{bmatrix} \Big(P^{\text{bin}}\, K^{\text{bin}} + I_m\Big) & \Big(P^{\text{bin}}\, K^{\text{bin}}\, P^{\text{bin}} + P^{\text{bin}}\Big) \\ K^{\text{bin}} & K^{\text{bin}}\, P^{\text{bin}} \end{bmatrix} \tag{20}$$

With this we get the following equivalent characterization of the set $\mathbb{Q}$ from (20):

$$\mathbb{Q} = \Big\{ Q \in \mathcal{A}^{(n_u+n_y)\times(n_u+n_y)} \Big| \text{Pattern}(Q) = Q^{\text{bin}} \Big\}. \tag{21}$$

Remark IV.4. *The alternative characterization of $\mathbb{Q}$ from (21) points out that $\mathbb{Q}$ is perfectly defined solely by the sparsity matrix Q^{bin}. The set $\mathbb{Q}$, contains only those square, stable Youla parameters Q from the set $\mathcal{A}^{(n_u+n_y)\times(n_u+n_y)}$, that have the specific sparsity pattern induced by Q^{bin}.*

D. Quadratic Invariance

From this point on, we take into account the Banach space structure of the linear spaces $\mathcal{F}^{n_u}$ and $\mathcal{F}^{n_y}$. For this, we can consider for instance the $\mathcal{H}^2$ or $\mathcal{H}^\infty$ norms. The definition of these norms for matrices with entries real rational functions of multivariate polynomials is done by a natural extension from the classical case of real rational matrix functions (which are the input/output operator of LTI systems).

Furthermore, under these assumptions, the set $\mathcal{S}$ defined in (13), is a closed linear subspace of $\mathcal{F}^{n_u \times n_y}$.

The following definition is a slight variation of the notion of *quadratic invariance* introduced in [7].

Definition IV.5. *Suppose $P \in \mathcal{F}^{n_y \times n_u}$ and $\mathcal{S} \subset \mathcal{F}^{n_u \times n_y}$. The set $\mathcal{S}$ is called quadratically invariant under P if*

$$K_1 P K_2 \in \mathcal{S} \qquad \text{for all } \; K_1, K_2 \, \in \mathcal{S}$$

Define the set

$$M \overset{def}{=} \Big\{ K \in \mathcal{F}^{n_u \times n_y} \Big| \big(I - PK\big) \text{is invertible} \Big\}.$$

With the same notation as in [7], for any $P \in \mathcal{F}^{n_y \times n_u}$ and any $K \in \mathcal{F}^{n_u \times n_y}$ define the resolvent set as

$$\rho(PK) \overset{def}{=} \big\{ \lambda \in \mathbb{C} \big| \big(\lambda I - PK\big) \text{is invertible} \big\}.$$

We denote by $\rho_{uc}(PK)$ the unbounded connected component of $\rho(PK)$. Clearly $1 \in \rho(PK)$ for all $K \in M$ and we define the subset $N \subseteq M$ as follows

$$N \overset{def}{=} \Big\{ K \in \mathcal{F}^{n_u \times n_y} \Big| 1 \in \rho_{uc}(PK) \Big\}.$$

For any $P \in \mathcal{F}^{n_y \times n_u}$ and any $K \in \mathcal{F}^{n_u \times n_y}$ define the function

$$h_{K,P}(K_1) \overset{def}{=} K_1 \big(I - PK\big)^{-1}. \tag{22}$$

The next theorem follows *mutatis mutandis* on the exact lines of proof of Theorem 7 in [7].

