ABSTRACT

Title of Document: ESTIMATION OF EXTREME BENDING MOMENTS ON SHIPS FROM LIFETIME FATIGUE LOADS

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The U. S. Coast Guard uses a U. S. Navy methodology and a computer program called SPECTRA to probabilistically characterize wave-induced bending moments on surface vessels. SPECTRA is primarily used for fatigue design based on defined cells of vessel operation with specified heading, sea condition and speed in order to calculate bending response using the probability a ship is within each cell in a specified time period. In this study, the SPECTRA output for a hypothetical ship was obtained to examine its appropriateness to be used as a basis to characterize lifetime extreme design bending moments on ship hulls. The objective was to develop a method to utilize the SPECTRA fatigue load output to estimate the parameters of an extreme value distribution, such as the Weibull probability distribution, for the largest bending moment of k years. The study examined how to appropriately interpret and use the mean and variance of the bending moments obtained from SPECTRA for this purpose. A four step method is proposed in this thesis involving first getting the statistical moments of the data from the SPECTRA histograms, estimating the

parameters of the Weibull using these moments, finding the moments of the largest in k years from the generated distribution, and finally estimating the parameters of the Weibull for the largest in k years from these moments. The study also includes the development of an efficient and robust method of estimating the parameters and moments that is called the adaptive technique, involving exact calculation and numerical integration. The method is illustrated using a hypothetical case and verified using extreme value computations. It is also observed that the SPECTRA output based on specifying two or more years produces only minor enhancements in the estimated moments for one year and does not produce the statistical moments of extreme loading.

ESTIMATION OF EXTREME BENDING MOMENTS ON SHIPS FROM LIFETIME FATIGUE LOADS

By

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Table of Contents

List of Tables

List of Figures

1. Introduction

1.1. Ship Reliability under Wave Loading

Ships at sea are subjected to a wide variety of loading conditions due to waves, wind, mechanical loadings, and many other forms; however wave loading is often the most paramount concern due to the fact that they produce both cyclic loadings leading to possible fatigue failures, as well as the possibility of an excessive load on the ship structure due to large waves. Compounding this problem is the difficulty to predict wave loading for a given ship due to the random nature of weather patterns and sea conditions. This makes ship design and lifetime analysis challenging, as the maximum loads, as well as cyclic loading, are required to make sure a vessel is capable of surviving through its design life. This study focuses on the extreme wave loading aspect of naval engineering, specifically the bending moments induced by wave loading conditions.

Examining the extreme loading events due to waves allows ship analysis to be handled by looking at the two most extreme cases, hogging and sagging. Hogging loads occur when a ship rides the crest of a wave Figure 1.1-a while sagging results from a ship being supported bow to stern between two waves Figure 1.1-b.
In these cases the ship can be viewed as acting much like a beam under self-weight loading. Coupled with the wave loading loading is still-water bending moment, or the bending moment of the ship in flat seas due to the ships self-weight being supported by a liquid medium. There are other considerations including how the weight is

distributed along the ship, as well as whipping effects which are a result of impulsive loading, and torsional loading among others, but the large length-to-width ratio of a ship usually means that the bending moments will control, with additional bending due to whipping effects being included in those values.

(a) Hogging Moment Condition **(b) Sagging Moment Condition**

Figure 1.1. Sea Conditions Resulting in Bending Moments

In response to the difficulties in dealing with wave loading various probabilistic and statistical methods have been developed and used to estimate loading and serve as design aids. One such method put forth by uses a Poisson like process reliability equation to determine the likelihood of a ships yearly survival (Ayyub B. M., 2011).

$$R(t) = \int_0^s e^{\left(-\lambda t\left(1-1/t\left(\int_0^\tau P_S(c(\tau)s)d\tau\right)\right)} f_S(s)ds \tag{1-1}$$

The purpose of the interior integral is to calculate the survival probability of the ship under the sea conditions and subjected to possible corrosion degradation. Equation 1-2 (Ayyub 2011b) provides the specific probability model used in the calculation.

$$P(c(\tau)s) = P(c(\tau)S_u(s) - L_{sw}(t) - L_w(t) \leq 0) \tag{1-2}$$

In Equation 1-2 c(τ) represents a corrosion degradation model, the form of which need not be specified, as any justifiable form may be chosen. $S_u(s)$ is the strength value from the corresponding outer integral in terms of s while $L_{sw}(t)$ is the still-water bending moment as a function of time (Ayyub 2011b). Ordinarily the still-water bending moment is constant over time, however if changes to the superstructure of a ship are expected then this value could be time dependent. Lastly $L_w(t)$ is the wave loading, taken as the largest distribution of the wave loading for the year in question.

Inherent assumptions are made in Equation 1-2, the first is that the still-water bending moment is additive to the wave induced loads. The assumption that the still-water bending moment can be uncoupled from the wave loading was be used in the analysis presented here.

Reliability methods based on probability and statistics have inherent complications as well. First of all, the reliability of a ship over the course of its design life, which can be upwards of 40 years, requires extensive computations and models to predict lifetime extreme loads and cumulative effects. Also the closed form solutions may be non-existent. In these cases numerical or simulation methods may be used to solve the problems, and in the case of a ship over a 50 year lifespan subjected to large numbers of wave loadings every year requiring the use of computer programs, such as the commonly used program SPECTRA (Michaelson 2000) by the U. S. Navy and Coast Guard, that will be the focus of the research presented here.

1.2. Overview of the SPECTRA Computer Program

The history of SPECTRA dates back to the 1980’s and is based on the work of the Sikora et al. paper A Method for Estimating Lifetime Loads and Fatigue Lives for SWATH and Conventional Monohull Ships (Sikora 1983). A full summary of the paper is outside the scope of this analysis however a general synopsis is important in order to understand the purpose of this report. Sikora et al. (1983) uses a spectral analysis to estimate the lifetime loadings for monohull and Small Waterplane Area Twin Hull ships (SWATHS).

To analyze a ship its operating mode is defined as a block or cell with axes corresponding to its speed, heading relative to waves, and the sea condition (Nikolaidas 1993). These cells are then further subdivided into cells for a particular mode, which may consist of a range of speeds, sea heights, or headings with a certain probability of being in that range. Each of these incremental modes results in a characteristic response and all of these responses can be combined to calculate exceedance levels for the ship (Sikora 1983).

Sikora et al. (1983) provided the user guideline for the SPECTRA computer program developed by the Naval Surface Warfare Center, Carderock Division, used primarily for fatigue loading. Several other sea spectra, ship models and predictive models were subsequently added to the library of functions available for use (Sikora 2002). In the program implementation the generated cells were chosen so that they were statistically stationary, meaning the parameters of the distributions determining the relevant quantities of the cell were time invariant. The amount of time a ship is within each of these cells can be calculated based on probabilistic means and wave spectra equations (Sikora 1983).

SPECTRA takes several input parameters related to the ship's structure, the sea state, and the loadings of interest and runs them through a simulation to predict the lifetime loadings for a given ship. These inputs include (Michaelson 2000).

Ship Dimensions – Includes beam length, displacement, length between perpendiculars, and draft and sets up a rudimentary definition of the ship's structure.

Calculation Location – Defines where the extreme load calculations will be taken.

Still-Water Bending Moment – Calculated from a separate program.

Service Life – How long the ship can be expected to be in operation.

Ship Type – Includes different classes for Navy ships

Bow Form – The shape of the bow, used in whipping calculations.

Sea Spectrum – Several models of sea conditions are available including the Ochi 6 parameter model used in the 1983 paper.

Sea State Probabilities – The likelihood of being in a certain sea state based on the ocean the ship is expected to operate in.

Operational Profile – Details how often the ship can be expected to be at sea, and under what conditions, i.e. combat, slow cargo, etc.

