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Effective Bezout Identities In $Q[z_1,\ldots,z_n]$

by

C.A. Berenstein and A. Yger

EFFECTIVE BEZOUT IDENTITIES IN $\mathbf{Q}[z_1, ..., z_n]$

Carlos A. Berenstein*†

Mathematics Department and

Systems Research Center

University of Maryland

College Park, MD 20742, USA

and

Alain Yger*

UER Mathematiques

Lab. Ass. CNRS 226

Universite' de Bordaeux I

33405 Talence Cedex, FRANCE

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Abstract

If $p_1, \cdots, p_m$ are n-variate polynomials with integral coefficients and no common zeros in C^n, Brownawell has shown in 1986 that there exist $q_1, \cdots, q_m$ polynomials with integral coefficients and $\nu \in^{Z^+}$ such that

$$p_1\, q_1 + \cdots + p_m\, q_m = \nu,$$

and max deg $q_j \le$ (max deg $p_j)^n$. On the other hand if $h =$ logarithm of the largest coefficient of all the p_j, and h_1 is the corresponding quantity for the q_j, then there is no sharp estimate of h_1 in terms of h and max deg p_j. In this paper we show that when the variety of common zeros at infinity of the p_j is discrete then (essentially) we have:

$$h_1 \le D^{cn} h$$

for an absolute constant c.

If there were an algorithm to compute the q_j in D^{cn} time one would obtain exactly the above estimate. Current algorithms require about D^{n^2} operations.

§ 1. Introduction

Let $p_1,\dots,p_m \in \mathbb{Z}[z_1,\dots,z_n] = \mathbb{Z}[z]$ without common zeros in $\mathbb{C}^n$. Hilbert's Nullstellensatz ensures that there is $a \in \mathbb{Z}^*$ and polynomials $q_1,\dots,q_m \in \mathbb{Z}[z]$ such that for every $z \in \mathbb{C}^n$

$$(1.1) \qquad a = p_1(z)q_1(z) + \dots + p_m(z)q_m(z).$$

The explicit resolution of the Bezout equation (1.1) consists in giving an algorithm to find such polynomials $q_1,\dots,q_m$. One such algorithm has been implemented by Buchberger [10]. It is based on principles that go back to G. Hermann [14] and Seidenberg [24]. Masser-Wüstholz [21] used this method to estimate the degree and the size of the polynomials q_j, and the size of a. Denote by $h(P)$ the logarithmic size of a polynomial $P \in \mathbb{Z}[z]$, i.e., $h(P)$ = the logarithm of the modulus of the coefficient of P of largest absolute value. They showed that using the Hermann algorithm one could find $q_1,\dots,q_m$ satisfying:

$$(1.2) \qquad \max(\deg q_j) \le 2(2D)^{2^{n-1}}, \quad D = \max(\deg p_j)$$

$$(1.3) \qquad \max(\log|a|, h(q_j)) \le (8D)^{4\times 2^{n-1}-1} \cdot (h + 8d \log 8D),$$
$$h = \max h(p_j).$$

More recently, using a combination of methods from elimination theory and several complex variables, Brownawell [8] has obtained an essentially sharp bound for the degrees of polynomials q_j satisfying (1.1):

$$(1.4)\qquad \max(\deg q_j) \le \mu n D^{\mu} + \mu D, \quad \mu = \inf\{n,m\}.$$

To be able to compare the nature of these two results, a word is necessary about Brownawell's polynomials q_j. First one proves that there exist $q_j^* \in \mathbb{C}[z]$ satisfying the equation (1.1) with $a = 1$, with degrees bounded as in (1.4). These q_j^* are obtained as integrals over the whole space $\mathbb{C}^n$ of some conveniently constructed kernels. In some sense we should say the q_j^* are given by explicit formulas, but these formulas do not constitute an algorithm. Namely, how does one compute the numerical integrals over $\mathbb{C}^n$ appearing in this method? One also obtains an upper bound of the absolute value of the coefficients of the q_j^*, this follows from the effective bounds for the constant c_1 appearing in the Lojasiewicz' type inequality [8], [23]

$$(1.5)\qquad \left[\sum_{j=1}^{m} |p_j(z)|^2\right]^{1/2} \ge c_1(1+\|z\|)^{1-(n-1)D^n}.$$

Since the p_j have integral coefficients, the existence of q_j^* implies the existence of $a \in \mathbb{Z}^*$, $q_j \in \mathbb{Z}[z]$ satisfying (1.1) and (1.4) (Lefschetz' principle). This is simply linear algebra (cf. e.g. [21], Lemma 1). One could ask what is the size of a and of the polynomials q_j obtained this way? Setting $\delta = \mu n D^{\mu} + \mu D$, then the above mentioned lemma of Masser-Wüstholz yields the estimate

$$(1.6)\qquad \max(\log|a|, h(q_j)) \le m\binom{n+\delta}{\delta}\left\{h + \log m + \log\binom{n+\delta}{\delta}\right\}.$$

For $m \geq n$ the order of magnitude of the right hand side of (1.6) is essentially

$$(1.7) \qquad mn^n D^{n^2}(h + \log m + n \log n + n^2 \log D).$$

Note that in special cases where a better estimate than (1.4) is possible, then this same Lefschetz' principle provides a better bound in (1.6). Such is the case studied by Macaulay [17], [19] when the polynomials $p_1,\ldots,p_m$ have no common points at infinity. Then one can find q_j satisfying the estimate

$$(1.8) \qquad \deg q_j \leq n(D-1).$$

The corresponding estimate for a and $h(q_j)$ is essentially

$$(1.9) \qquad \max(\log|a|, h(q_j)) \leq mn^n D^n(h + \log m + n \log n + n \log D).$$

As soon as there is even a single common point at ∞ for $p_1,\ldots,p_m$, the estimate (1.8) is false. This is precisely the situation for the example of Masser-Philippon [8]

$$(1.10) \qquad p_1=z_1^D,\ p_2=z_1-z_2^D,\ \ldots,\ p_{n-1}=z_{n-2}-z_{n-1}^D,\ p_n=1-z_{n-1}z_n^{D-1},$$

for which the best estimate possible for $\deg q_j$ is $D^n - D^{n-1}$.

One of the objectives of this paper is to study the case where the set of common zeros of $p_1,\ldots,p_m$ at ∞ is finite. We obtain a better bound than (1.7) for the size of a and the q_j. Essentially one finds that by losing a little bit in the estimate of the degrees of q_j, $D^{\varkappa_1 n}$ instead of D^n, the size estimate is basically (1.7) where D^{n^2} is replaced by $D^{\varkappa_2 n}$. ($\varkappa_1, \varkappa_2$ absolute constants), cf. Theorem 4.1 below.

Our method depends on tools from complex function theory, except that we have succeeded in obtaining by this method a solution q_j, a lying directly in $\mathbb{Z}[z], \mathbb{Z}$ respectively. $\mathbb{Z}$ can be replaced by the ring of integers of any number field. The formulas we introduce can also be used to study the question of finding a division formula in $\mathbb{Z}[z]$. That is, if q belongs to the ideal I generated by $p_1, \ldots, p_m$ in $\mathbb{Z}[z]$, to obtain polynomials $q_1, \ldots, q_m \in \mathbb{Q}[z]$ of the smallest possible degree such that

$$q = q_1 p_1 + \ldots + q_m p_m. \tag{1.11}$$

We show in Theorem 3.3 that if the variety $V = \{z \in \mathbb{C}^n : p_1(z) = \ldots = p_m(z) = 0\}$ is discrete, then we one can estimate $\max(\deg q_j)$ by $(\max(D, \deg q))^{\varkappa n}$, for some absolute constant $\varkappa$. On the other hand, if the variety V is not discrete, this estimate cannot hold by an example of Mayr-Meyer [22]. What is shown in [22] is that for any $D \geq 5$, $k \in \mathbb{N}$, there are $n + 1$ polynomials $p_1, \ldots, p_{n+1} \in \mathbb{Z}[z]$, $n = 10k$, with $z_1 \in I$, $\max \deg p_j = D$ and if $q_1, \ldots, q_{n+1} \in \mathbb{Q}[z]$ satisfy

$$z_1 = q_1 p_1 + \ldots + q_{n+1} p_{n+1} \tag{1.12}$$

then $\max \deg q_j > (D-2)^{2^{k-1}}$.

Our hope is that our methods might eventually reconcile the estimates (1.4) for the degrees, with (1.9) for the size of the solutions a, q_j of (1.1) in the general case. One might guess that in this direction one might find an algorithm of lower complexity than plain linear algebra to solve (1.1).

We would especially like to thank Dale Brownawell for his many useful remarks.

§ 2. Residue Currents

We incorporate in this section some results of Complex Analysis which form the basis for the rest of the paper. We start by fixing some notation that will be used throughout.

Let $f = (f_1,\dots,f_n)$ be a $\mathbb{C}^n$-valued function, $m \in \mathbb{N}^n$ a multi-index of length $|m| = m_1 + \dots + m_n$. Then we denote

$$f^m = f_1^{m_1} \dots f_n^{m_n}, \quad F = f_1 \cdots f_n, \quad \|f\| = \left(\sum |f_j|^2\right)^{1/2}$$

$$(2.1) \qquad \partial f = \partial f_1 \wedge \dots \wedge \partial f_n = \bigwedge_1^n \partial f_j, \quad \partial f_j = \sum_{k=1}^n \frac{\partial f_j}{\partial z_k} dz_k$$

$$\bar\partial f = \bar\partial f_1 \wedge \dots \wedge \bar\partial f_n = \bigwedge_1^n \bar\partial f_j, \quad \bar\partial f_j = \sum_{k=1}^n \frac{\partial f_j}{\partial \bar z_k} d\bar z_k,$$

$$df = df_1 \wedge \dots \wedge df_n, \qquad df_j = \partial f_j + \bar\partial f_j,$$

where $\frac{\partial}{\partial z_k}$, $\frac{\partial}{\partial \bar z_k}$ are the standard first order complex derivative operators [13], [15], and the functions f_j are continuously differentiable. Note that $dz = dz_1 \wedge \dots \wedge dz_n$ and $d\bar z = d\bar z_1 \wedge \dots \wedge d\bar z_n$ are particular cases of (2.1). Also note $\bigwedge_1^n$ is always understood in increasing order.

If Q is a $(1,0)$ form, i.e. $Q(\zeta) = \sum_{j=1}^n Q_j(\zeta)\, d\zeta$, then $\bar\partial Q$ is a $(1,1)$ form, and there is no ambiguity in writing for $k \in \mathbb{N}$

$$(2.2) \qquad (\bar\partial Q)^k = \bar\partial Q \wedge \dots \wedge \bar\partial Q \quad (k \text{ times})$$

since (1,1) forms commute for the wedge product. $((\bar\partial Q)^0 = 1.)$

The space of differential forms of type (j,k) with smooth coefficients of compact support in $\mathbb{C}^n$ is denoted $\mathcal{D}_{j,k}$. $\varphi \in \mathcal{D}_{j,k}$ is called a test form. The dual space of $D_{n-j,n-k}$, $\mathcal{D}'_{n-j,n-k}$, is called the space of currents of type (j,k). It can be identified to the space of differential forms of type (j,k) with coefficients in the space $\mathcal{D}'$ of distributions in $\mathbb{C}^n$ [18].

Given n entire holomorphic functions f_j defining a discrete variety $V = V(f)$, $V := \{z \in \mathbb{C}^n : f_1(z) = \dots = f_n(z) = 0\}$, we can define the *Grothendieck residue current* $\bar\partial\frac{1}{f}$ as the current of type $(0,n)$ defined on test forms $\varphi \in \mathcal{D}_{n,0}$ by

$$(2.3) \qquad \langle \bar\partial\frac{1}{f}, \varphi\rangle = \lim_{\lambda\to 0} \frac{(-1)^{\frac{n(n-1)}{2}}}{(2\pi i)^n}\, \lambda^n \int_{\mathbb{C}^n} |F|^{2(\lambda-1)}\bar\partial f \wedge \varphi,$$

where the meaning of the integral on the right hand side of (2.2) is the following. First, it is well defined as a holomorphic function of λ for $\operatorname{Re}\lambda > 1$. Then, the product

$\lambda^n \int_{\mathbb{C}^n} |F|^{2(\lambda-1)}\bar\partial f \wedge \varphi$ can be analytically continued to the whole complex plane to become a meromorphic function of λ, which is holomorphic in a neighborhood of $\lambda = 0$. In fact, the limit in (2.3) is just the evaluation of this analytically continued function at $\lambda = 0$. (cf [7]). This coincides with the usual definition of the Grothendieck residue current [12]. If we want to emphasize the components of f we will write

$$\bar{\partial}\frac{1}{f} = \bar{\partial}\frac{1}{f_1} \wedge \ldots \wedge \bar{\partial}\frac{1}{f_n}.$$

In particular $\bar{\partial}\frac{1}{f^m} = \bar{\partial}\frac{1}{f_1^{m_1}} \wedge \ldots \wedge \bar{\partial}\frac{1}{f_n^{m_n}}$. Note there is no contradiction between this notation and (2.1). If the holomorphic function f_j is such that $\frac{1}{f_j}$ is differentiable, it means that f_j has no zeros. Therefore the usual differential form $\bar{\partial}(\frac{1}{f_j}) = 0$, but the Grothendieck residue will also be zero since $V = \emptyset$. Furthermore, this observation holds in a local sense also, that is, if $\operatorname{supp} \varphi \cap V = \emptyset$ we have $\langle\bar{\partial}\frac{1}{f},\varphi\rangle = 0$.

If $f_1,\ldots,f_n$ are polynomials defining a discrete (hence finite) variety V and if h is a function which is C^∞ in a neighborhood of V we can define the action of $\bar{\partial}\frac{1}{f}$ on the form $h\,dz$ by

$$\langle\bar{\partial}\frac{1}{f},h\,dz\rangle := \langle\bar{\partial}\frac{1}{f},\varphi\,dz\rangle,$$

where $\varphi \in \mathcal{D}$, $\varphi = 1$ on a (small) neighborhood of V. When h is actually holomorphic in a neighborhood of V then

$$(2.4) \qquad \langle\bar{\partial}\frac{1}{f},h\,dz\rangle := \lim_{\varepsilon\to 0^+} \frac{1}{(2\pi i)^n} \int_{|f|=\varepsilon} h(z)\,\frac{dz}{f_1(z)\ldots f_n(z)}$$

$$= \lim_{\varepsilon\to 0} \frac{1}{(2\pi i)^n} \int_{|f|=\varepsilon} h\frac{dz}{F},$$

where $\{|f| = \varepsilon\}$ is the smooth cycle $\{z \in \mathbb{C}^n: |f_j(z)| = \varepsilon, 1 \le j \le n)\}$ defined (by Sard's theorem) for $0 < \varepsilon$ outside a negligible set, and it is taken to be positively oriented (that is

$d(\arg f_1) \wedge \ldots \wedge d(\arg f_n) \geq 0$ on $|f| = \varepsilon$) (cf. [13], [27]). Furthermore, once $0 < \varepsilon << 1$, the limit coincides with the integral over $\{|f| = \varepsilon\}$.