Theorem IV.6. *Suppose $P \in \mathcal{F}^{n_y \times n_u}$ strictly causal and $\mathcal{S} \subseteq \mathcal{F}^{n_u \times n_y}$ is a closed subspace. Suppose as well that $N \cap \mathcal{S} = M \cap \mathcal{S}$. Then*

$$\mathcal{S} \text{ is quadratically invariant under } P \iff h_{K,P}(\mathcal{S} \cap M) = \mathcal{S} \cap M$$

Remark IV.7. *Notice that, according to Theorem 28 in [5], for the case of transfer function matrices, when the subspace $\mathcal{S}$ is defined by block sparsity constraints, quadratic invariance as introduced in Definition (IV.5) is equivalent with the usual definition of quadratic invariance from [5], [6], [7], namely:*

$$KPK \in \mathcal{S} \qquad \text{for all} \ \ K \ \in \mathcal{S}.$$

V. MAIN RESULT

As stated in Theorem III.2 (see also Lemma III.1), *any* stabilizing controller K of G has the form

$$K(Q) \overset{(7)}{=} \Omega_{21}(Q)\Omega_{11}^{-1}(Q) \overset{(24)}{=} \Omega_{21}(Q)\Big(I - P\Omega_{21}(Q)\Big)^{-1} \tag{23}$$

for some stable $Q \in \mathcal{A}^{(n_u+n_y)\times(n_u+n_y)}$. In a completely similar manner with how the strongly–stabilizable case was dealt with in [6], in this section we prove that the information constraints on the controller ($K \in \mathcal{S}$), are equivalent to constraints on the Youla parameter Q (namely that $Q \in \mathbb{Q}$). More specifically, if we impose the constraint $K \in \mathcal{S}$ then *any* such stabilizing controller $K(Q)$ of G is going to be of the form $K(Q) = \Omega_{21}(Q)\Big(I - P\Omega_{21}(Q)\Big)^{-1}$ and it will belong to the set $\Big(\mathcal{C} \ \cap \ \mathcal{S}\Big)$ for some $Q \in \mathbb{Q}$. These important facts are precisely stated in Lemma V.3 and Theorem V.4.

We summarize here the hypothesis and notations that we assume for our main result:

- The set $\mathcal{F}^{n\times r}$ of transfer functions matrices over $\mathcal{F}$ along with the invoked norm is a Banach space. (This will hold true for all the particular instances of $\mathcal{A}$ that we are interested in);
- The given generalized plant G is stabilizable by a controller $K_0 \in \mathcal{S}$;
- The block G_{yu} is strictly causal (see Remark III.5);
- The set $\mathcal{S}$ is quadratically invariant under G_{yu};
- We will denote by P the block $-G_{yu}$ as we refer to it repeatedly, hence $P \overset{def}{=} -G_{yu}$.

The following preliminary result will be needed in the sequel.

Proposition V.1. *For any $Q \in \mathcal{A}^{(n_u+n_y)\times(n_u+n_y)}$, the first block column of $\Omega(Q)$ from (6) has the following expression:*

$$\begin{bmatrix} \Omega_{11}(Q) \\ \Omega_{21}(Q) \end{bmatrix} = \begin{bmatrix} I_{n_y} - P\Omega_{21}(Q) \\ \Omega_{21}(Q) \end{bmatrix} \quad \text{where} \tag{24}$$

$$\Omega_{21}(Q) = \Big(I+K_0P\Big)^{-1}\Big(K_0+K_0PK_0+K_0Q_{11}+K_0Q_{12}K_0+Q_{21}+Q_{22}K_0\Big)\Big(I+PK_0\Big)^{-1} \quad (25)$$

Proof: The proof is purely algebraic and is deferred to the Appendix. ■

Under the strict causality assumption of the block P (see also Remark III.5), an immmediate consequence of the previous proposition is the next Corollary:

Corollary V.2. *The block $\Omega_{11}(Q)$ is invertible for any causal Q in the set $\mathcal{A}^{(n_u+n_y)\times(n_u+n_y)}$.*

Proof: The invertibility of (see (24)) $\Omega_{11}(Q) = I_{n_y} - P\Omega_{21}(Q)$ follows from the causality of the $\Omega_{21}(Q)$ block and the strict causality of P, (see also the statement at the end of first column on page 232 in [4]). ■

Lemma V.3. *Let G be a causal, generalized plant stabilizable with a controller $K_0 \in \mathcal{S}$. Assume that $\mathcal{S}$ is quadratically invariant under the strictly causal block P. The function K : $\mathbb{Q} \longmapsto \Big(\mathcal{C} \, \cap \, \mathcal{S}\Big)$ with $K(Q) \overset{def}{=} \Omega_{21}(Q)\Omega_{11}^{-1}(Q)$ is* onto.