Response Amplitude Operators (RAOs) – Represents the means by which the seaway produces the bending moments.

Bending Type – Vertical, lateral, or torsional.

Whipping – How are whipping loads induced, several options of slamming are included.

Average Time Between Slams – Time between slams assuming the ship is in conditions where slamming can occur.

Whipping Frequency – Frequency at which the ship will vibrate under whipping conditions.

Log Decrement of Whipping – Defines how the vibrations due to whipping dissipate through damping.

Whipping Phase Angle – The point in the time history of bending moment where slam induced whipping begins.

After the ship is defined by the input the program is run resulting in output similar to that in Figure 1.2.

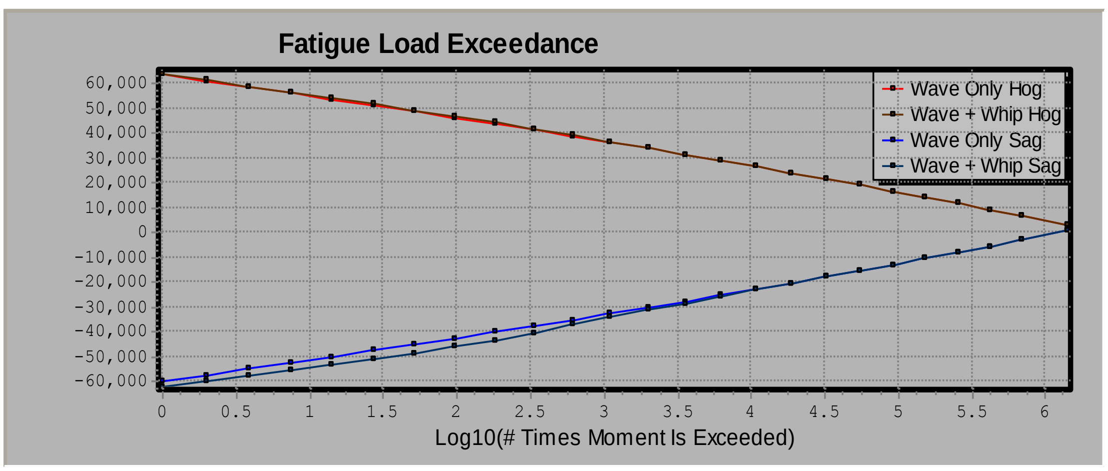

Figure 1.1. SPECTRA Example Exceedance Histogram Output

Figure 1.2 gives an example of a Fatigue Load Exceedance Histogram. The values are determined from values in Tables 1.1-a and 1.1-b, example tables for exceedance values from SPECTRA. These values are divided into two tables the first is without whipping effects included and the second table with whipping. Exceedance in this case means the number of times a particular load has been exceeded.

Table 1.1. Example Exceedance Results

(a) Vertical Bending (ft-lton) without Whipping

Hogging	Sagging	# Times Exceeded
63573	-59573	1
61110	-57110	1.965
58647	-54647	3.834
56184	-52184	7.423
53721	-49721	14.268
51258	-47258	27.217
48795	-44795	51.518
46332	-42332	96.748
43869	-39869	180.207
41407	-37407	332.836
38944	-34944	609.363
36481	-32481	1105.513
34018	-30018	1986.789
31555	-27555	3536.104
29092	-25092	6231.869
26629	-22629	10875.225
24166	-20166	18796.656
21703	-17703	32190.028
19240	-15240	54645.685
16777	-12777	91982.345
14315	-10315	153520.53
11852	-7852	254028.004
9389	-5389	417104.288
6926	-2926	685964.157
3231	769	1413835.687

(b) Vertical Bending (ft-lton) with Whipping

Hogging	Sagging	# Times Exceeded
63794	-61899	1
61374	-59752	1.965
58954	-57618	3.834
56541	-55484	7.423
54139	-53362	14.268
51731	-51122	27.217
49293	-48657	51.518
46765	-45986	96.748
44218	-43091	180.207
41671	-40108	332.836
39137	-37125	609.363
36614	-34143	1105.513
34104	-31210	1986.789
31606	-28339	3536.104
29119	-25569	6231.869
26645	-22886	10875.225
24171	-20290	18796.656
21709	-17750	32190.028
19241	-15254	54645.685
16779	-12777	91982.345
14311	-10318	153520.53
11848	-7847	254028.004
9386	-5388	417104.288
6924	-2930	685964.157
3231	764	1413835.687

A similar set up is used for the fatigue loading, the only difference is that the cycles are counted for a specific moment. The fatigue loads were the ones used for the analysis, as they were more in line with a traditional histogram.

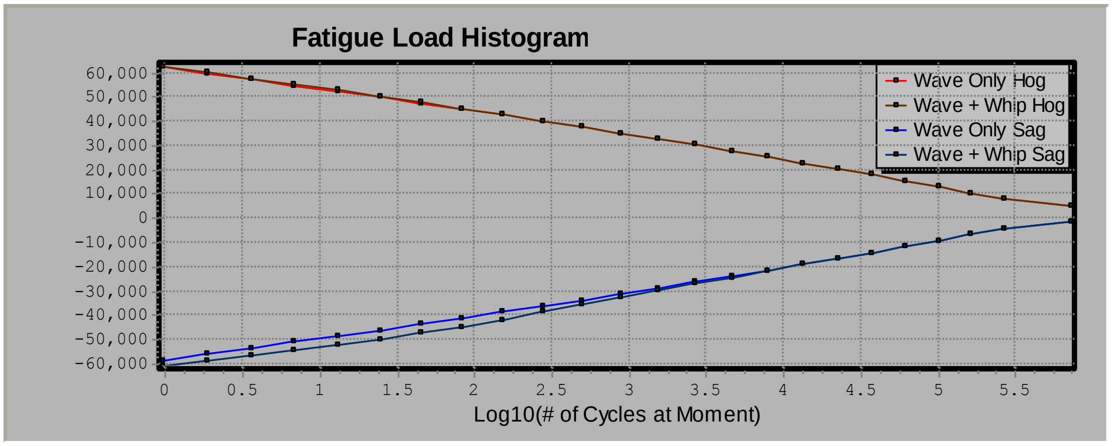

Figure 1.3. SPECTRA Example Fatigue Load Histogram Output

Table 1.2. Example Fatigue Load Results

(a) Vertical Bending (ft-lton)

Hogging	Sagging	# Cycles at Moment
62341	-58341	0.965
59878	-55878	1.868
57416	-53416	3.59
54953	-50953	6.845
52490	-48490	12.949
50027	-46027	24.301
47564	-43564	45.23
45101	-41101	83.459
42638	-38638	152.629
40175	-36175	276.527
37712	-33712	496.15
35249	-31249	881.276
32786	-28786	1549.315
30323	-26323	2695.764
27861	-23861	4643.357
25398	-21398	7921.431
22935	-18935	13393.373
20472	-16472	22455.656
18009	-14009	37336.66
15546	-11546	61538.185
13083	-9083	100507.474
10620	-6620	163076.284
8157	-4157	268859.869
5079	-1079	727871.53

(b) Vertical Bending with Whipping

Hogging	Sagging	# Cycles at Moment
62584	-60825	0.965
60164	-58685	1.868
57748	-56551	3.59
55340	-54423	6.845
52935	-52242	12.949
50512	-49890	24.301
48029	-47322	45.23
45491	-44539	83.459
42945	-41600	152.629
40404	-38617	276.527
37876	-35634	496.15
35359	-32676	881.276
32855	-29775	1549.315
30363	-26954	2695.764
27882	-24227	4643.357
25408	-21588	7921.431
22940	-19020	13393.373
20475	-16502	22455.656
18010	-14015	37336.66
15545	-11547	61538.185
13080	-9082	100507.474
10617	-6618	163076.284
8155	-4159	268859.869
5078	-1083	727871.53

Reliability results are also provided and are given for both the hogging case Table 1.3-a and the sagging case in Table 1.3-b. These include a table for the reliability fit at specified reliability values as well as the parameters of the Weibull used to generate them.

