ABSTRACT

Title of dissertation:	EXISTENCE AND WEAK-STRONG UNIQUENESS FOR THE NAVIER-STOKES-SMOLUCHOWSKI SYSTEM OVER MOVING DOMAINS
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This dissertation concerns the well-posedness of the Navier-Stokes-Smoluchowski system. The system models a mixture of fluid and particles in the so-called bubbling regime. The compressible Navier-Stokes equations governing the evolution of the fluid are coupled to the Smoluchowski equation for the particle density at a continuum level.

First, working on fixed domains, the existence of weak solutions is established using a three-level approximation scheme and based largely on the Lions-Feireisl theory of compressible fluids.

The system is then posed over a moving domain. By utilizing a Brinkman-type penalization as well as penalization of the viscosity, the existence of weak solutions of the Navier-Stokes-Smoluchowski system is proved over moving domains. As a corollary the convergence of the Brinkman penalization is proved.

Finally, a suitable relative entropy is defined. This relative entropy is used to establish a weak-strong uniqueness result for the Navier-Stokes-Smoluchowski

system over moving domains, ensuring that strong solutions are unique in the class of weak solutions.

Existence and weak-strong uniqueness for the Navier-Stokes-Smoluchowski system over moving domains

by

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Dedication

To my parents and family.

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List of Abbreviations

ALE	Arbitrary Lagrangian-Eulerian
NSS	Navier-Stokes-Smoluchowski

Chapter 1: Introduction

Models of fluids arise in a variety of physical phenomena including combustion, atmospheric dynamics, pollution control, amongst others. When mixtures of liquid, gas, or solid phases appear, the resulting flow is called *multiphase flow*. In this dissertation, we consider a two-phase flow consisting of a bulk fluid phase (liquid or gas), as well as a dispersed (particulate) phase. The particles are assumed to be light compared to the fluid and therefore due to buoyancy effects they tend to 'bubble' upwards.

Both fluid and particles are modeled at a macroscopic level, that is, we ignore the individual behavior of particles at a microscopic scale. The fluid will be governed by the compressible Navier-Stokes equations, describing the evolution of a linearly viscous compressible fluid. The dispersed phase is governed by a convection-diffusion equation, the so-called Smoluchowski equation, and coupled to the fluid via a drag force. The modeling of the Navier-Stokes-Smoluchowski (NSS) system is described in Section 1.1.

Applications of fluid-particle systems are numerous: hemodynamics ([43], [53], [62]), swarms ([8], [30]), suspensions ([16], [68]), sprays and aerosols ([3], [4], [10], [49], [59]), combustion [71], and sedimentation [11]. The macroscopic NSS model presented in Section 1.1 is in part[1] introduced by Carrillo et al. [18] as a formal hydrodynamic limit from a kinetic-fluid system. See [12], [13], and [69] for other fluid-particle models and modeling considerations. Some numerical schemes for models similiar to the NSS system can be found in [19], [48]. For a practical treatment of multiphase flows, see [14].

1.1 The Navier-Stokes-Smoluchowski system

The NSS equations describe the evolution of a fluid-particle system, and consist of a set of PDE for the fluid velocity $\mathbf{u} = \mathbf{u}(t, x)$, fluid density $\rho = \rho(t, x)$, and particle density $\eta = \eta(t, x)$:

$$\partial_t \rho + \mathrm{div}_x(\rho \mathbf{u}) = 0, \tag{1.1a}$$

$$\partial_t \eta + \mathrm{div}_x(\eta(\mathbf{u} - \nabla_x \Phi)) - \Delta_x \eta = 0, \tag{1.1b}$$

$$\partial_t(\rho \mathbf{u}) + \mathrm{div}_x(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla_x(p(\rho) + \eta) = \mathrm{div}_x \mathbb{S} - (\eta + \beta\rho)\nabla_x \Phi. \tag{1.1c}$$

These are respectively equations for the conservation of fluid mass, conservation of particle mass, and conservation of momentum. Except for the parameter β, all physical constants in the equations have been set to unity. The system (1.1) is posed over the space-time domain $((0, T) \times \Omega_t) \subset \mathbb{R} \times \mathbb{R}^3$, where $T \geq 0$ is fixed. Note that the spatial domain $\Omega_t \equiv \Omega(t)$ is in general time-dependent.

We assume the fluid pressure $p(\rho)$ follows the isentropic pressure law

$$p(\rho) = a\rho^\gamma, \quad \gamma > \frac{3}{2}, \tag{1.2}$$

[1] Viscous terms are ignored due to modeling considerations, while in this work they are retained.

where $a > 0$ is a constant, which we set to unity in the sequel. The restriction on the value of γ is a technicality needed for the existence theory presented in Chapter 2. The stress tensor is assumed to obey Newton's law of viscosity,

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu(\nabla_x \mathbf{u} + (\nabla_x \mathbf{u})^T) + \lambda \mathrm{div}_x \mathbf{u} \mathbb{I}. \tag{1.3}$$

The viscosity coefficients are assumed to be constant and satisfy

$$\mu > 0, \quad \lambda + \frac{2}{3}\mu \geq 0,$$

where μ is the shear (dynamic) viscosity coefficient and $\zeta \equiv \lambda + (2/3)\mu$ is the bulk viscosity coefficient.

The external potential Φ, incorporating external forces through $\nabla_x \Phi$, is determined up to a constant. We therefore assume Φ is non-negative and in addition suppose $\Phi \in C^1(\overline{\Omega})$.

The Navier-Stokes equations are well-known and well-studied. Issues of modeling and applications can be found in [66]. Let us make some comments regarding the particle contribution.

First, the pressure term in the momentum equation,

$$P(\rho, \eta) = p(\rho) + p_\eta(\eta) = \rho^\gamma + \eta,$$

contains contributions from the fluid and the particle density. Therefore the particle density gradient also drives the fluid evolution. Such a 'particle pressure' is oftentimes neglected in particle-fluid dynamics, though it can influence the overall behavior of the system (cf. [17], [42]).

In order to explain the coupling between the particles and fluid, the kinetic-fluid origin of the NSS system (1.1) is examined next.

The NSS system is formally derived from a Vlasov-Fokker-Planck type equation in [18], describing the evolution of the particle mass density function $f(t,x,v)$, and coupled to a compressible Euler system for the fluid, where v represents the microscopic 'actual' velocity of a particle. Assume the particles have constant mass m_p. The function $f(t,x,v)$ has the interpretation that $f(t,x,v)\mathrm{d}x\mathrm{d}v$ is the total mass of particles enclosed at time $t \geq 0$ in the infinitesimal domain of phase space centered on $(x,v) \in \mathbb{R}^3 \times \mathbb{R}^3$ with volume $\mathrm{d}x\mathrm{d}v$, standard in the kinetic theory of gases. Macroscopic observables are then obtained as averages with respect to the velocity v. For instance, the macroscopic particle mass density $\eta(t,x)$ is defined by

$$\eta(t,x) = \int_{\mathbb{R}^3} f(t,x,v)\,\mathrm{d}v.$$

The coupling between the fluid and particles is obtained from the mutual drag force, $\mathcal{F}(t,x,v)$, modeled as the linear Stokes' drag

$$\mathcal{F}(t,x,v) = 6\pi\mu a(\mathbf{u}(t,x) - v), \tag{1.4}$$

where a is the typical particle radius (assumed constant) and $\mathbf{u}(t,x)$ is the local velocity of the fluid. We assume the particle radius is small so that the low Reynold's number assumption for the Stokes' drag is satisfied. The force exerted by the particles on the fluid, denoted $\mathcal{F}_{fl}$, is then obtained by taking moments as

$$\mathcal{F}_{fl}(t,x) = 6\pi\mu a \int_{\mathbb{R}^3} (v - \mathbf{u}(t,x))f(t,x,v)\,\mathrm{d}v. \tag{1.5}$$

In fact, the NSS system (1.1) describes fluid-particle systems in the so-called *bubbling* *regime*. To make this more precise, we will now (formally) take the hydrodynamic limit of the governing kinetic-fluid model. For rigorous hydrodynamic limits in a similar context, see [46], [47], [56].

As our starting point, we consider the dimensionless system

$$\partial_t f + \sigma v \cdot \nabla_x f - \kappa \nabla_x \Phi \cdot \nabla_v f = \frac{1}{\varepsilon} \mathrm{div}_v((v - \sigma^{-1} u) f + \nabla_v f), \tag{1.6a}$$

$$\partial_t \rho + \mathrm{div}_x(\rho \mathbf{u}) = 0, \tag{1.6b}$$

$$\partial_t(\rho \mathbf{u}) + \mathrm{div}_x(\rho \mathbf{u} \otimes \mathbf{u}) + \chi \nabla_x p(\rho) + \alpha\sigma\kappa\rho\nabla_x \Phi = \frac{1}{\varepsilon}\frac{\rho_p}{\rho_f}(J - \eta \mathbf{u}) + \frac{2}{9}\left(\frac{a}{L}\right)^2 \frac{1}{\varepsilon}\frac{\rho_p}{\rho_f} \mathrm{div}_x \mathbb{S}, \tag{1.6c}$$

which can be found in [18]. This system consists of the Vlasov-Fokker-Planck equation (1.6a) coupled to the compressible Navier-Stokes system (1.6b), (1.6c). Let the parameters L, T, and U, denote typical length, time, and velocity scales respectively. Let ρ_p denote a typical particle density and let ρ_f denote a typical fluid density. The parameter χ represents the ratio of a typical pressure to the dynamic pressure $\rho_f U^2$, and will be set equal to unity. The particle momentum J is defined by

$$J = \sigma \int_{\mathbb{R}^3} v f \, \mathrm{d}v,$$

where $\sigma := \dfrac{v_{th}}{U}$. The quantity v_{th} is a measure of the fluctuation of particle velocity. For instance, if the particles' velocities are described by a Maxwell-Boltzmann distribution, then one possible definition is $v_{th} = \sqrt{k\theta/m_p}$, the root mean square in one direction, where k is the Boltzmann constant and θ is the temperature.

The parameter α is supposed to represent the different effects the potential Φ has on the fluid and dispersed phases. For instance, consider Φ incorporating gravity in the vertical direction. The dispersed phase is then acted upon by a gravitational and buoyancy force. The force on a particle is given by $-m_p \nabla_x \Phi$, where m_p is a typical particle mass, and $\Phi = (1 - \rho_f/\rho_p)\, g x_3$. The force exerted on the fluid should then be $\rho_f g \hat{e}_3 = \alpha \rho_f \nabla_x \Phi$, with $\alpha = (1 - \rho_f/\rho_p)^{-1}$. A typical situation in this context is air bubbles in water, in which case $\rho_f/\rho_p \approx 10^3$, and $\alpha \approx -10^{-3}$.

The rest of the parameters are defined as follows:

$$\frac{1}{\varepsilon} = \frac{T}{\tau_s}, \qquad \kappa = \frac{v_s}{v_{th}} \frac{T}{\tau_s}.$$

The quantities τ_s and v_s are the Stokes settling time and settling velocity of a particle. Under the effects of drag, buoyancy, and weight, a particle's vertical position $X(t)$ relative to rest will evolve like

$$\frac{\mathrm{d}X(t)}{\mathrm{d}t} = v_s \left(1 - e^{-t/\tau_s}\right),$$

as can be seen by a force balance. The velocity v_s is therefore the terminal speed of a particle settling and τ_s is a natural relaxation time (the time it takes for the particle to reach 63% of its terminal speed). With $\varepsilon << 1$ denoting a small parameter, we therefore assume τ_s is small.

In order for the energy of the system to dissipate, it is required that

$$\sigma = \kappa, \qquad \frac{\rho_p}{\rho_f} = \frac{1}{\sigma^2}.$$

See [18] for details on this condition. We can now scale the system by setting

$$\sigma = \frac{1}{\sqrt{\varepsilon}}.$$

Using this scaling in the definitions above implies the following:

$$\frac{\rho_p}{\rho_f} = \varepsilon,$$

and

$$v_s \approx U << v_{th}.$$

The system bears the name bubbling in particular due to the fact that the particle density is assumed to be much less than the fluid density.

Taking moments of equation (1.6a) with respect to 1 and v, using the ε scalings, and employing Corollary 1 of [18] yields

$$\partial_t \eta + \mathrm{div}_x J = 0, \tag{1.7a}$$

$$\varepsilon \partial_t J + \nabla_x \eta + \eta \nabla_x \Phi = -J + \eta \mathbf{u}. \tag{1.7b}$$

Formally letting $\varepsilon \to 0$ in (1.7b) yields

$$J = \eta \mathbf{u} - \nabla_x \eta - \eta \nabla_x \Phi.$$

Inserting this J into (1.7a) we then recover the Smoluchowski equation (1.1b), while inserting into (1.6c) we recover the momentum equation (1.1c).

Remark 1: The model (1.6) slightly differs from that of [18], where the Euler equations are considered. There they neglect the viscous terms (the last term in (1.6c)), due to the assumption $a << L$, where a is the particle radius and L is a typical length scale for the problem. The system (1.1) as stated is then more similar to the one analyzed in [21], and keeping the viscous term is necessary to stay in the Navier-Stokes regime.

Remark 2: In taking the hydrodynamic limit as $\varepsilon \to 0$, we have overlooked the term $\frac{1}{\varepsilon} \alpha \rho \nabla_x \Phi$, where an assumption needs to be made on the coefficient. In [18], it is assumed that $\alpha = \mathrm{sign}(\alpha)\varepsilon$, in which case $\beta = \mathrm{sign}(\alpha)$ in equation (1.1c). This is in contrast to [21] where β seems to be defined as a more general parameter. We choose the setting of [21] and include a generic β in this term. In any case, the hydrodynamic limit only requires that $\alpha = O(\varepsilon)$ as $\varepsilon \to 0$. This is verified for the case of gravitational forces as $\alpha = \varepsilon/(\varepsilon - 1)$ when $\rho_p/\rho_f = \varepsilon$

1.2 Existence theory

Regarding the compressible Navier-Stokes equations, the existence of weak solutions is originally due to Lions [51] for γ-type pressure laws with $\gamma \geq 9/5$ (in three dimensions). Feireisl [34] later extended Lions' results to the range $\gamma > 3/2$, and the existence of weak solutions in this case was proven by Feireisl et al. [38]. The existence theory nowadays is often referred to as the Lions-Feireisl theory.

The existence of weak solutions for the Navier-Stokes-Smoluchowski system is originally proved in Carrillo et al. [21] using a time-discretization scheme and arguments of Lions. Another proof in the context of the Feireisl theory is provided by Ballew et al. [9]. For some work related to the $1d$ model, see [32], [65].

Let us now briefly mention the key ideas regarding the proof of the existence of weak solutions to the NSS system, leaving the details to Chapter 2.

The proof relies on a three-level approximation scheme and weak compactness arguments ensuring *weak stability* of the class of weak solutions. The first level approximation relies on a Galerkin method, with the density and fluid velocity approximated by a smooth-enough family dense in a suitable function space. At the second level of the approximation, the continuity equation (1.1a) is modified by the addition of a Laplacian term (i.e. *artificial viscosity*), regularizing the fluid density and providing spatial compactness in the fluid density. To save an energy estimate, the momentum equation (1.1c) is also modified at this level. Finally, the third level of the approximation consists of the addition of an *artificial pressure* term, providing us with increased integrability of the fluid density and necessary to save an estimate at the second level of the approximation.

Having set the approximation scheme, the existence of solutions is proved locally in time using a fixed-point argument and then extended to the full time interval $[0, T]$ using uniform-in-time estimates. Uniform estimates provided by an energy inequality at the Galerkin level then allow us to pass to the limit in the first level of the approximation. At each level, weak lower semicontinuity of the norms allow us to keep the energy inequality valid.

In passing to the limit at the second and third levels of the approximation, the linear terms in (1.1) cause no difficulty. The nonlinear convective terms ($\rho\mathbf{u}, \eta\mathbf{u}$, and $\rho\mathbf{u} \otimes \mathbf{u}$) are handled in a natural way as the NSS system provides estimates on the time derivatives ($\partial_t \rho, \partial_t \eta$, and $\partial_t(\rho\mathbf{u})$). Along with a priori estimates on the velocity $\mathbf{u} \in L^2(0, T; H_0^1(\Omega; \mathbb{R}^3))$, the passage to the limit in the convective terms follows.

The difficuly is passing to the limit in the pressure term $p(\rho) = \rho^\gamma$. First, the a priori estimates only bound the pressure in $L^\infty(0, T; L^1(\Omega))$, which only allows passage weakly as a measure. It is therefore necessary to prove estimates on the pressure of the form

$$\int_0^T \int_\Omega \rho^{\gamma+\omega}\,\mathrm{d}x\mathrm{d}t \leq c,$$

where $\omega > 0$ is small. This type of estimate, first shown by P.L. Lions, allows us to pass to the limit weakly in the pressure to deduce

$$\rho_{\varepsilon,\delta}^{\gamma} \rightharpoonup \overline{\rho^{\gamma}},$$

in a suitable Lebesgue space, where the subscripts ε, δ indicate the second and third levels of the approximation respectively, and the overbar indicates a weak limit. The trick is now to show that in fact $\rho^\gamma = \overline{\rho^\gamma}$ almost everywhere, which requires strong convergence of the fluid density.

To this end, two tools are employed: renormalization of the continuity equation (1.1a) and weak-continuity of the so-called *effective viscous pressure*, defined as

$$P_{eff} = \rho^\gamma - (\lambda + 2\mu)\mathrm{div}_x\mathbf{u}.$$

The renormalization property essentially says that provided the density ρ is square integrable, we are allowed to conclude that for a suitable function $B(\rho)$ that

$$\partial_t B(\rho) + \mathrm{div}_x(B(\rho)\mathbf{u}) + (B'(\rho)\rho - B(\rho))\mathrm{div}_x\mathbf{u} = 0$$

holds in the sense of distributions. This is where Lions requires that $\gamma \geq 9/5$, ensuring the square-integrability of the density in light of the pressure estimates. Choosing the convex function $B(\rho) = \rho\log\rho$ in the renormalized equation for both $\rho_{\varepsilon,\delta}$ and the limit density ρ, we can show that

$$0 \leq \int_\Omega \left(\overline{\rho\log\rho} - \rho\log\rho\right)(t)\,\mathrm{d}x \leq \int_0^t \int_\Omega \rho\mathrm{div}_x\mathbf{u} - \overline{\rho\mathrm{div}_x\mathbf{u}}\,\mathrm{d}x\mathrm{d}t. \tag{1.8}$$

The left inequality follows from the convexity of the map $z \mapsto z \log z$. Provided we can show the upper bound of this inequality is non-positive, it follows that

$$\rho \log \rho = \overline{\rho \log \rho} \quad a.a.$$

From this equality we deduce strong convergence of the density $\rho_{\varepsilon,\delta}$ again due to convexity of the map $z \mapsto z \log z$. To close the argument, the weak continuity of the effective viscous pressure is used to ensure the non-positivity of the upper bound in (1.8) by transferring information from the monotonicity of the pressure ρ^γ to the terms $\rho \mathrm{div}_x \mathbf{u}$. In particular the result on the effective viscous pressure reads[2]

$$\begin{aligned} &\lim_{\varepsilon,\delta \to 0} \int_0^T \int_\Omega (\rho_{\varepsilon,\delta}^\gamma - (\lambda + 2\mu)\mathrm{div}_x \mathbf{u}_{\varepsilon,\delta})\rho_{\varepsilon,\delta} \, \mathrm{d}x\mathrm{d}t \\ &= \int_0^T \int_\Omega (\overline{\rho^\gamma} - (\lambda + 2\mu)\mathrm{div}_x \mathbf{u})\rho \, \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{1.9}$$

This highly nontrivial equality, along with the monotonicity of the pressure implying

$$\int_0^T \int_\Omega \overline{\rho^\gamma}\rho \, \mathrm{d}x\mathrm{d}t \le \liminf_{\varepsilon,\delta \to 0} \int_0^T \int_\Omega \rho_{\varepsilon,\delta}^{\gamma+1} \, \mathrm{d}x\mathrm{d}t,$$

allows us to conclude the non-positivity of the upper bound in (1.8), and therefore deduce strong convergence of the density. This allows us to pass to the limit in the pressure term.

Finally, Feireisl [34] showed that $(\rho, \mathbf{u})$ is a renormalized solution even if the density is not square integrable, by obtaining estimates on the possible density oscillations that can occur in the limit passage. In particular, introducing the *oscillations defect measure*

$$\mathbf{osc}_p[\rho_n \to \rho](O) := \sup_{k \ge 1} \left(\limsup_{n \to \infty} \int_O |T_k(\rho_n) - T_k(\rho)|^p \, \mathrm{d}x\mathrm{d}t \right),$$

[2] This is essentially correct though some truncation and localization is necessary in general. See Chapter 2 for more details.

where the T_k are suitable cutoff functions, and providing appropriate uniform bounds on the oscillations defect measure via the effective viscous pressure, we can show that $(\rho, \mathbf{u})$ are a renormalized solution of the equation of continuity. Similar arguments can then be used to extend the theory to pressures satisfying $\gamma > 3/2$.

1.3 Moving domains

Oftentimes a set of PDE is described on a fixed domain Ω. This is the situation for instance when studying air flow in a fixed cylinder. On the other hand, if for example the cylinder changes volume due to the compressive effect of a piston, the domain is no longer fixed but changes in time. A similar situation holds for objects moving through a fluid.

Details on the moving domain will be given in Chapter 3 but let us briefly mention some of the key ideas. We will denote by $\Omega_t \equiv \Omega(t)$ a time-dependent domain depending on $t \in [0, T]$. Unlike free boundary problems, where Ω_t is an unknown, we assume the boundary behavior is given, perhaps from some experimental data. Let $\mathbf{V}(t, \mathbf{x})$ be the boundary velocity and $\mathbf{X}(t, \mathbf{x})$ be the associated boundary position. The transport of the domain by $\mathbf{V}$ is expressed by the following ODE

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{X}(t, \mathbf{x}) = \mathbf{V}(t, \mathbf{X}(t, \mathbf{x})), \ t > 0, \\ &\mathbf{X}(0, \mathbf{x}) = \mathbf{x}. \end{aligned} \tag{1.10}$$

We then set

$$\Omega_t = \mathbf{X}(t, \Omega_0), \tag{1.11}$$

and define the non-cylindrical space-time domain Q^f by

$$Q^f = \{(t, \mathbf{x}) \,|\, t \in (0, T), \mathbf{x} \in \Omega_t\}.$$

In fact, the definition (1.11) requires a slight correction. The velocity $\mathbf{V}$ only describes movement of boundary points. Hence it is unclear in which manner the map $\mathbf{X}$ transports points interior to the domain. In other words, extension of the map $\mathbf{X}(t, \mathbf{x})$ to the interior of the domain needs to be defined as well.

This problem has been studied before in the context of Arbitrary Lagrangian-Eulerian (ALE) methods. In this context we denote by T_t the ALE map such that

$$T_t(\mathbf{x}) = \mathbf{X}(t, \mathbf{x}), \qquad \mathbf{x} \in \partial\Omega_0,$$

and ask how to define $T_t(\mathbf{x})$ for $\mathbf{x} \in \Omega_0$. Two techniques for constructing this extension are given in [44], along with additional detail on the ALE method. Since this dissertation makes no specific use of this interior extension, it suffices for our purposes that a suitable extension is assumed known and we assume that the velocity $\mathbf{V}$, as well as the position $\mathbf{X}$ is known for the entire domain.

A popular class of methods to deal with moving domains are the so-called *penalization methods*. We explain the basic idea by considering the incompressible Navier-Stokes equations in a domain containing an obstacle. Let D be an open set containing the 'obstacle' $\widetilde{\Omega}$, that is, $\widetilde{\Omega} \subset D$ and the set $D \backslash \widetilde{\Omega}$ is filled with incompressible fluid. Whereas normally the fluid equations are posed strictly in the fluid domain $D \backslash \widetilde{\Omega}$, we instead add a singular term to the momentum equation and pose the Navier-Stokes equations over the entire domain D as follows,

$$\begin{aligned} &\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla P = \Delta\mathbf{u} - \frac{1}{\varepsilon}\mathbb{1}_{\widetilde{\Omega}}\mathbf{u}, \\ &\operatorname{div}_x\mathbf{u} = 0, \qquad \text{in } (0,T)\times D. \end{aligned} \tag{1.12}$$

The term $-(1/\varepsilon)\mathbb{1}_{\widetilde{\Omega}}\mathbf{u}$ added in the momentum equation is a penalty term, where ε is a small parameter tending to zero and $\mathbb{1}_{\widetilde{\Omega}}$ denotes the characteristic function of the domain $\widetilde{\Omega}$.

In the limit as $\varepsilon \to 0$, the penalization forces $\mathbf{u} = 0$ in the obstacle domain $\widetilde{\Omega}$ and $\mathbf{u}$ satisfies the standard Navier-Stokes in the true fluid domain $D\backslash\widetilde{\Omega}$. Indeed, formally expanding the velocity as

$$\mathbf{u} = \mathbf{u}_0 + \varepsilon\mathbf{u}_1 + O(\varepsilon^2),$$

and plugging into the momentum equation in (1.12), we can match orders of ε to deduce

$$\begin{aligned} O(1/\varepsilon): &\quad \mathbb{1}_{\widetilde{\Omega}}\mathbf{u}_0 = 0, \\ O(1): &\quad \partial_t\mathbf{u}_0 + \mathbf{u}_0\cdot\nabla\mathbf{u}_0 + \nabla P = \Delta\mathbf{u}_0 - \mathbb{1}_{\widetilde{\Omega}}\mathbf{u}_1. \end{aligned} \tag{1.13}$$

This implies the leading order term $\mathbf{u}_0$ vanishes in $\widetilde{\Omega}$, satisfies the standard Navier-Stokes equations in $D\backslash\widetilde{\Omega}$, and in addition the correction $\mathbf{u}_1$ satisfies the Darcy law in the obstacle domain

$$\mathbf{u}_1 + \nabla P = 0, \qquad \text{in } (0,T)\times\widetilde{\Omega}.$$

The penalty approach therefore exchanges information about a possibly complicated domain setting for a global description on a 'nicer' domain. For numerical methods, this has the benefit of being able to use finite-difference methods without needing body-fitted non-Cartesian meshes. The penalization technique is used in [50] in the context of nonlinear evolution equations on moving domains. Some other presentations include [5], [6], [52] and [67].

Leray-Hopf weak solutions for the incompressible Navier-Stokes equations on moving domains are first established by Sather [64]. There the function spaces defined over moving domains are employed directly. A proof involving penalization methods is found partly in [45], while a proof based on mapping back to cylindrical domains is found in [7].

For compressible Navier-Stokes equations on moving domains, penalization methods were used to prove the existence of weak solutions in [36] in the context of no-slip boundaries, and in [39] for the case of slip boundaries. The NSS system (1.1) over moving domains is analyzed in [25], as well as Chapter 3 of this dissertation. In the context of cancer dynamics models, global weak solutions are proven to exist in [70] for the context of symmetrical tumors with free boundary. With boundary behavior given, the weak solutions have been analyzed via the penalty method in various contexts in [27], [28], [29].

1.4 Relative entropy and weak-strong uniqueness

Relative entropies are functionals that measure a sort of distance between two solutions in a given function space. Relative entropies originate in the works of Dafermos [22] and DiPerna [23], see also the review article [20]. These functionals are often used to compare a weak solution with a (possibly hypothetical) strong or classical solution. This is particularly relevant for the case of Navier-Stokes equations, in which typically only weak solutions are known to exist. The principle of weak-strong uniqueness establishes that a weak and strong solution coincide, provided they both exist and have the same initial data. This principle can be deduced from the relative entropy inequality.