Proof: See Appendix. ■

The previous Lemma is the centerpiece of our main result, as it proves that the constraint $K(Q) \in \mathcal{S}$ (equivalent via (7) with $\Omega_{21}(Q)\Omega_{11}^{-1}(Q) \in \mathcal{S}$) is actually equivalent in problem (16) with the constraint $Q \in \mathbb{Q}$. Hence, the next Theorem is the extension to the general case (non–strongly stabilizable) of the optimal controller design procedure proposed in [6] for strongly stabilizable plants. The equivalent convex program we obtain is utterly similar with the one from ((6), pp. 281 in [6]), only that here, the strong–stabilizability assumption has been removed.

Theorem V.4. *Let G be a causal, generalized plant stabilizable with a controller $K_0 \in \mathcal{S}$. Assume that $\mathcal{S}$ is quadratically invariant under the strictly causal block P. The decentralized optimal control problem (15) is equivalent with the problem*

$$\min_{Q \in \mathbb{Q}} \quad \Big\| \, T_1 \; - \; T_2 \, Q \, T_3 \, \Big\| \, . \quad (26)$$

Proof: It follows from Corollary IV.3 and the previous Lemma. ■

The convex problem (26) is completely similar with problem (6) pp. 281 in [6] which is the equivalent convex problem to solve the decentralized optimal $\mathcal{H}^2$ synthesis for strongly stabilizable plants. If we consider $\mathcal{A} = \mathcal{RH}^2$ then the vectorization techniques from Section VI of [6] are readily applicable for solving (26) and get the optimal $\mathcal{H}^2$ controller, without any strong–stabilizability assumption.

APPENDIX

Proof of Proposition V.1 The following algebraic identities will prove to be useful. They hold true in any ring provided the inverses involved exist:

$$(I + PK)^{-1}P = P(I + KP)^{-1}, \tag{27}$$

$$(I + PK)^{-1} = I - P(I + KP)^{-1}K \tag{28}$$

and their duals

$$(I + KP)^{-1}K = K(I + PK)^{-1}, \tag{29}$$

$$(I + KP)^{-1} = I - K(I + PK)^{-1}P. \tag{30}$$

We start with the Youla–like parametrization (5) of the stabilizable controllers from Theorem III.2. After expanding the expression of the first block–column of $\Omega(Q)$ in (5) we get that

$$\Omega(Q)\begin{bmatrix} I_{n_y} \\ O_{n_u \times n_y} \end{bmatrix} = \\ = \begin{bmatrix} \big(I + PK_0\big)^{-1} - I & -P\big(I + K_0P\big)^{-1} \\ K_0\big(I + PK_0\big)^{-1} & \big(I + K_0P\big)^{-1} \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \times \\ \times \begin{bmatrix} \big(I + PK_0\big)^{-1} \\ K_0\big(I + PK_0\big)^{-1} \end{bmatrix} + H(P, K_0) \begin{bmatrix} I_{n_y} \\ O_{n_u \times n_y} \end{bmatrix} \tag{31}$$