It follows from the fact that the current $\bar{\partial}\frac{1}{f}$ has support in V that for $\varphi \in \mathcal{D}$

$$(2.5) \qquad \langle \bar{\partial}\tfrac{1}{f}, \varphi dz \rangle = \Big[\sum_{\zeta \in V} \sum_{\alpha} c_{\alpha,\zeta} \delta_{\zeta}^{(\alpha)} \Big](\varphi),$$

where the interior sum takes place over multi-indices α, $|\alpha| \leq N$, $c_{\alpha,\zeta} \in \mathbb{C}$. In case the point $\zeta \in V$ is a simple zero then $c_{\alpha,\zeta} = 0$ for $\alpha \neq 0$ and $c_{0,\zeta} = \frac{1}{J(\zeta)}$, $J(\zeta)$ = determinant Jacobian $\frac{\partial(f_1 \ldots f_n)}{\partial(z_1 \ldots z_n)}$ at $z = \zeta$. More generally, we have the identity ([11], § 1.9) for $\varphi \in \mathcal{D}$:

$$(2.6) \qquad \langle J\, \bar{\partial}\tfrac{1}{f}, \varphi dz \rangle = \Big[\sum_{\zeta \in V} m_{\zeta} \delta_{\zeta} \Big](\varphi) = \sum_{\zeta \in V} m_{\zeta} \varphi(\zeta)$$

where m_{ζ} is the multiplicity of ζ as a common zero of $f_1, \ldots, f_n$. Here we use the fact that a current can be multiplied by a smooth function g by the rule $\langle g\bar{\partial}\frac{1}{f}, \varphi\rangle := \langle \bar{\partial}\frac{1}{f}, g\varphi\rangle$. Note this multiplication will also make sense if g is of class C^N in a neighborhood of V, N the integer from (2.5).

The formula (2.6) allows us to write Cauchy's formula in terms of residues. Namely, let $\varphi \in C_0^1(\mathbb{C}^n)$ and consider the functions $f_j(\zeta) = \zeta_j - z_j$, $j = 1, \ldots, n$, for $z \in \mathbb{C}^n$ fixed. Then we have

$$\langle \bar{\partial}\frac{1}{\zeta - z}, \varphi(\zeta) d\zeta \rangle = \varphi(z).$$

In fact this is a particular case of (2.6), where $V = \{z\}$, $m_z = 1$, $J \equiv 1$.

Another property that will play a role is that

$$(2.8) \qquad f_j \bar{\partial}\frac{1}{f} = 0, \quad j = 1,\dots,n.$$

Therefore, $\bar{\partial}\frac{1}{f}$ vanishes on the C_0^{∞}-submodule of $\mathcal{D}_{(n,0)}$ generated by $f_1,\dots,f_n$.

The three properties (2.5) (conveniently modified), (2.6), and (2.8) hold also for entire functions f_j.

<u>Lemma 2.1</u>. Let K be a subfield of $\mathbb{C}$, $f_1,\dots,f_n \in K[z]$ defining a discrete variety V, $g \in K[z]$. Then

$$\sum_{\zeta \in V} m_\zeta g(\zeta) \in K.$$

<u>Proof</u>. By (2.6) we have

$$\sum_{\zeta \in V} m_\zeta g(\zeta) = \langle J\bar{\partial}\frac{1}{f}, g dz\rangle$$

By elimination theory [26] there are polynomials $q_1,\dots,q_n \in K[z]$, q_j a polynomial depending only on the jth variable such that

$$(2.9) \qquad q_k = \sum_{j=1}^{n} h_{k,j} f_j, \quad h_{k,j} \in K[z].$$

Let us denote $\Delta = \det[h_{k,j}]_{k,j}$. The transformation law for the residue yields for any h smooth in $\mathbb{C}^n$, holomorphic in a neighborhood of V (cf. [13], [7, Prop. 2.3]) :

$$<\bar\partial\frac{1}{f},hdz> = <\Delta\bar\partial\frac{1}{q},hdz>.$$

In particular

$$\sum m_\zeta g(\zeta) = <\bar\partial\frac{1}{q},\Delta Jgdz>.$$

To finish the proof it is enough now to show that for any monomial $z^\alpha = z_1^{\alpha_1}\ldots z_n^{\alpha_n}$ we have $<\bar\partial\frac{1}{q},zdz> \in K$. To compute this value we can apply (2.4):

$$<\bar\partial\frac{1}{q},z^\alpha dz> = \lim_{\varepsilon\to 0} \frac{1}{(2\pi i)^n}\int_{|q|=\varepsilon} z^\alpha \frac{dz}{q_1\ldots q_n}$$

$$= \prod_{j=1}^{n} \lim_{\varepsilon\to 0} \frac{1}{2\pi i}\int_{|q_j|=\varepsilon} z_j^{\alpha_j}\frac{dz_j}{q_j(z_j)}$$

$$= \prod_{j=1}^{n}\left[\sum_{q_j(\beta)=0} \operatorname*{res}_\beta t^{\alpha_j}/q_j(t)\right],$$

where $\operatorname*{res}_\beta h(t)$ denotes the usual one variable residue of the function h at the point β. The easiest way to compute the inner sums is to recall that for rational functions of one variable the sum of the residues over all the poles plus the point at ∞ is zero. Therefore

$$\sum_{q_j(\beta)=0} \operatorname*{res}_\beta t^{\alpha_j}/q_j(t) = -\operatorname*{res}_\infty t^{\alpha_j}/q_j(t) = a_{-1}$$

where $t^{\alpha_j}/q_j(t) = a_1 t^1 + \ldots + a_0 + \frac{a_{-1}}{t} + \frac{a_{-2}}{t^2} + \ldots$ in a neighborhood of ∞. The coefficients a_k are rational linear combinations of the coefficients of q_j. Hence each sum is in K.

□

Corollary 2.2. Let K be a number field of degree e, $f_1,\ldots,f_n,g$ as in Lemma 2.1. Let $\zeta_0 = (\zeta_1^0,\ldots,\zeta_n^0) \in V$ then $g(\zeta_0)$ is an algebraic number of degree $\le e\left[\sum_{\zeta\in V} m_\zeta\right]$. If $\max_j \deg f_j = D$ then the degree of $g(\zeta_0) \le eD^n$.

Proof. Let $M = \sum_{\zeta\in V} m_\zeta$ = total number of finite zeros of $f_1,\ldots,f_n$, and denote $\zeta_1,\ldots,\zeta_M$ these zeros, each repeated according to its multiplicity. Then the polynomial $\prod_1^M (x-g(\zeta_j))$ has coefficients in K. In fact, the symmetric functions of $g(\zeta_j)$ can be written as rational combinations of the elementary symmetric functions (Newton Sums) [26], i.e., as rational combinations of $\sum_{j=1}^M g(\zeta_j)^p = \sum_{j=1} m_\zeta (g(\zeta_j))^p \in K$ by Lemma 2.1. The last statement follows from Bezout's theorem.

□

Lemma 2.3. Let K, $f_1,\ldots,f_n$ as in Lemma 2.1. Let $r \in K(z)$ without any poles on V, then $\langle\bar{\partial}\frac{1}{f},rdz\rangle \in K$.

Proof. Let $q_1,\ldots,q_n$ be the same as in the proof of Lemma 2.1. Let $r = g/p$, g,p coprime polynomials in $K[z]$, $V(p,f_1,\ldots,f_n) = \emptyset$. The difficulty in carrying over the proof as in Lemma 2.1 consists in that p could vanish on some points of $V(q_1,\ldots,q_n)\backslash V$. (In the application of the transformation law for the residue one had to assume h was globally smooth, it would be enough to know it is smooth in a neighborhood of $V(q_1,\ldots,q_n)$ but if r has a pole there we cannot apply that formula). We first show we can in fact assume this is not the case.

Let N be the integer defined by (2.5) and consider the polynomial

$$P = \lambda_0 p + \lambda_1 f_1^{N+1} + \ldots + \lambda_n f_n^{N+1}. \tag{2.10}$$

By Lemma 1, [21] we can choose $\lambda_0,\ldots,\lambda_n \in \mathbb{Z}$ such that P does not vanish on $V(q_1,\ldots,q_n)$. In particular $\lambda_0 \neq 0$. Therefore we can set $\lambda_0 = 1$ and $\lambda_1,\ldots,\lambda_n \in \mathbb{Q}$. From (2.5) it follows now that

$$\langle \bar\partial \frac{1}{f}, \frac{g}{p}\, dz\rangle = \langle \bar\partial \frac{1}{f}, \frac{g}{P}\, dz\rangle$$

since g/p and g/P coincide and have same derivatives up to total order N at each point of V.

Since we are now assuming that r has no poles on $V(q_1,\ldots,q_n)$ we have, as in Lemma 2.1,

$$\langle \bar\partial \frac{1}{f}, r dz\rangle = \langle \bar\partial \frac{1}{q}, \Delta r dz\rangle.$$

This time Δr is a rational function, hence we cannot reduce ourselves to the case of monomials as in Lemma 2.1. To overcome this difficulty let us factorize q_j in $K[t]$ into irreducible factors:

$$(2.10) \qquad q_j = q_{j,1}^{n_1}\cdots q_{j,s}^{n_s}, \quad q_{j,k} \in K[t], \quad s = s(j), \quad n_k \in \mathbb{Z}^+.$$

From (2.4) we can take $0 < \varepsilon << 1$ so that $\Delta(z)r(z)$ is holomorphic in $\{|q_j| \le \varepsilon,\ 1 \le j \le n\} = \{|q| \le \varepsilon\}$ and

$$\langle\bar\partial\frac{1}{q},\Delta r dz\rangle = \frac{1}{(2\pi i)^n}\int_{|q|=\varepsilon} \frac{\Delta(z)r(z)\,dz}{q_1(z_1)\cdots q_n(z_n)} .$$

This integral can be computed one variable at a time. Fixing z', $z' = (z_2,\ldots,z_n)$, we have

$$(2.11) \qquad \frac{1}{2\pi i}\int_{|q_1(z_1)|=\varepsilon} \frac{h(z_1,z')}{q_1(z_1)}\,dz_1$$

$$= \sum_{k=1}^{s(1)}\left[\sum_{q_{1,k}(\alpha)=0} \operatorname*{res}_{z_1=\alpha} \frac{h(z_1,z')/[q_1(z_1)/q_{1,k}^{n_k}(z_1)]}{(q_{1,k}(z_1))^{n_k}}\right].$$

Fix k, let $\nu = n_k$, $Q = q_{1,k}$, P = numerator in the interior sum of (2.11). The zeros of Q are all simple, let them be $\alpha_1,\ldots,\alpha_\mu$. We can factorize $Q(t)$ as follows:

$$Q(t) = (t-\alpha_1)(Q'(\alpha_1)+\ldots) = (t-\alpha_1)R_1(t),$$

$R_1(t)$ is a polynomial in t with coefficients in $K[\alpha_1]$. For a different root α_j we will have $Q(t) = (t-\alpha_j)R_j(t)$, where the coefficients of R_j are obtained by replacing α_1 to α_j everywhere in the computation of R_1. The function P is holomorphic at $t = \alpha_1,\ldots,\alpha_\mu$ since the different irreducible factors of q_1 have no common zeros. Therefore $P(t)/Q(t)$ has a pole of order exactly ν at $t = \alpha_1$.

$$(2.12) \qquad \operatorname*{res}_{t=\alpha_1} \frac{P(t)}{(Q(t))^{\nu}} = \frac{1}{(\nu-1)!} \frac{d^{\nu-1}}{dt^{\nu-1}} \frac{P(t)}{(R_1(t))^{\nu}} \Bigg|_{t=\alpha_1}$$

This expression is now a rational expression in α_1 (and z') with coefficients in K, such that the residue at $t = \alpha_j$ is obtained simply by replacing α_1 by α_j everywhere. Therefore

$$\sum_{q_{1,k}(\alpha)=0} \operatorname*{res}_{\alpha} \frac{P(t)}{(q_{1,k}(t))^{n_k}}$$ is a rational function in $K(z')$.

Furthermore, we note that the portion of the denominator of $P(t)$ which depends on z' is $p(t,z')$. The expression (2.12) will have a common denominator which is $p(\alpha_1,z')^{\nu}$. Hence the inner sum of (2.11) has no poles for z' a zero of the product $q_2(z_2)\ldots q_n(z_n)$. The same thing holds therefore for the expression (2.11). Now we can iterate the procedure and conclude that $\langle\bar\partial\frac{1}{q},\Delta r dz\rangle \in K$. Hence $\langle\bar\partial\frac{1}{f},r dz\rangle \in K$.

□

<u>Remark 2.4</u>. Later on we will need a quantitative version of the fact that $\langle\bar\partial\frac{1}{f},r dz\rangle \in K$. For this purpose we will use the local character of the residue current $\bar\partial\frac{1}{f}$. That is, by using a partition of unity (φ_ζ) we have $\langle\bar\partial\frac{1}{f},r dz\rangle = \sum_{\zeta\in V} \langle\bar\partial\frac{1}{f},\varphi_\zeta r dz\rangle$, $\varphi_\zeta \equiv 1$ near ζ. We further can assume that ζ is the only zero of $V(q_1,\ldots,q_n)$ lying in the support of φ_ζ and that r is holomorphic on $\operatorname{supp} \varphi_\zeta$. Therefore for each term of this sum we

can apply the transformation law for residues without changing r at all, i.e.

$$\langle\bar\partial\frac{1}{f},r dz\rangle = \sum_{\zeta\in V}\langle\bar\partial\frac{1}{q},\Delta r\varphi_\zeta dz\rangle = \langle\bar\partial\frac{1}{q},\Delta r dz\rangle_V \ , \tag{2.13}$$

where we have introduced the last notation to indicate it is only the points of V that count. Note there are many less points in V than in $V(q_1,\ldots,q_n)$. In the first case one has at most D^n points, while in the second one might have as many as D^{n^2} points.

In Section 3 we will need the following results from [7] to compute residues.

<u>Theorem 2.5</u>. (cf. [7, Theorem 2.2]). Let $f_1,\ldots,f_n$ be n polynomials in $\mathbb{C}^n$ defining a discrete variety V, φ a test function which is holomorphic in a neighborhood of V, m an n-tuple of non-negative integers. Then the function defined for $\operatorname{Re}\lambda$ sufficiently large by

$$\lambda \mapsto \frac{(-1)^{\frac{n(n-1)}{2}}}{(2\pi i)^n}\, n\lambda \int \frac{|F|^{2(n+|m|)\lambda}}{\|f\|^{2(n+|m|)}}\,\bar f^m\,\overline{\partial f}\wedge\varphi\, d\zeta \tag{2.14}$$

has an analytic continuation to the whole plane as a meromorphic function. Moreover, this continuation is holomorphic at $\lambda = 0$ and its value at this point is given by

$$\frac{m!}{(n+|m|)!}\langle\bar\partial\frac{1}{f^{m+\underline{1}}},\varphi\ d\zeta\rangle, \tag{2.15}$$

where $m! = m_1!\ldots m_n!$, $m + \underline{1} = (m_1+1,\ldots,m_n+1)$.

§ 3. Division Formulas

The division formula we obtain here generalizes our previous representation formulas for solutions of the algebraic Bezout equation. We had originally considered them from the point of view of deconvolution (cf. [3], [5], [6]). The same techniques can be applied to entire functions, but to simplify we will only consider the algebraic case [7].

Throughout this section we will assume we have M polynomials $p_1,\dots,p_M \in \mathbb{C}[z]$ such that

(3.1) $$M \geq n,$$

(3.2) $\exists\, \varkappa > 0$, $c > 0$ and $d > 0$ such that when $\|\zeta\| \geq \varkappa$ we have

$$\left(\sum_{j=1}^{n} |p_j(\zeta)|^2\right)^{1/2} \geq c\|\zeta\|^d,$$

(3.4) $$\max_{1\leq j\leq n} (\deg p_j) = D.$$

We will adopt the notation $f = (f_1,\dots,f_n) = (p_1,\dots,p_n)$, hence (3.2) can be written as $\|f(\zeta)\| \geq c\|\zeta\|^d$ and it implies that the variety $V = V(f)$ is discrete.