A relative entropy and weak-strong result in the context of compressible Navier-Stokes on fixed domains is established in [35], and over moving domains in [26]. For similar results on the Navier-Stokes-Smoluchowski system, see [9]. Relative entropies in the context of hydrodynamic limits can be found in [49], [56], [63], amongst others.

1.5 Outline of thesis

The contents of this dissertation are outlined as follows.

1. In Chapter 2, the existence of weak solutions to the NSS system is proved in the context of fixed spatial domains. The proof relies on a three-level approximation scheme involving Galerkin approximation, the addition of a vanishing fluid viscosity and the addition of an artificial pressure. The results presented are largely rooted in the Lions-Feireisl theory for viscous compressible fluids, and adapted for the NSS system in [9], [21]. In this dissertation some alternate results are presented regarding the functional setting for the particle density.

2. In Chapter 3, the existence of weak solutions to the NSS system is proved in the context of moving spatial domains. This is the content of the candidate's work in [25]. The proof utilizes a Brinkman-type penalization and penalization of the viscosity, both penalizing the momentum equation through the addition

of singular terms. As a corollary, convergence of the Brinkman penalization is proved.

3. In Chapter 4, the weak-strong uniqueness property of the NSS system is proved on moving domains. In the context of compressible Navier-Stokes equations, this is the content of the candidate's work in [26]. The proof involves establishing an appropriate relative entropy, and working in a functional framework whereby functions are extended by zero outside the moving domain.

Chapter 2: Global existence: fixed domains

In this chapter we prove the existence of global-in-time weak solutions of the compressible Navier-Stokes-Smoluchowski system (1.1) where the domain Ω is assumed to be fixed.

We assume the spatial domain $\Omega \subset \mathbb{R}^3$ is bounded, with boundary of class $C^{2,\nu}$, with $0 < \nu \leq 1$, and the time interval is fixed at $[0, T]$ for some $T > 0$. Boundary conditions are prescribed such that

$$\mathbf{u}(t, x) = 0 \quad \text{for } (t, x) \in [0, T] \times \partial\Omega, \tag{2.1}$$

and

$$(\nabla_x \eta + \eta \nabla_x \Phi) \cdot \mathbf{n} = 0 \quad \text{for } (t, x) \in [0, T] \times \partial\Omega, \tag{2.2}$$

where $\mathbf{n}$ is the outward unit normal to the boundary. These are respectively the no-slip and no-flux conditions for the fluid velocity and particle density. In particular the boundary conditions are conservative in the sense that they preserve the total fluid and particle mass.

Initial data $(\rho_0, \eta_0, \mathbf{m}_0)$ are prescribed such that

$$\begin{aligned} &\rho_0 \in L^\gamma(\Omega), \quad \rho_0 \geq 0 \ a.e. \ in \ \Omega \\ &\eta_0 \in L^1(\Omega), \quad \eta_0 \geq 0 \ a.e. \ in \ \Omega \\ &\mathbf{m}_0 \in L^1(\Omega; \mathbb{R}^3), \quad \frac{|\mathbf{m}_0|^2}{\rho_0} \in L^1(\Omega). \end{aligned} \tag{2.3}$$

The initial data should satisfy the compatibility condition

$$\mathbf{m}_0 = \frac{1}{2}\frac{|\mathbf{m}_0|^2}{\rho_0} = 0 \quad \text{for a.a. } x \in \{\rho_0 = 0\}.$$

The function

$$P(\rho) := \int_1^\rho \frac{p(z)}{z^2} \, \mathrm{d}z,$$

sometimes called the elastic pressure potential, will be used frequently in the sequel. Often $P(\rho)$ is taken to be $\frac{a}{\gamma-1}\rho^{\gamma-1}$. In particular the quantity $\int_\Omega \rho P(\rho)$ represents the total potential energy of the fluid in Ω.

2.1 Weak formulation and main result

In this section we state the definition of weak solutions of the Navier-Stokes-Smoluchowski system (1.1), and state the main result of this chapter.

Definition 1. *We say that $(\rho, \mathbf{u}, \eta)$ comprise a weak solution of the Navier-Stokes-Smoluchowski system* (1.1)*, along with the boundary conditions* (2.1) *and* (2.2)*, and the initial data* (2.3) *provided*

- *The density $\rho = \rho(t,x)$ and velocity $\mathbf{u} = \mathbf{u}(t,x)$ represent a weak renormalized solution of equation* (1.1a) *over $(0,T) \times \Omega$, that is, for any test function $\varphi \in$*

$\mathcal{D}([0,T)\times\overline{\Omega})$ *and any* b *such that*

$$b \in L^\infty \cap C[0,\infty), \;\; B(\rho) = B(1)\rho + \rho \int_1^\rho \frac{b(z)}{z^2}\,\mathrm{d}z,$$

the following integral identity holds:

$$\int_0^T \int_\Omega \Big(B(\rho)\partial_t\varphi + B(\rho)\mathbf{u}\cdot\nabla_x\varphi - b(\rho)\mathrm{div}_x\mathbf{u}\varphi \Big)\,\mathrm{d}x\mathrm{d}t = -\int_\Omega B(\rho_0)\varphi(0,\cdot)\,\mathrm{d}x. \tag{2.4}$$

The density, velocity, and momentum are required to have the following regularity

$$\rho \in L^\infty(0,T;L^\gamma(\Omega)), \quad \rho \geq 0 \text{ a.e. in } (0,T)\times\Omega,$$

$$\mathbf{u} \in L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^3)),$$

$$\rho\mathbf{u} \in L^\infty((0,T);L^{2\gamma/(\gamma-1)}(\Omega;\mathbb{R}^3)).$$

- *The particle density* $\eta = \eta(t,x)$ *and the velocity* $\mathbf{u} = \mathbf{u}(t,x)$ *represents a weak solution of equation* (1.1c)*. In particular, for all* $\varphi \in \mathcal{D}([0,T)\times\overline{\Omega})$

$$\int_0^T \int_\Omega \eta\partial_t\varphi + \eta\mathbf{u}\cdot\nabla_x\varphi - \eta\nabla_x\Phi\cdot\nabla_x\varphi - \nabla_x\eta\cdot\nabla_x\varphi\,\mathrm{d}x\mathrm{d}t = -\int_\Omega \eta_0\varphi(0,\cdot)\,\mathrm{d}x. \tag{2.5}$$

The particle density is required to have the following regularity

$$\eta \in L^2(0,T;W^{1,1}(\Omega)) \cap L^1(0,T;W^{1,3/2}(\Omega)),$$

$$\eta \geq 0 \text{ a.e. in } (0,T)\times\Omega.$$

- *The momentum equation holds in distributional sense. In particular, for all* $\varphi \in \mathcal{D}([0,T);\mathcal{D}(\Omega;\mathbb{R}^3))$*, the following integral identity holds*

$$\int_0^T \int_\Omega \Big(\rho\mathbf{u}\cdot\partial_t\varphi + \rho\mathbf{u}\otimes\mathbf{u} : \nabla_x\varphi + (p(\rho)+\eta)\mathrm{div}_x\varphi\Big)\,\mathrm{d}x\mathrm{d}t$$
$$= \int_0^T \int_\Omega \mathbb{S} : \nabla_x\varphi + (\eta+\beta\rho)\nabla_x\Phi\cdot\varphi\,\mathrm{d}x\mathrm{d}t - \int_\Omega (\rho\mathbf{u})_0\cdot\varphi(0,\cdot)\,\mathrm{d}x.$$

- *Defining the total energy of the system by*

$$E(\rho, \mathbf{u}, \eta)(t) := \int_\Omega \left(\frac{1}{2}\rho|\mathbf{u}|^2 + \rho P(\rho) + \eta \log \eta + \eta\Phi \right) \mathrm{d}x(t),$$

  the energy inequality

$$E(\rho, \mathbf{u},\eta)(t) + \int_0^t \int_\Omega \mathbb{S} : \nabla_x \mathbf{u} + |2\nabla_x\sqrt{\eta} + \sqrt{\eta}\nabla_x\Phi|^2 \,\mathrm{d}x\mathrm{d}s$$
$$\leq E(\rho, \mathbf{u}, \eta)(0) - \beta \int_0^t \int_\Omega \rho\mathbf{u} \cdot \nabla_x\Phi \,\mathrm{d}x\mathrm{d}s.$$

  holds for a.a. $t \in [0, T]$.

The main result of this chapter is the following.

Theorem 1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain, with boundary of class $C^{2,\nu}$, $0 < \nu \leq 1$. Suppose the pressure is given by* (1.2)*, and the stress tensor given by* (1.3)*. Then the system* (1.1)*, along with the boundary conditions* (2.1) *and* (2.2)*, and initial data* (2.3)*, has at least one weak solution $(\rho, \mathbf{u}, \eta)$ over $(0, T) \times \Omega$ in the sense of Definition 1.*

2.2 Approximation scheme

In this section the scheme used to construct the weak solutions to the NSS system is presented.

A three-level approximation scheme is employed to construct weak solutions of the NSS system. Let $n, \varepsilon, \delta > 0$ and let $\alpha > 1$. Rather than use the cumbersome notation $(\rho_{n,\varepsilon,\delta}, \mathbf{u}_{n,\varepsilon,\delta}, \eta_{n,\varepsilon,\delta})$, the subscript will denote the level of approximation. The Galerkin (first) level is denoted $(\rho_n, \mathbf{u}_n, \eta_n)$, the vanishing viscosity (second) level is denoted $(\rho_\varepsilon, \mathbf{u}_\varepsilon, \eta_\varepsilon)$, and the artificial pressure (third) level is denoted $(\rho_\delta, \mathbf{u}_\delta, \eta_\delta)$.

In other words, during the construction, the chain of convergences that we take are as follows: $\rho_n \to \rho_\varepsilon \to \rho_\delta \to \rho$, as $n \to \infty, \varepsilon \to 0$, and $\delta \to 0$ respectively (similarly for $\mathbf{u}, \eta$.)

The approximating system we use is

$$\partial_t \rho_n + \mathrm{div}_x(\rho_n \mathbf{u}_n) = \varepsilon \Delta_x \rho_n \tag{2.6a}$$

$$\partial_t \eta_n + \mathrm{div}_x(\eta_n(\mathbf{u}_n - \nabla_x \Phi)) = \Delta_x \eta_n \tag{2.6b}$$

$$\begin{aligned} \partial_t(\rho_n \mathbf{u}_n) + \mathrm{div}_x(\rho_n \mathbf{u}_n \otimes \mathbf{u}_n) + \nabla_x(p(\rho_n) + \eta_n + \delta \rho_n^\alpha) + \varepsilon \nabla_x \mathbf{u}_n \nabla_x \rho_n \\ = \mathrm{div}_x \mathbb{S}_n - (\eta_n + \beta \rho_n) \nabla_x \Phi, \end{aligned} \tag{2.6c}$$

considered over $(0,T) \times \Omega$, and where $\mathbb{S}_n$ denotes $\mathbb{S}(\nabla_x \mathbf{u}_n)$. The boundary conditions imposed are

$$\nabla_x \rho_n \cdot \mathbf{n} = 0, \quad \mathbf{u}_n = (\nabla_x \eta_n + \eta_n \nabla_x \Phi) \cdot \mathbf{n} = 0 \quad \text{on } (0,T) \times \partial\Omega. \tag{2.7}$$

Initial data $\{\rho_{0,\delta}, \mathbf{m}_{0,\delta}, \eta_{0,\delta}\}$ is prescribed over Ω, and modified such that

1. The density $\rho_{0,\delta} \in C^{2,\nu}(\overline{\Omega})$ satisfies

$$0 < \delta \leq \rho_{0,\delta}(\mathbf{x}) \leq \delta^{-1\backslash 2\alpha}, \quad \rho_{0,\delta} \to \rho_0 \text{ in } L^\gamma(\Omega), \tag{2.8}$$

   and

$$|\{\mathbf{x} \in \Omega \,:\, \rho_{0,\delta}(\mathbf{x}) < \rho_0(\mathbf{x})\}| \to 0 \text{ as } \delta \to 0. \tag{2.9}$$

2. The momenta $\mathbf{m}_{0,\delta}$ are defined as

$$\mathbf{m}_{0,\delta} = \begin{cases} \mathbf{m}_0 & \text{if } \rho_{0,\delta}(\mathbf{x}) \geq \rho_0(\mathbf{x}), \\ 0 & \text{else.} \end{cases} \tag{2.10}$$

3. The particle density $\eta_{0,\delta}$ satisfies

$$0 < \delta \le \eta_{0,\delta} \le C, \quad \eta_{0,\delta} \to \eta_0 \text{ in } L^2(\Omega). \tag{2.11}$$

In part, these hypotheses ensure that the initial energy

$$E(0) = E_\delta(0) := \int_\Omega \frac{1}{2}\frac{|\mathbf{m}_{0,\delta}|^2}{\rho_{0,\delta}} + \rho_{0,\delta}P(\rho_{0,\delta}) + \frac{\delta}{\alpha-1}\rho_{0,\delta}^\alpha + \eta_{0,\delta}\log\eta_{0,\delta} + \eta_{0,\delta}\Phi \,\mathrm{d}x \tag{2.12}$$

is finite.

The approximating system is motivated as follows. The fluid density equation (2.6a) contains the additional Laplacian term $\varepsilon\Delta_x\rho$, known as *vanishing viscosity*, in order to increase the regularity of the density ρ and obtain strong compactness of the density at the first level of the approximation. In order to keep the energy estimate satisfied, the $\varepsilon\nabla_x\mathbf{u}\nabla_x\rho$ term in the modified momentum equation (2.6c) is introduced to balance the vanishing viscosity term. Finally, the $\delta\rho^\alpha$ term in the momentum equation serves to increase the integrability of the pressure during the first two levels of approximation. This is called the *artificial pressure*.

Proposition 1. *Fix any n and T. Then there exist functions $\{\rho_n, \mathbf{u}_n, \eta_n\}$ solving the modified system (2.6) on the interval $[0,T]$, along with boundary conditions (2.7) and initial data (2.8)-(2.11) such that*

$$\rho_n, \eta_n \in C([0,T]; C^{2,\nu}(\overline{\Omega})), \quad \partial_t\rho_n, \partial_t\eta_n \in C([0,T]; C^\nu(\overline{\Omega})),$$

$$\mathbf{u}_n \in C^1([0,T]; X_n),$$

In addition, the following energy equality is satisfied

$$
\begin{aligned}
&\int_\Omega \left(\frac{1}{2}\rho_n|\mathbf{u}_n|^2 + \rho_n P(\rho_n) + \frac{\delta}{\alpha-1}\rho_n^\alpha + \eta_n \log\eta_n + \eta_n\Phi\right)(t)\,\mathrm{d}x \\
&\quad + \int_0^t\int_\Omega \mathbb{S}_n : \nabla_x \mathbf{u}_n + |2\nabla_x\sqrt{\eta_n} + \sqrt{\eta_n}\nabla_x\Phi|^2\,\mathrm{d}x\mathrm{d}s \\
&\quad + \varepsilon\int_0^t\int_\Omega |\nabla_x\rho_n|^2\left(\frac{p'(\rho_n)}{\rho_n} + \delta\alpha\rho_n^{\alpha-2}\right)\mathrm{d}x\mathrm{d}s \\
&= \int_\Omega \frac{1}{2}\mathbf{m}_{0,\delta}\cdot\mathbf{u}_{0,\delta,n} + \rho_{0,\delta}P(\rho_{0,\delta}) + \frac{\delta}{\alpha-1}\rho_{0,\delta}^\alpha + \eta_{0,\delta}\log\eta_{0,\delta} + \eta_{0,\delta}\Phi\,\mathrm{d}x \\
&\quad - \beta\int_0^t\int_\Omega \rho_n\mathbf{u}_n\cdot\nabla_x\Phi\,\mathrm{d}x\mathrm{d}s,
\end{aligned}
\tag{2.13}
$$

for any $t \in [0,T]$.

The next two sections, 2.3 and 2.4, are dedicated to a proof of Proposition 1, concerning existence of the Galerkin approximate solutions over the time inerval $[0,T]$. The results follow the general framework of the compressible Navier-Stokes and Navier-Stokes-Fourier framework of [33], [37], and [58].

We first setup some preliminaries. Let $X_n = \mathrm{span}\{\pi_j\}_{j=1}^n$, for some integer n, where $\pi_j \in [\mathcal{D}(\Omega)]^3$ are linearly independent functions ranging in $\mathbb{R}^3$. Equipped with the $L^2(\Omega;\mathbb{R}^3)$ inner product, the space X_n is a finite dimensional Hilbert space.

We rewrite the momentum equation (2.6c) in integral form as

$$
\begin{aligned}
&\int_\Omega \rho\mathbf{u}_n(t)\cdot\pi\,\mathrm{d}x - \int_\Omega \mathbf{m}_{0,\delta}\cdot\pi\,\mathrm{d}x \\
&\quad = \int_0^t\int_\Omega (\rho\mathbf{u}_n\otimes\mathbf{u}_n - \mathbb{S}_n) : \nabla_x\pi + [p(\rho) + \eta + \delta\rho^\alpha]\mathrm{div}_x\pi\,\mathrm{d}x\mathrm{d}s \\
&\qquad - \int_0^t\int_\Omega [\varepsilon\nabla_x\mathbf{u}_n\nabla_x\rho + (\beta\rho + \eta)\nabla_x\Phi]\cdot\pi\,\mathrm{d}x\mathrm{d}s,
\end{aligned}
\tag{2.14}
$$

for any $\pi \in X_n$, and any $t \in [0,T]$. The goal is to seek a fixed point $\mathbf{u}_n \in C([0,T];X_n)$ of (A.1). In order to carry this out, we need information on the mappings assigning each $\mathbf{u}_n$ to the unique solutions ρ, η, via equations (2.6a) and (2.6b). The following two propositions address these maps.

Proposition 2. *Let $\Omega \in \mathbb{R}^3$ be a bounded domain of class $C^{2,\nu}, 0 < \nu \leq 1$. Suppose that $\rho_{0,\delta} \in C^{2,\nu}(\overline{\Omega})$ is positive, and satisfies the condition $\nabla_x \rho_{0,\delta} \cdot \mathbf{n} = 0$ on $\partial\Omega$. Let $\mathbf{u} \mapsto \rho[\mathbf{u}]$ assign to any $\mathbf{u} \in C([0,T]; C_0^2(\overline{\Omega}; \mathbb{R}^3))$ the unique solution ρ of the modified fluid density equation (2.6a). Then this map takes bounded sets in the space $C([0,T]; C_0^2(\overline{\Omega}; \mathbb{R}^3))$, into bounded sets of the space*

$$V := \begin{cases} \partial_t \rho \in C([0,T]; C^{\nu}(\overline{\Omega})) \\ \rho \in C([0,T]; C^{2,\nu}(\overline{\Omega})), \end{cases}$$

and the map $\mathbf{u} \in C([0,T]; C_0^2(\overline{\Omega}; \mathbb{R}^3)) \mapsto \rho[\mathbf{u}] \in C^1([0,T] \times \overline{\Omega})$ is continuous.

Proof. See Proposition 7.1 in [33]. □

Proposition 3. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $C^{2,\nu}$, $0 < \nu \leq 1$. Assume that $\eta_{0,\delta} \in C^{2,\nu}(\overline{\Omega})$, and $\mathbf{u} \in C([0,T]; C_0^2(\overline{\Omega}; \mathbb{R}^3))$. Let the compatibility condition*

$$(\nabla_x \eta_{0,\delta}(\mathbf{x}) + \eta_{0,\delta}(\mathbf{x}) \nabla_x \Phi(\mathbf{x})) \cdot \mathbf{n}(\mathbf{x}) = 0, \quad \mathbf{x} \in \partial\Omega$$

be satisifed. Then the problem (2.6b) has a unique classical solution η such that $\eta \in V$, where V is the space introduced in Proposition 2. The solution operator $\mathbf{u} \mapsto \eta[\mathbf{u}]$, assigning to any $\mathbf{u} \in C([0,T]; C_0^2(\overline{\Omega}; \mathbb{R}^3))$ the unique solution η of (2.6b), takes bounded sets of $C([0,T]; C_0^2(\overline{\Omega}; \mathbb{R}^3))$ into bounded sets of V.

Proof. See Theorem 5.1.21 in [54]. □

2.3 Local-in-time existence of approximate solutions

Define the linear operators $\mathcal{M}[\rho] : X_n \to X_n^*$ such that

$$\langle \mathcal{M}[\rho]\mathbf{v}, \mathbf{w} \rangle := \int_\Omega \rho \mathbf{v} \cdot \mathbf{w} \, \mathrm{d}x.$$

Let $\overline{\rho} > 0$ be a positive constant, and let $\rho \geq \overline{\rho}$. Then $\mathcal{M}[\rho]$ is invertible and

$$\|\mathcal{M}^{-1}[\rho]\|_{\mathcal{L}(X_n^*;X_n)} \leq \frac{1}{\inf_\Omega \rho}.$$

Also, for any $\rho^1, \rho^2 \geq \overline{\rho}$, the identity

$$\mathcal{M}^{-1}[\rho^1] - \mathcal{M}^{-1}[\rho^2] = \mathcal{M}^{-1}[\rho^2] \left(\mathcal{M}[\rho^2] - \mathcal{M}[\rho^1] \right) \mathcal{M}^{-1}[\rho^1],$$

implies the continuity result

$$\|\mathcal{M}^{-1}[\rho^1] - \mathcal{M}^{-1}[\rho^2]\|_{\mathcal{L}(X_n^*;X_n)} \leq c(n, \underline{\rho}) \|\rho^2 - \rho^1\|_{X_n}. \tag{2.15}$$

Since all norms are equivalent on X_n, the X_n-norm can be taken to be, for instance, the $L^1(\Omega)$ or $L^\infty(\Omega)$ norm.

Now we can rewrite the integral form (A.1) as an implicit equation in X_n for $\mathbf{u}_n$,

$$\mathbf{u}_n(t) = \mathcal{M}^{-1}[\rho(t)] \left(\mathbf{m}_{0,\delta}^* + \int_0^t \mathcal{N}[\mathbf{u}_n(s), \rho(s), \eta(s)] \, \mathrm{d}s \right), \tag{2.16}$$

where

$$\mathbf{m}_{0,\delta}^* \in X_n^*, \quad \langle \mathbf{m}_{0,\delta}^*, \pi \rangle := \int_\Omega \mathbf{m}_{0,\delta} \cdot \pi \, \mathrm{d}x \quad \text{for any } \pi \in X_n$$

and

$$\begin{aligned}
&\mathcal{N} : X_n \to X_n^* \\
&\langle \mathcal{N}[\mathbf{u}_n, \rho, \eta], \pi \rangle := \int_\Omega (\rho \mathbf{u}_n \otimes \mathbf{u}_n - \mathbb{S}_n) : \nabla_x \pi + [p(\rho) + \eta + \delta \rho^\alpha] \mathrm{div}_x \pi \, \mathrm{d}x \\
&\qquad\qquad - \int_\Omega [\varepsilon \nabla_x \mathbf{u}_n \nabla_x \rho + (\beta \rho + \eta) \nabla_x \Phi] \cdot \pi \, \mathrm{d}x.
\end{aligned}$$

Recall that $\rho = \rho[\mathbf{u}_n]$ and $\eta = \eta[\mathbf{u}_n]$ are uniquely determined by $\mathbf{u}_n$ through Propositions 2 and 3.

Now let $\mathcal{B}$ be the unit ball in $C([0,T]; X_n)$,

$$\mathcal{B} := \left\{ \mathbf{v} \in C([0,T]; X_n) \,\middle|\, \sup_{t\in[0,T]} \|\mathbf{v}(t) - \mathbf{u}_{0,\delta,n}\|_{X_n} \leq 1 \right\}.$$

Since $\rho_{0,\delta} > 0$, the value $\mathbf{u}_{0,\delta,n}$ is uniquely determined from $\mathbf{m}_{0,\delta}$. It is easy to check that $\mathcal{B}$ is closed, bounded, and convex.

We now rewrite (2.16) as the fixed point problem $\mathbf{u}_n = \mathcal{T}[\mathbf{u}_n]$, where the mapping

$$\mathcal{T} : \mathcal{B} \to C([0,T]; X_n),$$

is defined by the right-hand side of (2.16). To determine whether this fixed point problem has a solution, we check the conditions of the Schauder fixed point theorem, stated below (for a proof, see [57]):

Theorem 2. *Let $\mathcal{B}$ be a closed, convex, bounded subset of a Banach space X, and $\mathcal{T} : \mathcal{B} \to B$ a compact operator. Then $\mathcal{T}$ has a fixed point.*

Using (2.15), and Propositions 2 and 3, it is easy to check the following inequality holds:

$$\sup_{t\in[0,T]} \|\mathcal{T}[\mathbf{u}] - \mathbf{u}_{0,\delta,n}\|_{X_n} \leq c \sup_{t\in[0,T]} \left(\|\rho(t) - \rho_{0,\delta}\|_{L^1(\Omega)} + t \right),$$

where $c = c(n, \underline{\rho}, \|\mathbf{m}^*_{0,\delta}\|_{X^*_n}, \|\mathcal{N}\|_{\mathcal{L}(X_n; X^*_n)})$. Then the continuity in time for $\rho(t)$ implies the right-hand side is made small provided $T = T(n)$ is small. We conclude $\mathcal{T}$ maps $\mathcal{B}$ into itself over a possibly short time interval.

It remains to show that $\mathcal{T}$ is a compact operator. To this end, it suffices to demonstrate the family $\{\mathcal{T}\mathbf{u}_n\}_{n\geq 1}$ is equicontinuous in $C([0,T];X_n)$, after which the Arzelá-Ascoli theorem applies (Theorem 2.1 in [33]). Without loss of generality, let $s<t$. A simple computation reveals that

$$\|\mathcal{T}\mathbf{u}_n(t)-\mathcal{T}\mathbf{u}_n(s)\|_{X_n}\leq c\left(\|\mathcal{M}^{-1}[\rho(t)]-\mathcal{M}^{-1}[\rho(s)]\|_{\mathcal{L}(X_n^*;X_n)}+|t-s|\right)\leq c|t-s|,$$

where we used the definition of $\mathcal{T}$, the inequality (2.15), and the time-continuity of $\rho(t)$. We can now apply the Schauder fixed point theorem to conclude the existence of a fixed point $\mathbf{u}_n$ of $\mathbf{u}_n=\mathcal{T}[\mathbf{u}_n]$ on the time interval $[0,T(n)]$.

2.4 Global-in-time existence of approximate solutions

In this section, uniform bounds in time are derived that extend the local solutions constructed in the previous section to the time interval $[0,T]$.

At this point, we can integrate (2.6a) and (2.6b) over Ω to deduce that masses are conserved for all time, provided the solutions exist. In particular,

$$\int_\Omega \rho(t)\,\mathrm{d}x=\int_\Omega \rho_{0,\delta}\,\mathrm{d}x \qquad \text{for any } t\geq 0, \tag{2.17}$$

and

$$\int_\Omega \eta(t)\,\mathrm{d}x=\int_\Omega \eta_{0,\delta}\,\mathrm{d}x \qquad \text{for any } t\geq 0. \tag{2.18}$$

In addition, the following proposition provides bounds from below on the densities which imply that (2.17) and (2.18) provide $L^\infty(0,T;L^1(\Omega))$ estimates.

Proposition 4. *Let ρ and η be solutions of problems* (2.6a) *and* (2.6b)*. Assume*

$\rho(0,\cdot), \eta(0,\cdot) \geq 0$. *Then*

$$\rho(t,x) \geq 0, \qquad \eta(t,x) \geq 0,$$

for any $t \geq 0$ *and almost every* $x \in \Omega$.