Table 1.3. Example Hogging Reliability Output

(a) Hogging Reliability Output

Hog (ft-lton)	Reliability	Weibull Fit
63794	0.367746	64032
65648	0.55214	65794
67502	0.703694	67656
69356	0.812882	69542
71210	0.885424	71410
73063	0.931266	73241
74917	0.959339	75029
76771	0.976188	76775
78625	0.986163	78480
80479	0.992011	80149
82332	0.995414	81786

Weibull Slope	=	1.353
Truncation Value	=	60605
Scale	=	6101
Mean X	=	5593
Variance X	=	17472456

(b) Sagging Reliability Output

Sag (ft-lton)	Reliability	Weibull Fit
-61899	0.367802	-61978
-63816	0.579327	-63878
-65733	0.742693	-65874
-67650	0.850483	-67866
-69567	0.915644	-69810
-71483	0.953184	-71689
-73400	0.974251	-73501
-75317	0.985909	-75252
-77234	0.992311	-76946
-79151	0.995813	-78592
-81068	0.997724	-80195

Weibull Slope	=	1.355
Truncation Value	=	-58804
Scale	=	-5643
Mean X	=	-5171
Variance X	=	14884569

Table 1.3 shows that Weibull parameters are given in SPECTRA. The problem that was encountered was whether or not the given parameters could be used as the values for a largest distribution. Knowing if the parameters were for a largest distribution was important as the ability to use the given parameters directly would save time and allow a direct extension of SPECTRA. If they could not then a methodology had to be used to take the histogram data and forecast distributions for the following years.

1.3. Research Objectives

The objectives of the research presented herein were to firstly characterize the underlying nature of SPECTRA's output and meaning of the Weibull distribution parameters produced by going through a statistical analysis, and secondly to develop a method of using the SPECTRA fatigue load output and extending it for forecasting extreme wave loading distributions to be used in reliability calculations similar to Equation 1-1. Extending the utilization of SPECTRA beyond the current analysis to forecasting and ship reliability would provide a more powerful tool that could be used in extreme lifetime reliability analysis, not just the cumulative lifetime load effect.

2. Methodology

2.1. Methodology Summary

2.1.1. Methodology Steps

The methodology consisted of the following steps:

1) Define assumptions
2) Make initial data observations
3) Determine parent distributions
4) Analyze parent distributions
5) Estimate Moments of Extreme Load Distributions based on Step 4
6) Analyze Results of Forecasting

The steps utilized entailed statistical analysis methods. The Excel add-on program @Risk (Palisade 2010) was used to expedite this process. @Risk extends Excel functionality by adding a wide range of statistical and probabilistic functions, and simulation capabilities. For the purposes of this analysis only the statistical functions were required.

2.1.2. Overview of Extreme Value Analysis and the Weibull Distribution

A primary consideration in life-time reliability assessment is the definition of the extreme value distribution of loads for which a brief overview is provided in this section. The premise behind extreme loads is that in designing buildings, vessels, or other structures which will be subjected to natural loads that vary widely it is important to design for the likely largest loading anticipated. Historical data are available for certain cases however most are system dependent. Accounting for and predicting largest loads usually involves probabilistic techniques. Common examples of largest loads include 100 year floods, 500 year earthquakes, and so on. Finding such quantities necessitates that the distributions for those years be forecasted. In general their PDF and CDF will take the respective forms of (Ayyub 2011a).

$$f_{M_k}(x) = k f_X(x)[F_X(x)]^{k-1} \tag{2-1}$$

$$F_{M_k}(x) = [F_X(x)]^k \tag{2-2}$$

where k is the number of observations or, in the case of this report, years. Using these the moments can be found and the design loadings determined. The calculation of the moments is generally non-trivial, and the use of computer software to determine them is commonplace. Most regularly used cumulative distribution functions and probability density functions cannot be integrated into closed form solutions, and numeric procedures have to be used to evaluate them. In the case of wave loading the Weibull smallest distribution is frequently chosen (Ayyub 2011b). Although it is called the smallest distribution it is lower bounded, making it better suited than the Weibull largest distribution for this case. The Weibull is flexible in the shape its PDF can represent by changing the shape parameter. Another advantage is when the Weibull is reduced to a two parameter distribution the method of moments (Al-Fawzan 2000) provides an extremely effective parameter estimation process.

2.2. Assumptions

The primary assumption in the analysis was that the Weibull was a good model for the data. The Weibull is often used for modeling extreme events and SPECTRA gave parameters for such a distribution as part of its output. Using a Weibull allowed a direct comparison for the SPECTRA values and modeled parameters. Another assumption made in using the Weibull was that it was unshifted so its lower bound was zero. Doing so was equivalent to saying that the still-water bending moment was zero. The two previous assumptions led to the following model which was used throughout this report.

$$f_X(x) = \frac{\alpha}{\beta^\alpha} x^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)^\alpha} \tag{2-3}$$

$$F_X(x) = 1 - e^{-\left(\frac{x}{\beta}\right)^\alpha} \tag{2-4}$$

with

shape factor = $\alpha \geq 0$

scale factor = $\beta \geq 0$

Equation 2-3 is the most regularly used form of the Weibull and was chosen mainly for computational ease (Ayyub 2011a). Its use was to be proven justifiable through the process of analysis. For the purposes of verifying the moment estimation methodology the location of the lower bound was not paramount. The measure of effectiveness was whether or not the predictions for any initial mean and standard deviation converge with the actual distribution of interest. Making the zero shift assumption presented a risk of skewing the mean and standard deviation of the histogram data away from the actual values. Any concerns from the shift assumption would be dealt with in the analysis and recommendations.

If a shift was required a general procedure was determined to allow a two parameter Weibull distribution to be used. Appendix A it illustrates that the scale factor for a distribution taken from some shifted value "ω" to the origin is

$$\beta_s = \beta_0 + \omega \tag{2-5}$$

where β_0 is the scale parameter from a distribution fitted to the histogram data with the moment values shifted as follows.

$$x_0 = x_s - \omega \tag{2-6}$$

The assumption made in Equations 2-3 and 2-4 was that the moments induced on a ship remained in the linear elastic range. Assuming they were in a linear elastic range meant they remained additive. Considering the nature of the reliability model given in Equation 1-2, this assumption was taken as valid. Any extension of uncoupling moments in other circumstances however would require a more rigorous examination. An advantage of uncoupling the still-water bending moment is that it makes the distribution based on β_0 equivalent to the wave loading only. Therefore a distribution based on β_0 allowed the resulting distributions to be used directly in Equation 1-1.

The histogram served as the statistical basis for representing the underlying distribution. Assuming that the fatigue loading histogram could be used similar to a loading one meant a final validation of the methodology would be required. Fatigue loading gives the number of cycles a ship experiences at a given bending moment, so it seemed reasonable to expect that it could represent a general wave loading distribution.

The purpose of a histogram is to visually approximate the underlying distribution in a data set. Accuracy of a model fitted to those histogram points is dependent on the number of bins chosen. Too few bins results in insufficient points to define a distribution adequately, while too many allows noise in the data to affect the distribution (Ayyub 2011a). It was assumed out of necessity that the histogram bins were properly defined as their generation method was unknown.

In order to use the @Risk software the sagging moments were all made positive in calculations. Doing so had no actual bearing on the shape and scale factors of the Weibull and simply mirrored the distribution about the y-axis. Since @Risk runs through Excel, Excel was also used for any non @Risk calculations to maintain commensurate precision in any results.