For every polynomial p_j we can find polynomials $g_{j,k}$ in $2n$ variables, of degree $\leq \deg p_j - 1$ in each variable, such that for every $z,\zeta \in \mathbb{C}^n$ we have

(3.4) $$p_j(z) - p_j(\zeta) = \sum_{k=1}^{n} g_{j,k}(z,\zeta)(z_k-\zeta_k).$$

For instance we can take

$$g_{j,k}(z,\zeta) = \int_0^1 \frac{\partial p_j}{\partial \zeta_k} (\zeta+t(z-\zeta))\ dt.$$

If $p_j \in \mathbb{Z}[z]$, $\deg p_j = D_j$, then with this choice of $g_{j,k}$ we have $D_j! g_{j,k} \in \mathbb{Z}[z]$ and $h(D_j! g_{j,k}) \le h(p_j) + \log D_j!$.

Theorem 3.1. Assume (3.1), (3.2), and (3.3) hold. Let P be a polynomial in $I(p_1,\ldots,p_M)$ and let $u_1,\ldots,u_M$ by any functions holomorphic in a neighborhood Ω of V such that

$$P = u_1 p_1 + \ldots + u_M p_M \quad \text{in } \Omega. \tag{3.5}$$

Then for $q \in \mathbb{N}$ satisfying

$$dq > \deg P + (n-1)(2D-d) + 1, \tag{3.6}$$

and for any $z \in \mathbb{C}^n$ we have

$$P(z) = \sum_{|m| \le q-n} \left\langle \bar\partial \frac{1}{f^{m+\underline{1}}}, \sum_{j=1}^{M} u_j \begin{vmatrix} g_{1,1}(z,\cdot) & \ldots\ g_{n,1}(z,\cdot) & g_{j,1}(z,\cdot) \\ \vdots & & \\ g_{1,n}(z,\cdot) & \ldots\ g_{n,n}(z,\cdot) & g_{j,n}(z,\cdot) \\ f_1(z)-f_1(\cdot) & \ldots\ f_n(z)-f_n(\cdot) & p_j(z) \end{vmatrix} d\zeta \right\rangle f^m(z)$$

where $m \in \mathbb{N}^n$, $m + \underline{1} = (m_1+1, m_2+1, \ldots, m_n+1)$, and the dot in the determinant represents the variable ζ on which the residue current $\bar\partial \frac{1}{f^{m+\underline{1}}}$ acts.

Remark 3.2. (i) The only term in the sum (3.7) that a priori might not belong to $I(p_1,\ldots,p_M)$ is that one corresponding to $m = (0,\ldots,0)$. In that case the development of the determinants along the last row shows that either one has a multiple of $p_j(z)$ for some j, $1 \le j \le M$, or a multiple of $f_j(\zeta)$ for some j, $1 \le j \le n$. This last type of term vanishes since $\bar{\partial}\frac{1}{f}$ annihilates the ideal generated by the f_j. Therefore (3.7) has the form

$$P(z) = A_1(z)p_1(z) + \dots + A_M(z)p_M(z).$$

(ii) In the case $M = n + 1$ and $V(p_1,\dots,p_M) = \emptyset$ this theorem improves upon Theorem 3 [6] and its applications in [3].

(iii) Note that the conditions (3.5) and $P \in I(p_1,\dots,p_M)$ are equivalent by Cartan's Theorem B [15].

<u>Example 3.3</u>. Let $M = n + 1$, $V(p_1,\dots,p_{n+1}) = \emptyset$, $p_j \in \mathbb{Z}[z]$. For $P = 1$ we can take $u_1 = \dots = u_n = 0$, $u_{n+1} = \frac{1}{p_{n+1}}$. In that case Lemma 2.3 implies that (3.7) gives a Bezout formula in $\mathbb{Q}[z]$, that is

$$1 = p_1(z)A_1(z) + \dots + p_{n+1}(z)A_{n+1}(z)$$

with $A_j \in \mathbb{Q}[z]$. Note that the result remains true if $\mathbb{Q}$ is replaced by a number field K and $\mathbb{Z}$ by the field of integers $\mathcal{O}_K$ of K.

<u>Proof of Theorem 3.1</u>. The germ of the idea of this proof goes back to our paper on deconvolution [5], [6] except that here we have to deal inevitably with multiple zeros in V. In the recent past we have found that the best way to deal with this

question is through the principle of analytic continuation of the distributions $|f|^{2m\lambda}$ as functions of λ [7]. We also use the recent work of Andersson-Passare on integral representation formulas [2].

Let us fix once and for all $\vartheta \in \mathcal{D}(\Omega)$, $\vartheta = 1$ in a neighborhood of V.

Let $\rho > 1$ so that $\Omega_\rho = \{\zeta \in \mathbb{C}^n : \|\zeta\| < \rho\} \supseteq \operatorname{supp} \vartheta \cup \{z\}$. Let $\chi \in \mathcal{D}(\Omega_\rho)$ such that $\chi = 1$ in a neighborhood of $\operatorname{supp} \vartheta \cup \{z\}$, $0 \le \chi \le 1$.

Consider the differential form $Q_0 = Q_0(z,\zeta)$ given by

$$(3.8) \qquad Q_0 := (1-\chi(\zeta)) \frac{\sum_{j=1}^{n} (\bar{\zeta}_j - \bar{z}_j)\, d\zeta_j}{\|\zeta - z\|^2} .$$

If ω is an open set such that $z \in \omega$ and $\chi = 1$ on ω then Q_0 is C^∞ in $\omega \times \mathbb{C}^n$. Let

$$(3.9) \qquad \Gamma_0(t) = (1+t)^N,$$

with N any integer $> n$.

For $\lambda \in \mathbb{C}$, $\operatorname{Re} \lambda > 1 + \frac{1}{n}$, let $Q_1 = Q_1(z,\zeta,\lambda)$ be the differential form (with the notation of § 1):.

$$(3.10) \qquad Q_1 := |F(\zeta)|^{2\lambda} \frac{\sum_{j=1}^{n} \bar{f}_j(\zeta) G_j}{\|f(\zeta)\|^2},$$

where the differential forms $G_j = G_j(z,\zeta)$ are given by

$$G_j := \sum_{k=1}^{n} g_{j,k}\, d\zeta_k. \tag{3.11}$$

The coefficients of G_j are therefore polynomials in z and ζ. Q_1 is of class C^1 and polynomial in z. If we let $\mathrm{Re}\,\lambda >> 1$ we can make Q_1 of class C^ℓ for any ℓ given. With q as in (3.6) let

$$\Gamma_1(t) = (1 + t)^q. \tag{3.12}$$

Finally, define a third differential form $Q_2 = Q_2(z,\zeta)$ by

$$Q_2 := \vartheta(\zeta) \sum_{j=1}^{M} u_j(\zeta) G_j. \tag{3.13}$$

$$\Gamma_2(t) := t. \tag{3.14}$$

These three differential forms are of type $(1,0)$ in ζ, hence they can be associated to $\mathbb{C}^n$-valued functions, simply take the coefficient of $d\zeta_j$ as its jth component. Using their bilinear products with the vector valued function $z - \zeta$ we can construct three auxiliary functions Φ_j. We have

$$\Phi_0 := \langle Q_0(z,\zeta), z-\zeta \rangle = \tag{3.15}$$

$$= \frac{(1-\chi(\zeta))}{\|\zeta - z\|^2} \sum_{j=1}^{n} (\bar\zeta_j - \bar z_j)(z_j - \zeta_j) = \chi(\zeta) - 1,$$

$$(3.16) \qquad \Phi_1 := \langle Q_1(z,\zeta,\lambda), z-\zeta \rangle$$

$$= \frac{|F(\zeta)|^{2\lambda}}{\|f(\zeta)\|^2} \sum_{j=1}^{n} \overline{f_j}(\zeta) \langle G_j, z-\zeta \rangle$$

$$= \frac{|F(\zeta)|^{2\lambda}}{\|f(\zeta)\|^2} \sum_{j=1}^{n} \overline{f_j}(\zeta) \left[\sum_{k=1}^{n} g_{j,k}(z,\zeta)(z_j-\zeta_j) \right]$$

$$= \frac{|F(\zeta)|^{2\lambda}}{\|f(\zeta)\|^2} \sum_{j=1}^{n} \overline{f_j}(\zeta) \left[f_j(z) - f_j(\zeta) \right],$$

by (3.3).

The last one is given by

$$(3.17) \qquad \Phi_2 := \langle Q_2(z,\zeta), z-\zeta \rangle + P(\zeta)$$

$$= \vartheta(\zeta) \sum_{j=1}^{M} u_j(\zeta) \left[p_j(z) - p_j(\zeta) \right] + P(\zeta).$$

Note that in a neighborhood of V we have $\Phi_2 = \sum_{j=1}^{M} u_j(\zeta) p_j(z)$.

As a function of ζ consider the product

$$(3.18) \qquad \zeta \longmapsto \varphi = \Gamma_0(\Phi_0)\Gamma_1(\Phi_1)\Gamma_2(\Phi_2),$$

for z fixed and λ fixed, $\operatorname{Re} \lambda >> 1$, this is a C^{n+1} function of compact support since $\Gamma_0(\Phi_0) = \chi(\zeta)^N$. Furthermore

$$(3.19) \qquad \varphi(z) = P(z).$$

We need one more piece of notation: for $0 \leq j \leq 2$, and α a non-negative integer denote

$$(3.20) \qquad \Gamma_j^{(\alpha)} = \Gamma_j^{(\alpha)}(z,\zeta) := \frac{d^\alpha}{dt^\alpha}\Gamma_j(t)\Big|_{t=\Phi_j(z,\zeta)} .$$

(Recall that Φ_1 depends also on λ.)

The following lemma will allow us to compute $P(z)$ with the help of Cauchy's formula (2.7) applied to φ (cf. [2]). Its proof will be postponed to the end of the proof of Theorem 2.1.

Lemma 3.2. With the above notation we have, for $\mathrm{Re}\,\lambda \gg 1$,

$$P(z) = \frac{1}{(2\pi i)^n}\int_{\Omega_\rho} \Phi_2(z,\zeta) \sum_{\alpha_0+\alpha_1=n} \frac{\Gamma_0^{(\alpha_0)}\Gamma_1^{(\alpha_1)}}{\alpha_0!\alpha_1!}\left(\bar\partial_\zeta Q_0(z,\zeta)\right)^{\alpha_0}\wedge\left(\bar\partial_\zeta Q_1(z,\zeta,\lambda)\right)^{\alpha_1}$$

(3.21)

$$+\frac{1}{(2\pi i)^n}\int_{\Omega_\rho} \sum_{\alpha_0+\alpha_1=n-1} \frac{\Gamma_0^{(\alpha_0)}\Gamma_1^{(\alpha_1)}}{\alpha_0!\alpha_1!}\left(\bar\partial_\zeta Q_0(z,\zeta)\right)^{\alpha_0}\wedge\left(\bar\partial_\zeta Q_1(z,\zeta,\lambda)\right)^{\alpha_1}\wedge\bar\partial_\zeta Q_2(z,\zeta).$$

The next step will be to study the analytic continuation of this formula as a function of λ. For that purpose, we compute explicitly $(\bar\partial_\zeta Q_1)^\alpha$, $1 \le \alpha \le n$, always for $\mathrm{Re}\,\lambda \gg 1$. To simplify we simply write $\bar\partial$ for $\bar\partial_\zeta$. Let us write first

$$A = \frac{\sum_1^n \bar f_j G_j}{\|f\|^2}, \quad Q_1 = |F|^{2\lambda}A.$$

Then

$$(3.22)\qquad (\bar\partial Q_1)^k = |F|^{2\lambda k}(\bar\partial A)^k + \lambda k|F|^{2(k\lambda-1)}F\ \bar\partial \bar F\wedge A\wedge(\bar\partial A)^{k-1}.$$

A is C^∞ form off the variety V. The form $(\bar\partial Q_1)^n$ can be written in a slightly different way by writing $Q_1 = \sum_{j=1}^{n} \psi_j G_j$, $\psi_j = \frac{|F|^{2\lambda}}{\|f\|^2}\bar f_j$. Then

$$(\bar\partial Q_1)^n = \left[\sum_{j=1}^{n} \bar\partial\psi_j\wedge G_j\right]^n = (-1)^{\frac{(n-1)n}{2}}\, n!\bigwedge_{j=1}^{n}\bar\partial\psi_j \wedge \bigwedge_{j=1}^{n} G_j.$$

We have

$$\bar\partial\psi_j = \frac{|F|^{2\lambda}}{\|f\|^2}\bar\partial\bar f_j + \bar f_j\left[\frac{\lambda|F|^{2(\lambda-1)}}{\|f\|^2}F\bar\partial\bar F - \frac{|F|^{2\lambda}}{\|f\|^4}\bar\partial\|f\|^2\right].$$

Hence

$$\bigwedge_{j=1}^{n}\bar\partial\psi_j = \frac{|F|^{2\lambda n}}{\|f\|^{2n}}\bar\partial\bar f + \lambda\frac{|F|^{2(\lambda n-1)}}{\|f\|^{2n}}F\sum_{j=1}^{n}\bar f_j\bigwedge_{k<j}\bar\partial\bar f_k\wedge\bar\partial\bar F\wedge\bigwedge_{j<k}\bar\partial\bar f_k$$

$$- \frac{|F|^{2\lambda n}}{\|f\|^{2(n+1)}}\sum_{j=1}^{n}\bar f_j\bigwedge_{k<j}\bar\partial\bar f_k\wedge\bar\partial\|f\|^2\wedge\bigwedge_{j<k}\bar\partial\bar f_k$$

Note that $\bar\partial\bar F = \sum(\bar F/\bar f_h)\bar\partial\bar f_h$ and $\bar\partial\|f\|^2 = \sum f_h\bar\partial\bar f_h$. Since $\bar\partial\bar f_k\wedge\bar\partial\bar f_k = 0$ we have

$$(3.23)\qquad \bigwedge_1^n\bar\partial\psi_j = \frac{|F|^{2\lambda n}}{\|f\|^{2n}}\bar\partial\bar f + n\lambda\frac{|F|^{2\lambda n}}{\|f\|^{2n}}\bar\partial\bar f - \frac{|F|^{2\lambda n}}{\|f\|^{2(n+1)}}\|f\|^2\,\bar\partial\bar f$$

$$= n\lambda\frac{|F|^{2\lambda n}}{\|f\|^{2n}}\bar\partial\bar f.$$

$$(3.24)\qquad (\bar\partial Q_1)^n = (-1)^{\frac{(n-1)n}{2}} n!n\lambda \frac{|F|^{2\lambda n}}{\|f\|^{2n}} \overline{\partial f} \wedge \bigwedge_1^n G_j .$$

Following the principle we introduced in [5] we have to transform, using Stokes' theorem, some terms in (3.21) to make them more singular. In this case consider in the second term of (3.21) the term with $\alpha_0 = 0$, $\alpha_1 = n - 1$. then

$$\bar\partial\left(\Gamma_0^{(0)}\Gamma_1^{(n-1)}(\bar\partial Q_1)^{n-1}\wedge Q_2\right) = \Gamma_0^{(0)}\Gamma_1^{(n-1)}(\bar\partial Q_1)^{n-1}\wedge \bar\partial Q_2$$

$$+ \Gamma_0^{(1)}\Gamma_1^{(n-1)}\bar\partial\chi \wedge (\bar\partial Q_1)^{n-1}\wedge Q_2 + \Gamma_0^{(0)}\Gamma_1^{(n)}\bar\partial\Phi_1 \wedge (\bar\partial Q_1)^{n-1}\wedge Q_2 .$$

Recall that $\Gamma_0^{(0)}$ has compact support in Ω_ρ and that $\bar\partial\chi \wedge Q_2 = 0$, since $\chi = 1$ on supp ϑ. Therefore

$$\int_{\Omega_\rho} \Gamma_0^{(0)}\Gamma_1^{(n-1)}(\bar\partial Q_1)^{n-1}\wedge \bar\partial Q_2 = - \int_{\Omega_\rho} \Gamma_0^{(0)}\Gamma_1^{(n)}\bar\partial\Phi_1 \wedge (\bar\partial Q_1)^{n-1}\wedge Q_2 .$$