Proof. The proof is similar for both cases. Here we prove it for the particle density η. Let

$$G(\eta) = \eta^{-} = \max\{-\eta, 0\} = \begin{cases} 0, & \eta \geq 0 \\ -\eta, & \eta < 0. \end{cases}$$

Note that G is convex, nonnegative, and $G(\eta) = G'(\eta)\eta$ with weak derivative

$$G'(\eta) = \begin{cases} 0, & \eta \geq 0 \\ -1, & \eta < 0 \end{cases}.$$

Multiplying (2.6b) by G' and integrating by parts yields

$$\int_\Omega G'\partial_t \eta \,\mathrm{d}x + G' \int_\Omega \mathrm{div}_x(\eta \mathbf{u} - \eta \nabla_x \Phi - \nabla_x \eta) \,\mathrm{d}x.$$

The second term is equal to zero by the boundary conditions, while integrating the first term in time gives

$$\int_\Omega G(\eta(t,\cdot)) \,\mathrm{d}x = \int_\Omega G(\eta(0,\cdot)) \,\mathrm{d}x = 0,$$

for any $t \geq 0$, provided that $\eta(0,\cdot) \geq 0$. Since $G(\eta) \geq 0$, this implies that $G(\eta)$ is identically zero, yielding the desired result $\eta \geq 0$. □

Since $\mathbf{u}_n$ satisfies the integral equation (A.1), $\mathbf{u}_n$ is continuously differentiable in time. Then differentiating (A.1) and setting $\pi = \mathbf{u}_n(t)$ yields

$$\begin{aligned}
&\int_\Omega \partial_t(\rho_n \mathbf{u}_n)\cdot \mathbf{u}_n \,\mathrm{d}x(t) \\
&\quad = \int_\Omega (\rho_n \mathbf{u}_n \otimes \mathbf{u}_n - \mathbb{S}_n) : \nabla_x \mathbf{u}_n + [p(\rho_n) + \eta_n + \delta \rho_n^\alpha]\mathrm{div}_x \mathbf{u}_n \,\mathrm{d}x(t) \\
&\qquad - \int_\Omega [\varepsilon \nabla_x \mathbf{u}_n \nabla_x \rho_n + (\beta \rho_n + \eta_n)\nabla_x \Phi]\cdot \mathbf{u}_n \,\mathrm{d}x(t).
\end{aligned} \tag{2.19}$$

The above identity holds for any $t \in (0, T(n))$. Recall we set $\rho_n = \rho[\mathbf{u}_n]$ and $\eta_n = \eta[\mathbf{u}_n]$.

The following lemma contains some preliminary computations.

Lemma 1. *The following identities hold:*

1. $$\begin{aligned}&\int_\Omega \partial_t(\rho_n \mathbf{u}_n)\cdot \mathbf{u}_n - (\rho_n \mathbf{u}_n \otimes \mathbf{u}_n) : \nabla_x \mathbf{u}_n \,\mathrm{d}x \\ &\quad = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}\rho_n |\mathbf{u}_n|^2 \,\mathrm{d}x + \frac{\varepsilon}{2}\int_\Omega \Delta_x \rho_n |\mathbf{u}_n|^2 \,\mathrm{d}x\end{aligned}$$
2. $$\int_\Omega p(\rho_n)\mathrm{div}_x \mathbf{u}_n \,\mathrm{d}x = -\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho_n P(\rho_n)\,\mathrm{d}x - \varepsilon \int_\Omega \frac{p'(\rho_n)}{\rho_n}|\nabla_x \rho_n|^2 \,\mathrm{d}x,$$
3. $$\int_\Omega \rho_n^\alpha \mathrm{div}_x \mathbf{u}_n \,\mathrm{d}x = -\frac{1}{\alpha - 1}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho_n^\alpha \,\mathrm{d}x - \alpha\varepsilon \int_\Omega \rho_n^{\alpha-2}|\nabla_x \rho_n|^2 \,\mathrm{d}x,$$
4. $$\begin{aligned}&\int_\Omega \eta_n \mathrm{div}_x \mathbf{u}_n \,\mathrm{d}x - \int_\Omega \eta_n \nabla_x \Phi \cdot \mathbf{u}_n \,\mathrm{d}x \\ &\quad = -\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \eta_n \log \eta_n \,\mathrm{d}x - \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \eta_n \Phi \,\mathrm{d}x - \int_\Omega |2\nabla_x \sqrt{\eta_n} + \sqrt{\eta_n}\nabla_x \Phi|^2 \,\mathrm{d}x.\end{aligned}$$

Proof. Since all quantities have the appropriate regularity, the proof consists of repeated applications of integration by parts and using (2.6a), (2.6b). The second identity requires the following preliminary equality,

$$p(\rho_n)\mathrm{div}_x \mathbf{u}_n = \mathrm{div}_x(p_n P(\rho_n)\mathbf{u}_n) - \partial_t(\rho_n P(\rho_n)) + \varepsilon \Delta_x \rho_n \left(P(\rho_n) + \frac{p(\rho_n)}{\rho_n}\right).$$

□

Using Lemma 1 and integrating (2.19) in time we deduce the energy *equality*

$$\begin{aligned}
&\int_\Omega \left(\frac{1}{2}\rho_n|\mathbf{u}_n|^2 + \rho_n P(\rho_n) + \frac{\delta}{\alpha-1}\rho_n^\alpha + \eta_n\log\eta_n + \eta_n\Phi\right)(t)\,\mathrm{d}x \\
&\quad + \int_0^t\int_\Omega \mathbb{S}_n : \nabla_x\mathbf{u}_n + |2\nabla_x\sqrt{\eta_n} + \sqrt{\eta_n}\nabla_x\Phi|^2\,\mathrm{d}x\mathrm{d}s \\
&\quad + \varepsilon\int_0^t\int_\Omega |\nabla_x\rho_n|^2\left(\frac{p'(\rho_n)}{\rho_n} + \delta\alpha\rho_n^{\alpha-2}\right)\mathrm{d}x\mathrm{d}s \qquad (2.20)\\
&= \int_\Omega \frac{1}{2}\mathbf{m}_{0,\delta}\cdot\mathbf{u}_{0,\delta,n} + \rho_{0,\delta}P(\rho_{0,\delta}) + \frac{\delta}{\alpha-1}\rho_{0,\delta}^\alpha + \eta_{0,\delta}\log\eta_{0,\delta} + \eta_{0,\delta}\Phi\,\mathrm{d}x \\
&\quad - \beta\int_0^t\int_\Omega \rho_n\mathbf{u}_n\cdot\nabla_x\Phi\,\mathrm{d}x\mathrm{d}s,
\end{aligned}$$

holds for any $t \in [0, T(n)]$. The terms on the left-hand side of (2.20) are all non-negative with the potential exception of $\eta_n\log\eta_n$. The following lemma provides bounds on the negative contribution $\eta_n\log^-\eta_n$.

Lemma 2. *Suppose Ω is a bounded domain, and $\eta \in L^1_+(\Omega)$. Assume*

$$\int_\Omega \eta(x)\log\eta(x)\,\mathrm{d}x \le C_1$$

for some constant C_1. Then $\eta\log\eta \in L^1(\Omega)$ and

$$\int_\Omega |\eta(x)\log\eta(x)|\,\mathrm{d}x \le c(C_1, |\Omega|).$$

Proof. We define as usual $f^+ = \max\{f, 0\}$ and $f^- = \min\{-f, 0\}$. Note that

$$\int_\Omega |\eta(x)\log\eta(x)|\,\mathrm{d}x = \int_\Omega \eta(x)\log\eta(x)\,\mathrm{d}x + 2\int_\Omega \eta(x)\log^-\eta(x)\,\mathrm{d}x.$$

Therefore it suffices to bound the quantity $\int_\Omega \eta(x)\log^-\eta(x)\,\mathrm{d}x$. Let $\overline{\eta} := \eta\mathbb{1}_{\{\eta\le 1\}}$. Since $z \mapsto z\log z$ is convex for $z > 0$, an application of Jensen's inequality yields

$$⨍_\Omega \overline{\eta}(x)\log\overline{\eta}(x)\,\mathrm{d}x \ge \left(⨍_\Omega \overline{\eta}(x)\,\mathrm{d}x\right)\log\left(⨍_\Omega \overline{\eta}(x)\,\mathrm{d}x\right),$$

which proves the lemma after observing that $\overline{\eta}\log\overline{\eta} = -\eta\log^-\eta$, and using $z\log z \ge -1/e$ for $z > 0$. $\square$

Now, the first term on the right-hand side of (2.20) is bounded due to boundedness of the initial energy (2.12). Upon splitting $\rho_n \mathbf{u}_n = \sqrt{\rho_n}\sqrt{\rho_n}\mathbf{u}_n$ and using Cauchy's inequality with ε, the second term on the right-hand side of (2.20) is bounded as follows:

$$\int_\Omega \rho_n \mathbf{u}_n \cdot \nabla_x \Phi \, \mathrm{d}x \le \frac{1}{2} \operatorname*{ess\,sup}_{\mathbf{x}\in\Omega} |\nabla_x \Phi(\mathbf{x})| \left(\frac{1}{4\varepsilon} \int_\Omega \rho_{0,\delta} \, \mathrm{d}x + \varepsilon \int_\Omega \rho_n |\mathbf{u}_n|^2 \, \mathrm{d}x \right).$$

Making ε small enough, the kinetic energy term can be absorbed into the left-hand side of (2.20). This yields estimates independent of n and $T(n) \le T$.

Based on these observations and Lemma 2, the energy equality (2.20) implies that $\mathbf{u}_n$ is uniformly bounded in the space $L^2(0, T(n); W_0^{1,2}(\Omega; \mathbb{R}^3))$ independent of n. Since $\mathbf{u}_n(t) \in X_n$, and all norms are equivalent on X_n, we get that $\mathbf{u}_n$ is uniformly bounded in $L^1(0, T(n); W^{1,\infty}(\Omega; \mathbb{R}^3))$. This implies

$$\frac{1}{c} \le \rho_n(t, \mathbf{x}) \le c,$$

for some constant c independent of $T(n) \le T$ (see Section 7.3.1 of [33] for details). Going back to the energy equality (2.20), this implies that $\mathbf{u}_n(t)$ is uniformly bounded in $L^2(\Omega; \mathbb{R}^3)$, and so in X_n, for any t. We can now iterate the local-in-time existence to the full interval $[0, T]$, concluding the proof of Proposition 1.

2.5 Faedo-Galerkin limit

In this section we take the limit $n \to \infty$. To carry out this feat, estimates independent of the dimension n must first be established. The bounds obtained in this section will therefore be uniform in n, but potentially depend on the parameters ε and δ.

2.5.1 Uniform bounds

The energy equality (2.13) provides the following simple estimates

$$
\begin{aligned}
&\sqrt{\rho_n}\mathbf{u}_n \in L^\infty(0,T;L^2(\Omega;\mathbb{R}^3)),\\
&\rho_n \in L^\infty(0,T;L^\gamma \cap L^\alpha(\Omega)),\\
&\mathbf{u}_n \in L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^3)),
\end{aligned}
\tag{2.21}
$$

where the last estimate follows from a Poincaré inequality. Using Lemma 2, we also have that

$$
\eta_n \in L^\infty(0,T;L\log L(\Omega)). \tag{2.22}
$$

From (2.13), and the equality

$$
|\nabla_x \rho_n^{\alpha/2}|^2 = \frac{\alpha^2}{4}\rho_n^{\alpha-2}|\nabla_x \rho_n|^2,
$$

it is easy to see that $\nabla_x \rho_n^{\alpha/2}$ is bounded in $L^2((0,T)\times\Omega)$. Along with available estimates on ρ_n, a Poincaré-type inequality (Proposition 2.2 in [37]) then yields

$$
\rho_n^{\alpha/2} \in L^2(0,T;W^{1,2}(\Omega)). \tag{2.23}
$$

The energy equality (2.13) also provides the estimate

$$
2\nabla_x\sqrt{\eta_n} + \sqrt{\eta_n}\nabla_x\Phi \in L^2(0,T;L^2(\Omega;\mathbb{R}^3)).
$$

Since $\eta_n \in L^\infty(0,T;L^1(\Omega))$ and $\nabla_x\Phi$ is uniformly bounded, this implies

$$
\nabla_x\sqrt{\eta_n} \in L^2(0,T;L^2(\Omega;\mathbb{R}^3)),
$$

and so

$$
\sqrt{\eta_n} \in L^2(0,T;W^{1,2}(\Omega)) \hookrightarrow L^2(0,T;L^6(\Omega))
$$

Using these estimates, the equality

$$\nabla_x \eta_n = 2\sqrt{\eta_n}\nabla_x\sqrt{\eta_n},$$

and Hölder's inequality, the particle density η_n satisfies

$$\begin{aligned} &\eta_n \in L^1(0,T;W^{1,3/2}(\Omega)) \hookrightarrow L^1(0,T;L^3(\Omega)),\\ &\eta_n \in L^2(0,T;W^{1,1}(\Omega)) \hookrightarrow L^2(0,T;L^{3/2}(\Omega)). \end{aligned} \tag{2.24}$$

Although at this stage the regularized equation (2.6a) can be shown to be satisfied in $\mathcal{D}'((0,T)\times\Omega)$ (indeed we can estimate $\partial_t \rho_n$ and apply the Aubin-Lions lemma), we can show $(\rho_n, \mathbf{u}_n)$ is in fact a strong solution. This result is obtained via the $L^p - L^q$ theory of parabolic equations followed by a bootstrap argument. The following computations for ρ_n follow Section 3.5.2 of [37].

To begin, we renormalize the equation (2.6a) with $G(\rho_n)$ (meaning multiplying by $G'(\rho_n)$ and integrating by parts), to find

$$\partial_t \int_\Omega G(\rho_n)\,\mathrm{d}x + \varepsilon \int_\Omega G''(\rho_n)|\nabla_x \rho_n|^2\,\mathrm{d}x = \int_\Omega (G(\rho_n) - G'(\rho_n)\rho_n)\mathrm{div}_x \mathbf{u}_n\,\mathrm{d}x.$$

Letting $G(\rho_n) = \rho_n \log \rho_n$, we see that

$$\varepsilon \int_0^T \int_\Omega \frac{|\nabla_x \rho_n|^2}{\rho_n}\,\mathrm{d}x\mathrm{d}t \le C.$$

Therefore,

$$\|\nabla_x \rho_n \cdot \mathbf{u}_n\|_{L^1(\Omega)} \le \left\| \frac{\nabla_x \rho_n}{\sqrt{\rho_n}} \right\|_{L^2(\Omega;\mathbb{R}^3)} \|\sqrt{\rho_n}\mathbf{u}_n\|_{L^2(\Omega;\mathbb{R}^3)}, \tag{2.25}$$

with the right hand side bounded in $L^2(0,T)$. In addition, the estimates (2.21) and (2.23) yield

$$\nabla_x \rho_n \cdot \mathbf{u}_n \in L^1(0,T;L^{3/2}(\Omega)). \tag{2.26}$$

Interpolating the estimates (2.25) and (2.26), we get

$$\nabla_x \rho_n \cdot \mathbf{u}_n \in L^p(0,T;L^q(\Omega)), \qquad p \in \left(1, \frac{3}{2}\right), \; q(p) \in (1,2). \tag{2.27}$$

Using the $L^p - L^q$ theory for parabolic equations (see Section 7.3.1 in [33]), the estimate on $\rho_n \mathrm{div}_x \mathbf{u}_n$ provided by (2.21), and (2.27), the density ρ_n satisfies

$$\partial_t \rho_n, \; \partial_{x_i} \partial_{x_j} \rho_n \in L^p(0,T;L^q(\Omega)) \quad \text{for } i,j = 1,2,3, \tag{2.28}$$

where $p, q > 1$.

Next, since (2.6b) is satisfied in $\mathcal{D}'((0,T) \times \Omega)$ we get estimates on the time derivative $\partial_t \eta_n$ of the form

$$\partial_t \eta_n \in L^p(0,T;W^{-1,q}(\Omega)), \tag{2.29}$$

for some $p, q > 1$. This follows from interpolating

$$\eta_n \in L^2(0,T;L^{3/2}(\Omega)) \cap L^\infty(0,T;L^1(\Omega))$$

and

$$\nabla_x \eta_n \in L^1(0,T;L^{3/2}(\Omega)) \cap L^2(0,T;L^1(\Omega))$$

and using the embeddings of Lebesgue spaces into Sobolev duals (cf. [33], Theorem 2.8).

2.5.2 Convergent subsequences

We next extract the relevant converging sequences from the estimates previously derived. Subsequences (without relabeling) are taken when necessary.

First, we make sense of the initial condition by showing that the density ρ_n is continuous in time with values in an appropriate function space. Consider the map

$$t \mapsto \left[\int_\Omega \rho_n \varphi \,\mathrm{d}x\right](t) =: \mathcal{L}_{\rho_n}(t), \qquad \forall \varphi \in C_c^\infty(\Omega).$$

By virtue of Proposition 1, we can multiply (2.6a) by a smooth function $\varphi \in C_c^\infty(\Omega)$ and integrate in space and time to find that for all $t, t' \in [0, T]$,

$$\left[\int_\Omega \rho_n \varphi \,\mathrm{d}x\right](t') - \left[\int_\Omega \rho_n \varphi \,\mathrm{d}x\right](t) = \int_t^{t'} \int_\Omega [\varepsilon \Delta_x \rho_n - \mathrm{div}_x(\rho_n \mathbf{u}_n)]\varphi \,\mathrm{d}x\mathrm{d}s. \quad (2.30)$$

Since the integrand of the right hand side of (2.30) is integrable, this equation yields the equicontinuity of the family $\mathcal{L}_{\rho_n}(t)$. This equicontinuity, along with $\rho_n \in L^\infty(0, T; L^\alpha(\Omega))$, and the fact that $C_c^\infty(\Omega)$ is dense in $L^\alpha(\Omega)$, implies (by Corollary 2.1 in [33]) that

$$\rho_n \in C([0, T]; L^\alpha_{weak}(\Omega)), \quad (2.31)$$

and therefore there exists ρ_ε such that

$$\rho_n \to \rho_\varepsilon \quad \text{in } C([0, T]; L^\alpha_{weak}(\Omega)). \quad (2.32)$$

By Lemma 6.4 in [58], we conclude that

$$\rho_n \to \rho_\varepsilon \quad \text{in } L^p(0, T; W^{-1,q}(\Omega)), \quad 1 \le p < \infty, \quad \frac{3}{2} < q < \infty, \quad (2.33)$$

provided $6/5 < \alpha < \infty$. The strong convergence (2.33), along with

$$\mathbf{u}_n \rightharpoonup \mathbf{u}_\varepsilon \quad \text{in } L^2(0, T; W_0^{1,2}(\Omega)), \quad (2.34)$$

yields

$$\rho_n \mathbf{u}_n \rightharpoonup^* \rho_\varepsilon \mathbf{u}_\varepsilon \quad \text{in } L^\infty(0, T; L^{2\gamma/(\gamma+1)}(\Omega; \mathbb{R}^3)). \quad (2.35)$$

The time continuity of the momentum $\rho_n\mathbf{u}_n$ is treated in a similar way. Define the map

$$t \mapsto \left[\int_\Omega \rho_n\mathbf{u}_n\cdot\varphi\,\mathrm{d}x\right](t) =: \mathcal{L}_{\rho_n u_n}(t), \qquad \forall\varphi\in C_c^\infty(\Omega).$$

Multiplying the momentum equation (2.6b) by a function $\varphi\in[C_c^\infty(\Omega)]^3$ and integrating over space and time we find

$$\begin{aligned}\left[\int_\Omega \rho_n\mathbf{u}_n\cdot\varphi\,\mathrm{d}x\right](t') - \left[\int_\Omega \rho_n\mathbf{u}_n\cdot\varphi\,\mathrm{d}x\right](t)&\\ = \int_t^{t'}\int_\Omega [-\mathrm{div}_x(\rho_n\mathbf{u}_n\otimes\mathbf{u}_n) - \nabla_x(p(\rho_n)+\eta_n+\delta\rho_n^\alpha)&\\ -\varepsilon\nabla_x\mathbf{u}_n\nabla_x\rho_n + \mathrm{div}_x\mathbb{S}_n - (\eta_n+\beta\rho_n)\nabla_x\Phi]\cdot\varphi\,\mathrm{d}x\mathrm{d}s&\end{aligned} \tag{2.36}$$

Using available estimates, the integrand on the right hand side is integrable, and so this equation yields the equicontinuity of $\mathcal{L}_{\rho_n u_n}(t)$. We deduce that

$$\rho_n\mathbf{u}_n \to \rho_\varepsilon\mathbf{u}_\varepsilon \qquad \text{in } C([0,T];L^{2\gamma/(\gamma+1)}_{weak}(\Omega;\mathbb{R}^3)). \tag{2.37}$$

Invoking the compact embedding of $L^{2\gamma/(\gamma+1)}(\Omega)$ into $W^{-1,2}(\Omega)$ we deduce that

$$\rho_n\mathbf{u}_n\otimes\mathbf{u}_n \to \rho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon \qquad \text{in } L^2(0,T;L^{6\gamma/(4\gamma+3)}(\Omega;\mathbb{R}^3)). \tag{2.38}$$

Finally we show strong convergence of the density gradient. First, ρ_n and ρ_ε, being strong solutions of (2.6a), satisfy the energy equalities

$$\|\rho_n(t)\|^2_{L^2(\Omega)} + 2\varepsilon\int_0^t \|\nabla_x\rho_n\|^2_{L^2(\Omega)}\,\mathrm{d}s = -\int_0^t\int_\Omega \mathrm{div}_x\mathbf{u}_n\rho_n^2\,\mathrm{d}x\mathrm{d}s + \|\rho_{0,\delta}\|^2_{L^2(\Omega)}, \tag{2.39}$$

and

$$\|\rho_\varepsilon(t)\|^2_{L^2(\Omega)} + 2\varepsilon\int_0^t \|\nabla_x\rho_\varepsilon\|^2_{L^2(\Omega)}\,\mathrm{d}s = -\int_0^t\int_\Omega \mathrm{div}_x\mathbf{u}_\varepsilon\rho_\varepsilon^2\,\mathrm{d}x\mathrm{d}s + \|\rho_{0,\delta}\|^2_{L^2(\Omega)} \tag{2.40}$$

for any $t \in [0,T]$. Next, we demonstrate strong compactness of the sequence $\{\rho_n\}_{n\geq 1}$. Note that

$$\rho_n \in L^2(0,T;W^{1,2}(\Omega)) \cap L^\infty(0,T;L^\alpha(\Omega)),$$

and from equation (2.6a) it is easy to see that

$$\partial_t \rho_n \in L^2(0,T;W^{-1,2}(\Omega)).$$

Provided $1 \leq \alpha < 6$, the compact embedding $W^{1,2}(\Omega) \subset\subset L^\alpha(\Omega)$ holds, while $L^\alpha(\Omega) \hookrightarrow W^{-1,2}(\Omega)$ provided $\alpha > 6/5$. A version of Aubin-Lions lemma (Theorem 1.71 in [58]) then implies that

$$\rho_n \to \rho_\varepsilon \quad \text{in } L^p(0,T;L^\alpha(\Omega)), \quad 1 \leq p < \infty. \tag{2.41}$$

In particular, we can take $\alpha \geq 4$ to conclude ρ_n converges strongly in $L^4((0,T)\times\Omega)$. This implies that $\rho_n^2 \to \rho_\varepsilon^2$ strongly in $L^2((0,T)\times\Omega)$ and $\|\rho_n(t)\|_{L^2(\Omega)} \to \|\rho_\varepsilon(t)\|_{L^2(\Omega)}$. These observations, along with the equalities (2.39) and (2.40) imply that

$$\|\nabla_x \rho_n(t)\|_{L^2((0,T)\times\Omega)} \to \|\nabla_x \rho_\varepsilon(t)\|_{L^2((0,T)\times\Omega)}.$$

Since $\nabla_x \rho_n$ also converges weakly, we deduce that

$$\nabla_x \rho_n \to \nabla_x \rho_\varepsilon \quad \text{in } L^2((0,T)\times\Omega). \tag{2.42}$$

We can then pass the limit into the momentum correction term, namely,

$$\nabla_x \mathbf{u}_n \nabla_x \rho_n \to \nabla_x \mathbf{u}_\varepsilon \nabla_x \rho_\varepsilon \quad \text{in } [\mathcal{D}'((0,T)\times\Omega)]^3. \tag{2.43}$$

The strong convergence in (2.41) allows us to conclude the pressure terms converge, that is,

$$\rho_n^\gamma + \rho_n^\alpha \to \rho_\varepsilon^\gamma + \rho_\varepsilon^\alpha \quad \text{in } L^1((0,T)\times\Omega). \tag{2.44}$$

The estimates (2.24) and (2.29), along with the chain of embeddings

$$W^{1,1}(\Omega) \hookrightarrow\hookrightarrow L^{6/5}(\Omega) \hookrightarrow W^{-1,q}(\Omega),$$

for some $q > 1$, allows us conclude via Aubin-Lions lemma that

$$\eta_n \to \eta_\varepsilon \quad \text{in } L^2(0,T;L^{6/5}(\Omega)) \tag{2.45}$$

The time continuity of η_n follows from the control on $\partial_t \eta_n$ in $L^p(0,T;W^{-1,q}(\Omega))$ provided by (2.29), for some $p, q > 1$. In particular, along with the integrability of $\eta_n \in L^\infty(0,T;L^1(\Omega))$ we deduce that

$$\eta_n \in W^{1,p}(0,T;W^{-1,q}(\Omega)) \hookrightarrow C([0,T];W^{-1,q}(\Omega))$$

for some $p, q > 1$ (see for instance Theorem 2 in [31]). This allows us in particular to make sense of the initial condition for the particle density.

The previous discussion is summarized in the following proposition.

Proposition 5. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $C^{2,\nu}$, $\nu > 0$. Let $\varepsilon, \delta > 0$ be fixed. Assume $\cup_{n\geq 1} X_n$ is dense in the space $W_0^{1,2}(\Omega;\mathbb{R}^3)$ Then problem (2.6a)-(2.6c) admits a solution $(\rho_\varepsilon, \mathbf{u}_\varepsilon, \eta_\varepsilon)$ in the following sense:*

1. *The density $\rho_\varepsilon \geq 0$ has the regularity*

$$\rho_\varepsilon \in L^r(0,T;W^{2,r}(\Omega)), \quad \partial_t \rho_\varepsilon \in L^r((0,T)\times\Omega) \quad \textit{for some } r > 1,$$

the velocity $\mathbf{u}_\varepsilon$ belongs to $L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^3))$, and the particle density $\eta_\varepsilon \geq 0$ belongs to the following spaces

$$\eta_\varepsilon \in L^2(0,T;W^{1,1}(\Omega)) \cap L^1(0,T;W^{1,3/2}(\Omega)), \quad \partial_t \eta_\varepsilon \in L^p(0,T;W^{-1,q}(\Omega)),$$

for some $p, q > 1$.

2. *Equation* (2.6a) *holds a.a. on* $(0,T)\times\Omega$, *and the boundary conditions* (2.7) *are satisfied in the sense of traces. Equation* (2.6b) *is satisfied in* $\mathcal{D}'((0,T)\times\Omega)$. *Moreover,*

$$\rho_\varepsilon \in C([0,T]; L^\alpha_{weak}(\Omega)), \quad \eta_\varepsilon \in C([0,T]; W^{-1,q}(\Omega))$$

for some $q>1$ *and the initial conditions* $\rho_\varepsilon(0,\cdot)=\rho_{0,\delta}$ *and* $\eta_\varepsilon(0,\cdot)=\eta_{0,\delta}$ *are satisfied. In addition, the total masses are conserved,*

$$\int_\Omega \rho_\varepsilon(t,\cdot)\,\mathrm{d}x = \int_\Omega \rho_{0,\delta}\,\mathrm{d}x, \qquad \int_\Omega \eta_\varepsilon(t,\cdot)\,\mathrm{d}x = \int_\Omega \eta_{0,\delta}\,\mathrm{d}x, \tag{2.46}$$

for all $t\in[0,T]$.