Two final assumptions served as a check of validity of the results in general. Navy guidelines require that the coefficient of variation of wave loading at 15 years be roughly 0.25 (Ayyub 2011b). All values obtained were checked against this benchmark in the final methodology Validation. The standard deviation would be expected to converge as the value of k increases, as the tails of a distribution die out under repeated multiplications.

2.3. Initial Data Observations

The analysis contained here was contingent upon hypothetical data provided from SPECTRA. Hogging and sagging moment histograms from SPECTRA were provided for 1 year, as well as for 15 to 50 years in 5 year increments. The reliability calculations included the Weibull parameters in question. All data was supplied "as is" and contained none of the input used to generate it. Using "as is" data lead to certain aspects regarding the accuracy of the program to be taken as given for lack of a way to independently validate them. Program code for SPECTRA was not accessible due to it being proprietary.

Prior to a more rigorous analysis of the data it was examined for any obvious insights that could provide guidelines or checks. Reviewing the histogram data from the output it appears that the sagging and hogging moment distributions were roughly symmetric about the still water bending moment. The symmetry in the data provided a sanity check of any generated parent distributions. Secondly the given parameters for the Weibull fit seem off. For the years of interest they are almost constant which would not be expected of a largest distribution. The mean and standard deviation appeared to be near constant as well. Constant values suggested that they are either predefined in doing the programs Weibull calculations, or indicated the distribution for each year is not actually altering any of the histogram statistics. Thus by adding more years more refinement in statistical moment estimation was added. The nature of these problems warranted investigation in the course of the analysis.

The maximum wave loading also increased from year to year due to an increased exposure to the random sea environment. This trend makes sense as the more years a ship is at sea the more likely it is that it will experience an extreme wave loading, while not surprising it did add some confidence that the wave loading calculations were correct. Thus the hypothetical data provided was assumed to represent a valid SPECTRA result output.

Some observations were made on the way in which SPECTRA displayed the output. SPECTRA used both exceedance levels which are the probability of a wave inducing a higher bending moment than the one in question and fatigue loads. This division proved useful in determining how to proceed with the creation of the parent distributions. SPECTRA also handled whipping as a separate variable and added it through some mathematical means to the original bending moments independently (Nikolaidas 1993). In this analysis the histograms including whipping effects were utilized.

2.4. Basic Statistical Analysis

2.4.1. Mean and Standard Deviation

Estimation of the mean and standard deviation of the given histogram data were calculated independently of the values given in SPECTRA to validate them. The calculations were done assuming that the bin count could be treated as weight factors. Table 2.1 displays that the mean and standard deviation show little deviation as time increases. A general trend of increasing average can be seen in the results but it is not significant. Table 2.1 indicates additional years only add slightly more refinement to the statistical measures, and not any real additional information. The standard deviation showed no trend and remained in a relatively tight spread, suggesting it was only affected by the randomness of the probabilistic techniques used in SPECTRA, and not by additional time.

Table 2.1. Histogram Mean and Standard Deviations

	Hogging		Sagging	
Year	Mean	Standard Deviation	Mean	Standard Deviation
1	8396.017	4848.457	4407.853	4887.619
15	8647.731	4794.381	4647.663	4821.567
20	8672.877	4788.977	4675.995	4811.643
25	8692.673	4785.624	4698.201	4805.367
30	8709.832	4781.479	4712.524	4799.522
35	8722.692	4777.891	4725.762	4795.668
40	8735.085	4775.172	4739.057	4791.727
45	8746.725	4772.5	4751.861	4787.59
50	8754.203	4771.591	4757.644	4785.634

The calculated means and standard deviations based on the histogram data were off when compared to the given mean and standard deviation by SPECTRA. There is the possibility that the differences in the means was a result of the mean in SPECTRA being calculated using the individual point data as opposed to the histogram data. Still the discrepancy placed sufficient doubt that the mean given in the output refers to the given data that it was neglected for actual use.

2.4.2. Examination of SPECTRA Weibull Parameters

The check of the mean and standard deviation indicated that they may be related to the given Weibull parameters. Since the moments do not agree with the histogram and that they are positioned in the same area of the output guided the decision. It appeared that the given statistics were for the Weibull distribution but they initially appeared to be far too low for that to be the case. There was also the given shift parameter whose determination is not explained in the program documentation (Michaelson 2000). As noted before the shape parameter is independent of any shift so it was decided to estimate a Weibull distribution using the given means and standard deviations. Using the method of moments, a table look-up method, Weibull parameters were calculated based on their statistical moments, resulting in Table 2.2 (Al-Fawzan 2000).

Table 2.2. Weibull Parameters for Given SPECTRA Moments versus Weibull Parameters Given by SPECTRA

		Hogging		Sagging	
	Year	Shape	Scale	Shape	Scale
SPECTRA	15	1.426	6405	1.327	5383
Approximation		1.425	6404	1.328	5384
SPECTRA	20	1.434	6440	1.334	5434
Approximation		1.433	6439	1.335	5434
SPECTRA	25	1.441	6468	1.338	5476
Approximation		1.440	6467	1.339	5477
SPECTRA	30	1.446	6489	1.343	5503
Approximation		1.445	6489	1.344	5504
SPECTRA	35	1.451	6507	1.347	5525
Approximation		1.450	6506	1.346	5525
SPECTRA	40	1.323	5750	1.350	5544
Approximation		1.323	5749	1.349	5544
SPECTRA	45	1.326	5761	1.352	5565
Approximation		1.326	5761	1.352	5565
SPECTRA	50	1.329	5772	1.352	5592
Approximation		1.328	5771	1.352	5592

The agreement between the values in Table 2.2 suggests that the proposed reasoning is correct. Therefore the given Weibull parameters were based on a shifted distribution which had the SPECTRA provided mean and standard deviation of a zeroed distribution and the given shift value. A check of this reasoning was done by examining how the distribution fitted the given values in the CDF table provided in the SPECTRA reliability section. The 30 year hogging data was used for illustration purposes in Table 2.3 though the procedure was repeated for all available years.

Table 2.3. Analysis of 30 Year Reliability Data

Moment	Reliability	SPECTRA Fit	Generated Fit
76153	0.361834	76131	76075.69
78378	0.596576	78414	78413.96
80602	0.767188	80765	80765.32
82827	0.873677	83073	83073.36
84041	0.933958	85299	85299.76
87276	0.96627	87437	87438.44
89501	0.983047	89494	89496.01
91725	0.991584	97480	91482.95
93950	0.995866	93406	93409.79
96174	0.997989	95282	95285.78
98399	0.999031	97115	97119.12

Table 2.3 indicates that using the Weibull parameters as prescribed in this section would provide a distribution that matches the SPECTRA fit. Differences in the results were considered to be due to slight rounding errors.

The SPECTRA provided Weibull parameters do not follow the steps required for a valid extreme value analysis. The remainder of the analysis focused on a true extreme value analysis and a method to take the SPECTRA data and utilize it in reliability calculations.

2.5. Estimation of Parent Distributions

2.5.1. General Procedure

The first step was to create the parent distributions to be used in developing the forecasted distributions. Previous analysis suggested that the mean and standard deviation of any year could theoretically be chosen, however to conform with general practices and for future calculation purposes the first year data was chosen to be the basis of the parent distribution. Creating the parent distributions required using the histogram data taken from the SPECTRA for the one-year data and converting the "number of cycles at moment" column to CDF values. The results of the conversion step are given in Table 2.4. For calculation purposes frequency is the number of cycles at the particular moment over the total number of cycles, and the CDF value is the sum of all preceding frequencies including the current one.