$$
\begin{aligned}
&\overset{(28,29)}{=} \begin{bmatrix} -PK_0\big(I+PK_0\big)^{-1} & -P\big(I+K_0P\big)^{-1} \\ K_0\big(I+PK_0\big)^{-1} & \big(I+K_0P\big)^{-1} \end{bmatrix} \times \\
&\qquad \times \begin{bmatrix} Q_{11}\big(I+PK_0\big)^{-1} + Q_{12}K_0\big(I+PK_0\big)^{-1} \\ Q_{21}\big(I+PK_0\big)^{-1} + Q_{22}K_0\big(I+PK_0\big)^{-1} \end{bmatrix} + H(P,K_0) \begin{bmatrix} I_{n_y} \\ O_{n_u \times n_y} \end{bmatrix} \\
&\overset{(27,28,29,30,3)}{=} \begin{bmatrix} -P\Big(I+K_0P\Big)^{-1}\Big(K_0Q_{11} + K_0Q_{12}K_0 + Q_{21} + Q_{22}K_0\Big)\Big(I+PK_0\Big)^{-1} \\ \Big(I+K_0P\Big)^{-1}\Big(K_0Q_{11} + K_0Q_{12}K_0 + Q_{21} + Q_{22}K_0\Big)\Big(I+PK_0\Big)^{-1} \end{bmatrix} + \\
&\qquad + \begin{bmatrix} I - P\big(I+K_0P\big)^{-1}K_0 \\ K_0\big(I+PK_0\big)^{-1} \end{bmatrix} \\
&= \begin{bmatrix} I - P\Big(I+K_0P\Big)^{-1}\Big(K_0 + K_0PK_0 + K_0Q_{11} + K_0Q_{12}K_0 + Q_{21} + Q_{22}K_0\Big)\Big(I+PK_0\Big)^{-1} \\ \Big(I+K_0P\Big)^{-1}\Big(K_0 + K_0PK_0 + K_0Q_{11} + K_0Q_{12}K_0 + Q_{21} + Q_{22}K_0\Big)\Big(I+PK_0\Big)^{-1} \end{bmatrix}
\end{aligned}
\tag{32}
$$

which is the desired expression.

Proof of Lemma V.3 We divide the proof in two parts: in part **I)** we prove that the function $K(\cdot)$ is a well–defined function indeed, from $\mathbb{Q}$ to $\big(\mathcal{C} \ \cap \ \mathcal{S}\big)$. In part **II)** we show that the function $K(\cdot)$ is *onto*.

I) Let $Q \in \mathbb{Q}$ be arbitrary but fixed. Since $Q \in \mathcal{A}^{(n_u+n_y)\times(n_u+n_y)}$ then by Theorem III.2 ii) we get that $K(Q) \in \mathcal{C}$, so it only remains to show that $K(Q) \in \mathcal{S}$.

We expand the product in (25) to get that $\Omega_{21}(Q)$ (in the form provided by Proposition V.1) is the sum of the following six terms:

$$
\begin{aligned}
\Omega_{21}(Q) = &\underbrace{\big(I+K_0P\big)^{-1}K_0\big(I+PK_0\big)^{-1}}_{t_1} + \underbrace{\big(I+K_0P\big)^{-1}K_0PK_0\big(I+PK_0\big)^{-1}}_{t_2} + \\
&\underbrace{\big(I+K_0P\big)^{-1}K_0Q_{11}\big(I+PK_0\big)^{-1}}_{t_3} + \underbrace{\big(I+K_0P\big)^{-1}K_0Q_{12}K_0\big(I+PK_0\big)^{-1}}_{t_4} +
\end{aligned}
\tag{33}
$$

$$\underbrace{\big(I+K_0P\big)^{-1}Q_{21}\big(I+PK_0\big)^{-1}}_{t_5}+\underbrace{\big(I+K_0P\big)^{-1}Q_{22}K_0\big(I+PK_0\big)^{-1}}_{t_6}$$

To start with, we prove that $\Omega_{21}(Q)$ is in $\mathcal{S}$. We prove this by showing that each of the six terms of the sum in (33) are in $\mathcal{S}$. Since $\mathcal{S}$ is a (closed) linear subspace it will follow that $\Omega_{21}(Q)$ is in $\mathcal{S}$ indeed. Denote

$$\Delta_0 \stackrel{def}{=} K_0(I+PK_0)^{-1} \tag{34}$$

which belongs to $\mathcal{S}$ by Theorem 7 in [7], and the fact that $\mathcal{S}$ is quadratically invariant under P.