3. *All quantities appearing in equation* (2.6c) *are locally integrable, and the equation is satisfied in* $\mathcal{D}'((0,T)\times\Omega)$. *In addition,*

$$\rho_\varepsilon\mathbf{u}_\varepsilon \in C([0,T]; L^{2\gamma/\gamma+1}_{weak}(\Omega;\mathbb{R}^3)),$$

and $\rho_\varepsilon\mathbf{u}_\varepsilon$ *satisfies the initial condition* $(\rho_\varepsilon\mathbf{u}_\varepsilon)(0)=\mathbf{m}_{0,\delta}$.

4. *The energy inequality*

$$\begin{aligned}
&\int_\Omega \left(\frac{1}{2}\rho_\varepsilon|\mathbf{u}_\varepsilon|^2 + \rho_\varepsilon P(\rho_\varepsilon) + \frac{\delta}{\alpha-1}\rho_\varepsilon^\alpha + \eta_\varepsilon\log\eta_\varepsilon + \eta_\varepsilon\Phi\right)(t)\,\mathrm{d}x \\
&\quad + \int_0^t\int_\Omega \mathbb{S}_\varepsilon : \nabla_x\mathbf{u}_\varepsilon + |2\nabla_x\sqrt{\eta_\varepsilon} + \sqrt{\eta_\varepsilon}\nabla_x\Phi|^2\,\mathrm{d}x\mathrm{d}s \\
&\leq \int_\Omega \frac{1}{2}\mathbf{m}_{0,\delta}\cdot\mathbf{u}_{0,\delta} + \rho_{0,\delta}P(\rho_{0,\delta}) + \frac{\delta}{\alpha-1}\rho_{0,\delta}^\alpha + \eta_{0,\delta}\log\eta_{0,\delta} + \eta_{0,\delta}\Phi\,\mathrm{d}x \\
&\quad - \beta\int_0^t\int_\Omega \rho_\varepsilon\mathbf{u}_\varepsilon\cdot\nabla_x\Phi\,\mathrm{d}x\mathrm{d}s,
\end{aligned} \tag{2.47}$$

holds for a.a. $t\in[0,T]$.

Proof. It only remains to check the validity of the energy inequality (2.47). First, it is clear from the weak continuity of the momentum that

$$\lim_{n\to\infty}\int_0^t\int_\Omega \rho_n\mathbf{u}_n\cdot\nabla_x\Phi\,\mathrm{d}x\mathrm{d}s=\int_0^t\int_\Omega \rho_\varepsilon\mathbf{u}_\varepsilon\cdot\nabla_x\Phi\,\mathrm{d}x\mathrm{d}s.$$

The other terms are treated by the weak lower-semicontinuity of convex functions on $L^1((0,T)\times\Omega)$, see [33] Theorem 2.11 and Corollary 2.2. In particular, the functions $z\mapsto z^2$, $z\mapsto z^\alpha$, and $z\mapsto z\log z$ are convex. Along with the convergences established in this section, we conclude that

$$\begin{aligned}
&\int_\Omega\left(\frac{1}{2}\rho_\varepsilon|\mathbf{u}_\varepsilon|^2+\rho_\varepsilon P(\rho_\varepsilon)+\frac{\delta}{\alpha-1}\rho_\varepsilon^\alpha+\eta_\varepsilon\log\eta_\varepsilon+\eta_\varepsilon\Phi\right)(t)\,\mathrm{d}x\\
&\qquad+\int_0^t\int_\Omega \mathbb{S}_\varepsilon:\nabla_x\mathbf{u}_\varepsilon+|2\nabla_x\sqrt{\eta_\varepsilon}+\sqrt{\eta_\varepsilon}\nabla_x\Phi|^2\,\mathrm{d}x\mathrm{d}s\\
&\leq\liminf_{n\to\infty}\left[\int_\Omega\left(\frac{1}{2}\rho_n|\mathbf{u}_n|^2+\rho_n P(\rho_n)+\frac{\delta}{\alpha-1}\rho_n^\alpha+\eta_n\log\eta_n+\eta_n\Phi\right)(t)\,\mathrm{d}x\right.\\
&\qquad\left.+\int_0^t\int_\Omega \mathbb{S}_n:\nabla_x\mathbf{u}_n+|2\nabla_x\sqrt{\eta_n}+\sqrt{\eta_n}\nabla_x\Phi|^2\,\mathrm{d}x\mathrm{d}s\right].
\end{aligned}$$

□

2.6 Vanishing viscosity limit

The next step is to let the parameter ε vanish and demonstrate that the solution $(\rho_\varepsilon,\eta_\varepsilon,\mathbf{u}_\varepsilon)$, constructed in the previous section, converges to $(\rho_\delta,\eta_\delta,\mathbf{u}_\delta)$. At this stage, we lose control of ρ_ε in a positive Sobolev space. Because of this, demonstrating strong compactness of ρ_ε is key in order to pass to the limit in the nonlinear terms.

First, the energy inequality (2.47) provides bounds for the pressure $p(\rho_\varepsilon)+\delta\rho_\varepsilon^\alpha$ in $L^1((0,T)\times\Omega)$. This is estimate is not strong enough to prevent concentration phenomena from occurring in the pressure terms. The following lemma provides the necessary *pressure estimates*.

Lemma 3. *There exists a nonnegative constant c, independent of ε, such that*

$$\int_0^T \int_\Omega \rho_\varepsilon^{\alpha+1} \,\mathrm{d}x\mathrm{d}t \leq c.$$

Proof. The proof is given in Appendix A. $\square$

Lemma 3 implies that $\rho_\varepsilon^\alpha \in L^{1+1/\alpha}((0,T)\times\Omega)$, and in particular guarantees that there exists a weak limit[1] $\overline{p}$ of $p(\rho_\varepsilon) + \delta\rho_\varepsilon^\alpha$, i.e.

$$p(\rho_\varepsilon) + \delta\rho_\varepsilon^\alpha \rightharpoonup \overline{p} \quad \text{in } L^p((0,T)\times\Omega), \quad p>1. \tag{2.48}$$

Next, from the estimates provided by the energy inequality (2.47) we deduce that

$$\rho_\varepsilon \rightharpoonup^* \rho_\delta \quad \text{in } L^\infty(0,T;L^\alpha(\Omega)), \tag{2.49}$$

$$\mathbf{u}_\varepsilon \rightharpoonup \mathbf{u}_\delta \quad \text{in } L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^3)), \tag{2.50}$$

$$\eta_\varepsilon \rightharpoonup \eta_\delta \quad \text{in } L^2(0,T;W^{1,1}(\Omega)) \cap L^1(0,T;W^{1,3/2}(\Omega)). \tag{2.51}$$

In the same way as in the previous section, it follows that

$$\eta_\varepsilon \to \eta_\delta \quad \text{in } L^2(0,T;L^{6/5}(\Omega)). \tag{2.52}$$

Then (2.50) and (2.52) imply that

$$\eta_\varepsilon \mathbf{u}_\varepsilon \rightharpoonup \eta_\delta \mathbf{u}_\delta \quad \text{in } L^1((0,T)\times\Omega). \tag{2.53}$$

By interpolation and using (2.51), we also deduce that

$$\nabla_x \eta_\varepsilon \rightharpoonup \nabla_x \eta_\varepsilon \quad \text{in } L^p(0,T;L^q(\Omega;\mathbb{R}^3)),$$

[1] An overbar will denote a weak limit unless otherwise stated.

for some $p, q > 1$. Next, in a similar way as in the previous section, the equicontinuity of the map $\mathcal{L}_{\rho_\varepsilon}(t)$ (defined in Section 2.5.2) and boundedness of ρ_ε in $L^\infty(0,T;L^\alpha(\Omega))$ imply

$$\rho_\varepsilon \to \rho_\delta \quad \text{in } C([0,T];L^\alpha_{weak}(\Omega)). \tag{2.54}$$

Provided $\alpha > 6/5$, this implies

$$\rho_\varepsilon \to \rho_\delta \quad \text{in } L^p(0,T;W^{-1,q}(\Omega)), \quad 1 \le p < \infty,\ 3/2 < q < \infty. \tag{2.55}$$

In a similar way, equicontinuity of the map $\mathcal{L}_{\rho_\varepsilon \mathbf{u}_\varepsilon}(t)$, and boundedness of $\rho_\varepsilon \mathbf{u}_\varepsilon$ in $L^\infty(0,T;L^{2\alpha/(\alpha+1)}(\Omega;\mathbb{R}^3))$ imply

$$\rho_\varepsilon \mathbf{u}_\varepsilon \to \rho_\delta \mathbf{u}_\delta \quad \text{in } C([0,T];L^{2\alpha/(\alpha+1)}_{weak}(\Omega;\mathbb{R}^3)). \tag{2.56}$$

Since $L^{2\alpha/(\alpha+1)}(\Omega) \subset\subset W^{-1,2}(\Omega)$ provided $\alpha > 3/2$, (2.56) implies that

$$\rho_\varepsilon \mathbf{u}_\varepsilon \to \rho_\delta \mathbf{u}_\delta \quad \text{in } L^p(0,T;W^{-1,2}(\Omega;\mathbb{R}^3)),\ 1 \le p < \infty. \tag{2.57}$$

Along with (2.50), the last observation implies

$$\rho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon \rightharpoonup \rho_\delta \mathbf{u}_\delta \otimes \mathbf{u}_\delta \quad \text{in } L^2(0,T;L^{6\alpha/(4\alpha+3)}(\Omega;\mathbb{R}^{3\times 3})). \tag{2.58}$$

Next we show convergence of the $\nabla_x \rho_\varepsilon$ term. By virtue of Proposition 5, equation (2.6a) is satisfied almost everywhere. Hence, upon multiplying (2.6a) by ρ_ε and integrating by parts we get

$$\begin{aligned} \frac{1}{2}\int_\Omega \rho_\varepsilon^2(T,\cdot)\,\mathrm{d}x + \varepsilon \int_0^T \int_\Omega |\nabla_x \rho_\varepsilon|^2 \,\mathrm{d}x\mathrm{d}t \\ = \frac{1}{2}\int_\Omega \rho_{0,\delta}^2 \,\mathrm{d}x - \frac{1}{2}\int_0^T \int_\Omega \rho_\varepsilon^2 \mathrm{div}_x \mathbf{u}_\varepsilon \,\mathrm{d}x\mathrm{d}t \end{aligned} \tag{2.59}$$

Using (2.49) and (2.50), this implies $\sqrt{\varepsilon}\nabla_x \rho_\varepsilon$ is bounded in $L^2(0,T;L^2(\Omega;\mathbb{R}^3))$ and so

$$\varepsilon \nabla_x \rho_\varepsilon \to 0 \quad \text{in } L^2(0,T;L^2(\Omega;\mathbb{R}^3)).$$

Along with (2.50), we deduce that

$$\varepsilon \nabla_x \mathbf{u}_\varepsilon \nabla_x \rho_\varepsilon \to 0 \quad \text{in } L^1((0,T)\times\Omega). \tag{2.60}$$

As a result of the convergences established in (2.48)-(2.60), and passing to the limit in the weak formulations, it follows that

1. For all $\varphi \in \mathcal{D}([0,T)\times\overline{\Omega})$,

$$\int_0^T \int_\Omega \rho_\delta \partial_t \varphi + \rho_\delta \mathbf{u}_\delta \cdot \nabla_x \varphi \,\mathrm{d}x\mathrm{d}t = -\int_\Omega \rho_{0,\delta}\varphi(0,\cdot)\,\mathrm{d}x. \tag{2.61}$$

2. For all $\varphi \in \mathcal{D}([0,T)\times\overline{\Omega})$,

$$\int_0^T \int_\Omega \eta_\delta \partial_t \varphi + \eta_\delta \mathbf{u}_\delta \cdot \nabla_x \varphi - \eta_\delta \nabla_x \Phi \cdot \nabla_x \varphi - \nabla_x \eta_\delta \cdot \nabla_x \varphi \,\mathrm{d}x\mathrm{d}t = -\int_\Omega \eta_{0,\delta}\varphi(0,\cdot)\,\mathrm{d}x. \tag{2.62}$$

3. For all $\varphi \in \mathcal{D}([0,T);\mathcal{D}(\Omega;\mathbb{R}^3))$,

$$\begin{aligned}\int_0^T \int_\Omega &\Big(\rho_\delta \mathbf{u}_\delta \cdot \partial_t \varphi + \rho_\delta \mathbf{u}_\delta \otimes \mathbf{u}_\delta : \nabla_x \varphi + (\overline{p}+\eta_\delta)\mathrm{div}_x \varphi\Big)\,\mathrm{d}x\mathrm{d}t \\ &= \int_0^T \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}_\delta) : \nabla_x \varphi + (\eta_\delta + \beta\rho_\delta)\nabla_x \Phi \cdot \varphi \,\mathrm{d}x\mathrm{d}t - \int_\Omega (\rho\mathbf{u})_{0,\delta}\cdot \varphi(0,\cdot)\,\mathrm{d}x.\end{aligned} \tag{2.63}$$

It remains to show that $\overline{p} = p(\rho_\delta) + \delta\rho_\delta^\alpha$. This will be carried out in the next section where we show strong convergence of the density ρ_ε.

2.6.1 Strong convergence of the density

There are two key results needed to obtain the strong convergence of the density: 1) establishing the weak continuity of the effective viscous pressure, and 2) renormalizing the continuity equation both at the level of the approximate solution and the limiting solution. The former is originally due to Lions [51] and asserts that the effective viscous pressure, defined as

$$P_{eff} = p - (2\mu + \lambda)\mathrm{div}_x \mathbf{u},$$

satisfies a weak continuity property in the sense that its product with another weakly converging sequence converges to the product of the weak limits. The latter result allows us to deduce that if ρ satisfies the continuity equation, then so does a suitable nonlinear composition $B(\rho)$, up to minor modification of the equation. This so-called renormalization property is originally due to DiPerna and Lions in the context of linear transport PDE [24].

The theory of the strong convergence of the fluid density is well-established (cf. [51], [37]). There are some additional terms to consider when the Smoluchowki equation is included, in particular in proving the weak continuity of the effective viscous pressure. These computations can be found in [9]. For the details on the following lemmas we refer the reader to [51], [37], and [9].

Lemma 4. *Let $\alpha > \max\{4, \gamma\}$. For any $\psi \in \mathcal{D}(0,T)$ and $\zeta \in \mathcal{D}(\Omega)$, it holds that*

$$\lim_{\varepsilon \to 0} \int_0^T \int_\Omega \psi\zeta \left(p(\rho_\varepsilon) + \delta\rho_\varepsilon^\alpha - (\lambda + 2\mu)\mathrm{div}_x \mathbf{u}_\varepsilon\right) \rho_\varepsilon \,\mathrm{d}x\mathrm{d}t$$
$$= \int_0^T \int_\Omega \psi\zeta \left(\overline{p} - (\lambda + 2\mu)\mathrm{div}_x \mathbf{u}_\delta\right) \rho_\delta \,\mathrm{d}x\mathrm{d}t.$$

Lemma 5. *The pair $(\rho_\delta, \mathbf{u}_\delta)$ is a renormalized solution of equation (1.1a) on $(0,T)\times\mathbb{R}^3$ after extending by zero outside Ω. In particular, for any $B \in C[0,\infty)\cap C^1(0,\infty)$, $b(z) = B'(z)z - B(z)$ such that $b \in C[0,\infty) \cap L^\infty[0,\infty)$, and $B(0) = b(0) = 0$, the equation*

$$\partial_t B(\rho_\delta) + \mathrm{div}_x(B(\rho_\delta)\mathbf{u}_\delta) + b(\rho_\delta)\mathrm{div}_x\mathbf{u}_\delta = 0$$

holds in $\mathcal{D}'((0,T)\times\mathbb{R}^3)$.

Since the pair $(\rho_\varepsilon, \mathbf{u}_\varepsilon)$ satisfies equation (2.6a) a.a. on $(0,T)\times\Omega$ by Proposition 5, we can easily renormalize to find that for any twice-differentiable B,

$$\begin{aligned}&\partial_t B(\rho_\varepsilon) + \mathrm{div}_x(B(\rho_\varepsilon)\mathbf{u}_\varepsilon) + b(\rho_\varepsilon)\mathrm{div}_x\mathbf{u}_\varepsilon \\ &\quad = \varepsilon\mathrm{div}_x(\mathbb{1}_\Omega\nabla_x B(\rho_\varepsilon)) - \varepsilon\mathbb{1}_\Omega B''(\rho_\varepsilon)|\nabla_x\rho_\varepsilon|^2 \quad \text{in } \mathcal{D}'((0,T)\times\mathbb{R}^3).\end{aligned} \tag{2.64}$$

The goal now is to use the renormalized equations for ρ_ε and ρ_δ with the renormalization $B(z) = z\log z$ and argue that

$$\overline{\rho_\delta \log\rho_\delta} = \rho_\delta\log\rho_\delta \quad \text{a.a. on } (0,T)\times\Omega. \tag{2.65}$$

This equivalence would imply, by virtue of Theorem 2.11 [33], that $\rho_n \to \rho$ almost everywhere and therefore strongly in $L^1((0,T)\times\Omega)$. The Dominated Convergence Theorem, along with the integrability estimates on the pressure, then allows us to conclude that the weak limit $\overline{p}$ in (2.63) is in fact equal to $p(\rho_\delta) + \delta\rho_\delta^\alpha$. We next explain the details of this approach based on Lemma 4.

First, let $B(z) = z\log z$ and choose test functions $\varphi(t,x) = \psi_n(t)\zeta_n(x)$ in Lemma 5 such that $\psi \in \mathcal{D}(0,T)$ and $\zeta \in \mathcal{D}(\mathbb{R}^3)$ are nonnegative. Letting $\psi_n \to \mathbb{1}_{(0,t)}$ and $\zeta_n \to \mathbb{1}_\Omega$ it follows that for any $t\in(0,T)$,

$$\int_0^t\int_\Omega \rho_\delta\mathrm{div}_x\mathbf{u}_\delta\,\mathrm{d}x\mathrm{d}s = \int_\Omega \rho_{0,\delta}\log\rho_{0,\delta}\,\mathrm{d}x - \int_\Omega[\rho_\delta\log\rho_\delta](t,\cdot)\,\mathrm{d}x. \tag{2.66}$$

Next, choosing $B(z) = z \log z$ in (2.64), along with $\varphi(t,x)$ as used in deriving (2.66) and letting $\varepsilon \to 0$, it follows that for any Lebesgue point t,

$$\int_0^t \int_\Omega \overline{\rho_\delta \mathrm{div}_x \mathbf{u}_\delta}\, \mathrm{d}x\mathrm{d}s \le \int_\Omega \rho_{0,\delta} \log \rho_{0,\delta}\, \mathrm{d}x - \int_\Omega \overline{\rho_\delta \log \rho_\delta}(t,\cdot)\, \mathrm{d}x. \tag{2.67}$$

Here we used that $z \log z$ is convex. Now combining (2.66) with (2.67), the following inequality holds

$$\int_\Omega \left(\overline{\rho_\delta \log \rho_\delta} - \rho_\delta \log \rho_\delta\right)(t)\, \mathrm{d}x \le \int_0^t \int_\Omega \rho_\delta \mathrm{div}_x \mathbf{u}_\delta - \overline{\rho_\delta \mathrm{div}_x \mathbf{u}_\delta}\, \mathrm{d}x\mathrm{d}s, \tag{2.68}$$

for a.a. $t \in (0,T)$. Note the left-hand side of this inequality is nonnegative due to convexity of the map $z \mapsto z \log z$. Provided the right-hand side of (2.68) is nonpositive, the equality (2.65) is established and therefore also strong convergence of the density. But this follows directly from the continuity of the effective viscous pressure via Lemma 4. Indeed, rearranging Lemma 4 we have that for any $O \subset\subset (0,T) \times \Omega$,

$$\begin{aligned}\int_O &\overline{\rho_\delta \mathrm{div}_x \mathbf{u}_\delta} - \rho_\delta \mathrm{div}_x \mathbf{u}_\delta\, \mathrm{d}x\mathrm{d}t \\ &\ge \frac{1}{2\mu + \lambda} \liminf_{\varepsilon \to 0} \int_O \left(p(\rho_\varepsilon)\rho_\varepsilon + \delta \rho_\varepsilon^{\alpha+1}\right) - \left(\overline{p(\rho_\delta)} + \delta \overline{\rho_\delta^\alpha}\right) \rho_\delta\, \mathrm{d}x\mathrm{d}t.\end{aligned} \tag{2.69}$$

By appealing to the weak lower semi-continuity of convex functions and using that $z \mapsto z^p$ is increasing for $p > 0$, a monotonicity argument implies that the right-hand side of (2.69) is nonnegative. In particular,

$$\int_O \delta \overline{\rho_\delta^\alpha} \rho_\delta + a \overline{\rho_\delta^\gamma} \rho_\delta \le \liminf_{\varepsilon \to 0} \int_O \delta \rho_\varepsilon^{\alpha+1} + a \rho_\varepsilon^{\gamma+1}\, \mathrm{d}x\mathrm{d}t.$$

We conclude that the right-hand side of (2.68) is nonpositive and the strong convergence of the density at the ε-level approximation has been proved.

The results of this section are summarized in the following proposition.

Proposition 6. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $C^{2,\nu}$, $\nu \in (0,1)$. Let $\alpha > \max\{5, \gamma\}$ and $\delta > 0$ be given. Then there exists a solution $(\rho_\delta, \eta_\delta, \mathbf{u}_\delta)$ such that*

1. *The density $\rho_\delta \geq 0$ belongs to $L^{\alpha+1}(O)$ such that $O \subset\subset ((0,T) \times \Omega)$. In addition,*

   $$\rho_\delta \in C([0,T]; L^\alpha_{weak}(\Omega)).$$

   The velocity $\mathbf{u}_\delta$ belongs to $L^2(0,T; W_0^{1,2}(\Omega; \mathbb{R}^3))$ and the particle density $\eta_\delta \geq 0$ satisfies

   $$\eta_\delta \in L^2(0,T; W^{1,1}(\Omega)) \cap L^1(0,T; W^{1,3/2}(\Omega)), \quad \partial_t \eta_\delta \in L^p(0,T; W^{-1,q}(\Omega)),$$

   for some $p, q > 1$. In addition,

   $$\rho_\delta \mathbf{u}_\delta \in C([0,T]; L^{2\gamma/(\gamma+1)}_{weak}(\Omega; \mathbb{R}^3)) \quad \text{and} \quad \eta_\delta \in C([0,T]; W^{-1,q}(\Omega)),$$

   for some $q > 1$.

2. *The initial conditions $\rho_\delta(0,\cdot) = \rho_{0,\delta}$, $\eta_\delta(0,\cdot) = \eta_{0,\delta}$, and $(\rho_\delta \mathbf{u}_\delta)(0,\cdot) = \mathbf{m}_{0,\delta}$ are satisfied. Upon extending by zero, the pair $(\rho_\delta, \mathbf{u}_\delta)$ represents a renormalized solution of (1.1a) on $(0,T) \times \mathbb{R}^3$ in the sense that*

   $$\partial_t B(\rho_\delta) + \operatorname{div}_x(B(\rho_\delta)\mathbf{u}_\delta) + b(\rho_\delta)\operatorname{div}_x \mathbf{u}_\delta = 0, \quad \text{in } \mathcal{D}'((0,T) \times \mathbb{R}^3),$$

   for any $B \in C[0,\infty) \cap C^1(0,\infty)$, $b \in C[0,\infty) \cap L^\infty[0,\infty)$, such that $B(0) = b(0) = 0$ and $b(z) = B'(z)z - B(z)$.

   The pair $(\eta_\delta, \mathbf{u}_\delta)$ satisfies equation in $\mathcal{D}'((0,T) \times \Omega)$ and the triple $(\rho_\delta, \eta_\delta, \mathbf{u}_\delta)$ satisfies (1.1c) in $\mathcal{D}'((0,T) \times \Omega)$ where $p(\rho_\delta)$ is taken to be $a\rho_\delta^\gamma + \delta\rho_\delta^\alpha$.

3. The energy inequality holds for almost every $t \in [0,T]$,

$$\begin{aligned}
&\int_\Omega \left(\frac{1}{2}\rho_\delta|\mathbf{u}_\delta|^2 + \rho_\delta P(\rho_\delta) + \frac{\delta}{\alpha-1}\rho_\delta^\alpha + \eta_\delta \log \eta_\delta + \eta_\delta \Phi\right)(t)\,\mathrm{d}x \\
&\quad + \int_0^t \int_\Omega \mathbb{S}_\delta : \nabla_x \mathbf{u}_\delta + |2\nabla_x \sqrt{\eta_\delta} + \sqrt{\eta_\delta}\nabla_x \Phi|^2 \,\mathrm{d}x\mathrm{d}s \\
&\quad \leq \int_\Omega \frac{1}{2}\mathbf{m}_{0,\delta} \cdot \mathbf{u}_{0,\delta} + \rho_{0,\delta} P(\rho_{0,\delta}) + \frac{\delta}{\alpha-1}\rho_{0,\delta}^\alpha + \eta_{0,\delta}\log \eta_{0,\delta} + \eta_{0,\delta}\Phi \,\mathrm{d}x \\
&\quad - \beta \int_0^t \int_\Omega \rho_\delta \mathbf{u}_\delta \cdot \nabla_x \Phi \,\mathrm{d}x\mathrm{d}s,
\end{aligned} \tag{2.70}$$

2.7 Artificial pressure limit

Having taken the artificial diffusion limit $\varepsilon \to 0$ in the previous section, it remains to let $\delta \to 0$ in the artificial pressure and obtain a weak solution to the full system (1.1a) in the sense of Definition 1, thereby proving Theorem 1. The approach is similar to that of the last section. The largest difference is that we lose integrability of the fluid density due to loss of the artificial pressure term.

At this stage, the assumption that the pressure satisfies $p(\rho) = a\rho^\gamma$ with $\gamma \geq 9/5$ ensures that the density would be square-integrable (a consequence of the pressure estimates), and we can conclude in much the same way as before. This constitutes the original approach of Lions. Our assumption that the pressure is given instead with $\gamma > 3/2$ requires additional tools, in particular the so-called oscillations defect measure introduced by Feireisl.

To begin, the energy inequality in Proposition 6 provide various estimates independent of δ. This is a consequence of the choice of initial data (2.8)-(2.11). Note that the assumption (2.8) implies that the initial data corresponding to the artificial pressure in (2.70) vanishes as $\delta \to 0$. In addition, the initial data $(\rho_{0,\delta}, \eta_{0,\delta}, \mathbf{m}_{0,\delta})$ converges to $(\rho_0, \eta_0, \mathbf{m}_0)$. The other term on the right-hand side of the energy inequality is independent of δ after invoking Hölder's inequality. As a consequence the energy inequality (2.70) provides estimates uniformly in $\delta > 0$.