Table 2.4. Conversion of Histogram Data to CDF Values

Hogging (ft-lton)	# Cycles @ Moment	Sagging (ft-lton)	# Cycles @ Moment	% Frequency	CDF
0	0	0	0	0	0
5078	727871.5	1083	727871.5	0.514820818	0.514821
8155	268859.9	4159	268859.9	0.190163582	0.704984
10617	163076.3	6618	163076.3	0.115343247	0.820328
13080	100507.5	9082	100507.5	0.071088561	0.891416
15545	61538.19	11547	61538.19	0.043525729	0.934942
18010	37336.66	14015	37336.66	0.02640808	0.96135
20475	22455.66	16502	22455.66	0.015882802	0.977233
22940	13393.37	19020	13393.37	0.009473083	0.986706
25408	7921.431	21588	7921.431	0.005602799	0.992309
27882	4643.357	24227	4643.357	0.003284229	0.995593
30363	2695.764	26954	2695.764	0.001906704	0.9975
32855	1549.315	29775	1549.315	0.001095825	0.998595
35359	881.276	32676	881.276	0.000623323	0.999219
37876	496.15	35643	496.15	0.000350925	0.99957
40404	276.527	38617	276.527	0.000195587	0.999765
42945	152.629	41600	152.629	0.000107954	0.999873
45491	83.459	44539	83.459	5.90302E-05	0.999932
48029	45.23	47322	45.23	3.1991E-05	0.999964
50512	24.301	49890	24.301	1.7188E-05	0.999981
52935	12.949	52242	12.949	9.15878E-06	0.999991
55340	6.845	54423	6.845	4.84144E-06	0.999995
57748	3.59	56551	3.59	2.53919E-06	0.999998
60164	1.868	58685	1.868	1.32123E-06	0.999999
62584	0.965	60825	0.965	6.82541E-07	1

Table 2.5. Comparison of Fitted Parent Distribution with SPECTRA Source Data

Case		Shape	Scale	Mean (ft-lton)	Std Dev (ft-lton)	COV	RMS Error
Hogging	Fitted	1.1903	6750.5	6363.353	5367.018	0.843426	0.00224
	Histogram	-	-	8396.017	4848.457	0.834023	-
	SPECTRA	1.353	6101	5592.348	4178.457	0.747174	-
Sagging	Fitted	0.58239	2269.2	3553.411	6485.456	1.825135	0.0167
	Histogram	-	-	4407.853	4887.619	1.320361	-
	SPECTRA	1.355	5643	5171.186	3858.442	0.746143	-

Figure 2.1 plots the parent distributions alongside of the SPECTRA histogram data allowing for a visual comparison.

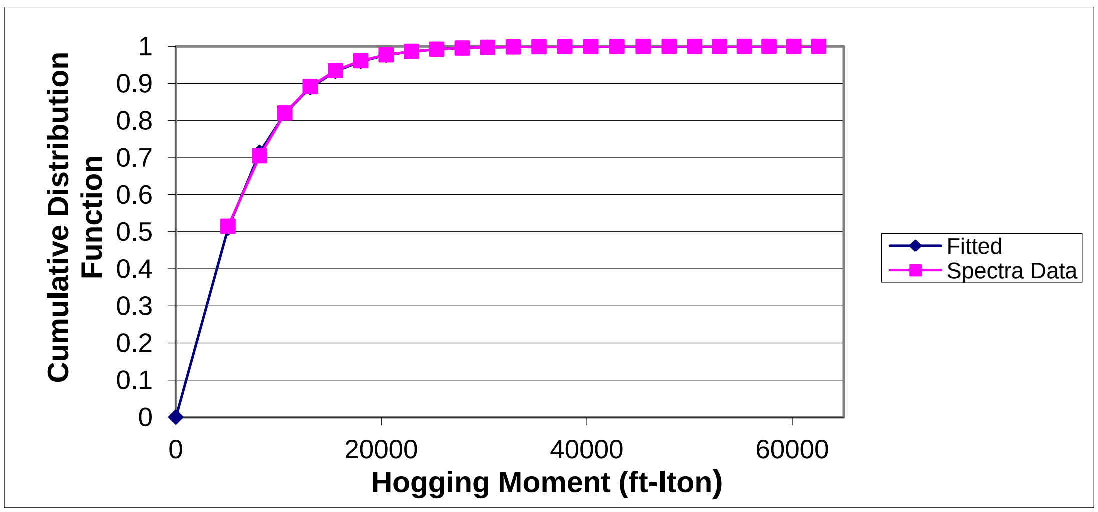

Figure 2.1. (a) Fitted Hogging Moment Parent Distribution versus the SPECTRA Histogram Data

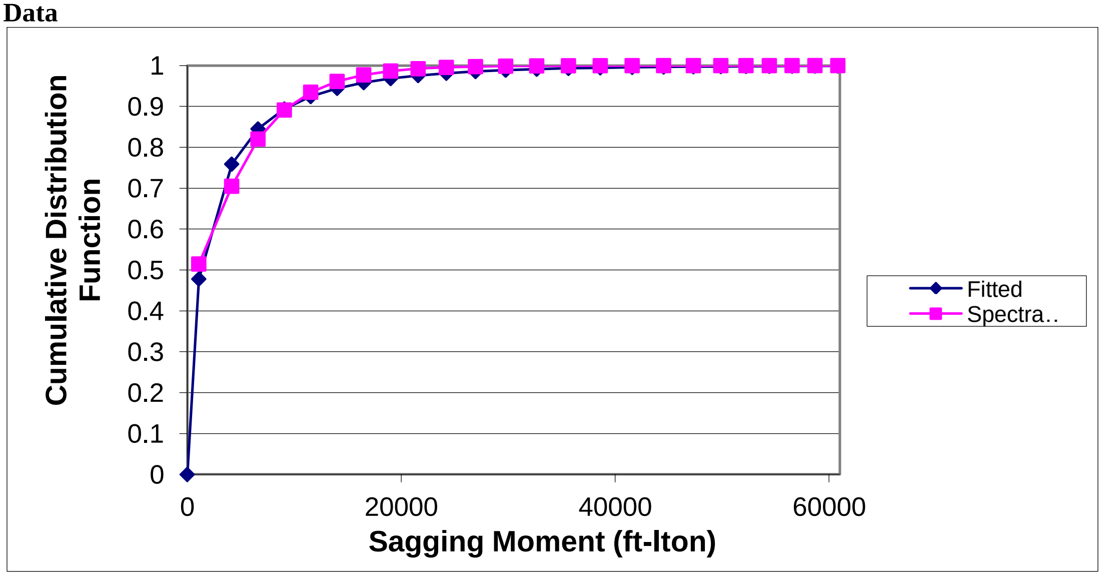

Figure 2.1. (b) Fitted Sagging Moment Parent Distribution versus the SPECTRA Histogram Data

2.5.2. Parent Distribution Analysis

An initial inspection of the results of Figure 2.1 and Table 2.5 was done to check their validity. The root mean square error (RMS) with the SPECTRA histogram output was the first measure and in all cases it was fairly low. A low RMS leant credence to the use of the Weibull as a valid model. Comparing the statistical moments of each distribution there was a large discrepancy. Differences were a result of @Risk treating the data as histogram points, while the histogram moments were done using a weighted average and standard deviation. Despite the fitted distributions being off it was decided to use them. Using the generated parent distribution was the only way to compare the accuracy of the estimation methods. Any estimation comparison required a baseline and the fitted distributions provided that. Corrections to the parent distribution generation process would be accounted for once a valid estimation procedure was obtained.

Two items had to be considered, the significant difference between the SPECTRA parameters and parent distribution parameters, and the loss of symmetry in the data. It is possible that the mean reported by SPECTRA was calculated based on all generated data points causing the difference. Still these values cannot be used confidently and were kept only for comparison purposes. The decision as to whether or not to proceed with the fitted distributions needed to be made. Due to the closeness of fit with the histogram data they could be used for the purposes of estimation method comparison. The overall accuracy in terms of a final methodology would be checked after a complete evaluation of the proposed approximation process.