To simplify the computation of this last integral let us introduce polynomials $\Delta_{j,l}$, $1 \le j \le n$, $1 \le l \le M$, by

$$(3.25)\qquad G_1 \wedge \dots \wedge \hat G_j \wedge \dots \wedge G_n \wedge G_l = \Delta_{j,l}\, d\zeta , \quad \text{and}$$

$$(3.26)\qquad G_1 \wedge \dots \wedge G_n = \Delta_0\, d\zeta .$$

Now we can compute the integrand above as follows

$$- \bar\partial\Phi_1 \wedge (\bar\partial Q_1)^{n-1}\wedge Q_2 = -\left(\sum_{j=1}^n \left(f_j(z)-f_j(\zeta)\right)\bar\partial\psi_j\right) \wedge \left(\sum_1^n \bar\partial\psi_j \wedge G_j\right)^{n-1}\wedge Q_2$$

$$= (-1)^{\frac{(n-2)(n-1)}{2}} (n-1)! \left[\bigwedge_1^n \bar\partial \psi_j\right] \wedge \sum_{j=1}^{n} (-1)^j \left(f_j(z) - f_j(\zeta)\right) \bigwedge_{k \neq j}^{n} G_k \wedge Q_2$$

$$= (-1)^{\frac{(n-2)(n-1)}{2}} n! \lambda \frac{|F|^{2\lambda n}}{\|f\|^{2n}} \bar\partial \bar f \wedge \left[\sum_{j=1}^{n} \sum_{l=1}^{M} (-1)^j \left(f_j(z) - f_j(\zeta)\right) \Delta_{j,l}(z,\zeta) \vartheta(\zeta) u_l(\zeta)\right] d\zeta$$

Note that we have already computed in (3.24) the term with $\alpha_0 = 0$ in the first integral of (3.21). Let us write now (3.21) as a sum of the contributions from $\alpha_0 = 0$ in both integrals and the other terms put together:

(3.27) $$P(z) = \frac{(-1)^{\frac{(n-1)n}{2}}}{(2\pi i)^n} n\lambda \int_{\Omega_\rho} \Gamma_0^{(0)} \Gamma_1^{(n)} \frac{|F|^{2\lambda n}}{\|f\|^{2n}} \bar\partial \bar f \wedge$$

$$\left[\Phi_2 \Delta_0 + (-1)^{n-1} \vartheta \left[\sum_{j,l} (-1)^j \left(f_j(z) - f_j\right) \Delta_{j,l} u_l\right]\right] d\zeta$$

$$+ R(\lambda, z)$$

Let us call $T(z,\zeta)$ the term between brackets in (3.27). Let us show that in the set where $\vartheta = 1$ this term is exactly the determinant that appears in the final formula (3.7). First we observe that since on supp ϑ we have $P(\zeta) = \sum_{j=1}^{M} u_j(\zeta) p_j(\zeta)$ then

$$\Phi_2(z,\zeta) = \sum_{j=1}^{M} u_j(\zeta) p_j(z).$$

Now we can expand the determinants in (3.7) by the last row and obtain $T(z,\zeta)$. This function is therefore holomorphic on ζ in a neighborhood of V.

To evaluate (3.27) we will use the fact that both terms are holomorphic functions of λ for $\mathrm{Re}\,\lambda >> 1$ and they have analytic continuations to the whole plane as meromorphic functions. We will further see that they are both holomorphic at $\lambda = 0$, hence $P(z)$ will appear as $\lim_{\lambda\to 0} R(\lambda,z) + \lim_{\lambda\to 0}$ (of the first term in (3.27)).

We proceed now to verify these statements for the first term of (3.27). We have

$$\Gamma_1^{(n)} = \frac{q!}{(q-n)!}\,(\Phi_1+1)^{q-n} = \frac{q!}{(q-n)!}\left[(1-|F|^{2\lambda}) + \sum_1^n \psi_j(\zeta)f_j(z)\right]^{q-n}$$

$$= \frac{q!}{(q-n)!}\sum_{k=0}^{q-n}\binom{q-n}{k}(1-|F|^{2\lambda})^{q-n-k}\frac{|F|^{2\lambda k}}{\|f\|^{2k}}\sum_{|m|=k}\frac{k!}{m!}\bar{f}^m(\zeta)f^m(z)$$

$$= \frac{q!}{(q-n)!}\sum_{k=0}^{q-n}\binom{q-n}{k}\frac{|F|^{2\lambda k}}{\|f\|^{2k}}\left[\sum_{j=0}^{q-n-k}\binom{q-n-k}{j}(-1)^j|F|^{2\lambda j}\right]\left[\sum_{|m|=k}\frac{k!}{m!}\bar{f}^m(\zeta)f^m(z)\right].$$

In order to apply Theorem 2.5, we fix a k, a multi-index m $|m| = k$, and an index j in the expansion of $\Gamma_1^{(n)}$. The corresponding term in (3.27) is then, up to a factor $f^m(z)$,

$$(3.28)\qquad \frac{(-1)^{\frac{n(n-1)}{2}}}{(2\pi i)^n}\, n\lambda\, c_{k,m,j}\int_{\Omega_\rho}\frac{|F|^{2\lambda(j+k+n)}}{\|f\|^{2(n+|m|)}}\bar{f}^m\,\overline{\partial f}\wedge(\chi^N T)\,d\zeta,$$

where $c_{k,m,j} = (-1)^j\frac{q!}{(q-n)!}\frac{k!}{m!}\binom{q-n}{k}\binom{q-n-k}{j}$. Replacing λ by

$\left(\frac{n+|m|}{j+k+n}\right)\lambda$, we are in the situation of (2.14) up to the new constant $c'_{k,m,j} = \left(\frac{n+k}{n+k+j}\right) c_{k,m,j}$. (Note $\chi = 1$ in a neighborhood of V, hence $\chi^N T$ is holomorphic there.) Therefore, by Theorem 2.5, the analytic continuation exists, it is holomorphic at $\lambda = 0$ and its value at this point is

$$(3.29) \qquad c'_{k,m,j} \frac{m!}{(n+|m|)!} \langle \bar{\partial} \frac{1}{f^{m+\underline{1}}}, \chi^N T \, d\zeta \rangle .$$

Note that the value in (3.29) is independent of the choice of χ. We need to evaluate the constant obtained by adding over all values of j.

$$(3.30) \qquad c_{k,m} = \sum_{j=0}^{q-n-k} c'_{k,m,j} = \sum_{j=0}^{q-n-k} \left(\frac{n+k}{n+k+j}\right) c_{k,m,j}$$

$$= \frac{q!}{(q-n)!} \frac{k!}{m!} \binom{q-n}{k} \sum_{j=0}^{q-n-k} (-1)^j \binom{q-n-k}{j} \frac{m+k}{n+k+j} .$$

This sum can be computed in terms of the Beta function. Namely,

$$\sum_{j=0}^{q-n-k} (-1)^j \binom{q-n-k}{j} \frac{1}{n+k+j} = \int_0^1 (1-u)^{q-n-k} u^{n+k-1} \, du$$

$$= B(n+k, q-n-k+1) = \frac{(n+k-1)!(q-n-k)!}{q!} .$$

We find

$$(3.31) \qquad \frac{m!}{(n+|m|)!} c_{k,m} = 1 .$$

Therefore, the value at $\lambda = 0$ of the first term in (3.27) is exactly the right hand side of (3.7). We stress once more the value we obtained is independent of the choice of χ.

To end the proof we need to study the analytic continuation of $R(\lambda, z)$ and evaluate it at $\lambda = 0$. Recall we work first with $\operatorname{Re} \lambda > 1 + \frac{1}{n}$. In $R(\lambda, z)$ we have all terms (3.21) where $\alpha_0 > 0$. Introducing the auxiliary differential forms

$$S = \sum_{j=1}^{n} (\overline{\zeta}_j - \overline{z}_j) d\zeta_j, \qquad \overline{S} = \sum_{j=1}^{n} (\zeta_j - z_j) d\overline{\zeta}_j$$

we have

$$(3.32) \qquad \overline{\partial} Q_0 = -\frac{\overline{\partial}\chi \wedge S}{\|\zeta - z\|^2} + (1-\chi)\left[\frac{\sum_{j=1}^{n} d\overline{\zeta}_j \wedge d\zeta_j}{\|\zeta - z\|^2} - \frac{\overline{S} \wedge S}{\|\zeta - z\|^4}\right].$$

This shows that $\overline{\partial} Q_0$ is identically zero in a neighborhood of $\operatorname{supp} \vartheta \cup \{z\}$ by the conditions imposed on χ. Since there is a factor ϑ in Q_2, it follows that all the terms with $\alpha_0 > 0$ in the second integral of (3.21) are identically zero.

Consider now the term with $\alpha_0 = n$ in the first integral. Let us rewrite first

$$(3.33) \qquad \Phi_1 + 1 = 1 - |F|^{2\lambda} + |F|^{2\lambda} \sum_{j=1}^{n} \theta_j f_j(z)$$

$$= 1 - |F|^{2\lambda} + |F|^{2\lambda} B,$$

where $\theta_j = \theta_j(\zeta) = \overline{f}_j(\zeta)/\|f(\zeta)\|^2$. On the support of $\overline{\partial} Q_0$ we have that B is C^∞, since $\chi = 1$ on a neighborhood of the singular points of $\|f(\zeta)\|^{-2}$, namely V. Since F is a polynomial, it follows (for instance by the Weierstrass'

Preparation Theorem or Hironaka's Resolution of Singularities) that on the ball $\overline{\Omega}_\rho$ we have that $|F|^{-\varepsilon}$ is integrable for some $\varepsilon > 0$. Whence, this term with $\alpha_0 = n$ which is given by

$$(3.34)\qquad \frac{1}{(2\pi i)^n}\begin{bmatrix}N\\n\end{bmatrix}\int_{\Omega_\rho} \Phi_2\chi^{N-n}(1+\Phi_1)^q\,(\bar\partial Q_0)^n$$

for $\operatorname{Re}\lambda > 1 + \frac{1}{n}$ and depends on λ only in the term $(1+\Phi_1)^q$, is holomorphic for $\operatorname{Re}\lambda > -\varepsilon$. Its value at $\lambda = 0$ is obtained simply by taking $\lambda = 0$ in the expression of Φ_1. That is, the value at $\lambda = 0$ of (3.34) is

$$(3.35)\qquad \frac{1}{(2\pi i)^n}\begin{bmatrix}N\\n\end{bmatrix}\int_{\Omega_\rho} \Phi_2\chi^{N-n}B^q(\bar\partial Q_0)^n.$$

We now have left the case $0 < \alpha_0 < n$, $\alpha_1 = n - \alpha_0$, to consider. By (3.22) we have $(\bar\partial Q_1)^{\alpha_1}$ as the sum of two terms. We study first the one that does not contain the factor λ. As we have just shown, A is smooth on the support of $\bar\partial Q_0$ and the whole integral is holomorphic for $\lambda = 0$. Its value, obtained by simply setting $\lambda = 0$, is the following

$$(3.36)\qquad \frac{1}{(2\pi i)^n}\begin{bmatrix}N\\\alpha_0\end{bmatrix}\begin{bmatrix}q\\\alpha_1\end{bmatrix}\int_{\Omega_\rho} \Phi_2\chi^{N-\alpha_0}B^{q-\alpha_1}(\bar\partial Q_0)^{\alpha_0}\wedge(\bar\partial A)^{\alpha_1}.$$

The other term can be written as a linear combination of integrals of the form

$$\lambda \int_{\Omega_\rho} |F|^{2(p\lambda-1)} F\, \overline{\partial F} \wedge C,$$

p an integer $\geq \alpha_1$, C a smooth form of compact support. By Theorem 1.3 [7], this function has an analytic continuation as a meromorphic function of λ, whose value at $\lambda = 0$ is, up to multiplicative constants

$$\langle \bar{\partial}\tfrac{1}{F}, FC \rangle$$

which is the residue on the hypersurface $F = 0$. Since F appears in the test form FC, this residue is zero.

At this point we can summarize what we have just done by saying that $\lambda \mapsto R(\lambda,z)$ has an analytic continuation which is holomorphic at $\lambda = 0$, and

$$(3.37) \qquad R_0 = R(\lambda,z)|_{\lambda=0} =$$

$$= \frac{1}{(2\pi i)^n} \int_{\Omega_\rho} \sum_{j=1}^{n} \binom{N}{n} \binom{q}{n-j} \Phi_2 \chi^{N-j} B^{q-(n-j)} (\bar{\partial} Q_0)^j \wedge (\bar{\partial} A)^{n-j}.$$

By now we are essentially in the same situation as in the new Andersson-Passare proof of the Andersson-Bendtsson integral representation formula (cf. formula (6), proof of Theorem 2, [2]). They show we can let χ tend to the characteristic function of Ω_ρ and use the fact that for a smooth form φ, and p integral ≥ 1, one has

$$\int_{\Omega_\rho} \bar\partial \chi^p \wedge \varphi \longrightarrow -\int_{\partial\Omega_\rho} \varphi .$$

Since $B = (\Phi_1+1)|_{\lambda=0}$ and $A = Q_1|_{\lambda=0}$, the formula (3.37) is just the boundary term in the Andersson-Berndtsson formula for the single pair (A,t^q) (cf.[2],[7]):

$$(3.38) \qquad R_0 = \frac{1}{(2\pi i)^n} \int_{\partial\Omega_\rho} \sum_{j=0}^{n-1} \Phi_2 \frac{1}{j!} \binom{q}{n-1-j} B^{q+j+1-n} \frac{S\wedge(\bar\partial S)^j \wedge (\bar\partial A)^{n-1-j}}{\|\zeta-z\|^{2(j+1)}},$$

where $\bar\partial = \bar\partial_\zeta$.

The last step of the proof is to verify that the estimates on A, B we can obtain from the hypotheses are enough to let $\rho \longrightarrow \infty$ in (3.38).

Since $\|f(\zeta)\| \geq c\|\zeta\|^d$ if $\|\zeta\| \geq \varkappa$ we have that for $\rho > \varkappa$ the following estimates hold:

$$|B| \leq \text{const.}\ \|\zeta\|^{-d},$$

$$|\text{largest coefficient of } (\bar\partial A)^{n-1-j}| \leq \text{const.}\ \|\zeta\|^{2(D-d-1)(n-1-j)}.$$

Furthermore $\Phi_2 = P$ on $\partial\Omega_\rho$. If follows that the worst term in the sum corresponds to $j = 0$. From this we conclude that since

$$(3.39) \qquad \deg P + (n-1)(2D-d) + 1 < dq$$

the integral in (3.38) tends to zero when $\rho \longrightarrow \infty$.

This concludes the proof of Theorem 3.1, except for the proof of Lemma 3.2.

□

Proof of Lemma 3.2. From its definition (3.18) and (3.19) we have a C^{n+1} function φ of compact support in Q_ρ which for a fixed z satisfies $\varphi(z) = P(z)$. Cauchy's formula (2.7) states that

$$\langle \bar\partial \frac{1}{\zeta - z}, \varphi(\zeta)d\zeta \rangle = \varphi(z) = P(z). \tag{3.40}$$

The proof of this lemma consists in evaluating the residue in the left hand side of (3.40) using the particular form of φ. It simplifies the computation of this residue to consider the slightly more general form of φ:

$$\varphi(\zeta) = \Gamma(\zeta, \langle Q(z,\zeta), z-\zeta\rangle), \tag{3.41}$$

where Γ is an entire function of $n + \nu$ variables (ζ, t), $Q = (Q_1, \ldots, Q_\nu)$ a vector of $(1,0)$-differential forms in ζ, of class C^{n+1}, $\langle Q, z-\zeta\rangle := (\langle Q_1, z-\zeta\rangle, \ldots, \langle Q_\nu, z-\zeta\rangle)$. For a multi-index α of ν components, we write, as above,

$$\Gamma^{(\alpha)} := D_1^{\alpha_1} \ldots D_\nu^{\alpha_\nu}\Gamma := \frac{\partial^\alpha}{\partial t^\alpha} \Gamma \Big|_{t=\langle Q(z,\zeta), z-\zeta\rangle}. \tag{3.42}$$

From (2.3) we have $\left[c_n = \dfrac{(-1)^{\frac{(n-1)n}{2}}}{(2\pi i)^n}\right]$:

$$\langle \bar\partial \frac{1}{\zeta - z}, \varphi(\zeta)d\zeta \rangle = \lim_{\mu\to 0} c_n \mu^n \int \left|\prod_1^n (\zeta_j - z_j)\right|^{2(\mu-1)} \varphi(\zeta)\, d\bar\zeta \wedge d\zeta .$$

We compute the analytic continuation of this integral which is originally defined for $\operatorname{Re} \mu > 0$.