The first term in (33) is

$$t_1=\Big((I+K_0P)^{-1}K_0\Big)(I+PK_0)^{-1} \stackrel{(29)}{=} \Delta_0(I+PK_0)^{-1}$$

which is in $\mathcal{S}$ by Theorem IV.6.

The second term

$$t_2=\Big((I+K_0P)^{-1}K_0\Big)P\Big(K_0(I+PK_0)^{-1}\Big) \stackrel{(29)}{=} \Delta_0P\Delta_0$$

which is in $\mathcal{S}$ because $\Delta_0\in\mathcal{S}$ and $\mathcal{S}$ is quadratically invariant under P.

Since $Q_{11}\in\mathbb{Q}_{11}$ then from (18) we get that $\mathrm{Pattern}(Q_{11})=P^{\mathrm{bin}}K^{\mathrm{bin}}+I_m$. Furthermore

$$\begin{aligned}&\mathrm{Pattern}\big(\Delta_0Q_{11}\big)=\mathrm{Pattern}(\Delta_0)\mathrm{Pattern}(Q_{11})=K^{\mathrm{bin}}\big(P^{\mathrm{bin}}K^{\mathrm{bin}}+I_m\big)\\&=K^{\mathrm{bin}}P^{\mathrm{bin}}K^{\mathrm{bin}}+K^{\mathrm{bin}}=K^{\mathrm{bin}}+K^{\mathrm{bin}}=K^{\mathrm{bin}}\end{aligned} \tag{35}$$

because $K^{\mathrm{bin}}P^{\mathrm{bin}}K^{\mathrm{bin}}=K^{\mathrm{bin}}$ due to quadratic invariance and obviously $K^{\mathrm{bin}}+K^{\mathrm{bin}}=K^{\mathrm{bin}}$ due to the way addition is defined for binary matrices.

Define

$$W_{11} \stackrel{def}{=} \big(\Delta_0Q_{11}\big)$$

and since $\mathrm{Pattern}(W_{11})=K^{\mathrm{bin}}$ we conclude $W_{11}\in\mathcal{S}$. The third term is

$$t_3=\Delta_0Q_{11}(I+PK_0)^{-1}=W_{11}(I+PK_0)^{-1}$$

and it belongs to $\mathcal{S}$ by Theorem IV.6.

Since $Q_{12}\in\mathbb{Q}_{12}$ then from (18) we know that

$$\text{Pattern}(Q_{12}) = P^{\text{bin}}K^{\text{bin}}P^{\text{bin}} + P^{\text{bin}}$$

The fourth term is $t_4 = \Delta_0 Q_{12} \Delta_0$. It follows that

$$\begin{aligned}
&\text{Pattern}\big(\Delta_0 Q_{12}\Delta_0\big) = \text{Pattern}\big(\Delta_0\big)\text{Pattern}\big(Q_{12}\big)\text{Pattern}\big(\Delta_0\big) \\
&= K^{\text{bin}}\big(P^{\text{bin}}K^{\text{bin}}P^{\text{bin}} + P^{\text{bin}}\big)K^{\text{bin}} = K^{\text{bin}}\big(P^{\text{bin}}K^{\text{bin}}P^{\text{bin}}\big)K^{\text{bin}} + K^{\text{bin}}P^{\text{bin}}K^{\text{bin}} \\
&= \big(K^{\text{bin}}P^{\text{bin}}K^{\text{bin}}\big)P^{\text{bin}}K^{\text{bin}} + K^{\text{bin}} = K^{\text{bin}}P^{\text{bin}}K^{\text{bin}} + K^{\text{bin}} = K^{\text{bin}} + K^{\text{bin}} = K^{\text{bin}}
\end{aligned} \tag{36}$$

Since $\text{Pattern}(t_4) = K^{\text{bin}}$ we get that $t_4 \in \mathcal{S}$ as well.