As in the previous sections, Proposition 6 allows us to secure subsequences such that

$$\rho_\delta \to \rho \quad \text{in } C([0,T]; L^\gamma_{weak}(\Omega)), \tag{2.71a}$$

$$\mathbf{u}_\delta \rightharpoonup \mathbf{u} \quad \text{in } L^2(0,T; W_0^{1,2}(\Omega; \mathbb{R}^3)), \tag{2.71b}$$

$$\eta_\delta \to \eta \quad \text{in } L^2(0,T; L^{6/5}(\Omega)) \tag{2.71c}$$

$$\nabla_x \eta_\delta \rightharpoonup \nabla_x \eta \quad \text{in } L^p((0,T)\times\Omega), \text{ for some } p > 1. \tag{2.71d}$$

The convective nonlinear terms are handled as in previous sections and we can easily conclude that

$$\rho_\delta \mathbf{u}_\delta \rightharpoonup^* \rho\mathbf{u} \quad \text{in } L^\infty(0,T; L^{2\gamma/(\gamma+1)}(\Omega; \mathbb{R}^3)), \tag{2.72a}$$

$$\rho_\delta \mathbf{u}_\delta \to \rho\mathbf{u} \quad \text{in } C([0,T]; L^{2\gamma/(\gamma+1)}(\Omega; \mathbb{R}^3)), \tag{2.72b}$$

$$\eta_\delta \mathbf{u}_\delta \rightharpoonup \eta\mathbf{u} \quad \text{in } L^1((0,T)\times\Omega), \tag{2.72c}$$

$$\rho_\delta \mathbf{u}_\delta \otimes \mathbf{u}_\delta \rightharpoonup \rho\mathbf{u}\otimes\mathbf{u} \quad \text{in } L^2(0,T; L^{6\gamma/(4\gamma+3)}(\Omega; \mathbb{R}^{3\times 3})). \tag{2.72d}$$

Next, in a similar way as Lemma 3, we get pressure estimates of the form

$$\int_K \rho_\delta^{\gamma+\omega} + \delta\rho_\delta^{\alpha+\omega}\,\mathrm{d}x\mathrm{d}t \le c, \quad 0 < \omega < \min\left\{\frac{1}{3}, \frac{2}{3}\gamma - 1\right\}, \tag{2.73}$$

for some constant c independent of δ, where $K \subset\subset ((0,T)\times\Omega)$. Note that in order to get estimates uniformly in δ, we must restrict the integrability gains more than that of Lemma 3. For details see [33], and also the remark in Appendix A.

The pressure estimates (2.73) along with a Hölder inequality ensures that

$$\delta \rho_\delta^\alpha \to 0 \quad \text{in } L^1((0,T)\times\Omega), \tag{2.74}$$

and therefore the corresponding term in the weak formulation of the momentum equation vanishes as $\delta \to 0$. The final step is to again prove the strong convergence of the density ρ_δ in $L^1((0,T)\times\Omega)$ and invoke the pressure estimates to pass to the limit in the pressure term $a\rho_\delta^\gamma$ when $\gamma > 3/2$.

Recall that in the previous section, the key in renormalizing the equation of continuity was using the integrability gain from the artificial pressure to ensure that ρ_ε was bounded in $L^2(0,T;L^2(\Omega))$. Having lost this integrability through the passage $\delta \to 0$ and since we require that $\gamma > 3/2$, we proceed by first defining the *oscillations defect measure*, a tool introduced by Feireisl. The oscillations defect measure is defined as

$$\mathbf{osc}_p[\rho_\delta \to \rho](O) := \sup_{k\geq 1}\left(\limsup_{n\to\infty}\int_O |T_k(\rho_\delta) - T_k(\rho)|^p \,\mathrm{d}x\mathrm{d}t\right).$$

A bound on the oscillations defect measure will replace the requirement that ρ_δ needs to be bounded in $L^2(0,T;L^2(\Omega))$. The functions T_k are cutoff functions defined by

$$T_k(z) := kT\left(\frac{z}{k}\right)$$

where T is such that for nonnegative arguments, $T(z) = z$ for $z \in [0,1]$, $T(z) = 2$ for $z \geq 3$, and a smooth concave extension is used over the interval $[0,2]$.

At this level of the approximation, the weak continuity of the effective viscous pressure reads

$$\begin{aligned}\lim_{\delta\to 0}\int_0^T\int_\Omega \psi\zeta\left(p(\rho_\delta)-(\lambda+2\mu)\mathrm{div}_x\mathbf{u}_\delta\right)T_k(\rho_\delta)\,\mathrm{d}x\mathrm{d}t\\ =\int_0^T\int_\Omega \psi\zeta\left(\overline{p(\rho)}-(\lambda+2\mu)\mathrm{div}_x\mathbf{u}\right)\overline{T_k(\rho)}\,\mathrm{d}x\mathrm{d}t,\end{aligned}\tag{2.75}$$

where $\psi\in\mathcal{D}(0,T)$ and $\zeta\in\mathcal{D}(\Omega)$. Again, this is a highly nontrivial observation and we refer to [33] and [9] for further details.

The validity of the weak continuity of the effective viscous pressure in fact implies that the oscillations defect measure is bounded:

$$\mathbf{osc}_{\gamma+1}[\rho_\delta\to\rho](O)\le c(|O|).\tag{2.76}$$

For details see Proposition 6.2 of [33]. In turn, the boundedness of the oscillations defect measure implies that $(\rho,\mathbf{u})$ is a renormalized solution of the equation of continuity. This is the contents of Proposition 6.3 in [33]. Given that the limiting functions $(\rho,\mathbf{u})$ are renormalized, we can proceed in almost the exact same way as in Section 2.6.1 to conclude that

$$\rho_\delta\to\rho\quad\text{strongly in } L^1((0,T)\times\Omega).$$

Therefore we can pass to the limit in the pressure term in the weak formulation of the momentum equation. Theorem 1 has been proved.

Chapter 3: Global existence: moving domains

In Chapter 2, the existence of weak solutions to the NSS system on fixed domains was proved using the Lions-Feireisl theory of compressible Navier-Stokes equations. In this chapter, the NSS system is posed on a moving domain and the existence of weak solutions is established by the penalization technique introduced in Section 1.3. The main components of this method are the introduction of a singular term in the momentum equation (the so-called Brinkman penalization), and penalizing the viscosity. From a modeling perspective these terms model the solid portions of domain as porous media, with permeability approaching zero. Effectively, the problem is reformulated over a fixed domain such that the fluid is allowed to 'flow' through solid obstacles. Penalization of the viscosity is used to get rid of extra shear terms that appear in the solid portion of the domain. A key ingredient is getting rid of the terms supported on the 'solid' part of the domain. This part of the analysis will make use of the *level set method* (cf. [60]). As a straightforward corollary, convergence of the Brinkman penalization is established. As a remark, the original work of Brinkman involves a Laplacian regularization of the Darcy law, rather than a singular penalization as here. See [15] for details.

3.1 Moving domains

We continue the discussion of Section 1.3 and describe the mathematical framework of moving domains in more detail. Let $\Omega_0 \subset\subset D \subset \mathbb{R}^3$ denote a domain contained in the fixed domain D, sometimes called the universal domain. At a later time $t > 0$, the initial domain Ω_0 has moved to the new position Ω_t. The family $\{\Omega_t\}_{t=0}^{T}$ then forms a one-parameter transformation of the domain Ω_0. We assume that each image is compactly contained within D. The boundary $\partial\Omega_t$ is denoted by Γ_t.

When viewed as a subset of $[0, T] \times D$, moving spatial domains form *non-cylindrical* space-time domains. In this context, we define the 'fluid' space-time domain Q^f by

$$Q^f := \bigcup_{t \in (0,T)} (\{t\} \times \Omega_t).$$

The set

$$Q^s := ((0, T) \times D) \backslash \overline{Q^f},$$

in many contexts is often called the 'solid' domain. The evolution of the domain is characterized by a prescribed velocity field $\mathbf{V}(t, \mathbf{x})$ defined over $(0, T) \times D$. Note in applications typically only boundary behavior is known (cf. Section 1.3). The velocity field allows us to define the position $\mathbf{X}(t, \mathbf{x})$ as the solution to

$$\begin{cases} \dfrac{d}{dt}\mathbf{X}(t, \mathbf{x}) = \mathbf{V}(t, \mathbf{X}(t, \mathbf{x})), \; t > 0, \\ \mathbf{X}(0, \mathbf{x}) = \mathbf{x}, \qquad \mathbf{x} \in \Omega_0. \end{cases} \tag{3.1}$$

The domains therefore evolve according to

$$\Omega_t = \mathbf{X}(t, \Omega_0).$$

We also recall the definition of the ALE map T_t, defined such that

$$T_t(\mathbf{x}) = \mathbf{X}(t, \mathbf{x}), \quad \text{for all } \mathbf{x} \in \Omega_0. \tag{3.2}$$

Furthermore, the velocity $\mathbf{V}$ is assumed to have the following regularity,

$$\mathbf{V} \in C^{2,\nu}([0, T] \times \overline{D}; \mathbb{R}^3).$$

3.1.1 Function spaces

With an evolving spatial domain, the function spaces in which solutions are looked for need to be modified accordingly. Consider for example the heat equation

$$u_t - \Delta u = 0 \quad \text{in } (0, T) \times \Omega,$$

supplemented with Dirichlet boundary data. Weak solutions are typically sought in the Bochner spaces

$$u \in L^2(0, T; H_0^1(\Omega)), \qquad u_t \in L^2(0, T; H^{-1}(\Omega)).$$

The functions $u : (0, T) \to H_0^1(\Omega)$ therefore are valued in the fixed space $H_0^1(\Omega)$. Naively replacing Ω with Ω_t, we would obtain $u : (0, T) \to H_0^1(\Omega_t)$, which isn't exactly correct since the interval $(0, T)$ should get mapped to the full range of spaces, for instance $\bigcup(\{t\} \times H_0^1(\Omega_t))$. We therefore can not directly use the standard Bochner spaces when dealing with moving domains.

The way out of this situation is simple: embed functions into the 'global' space and extend by zero outside the moving domains Ω_t.

Let $p \in [1,\infty], q \in [1,\infty]$ be given exponents, with dual exponents p', q', respectively. We define

$$\begin{aligned} L^{p,q}(Q^f) &\equiv L^p(0,T;L^q(\Omega_t)) \\ &:= \left\{ u \in L^p(0,T;L^q(D)) \,\middle|\, u(t,\cdot) = 0 \text{ over } D\backslash\Omega_t \text{ for a.e. } t \in (0,T) \right\}, \end{aligned}$$

with the norm

$$\|u\|_{L^{p,q}(Q^f)} := \begin{cases} \left(\displaystyle\int_0^T \|u(t)\|^p_{L^q(\Omega_t)} \,\mathrm{d}t \right)^{\frac{1}{p}}, & \text{if } p < \infty. \\ \operatorname*{ess\,sup}_{t\in(0,T)} \|u(t)\|_{L^q(\Omega_t)}, & \text{if } p = \infty. \end{cases}$$

If $p = q$, we write $L^p(Q^f) \equiv L^{p,p}(Q^f)$.

Remark. The spaces $L^{p,q}(Q^f)$ are Banach spaces and enjoy similar properties to the standard Bochner spaces. See [55] for details.

Let $l \in \mathbb{N}$, and let α be a multi-index. We define

$$W^l_{p,q}(Q^f) \equiv L^p(0,T;W^{l,q}(\Omega_t)) := \left\{ u \in L^{p,q}(Q^f) \,\middle|\, \partial^\alpha u \in L^{p,q}(Q^f),\ \forall |\alpha| \le l \right\},$$

with the norm

$$\|u\|_{W^l_{p,q}(Q^f)} := \sum_{|\alpha|\le l} \|\partial^\alpha u\|_{L^{p,q}(Q^f)}.$$

The definition of $W^{0,l}_{p,q}(Q^f)$ is similar, with the addition of a zero trace condition.

The space of functions continuous with respect to the weak-topology of $L^\gamma(\Omega_t)$ is defined by

$$C([0,T];L^\gamma_{wk}(\Omega_t)) := \left\{ u \in C([0,T];L^\gamma_{wk}(D)) \,\middle|\, u(t,\cdot) = 0 \text{ on } D\backslash\Omega_t \text{ for all } t \in (0,T) \right\}.$$

In order to define the space of test functions we need to make use of the map T_t, defined in (3.2). The homeomorphism T_t is used to push the test functions $\mathcal{D}([0,T]\times\Omega_0;\mathbb{R}^N)$ to the set $(0,T)\times\Omega_t$. The space $\mathcal{D}([0,T]\times\Omega_t;\mathbb{R}^N)$ is defined as

$$\mathcal{D}([0,T]\times\Omega_t;\mathbb{R}^N) := \left\{ u : Q^f \to \mathbb{R} \;\middle|\; u(t,x) = \hat{u}(t,T_t^{-1}(x)),\ \hat{u} \in \mathcal{D}([0,T]\times\Omega_0;\mathbb{R}^N) \right\}.$$

The spaces as defined here are introduced in various contexts in [50], [41], [55] and [61]. A more general framework of evolving function spaces (rather than spaces defined on evolving domains), can be found in [1], [2].

3.2 Weak formulation and main result

Let us recall the governing equations, introduced in Chapter 1. The fluid-particle system is given by

$$\partial_t\rho + \mathrm{div}_x(\rho\mathbf{u}) = 0, \tag{3.3a}$$

$$\partial_t\eta + \mathrm{div}_x(\eta(\mathbf{u} - \nabla_x\Phi)) - \Delta_x\eta = 0, \tag{3.3b}$$

$$\partial_t(\rho\mathbf{u}) + \mathrm{div}_x(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla_x(p(\rho)+\eta) = \mathrm{div}_x\mathbb{S} - (\eta+\beta\rho)\nabla_x\Phi. \tag{3.3c}$$

The system (3.3) is posed on the space-time domain Q^f. The no-slip boundary conditions are imposed on the velocity,

$$\mathbf{u}(t,\cdot)\big|_{\Gamma_t} = \mathbf{V}(t,\cdot)\big|_{\Gamma_t}, \text{ for any } t \geq 0, \tag{3.4}$$

while the no-flux condition for particle density holds,

$$(\nabla_x\eta + \eta\nabla_x\Phi)\cdot\nu = 0 \quad \text{on } (0,T)\times\Gamma_t, \tag{3.5}$$

with $\nu(t,x)$ denoting the outer normal vector to the boundary Γ_t.

Initial data are prescribed such that

$$
\begin{aligned}
&\rho_0 \in L^\gamma(D), \quad \rho_0 \geq 0 \textit{ a.e. in } \Omega_0 \\
&\eta_0 \in L^1(D), \quad \eta_0 \geq 0 \textit{ a.e. in } \Omega_0 \\
&\mathbf{m}_0 \in L^1(D;\mathbb{R}^3), \quad \frac{|\mathbf{m}_0|^2}{\rho_0} \in L^1(D),
\end{aligned}
\tag{3.6}
$$

and all initial data is assumed to vanish on $D\backslash\Omega_0$.

Definition 2. *We say that $(\rho, \mathbf{u}, \eta)$ comprise a weak solution of the NSS system (3.3) over the noncylindrical domain Q^f, along with the boundary conditions (3.4) and (3.5), and the initial data (3.6) provided*

- *The density $\rho = \rho(t,x)$ and velocity $\mathbf{u} = \mathbf{u}(t,x)$ represent a weak solution of equation (3.3a) over Q^f. In particular, for any test function $\varphi \in \mathcal{D}([0,T)\times\overline{\Omega_t})$, the following integral identity holds:*

$$
\begin{aligned}
&\int_0^T \int_{\Omega_t} \rho\partial_t\varphi + \rho\mathbf{u}\cdot\nabla_x\varphi \,\mathrm{d}x\mathrm{d}t \\
&\qquad = \int_{\Omega_T} \rho(T,\cdot)\varphi(T,\cdot)\,\mathrm{d}x - \int_{\Omega_0} \rho_0\varphi(0,\cdot)\,\mathrm{d}x.
\end{aligned}
\tag{3.7}
$$

  The density, velocity, and momentum are required to have the following regularity

$$
\begin{aligned}
&\rho \in L^\infty(0,T;L^\gamma(\Omega_t)), \quad \rho \geq 0 \textit{ a.e. in } Q^f, \\
&\mathbf{u} \in L^2(0,T;W_0^{1,2}(\Omega_t;\mathbb{R}^3)), \\
&\rho\mathbf{u} \in L^\infty((0,T);L^{2\gamma/(\gamma-1)}(\Omega_t;\mathbb{R}^3)).
\end{aligned}
$$

- *The particle density $\eta = \eta(t,x)$ and the velocity $\mathbf{u} = \mathbf{u}(t,x)$ represents a weak solution of equation (3.3b). In particular, for all $\varphi \in \mathcal{D}([0,T)\times\overline{\Omega_t})$*

$$
\begin{aligned}
&\int_0^T \int_{\Omega_t} \eta\partial_t\varphi + \eta\mathbf{u}\cdot\nabla_x\varphi - \eta\nabla_x\Phi\cdot\nabla_x\varphi - \nabla_x\eta\cdot\nabla_x\varphi \,\mathrm{d}x\mathrm{d}t \\
&\qquad = \int_{\Omega_T} \eta(T,\cdot)\varphi(T,\cdot)\,\mathrm{d}x - \int_{\Omega_0} \eta_0\varphi(0,\cdot)\,\mathrm{d}x.
\end{aligned}
\tag{3.8}
$$

The particle density is required to have the following regularity

$$\eta \in L^2(0,T;W^{1,1}(\Omega_t)) \cap L^1(0,T;W^{1,3/2}(\Omega_t)),$$

$$\eta \geq 0 \text{ a.e. in } Q^f.$$

- *The momentum equation holds in distributional sense. In particular, for all* $\varphi \in \mathcal{D}([0,T) \times \overline{\Omega_t}; \mathbb{R}^3)$ *such that* $\varphi|_{\Gamma_t} = 0$*, the following integral identity holds*

$$\begin{aligned} \int_0^T \int_{\Omega_t} \Big(\rho\mathbf{u} \cdot \partial_t \varphi + \rho\mathbf{u} \otimes \mathbf{u} : \nabla_x \varphi + (p(\rho) + \eta)\mathrm{div}_x \varphi\Big) \,\mathrm{d}x\mathrm{d}t \\ = \int_0^T \int_{\Omega_t} \mathbb{S} : \nabla_x \varphi + (\eta + \beta\rho)\nabla_x \Phi \cdot \varphi \,\mathrm{d}x\mathrm{d}t \\ + \int_{\Omega_T} (\rho\mathbf{u} \cdot \varphi)(T, \cdot)\,\mathrm{d}x - \int_{\Omega_0} (\rho\mathbf{u})_0 \cdot \varphi(0, \cdot)\,\mathrm{d}x. \end{aligned} \tag{3.9}$$

- *Defining the total energy of the system by*

$$E(\rho, \mathbf{u}, \eta)(t) := \int_{\Omega_t} \left(\frac{1}{2}\rho|\mathbf{u}|^2 + \rho P(\rho) + \eta \log \eta + \eta\Phi\right) \mathrm{d}x(t),$$

the energy inequality

$$\begin{aligned} &\int_{\Omega_\tau} \left(\frac{1}{2}\rho|\mathbf{u}|^2 + P(\rho) + \eta \log \eta + \eta\Phi\right)(\tau, \cdot)\,\mathrm{d}x \\ &\quad + \int_0^\tau \int_{\Omega_t} \mathbb{S} : \nabla_x \mathbf{u} + |2\nabla_x \sqrt{\eta} + \sqrt{\eta}\nabla_x \Phi|^2 \,\mathrm{d}x\mathrm{d}t \\ &\leq \int_{\Omega_\tau} \left(\frac{1}{2}\rho_0|\mathbf{u}_0|^2 + P(\rho_0) + \eta_0 \log \eta_0 + \eta_0\Phi\right) \mathrm{d}x \\ &\quad + \int_{\Omega_\tau} (\rho\mathbf{u} \cdot \mathbf{V})(\tau, \cdot) - (\rho\mathbf{u})_0 \cdot \mathbf{V}(0, \cdot)\,\mathrm{d}x \\ &\quad + \int_0^\tau \int_{\Omega_t} \mathbb{S} : \nabla_x \mathbf{V} - \rho\mathbf{u} \cdot \partial_t \mathbf{V} - \varrho\mathbf{u} \otimes \mathbf{u} : \nabla_x \mathbf{V} \\ &\quad + (\eta + \beta\rho)\nabla_x \Phi \cdot \mathbf{V} - (p(\rho) + \eta)\,\mathrm{div}_x \mathbf{V}\,\mathrm{d}x\mathrm{d}t \\ &\quad - \beta \int_0^\tau \int_{\Omega_\tau} \rho\nabla_x \Phi \cdot \mathbf{u}\,\mathrm{d}x\mathrm{d}t \end{aligned} \tag{3.10}$$

holds for a.a. $t \in [0,T]$.

Remark. In anticipation of proving the weak-strong uniqueness in Chapter 4, the weak formulations in Definition 2 include integral terms at the fixed time T. By the results of Chapter 2, we in fact have that ρ, η, and $\rho\mathbf{u}$ are continuous in time with values in a negative Sobolev space. Therefore these integrals are well-defined.

We now state the main result of this chapter.

Theorem 3. *Let $\Omega_0 \subset\subset D \subset \mathbb{R}^3$ be a bounded domain with boundary of class $C^{2,\nu}, 0 < \nu \le 1$. Assume that the pressure is given by*

$$p(\rho) = \rho^\gamma, \quad \gamma > 3/2,$$

and the stress tensor is given by

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu(\nabla_x \mathbf{u} + (\nabla_x \mathbf{u})^T) + \lambda \mathrm{div}_x \mathbf{u} \mathbb{I}.$$

Let $\mathbf{V}$ be a given vector field belonging to $C^{2,\nu}([0,T] \times \overline{D}; \mathbb{R}^3)$, such that

$$\mathbf{V}\Big|_{\partial D} = 0.$$

Suppose the initial data $(\rho_0, \mathbf{m}_0, \eta_0)$ satisfy (3.6), and all initial data vanishes on $D \backslash \Omega_0$. Then there exists a weak solution $(\rho, \mathbf{u}, \eta)$ of problem (3.3) in the sense of Definition 2.

The rest of Chapter 3 is devoted to the proof of Theorem 3. The proof follows some of the main ideas of [36], and is based on the work of the candidate in [25].

3.3 Penalization scheme

The proof of Theorem 3 relies on a two-level approximation scheme. In the first level, we penalize the momentum equation by addition of the singular term

$-(1/\varepsilon)\chi(\mathbf{u}-\mathbf{V})$ (Brinkman's penalization), and in the second level we penalize the viscosities $\mu = \mu_\omega$ and $\lambda = \lambda_\omega$. Theorem 3 will be proven after taking the limits $\varepsilon \to 0$, followed by $\omega \to 0$.

Denote by $\chi = \chi(t, \mathbf{x})$ the characteristic function of Q^s, that is,

$$\chi(t,\mathbf{x}) = \begin{cases} 0, & \text{if } t \in (0,T),\ \mathbf{x} \in \Omega_t \\ 1, & \text{otherwise.} \end{cases}$$

The function χ represents a distributional solution of the transport equation

$$\begin{cases} \partial_t \chi + \mathbf{V} \cdot \nabla_x \chi = 0 \\ \chi(0,\cdot) = \mathbb{1}_D - \mathbb{1}_{\Omega_0}. \end{cases} \tag{3.11}$$

The system (3.3) is then replaced by the penalized problem

$$\partial_t \rho + \mathrm{div}_x(\rho \mathbf{u}) = 0, \tag{3.12a}$$

$$\partial_t \eta + \mathrm{div}_x(\eta(\mathbf{u} - \nabla_x \Phi)) - \Delta_x \eta = 0. \tag{3.12b}$$

$$\begin{aligned} &\partial_t(\rho\mathbf{u}) + \mathrm{div}_x(\rho\mathbf{u} \otimes \mathbf{u}) + \nabla_x(p(\rho) + \eta) \\ &\qquad = \mathrm{div}_x \mathbb{S}_\omega - (\eta + \beta\rho)\nabla_x \Phi - \frac{1}{\varepsilon}\chi(\mathbf{u} - \mathbf{V}) \end{aligned} \tag{3.12c}$$

considered in the cylinder $(0,T) \times D$. The penalized problem is supplemented with boundary conditions

$$\mathbf{u}\big|_{\partial D} = \mathbf{V}\big|_{\partial D} = 0, \tag{3.13}$$

$$(\nabla_x \eta + \eta \nabla_x \Phi) \cdot \nu\big|_{\partial D} = 0, \tag{3.14}$$

with ν denoting the outer normal vector to the boundary ∂D, and initial conditions

$(\rho_{0,\varepsilon}, \mathbf{m}_{0,\varepsilon}, \eta_{0,\varepsilon})$ such that

$$\begin{aligned}
&\rho_{0,\varepsilon} \to \rho_0 \ in \ L^\gamma(D), \quad \rho_0|_{\Omega_0} > 0, \quad \rho_0|_{D\setminus\Omega_0} = 0, \\
&\mathbf{m}_{0,\varepsilon} \to \mathbf{m}_0 \ in \ L^1(D;\mathbb{R}^3), \quad \mathbf{m}_0|_{D\setminus\Omega_0} = 0, \\
&\eta_{0,\varepsilon} \to \eta_0 \ in \ L^2(D), \quad \eta_0|_{\Omega_0} > 0, \quad \eta_0|_{D\setminus\Omega_0} = 0, \\
&\int_D \frac{|\mathbf{m}_{0,\varepsilon}|^2}{\rho_{0,\varepsilon}} \,\mathrm{d}x < c.
\end{aligned} \tag{3.15}$$

In order to eliminate extra stresses that appear due to the moving domain, the variable viscosity coefficients $\mu = \mu_\omega(t,\mathbf{x})$ and $\lambda = \lambda_\omega(t,\mathbf{x})$ are defined such that the modified viscosity vanishes in the solid domain Q^s as $\omega \to 0$. In particular,

$$\mu_\omega \in C_c^\infty\left([0,T]\times\mathbb{R}^3\right), \ \ 0 < \underline{\mu}_\omega \le \mu_\omega(t,\mathbf{x}) \le \mu \text{ in } [0,T]\times D,$$

$$\mu_\omega = \begin{cases} \mu = const > 0 & \text{in } Q^f \\ \\ \mu_\omega \to 0 & \text{a.e. in } ((0,T)\times D)\setminus Q^f, \end{cases}$$

such that $\underline{\mu}_\omega \to 0$ as $\omega \to 0$. The coefficient $\lambda = \lambda_\omega(t,x)$ is penalized in exactly the same way. The initial data is also assumed to satisfy the compatibility condition

$$(\rho\mathbf{u})_{0,\varepsilon} = \frac{|(\rho\mathbf{u})_{0,\varepsilon}|^2}{\rho_{0,\varepsilon}} = 0, \quad \text{whenever } \rho_{0,\varepsilon} = 0.$$

The weak formulation of the penalized problem reads as follows.