The loss of symmetry in the Weibull parameters could be corrected by removing the shift as shown in Appendix B. Due to the difference in how @Risk viewed the histogram from the weighted view these parameters were considered equally invalid. As the SPECTRA Weibull parameters were found to be invalid for use in a largest distribution the only question remaining was how to take the histogram data and use it to generate largest distributions. Any of the steps pertaining to the estimation process were going to determine which moment determination method was most suitable for use. Therefore the only requirement was that the methods all use the same initial data and that the distributions fit to them were valid. So long as the initial parameters were consistent for all methods a justifiable comparison could be made. One of the shape parameters being less than one also proved useful. Since the Weibull is very sensitive to the shape parameter it was deemed beneficial to see how the difference would affect the forecasting results. Loss of symmetry in the parent distributions would be handled in the final proposed process. An accuracy check based on the required 0.25 coefficient of variation would be done once the estimation process for forecasting was determined.

2.6. Extreme Value Analysis and Estimation Method

2.6.1. Estimation Methods

The point of forecasting is to get the mean and standard deviation for a given year in the future. Thus the parent distributions were used to get the estimated distributions. Determining the most efficient method to do so utilized four approaches. All methods used the same parent distributions to facilitate comparison. As such the moments obtained should trend towards the same values. The four methods examined were the distribution fitting method, which was similar to the SPECTRA program, integration, both numeric and symbolic, and an approximation.

a. The initial method was the same as in the parent distribution estimation. This technique is based on the concept that if data points are generated point-wise from a distribution then the distribution fit to them should be the same. Accuracy is dependent upon the number of generated points and errors can be introduced through precision losses. Many of the values that are manipulated are close to the upper and lower bounds of Excel's precision limit. With sufficient points an accurate model could be produced. The main purpose of using this method was to supply a base line for comparison for the subsequent methods.

Front end calculations were required to be able to fit the distribution. First the parent distribution was used generate CDF values for discreet values of the moment in increments of 1000 ft-lton. The generated points were then raised to the power of the year of interest to produce the CDF value of the largest distribution. A distribution was fit to the newly created CDF points using @Risk. Appendix C-1 provides a numeric example of the process used here.

b. The second method a numeric integration technique to determine the mean and standard deviation. For a continuous distribution the mean and variance are defined respectively as

$$\mu = \int_0^{\infty} x f_X(x) dx \tag{2-7}$$

$$\sigma^2 = \int_0^{\infty} (x-\mu)^2 f_X(x) dx \tag{2-8}$$

where the standard deviation is the square root of the variance. In the case of the extreme value distribution equation with the Weibull function as the parent distribution Equations 2-7 and 2-8 become

$$\mu_{M_k} = \frac{k\alpha}{\beta^\alpha} \int_0^{\infty} x^\alpha e^{-\left(\frac{x}{\beta}\right)^\alpha} \left[1 - e^{-\left(\frac{x}{\beta}\right)^\alpha}\right]^{k-1} dx \tag{2-9}$$

$$\sigma_{M_k}{}^2 = \frac{k\alpha}{\beta^\alpha} \int_0^{\infty} \left(x - \mu_{M_k}\right)^2 e^{-\left(\frac{x}{\beta}\right)^\alpha} \left[1 - e^{-\left(\frac{x}{\beta}\right)^\alpha}\right]^{k-1} dx \tag{2-10}$$

Equations 2-9 and 2-10 come courtesy of Probability, Statistics, and Reliability for Engineers and Scientists (Ayyub 2011a).

Front-end calculations were required for numeric integration as well. An initial step size was chosen and the value of equation 2-9 was found using the trapezoid rule to find the approximate areas of each interval. Any valid numeric integration technique could be used, and the trapezoid rule was only picked due to its ease in implementation. The calculated values were summed to find the approximate integral.

The mean was calculated first and then used to calculate the variance. Appendix C-2 contains the process as it was done for the hogging parent distribution for 15 years.

An initial step size of 1000 ft-lton was chosen and adjusted until the relative error between step size iterations was less than 1% for both the mean and standard deviation. Such precision is not necessarily required and was only used for the sake of comparison. In the hogging case a 1000 ft-lton step size proved sufficient for a high degree of accuracy. The sagging moment was not as precise. Since the Weibull function is highly sensitive to changes in either the shape or scale factors providing a recommended number of intervals is not feasible. That is also true for the upper cutoff limit of the integration. Thus achieving a specific accuracy level in the general case is not feasible without multiple calculations.

c. Distribution fitting and numeric integration involved repeated calculations and several data points to ensure accuracy. An approximation that would reduce calculations was used to see if it was valid for the input. If so it would increase the efficiency of the estimation process. The reasoning behind such an approximation is that when raising a distribution to large values an extreme distribution will approach an asymptotic form that is independent of the exact parent distribution from which it was generated. Instead it will be more affected by the properties of the tail. If the parent distribution has exponential tails as the Weibull does it can be shown that the extreme value distribution will approach a double exponential asymptotic form (Ayyub 2011a). Therefore it may be possible to approximate the mean and standard deviation of the parent distribution generated from SPECTRA using Equations 2-11 and 2-12 (Ayyub 2011a).

$$\mu_{M_k} = \sigma\mu_k + \mu + \frac{\gamma\sigma}{\alpha_k} \tag{2-11}$$

$$\sigma_{M_k} = \frac{\pi}{\sqrt{6}}\frac{\sigma}{\alpha_k} \tag{2-12}$$

With

$$\alpha_k = [2ln(k)]^{0.5} \tag{2-13}$$

$$\mu_k = \alpha_k - \frac{\{ln[ln(k)] + ln(4\pi)\}}{(2\alpha_k)} \tag{2-14}$$

d. The exact mean and standard deviation were obtained through direct integration of equations 2-7 and 2-8, they are found in Equation 2-15 and 2-16.

$$\mu_{M_k} = k\beta\Gamma\left(1 + \frac{1}{\alpha}\right)\sum_i \binom{k-1}{i}\frac{(-1)^i}{(i+1)^{(1+1/\alpha)}} \tag{2-15}$$

$${\sigma_{M_k}}^2 = k\beta^2\Gamma\left(1 + \frac{2}{\alpha}\right)\sum_i \binom{k-1}{i}\frac{(-1)^i}{(i+1)^{(1+2/\alpha)}} - {\mu_{M_k}}^2 \tag{2-16}$$

with "μ" and "σ" being the mean and standard deviation of the parent distribution respectively and "k" being the number of years.

Equations 2-15 and 2-16 are only valid for a distribution with a zero lower bound. If the function is shifted in any way from this a numerical integration method will most likely be required unless the procedure followed in Appendices A and B is used. The derivation of 2-15 and 2-16 can be found in Appendix C-3.

A full comparison of all results is given in Table 2.6 on the following pages. In analyzing the tables note that the “Fitted” rows correspond to section 2.6.1a, “Numeric Integration” to 2.6.1b, “Exact” to 2.6.1d, and “Approximation” to 2.6.1c. The “SPECTRA” rows give the mean and the standard deviation of the SPECTRA reliability data. The “n” column is the power that any distributions were raised to; while dashes mean that the specific column did not apply to that method. Plots of the variation of mean and standard deviation of the mean with the number of years were also generated and can be found in Figure 2.2 immediately following the summary tables.