Because $Q_{21} \in \mathbb{Q}_{21}$, from (18) we get that $Q_{21} \in \mathcal{S}$. But then $Q_{21}(I + PK_0)^{-1} \in \mathcal{S}$ by Theorem IV.6. Denote

$W_{21} \overset{def}{=} Q_{21}(I + PK_0)^{-1}$. The fifth term is then

$$t_5 = (I + K_0P)^{-1}\Big(Q_{21}(I + PK_0)^{-1}\Big) = (I + K_0P)^{-1}W_{21} \overset{(29)}{=} W_{21}(I + PK_0)^{-1}$$

which is in $\mathcal{S}$ by Theorem IV.6.

Finally, $Q_{22} \in \mathbb{Q}_{22}$ and (18) implies that $\text{Pattern}(Q_{22}) = K^{\text{bin}}P^{\text{bin}}$ and so

$$\begin{aligned}
&\text{Pattern}\big(Q_{22}\Delta_0\big) = \text{Pattern}\big(Q_{22}\big)\text{Pattern}\big(\Delta_0\big) = \big(K^{\text{bin}}P^{\text{bin}}\big)K^{\text{bin}} \\
&= K^{\text{bin}}P^{\text{bin}}K^{\text{bin}} = K^{\text{bin}}
\end{aligned} \tag{37}$$

Denote

$$W_{22} \overset{def}{=} Q_{22}\Delta_0.$$

Since $\text{Pattern}(W_{22}) = K^{\text{bin}}$ we get that $W_{22} \in \mathcal{S}$. Therefore the sixth and last term

$$t_6 = (I + K_0P)^{-1}\big(Q_{22}\Delta_0\big) = (I + K_0P)^{-1}W_{22} \overset{(29)}{=} W_{22}(I + PK_0)^{-1}$$

and it belongs to $\mathcal{S}$ by Theorem IV.6.

We have just proved that $\Omega_{21}(Q) \in \mathcal{S}$ for any $Q \in \mathbb{Q}$. It follows then by Theorem 7 in [7] that

$$K(Q) \stackrel{(7)}{=} \Omega_{21}(Q)\Omega_{11}^{-1}(Q) \stackrel{(24)}{=} \Omega_{21}(Q)\Big(I - P\Omega_{21}(Q)\Big)^{-1} \in \mathcal{S}.$$

and the first part of the proof ends.

II) Let be $\overline{K} \in \Big(\mathcal{C} \ \cap \ \mathcal{S}\Big)$, arbitrary chosen. We will prove that there exists a $\overline{Q} \in \mathbb{Q}$ such that $K(\overline{Q}) = \overline{K}$. We show that such a $\overline{Q}$ is given by

$$\overline{Q} = \begin{bmatrix} -\big(I + P\overline{K}\big)^{-1} & -P\big(I + \overline{K}P\big)^{-1} \\ \overline{K}\big(I + P\overline{K}\big)^{-1} & I - \big(I + \overline{K}P\big)^{-1} \end{bmatrix}. \tag{38}$$

Note that

$$\overline{Q} \stackrel{(3)}{=} \begin{bmatrix} -I & O \\ O & I \end{bmatrix} H(P, \overline{K}) \begin{bmatrix} I & O \\ O & -I \end{bmatrix} + \begin{bmatrix} O & O \\ O & I \end{bmatrix}$$

and because $\overline{K} \in \mathcal{C}$ implies that $H(P, \overline{K}) \in \mathcal{A}^{(n_u+n_y)\times(n_u+n_y)}$, we get that $\overline{Q}$ is in the set $\mathcal{A}^{(n_u+n_y)\times(n_u+n_y)}$ as well.