One can easily verify that:

$$d\left[\frac{\mu^{n-1}}{(\zeta_1-z_1)}|\zeta_1-z_1|^{2\mu}\left|\prod_2^n(\zeta_j-z_j)\right|^{2(\mu-1)}\varphi(\zeta)d\bar\zeta_2\wedge\ldots\wedge d\bar\zeta_n\wedge d\zeta\right]$$

$$=\mu^n\left|\prod_1^n(\zeta_j-z_j)\right|^{2(\mu-1)}\varphi(\zeta)d\bar\zeta\wedge d\zeta$$

$$+(-1)^{n-1}\frac{\mu^{n-1}}{\zeta_1-z_1}|\zeta_1-z_1|^{2\mu}\left|\prod_2^n(\zeta_j-z_j)\right|^{2(\mu-1)}d\bar\zeta_2\wedge\ldots\wedge d\bar\zeta_n\wedge\bar\partial\varphi\wedge d\zeta .$$

Here $d,\bar\partial$ are only computed with respect to ζ. Since the first term is the exact differential of a form of compact support, we have by Stokes' Theorem:

$$\int\mu^n\left|\prod_1^n(\zeta_j-z_j)\right|^{2(\mu-1)}\varphi(\zeta)d\bar\zeta\wedge d\zeta= \tag{3.43}$$

$$=(-1)^n\int\frac{\mu^{n-1}}{\zeta_1-z_1}|\zeta_1-z_1|^{2\mu}\left|\prod_2^n(\zeta_j-z_j)\right|^{2(\mu-1)}d\bar\zeta_2\wedge\ldots\wedge d\bar\zeta_n\wedge\bar\partial\varphi\wedge d\zeta .$$

From (3.41) we have

$$\bar\partial\varphi=\sum_{k=1}^{\nu}D_k\Gamma\left[\sum_{j=1}^{n}(z_j-\zeta_j)\bar\partial Q_{k,j}(z,\zeta)\right],$$

where we recall $Q_k=\sum_{j=1}^{n}Q_{k,j}d\zeta_j$. Let us rewrite $\bar\partial\varphi$ as follows

$$\bar\partial\varphi=-(\zeta_1-z_1)\sum_{k=1}^{\nu}D_k\Gamma\ \bar\partial Q_{k,1}+R_1. \tag{3.44}$$

The analytic continuation of the two separate terms obtained by replacing (3.44) into (3.43) exists by Theorem 1.3 [7]. The second one is a sum of integrals of the form: ($1 \le k \le \nu$, $2 \le i \le n$).

$$\mu^{n-1}\int D_k\Gamma\frac{|\zeta_1-z_1|^{2\mu}}{\zeta_1-z_1}(z_i-\zeta_i)\left|\prod_2^n(\zeta_j-z_j)\right|^{2(\mu-1)} d\overline{\zeta}_2\wedge\ldots\wedge d\overline{\zeta}_n\wedge\overline{\partial}Q_{k,i}\wedge d\zeta .$$

Since the two distributions $\frac{|\zeta_1-z_1|^{2\mu}}{\zeta_1-z_1}$, $\mu^{n-1}\left|\prod_2^n(\zeta_j-z_j)\right|^{2(\mu-1)}$ depend on different variables, their analytic continuations as distribution-valued meromorphic functions can be multiplied (this is just their tensor product). The first one is holomorphic for $\mu = 0$, the second one leads to the residue current $\overline{\partial}\frac{1}{\zeta_2-z_2}\wedge\ldots\wedge\overline{\partial}\frac{1}{\zeta_n-z_n}$. But the remaining differential form is in the ideal generated by the functions defining this current. Therefore the value of this product at $\mu = 0$ is null.

We can therefore forget R_1 and consider only

$$(-1)^{n-1}\mu^{n-1}\int|\zeta_1-z_1|^{2\mu}\left|\prod_2^n(\zeta_j-z_j)\right|^{2(\mu-1)} d\overline{\zeta}_2\wedge\ldots\wedge d\overline{\zeta}_n\wedge\left(\sum_{k=1}^{\nu} D_k\Gamma\ \overline{\partial}Q_{k,1}\right)\wedge d\zeta$$

(3.45)

In ([7], Proof of Theorem 1.3), we have shown, in a much more general situation, not only that the analytic continuation of (3.45) is holomorphic at $\mu = 0$ but its value is exactly the same as the one obtained from

$$(3.46) \qquad (-1)^{n-1}\mu^{n-1}\int \left|\prod_{2}^{n}(\zeta_j - z_j)\right|^{2(\mu-1)} d\bar{\zeta}_2\wedge\ldots d\bar{\zeta}_n\wedge\left[\sum_{k=1}^{\nu} D_k\Gamma\,\bar{\partial}Q_{k,1}\right]\wedge d\zeta$$

(This also follows from the above remark on the product of the distributions of separate variables). It is clear now what the general procedure is, the only point to verify is that the factor $(z_1-\zeta_1)$ does not reappear when we apply Stokes' theorem. For this, it is enough to compute $\sum_{k=1}^{\nu}(\bar{\partial}D_k\Gamma\wedge\bar{\partial}Q_{k,1})$.

$$\sum_{k=1}^{\nu}\bar{\partial}D_k\Gamma\wedge\bar{\partial}Q_{k,1} = \sum_{k=1}^{\nu}\sum_{j=1}^{\nu} D_jD_k\Gamma\left[\sum_{i=1}^{n}(z_i-\zeta_i)\bar{\partial}Q_{j,i}\right]\wedge\bar{\partial}Q_{k,1}.$$

The term $(z_1-\zeta_1)$ is the coefficient of $\sum_{j,k=1}^{\nu} D_jD_k\Gamma\,\bar{\partial}Q_{j,1}\wedge\bar{\partial}Q_{k,1} = 0$ by the anticommutativity of the wedge product.

After iterating this procedure a total of n times, and some algebra, one obtains, $(\alpha = (\alpha_1,\alpha_2,\ldots,\alpha_\nu),\ \alpha! = \alpha_1!\ldots\alpha_\nu!)$

$$\langle\bar{\partial}\frac{1}{\zeta-z},\varphi(\zeta)d\zeta\rangle = \frac{1}{(2\pi i)^n}\int\sum_{|\alpha|=n}\frac{1}{\alpha!}\Gamma^{(\alpha)}(\bar{\partial}Q_1)^{\alpha_1}\wedge\ldots\wedge(\bar{\partial}Q_\nu)^{\alpha_\nu}.$$

Note that $\bar{\partial}Q_j$ are $(1,1)$ forms which absorb the $d\zeta$ term from (3.44). For a detailed version of this algebraic computation see ([2], Proof of Theorem 1). The statement of the Lemma follows from the explicit form of Γ in this case, we just use that $D_3^2\Gamma = 0$. □

As a first application of Theorem 3.1, we will show the impossibility of the Mayr-Meyer [22] example (1.12) for discrete varieties as mentioned in the Introduction.

<u>Theorem 3.3</u>. Let $p_1,\dots,p_M \in \mathbb{C}[z]$ defining a discrete variety V_0, $V_0 = \{z \in \mathbb{C}^n: p_j(z) = 0, \quad 1 \le m \le M\}$. Let $P \in \mathbb{C}[z]$ which belongs to the ideal I generated by $p_1,\dots,p_M$. Let $D = \max(\deg p_j \ (1 \le j \le M), \deg P)$. Then there are polynomials $A_1,\dots,A_M$ such that $P = p_1A_1 + \dots + p_MA_M$ and

$$(3.47) \qquad \max \deg A_j \le (n+1)^3 D^{3n+1}.$$

<u>Proof</u>. If $V_0 = \emptyset$ then by [8] there are polynomials q_j, $\max \deg q_j \le \mu n D^{\mu} + \mu D$, $\mu = \inf\{n,M\}$ such that

$$1 = p_1q_1 + \dots + p_Mq_M.$$

Therefore $A_j = Pq_j$ will satisfy the conclusion of the Theorem.

Hence we assume $V_0 \neq \emptyset$. In this case we have the following

<u>Lemma 3.4</u>. If $p_1,\dots,p_M$ define a discrete, non-empty, variety V_0 in $\mathbb{C}^n$, there exist polynomials $h_1,\dots,h_n \in I$ which are linear combinations of the p_j with integral coefficients such that $h_1,\dots,h_n$ is a regular sequence. Furthermore, these integers will have absolute value $\le D_0^{n-1}$, where $D_0 = \max\{\deg p_j\}$.

Let us accept this Lemma for the time being. By Bezout's Theorem [26], the cardinality of V_1, $V_1 = \{z \in \mathbb{C}^n: h_1(z) = \ldots = h_n(z) = 0\}$, is at most D^n. Therefore we can find polynomials $g_1,\ldots,g_n$, $g_j \in \mathbb{C}[z_j]$, $\deg g_j \le D^n$ and g_j vanishing on V_1. We can use Rabinowitsch's trick to estimate the power one needs to raise a g_j to be in the ideal J generated by $h_1,\ldots,h_n$. Namely, for fixed j consider the polynomials of $n + 1$ variables.

$$h_1(z),\ldots,h_n(z), \quad 1 - z_0 g_j(z_j)$$

which form a regular sequence in $\mathbb{C}[Z]$, $Z = (z_0,z)$, $z = (z_1,\ldots,z_n)$, and have no common zeros. By ([8], Proposition 8) one has the inequality $(c > 0)$

$$(\|h(z)\|^2 + |1 - z_0 g_j(z_j)|^2)^{1/2} \ge c\|Z\|^{1-nD^{2n+1}}.$$

By the classical division formulas of Skoda [25], one obtains polynomials $B_{j,1},\ldots,B_{j,n+1}$ of the $n + 1$ variables such that

$$B_{j,1}h_1 + \ldots + B_{j,n}h_n + B_{j,n+1}(1-z_0 g_j) = 1 \tag{3.48}$$

with $\deg B_{j,i} \le n(n+1)D^{2n+1} + nD^{n+1} = e$. Multiplying (3.48) by g_j^e and replacing z_0 by $1/g_j$ we obtain

$$f_j := g_j^e = C_{j,1}h_1 + \ldots + C_{j,n}h_n$$

we have the estimates

$$\deg f_j \le eD^n, \quad \deg C_{j,i} \le e(1+D^n). \tag{3.50}$$

These degrees are essentially D^{3n}. Brownawell has announced a more precise estimate of the exponent e and the $\deg C_j$ [9]. From this it would follow an estimate of the order of magnitude D^{2n} in (3.50). There will be a corresponding improvement in (3.47). Multiplying, if necessary, f_j by a power of z_j we can assume

$$(3.51) \qquad \deg f_1 = \dots = \deg f_n = eD^n = D_1$$

this will make $\deg C_{j,i} \le 2eD^n = 2D_1$.

We will apply Theorem 3.1 for the sequence of $M + n$ polynomials $f_1,\dots,f_n$, $p_1,\dots,p_M$. In this case $d = D_1$ from (3.51) since the variables are separated. Let u_j be any polynomials such that $P = u_1p_1 + \dots + u_Mp_M$, and, to avoid any indexation problems, denote $\gamma_{j,k}$ the polynomials defined by (3.4) for the p_j. It follows from (3.6) that $q = n$ suffices. Therefore the formula (3.7) will have a single term:

$$(3.52) \quad P(z) = \langle \bar\partial\tfrac{1}{f}\ , \sum_{j=1}^{M} u_j(\zeta) \begin{vmatrix} g_{1,1}(z,\zeta) & \dots & g_{n,1}(z,\zeta) & \gamma_{j,1}(z,\zeta) \\ \vdots & & & \\ g_{1,n}(z,\zeta) & \dots & g_{n,n}(z,\zeta) & \gamma_{j,n}(z,\zeta) \\ f_1(z)-f_1(\zeta) & \dots & f_n(z)-f_n(\zeta) & p_j(z) \end{vmatrix} d\zeta \rangle .$$

Since the current $\bar\partial\frac{1}{f}$ annihilates the elements in the ideal of $f_1,\dots,f_n$, we can eliminate the terms $f_j(\zeta)$ from (3.52) when developing the determinant. Hence (3.52) simplifies to

$$(3.52)\qquad P(z) = \langle \bar\partial \frac{1}{f}\ , \sum_{j=1}^{M} u_j \begin{vmatrix} g_{1,1}(z,\cdot) & \dots & g_{n,1}(z,\cdot) & \gamma_{j,1}(z,\cdot) \\ \vdots & & & \\ g_{1,n}(z,\cdot) & \dots & g_{n,n}(z,\cdot) & \gamma_{j,n}(z,\cdot) \\ f_1(z) & & f_n(z) & p_j(z) \end{vmatrix} d\zeta \rangle$$

$$= q_1 f_1 + \dots + q_n f_n + \sum_{j=1}^{M} r_j p_j ,$$

with $\deg q_j \le (n-1)(D_1-1) + D - 1$, $\deg r_j \le n(D_1 - 1)$.

Replacing f_j by $\sum_i C_{j,i} p_i$, $\deg C_{j,i} \le 2D_1$ we have

$$P(z) = p_1 A_1 + \dots + p_M A_M ,$$

with

$$\deg A_j \le (n+1)D_1 + D - n \le (n+1)^3 D^{3n+1} .$$

□

<u>Proof of Lemma 3.4</u>. This is a very small modification of the proof of Lemma 2, [21].

We can assume that $n > 1$ and that none of the p_j is a constant. Define $h_1 := p_1$.

Let $\mathfrak{I}_1, \dots, \mathfrak{I}_r$ be the prime ideals associated to the principal ideal $h_1\mathbb{C}[z]$. We have $1 \le r \le D_0$. For each j, $1 \le j \le r$, there must exist $p_k \notin \mathfrak{I}_j$, otherwise V_0 could not be discrete. By Lemma 1, [21], there are integers $\lambda_1, \dots, \lambda_M$, $|\lambda_j| \le D_0$ such that

$$h_2 := \lambda_1 p_1 + \dots + \lambda_M p_M \not\subset \mathfrak{I}_i, \quad 1 \le i \le r.$$

The ideal $((h_1, h_2))$ generated by h_1, h_2 is proper, otherwise $V_0 = \emptyset$. By intersection theory [28], the degree of the ideal $((h_1, h_2))$ is $\le D_0^2$. This ideal is unmixed of rank 2 (Macaulay's Theorem [28]), hence it has at most D_0^2 associated prime ideals $\mathfrak{I}_i$. One can restart the argument above if $n > 2$. In this case we will have integers $\mu_1, \dots, \mu_M$, $|\mu_j| \le D_0^2$ such that

$$h_3 := \mu_1 p_1 + \dots + \mu_M p_M \not\subset \mathfrak{I}_i, \quad \text{for any} \quad i.$$

The ideal $((h_1, h_2, h_3))$ is again proper. Continuing this way we obtain the polynomials $h_1, \dots, h_n$ we were looking for.

□

§ 4. Effective bounds for sizes in Bezout identities.