Table 2.6. (a) Summary of Moment Estimation Method Results for Hogging Moment

Year		n	Mean (ft-lton)	Std Dev (ft-lton)	COV	RMS Error$_a$	RMS Error$_b$
15	Fitted	15	17849.62	5408.718	0.303016	0.0174	0.2869
	Numeric	15	18325.7	5771.4	0.314935	-	-
	Exact	15	18325.7	5771.4	0.314935	-	-
	Approximation	-	16117.7	2957.767	0.18351	-	-
	SPECTRA	-	5821	4143.183	0.711765	-	-
20	Fitted	20	19146.54	5384.972	0.28125	0.0183	0.3113
	Numeric	20	19638.74	5730.962	0.291819	-	-
	Exact	20	19638.74	5730.962	0.291819	-	-
	Approximation	-	16788.4	2812.165	0.167506	-	-
	SPECTRA	-	5848	4138.875	0.707742	-	-
25	Fitted	25	20130.02	5354.615	0.266001	0.0189	0.3289
	Numeric	25	20651.46	5697.971	0.275911	-	-
	Exact	25	20651.46	5697.971	0.275911	-	-
	Approximation	-	17288.6	2712.94	0.156921	-	-
	SPECTRA	-	5870	4137.097	0.704787	-	-
30	Fitted	30	20938.12	5315.123	0.253849	0.0193	0.3429
	Numeric	30	21474.79	5670.462	0.264052	-	-
	Exact	30	21474.79	5670.462	0.264052	-	-
	Approximation	-	17685.41	2639.225	0.149232	-	-
	SPECTRA	-	5886	4133.717	0.702297	-	-
35	Fitted	35	21617.78	5305.306	0.245414	0.0197	0.354
	Numeric	35	22167.88	5647.032	0.254739	-	-
	Exact	35	22167.88	5647.041	0.25474	-	-
	Approximation	-	18013.15	2581.376	0.143305	-	-
	SPECTRA	-	5899	4130.835	0.70026	-	-
40	Fitted	40	22209.37	5283.498	0.237895	0.0199	0.3634
	Numeric	40	22765.96	5626.716	0.247155	-	-
	Exact	40	22765.96	5626.686	0.247153	-	-
	Approximation	-	18291.63	2534.224	0.138546	-	-
	SPECTRA	-	5292	4037.438	0.762932	-	-
45	Fitted	45	22729.37	5269.292	0.231827	0.0201	0.3715
	Numeric	45	23291.69	5608.84	0.240809	-	-
	Exact	45	23293.72	5600.754	0.240441	-	-
	Approximation	-	18533.28	2494.71	0.134607	-	-
	SPECTRA	-	5300	4034.893	0.761301	-	-
50	Fitted	50	23188.34	5258.44	0.226771	0.0202	0.3784
	Numeric	50	23760.54	5592.919	0.235387	-	-
	Exact	50	23862.68	5152.75	0.215933	-	-
	Approximation	-	18746.4	2460.886	0.131272	-	-
	SPECTRA	-	5308	4034.813	0.760138	-	-

a RMS error with respect to generated points to fit distribution
b RMS error with respect to SPECTRA Data

Table 2.6. (b) Summary of Moment Estimation Method Results for Sagging Moment

Year		n	Mean (ft-lton)	Std Dev (ft-lton)	COV	RMS $Error_a$	RMS $Error_b$
15	Fitted	15	18330.65	11137.58	0.607594	0.013	0.3186
	Numeric	15	19321.62	13451.77	0.696203	-	-
	Exact	15	19325.93	13482.66	0.697646	-	-
	Approximation	-	15340.48	3574.139	0.232987	-	-
	SPECTRA	-	4952	3768.508	0.761007	-	-
20	Fitted	20	20822.39	11710.93	0.56242	0.0143	0.3554
	Numeric	20	21953.8	14147.65	0.644428	-	-
	Exact	20	21959.55	14185.83	0.645998	-	-
	Approximation	-	16150.95	3398.195	0.210402	-	-
	SPECTRA	-	4993	3780.586	0.757177	-	-
25	Fitted	25	22894.18	12158.52	0.531075	0.0152	0.3818
	Numeric	25	24113.34	14676.53	0.608648	-	-
	Exact	25	24120.52	14721.58	0.610334	-	-
	Approximation	-	16755.38	3278.292	0.195656	-	-
	SPECTRA	-	5030	3797.474	0.754965	-	-
30	Fitted	30	24642.77	12520.99	0.5081	0.016	0.3535
	Numeric	30	25952.55	15101.63	0.581894	-	-
	Exact	30	25961.17	15153.22	0.583688	-	-
	Approximation	-	17234.89	3189.215	0.185044	-	-
	SPECTRA	-	5051	3801.111	0.752546	-	-
35	Fitted	35	26182.19	12840.89	0.490444	0.0166	0.3721
	Numeric	35	27559.09	15456.14	0.560837	-	-
	Exact	35	27569.14	15514.03	0.562732	-	-
	Approximation	-	17630.93	3119.311	0.176923	-	-
	SPECTRA	-	5069	3804.434	0.75053	-	-
40	Fitted	40	27559.67	13135.61	0.476624	0.0172	0.3883
	Numeric	40	28988.3	15759.64	0.543655	-	-
	Exact	40	28999.8	15823.59	0.545645	-	-
	Approximation	-	17967.43	3062.334	0.170438	-	-
	SPECTRA	-	5084	3807.607	0.748939	-	-
45	Fitted	45	28793.95	13345.6	0.463486	0.0176	0.403
	Numeric	45	30277.55	16024.58	0.529256	-	-
	Exact	45	30290.68	16094.04	0.53132	-	-
	Approximation	-	18259.44	3014.585	0.165097	-	-
	SPECTRA	-	5102	3814.669	0.747681	-	-
50	Fitted	50	29912.66	13576.14	0.453859	0.018	0.4156
	Numeric	50	31453.27	16259.4	0.516938	-	-
	Exact	50	31473.06	16324.55	0.518683	-	-
	Approximation	-	18516.98	2973.713	0.160594	-	-
	SPECTRA	-	5126	3831.684	0.7475	-	-

a RMS with respect to generated points to fit distribution
b RMS with respect to SPECTRA Data