Next, denote

$$\overline{\Delta} \stackrel{not}{=} \overline{K}\big(I + P\overline{K}\big)^{-1}. \tag{39}$$

By the quadratic invariance of $\mathcal{S}$ under P and then by Theorem 7 in [7], it follows that $\overline{\Delta} \in \mathcal{S}$. Furthermore,

$$\overline{Q} = \begin{bmatrix} \big(P\overline{\Delta} - I\big) & \big(P\overline{\Delta}P - P\big) \\ \overline{\Delta} & \overline{\Delta}P \end{bmatrix} \tag{40}$$

because

$$\begin{aligned} &P\overline{\Delta} - I \stackrel{(39)}{=} P\overline{K}\big(I+P\overline{K}\big)^{-1} - I \stackrel{(29)}{=} P\big(I+\overline{K}P\big)^{-1}\overline{K} - I \stackrel{(28)}{=} I - \big(I+P\overline{K}\big)^{-1} - I = -\big(I+P\overline{K}\big)^{-1}, \\ &P\overline{\Delta}P - P = -\big(I + P\overline{K}\big)^{-1}P \stackrel{(27)}{=} -P\big(I + P\overline{K}\big)^{-1}, \\ &\overline{\Delta}P \stackrel{(39)}{=} \overline{K}\big(I + P\overline{K}\big)^{-1}P \stackrel{(30)}{=} I - \big(I + \overline{K}P\big)^{-1}. \end{aligned} \tag{41}$$

It is pretty straightforward now via the definitions in (18) that $\overline{Q}_{11} = (P\overline{\Delta} - I) \in \mathbb{Q}_{11}$, $\overline{Q}_{12} = (P\overline{\Delta}P - P) \in \mathbb{Q}_{12}$, $\overline{Q}_{21} = \overline{\Delta} \in \mathbb{Q}_{21}$ and $\overline{Q}_{22} = \overline{\Delta}P \in \mathbb{Q}_{22}$.

We have just proved that $\overline{Q} \in \mathbb{Q}$ so it only remains to prove that $K(\overline{Q}) = \overline{K}$. By plugging (40) in (25) we get

$$\begin{aligned}
\Omega_{21}(\overline{Q}) &= \Big(I+K_0P\Big)^{-1}\Big(K_0+K_0PK_0+K_0(P\overline{\Delta}-I)+K_0(P\overline{\Delta}P-P)K_0+\overline{\Delta}+\overline{\Delta}PK_0\Big)\Big(I+PK_0\Big)^{-1} \\
&= \Big(I + K_0P\Big)^{-1}\Big(K_0P\overline{\Delta} + K_0P\overline{\Delta}PK_0 + \overline{\Delta} + \overline{\Delta}PK_0\Big)\Big(I + PK_0\Big)^{-1} \\
&= \Big(I + K_0P\Big)^{-1}\Big(K_0P\overline{\Delta}(I + PK_0) + \overline{\Delta}(I + PK_0)\Big)\Big(I + PK_0\Big)^{-1} \\
&= \Big(I + K_0P\Big)^{-1}\Big(K_0P\overline{\Delta} + \overline{\Delta}\Big) \\
&= \Big(I + K_0P\Big)^{-1}\Big(I + K_0P\Big)\overline{\Delta} \\
&= \overline{\Delta}
\end{aligned} \tag{42}$$

Finally

$$K(\overline{Q}) \overset{(24)}{=} \Omega_{21}(\overline{Q})\Big(I - P\Omega_{21}(\overline{Q})\Big)^{-1} \overset{(42)}{=} \overline{\Delta}\Big(I - P\overline{\Delta}\Big)^{-1} \overset{(39)}{=} \overline{K}$$

hence the proof.

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