In this section we will study the Bezout equation for polynomials in $\mathcal{O}_K[z]$. $\mathcal{O}_K$ = ring of integers of a number field $K, [K:\mathbb{Q}] = e$. We need to recall a few definitions from algebraic number theory [16]. Given an algebraic number α denote

$$|\overline{\alpha}| = \max \{|\alpha'| : \alpha' \text{ conjugate of } \alpha \text{ over } \mathbb{Q}\}$$

$$s(\alpha) = \max \{\log \operatorname{den}(\alpha), \log |\overline{\alpha}|\},$$

where $\operatorname{den}(\alpha)$ = denominator of α = smallest integer $d > 0$ such that $d\alpha$ is an algebraic integer. For a polynomial $p \in \mathcal{O}_K[z]$, $p(z) = \sum c_k z^k$, denote

$$H(p) = \max |\overline{c_k}|, \ h(p) = \log H(p) = \max \log s(c_k).$$

Note that $h(p)$ can be defined for $p \in K[z]$ using the last term as its definition.

Let $C'_d = \binom{n+d-1}{n-1}$ = number of monomials of degree exactly d, $C_d \le (1+d)^{n-1}$, and $C_d = \binom{n+d}{n}$ = dimension of the vector space of polynomials of degree at most d, $C_d \le (1+d)^n$. Then if $p, q \in \mathcal{O}_K[z]$, $\deg p \le d$, we have

$$(4.1) \qquad H(pq) \le C_d H(p) H(q).$$

If one changes coordinates as follows:

$$z_j = a_{j1}w_1 + \dots + a_{jn}w_n, \quad 1 \le j \le n,$$

$a_{jk} \in \mathbb{Z}$, $|a_{jk}| \le M$, $\det[a_{jk}] \ne 0$, and define a polynomial $q \in \mathcal{O}_K[w]$ by the formula

$$q(w) = p(z),$$

we have the estimate

$$H(q) \le C'_d(nM)^d H(p). \tag{4.2}$$

Finally, let $p \in \mathcal{O}_K[z_1,\ldots,z_n]$, $\alpha_1,\ldots,\alpha_n$ algebraic numbers, and $\beta = p(\alpha_1,\ldots,\alpha_n)$. To estimate $\mathrm{den}(\beta)$ and $s(\beta)$, let $r_j \ge \deg_{z_j} p$ = degree of p with respect to the variable z_j. Then

(4.3) $\mathrm{den}(\beta)$ is a divisor of $\prod_{j=1}^{n} (\mathrm{den}(\alpha_j))^{r_j}$, and

$$s(\beta) \le h(p) + \sum_{j=1}^{n} (r_j s(\alpha_j) + \log(r_j+1)). \tag{4.4}$$

Later on we will need to estimate the denominator of the inverse of an algebraic number α $(\alpha \ne 0)$. If $N(\alpha)$ denotes its norm and d a denominator of α one can use that $N(d\alpha)$ is a denominator for $(d\alpha)^{-1}$. Therefore

$$\begin{aligned} \log \mathrm{den}(\alpha^{-1}) &\le \log \mathrm{den}((d\alpha)^{-1}) \le \log N(d\alpha) \\ &\le (\deg \alpha) s(d\alpha) \le 2(\deg \alpha) s(\alpha) \end{aligned} \tag{4.5}$$

Given polynomials $p_1,\ldots,p_m$ of $\deg p_j = d_j$, consider p_j^o = leading homogeneous term of p_j. Then the variety at ∞ of the p_j, V_∞, is the conic subvariety of $\mathbb{C}^n$ defined by

$$V_\infty = \{z \in \mathbb{C}^n : p_j^o(z) = 0, \quad 1 \le j \le m\}.$$

We will say V_∞ is *discrete*, if $\dim V_\infty \le 1$. (The terminology is justified by considering V_∞ as a subvariety of

$\mathbb{P}^{n-1}$.) From the algebraic point of view, V_∞ is discrete is equivalent to the fact that the rank of the ideal $((p_1^o,\dots,p_m^o))$ generated in $\mathbb{C}[z_1,\dots,z_n]$ by $p_1^o,\dots,p_m^o$, is at least $n-1$, cf. [28].

Let π_j be the homogeneous polynomial in the $n+1$ variables (z_0,z) associated to p_j. Namely, let $Z = (z_0,z)$ and $\pi_j(Z) = z_0^{d_j} p_j(z/z_0)$. Then, if V_∞ is discrete, the rank of the ideal $((\pi_1,\dots,\pi_m))$ in $\mathbb{C}[Z]$ is at least n, and conversely.

The main theorem of this paper is the following:

<u>Theorem 4.1</u>. Let $p_1,\dots,p_m \in \mathcal{O}_K[z_1,\dots,z_n]$, $\max \deg p_j = D$, $\max h(p_j) = h$. Assume $p_1,\dots,p_m$ do not have any common zeros in $\mathbb{C}^n$ and that V_∞ is discrete. Then there are polynomials $q_1,\dots,q_m \in \mathcal{O}_K[z]$, $a \in \mathbb{Z}^+$ such that they satisfy the identity

(4.6) $$a = p_1q_1 + \dots + p_mq_m$$

and, the estimates

(4.7) $$\deg q_j \le 10n^3D^{2n}$$

(4.8)
$$\max(\log a, \max h(q_j)) \le e^2 3^{5n+2} n^{3n+15} D^{8n+3}(6n^2 D\log nD + \log m + h)$$

The proof of this theorem will depend on the use of formula (3.7) after we have found convenient polynomials $f_1,\dots,f_n$ in the ideal $((p_1,\dots,p_m)) \subseteq \mathcal{O}_K[z]$. We assume $D \ge 2$, $n \ge 2$, otherwise the theorem is trivial. The next few lemmas have as a purpose to find the f_j.

Lemma 4.2. There exists a family of homogeneous polynomials $r_{j,k} \in \mathbb{Z}[z]$, $1 \le j \le n-1$, $1 \le k \le m$, such that the polynomials $\tilde{p}_j$ given by

$$(4.9) \qquad \tilde{p}_j(z) = \sum_{k=1}^{m} r_{j,k} p_k^o, \quad 1 \le j \le n-1,$$

are homogeneous of degree exactly D, and the rank of $((\tilde{p}_1, \ldots, \tilde{p}_{n-1}))$ in $\mathbb{C}[z]$ is exactly $n-1$. Moreover

$$(4.10) \qquad \max h(r_{j,k}) \le (D-1)\log n + (n-2)D\log D.$$

Proof. Let $\tilde{p}_1$ be a p_j^o of degree D. We can assume that $n > 2$. Let $\mathfrak{I}_1, \mathfrak{I}_2, \ldots$ the prime ideals associated to $((\tilde{p}_1))$. There are at most D of them. It is not possible that all the monomials $z_1, \ldots, z_n$ belong to $\mathfrak{I}_1$. The same holds for $\mathfrak{I}_2$, etc. By ([21], Lemma 1) there are integers $u_1, \ldots, u_n$, $|u_j| \le D$, such that the linear form

$$L_1 = u_1 z_1 + \ldots + u_n z_n$$

does not belong to any $\mathfrak{I}_i$.

Let us now multiply p_j^o by $L_1^{D-d_j}$, $2 \le j \le n$. The new polynomials p_j' are homogeneous of degree exactly D. For any $\mathfrak{I}_i$, it is not possible that all $p_1', p_2', \ldots, p_m' \in \mathfrak{I}_i$. Otherwise, since $\mathfrak{I}_i$ is prime and $L_1 \not\in \mathfrak{I}_i$, then $p_1^o, \ldots, p_m^o$ would all belong to $\mathfrak{I}_i$ and, hence, rank of $((p_1^o, \ldots, p_m^o)) = 1 < n-1$. One can apply again ([21], Lemma 1) and find $\lambda_1, \ldots, \lambda_m \in \mathbb{Z}$, $|\lambda_j| \le D$ such that

$$\tilde{p}_2 = \lambda_1 p_1' + \dots + \lambda_m p_m'$$

does not belong to any $\mathfrak{Z}_i$.

It follows that $((\tilde{p}_1, \tilde{p}_2))$ is a proper ideal of rank 2, degree $\le D^2$. Therefore by Macaulay's Theorem [28] it has at most D^2 associated prime ideals. If $n = 3$ we are done. If $n > 3$ we can continue the same way. (Construct a new linear L_2 and define $p_j' = L_2^{D-d_j} p_j^o$ for the next step.)

To end the proof of Lemma 4.2 we only need to estimate $h(r_{j,k})$, where we have written $\tilde{p}_j$ as in (4.9). We had $r_{1,1} = 1$, $r_{1,j} = 0$ for $2 \le j \le m$; $r_{2,j} = \lambda_j L_1^{D-d_j}$. hence by (4.1) (and the fact that they are all homogeneous)

$$H(r_{2,j}) \le D(C_1')^{D-d_j} H(L_1)^{D-d_j}.$$

Therefore

$$h(r_{2,j}) \le (D-1)\log n + D \log D.$$

For $r_{3,j}$ the size of the corresponding λ_j is $\le D^2$, $H(L_2) \le D^2$ since this depends only on the number of components of $((\tilde{p}_1, \tilde{p}_2))$. The worst case estimate occurs for $r_{n-1,j}$. We have

$$h(r_{k,j}) \le (D-1)\log n + (n-2)D \log D. \quad (1 \le k \le n-1,\ 1 \le j \le m).$$

This proves the lemma.

□

Let us now define $P_1, \dots, P_{n-1} \in ((p_1, \dots, p_m)) \cap \mathcal{O}_K[z]$ by

$$P_j = \sum_{k=1}^{m} r_{j,k} p_k, \quad 1 \le j \le n-1, \tag{4.11}$$

with $r_{j,k}$ given by Lemma 4.2. Then we have P_j^o = leading term of $P_j = \tilde{p}_j$. Let σ_j be the corresponding homogeneous polynomials in $\mathcal{O}_K[Z]$

$$\sigma_j(Z) = z_0^D P_j(z/z_0). \tag{4.12}$$

The ideal $((\sigma_1,\ldots,\sigma_{n-1}))$ has rank $n-1$ in $\mathbb{C}[Z]$, that is $\sigma_1,\ldots,\sigma_{n-1}$ is a regular sequence. If that were not the case $\dim W > 2$, where $W = \{Z \in \mathbb{C}^{n+1} : \sigma_j(Z) = 0,\ 1 \le j \le n-1\}$. Therefore

$$\dim(W \cap \{z_0 = 0\}) \ge 2.$$

But $W \cap \{z_0 = 0\} = \{z \in \mathbb{C}^n : \tilde{p}_1(z) = \ldots = \tilde{p}_{n-1}(z) = 0\}$ has dim 1 since $((\tilde{p}_1,\ldots,\tilde{p}_{n-1}))$ was of rank $n-1$ in $\mathbb{C}[z]$.

It follows that the ideal $((\sigma_1,\ldots,\sigma_{n-1}))$ is unmixed and it has at most D^{n-1} associated prime ideals $\mathfrak{J}_i$.

Recall that $\pi_1,\ldots,\pi_m$ are the homogeneous versions of $p_1,\ldots,p_m$, and that the rank of $((\pi_1,\ldots,\pi_m))$ in $\mathbb{C}[Z]$ is at least n. By the same argument of Masser-Wüstholz as above, for each $\mathfrak{J}_i$ there is some π_j which does not belong to it. Therefore there are integers μ_k, $|\mu_k| \le D^{n-1}$ such that

$$\sigma_n' = \mu_1\pi_1 + \ldots + \mu_m\pi_m \notin \mathfrak{J}_i \quad \text{for any } i.$$

Hence the rank of $((\sigma_1,\ldots,\sigma_n'))$ in $\mathbb{C}[Z]$ is exactly n.

We can define

$$P_n'(z) = \sigma_n'(1,z). \tag{4.13}$$

Hence $P_n' = \sum_1^m \mu_j p_j$, $|\mu_j| \le D^{n-1}$. Note that $X = \{z \in \mathbb{C}^n : P_j = 0$

$(1 \le j \le n-1)$, $P_n' = 0\}$ is discrete. We do not yet have the right choice of the nth polynomial to add to the list $P_1,\dots,P_{n-1}$. We need that the variety W_∞ defined by $W_\infty := \{P_1^o = \dots = P_n^o = 0\}$ in $\mathbb{C}^n$ does not change if we take away one of the equations. For that reason, let

$$(4.14) \qquad P_n = P_n' + \sum_{j=1}^{n-1} P_j^2.$$

We know that $\sum_1^{n-1} (P_j^o)^2 = \sum_1^{n-1} (\tilde{p}_j)^2 \not\equiv 0$. Otherwise the sequence $\tilde{p}_1,\dots,\tilde{p}_{n-1}$ could not have been a regular sequence. Therefore $\deg P_n = 2D$ and $P_n^o = \sum_{j=1}^{n-1} (P_j^o)^2$. Clearly, this polynomial satisfies the required condition on the variety W_∞. By Lemma 4.2, $\dim W_\infty = 1$.

Let us write $P_n = \sum_{k=1}^{m} r_{n,k} p_k$. The logarithmic height of $r_{n,k}$ can be estimated by

$$(4.15) \qquad h(r_{n,k}) \le h + \log m + 3nD \log nD.$$

For later use we summarize the different estimates of the size of $P_1,\dots,P_n$ into the single estimate

$$(4.16) \qquad h(P_j) \le 2(h + \log m) + 4nD \log nD \qquad (1 \le j \le n).$$

<u>Lemma 4.3</u>. There are constants $\varkappa_1, c_1 > 0$ such that for $\|\zeta\| \ge \varkappa_1$,

$$(4.17) \qquad \|P(\zeta)\| = \left[\sum_1^n |P_j(\zeta)|^2\right]^{1/2} \ge c_1 \|\zeta\|^{D-2(n-1)D^n}.$$

__Proof__. We follow the proof of ([8], Proposition 8). The only difference here is that the variety $X = \{P_1 = \dots = P_n = 0\}$ $= \{P_1 = \dots = P_{n-1} = P'_n = 0\}$ is not empty but discrete by the observation after (4.13).

One observes that the step H_n in that proof is valid for the regular sequence $P_1,\dots,P_n$. F_n^* is the product of Chow forms corresponding to points in $\mathbb{C}^n$. (Here we use the discreteness of X.) These Chow forms are $u_0 + b_1u_1 + \dots + u_nb_n$, $(b_1,\dots,b_n) \in \mathbb{C}^n$. From the definition of $\|F_n^*\|_\zeta$ ([8], Section III) we have

$$\|F_n^*\|_\zeta \geq \text{constant} > 0 \quad \text{if} \quad \|\zeta\| \geq \varkappa_1.$$

Part (iii) of H_n gives

$$\log\|P(\zeta)\| \geq \text{const.} + (\min \deg P_j)\log\|\zeta\| - \left[(n-1)\prod_1^n \deg P_j\right]\log\|\zeta\|$$
$$= \text{const.} + (D - 2(n-1)D^n)\log\|\zeta\|,$$

which is the inequality we were looking for.

□

The construction of L_1 is the proof of Lemma 2.4, this time with respect to the variety W_∞, shows there is a linear form $L \in \mathbb{Z}[z]$

$$\begin{aligned} L &= a_{1,1}z_1 + \dots + a_{1,n}z_n, \\ H(L) &= \max|a_{1,j}| \leq D^{n-1}, \end{aligned} \tag{4.18}$$

such that $W_\infty \cap \{L = 0\} = \{0\}$.