Definition 3 (Weak solutions of the penalized problem). *We say that $(\rho, \mathbf{u}, \eta)$ comprise a weak solution of the penalized NSS system (3.12), along with the boundary conditions (3.13) and (3.14), and the initial data (3.15) provided*

- *The density $\rho = \rho(t,x)$ and velocity $\mathbf{u} = \mathbf{u}(t,x)$ represent a weak renormalized solution of equation (1.1a) over $(0,T)\times D$, that is, for any test function $\varphi \in$*

$\mathcal{D}([0,T]\times\overline{D})$ *and any* b *such that*

$$b \in L^\infty \cap C[0,\infty),\ \ B(\rho) = B(1)\rho + \rho\int_1^\rho \frac{b(z)}{z^2}\,\mathrm{d}z,$$

the following integral identity holds:

$$\begin{aligned}\int_0^T\int_D \Big(B(\rho)\partial_t\varphi + B(\rho)\mathbf{u}\cdot\nabla_x\varphi - b(\rho)\mathrm{div}_x\mathbf{u}\varphi\Big)\,\mathrm{d}x\mathrm{d}t \\ = \int_D B(\rho)(T,\cdot)\varphi(T,\cdot)\,\mathrm{d}x - \int_D B(\rho_{0,\varepsilon})\varphi(0,\cdot)\,\mathrm{d}x.\end{aligned} \tag{3.16}$$

The density, velocity, and momentum are required to have the following regularity

$$\rho \in L^\infty(0,T;L^\gamma(D)),\quad \rho \geq 0 \text{ a.e. in } (0,T)\times D,$$

$$\mathbf{u} \in L^2(0,T;W_0^{1,2}(D;\mathbb{R}^3)),$$

$$\rho\mathbf{u} \in L^\infty((0,T);L^{2\gamma/(\gamma-1)}(D;\mathbb{R}^3)).$$

- *The particle density* $\eta = \eta(t,x)$ *and the velocity* $\mathbf{u} = \mathbf{u}(t,x)$ *represents a weak solution of equation (1.1c). In particular, for all* $\varphi \in \mathcal{D}([0,T]\times\overline{D})$

$$\begin{aligned}\int_0^T\int_D \eta\partial_t\varphi + \eta\mathbf{u}\cdot\nabla_x\varphi - \eta\nabla_x\Phi\cdot\nabla_x\varphi - \nabla_x\eta\cdot\nabla_x\varphi\,\mathrm{d}x\mathrm{d}t \\ = \int_D \eta(T,\cdot)\varphi(T,\cdot)\,\mathrm{d}x - \int_D \eta_0\varphi(0,\cdot)\,\mathrm{d}x.\end{aligned} \tag{3.17}$$

The particle density is required to have the following regularity

$$\eta \in L^2(0,T;W^{1,1}(D)) \cap L^1(0,T;W^{1,3/2}(D)),$$

$$\eta \geq 0 \text{ a.e. in } (0,T)\times D.$$

- *The momentum equation holds in distributional sense. In particular, for all*

$\varphi \in \mathcal{D}([0,T];\mathcal{D}(D;\mathbb{R}^3))$, *the following integral identity holds*

$$
\begin{aligned}
&\int_0^T \int_D \Big(\rho\mathbf{u}\cdot\partial_t\varphi + \rho\mathbf{u}\otimes\mathbf{u} : \nabla_x\varphi + (p(\rho)+\eta)\mathrm{div}_x\varphi\Big)\,\mathrm{d}x\mathrm{d}t \\
&= \int_0^T \int_D \mathbb{S} : \nabla_x\varphi + (\eta+\beta\rho)\nabla_x\Phi\cdot\varphi\,\mathrm{d}x\mathrm{d}t + \int_0^T\int_D \frac{\chi(\mathbf{u}-\mathbf{V})}{\varepsilon}\cdot\varphi\,\mathrm{d}x\mathrm{d}t \\
&+ \int_D (\rho\mathbf{u})(T,\cdot)\cdot\varphi(T,\cdot)\,\mathrm{d}x - \int_D (\rho\mathbf{u})_0\cdot\varphi(0,\cdot)\,\mathrm{d}x.
\end{aligned}
\tag{3.18}
$$

- *Defining the total energy of the system by*

$$
E(\rho,\mathbf{u},\eta)(t) := \int_D \left(\frac{1}{2}\rho|\mathbf{u}|^2 + \rho P(\rho) + \eta\log\eta + \eta\Phi\right)\mathrm{d}x(t),
$$

the energy inequality

$$
\begin{aligned}
E(\rho,\mathbf{u},\eta)(t) &+ \int_0^t\int_D \mathbb{S}:\nabla_x\mathbf{u} + |2\nabla_x\sqrt{\eta} + \sqrt{\eta}\nabla_x\Phi|^2\,\mathrm{d}x\mathrm{d}s \\
&\le E(\rho,\mathbf{u},\eta)(0) - \int_0^t\int_D \frac{\chi}{\varepsilon}(\mathbf{u}-\mathbf{V})\cdot\mathbf{u}\,\mathrm{d}x\mathrm{d}s \\
&\quad - \beta\int_0^\tau\int_D \rho\nabla_x\Phi\cdot\mathbf{u}\,\mathrm{d}x\mathrm{d}t.
\end{aligned}
\tag{3.19}
$$

holds for a.a. $t\in[0,T]$.

For any fixed ε, weak solutions in the sense of Definition 3 exist by the same methods of Chapter 2. Estimates from the energy inequality (3.19) are uniform after applying a Cauchy and Poincaré inequality to the extra term on the right hand side of (3.19) and absorbing into the left hand side. Passing to the limit in the momentum equation in Definition 3 causes no new difficulties as the penalty term is linear in $\mathbf{u}$.

3.4 Uniform estimates

We eventually will take the limits $\varepsilon \to 0$, and $\omega \to 0$. We therefore denote a solution at the $\varepsilon-$level by $(\rho_\varepsilon, \mathbf{u}_\varepsilon, \eta_\varepsilon)$, and at the ω-level by $(\rho_\omega, \mathbf{u}_\omega, \eta_\omega)$.

To obtain estimates uniform in ε, we first derive a modified energy inequality. Choosing as a test function $\varphi = \psi_n(t)\mathbf{V}$, $\psi_n \in C_c^\infty[0,T)$, $\psi_n \to \mathbb{1}_{[0,\tau)}$ in (3.18) and adding to the inequality (3.19), we find that

$$\begin{aligned}
&\int_D \left(\frac{1}{2}\rho_\varepsilon|\mathbf{u}_\varepsilon|^2 + \frac{a}{\gamma-1}\rho_\varepsilon^\gamma + \eta_\varepsilon \log \eta_\varepsilon + \eta_\varepsilon \Phi\right)(\tau,\cdot)\,\mathrm{d}x \\
&\quad + \int_0^\tau\!\!\int_D \left(\mu_\omega|\nabla_x \mathbf{u}_\varepsilon|^2 + \lambda_\omega|\mathrm{div}_x \mathbf{u}_\varepsilon|^2 + |2\nabla_x\sqrt{\eta_\varepsilon} + \sqrt{\eta_\varepsilon}\nabla_x\Phi|^2\right)\,\mathrm{d}x\mathrm{d}t \\
&\quad + \frac{1}{\varepsilon}\int_0^\tau \int_D \chi|\mathbf{u}_\varepsilon - \mathbf{V}|^2\,\mathrm{d}x\mathrm{d}t \\
&\le \int_D \left(\frac{1}{2}\frac{|(\rho\mathbf{u})_{0,\varepsilon}|^2}{\varrho_{0,\varepsilon}} + \frac{a}{\gamma-1}\varrho_{0,\varepsilon}^\gamma + \eta_{0,\varepsilon}\log\eta_{0,\varepsilon} + \eta_{0,\varepsilon}\Phi\right)\,\mathrm{d}x \\
&\quad + \int_D (\rho_\varepsilon \mathbf{u}_\varepsilon \cdot \mathbf{V})(\tau,\cdot) - (\rho\mathbf{u})_{0,\varepsilon}\cdot\mathbf{V}(0,\cdot)\,\mathrm{d}x \\
&\quad + \int_0^\tau \int_D \mathbb{S}_\varepsilon : \nabla_x\mathbf{V} - \rho_\varepsilon\mathbf{u}_\varepsilon\cdot\partial_t\mathbf{V} - \varrho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon : \nabla_x\mathbf{V} \\
&\quad - (\eta_\varepsilon + \beta\rho_\varepsilon)\nabla_x\Phi\cdot\mathbf{V} - (p(\rho_\varepsilon)+\eta_\varepsilon)\,\mathrm{div}_x\mathbf{V}\,\mathrm{d}x\mathrm{d}t \\
&\quad - \beta\int_0^\tau \int_D \rho_\varepsilon\nabla_x\Phi\cdot\mathbf{u}_\varepsilon\,\mathrm{d}x\mathrm{d}t
\end{aligned} \tag{3.20}$$

for a.a. $\tau \in (0,T)$. This yields uniform bounds on $(\rho_\varepsilon, \mathbf{u}_\varepsilon, \eta_\varepsilon)$ independent of $\varepsilon \to 0$ provided $\mathbf{V}$ is sufficiently smooth, and using the Cauchy and Grönwall inequalities.

In accordance with the boundary conditions (3.13) and (3.14), the total fluid and particle mass

$$M_{\rho,\varepsilon} = \int_D \rho_\varepsilon(t,\cdot)\,\mathrm{d}x = \int_D \rho_{0,\varepsilon}\,\mathrm{d}x \tag{3.21}$$

$$M_{\eta,\varepsilon} = \int_D \eta_\varepsilon(t,\cdot)\,\mathrm{d}x = \int_D \eta_{0,\varepsilon}\,\mathrm{d}x \tag{3.22}$$

are constants of motion (see [21], lemma 3.13). The following bounds, uniform in ε, ω, are evident from a quick inspection of (3.20):

$$\sqrt{\rho_\varepsilon}\mathbf{u}_\varepsilon \in L^\infty(0,T;L^2(D;\mathbb{R}^3)) \tag{3.23}$$

$$\rho_\varepsilon \in L^\infty(0,T;L^\gamma(D)) \tag{3.24}$$

$$\nabla_x \mathbf{u}_\varepsilon \in L^2(0,T;L^2(D;\mathbb{R}^3\times\mathbb{R}^3)) \tag{3.25}$$

$$\mathrm{div}_x \mathbf{u}_\varepsilon \in L^2(0,T;L^2(D)) \tag{3.26}$$

$$\nabla_x \sqrt{\eta_\varepsilon} \in L^2(0,T;L^2(D;\mathbb{R}^3)) \tag{3.27}$$

In addition,

$$\int_0^\tau \int_D \chi |\mathbf{u}_\varepsilon - \mathbf{V}|^2\,\mathrm{d}x\mathrm{d}t = \int_{Q^s} |\mathbf{u}_\varepsilon - \mathbf{V}|^2\,\mathrm{d}x\mathrm{d}t \le \varepsilon c, \tag{3.28}$$

for a.a. $\tau \in (0,T)$ with c independent of ε, ω, where we used the definition of $\chi(t,x)$.

Using the embedding of $W^{1,2}(D)$ in $L^6(D)$ (since $D \subset \mathbb{R}^3$) on the last bound listed above, it is clear that $\eta_\varepsilon \in L^1(0,T;L^3(D))$. This, and mass conservation implies

$$\eta_\varepsilon \in L^1(0,T;L^3(D)) \cap L^\infty(0,T;L^1(D)). \tag{3.29}$$

Using this result, and that

$$2\nabla_x\sqrt{\eta} = \frac{\nabla_x \eta}{\sqrt{\eta}},$$

it is also clear that

$$\eta_\varepsilon \in L^1(0,T;W^{1,\frac{3}{2}}(D)) \cap L^2(0,T;W^{1,1}(D)). \tag{3.30}$$

By Poincaré's inequality and (3.25), we get that

$$\mathbf{u}_\varepsilon \in L^2(0,T;W_0^{1,2}(D;\mathbb{R}^3)). \tag{3.31}$$

3.5 Pressure estimates and pointwise convergence of the fluid density

The detailed analysis in [36] yields the estimates needed to deal with the nonlinear pressure, $p(\rho) = a\rho^\gamma$, obtain pointwise convergence of the fluid density ρ, and pass to the limit in (3.16), (3.18). In particular,

$$\int_K p(\rho_\varepsilon)\rho_\varepsilon^\nu \,\mathrm{d}x\mathrm{d}t \le c(K) \text{ for any compact } K \subset Q^f, \tag{3.32}$$

and these estimates can be extended up to the boundary, and

$$\rho_\varepsilon \to \rho_\omega \text{ in } L^q((0,T)\times D) \text{ for any } 1 \le q < \gamma.$$

Though we omit the proof, let us simply remark that the appearance of extra terms involving the particle density η pose no additional difficulty in obtaining these estimates.

Observe that the penalization term, singular in ε over Q^s, doesn't allow any uniform pressure estimates in the solid portion of the domain. The estimates are therefore local in Q^f.

3.6 The limit $\varepsilon \to 0$

Combining (3.24), (3.31) with equation (3.16) we may infer that

$$\rho_{\varepsilon,\omega} \to \rho_\omega \quad \text{in } C([0,T];L^\gamma_{weak}(D)), \tag{3.33a}$$

$$\mathbf{u}_{\varepsilon,\omega} \rightharpoonup \mathbf{u}_\omega \quad \text{in } L^2(0,T;W_0^{1,2}(D;\mathbb{R}^3)), \tag{3.33b}$$

passing to subsequences if necessary. Moreover as a consequence of (3.28),

$$\mathbf{u}_\omega = \mathbf{V} \quad \text{a.e. in } Q^s, \tag{3.34}$$

again after passing to a subsequence. From (3.29) and interpolation we get that

$$\eta_{\varepsilon,\omega} \to \eta_\omega \quad \text{in } L^2(0,T;L^{\frac{3}{2}}(D)). \tag{3.35}$$

To deal with the $\nabla_x \eta_{\varepsilon,\omega}$ term in (3.17), we can interpolate in (3.30) and conclude that

$$\nabla_x \eta_{\varepsilon,\omega} \rightharpoonup \nabla_x \eta_\omega \ \text{ in } L^p(0,T;L^q(D)), \tag{3.36}$$

for some $p, q > 1$.

3.7 Convergence in the set Q^s

In this section we show that the densities in the 'solid' domain Q^s vanish in the limit. That

$$\rho(t,x) = 0 \quad \text{for a.e. } (t,\mathbf{x}) \in Q^s$$

holds has been worked out in [36]. The proof relies on regularizing the equation of continuity (3.12a) and employing the commutator lemma of DiPerna and Lions [24]. It remains to show that

$$\eta(t,x) = 0 \quad \text{for a.e. } (t,\mathbf{x}) \in Q^s.$$

Before proving the following lemmas, first we set some notation. Recall that the cutoff function $\chi(t,\mathbf{x})$ satisfies the transport equation (3.11). In anticipation of using a suitable (smooth) test function, consider instead the unique function $\overline{\chi} \in C^\infty(\mathbb{R}^3)$ solving

$$\partial_t \overline{\chi} + \mathbf{V} \cdot \nabla_x \overline{\chi} = 0 \quad t > 0,\, \mathbf{x} \in \mathbb{R}^3,$$

with the initial data satisfying

$$C^\infty(\mathbb{R}^3) \ni \overline{\chi}(0,\cdot) = \begin{cases} > 0 & x \in D \backslash \Omega_0 \\ \\ < 0 & x \in \Omega_0 \cup (\mathbb{R}^3 \backslash \overline{D}) \end{cases}, \quad \nabla_x \overline{\chi}_0 \neq 0 \quad \text{on } \partial\Omega_0.$$

We define the level-set test function,

$$\varphi_\xi = \begin{cases} 1 & \overline{\chi} \geq \xi \\ \\ \dfrac{\overline{\chi}}{\xi} & 0 \leq \overline{\chi} < \xi \\ \\ 0 & \overline{\chi} < 0 \end{cases} = \min\left\{\frac{\overline{\chi}}{\xi}, 1\right\}^+, \tag{3.37}$$

supported on $D \backslash \Omega_\tau$, see [39], [60].

Lemma 6. *Let* $\eta_{\varepsilon,\omega} \in L^2(0,T; W^{1,1}(D)) \cap L^1(0,T; W^{1,\frac{3}{2}}(D))$, $\eta_{\varepsilon,\omega} \geq 0$, *and* $\mathbf{u}_{\varepsilon,\omega} \in L^2(0,T; W^{1,2}(D;\mathbb{R}^3))$ *be a weak solution of* (3.17)*, that is,*

$$\begin{aligned} \int_0^T \int_D \eta_{\varepsilon,\omega} \partial_t \varphi + \eta_{\varepsilon,\omega} \mathbf{u}_{\varepsilon,\omega} \cdot \nabla_x \varphi - \eta_{\varepsilon,\omega} \nabla_x \Phi \cdot \nabla_x \varphi - \nabla_x \eta_{\varepsilon,\omega} \cdot \nabla_x \varphi \,\mathrm{d}x\mathrm{d}t \\ = - \int_D \eta_{0,\varepsilon,\omega} \varphi(0,\cdot)\,\mathrm{d}x, \end{aligned} \tag{3.38}$$

holds for all $\varphi \in \mathcal{D}([0,T) \times \overline{D})$ *and any* $T > 0$*. Let the initial data satisfy*

$$\eta_0 \in L^2(D) \cap L^1_+(D), \quad \eta_0\big|_{D \backslash \Omega_0} = 0.$$

Then for $\xi > 0$ *and* $\overline{\chi}$ *defined as above, it holds that*

$$\lim_{\xi \to 0} \frac{1}{\xi} \int_0^\tau \int_{\{0 \leq \bar{\chi} < \xi\}} (\eta \nabla_x \Phi + \nabla_x \eta) \cdot \nabla_x \bar{\chi} \,\mathrm{d}x\mathrm{d}t = 0, \tag{3.39}$$

for any $\tau > 0$.

Proof. Plugging (3.37) into (3.38) and rearranging we get that

$$\begin{aligned}\frac{1}{\xi}\int_0^\tau\int_{\{0\le\overline{\chi}<\xi\}}(\eta_{\varepsilon,\omega}\nabla_x\Phi+\nabla_x\eta_{\varepsilon,\omega})\cdot\nabla_x\overline{\chi}\,\mathrm{d}x\mathrm{d}t= \\ \frac{1}{\xi}\int_0^\tau\int_{\{0\le\overline{\chi}<\xi\}}\eta_{\varepsilon,\omega}(\mathbf{u}_{\varepsilon,\omega}-\mathbf{V})\cdot\nabla_x\overline{\chi}\,\mathrm{d}x\mathrm{d}t+\int_D\eta_{0,\varepsilon,\omega}\varphi_\xi(0,\cdot)\,\mathrm{d}x.\end{aligned} \tag{3.40}$$

Since we can pass $\varepsilon,\omega\to 0$ on the left side in (3.40), it suffices to show that right side vanishes as we take $\varepsilon,\omega\to 0$ and $\xi\to 0$ successively. First,

$$\lim_{\varepsilon,\omega\to 0}\int_D\eta_{0,\varepsilon,\omega}\varphi_\xi(0,\cdot)\mathrm{d}x=\int_{\Omega_0}\eta_0\varphi_\xi(0,\cdot)\,\mathrm{d}x=0,$$

since on Ω_0, we have $\overline{\chi}(0,\cdot)<0$ and so $\varphi_\xi(0,\cdot)=0$. Now,

$$\lim_{\varepsilon,\omega\to 0}\frac{1}{\xi}\int_0^\tau\int_{\{0\le\overline{\chi}<\xi\}}\eta_{\varepsilon,\omega}(\mathbf{u}_{\varepsilon,\omega}-\mathbf{V})\cdot\nabla_x\overline{\chi}\,\mathrm{d}x\mathrm{d}t=\frac{1}{\xi}\int_0^\tau\int_{\{0\le\overline{\chi}<\xi\}}\eta(\mathbf{u}-\mathbf{V})\cdot\nabla_x\overline{\chi}\,\mathrm{d}x\mathrm{d}t=0,$$

since $\mathbf{u}=\mathbf{V}$ a.e. in $D\backslash\Omega_0$, i.e. where $\overline{\chi}\ge 0$, using (3.34). Letting $\xi\to 0$ concludes the proof of the lemma. □

Lemma 7. *Under the same conditions as lemma 6, the following holds,*

$$\eta(\tau,\cdot)|_{D\backslash\Omega_\tau}=0\quad \textit{for a.a. } \tau\in[0,T].$$

Proof. First note that by choosing a test function having the form

$$\varphi_n=\psi_n(t)\varphi(t,x),\varphi\in C_c^\infty([0,T)\times\bar{D}),\psi_n\to\mathbb{1}_{[0,\tau)}\text{ as }n\to\infty,$$

and $\psi_n\in C^\infty[0,T)$, we can rewrite the weak form (3.38) as

$$\begin{aligned}\int_D\eta_{\varepsilon,\omega}(\tau,\cdot)\varphi(\tau,\cdot)-\eta_{0,\varepsilon,\omega}\varphi(0,\cdot)\,\mathrm{d}x=\int_0^\tau\int_D\eta_{\varepsilon,\omega}(\partial_t\varphi+\mathbf{u}_{\varepsilon,\omega}\cdot\nabla_x\varphi) \\ -(\eta_{\varepsilon,\omega}\nabla_x\Phi+\nabla_x\eta_{\varepsilon,\omega})\cdot\nabla_x\varphi\,\mathrm{d}x\mathrm{d}t,\end{aligned} \tag{3.41}$$

for any $\varphi\in C_c^\infty([0,T)\times\bar{D})$. It suffices to establish that

$$\int_{D\backslash\Omega_\tau}\eta(\tau,\cdot)\,\mathrm{d}x=0,\quad a.a\ \tau\in(0,T).$$

Inserting φ_ξ into (3.41), using the initial conditions, and letting $\varepsilon, \omega \to 0$ yields,

$$\int_D \eta(\tau,\cdot)\varphi_\xi(\tau,\cdot)\,\mathrm{d}x = \frac{1}{\xi}\int_0^\tau \int_{\{0\le\overline{\chi}<\xi\}} \eta(\mathbf{u}-\mathbf{V})\cdot\nabla_x\overline{\chi} - (\eta\nabla_x\Phi + \nabla_x\eta)\cdot\nabla_x\overline{\chi}\,\mathrm{d}x\,\mathrm{d}t. \tag{3.42}$$

Since $\varphi_\xi(\tau,\cdot) \to \mathbb{1}_{D\setminus\Omega_\tau}$ as $\xi \to 0$ in any $L^p(D), p < \infty$, and $\eta \in L^2(0,T;L^{3/2}(D))$, the left-hand side of (3.42) converges to

$$\int_{D\setminus\Omega_\tau} \eta(\tau,\cdot)\,\mathrm{d}x,$$

as $\xi \to 0$. Finally, using lemma 6 and that $\mathbf{u} = \mathbf{V}$ for any $\xi > 0$, it is clear the right hand side of (3.42) vanishes as $\xi \to 0$. □

3.8 The limit $\omega \to 0$

Performing the limit $\varepsilon \to 0$, we arrive at the weak formulation of the momentum satisfied, except for the following term

$$\int_0^\infty \int_D (\mu_\omega\nabla_x\mathbf{u}_\omega + \lambda_\omega\mathrm{div}_x\mathbf{u}_\omega\mathbb{I}) : \nabla_x\varphi\,\mathrm{d}x\mathrm{d}t. \tag{3.43}$$

Using that the viscosity penalization is assumed to vanish on $((0,T)\times D)\setminus Q^f$ and using that $\mathbf{u}_\omega = \mathbf{V}$ here, we conclude that

$$\int_0^T \int_{D\setminus\Omega_t} (\mu_\omega\nabla_x\mathbf{u}_\omega + \lambda_\omega\mathrm{div}_x\mathbf{u}_\omega\mathbb{I}) : \nabla_x\varphi\,\mathrm{d}x\mathrm{d}t \to 0 \text{ as } \omega \to 0.$$

We can now pass all terms in the weak formulation as $\omega \to 0$, using the same estimates in the previous sections.

Remark. In fact when letting $\varepsilon, \omega \to 0$ in the momentum equation (3.18), the penalization term remains as a weak limit,

$$\frac{\mathbf{u}_{\varepsilon,\omega} - \mathbf{V}}{\varepsilon} \rightharpoonup h \quad \text{in } L^1(Q^s),$$

where h is the weak limit. This term, which appears artificially in the solid domain, is then removed in the weak formulation by proper choice of test functions. For some remarks on this term see [6].

In order to obtain the limiting energy inequality, we first state the following lemma. See Corollary 2.2 in [33] for the proof.

Lemma 8. *Let $O \subset \mathbb{R}^m$ be a bounded measurable set, and $\{\mathbf{v}_n\}_{n=1}^{\infty}$ a sequence of functions such that*

$$\mathbf{v}_n \to \mathbf{v} \quad \textit{weakly in} \quad L^1(O;\mathbb{R}^n).$$

Let $\Phi : \mathbb{R}^n \to (\infty, \infty]$ be a convex lower semi-continuous function. Then $\Phi(\mathbf{v}) : O \to \mathbb{R}$ is integrable, and

$$\int_O \Phi(\mathbf{v})\,\mathrm{d}\mathbf{y} \leq \liminf_{n\to\infty} \int_O \Phi(\mathbf{v}_n)\,\mathrm{d}\mathbf{y}.$$

Using this lemma, the previously derived estimates, and lemma 7, it is now easy to pass $\epsilon, \omega \to 0$ in (3.20) to derive the energy inequality (3.10).

Chapter 4: Relative entropy and weak-strong uniqueness for NSS: moving domains

In Chapters 2 and 3 we discussed the existence of solutions for the NSS system in a weak sense. One of the many outstanding problems surrounding the Navier-Stokes equations, at least in three dimensions, concerns the question of uniqueness. Though a uniqueness result is in general not available, we can instead prove a partial result concerning the uniqueness of strong solutions in the class of weak solutions. This is known as a weak-strong uniqueness, and ensures that the class of weak solutions is somehow not too big. In this chapter we establish a relative entropy inequality for the compressible NSS system on moving domains. The relative entropy is then used to deduce the weak-strong uniqueness result.

4.1 Definitions of relative entropies

We define the energy contributions of the fluid and particle by

$$H_f(\rho) := \frac{a}{\gamma - 1}\rho^\gamma,$$

and

$$H_p(\eta) := \eta \log \eta$$

respectively.

Suppose that $(\rho, \eta, \mathbf{u})$ is a weak solution of the NSS system over moving domains in the sense of Definition 2, and suppose that $(r, s, \mathbf{U})$ are arbitrary smooth functions, with regularity to be made precise later. The relative entropy contribution of the fluid potential energy is defined by

$$\mathcal{E}_f(\rho|r) := H_f(\rho) - H_f'(r)(\rho - r) - H_f(r), \tag{4.1}$$

and the relative entropy contribution of the particles is defined by

$$\mathcal{E}_p(\eta|s) := H_p(\eta) - H_p'(s)(\eta - s) - H_p(s). \tag{4.2}$$

The total relative entropy $\mathcal{E}\left(\rho, \eta, \mathbf{u} \middle| r, s, \mathbf{U}\right)$ is defined by

$$\mathcal{E}\left(\rho, \eta, \mathbf{u} \middle| r, s, \mathbf{U}\right) = \int_{\Omega_t} \frac{1}{2}\rho|\mathbf{u} - \mathbf{U}|^2 + \mathcal{E}_f(\rho|r) + \mathcal{E}_p(\eta|s)\,\mathrm{d}x. \tag{4.3}$$

Having defined, but not yet motivated the definitions of relative entropy, we can now state the theorem to be proved in Section 4.2.