__Lemma 4.4__. Let $f_1,\dots,f_n$ be defined by

$$(4.19) \qquad f_j = (L + j-1)^{2(n-1)D^n} P_j.$$

There are constants $\varkappa, c > 0$ such that

$$(4.20) \qquad \|f(\zeta)\| \geq c\|\zeta\| \quad \text{if} \quad \|\zeta\| \geq \varkappa.$$

Proof. By (4.17) in the preceding Lemma we have

$$\|P(\zeta)\| \geq c_1\|\zeta\|^{D-2(n-1)D^n} \quad \text{if} \quad \|\zeta\| \geq \varkappa_1.$$

Since W_∞ can be defined by any $n-1$ among the P_i^o and, $W_\infty \cap \{L = 0\}$ is just the origin, it follows that there exist constants $c_2, \varkappa_2 > 0$ such that for $\|\zeta\| \geq \varkappa_2$ we have for any k, $1 \leq k \leq n$,

$$(4.21) \qquad \left[|L(\zeta)|^2 + \sum_{j \neq k} |P_i(\zeta)|^2\right]^{1/2} \geq c_2\|\zeta\|.$$

Let $\varkappa_3 = \max\{\varkappa_1, \varkappa_2, \frac{2n^{3/2}}{c_2}\}$. For any given ζ, $\|\zeta\| \geq \varkappa_3$, we can choose k such that

$$|P_k(\zeta)|^2 = \max_{1 \leq j \leq n} |P_j(\zeta)| \geq \frac{c_1}{\sqrt{n}} \|\zeta\|^{D-2(n-1)D^n}.$$

For this k we apply (4.21). Either

$$|L(\zeta)| \geq \frac{c_2}{\sqrt{n}} \|\zeta\|,$$

hence, for some constant $c_3 > 0$,

$$|L(\zeta) + k-1| \geq c_3\|\zeta\|,$$

in which case

$$|f_k(\zeta)| \geq \text{const. } \|\zeta\|^D.$$

Or, in the other case, there is an index $j \neq k$ such that

$$|P_j(\zeta)| \geq \frac{c_2}{\sqrt{n}} \|\zeta\|.$$

By the definition of k we also have

$$|P_k(\zeta)| \geq \frac{c_2}{\sqrt{n}} \|\zeta\|.$$

Since $k \neq j$ we have

$$|L(\zeta) + k-1| + |L(\zeta) + j-1| \geq |k-j| \geq 1.$$

It follows that

$$|f_j(\zeta)| + |f_k(\zeta)| \geq \left(\frac{1}{2}\right)^{2(n-1)D^n} \frac{c_2}{\sqrt{n}} \|\zeta\|.$$

This proves (4.20).

□

Note that though $\deg f_j = 2(n-1)D^n + D$, $1 \leq j \leq n-1$, $\deg f_n = 2(n-1)D^n + 2D$, the cardinality of $V = \{z \in \mathbb{C}^n: f_1(z) = \ldots = f_n(z) = 0\}$ is at most $(2D+1)(D+1)^{n-1} \leq 2(D+1)^n$ by Bezout's theorem.

Since the original polynomials $p_1,\ldots,p_m$ have no common zeros we can find $\nu_j \in \mathbb{Z}$, $|\nu_j| \leq 2(D+1)^n$, such that if

(4.21) $$P_{n+1} := \nu_1 p_1 + \ldots + \nu_m p_m,$$

then

$$V \cap \{z \in \mathbb{C}^n : P_{n+1}(z) = 0\} = \emptyset.$$

Before proceeding any further let us remark that one can estimate the logarithmic size of the f_j and of the polynomial coefficients representing them in terms of the original polynomials $p_1,\ldots,p_m$.

Let $s_{j,k} := (L + j-1)^{2(n-1)D^n} r_{j,k}$. Then

$$f_j = \sum_{k=1}^{m} s_{j,k} p_k. \tag{4.22}$$

Then

$$\deg s_{j,k} \le 2nD^n, \tag{4.23}$$

$$h(s_{j,k}) \le h + \log m + 6n^2D^n \log D \tag{4.24}$$

$$h(f_j) \le 2(h + \log m) + 8n^2D^n \log D. \tag{4.25}$$

We also know from above that

$$h(P_{n+1}) \le n \log(D+1) + \log 2m + h. \tag{4.26}$$

We have now performed the preliminary algebraic steps and we are ready to proceed to the proof of Theorem 4.1.

Proof of Theorem 4.1. The sequence $f_1,\ldots,f_n,P_{n+1}$ fits exactly in the situation of Example 3.3. From there it follows that there are polynomials $A_1,\ldots,A_{n+1} \in K[z]$ such that

$$f_1A_1 + \ldots + f_nA_n + P_{n+1}A_{n+1} = 1.$$

What we need to do now is to unravel (3.7) to obtain an estimate for the common denominator of the A_j and the size of their coefficients. Together with the estimates (4.23)-(4.25) we will then be able to achieve the proof of Theorem 4.1.

We now have $d = 1$ in (3.2), $P = 1$ (hence $\deg P = 0$), $u_1 = u_2 = \ldots = u_n = 0$, $u_{n+1} = \frac{1}{P_{n+1}}$ in (3.5). The quantity D in (3.4) is now $\max\{\deg f_j,\ 1 \le j \le n\} \le 2(n-1)D^n + 2D$. An easy estimate for q in (3.6) says we can take

$$(4.27)\quad q = 4n^2D^n.$$

With this estimate we can obtain an estimate of the degree of the polynomials A_j by

$$(4.28)\qquad \deg A_j \le 9n^3D^{2n}.$$

From this estimate, (4.7) follows immediately. Let us recall that in this case (3.7) has the form

(4.29)

$$1 = \sum_{|m|\le q-n} \langle \bar\partial \frac{1}{f^{m+\underline{1}}}, \frac{1}{P_{n+1}} \begin{vmatrix} g_{1,1} & \cdots\ g_{n,1} & g_{n+1,1} \\ \vdots & & \\ g_{1,n} & \cdots\ g_{n,n} & g_{n+1,n} \\ f_1(z)-f_1 & \cdots f_n(z)-f_n & P_{n+1}(z) \end{vmatrix} d\zeta \rangle\ f^m(z).$$

We are therefore obliged to estimate a common denominator as well as their largest possible absolute value for all the algebraic numbers of the form

$$(4.30)\qquad \langle \bar\partial \frac{1}{f^m}, \frac{\zeta^k}{P_{n+1}}\, d\zeta \rangle, \quad (|k| \le 2n^2D^n, \quad |m| \le q).$$

To compute these residues, we can use an observation from ([7], Proposition 2.3 and following remark) that shows it is enough to find n polynomials $g_1,\ldots,g_n \in \mathcal{O}_K[z]$, g_j a polynomial in the single variable z_j, which belong to the ideal generated by $f_1,\ldots,f_n$. With a little bit of care we could find g_j's with reasonable degrees as it was done in the previous section. Our problem is that we also need a control on the logarithmic sizes $h(g_j)$. For that purpose we will appeal to the following variation of a result of Macaulay [14], [19].

Lemma 4.5. Let $\gamma_1,\ldots,\gamma_l \in \mathcal{O}_K[z_1,\ldots,z_n]$, $\deg \gamma_j \le \delta$, $h(\gamma_j) \le M$. Assume $Y = \{z \in \mathbb{C}^n : \gamma_1(z) = \ldots = \gamma_l(z) = 0\}$ is discrete. Let $\tilde{\gamma}_1,\ldots,\tilde{\gamma}_l$ be the polynomials of $n-1$ variables defined by $\tilde{\gamma}_j(z_2,\ldots,z_n) = \gamma_j^o(0,z_2,\ldots,z_n)$, assume the origin of $\mathbb{C}^{n-1}$ is their only common zero (γ_j^o is the leading term of γ_j). Then there is a non-zero polynomial $\varphi \in ((\gamma_1,\ldots,\gamma_l)) \cap \mathcal{O}_K[z_1,\ldots,z_n]$, φ depends only on the variable z_1, $\varphi(z_1) = \sum_1^l b_j(z)\gamma_j(z)$, with

$$\max\,(\deg \varphi,\ \deg b_j) \le l\, n^n \delta^n, \tag{4.31}$$

$$\max(h(\varphi), h(b_j)) \le l\, n^n \delta^n (M + \log(l\, n^n \delta^n)). \tag{4.32}$$

Proof. Since Y is discrete there is at least one polynomial $\psi \ne 0$, ψ depending only on z_1 such that

$$\psi(z_1) = \sum_{j=1}^{l} c_j(z)\gamma_j(z), \quad c_j \in \mathcal{O}_K[z].$$

(See previous section or use elimination theory.) Let $\zeta_1 \in \mathbb{C}$ be fixed such that $\psi(\zeta_1) \ne 0$ and consider the polynomials γ'_j of $n-1$ variables given by $\gamma'_j(z_2,\ldots,z_n) = \gamma'_j(z') = \gamma_j(\zeta_1,z')$. The homogeneous variety $\{z' \in \mathbb{C}^{n-1} : (\gamma'_j)^o(z') = 0,\ 1 \le j \le l\} \subseteq$

$\{z' \in \mathbb{C}^{n-1}: \tilde{\gamma}_j(z') = 0, \quad 1 \le j \le l\} = \{0\}$. Moreover, the identity (4.32) evaluated at the point $z_1 = \zeta_1$ has the form

$$0 \ne \varphi(\zeta_1) = \sum_{j=1}^{l} c'_j(z')\gamma'_j(z').$$

That means that the polynomials $\gamma'_j(z')$ have no common zeros in $\mathbb{C}^{n-1}$. As pointed out in (1.8), a classical result of Macaulay tells us there are polynomials $c''_j \in \mathbb{C}[z']$ such that

$$\sum_{1}^{l} c''_j(z')\gamma'_j(z') - 1 = 0 \tag{4.33}$$

$$\deg c''_j \le (n-1)(\delta-1). \tag{4.34}$$

One can consider (4.33) as a homogeneous system of equations with $l\binom{(n-1)\delta}{n-1} + 1$ unknowns, which has a non-trivial solution. The coefficients of the matrix are polynomials in $\mathcal{O}_K[\zeta_1]$, of degree $\le \delta$, logarithmic size $\le M$. By linear algebra (cf. [21], Lemma 4) one can choose a particular solution using $r \times r$ minors of the matrix such that the last entry is different from zero in $\mathcal{O}_K[\zeta_1]$ $\left(r \le \ell\binom{(n-1)\delta}{n-1}\right)$. Though up to this point ζ_1 was considered a fixed value, the total number of solutions constructed by this method is finite. Therefore there is one of them that leads to a non-zero polynomial φ in $\mathcal{O}_K[z_1]$. We have therefore

$$\varphi(z_1) = \sum_{j=1}^{l} b_j(z)\gamma_j(z).$$

Using that φ is an $r \times r$ minor we have

$$\deg \varphi \le r\delta \le l\,\delta \binom{(n-1)\delta}{n-1} \le l\,\delta((n-1)\delta+1)^{n-1} \le l\, n^n \delta^n .$$

The same estimate holds for the degrees of the b_j. Similarly we obtain logarithmic size estimates (4.32).

□

The residue formula (4.30) is a sum of local residues at the points of the variety V of common zeros of $f_1,\dots,f_n$. At one such zero α, at most one of the functions $L, L+1, \dots, L+n-1$ could vanish. Let us say $L(\alpha) = 0$. Then the local form of (2.4) says that

$$(4.35) \qquad \left\langle \bar\partial \frac{1}{f^m}, \frac{\zeta^k}{P_{n+1}} \right\rangle_\alpha = \left\langle \bar\partial \frac{1}{f_1^{m_1}} \wedge \bar\partial \frac{1}{P_2^{m_2}} \wedge \dots \wedge \bar\partial \frac{1}{P_n^{m_n}}, \frac{\zeta^k}{Q} \right\rangle_\alpha ,$$

where $\langle\ ,\ \rangle_\alpha$ indicates the local residue at α, and $Q = P_{n+1}\left(\prod_2^n (L+j-1)^{m_j}\right)^{2(n-1)D^n}$. If one of the other affine functions $L+j$ vanishes at α, the expression one obtains is obviously similar. If none of them vanish we will nevertheless group them as in (4.35).

Our construction of $P_1,\dots,P_n,L$ is such that the variety at ∞ of the polynomials $f_1^{m_1},\dots,f_n^{m_n}$ is W_∞, which is discrete and has at most D^{n-1} points in common with the hyperplane at ∞. Before applying Lemma 4.5 we will introduce new coordinates $w_1,\dots,w_n$ by

$$(4.36) \qquad \langle a_j, z\rangle = \eta w_j, \quad a_j \in \mathbb{Z}^n, \quad \eta := \det[a_{j,k}]$$

so that W_∞ does not intersect any of the hyperplanes $\{w_j = 0\}$.

We can choose a_1 as in (4.18) since the required condition $W_\infty \cap \{w_1 = 0\} = \emptyset$ is already satisfied with this choice. The condition on the a_j is therefore

$$\eta \prod_{\zeta, j} \langle a_j, \zeta \rangle \neq 0,$$

where ζ are projective representatives of the points in W_∞. This is a polynomial condition on the $a_{j,k}$ $(2 \le j \le n)$ of degree $(n-1)(D^{n-1}+1)$. Theorem 1, [20] ensures that there is a solution $a_2, \ldots, a_n$ with

$$\max |a_{j,k}| \le n^3 D^{n-1}. \tag{4.37}$$

This estimate also holds for $j = 1$ by (4.18). It follows that

$$1 \le |\eta| = |\det[a_{j,k}]| \le n^{4n} D^{n^2}. \tag{4.38}$$

Using this change of coordinates, we have

$$z = A^* w, \tag{4.39}$$

A^* is the cofactor matrix of $[a_{j,k}]$; all its entries are integers

$$\max |a^*_{j,k}| \le n^{4n} D^{n^2}. \tag{4.40}$$

If we define

$$\gamma_j(w) = P_j(z) = P_j(A^* w), \quad \lambda(w) = L(A^* w),$$

then using (4.2) and (4.16), we find:

$$\lambda(w) = \eta w_1 \tag{4.41}$$

$$h(\gamma_j) \le 8n^2D \log nD + 2(\log m + h), \quad (1 \le j \le n).$$

(4.42)

$$h(\gamma_{n+1}) \le 4n^2D \log D + \log 2m + h.$$

We can now apply Lemma 4.5 to the system of polynomials $\lambda\gamma_1,\gamma_2,\ldots,\gamma_n$. We find polynomials $\varphi_j \in \mathcal{O}_K[w_j]$, $b_{j,i} \in \mathcal{O}_K[w]$ such that

$$\varphi_j(w_j) = b_{j,1}(w)\lambda(w)\gamma_1(w) + \sum_{i=2}^{n} b_{j,i}(w)\gamma_i(w), \tag{4.43}$$

$$\max(\deg \varphi_j, \deg b_{j,i}) \le n^{n+1}(2D+1)^n \le (3nD)^{n+1}, \tag{4.44}$$

$$\max(h(\varphi_j), h(b_{j,i})) \le (3nD)^{n+1} \quad (11n^2D \log nD + 2 \log m + 2h). \tag{4.45}$$

The estimates have been generous enough so that they will not change in the other possible groupings of (4.35), the polynomials φ_j will change.

To profit from the knowledge of the φ_j to compute (4.35) as we have done in Lemma 2.1, we need first to change coordinates. From (2.4) it is clear that

$$\langle\bar\partial\frac{1}{f_1^{m_1}}\wedge\bar\partial\frac{1}{P_2^{m_2}}\wedge\ldots\wedge\bar\partial\frac{1}{P_n^{m_n}},\frac{z^k}{Q}\,dz\rangle_\alpha = (\det A^*)\langle\bar\partial\frac{1}{\psi_1^{m_1}}\wedge\bar\partial\frac{1}{\gamma_2^{m_2}}\wedge\ldots\wedge\bar\partial\frac{1}{\gamma_n^{m_n}},\frac{R}{S}\,dw\rangle_\beta,$$

where $\psi_1(w) = f_1(A^*w)$, $R(w) = (A^*w)^k$, $S = Q(A^*w)$, $\beta = \frac{A\alpha}{\eta}$.