Theorem 4. *Let $(\rho, \eta, \mathbf{u})$ be a weak solution of the NSS system (3.3) in the sense of Definition 2. Let $(r, s, \mathbf{U})$ be a triple of smooth functions such that $\mathbf{U} \in \mathcal{D}([0,T] \times \Omega_t; \mathbb{R}^3)$, and $r, s \in \mathcal{D}([0,T] \times \overline{\Omega_t})$. Both $\mathbf{u}$ and $\mathbf{U}$ are required to agree with $\mathbf{V}$ on the boundary Γ_τ, for any $\tau \geq 0$. Then the relative entropy (4.3) satisfies the inequality*

$$\begin{aligned}
&\mathcal{E}\left(\rho, \eta, \mathbf{u} \middle| r, s, \mathbf{U}\right) + \int_0^\tau \int_{\Omega_t} (\mathbb{S}(\nabla_x \mathbf{u}) - \mathbb{S}(\nabla_x \mathbf{U})) : (\nabla_x \mathbf{u} - \nabla_x \mathbf{U})\,\mathrm{d}x\mathrm{d}t \\
&\leq \mathcal{E}\left(\rho_0, \eta_0, \mathbf{u}_0 \middle| r_0, s_0, \mathbf{U}_0\right) + \mathcal{R}(\rho, r, \eta, s, \mathbf{u}, \mathbf{U}),
\end{aligned} \tag{4.4}$$

for a.a. $\tau > 0$. The remainder term $\mathcal{R}$ appearing in (4.4) is defined as

$$
\begin{aligned}
&\mathcal{R}(\rho, r, \eta, s, \mathbf{u}, \mathbf{U}) \\
&= -\int_0^\tau \int_{\Omega_t} (\eta + \beta\rho)\nabla_x \Phi \cdot (\mathbf{u} - \mathbf{U}) \,\mathrm{d}x\mathrm{d}t \\
&- \int_0^\tau \int_{\Omega_t} (\rho - r)\partial_t H_f'(r) + (\rho\mathbf{u} - r\mathbf{U}) \cdot \nabla_x H_f'(r) + (p(\rho) - p(r))\mathrm{div}_x \mathbf{U} \,\mathrm{d}x\mathrm{d}t \\
&- \int_0^\tau \int_{\Omega_t} (\eta - s)\partial_t H_p'(s) + (\eta\mathbf{u} - s\mathbf{U}) \cdot \nabla_x H_p'(s) + (\eta - s)\mathrm{div}_x \mathbf{U} \,\mathrm{d}x\mathrm{d}t \\
&+ \int_0^\tau \int_{\Omega_t} \mathbb{S}(\nabla_x \mathbf{U}) : \nabla_x(\mathbf{U} - \mathbf{u}) + \rho(\mathbf{U} - \mathbf{u}) \cdot \partial_t \mathbf{U} + \rho\mathbf{u} \otimes (\mathbf{U} - \mathbf{u}) : \nabla_x \mathbf{U} \,\mathrm{d}x\mathrm{d}t \\
&+ \int_0^\tau \int_{\Omega_t} (\eta\nabla_x \Phi + \nabla_x \eta) \cdot \nabla_x (H_p'(s) - H_p'(\eta)) \,\mathrm{d}x\mathrm{d}t.
\end{aligned}
\tag{4.5}
$$

The proof of Theorem 4 will be given in Section 4.2. Let us next motivate the definitions of the relative entropy. Following the notation of [9] (see also [20]), the relative entropy is defined by

$$\mathcal{E}(V|\overline{V}) = \mathcal{E}(V) - \mathcal{E}(\overline{V}) - \nabla\mathcal{E}(\overline{V}) \cdot (V - \overline{V}), \tag{4.6}$$

where ∇ denotes the gradient $(\partial_\rho, \partial_{\mathbf{u}}, \partial_\eta)$, $\mathcal{E}$ is functional

$$\mathcal{E}(V) = \frac{1}{2}\rho|\mathbf{u}|^2 + \frac{a}{\gamma - 1}\rho^\gamma + \eta \log \eta, \tag{4.7}$$

and $V, \overline{V}$ are the vectors

$$V = (\rho, \mathbf{u}, \eta), \quad \overline{V} = (r, \mathbf{U}, s).$$

Therefore, the relative entropy essentially characterizes the quadratic correction to the Taylor expansion of $\mathcal{E}(V)$ about the state $\overline{V}$. Using the inequalities $1 - 1/x \leq \log x$ and $1 + \gamma(x - 1) \leq x^\gamma$, it is easy to check that $\mathcal{E}(V)$ is strictly convex and vanishes precisely when $V = \overline{V}$. This quantity therefore provides a measure of the 'distance' between two solutions. The definition (4.3) is then a straightforward consequence of substituting (4.7) into (4.6).

4.2 Relative entropy inequality

In this section we prove Theorem 4. The derivation of the relative entropy inequality consists of choosing appropriate test functions in the weak formulations of Definition 2 and combining with the energy inequality (3.10). We proceed in several steps.

Using $\frac{1}{2}|\mathbf{U}|^2$ as a test function in the weak formulation (2.64) yields

$$\begin{aligned} -\int_{\Omega_\tau} \frac{1}{2}\rho|\mathbf{U}|^2(\tau,\cdot)\,\mathrm{d}x = &-\int_{\Omega_0} \frac{1}{2}\rho_0|\mathbf{U}(0,\cdot)|^2\,\mathrm{d}x \\ &- \int_0^\tau \int_{\Omega_t} \rho\mathbf{U}\cdot\partial_t\mathbf{U} + \rho\mathbf{u}\otimes\mathbf{U} : \nabla_x\mathbf{U}\,\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{4.8}$$

Next, use $\mathbf{U}-\mathbf{V}$ as a test function in the momentum equation (3.9), which is valid since $\mathbf{U}=\mathbf{V}$ on Γ_t,

$$\begin{aligned} -\int_{\Omega_\tau} \rho\mathbf{u}\cdot\mathbf{U}(\tau,\cdot)\,\mathrm{d}x = &-\int_{\Omega_0} \rho_0\mathbf{u}_0\cdot\mathbf{U}(0,\cdot)\,\mathrm{d}x - \int_0^\tau\int_{\Omega_t} \rho\mathbf{u}\cdot\partial_t\mathbf{U}\,\mathrm{d}x\mathrm{d}t \\ &- \int_0^\tau\int_{\Omega_t} \rho\mathbf{u}\otimes\mathbf{u} : \nabla_x\mathbf{U} + (p(\rho)+\eta)\mathrm{div}_x\mathbf{U} - \mathbb{S} : \nabla_x\mathbf{U}\,\mathrm{d}x\mathrm{d}t \\ &+ \int_0^\tau\int_{\Omega_t} (\eta+\beta\rho)\nabla_x\Phi\cdot\mathbf{U}\,\mathrm{d}x\mathrm{d}t - \int_{\Omega_\tau} \rho\mathbf{u}\cdot\mathbf{V}(\tau,\cdot)\,\mathrm{d}x \\ &+ \int_{\Omega_0} \rho_0\mathbf{u}_0\cdot\mathbf{V}(0,\cdot)\,\mathrm{d}x + \int_0^\tau\int_{\Omega_t} \rho\mathbf{u}\cdot\partial_t\mathbf{V}\,\mathrm{d}x\mathrm{d}t \\ &+ \int_0^\tau\int_{\Omega_t} \rho\mathbf{u}\otimes\mathbf{u} : \nabla_x\mathbf{V} + (p(\rho)+\eta)\mathrm{div}_x\mathbf{V}\,\mathrm{d}x\mathrm{d}t \\ &- \int_0^\tau\int_{\Omega_t} \mathbb{S} : \nabla_x\mathbf{V} + (\eta+\beta\rho)\nabla_x\Phi\cdot\mathbf{V}\,\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{4.9}$$

Using that $H_f'(r)r - H_f(r) = p(r)$ for any time $\tau \geq 0$, and the equality

$$\int_{\Omega_\tau} p(r)(\tau,\cdot)\,\mathrm{d}x = \int_{\Omega_0} p(r)(0,\cdot)\,\mathrm{d}x + \int_0^\tau \int_{\Omega_t} \partial_t p(r) + \mathrm{div}_x(p(r)\mathbf{U})\,\mathrm{d}x\mathrm{d}t$$
$$= \int_0^\tau \int_{\Omega_t} r\partial_t H_f'(r) + p(r)\mathrm{div}_x\mathbf{U} + r\mathbf{U}\cdot\nabla_x H_f'(r)\,\mathrm{d}x\mathrm{d}t,$$

we deduce that

$$\int_{\Omega_t} [H_f'(r)r - H_f(r)](\tau,\cdot)\,\mathrm{d}x = \int_{\Omega_0} [H_f'(r)r - H_f(r)](0,\cdot)\,\mathrm{d}x\mathrm{d}t + \int_0^\tau \int_{\Omega_t} r\partial_t H_f'(r) + p(r)\mathrm{div}_x\mathbf{U} + r\mathbf{U}\cdot\nabla_x H_f'(r). \tag{4.10}$$

By a similar computation, using instead the equality $H_p'(s)s - H_p(s) = s$ (valid for any $\tau \geq 0$) we get that

$$\int_{\Omega_t} [H_p'(s)s - H_p'(s)](\tau,\cdot)\,\mathrm{d}x = \int_{\Omega_0} [H_p'(s)s - H_p'(s)](0,\cdot)\,\mathrm{d}x\mathrm{d}t + \int_0^\tau \int_{\Omega_t} s\partial_t H_p'(s) + s\mathrm{div}_x\mathbf{U} + s\mathbf{U}\cdot\nabla_x H_p'(s). \tag{4.11}$$

Using $H_p'(s)$ as a test function in equation (3.8) and combining with (4.11) yields

$$\begin{aligned} &-\int_{\Omega_\tau} [H_p'(s)(\eta - s) + H_p(s)](\tau,\cdot)\,\mathrm{d}x \\ &\quad= -\int_{\Omega_0} H_p'(s_0)(\eta_0 - s_0) + H_p(s_0)\,\mathrm{d}x \\ &\qquad - \int_0^\tau \int_{\Omega_t} (\eta - s)\partial_t H_p'(s) + (\eta\mathbf{u} - s\mathbf{U})\cdot\nabla_x H_p'(s)\,\mathrm{d}x\mathrm{d}t \\ &\qquad - \int_0^\tau \int_{\Omega_t} (\eta\nabla_x\Phi + \nabla_x\eta)\cdot\nabla_x H_p'(s) - s\mathrm{div}_x\mathbf{U}\,\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{4.12}$$

Next, test equation (3.8) against Φ to get

$$-\int_{\Omega_\tau} \eta(\tau,\cdot)\Phi\,\mathrm{d}x = -\int_{\Omega_0} \eta_0\Phi\,\mathrm{d}x - \int_0^\tau \int_{\Omega_t} \eta\mathbf{u}\cdot\nabla_x\Phi - (\eta\nabla_x\Phi + \nabla_x\eta)\cdot\nabla_x\Phi\,\mathrm{d}x\mathrm{d}t \tag{4.13}$$

Let us now put together the energy inequality (3.10), and the equalities (4.8)-(4.13), to deduce

$$\begin{aligned}
&\mathcal{E}\left(\rho,\eta,\mathbf{u}\middle|r,s,\mathbf{U}\right)+\int_0^\tau\int_{\Omega_t}(\mathbb{S}(\nabla_x\mathbf{u})-\mathbb{S}(\nabla_x\mathbf{U})):(\nabla_x\mathbf{u}-\nabla_x\mathbf{U})\,\mathrm{d}x\mathrm{d}t\\
&\quad+\int_0^\tau\int_{\Omega_t}|2\nabla_x\sqrt{\eta}+\eta\nabla_x\Phi|^2\,\mathrm{d}x\mathrm{d}t\\
&\le\mathcal{E}\left(\rho_0,\eta_0,\mathbf{u}_0\middle|r_0,s_0,\mathbf{U}_0\right)-\int_0^\tau\int_{\Omega_t}(\eta\nabla_x\Phi+\nabla_x\eta)\cdot\nabla_xH_p'(s)\,\mathrm{d}x\mathrm{d}t\\
&\quad+\int_0^\tau\int_{\Omega_t}\eta|\nabla_x\Phi|^2+\nabla_x\eta\cdot\nabla_x\Phi\,\mathrm{d}x\mathrm{d}t+\widetilde{\mathcal{R}}(\rho,r,\eta,s,\mathbf{u},\mathbf{U}).
\end{aligned}\tag{4.14}$$

The remainder term $\widetilde{\mathcal{R}}$ appearing in (4.14) is defined by

$$\begin{aligned}
&\widetilde{\mathcal{R}}(\rho,r,\eta,s,\mathbf{u},\mathbf{U})\\
&=-\int_0^\tau\int_{\Omega_t}(\eta+\beta\rho)\nabla_x\Phi\cdot(\mathbf{u}-\mathbf{U})\,\mathrm{d}x\mathrm{d}t\\
&-\int_0^\tau\int_{\Omega_t}(\rho-r)\partial_tH_f'(r)+(\rho\mathbf{u}-r\mathbf{U})\cdot\nabla_xH_f'(r)+(p(\rho)-p(r))\mathrm{div}_x\mathbf{U}\,\mathrm{d}x\mathrm{d}t\\
&-\int_0^\tau\int_{\Omega_t}(\eta-s)\partial_tH_p'(s)+(\eta\mathbf{u}-s\mathbf{U})\cdot\nabla_xH_p'(s)+(\eta-s)\mathrm{div}_x\mathbf{U}\,\mathrm{d}x\mathrm{d}t\\
&+\int_0^\tau\int_{\Omega_t}\mathbb{S}(\nabla_x\mathbf{U}):\nabla_x(\mathbf{U}-\mathbf{u})+\rho(\mathbf{U}-\mathbf{u})\cdot\partial_t\mathbf{U}+\rho\mathbf{u}\otimes(\mathbf{U}-\mathbf{u}):\nabla_x\mathbf{U}\,\mathrm{d}x\mathrm{d}t
\end{aligned}\tag{4.15}$$

To deal with the remaining terms in (4.14), we proceed by adding and subtracting the terms

$$\pm\int_0^\tau\int_{\Omega_t}4|\nabla_x\sqrt{\eta}|^2\,\mathrm{d}x\mathrm{d}t\pm\int_0^\tau\int_{\Omega_t}\nabla_x\eta\cdot\nabla_x\Phi\,\mathrm{d}x\mathrm{d}t\tag{4.16}$$

on the right hand side of (4.14). This allows us to create the term

$$-\int_0^\tau\int_{\Omega_t}4|\nabla_x\sqrt{\eta}|^2+\nabla_x\eta\cdot\nabla_x\Phi\,\mathrm{d}x\mathrm{d}t=-\int_0^\tau\int_{\Omega_t}(\nabla_x\eta+\eta\nabla_x\Phi)\cdot\nabla_xH_p'(\eta)\,\mathrm{d}x\mathrm{d}t\tag{4.17}$$

which when combined with the similar term on the right hand side of (4.14) yields

$$+\int_0^\tau\int_{\Omega_t}(\eta\nabla_x\Phi+\nabla_x\eta)\cdot\nabla_x(H_p'(s)-H_p'(\eta))\,\mathrm{d}x\mathrm{d}t.\tag{4.18}$$

Finally, the remaining terms in (4.16) combined with the remaining terms in (4.14) become

$$\int_0^\tau \int_{\Omega_t} 4|\nabla_x \sqrt{\eta}|^2 + 2\nabla_x \eta \cdot \nabla_x \Phi + \eta |\nabla_x \Phi|^2 \,\mathrm{d}x\mathrm{d}t = \int_0^\tau \int_{\Omega_t} |2\nabla_x \sqrt{\eta} + \sqrt{\eta}\nabla_x \Phi|^2 \,\mathrm{d}x\mathrm{d}t. \tag{4.19}$$

This term cancels with the same term also appearing on the left hand side of (4.14).

Combining (4.15) and (4.18), we define

$$\begin{aligned}\mathcal{R}(\rho, r, \eta, s, \mathbf{u}, \mathbf{U}) = &\widetilde{\mathcal{R}}(\rho, r, \eta, s, \mathbf{u}, \mathbf{U}) \\ &+ \int_0^\tau \int_{\Omega_t} (\eta \nabla_x \Phi + \nabla_x \eta) \cdot \nabla_x (H_p'(s) - H_p'(\eta)) \,\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{4.20}$$

Finally, we put together the results from (4.14)-(4.20) and deduce that the relative entropy inequality (4.4) is satisfied.

4.2.1 Regularity of smooth solutions

The smooth solutions $(r, s, \mathbf{U})$ can be extended to a larger class of less regular solutions by means of a density argument on the relative entropy inequality (4.4). Indeed, with known regularity of weak solutions $(\rho, \eta, \mathbf{u})$, we need only check what regularity of $(r, s, \mathbf{U})$ is necessary to make each term integrable. It is not difficult to check that $(r, s, \mathbf{U})$ can be extended to the following class:

$$
\begin{cases}
r \in L^\infty(0,T;L^\gamma(\Omega_t)) \\
\partial_t H_f'(r) \in L^1(0,T;L^{\gamma/(\gamma-1)}(\Omega_t)) \\
\nabla_x H_f'(r) \in L^1(0,T;L^{2\gamma/(\gamma-1)} \cap L^2(0,T;L^{6\gamma/(5\gamma-6)}(\Omega_t;\mathbb{R}^3)) \\
\mathbf{U} \in L^1(0,T;L^{\gamma/(\gamma-1)}(\Omega_t;\mathbb{R}^3)) \cap L^\infty(0,T;L^{2\gamma/(\gamma-1)}(\Omega_t;\mathbb{R}^3)) \cap L^2(0,T;W^{1,2}(\Omega_t;\mathbb{R}^3)) \\
\mathrm{div}_x \mathbf{U} \in L^1(0,T;L^\infty(\Omega_t)) \cap L^2(0,T;L^{3/2}(\Omega_t)), \quad \mathbf{U}\big|_{\partial\Omega_t} = \mathbf{V}\big|_{\partial\Omega_t} \\
\partial_t \mathbf{U} \in L^1(0,T;L^{2\gamma/(\gamma-1)}(\Omega_t;\mathbb{R}^3)) \cap L^2(0,T;L^{6\gamma/(5\gamma-6)}(\Omega_t;\mathbb{R}^3)) \\
\nabla_x \mathbf{U} \in L^2(0,T;L^2 \cap L^{6\gamma/(2\gamma-3)}(\Omega_t;\mathbb{R}^{3\times 3})) \\
s \in L^\infty(0,T;L\log L(\Omega_t)) \cap L^2(0,T;L^3(\Omega_t)) \\
\partial_t H_p'(s) \in L^2(0,T;L^{3/2}(\Omega_t)) \\
\nabla_x H_p'(s) \in L^\infty(0,T;L^3(\Omega_t \mathbb{R}^3)).
\end{cases}
\tag{4.21}
$$

This regularity can potentially be further weakened with a more careful analysis but this suffices for our purposes.

4.3 Weak-strong uniqueness

In order to obtain the weak-strong uniqueness result, we let the triple $(r, s, \mathbf{U})$ be a solution of the NSS system (3.3), originating from the same initial data. It will be shown that the relative entropy $\mathcal{E}\left(\rho, \eta, \mathbf{u} \middle| r, s, \mathbf{U}\right)$ vanishes for almost every time $\tau \geq 0$. We assume furthermore that the densities r, s are bounded strictly away from zero,

$$r(t,\cdot), s(t,\cdot) > 0, \quad \text{for any } t \geq 0.$$

First, we can rewrite the momentum equation for $(r, s, \mathbf{U})$ on condition that r is strictly positive to get

$$\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla_x \mathbf{U} = \frac{1}{r} \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{U}) - \left(\beta + \frac{s}{r}\right) \nabla_x \Phi - \nabla_x H_f'(r) - \frac{1}{r} \nabla_x s. \tag{4.22}$$

Next we manipulate the remainder term $\mathcal{R}(\rho, r, \eta, s, \mathbf{u}, \mathbf{U})$ in (4.4) in order to put it in a form suitable for the Grönwall inequality. We proceed in several steps.

1. By appropriately adding and subtracting the term $\mathbf{U} \cdot \nabla_x \mathbf{U}$, we deduce

$$\begin{aligned}
&\int_0^\tau \int_{\Omega_t} \mathbb{S}(\nabla_x \mathbf{U}) : \nabla_x (\mathbf{U} - \mathbf{u}) + \rho(\partial_t \mathbf{U} + \mathbf{u} \cdot \nabla_x \mathbf{U}) \cdot (\mathbf{U} - \mathbf{u}) \, \mathrm{d}x \mathrm{d}t \\
&\quad = \int_0^\tau \int_{\Omega_t} \mathbb{S}(\nabla_x \mathbf{U}) : \nabla_x (\mathbf{U} - \mathbf{u}) \, \mathrm{d}x \mathrm{d}t \\
&\qquad + \int_0^\tau \int_{\Omega_t} \rho(\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla_x \mathbf{U}) \cdot (\mathbf{U} - \mathbf{u}) \, \mathrm{d}x \mathrm{d}t \\
&\qquad + \int_0^\tau \int_{\Omega_t} \rho(\mathbf{u} - \mathbf{U}) \cdot \nabla_x \mathbf{U} \cdot (\mathbf{U} - \mathbf{u}) \, \mathrm{d}x \mathrm{d}t \\
&\quad = I_1 + I_2 + I_3.
\end{aligned} \tag{4.23}$$

2. Using the identity (4.22) we deduce that

$$\begin{aligned}
I_1 + I_2 = &- \int_0^\tau \int_{\Omega_t} \rho \nabla_x H_f'(r) \cdot (\mathbf{U} - \mathbf{u}) \, \mathrm{d}x \mathrm{d}t \\
&+ \int_0^\tau \int_{\Omega_t} \frac{1}{r} (\rho - r) \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{U}) \cdot (\mathbf{U} - \mathbf{u}) \, \mathrm{d}x \mathrm{d}t \\
&- \int_0^\tau \int_{\Omega_t} \left[\left(\beta\rho + \rho \frac{s}{r}\right) \nabla_x \Phi + \frac{\rho}{r} \nabla_x s \right] \cdot (\mathbf{U} - \mathbf{u}) \, \mathrm{d}x \mathrm{d}t
\end{aligned} \tag{4.24}$$

3. Next, the first integral in (4.24) combines with the fluid pressure terms in

$\mathcal{R}(\rho, r, \eta, s, \mathbf{u}, \mathbf{U})$ to yield

$$\begin{aligned}
&-\int_0^\tau \int_{\Omega_t} (\rho - r)\partial_t H_f'(r) + (\rho\mathbf{u} - r\mathbf{U}) \cdot \nabla_x H_f'(r) + (p(\rho) - p(r))\mathrm{div}_x \mathbf{U} \,\mathrm{d}x\mathrm{d}t \\
&\quad - \int_0^\tau \int_{\Omega_t} \rho \nabla_x H_f'(r) \cdot (\mathbf{U} - \mathbf{u}) \,\mathrm{d}x\mathrm{d}t \\
&= \int_0^\tau \int_{\Omega_t} (r - \rho)[\partial_t H_f'(r) + \mathbf{U} \cdot \nabla_x H_f'(r)] - (p(\rho) - p(r))\mathrm{div}_x \mathbf{U} \,\mathrm{d}x\mathrm{d}t \\
&= (1 - \gamma) \int_0^\tau \int_{\Omega_t} \mathcal{E}_f(\rho \,|\, r)\mathrm{div}_x \mathbf{U} \,\mathrm{d}x\mathrm{d}t,
\end{aligned} \tag{4.25}$$

where in the last step we made use of the equality

$$\partial_t H_f'(r) + \mathbf{U} \cdot \nabla_x H_f'(r) = -p'(r)\mathrm{div}_x \mathbf{U}.$$

4. To simplify things, let us combine the steps in (4.23)-(4.25) to write the remainder term as

$$\begin{aligned}
&\mathcal{R}(\rho, r, \eta, s, \mathbf{u}, \mathbf{U}) \\
&= \int_0^\tau \int_{\Omega_t} \rho(\mathbf{u} - \mathbf{U}) \cdot \nabla_x \mathbf{U} \cdot (\mathbf{U} - \mathbf{u}) \,\mathrm{d}x\mathrm{d}t + (1 - \gamma) \int_0^\tau \int_{\Omega_t} \mathcal{E}_f(\rho \,|\, r)\mathrm{div}_x \mathbf{U} \,\mathrm{d}x\mathrm{d}t \\
&+ \int_0^\tau \int_{\Omega_t} \frac{1}{r}(\rho - r)\mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{U}) \cdot (\mathbf{U} - \mathbf{u}) \,\mathrm{d}x\mathrm{d}t \\
&+ \int_0^\tau \int_{\Omega_t} \left[-\rho \frac{s}{r} \nabla_x \Phi - \frac{\rho}{r} \nabla_x s\right] \cdot (\mathbf{U} - \mathbf{u}) \,\mathrm{d}x\mathrm{d}t + \int_0^\tau \int_{\Omega_t} \eta \nabla_x \Phi \cdot (\mathbf{U} - \mathbf{u}) \,\mathrm{d}x\mathrm{d}t \\
&+ \int_0^\tau \int_{\Omega_t} (\eta \nabla_x \Phi + \nabla_x \eta) \cdot \nabla_x (H_p'(s) - H_p'(\eta)) \,\mathrm{d}x\mathrm{d}t \\
&- \int_0^\tau \int_{\Omega_t} (\eta - s)\partial_t H_p'(s) + (\eta\mathbf{u} - s\mathbf{U}) \cdot \nabla_x H_p'(s) + (\eta - s)\mathrm{div}_x \mathbf{U} \,\mathrm{d}x\mathrm{d}t \\
&= \sum_{n=1}^{7} J_n,
\end{aligned} \tag{4.26}$$

where J_n denotes the nth integral in the remainder.

5. By appropriately adding and subtracting the term $\eta\mathbf{U}$, we deduce that

$$
\begin{aligned}
J_7 = &-\int_0^\tau \int_{\Omega_t} (\eta - s)[\partial_t H_p'(s) + \mathbf{U}\cdot\nabla_x H_p'(s)]\,\mathrm{d}x\mathrm{d}t \\
&+\int_0^\tau \int_{\Omega_t} \eta H_p'(s)\cdot(\mathbf{U}-\mathbf{u}) - (\eta - s)\mathrm{div}_x\mathbf{U}\,\mathrm{d}x\mathrm{d}t.
\end{aligned}
\tag{4.27}
$$

Then, the equality

$$
\partial_t H_p'(s) + \mathbf{U}\cdot\nabla_x H_p'(s) = \frac{1}{s}\Delta_x s - \mathrm{div}_x\mathbf{U} + \frac{1}{s}\mathrm{div}_x(s\nabla_x\Phi),
$$

the condition that $(\nabla_x s + s\nabla_x\Phi)\cdot\hat{\mathbf{n}}$ vanishes on the boundary, and equation (4.27) imply that

$$
\begin{aligned}
J_4 + J_5 + J_6 + J_7 = &\int_0^\tau \int_{\Omega_t} (\eta\nabla_x\Phi + \nabla_x\eta) + \nabla_x(H_p'(s) - H_p'(\eta))\,\mathrm{d}x\mathrm{d}t \\
&-\int_0^\tau \int_{\Omega_t} \left[\rho\frac{s}{r}\nabla_x\Phi - \eta\nabla_x\Phi + \frac{\rho}{r}\nabla_x s - \frac{\eta}{s}\nabla_x s\right]\cdot(\mathbf{U}-\mathbf{u})\,\mathrm{d}x\mathrm{d}t \\
&-\int_0^\tau \int_{\Omega_t} \frac{\eta}{s}\mathrm{div}_x(\nabla_x s + s\nabla_x\Phi)\,\mathrm{d}x\mathrm{d}t.
\end{aligned}
\tag{4.28}
$$

Notice in writing (4.28) that we require the densities to be bounded away from zero.

6. By a simple computation (cf. Ballew et al. [9]), the first and third terms in equation (4.28) together are equal to

$$
-\int_0^\tau \int_{\Omega_t} \frac{1}{s}\left|\sqrt{\frac{\eta}{s}}\nabla_x s - \sqrt{\frac{s}{\eta}}\nabla_x\eta\right|^2 \mathrm{d}x\mathrm{d}t \le 0. \tag{4.29}
$$

Since this term has the correct sign, it can subsequently be ignored from the relative entropy inequality (4.4).

Finally, we rewrite the second term in equation (4.28) as

$$
\begin{aligned}
&-\int_0^\tau \int_{\Omega_t} (\rho - r)\left[\frac{s}{r}\nabla_x\Phi + \frac{1}{r}\nabla_x s\right]\cdot(\mathbf{U}-\mathbf{u})\,\mathrm{d}x\mathrm{d}t \\
&-\int_0^\tau \int_{\Omega_t} (s - \eta)\left[\nabla_x\Phi + \frac{1}{s}\nabla_x s\right]\cdot(\mathbf{U}-\mathbf{u})\,\mathrm{d}x\mathrm{d}t.
\end{aligned}
\tag{4.30}
$$

As a result of the previous computations, we can rewrite the remainder $\mathcal{R}$ in the relative entropy inequality (4.4) as

$$\begin{aligned}\mathcal{R}(\rho, r, \eta, s, \mathbf{u}, \mathbf{U}) = \int_0^\tau \int_{\Omega_t} \rho(\mathbf{u}-\mathbf{U})\cdot\nabla_x\mathbf{U}\cdot(\mathbf{U}-\mathbf{u})\,\mathrm{d}x\mathrm{d}t \\ + (1-\gamma)\int_0^\tau\int_{\Omega_t} \mathcal{E}_f(\rho\,|\,r)\mathrm{div}_x\mathbf{U}\,\mathrm{d}x\mathrm{d}t \\ + \int_0^\tau\int_{\Omega_t} \frac{1}{r}(\rho-r)\mathrm{div}_x\mathbb{S}(\nabla_x\mathbf{U})\cdot(\mathbf{U}-\mathbf{u})\,\mathrm{d}x\mathrm{d}t \\ - \int_0^\tau\int_{\Omega_t} \frac{1}{r}(\rho-r)\left[s\nabla_x\Phi+\nabla_x s\right]\cdot(\mathbf{U}-\mathbf{u})\,\mathrm{d}x\mathrm{d}t \\ - \int_0^\tau\int_{\Omega_t} \frac{1}{s}(s-\eta)\left[s\nabla_x\Phi+\nabla_x s\right]\cdot(\mathbf{U}-\mathbf{u})\,\mathrm{d}x\mathrm{d}t \\ = \sum_{n=1}^5 K_n.\end{aligned} \tag{4.31}$$

The relative entropy inequality (4.4) can be reformulated as

$$\begin{aligned}\mathcal{E}\left(\rho,\eta,\mathbf{u}\middle|r,s,\mathbf{U}\right) + \int_0^\tau\int_{\Omega_t} |\nabla_x(\mathbf{u}-\mathbf{U})|^2\,\mathrm{d}x\mathrm{d}t \\ \leq \mathcal{E}\left(\rho_0,\eta_0,\mathbf{u}_0\middle|r_0,s_0,\mathbf{U}_0\right) + \mathcal{R}(\rho,r,\eta,s,\mathbf{u},\mathbf{U}),\end{aligned} \tag{4.32}$$

on account of the following computation

$$\begin{aligned}\int_{\Omega_t} |\nabla_x(\mathbf{u}-\mathbf{U})|^2\,\mathrm{d}x \leq C\int_{\Omega_t} \mu|\nabla_x(\mathbf{u}-\mathbf{U})|^2 + (\mu+\lambda)|\mathrm{div}_x(\mathbf{u}-\mathbf{U})|^2\,\mathrm{d}x \\ = \int_{\Omega_t} \mathbb{S}(\nabla_x(\mathbf{u}-\mathbf{U})) : \nabla_x(\mathbf{u}-\mathbf{U})\,\mathrm{d}x.\end{aligned} \tag{4.33}$$

This is convenient, as the following Poincaré-type inequality for the velocity terms holds,

$$\|\mathbf{u}-\mathbf{U}\|^2_{L^6(\Omega_t;\mathbb{R}^3)} \leq C\|\nabla_x(\mathbf{u}-\mathbf{U})\|^2_{L^2(\Omega_t;\mathbb{R}^{3\times3})}, \tag{4.34}$$

on account that $\mathbf{u}-\mathbf{U}\in W_0^{1,2}(\Omega_t;\mathbb{R}^3)$. We can therefore eventually absorb velocity gradient terms to the left hand side of (4.32) using a Cauchy inequality with ε.

The following lemma concerning growth properties of the relative entropy will be used in the remaining computations.

Lemma 9. *Let $\rho, r, \eta, s \geq 0$ and let the relative entropies $\mathcal{E}_f$ and $\mathcal{E}_p$ be defined as in (4.1) and (4.2). Assume $\gamma > 3/2$. Then*

1. *If $\eta \leq 2s$, then $|\eta - s|^2 \lesssim s\mathcal{E}_p(\eta \,|\, s)$,*

2. *If $\eta \geq 2s$, then $|\eta - s|^2 \lesssim \eta\mathcal{E}_p(\eta \,|\, s)$,*

3. *If $r/2 \leq \rho \leq 2r$, then $|\rho - r|^2 \lesssim \mathcal{E}_f(\rho \,|\, r)$,*

4. *It $\rho \leq r/2$ or $\rho \geq 2r$, then $(1 + |\rho|^\gamma) \leq C(r)\mathcal{E}_f(\rho \,|\, r)$.*

Remark. The requirement on γ in Lemma 9 can be weakened. At the least it should be strictly greater than 1 to keep H_f strictly convex to avoid a trivial relative entropy.

Proof. The first two inequalities are proved in [47], Lemma 2. In particular, the second follows by writing

$$\mathcal{E}_p(\eta|s) = \int_s^\eta \frac{\eta - z}{z}\,\mathrm{d}z,$$

and using that η dominates s. Statement 3 follows from $\mathcal{E}_f$ being strongly convex over the set $r/2 \leq \rho \leq 2r$. To prove statement 4, note $\mathcal{E}_f(\rho \,|\, r) = H_f(r)h\left(\frac{\rho}{r}\right)$ where $h(z) := z^\gamma - \gamma(z-1) - 1$. Define

$$F(z) := \frac{h(z)}{|z|^\gamma + 1}.$$

It suffices to show $F(z)$ is bounded below by some positive constant C, after which setting $z = \rho/r$ completes the proof. On the interval $0 \leq z \leq 1/2$, it is easy to check F is decreasing, with $F(1/2) = 1 + (\gamma/2 - 1)2^\gamma$. On the interval $z \geq 2$, F is also decreasing with limit 1 as $z \to \infty$. Therefore we can choose

$$C \leq \min\{1, [1 + 2^\gamma(\gamma/2 - 1)](2^\gamma + 1)^{-1}\}.$$

$\square$

Going back to (4.31), we easily bound K_1, K_2 by

$$K_1 + K_2 \leq \int_0^\tau a_1(\tau) \int_{\Omega_t} \rho|\mathbf{u} - \mathbf{U}|^2 + \mathcal{E}_f(\rho\,|\,r)\,\mathrm{d}x\mathrm{d}t \\ \leq \int_0^\tau a_1(\tau)\mathcal{E}\left(\rho, \eta, \mathbf{u}\middle| r, s, \mathbf{U}\right)\,\mathrm{d}t, \tag{4.35}$$

where $a_1(\tau)$ is in $L^1(0,\tau)$. Next, following the approach of [35], the terms K_2 and K_3 will be split over the sets $\{r/2 \leq \rho \leq 2r\}$, $\{\rho \leq r/2\}$, and $\{\rho \geq 2r\}$. First considering K_2, using Hölder and Cauchy's inequalities, along with (4.34) and Lemma 9, we have

$$\begin{aligned} &\int_{\{r/2\leq\rho\leq 2r\}} (\rho - r)(r^{-1}\mathrm{div}_x\mathbb{S}(\nabla_x\mathbf{U}))\cdot(\mathbf{U} - \mathbf{u})\,\mathrm{d}x \\ &\leq \|r^{-1}\mathrm{div}_x\mathbb{S}(\nabla_x\mathbf{U})\|_{L^3(\Omega_t;\mathbb{R}^3)}\left(\int_{\{r/2\leq\rho\leq 2r\}}|\rho - r|^2\,\mathrm{d}x\right)^{1/2}\|\mathbf{U} - \mathbf{u}\|_{L^6(\Omega_t;\mathbb{R}^3)} \\ &\leq \frac{1}{4\varepsilon}\|r^{-1}\mathrm{div}_x\mathbb{S}(\nabla_x\mathbf{U})\|^2_{L^3(\Omega_t;\mathbb{R}^3)}\int_{\Omega_t}\mathcal{E}_f(\rho\,|\,r)\,\mathrm{d}x + \varepsilon\|\mathbf{U} - \mathbf{u}\|^2_{L^6(\Omega_t;\mathbb{R}^3)} \\ &\leq \int_{\Omega_t} a_2(t)\mathcal{E}_f(\rho\,|\,r)\,\mathrm{d}x + \varepsilon C\|\nabla_x(\mathbf{u} - \mathbf{U})\|^2_{L^2(\Omega_t;\mathbb{R}^{3\times 3})}, \end{aligned} \tag{4.36}$$

where $a_2(t) \in L^1(0,\tau)$ and $\varepsilon > 0$. It is also required that

$$\mathrm{div}_x\mathbb{S}(\nabla_x\mathbf{U}) \in L^2(0,T;L^3(\Omega_t;\mathbb{R}^{3\times 3})).$$

In a similar way, we observe that

$$\int_{\{\rho\leq r/2\}} (\rho - r)(r^{-1}\mathrm{div}_x\mathbb{S}(\nabla_x\mathbf{U}))\cdot(\mathbf{U} - \mathbf{u})\,\mathrm{d}x \\ \leq \int_{\Omega_t} a_3(t)\mathcal{E}_f(\rho\,|\,r)\,\mathrm{d}x + \varepsilon C\|\nabla_x(\mathbf{u} - \mathbf{U})\|^2_{L^2(\Omega_t;\mathbb{R}^{3\times 3})}, \tag{4.37}$$

using that ρ is uniformly bounded, and where $a_3(t) \in L^1(0,\tau)$. On the set $\{\rho \geq 2r\}$, and using that

$$\mathcal{E}\left(\rho, \eta, \mathbf{u}\middle| r, s, \mathbf{U}\right) \in L^\infty(0,T),$$

we observe that

$$\|\rho\|_{L^\gamma} \lesssim \mathcal{E}\left(\rho, \eta, \mathbf{u} \middle| r, s, \mathbf{U}\right)^{1/\gamma} \lesssim \mathcal{E}\left(\rho, \eta, \mathbf{u} \middle| r, s, \mathbf{U}\right)^{1/2}$$
$$\|\rho^{\gamma/2}\|_{L^2} \lesssim \mathcal{E}\left(\rho, \eta, \mathbf{u} \middle| r, s, \mathbf{U}\right)^{1/2}.$$

We now compute

$$\begin{aligned}
&\int_{\{\rho\geq 2r\}} (\rho - r)(r^{-1}\mathrm{div}_x\mathbb{S}(\nabla_x\mathbf{U}))\cdot(\mathbf{U}-\mathbf{u})\,\mathrm{d}x \\
&\leq \int_{\{\rho\geq 2r\}} \left|\frac{\rho - r}{\rho r}\right| \max\{\rho, \rho^{\gamma/2}\}|\mathrm{div}_x\mathbb{S}(\nabla_x\mathbf{U})||\mathbf{U}-\mathbf{u}|\,\mathrm{d}x \\
&\lesssim \|\mathbf{u}-\mathbf{U}\|_{L^6(\Omega_t;\mathbb{R}^3)}\|\mathrm{div}_x\mathbb{S}(\nabla_x\mathbf{U})\|_{L^q\cap L^3(\Omega_t;\mathbb{R}^3)}\left(\int_{\Omega_t}\mathcal{E}_f(\rho\,|\,r)\,\mathrm{d}x\right)^{1/2} \\
&\leq \int_{\Omega_t} a_4(t)\mathcal{E}_f(\rho\,|\,r)\,\mathrm{d}x + \varepsilon C\|\nabla_x(\mathbf{u}-\mathbf{U})\|^2_{L^2(\Omega_t;\mathbb{R}^{3\times 3})},
\end{aligned} \tag{4.38}$$

where $q = 6\gamma/(5\gamma - 6)$ and $a_4(t) \in L^1(0, \tau)$. It is therefore required that

$$\mathrm{div}_x\mathbb{S}(\nabla_x\mathbf{U}) \in L^2(0, T; L^q(\Omega_t; \mathbb{R}^{3\times 3})).$$

The term K_4 is treated in exactly the same way as K_3 and we deduce that

$$\begin{aligned}
&\int_{\Omega_t} (\rho - r)r^{-1}[s\nabla_x\Phi + \nabla_x s]\cdot(\mathbf{U}-\mathbf{u})\,\mathrm{d}x \\
&\leq \int_{\Omega_t} a_5(t)\mathcal{E}_f(\rho\,|\,r)\,\mathrm{d}x + \varepsilon C\|\nabla_x(\mathbf{u}-\mathbf{U})\|^2_{L^2(\Omega_t;\mathbb{R}^{3\times 3})}
\end{aligned} \tag{4.39}$$

where $a_5(t) \in L^1(0, \tau)$. Finally, we split K_5 over the sets $\{\eta \leq 2s\}$ and $\{\eta \geq 2s\}$. First, on the set $\{\eta \leq 2s\}$ we have

$$\begin{aligned}
&\int_{\{\eta\leq 2s\}} \frac{1}{s}(s-\eta)\,[s\nabla_x\Phi + \nabla_x s]\cdot(\mathbf{U}-\mathbf{u})\,\mathrm{d}x \\
&\leq \|\eta - s\|_{L^2(\Omega_t)}\|\nabla_x\Phi + \nabla_x s/s\|_{L^3(\Omega_t;\mathbb{R}^3)}\|\mathbf{u}-\mathbf{U}\|_{L^6(\Omega_t;\mathbb{R}^3)} \\
&\leq \frac{1}{4\varepsilon}\|\nabla_x\Phi + \nabla_x s/s\|^2_{L^3(\Omega_t;\mathbb{R}^3)}\|\eta - s\|^2_{L^2(\Omega_t)} + \varepsilon\|\mathbf{u}-\mathbf{U}\|^2_{L^6(\Omega_t;\mathbb{R}^3)} \\
&\leq a_6(t)\int_{\Omega_t}\mathcal{E}_p(\eta\,|\,s)\,\mathrm{d}x + \varepsilon C\|\nabla_x\mathbf{u} - \nabla_x\mathbf{U}\|^2_{L^2(\Omega_t;\mathbb{R}^{3\times 3})},
\end{aligned} \tag{4.40}$$

with $a_6(t)$ in $L^1(0,\tau)$. Now over the set $\{\eta \geq 2s\}$ we have

$$
\begin{aligned}
&\int_{\{\eta\geq 2s\}} \frac{1}{s}(s-\eta)\,[s\nabla_x\Phi + \nabla_x s]\cdot(\mathbf{U}-\mathbf{u})\,\mathrm{d}x \\
&\left\|\frac{s-\eta}{\sqrt{\eta}}\right\|_{L^2(\Omega_t)} \|\sqrt{\eta}\|_{L^3(\Omega_t)}\|\nabla_x\Phi + \nabla_x s/s\|_{L^\infty(\Omega_t;\mathbb{R}^3)}\|\mathbf{U}-\mathbf{u}\|_{L^6(\Omega_t;\mathbb{R}^3)} \\
&\leq \frac{1}{4\varepsilon}\left\|\frac{s-\eta}{\sqrt{\eta}}\right\|^2_{L^2(\Omega_t)} + \varepsilon\|\eta\|_{L^{3/2}(\Omega_t)}\|\nabla_x\Phi + \nabla_x s/s\|^2_{L^\infty(\Omega_t;\mathbb{R}^3)}\|\nabla_x(\mathbf{u}-\mathbf{U})\|^2_{L^2(\Omega_t;\mathbb{R}^{3\times 3})} \\
&\leq C\int_{\Omega_t} \mathcal{E}_p(\eta|s)\,\mathrm{d}x + \varepsilon\|\eta\|_{L^{3/2}(\Omega_t)}\|\nabla_x\Phi + \nabla_x s/s\|^2_{L^\infty(\Omega_t;\mathbb{R}^3)}\|\nabla_x(\mathbf{u}-\mathbf{U})\|^2_{L^2(\Omega_t;\mathbb{R}^{3\times 3})}.
\end{aligned}
\tag{4.41}
$$

Remark. This requires further the condition that

$$\frac{\nabla_x s}{s} \in L^\infty((0,T)\times\Omega_t;\mathbb{R}^3),$$

as well as

$$\eta \in L^\infty(0,T;L^{3/2}(\Omega_t)).$$

Note this estimate on η isn't known a priori for the weak solutions and must be taken as an assumption.

Now integrating (4.36)-(4.40) over the time-interval $(0,\tau)$, and using (4.35), we estimate the remainder term (4.31). Absorbing the small ε terms into the left hand side of (4.32), we deduce that the relative entropy satisfies

$$
\begin{aligned}
\mathcal{E}\left(\rho,\eta,\mathbf{u}\,\middle|\, r,s,\mathbf{U}\right)(\tau) \leq \mathcal{E}\left(\rho_0,\eta_0,\mathbf{u}_0\,\middle|\, r_0,s_0,\mathbf{U}_0\right)& \\
+\int_0^\tau a(t)\mathcal{E}\left(\rho,\eta,\mathbf{u}\,\middle|\, r,s,\mathbf{U}\right)(t)\,\mathrm{d}t&,
\end{aligned}
\tag{4.42}
$$

where $a(t) \in L^1(0,\tau)$. Since the initial data are assumed the same for both classes of solution, $\mathcal{E}(0) \equiv 0$ and an application of Grönwall's inequality allows us to conclude that for all time, $(r,s,\mathbf{U}) \equiv (\rho,\eta,\mathbf{u})$, after possibly modifying on a set of measure zero. We therefore have proved the following theorem.

Theorem 5. *Let $\Omega_t \subset\subset D \subset \mathbb{R}^3$ be a bounded domain for all $t \geq 0$ of class $C^{2,\nu}$, $0 < \nu \leq 1$. Assume the pressure is given by*

$$p(\rho) = \rho^\gamma, \quad \gamma > 3/2,$$

and the stress tensor is given by

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu(\nabla_x \mathbf{u} + (\nabla_x \mathbf{u})^T) + \lambda \mathrm{div}_x \mathbf{u} \mathbb{I}.$$

Assume the triple $(\rho, \eta, \mathbf{u})$ is a weak solution of the NSS system (3.3) in the sense of Definition 2. Assume furthermore that

$$\eta \in L^\infty(0,T; L^{3/2}(\Omega_t)).$$

Let $(r, s, \mathbf{U})$ be a strong solution of the same problem, with regularity (4.21), and in addition

$$0 < c_1 \leq s(t,\mathbf{x}), r(t,\mathbf{x}) \leq c_2 < \infty$$

$$\nabla_x H_p'(s) \in L^\infty((0,T) \times \Omega_t; \mathbb{R}^3), \quad \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{U}) \in L^2(0,T; L^3 \cap L^q(\Omega_t; \mathbb{R}^{3\times 3}))$$

where $q = 6\gamma/(5\gamma - 6)$ and $c_1, c_2 > 0$ are constant. Then, after possibly modifying $(\rho, \eta, \mathbf{u})$ on a set of measure zero, we have

$$\rho \equiv r, \ \eta \equiv s, \ \mathbf{u} \equiv \mathbf{U} \quad \text{in } Q^f.$$

Appendix A: Pressure estimates

This appendix is devoted to the proof of Lemma 3 of Section 2.6. This proof, based on the Bogovskii operator, was first established in [40] for the case of compressible Navier-Stokes equations. More details can be found in [37]. The local pressure estimates by different methods were originally established by Lions [51].

Let us recall the weak formulation of the momentum equation (2.6c), at the ε-level. For all $\varphi \in \mathcal{D}([0,T);\mathcal{D}(\Omega;\mathbb{R}^3))$, it holds that

$$\begin{aligned}
&\int_0^T\int_\Omega \Big(\rho_\varepsilon \mathbf{u}_\varepsilon\cdot\partial_t\varphi + \rho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon : \nabla_x\varphi + (p(\rho_\varepsilon)+\eta_\varepsilon+\delta\rho_\varepsilon^\alpha)\mathrm{div}_x\varphi\Big)\,\mathrm{d}x\mathrm{d}t \\
&\qquad = \int_0^T\int_\Omega \mathbb{S}(\nabla_x\mathbf{u}_\varepsilon):\nabla_x\varphi + (\eta_\varepsilon+\beta\rho_\varepsilon)\nabla_x\Phi\cdot\varphi + \varepsilon\nabla_x\mathbf{u}_\varepsilon\nabla_x\rho_\varepsilon\cdot\varphi\,\mathrm{d}x\mathrm{d}t \qquad (A.1)\\
&\qquad\quad - \int_\Omega (\rho\mathbf{u})_{0,\delta}\cdot\varphi(0,\cdot)\,\mathrm{d}x.
\end{aligned}$$

One notices that if we formally choose a test function φ such that $\varphi \approx \mathrm{div}_x^{-1}\rho_\varepsilon$, where div_x^{-1} is somehow the inverse of the divergence operator, then we obtain the desired estimates from (A.1) as long as all terms are sufficiently bounded. This is accomplished at a rigorous level by means of the Bogovskii operator, $\mathcal{B}\approx\mathrm{div}_x^{-1}$. Details on this operator can be found in Section 10.5 of [37]. We only mention that provided g is a smooth compactly supported function with zero mean, then so is $\mathcal{B}[g]$ and $\mathrm{div}_x\mathcal{B}[g]=g$. In addition, $\mathcal{B}$ is a bounded operator from $W^{m,q}(\Omega)$ to $W^{m+1,q}(\Omega)$ for any $m\geq 0$ and $1<q<\infty$, provided the domain is Lipschitz continuous.

Following [37], we choose

$$\varphi(t,x) = \psi(t)\phi(t,x),$$

where $\psi \in C_c^\infty(0,T)$ and $\phi = \mathcal{B}[\rho_\varepsilon - \overline{\rho}]$. Here $\overline{\rho}$ denotes the average of ρ_ε over the set Ω. Basic properties of the Bogovskii operator ensure that since $\mathcal{B}$ is linear, we have that

$$\partial_t \phi = -\mathcal{B}[\operatorname{div}_x(\rho_\varepsilon \mathbf{u}_\varepsilon - \varepsilon \nabla_x \rho_\varepsilon)],$$

where we also used that ρ_ε solves the mass conservation equation. In addition, for $a.a.\, t \in (0,T)$ and $1 < p < \infty$, the following estimates hold:

$$\begin{aligned} &\|\phi(t,\cdot)\|_{W^{1,p}(\Omega;\mathbb{R}^3)} \leq c\|\rho_\varepsilon\|_{L^p(\Omega)} \\ &\|\partial_t \phi(t,\cdot)\|_{L^p(\Omega;\mathbb{R}^3)} \leq c\|\rho_\varepsilon \mathbf{u}_\varepsilon + \varepsilon \nabla_x \rho_\varepsilon\|_{L^p(\Omega;\mathbb{R}^3)}. \end{aligned} \tag{A.2}$$

By a density argument and provided $\alpha \geq 5$, we can use $\varphi(t,x) = \psi(t)\phi(t,x)$ as a test function in (A.1). Rearranging, it follows that

$$\int_0^T \psi \int_\Omega (p(\rho_\varepsilon) + \delta \rho_\varepsilon^\alpha)\rho_\varepsilon \,\mathrm{d}x\mathrm{d}t = \sum_{j=1}^{8} I_j, \tag{A.3}$$

where

$$\begin{aligned} I_1 &= -\int_0^T \psi \int_\Omega \rho_\varepsilon \mathbf{u}_\varepsilon \cdot \partial_t \phi \,\mathrm{d}x\mathrm{d}t, \\ I_2 &= \int_0^T \psi \overline{\rho} \int_\Omega p(\rho_\varepsilon) + \delta \rho_\varepsilon^\alpha \,\mathrm{d}x\mathrm{d}t, \\ I_3 &= -\int_0^T \psi \int_\Omega \eta_\varepsilon \phi \,\mathrm{d}x\mathrm{d}t, \\ I_4 &= -\int_0^T \psi \int_\Omega \rho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla_x \phi \,\mathrm{d}x\mathrm{d}t, \end{aligned}$$

and

$$
\begin{aligned}
I_5 &= -\int_0^T \psi' \int_\Omega \rho_\varepsilon \mathbf{u}_\varepsilon \cdot \phi \,\mathrm{d}x\mathrm{d}t, \\
I_6 &= \int_0^T \psi \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}_\varepsilon) : \nabla_x \phi \,\mathrm{d}x\mathrm{d}t, \\
I_7 &= \int_0^T \psi \int_\Omega (\eta_\varepsilon + \beta\rho_\varepsilon) \nabla_x \Phi \cdot \phi \,\mathrm{d}x\mathrm{d}t, \\
I_8 &= \int_0^T \psi \int_\Omega \varepsilon \nabla_x \mathbf{u}_\varepsilon \nabla_x \rho_\varepsilon \cdot \phi \,\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Applying Hölder's inequality to the I_j and invoking (A.2) yields

$$
\begin{aligned}
|I_1| &\le c\|\psi\|_{L^\infty_T} \|\rho_\varepsilon \mathbf{u}_\varepsilon\|_{L^\infty_T L_X^{2\alpha/(\alpha+2)}} \|\rho_\varepsilon \mathbf{u}_\varepsilon + \varepsilon \nabla_x \rho_\varepsilon\|_{L^1_T L_X^{2\alpha/(\alpha-2)}}, \\
|I_2| &\le c\|\psi\|_{L^\infty_T} \|p(\rho_\varepsilon) + \delta\rho_\varepsilon^\alpha\|_{L^\infty_T L^1_X}, \\
|I_3| &\le \|\psi\|_{L^\infty_T} \|\eta_\varepsilon\|_{L^1_T L^3_X} \|\rho_\varepsilon\|_{L^\infty_T L_X^{3/2}}, \\
|I_4| &\le \|\psi\|_{L^\infty_T} \|\rho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon\|_{L^2_T L_X^{6\alpha/(4\alpha+3)}} \|\rho_\varepsilon\|_{L^2_T L_X^{6\alpha/(2\alpha-3)}}, \\
|I_5| &\le \|\psi'\|_{L^\infty_T} \|\rho_\varepsilon \mathbf{u}_\varepsilon\|_{L^\infty_T L_X^{2\alpha/(\alpha-2)}} \|\rho_\varepsilon\|_{L^\infty_T L^\alpha_X}, \\
|I_6| &\le \|\psi\|_{L^\infty_T} \|\mathbb{S}_\varepsilon\|_{L^2_{T,X}} \|\rho_\varepsilon\|_{L^2_{T,X}}, \\
|I_7| &\le \|\psi\|_{L^\infty_T} \|\nabla_x \Phi\|_{L^\infty_X} \|\eta_\varepsilon + \beta\rho_\varepsilon\|_{L^2_T L^3_X} \|\rho_\varepsilon\|_{L^2_T L_X^{3/2}}, \\
|I_8| &\le \varepsilon\|\psi\|_{L^\infty_T} \|\nabla_x \mathbf{u}_\varepsilon\|_{L^2_{T,X}} \|\nabla_x \rho_\varepsilon\|_{L^2_{T,X}} \|\rho_\varepsilon\|_{L^\infty_T L^\alpha_X}.
\end{aligned}
$$

Since the right hand sides are all finite given as a result of the known a priori estimates, Lemma 3 is proved.

Remark. The argument above is valid for any finite $\delta > 0$, and is the result of the integrability gains provided by the artificial pressure. In a similar way, we can modify the proof to show that the pressure estimates

$$
\int_0^T \int_\Omega \rho_\delta^{\gamma+\omega} \,\mathrm{d}x\mathrm{d}t \le c,
$$

and

$$\delta \int_0^T \int_\Omega \rho_\delta^{\alpha+\omega} \,\mathrm{d}x\mathrm{d}t \leq c$$

hold independently of $\delta > 0$, for any $0 < \omega < \min\left\{\frac{1}{3}, \frac{2}{3}\gamma - 1\right\}$.

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