It is easy to see that if $N = 8n^4D^{2n}$ then

$$\varphi_j^N \in ((\psi_1^{m_1},\gamma_2^{m_2},\ldots,\gamma_n^{m_n})), \quad |m| \le q, \quad 1 \le j \le n. \tag{4.46}$$

Let

$$\varphi_j^N = c_{j,1}\psi_1^{m_1} + \dots + c_{j,n}\gamma_n^{m_n}, \quad c_{j,i} \in \mathcal{O}_K[z].$$

Denote $\Delta := \det[c_{j,i}] \in \mathcal{O}_K[z]$. The $c_{j,i}$, Δ depend on the multi-index m. One can verify:

$$\text{(4.47)} \qquad \deg \Delta \le 6n^6(3nD^3)^n \quad \text{and}$$
$$h(\Delta) \le 10n^5D^{2n}(3nD)^{n+1}(11n^2D \log nD + 2(\log m + h)).$$

We also have

$$\text{(4.48)} \qquad h(R) \le 5n^4D^n \log nD, \quad \deg R = |k| \le 2n^2D^n.$$

$$\text{(4.49)} \qquad h(S) \le 18n^5D^{2n}\log nD + (\log m + h), \quad \deg S \le 8n^3D^{2n}.$$

The transformation law of residues (cf. Lemma 2.1) can now be used and it yields:

$$\xi = \xi(\beta) := (\det A^*)\langle\bar\partial\frac{1}{\psi_1^{m_1}}\wedge\bar\partial\frac{1}{\gamma_2^{m_2}}\wedge\dots\wedge\bar\partial\frac{1}{\gamma_n^{m_n}}, \frac{R}{S}\, dw\rangle_\beta, =$$

$$= (\det A^*)\langle\bar\partial\frac{1}{\varphi_1^N}\wedge\dots\wedge\bar\partial\frac{1}{\varphi_n^N}, \frac{\Delta R}{S}\, dw\rangle_\beta .$$

This last expression can be computed iteratively as we have done in Lemma 2.1. Let $\beta = (\beta_1,\dots,\beta_n)$, ν_j the multiplicity of β_j as a zero of φ. Define θ by the identity $\varphi_j(t) := (t-\beta_j)^{\nu_j}\theta_j(t)$. If we let $\vartheta(w) = \frac{\Delta R}{S}(\theta_1\dots\theta_n)^{-N}$, the above residue is given by

$$(4.50) \qquad \xi = \frac{(\deg A^*)}{(N\nu_1-1)!\dots(N\nu_n-1)!} \frac{\partial^{N(\nu_1+\dots+\nu_n)-n}}{\partial w_1^{N\nu_1-1}\dots\partial w_n^{N\nu_n-1}} \vartheta(w)\Big|_{w=\beta} .$$

The number ξ is in $K(\beta_1,\dots,\beta_n)$. In fact it depends also on the index k. We want to estimate first a common denominator for all the ξ that appear with a fixed β and an arbitrary k. Since ξ is obtained (up to a rational number) by evaluating a derivative of ϑ, we rewrite ϑ in order to use (4.3).

Leibniz' rule allows us to write

$$(4.51) \quad \frac{\partial^{N|\nu|-n}}{\partial w_1^{N\nu_1-1}\dots\partial w_n^{N\nu_n-1}} \vartheta(w) = \Phi\left[w_1,\dots,w_n,\frac{1}{\zeta(w)},\frac{1}{\theta_1(w_1)},\dots,\frac{1}{\theta_n(w_n)}\right],$$

$\Phi \in \mathcal{O}_K[X_1,\dots,X_{2n+1}]$. We have $r_j := \deg_{X_j} \Phi$,

$$(4.52) \qquad \begin{aligned} & r_j \le 6n^6(3nD^3)^n N|\nu| \quad \text{if} \quad 1 \le j \le n \\ & r_{n+1} \le N|\nu| \\ & r_{n+1+j} \le N\nu_j \quad \text{if} \quad 1 \le j \le n. \end{aligned}$$

By letting ν_j take the largest possible values we guarantee that

$$d = \prod_{j=1}^{n}(\mathrm{den}(\beta_j))^{r_j}(\mathrm{den}(1/S(\beta))^{r_{n+1}}\prod_{1}^{n}(\mathrm{den}(1/\theta_j(\beta_j)))^{r_{n+1+j}}(N\nu_1)!\dots(N\nu_n)!$$
(4.53)

is a denominator for $\xi(\beta)$ for all k (and all conjugates of β.) To simplify the notation let $\varepsilon_j := \theta_j(\beta_j)$, $\sigma := S(\beta)$. This shows that to estimate the quantity $\log d$ we need to estimate $\mathrm{den}(\beta_j)$, etc.

For that purpose we use the fact that β_j is a root of $\varphi_j(t) = 0$. A denominator for β_j is therefore given by product of all the conjugates (over $\mathbb{Q}$) of the leading term of φ_j, therefore (recall $e = [K:\mathbb{Q}]$)

$$(4.54) \quad \log \operatorname{den}(\beta_j) \le eh(\varphi_j) \le e(3nD)^{n+1}(11n^2D \log nD + 2(\log m + h)).$$

To use the estimate (4.5) for $\operatorname{den} \sigma^{-1}$, $\operatorname{den} \varepsilon_j^{-1}$ we need to estimate $s(\sigma)$, $s(\varepsilon_j)$. These are dependent on $s(\beta_1),\dots,s(\beta_n)$. To obtain $s(\beta_j)$ we use again the equation $\varphi_j(\beta_j) = 0$. Let us write this equation as

$$(4.55) \qquad a_0(\beta_j)^l + a_1(\beta_j)^{l-1} + \dots + a_l = 0.$$

We can assume that $|\beta_j| \ge 1$ since we are trying to compute $s(\beta_j) = \max\{\log \operatorname{den} \beta_j, \log|\beta_j|, \log|\beta_j'|, \dots\}$, where $\beta_j', \dots$ are the conjugates of β_j. Hence

$$|a_0||\beta_j| \le \sum_{i=1}^{l} |a_i|.$$

Multiplying by the conjugates of a_0 $(a_0 \in \mathcal{O}_K)$ we obtain

$$(4.56) \qquad |\beta_j| \le |\beta_j| N(a_0) \le \left(\sum_{1}^{l} |\overline{a_i}|\right)(|\overline{a_0}|)^{e-1}.$$

Since a conjugate of β_j will be an equation like (4.55) where each a_i is replaced by some conjugate and the estimate is independent of the conjugate of the a_i's we pick, we conclude

$$(4.57) \qquad \log |\overline{\beta_j}| \le \log \deg \varphi_j + eh(\varphi_j)$$

(4.58) $s(\beta_j) \le (n+1)\log(3nD) + e(3nD)^{n+1}(11n^2D \log nD + 2(\log m + h))$.

(The first term can be eliminated replacing e by $e(1 + 8\times 10^{-5})$.)

Now $\varepsilon_j = \dfrac{\varphi^{(\nu_j)}(\beta_j)}{\nu_j!} \in \mathcal{O}_K[\beta_j]$, its degree and logarithmic size can be easily estimated. Using (4.4) we obtain

$$(4.59) \quad s(\varepsilon_j) \le h(\varphi_j) + (\deg \varphi_j)\log 2 + (\deg \varphi_j)s(\beta_j) + \log(\deg \varphi_j)$$
$$\le e(1 + 10^{-3})(3nD)^{2(n+1)}(11n^2D \log nD + 2(\log m + h)).$$

We know that β is a solution of the equations $\lambda\gamma_1 = \gamma_2 = \ldots = \gamma_n = 0$. By Corollary 2.2 it follows that $\deg \beta_j \le e(2D+1)^n$ and $\deg \varepsilon_j \le e(2D+1)^n$. Using (4.5) again we have

$$(4.60) \quad \log \operatorname{den}(\varepsilon_j^{-1}) \le 2e(2D+1)^n s(\varepsilon_j).$$

Similarly,

$$(4.61) \quad s(\sigma) \le e(1 + 10^{-2})8n^4D^{2n}(3nD)^{n+1}(11n^2D \log nD + 2(\log m + h))$$
$$\log \operatorname{den} \sigma^{-1} \le 2e(2D+1)^n s(\sigma).$$

We can now use (4.53) with ν_j replaced by $(3nD)^{n+1}$, which is an upper bound for $\deg \varphi_j$. We have, $d = d(\beta)$,

$$(4.62) \quad \log d \le \tfrac{1}{2}e^2 3^{4n+3} n^{3n+15} D^{7n+3}(11n^2D \log nD + 2(\log m + h)).$$

A denominator d_0 for the sum of all the local residues appearing in the algebraic number (4.30) will be the product of the different denominators $d(\beta)$ (β corresponds to an α after a change of coordinates that depends on α), hence

$$(4.63)\qquad \log d_0 \le (2D+1)^n \max_{\alpha}\{\log d\}$$

$$\le \tfrac{1}{2}e^2 3^{5n+2} n^{3n+15} D^{8n+3}(11n^2 D \log nD + 2(\log m + h)).$$

What we want to do next is to estimate $|\overline{\xi(\beta)}|$. There are basically two ways to proceed. One way is to write $\xi(\beta)$ as an element in $\mathcal{O}_K(\beta_1,\ldots,\beta_n)$, and estimate the logarithmic size of the numerator and denominator polynomials. Another way, simpler in our view, is to estimate $|\xi(\beta)|$, $|\xi(\beta')|,\ldots$ analytically using the integral expression of a residue. For this purpose (in fact, this is also needed in the first approach), we need lower bounds for $|\sigma|$, $|\varepsilon_j|$, and the expressions obtained replacing β by β', etc. The size inequality ([16], Chapter 1) is

$$\log |\sigma^{-1}| \le 2s(\sigma)(\deg \sigma) \le 2e(2D+1)^n s(\sigma)$$

similarly for the other terms. These are precisely the expression in (4.60) and the second inequality in (4.61). Note we can obtain better inequalities using the Proposition in [23]. But in any case the worst estimates will eventually come from (4.63).

We observe that

$$M_j := \log \max\{|\varphi_j(t)| : |t| \le |\beta_j| + 1\}$$
$$\le \log(\deg \varphi_j + 1) + h(\varphi_j) + \deg(\varphi_j)(s(\beta_j)+1)$$

and

$$G = \log \max_{\|z-\beta\|\le 1} \|\mathrm{grad}\, S(z)\| \le (n+1)\log(\deg S)+h(S)+(\deg S)(\max s(\beta_j)+1).$$

If there is a zero t_0 of φ_j in $|\beta_j - t| \le 1$ then (cf. [4], Lemma 3)

$$\log |\beta_j - t| \ge -\log |\varepsilon_j^{-1}| - M_j.$$

We can define $\delta = \delta(\beta)$ by

$$\log \delta = \inf\{-\log |\varepsilon_j^{-1}| - M_j, \ (1 \le j \le n), -G - \log |\sigma^{-1}| - \log 2.$$

With this choice of δ we have that

$$\log |\varphi_j(\beta_j - t)| \ge \deg(\varphi_j) \log \delta, \qquad |t| = \delta$$

$$\log |S(\beta_1 - t_1, \dots, \beta_n - t_n)| \ge -\log |\sigma^{-1}| - \log 2, \qquad |t_j| = \delta.$$

By the remark after (2.4) we have

$$\xi(\beta) = (\deg A^*) \langle \bar\partial \frac{1}{\varphi_1^N} \wedge \dots \wedge \bar\partial \frac{1}{\varphi_n^N}, \frac{\Delta R}{S} dw \rangle_\beta$$

$$= (\det A^*) \frac{1}{(2\pi i)^n} \int_{|t_1|=\delta} \int_{|t_n|=\delta} \frac{\Delta(\beta - t) R(\beta - t)}{S(\beta - t)\varphi_1^N(\beta_1 - t_1) \dots \varphi_n^N(\beta_n - t_n)} dt_1 \dots dt_n.$$

Therefore

$$|\xi| \le |\eta^{n-1}| \delta^n \max_{|t_j|=\delta} \left| \frac{\Delta(\beta - t) R(\beta - t)}{S(\beta - t)\varphi_1^N(\beta_1 - t_1) \dots \varphi_n^N(\beta_n - t_n)} \right|.$$

Using the above estimates one obtains

$$(4.64) \quad \log|\xi| \le e^2 3^{5n+3} n^{3n+8} D^{7n+1} (11n^2 D \log nD + 2(\log m + h)).$$

In fact (4.64) is an estimate for $\log|\overline{\xi(\beta)}|$. The sum over all points in the variety V of zeros of $f_1, \dots, f_n$ gives us the element (4.30) of K. For fixed k, we will have

$$\log \left| \overline{\langle \bar\partial \frac{1}{f^m}, \frac{\zeta^k}{P_{n+1}} d\zeta} \right| \le \text{estimate } (4.64) + n \log (2D+1).$$

We can now return to the identity (4.29). A denominator common to all the entries in the determinants is $(\max \deg f_j)!$ Therefore a common denominator for all the determinants is $(2nD^n)!^n$. Hence we can take $a \in \mathbb{Z}^+$ for (4.6) to be

$$a := d_0(2nD^n)!^n. \tag{4.65}$$

This way we obtain polynomials $q_j \in \mathcal{O}_K[z]$ satsifying

$$a = p_1q_1 + \dots + p_mq_m$$

and

$$h(q_j) \le e^2 3^{5n+4} n^{3n+8} D^{7n+1}(11n^2D \log nD + 2(\log m + h)).$$

This concludes the proof of Theorem 4.1.

□

Let us denote

$$\mathfrak{h}(n,D,m,h,e) := e^2 3^{5n+2} n^{3n+15} D^{8n+3}(6n^2D \log nD + \log m + h).$$

<u>Corollary 4.7</u>. Let $p_1,\dots,p_m \in \mathcal{O}_K[z]$ without common zeros at infinity. Let $q \in \mathcal{O}_K[z]$ vanish on the variety V of common zeros of the p_j. Set $D = \max(\deg p_j, \deg q)$, $h = \max(h(p_j), h(q))$. Then there are $a, \mu \in \mathbb{Z}^+$, $q_1,\dots,q_m \in \mathcal{O}_K[z]$ such that

$$aq^\mu = p_1q_1 + \dots + p_mq_m,$$

$$\deg q_j \le 20(n+1)^3(D+1)^{2(n+1)}$$

$$\mu \le 10(n+1)^3(D+1)^{2(n+1)}$$

$$\max(\log a,\ h(q_j)) \le \mathfrak{h}(n+1,D+1,m+1,h,e).$$

Proof. Using Rabinowitsch's trick one considers $Q \in \mathcal{O}_K[z_0, z]$ given by $Q(z_0, z) = 1 - z_0 q(z)$. The polynomials $Q, p_1, \ldots, p_m$ considered in $\mathbb{C}^{n+1}$ have no common zeros and exactly one common zero at ∞. One can apply Theorem 4.1 to these polynomials and obtain

$$a = A(z_0, z)(1 - z_0 q(z)) + \sum_{j=1}^{m} B_j(z_0, z) p_j(z).$$

We have by (4.7)

$$\deg A,\ \deg B_j \le 10(n+1)^3 (D+1)^{2(n+1)}.$$

Let $\mu = 10(n+1)^3 (D+1)^{2(n+1)}$. Then, set as usual $z_0 = \frac{1}{q(z)}$,

$$a q^{\mu}(z) = \sum_{j=1}^{m} q^{\mu} B_j(\tfrac{1}{q}, z) p_j(z) = \sum_{1}^{m} q_j(z) p_j(z).$$

One computes the size of the q_j using (4.8) and obtains the desired bound.

$\square$

The inequalities can be improved by separating $\deg q$ from $\deg p_j$ and using the sharper new form of the Nullstellensatz found by Brownawell [9].

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