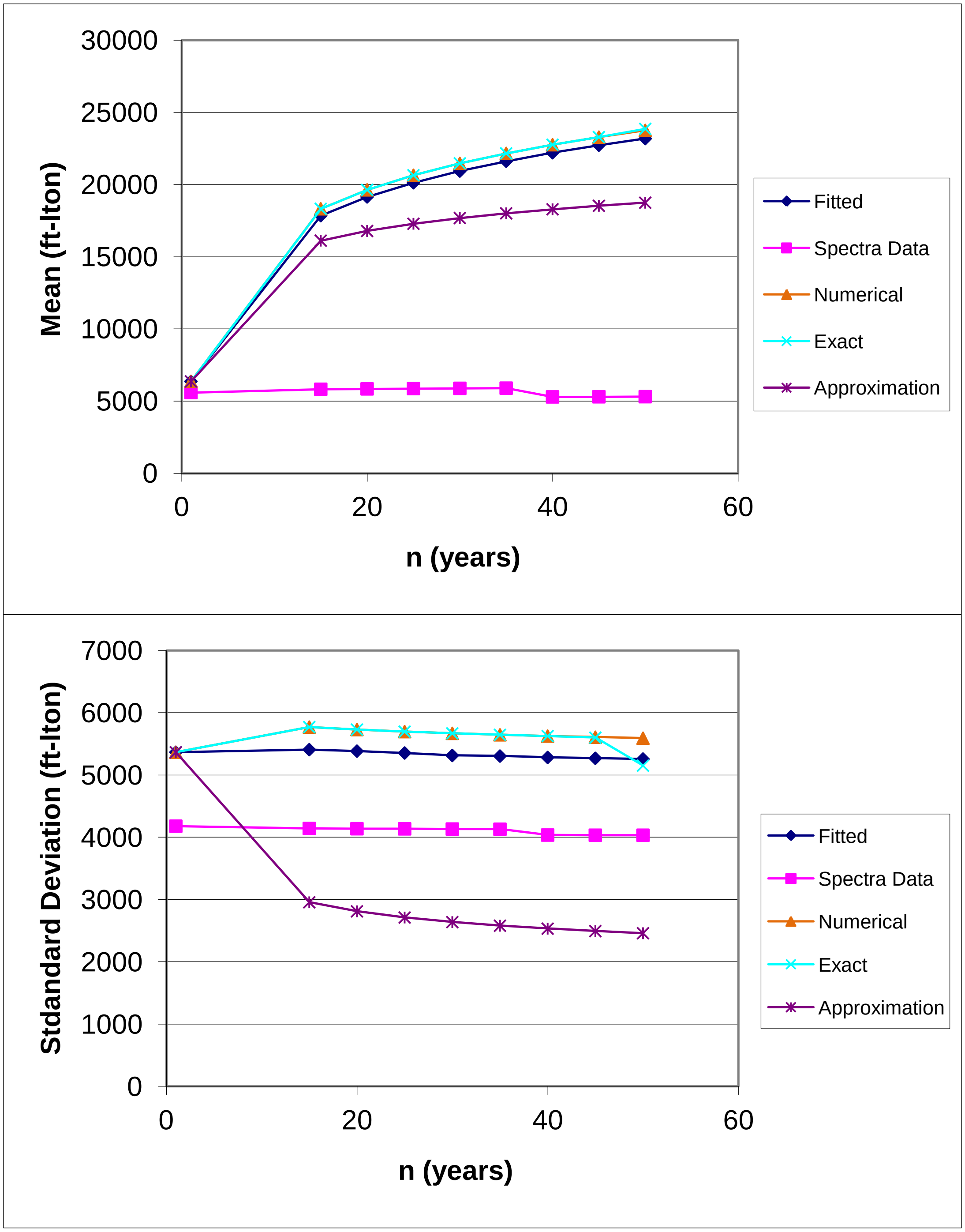

Figure 2.2. (a) Variation of mean and Standard Deviation with Number of Years for Hogging Moment

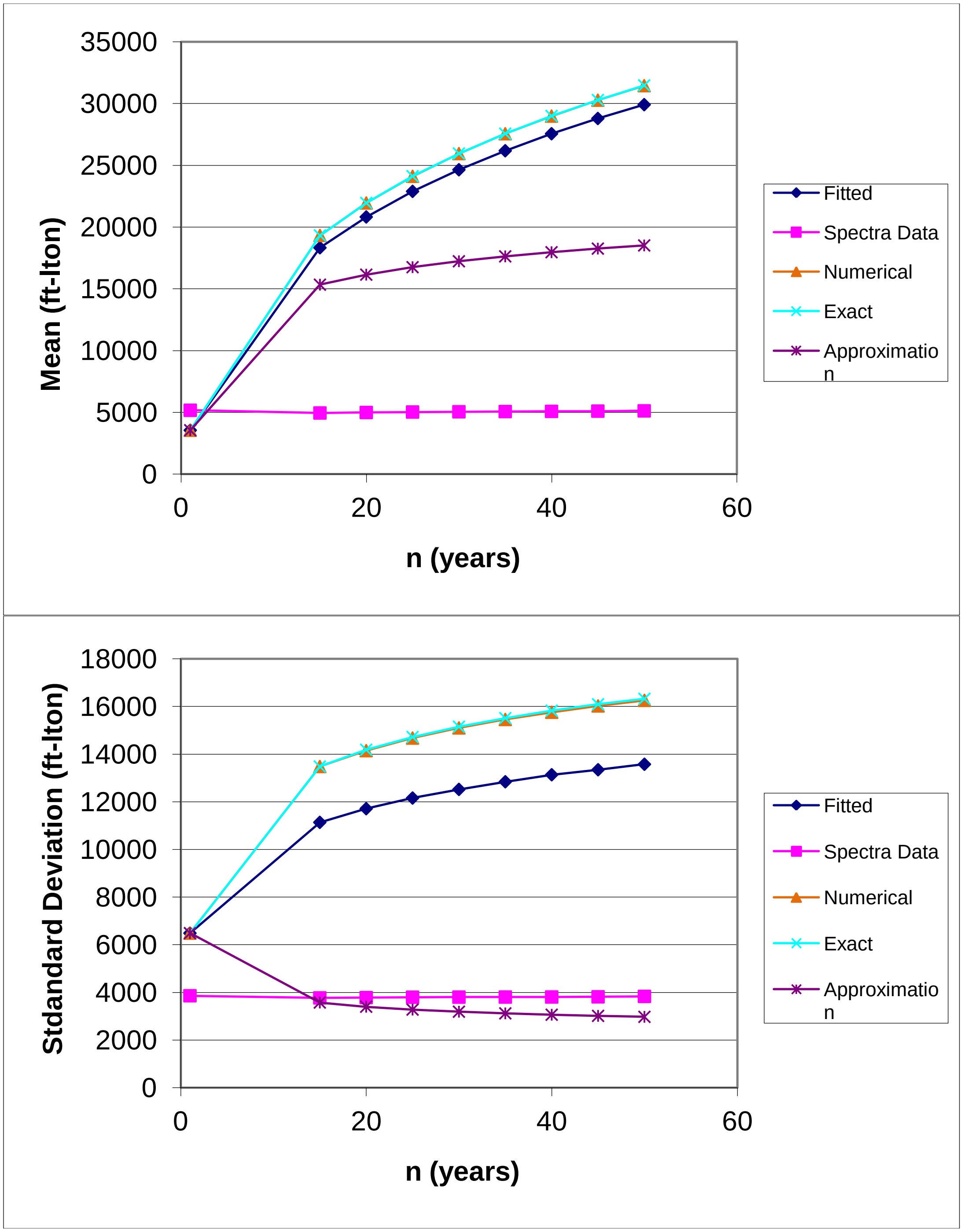

Figure 2.2. (b) Variation of Mean and Standard Deviation with Number of Years for Sagging Moment

2.6.2. Analysis of Moment Estimation Methods

The results of Section 2.6.1 could be directly compared since all methods were based off of the same parent distributions. A validation of the overall process is found later in Section 2.6.3. Initially the analysis focused on how accurate the forecasting results were in comparison to each other. All relative errors are relative to moments and parameters calculated from the numeric method.

The approximation method did not fit with any of the other methods of analysis. Its relative error of the mean with the integration value for the hogging moment started off at 12% at the 15-year mark and increased to 21.1% at the 50-year mark. Standard deviations calculated from this method had a minimum relative error of 48.8%. The results indicate that the proposed approximation was insufficient to use a forecasting tool. Its reported means and standard deviations were all under values and the rate of deterioration in the predictions is rapid.

The quality that the method of distribution fitting produced for the hogging distribution was also reviewed. Relative errors started off at 2.60% and 6.28% off the standard deviation. Unlike the approximate method distribution fitting results improved with an increase in years. Results of this nature were expected due to the fact that as the “k” value becomes large the exact form of the parent distribution becomes less important. For larger years, any minor discrepancies in the generated distribution become less noticeable. Figure 2.2-b displays a major discrepancy in the sagging distribution moments calculated from the integrated results. The standard deviation results are far off. The error in Figure 2.2-b was most likely a manifestation of the difference in the distribution fitting method minimizing the root mean square error while the other methods were based only off of the given mean and standard deviation. The sagging moment's parent distribution also began with error in the standard deviation. Therefore the divergence in the fitted distribution results was possibly propagation of the initial error.

The results of the exact method and numerical integration were the last be examined. Their results started off identical for the hogging case but starting with the 45th year minor differences emerged. These discrepancies eventually become significant, especially for the standard deviation for the 50th year. Figure 2.2-a illustrates that the exact method eventually diverges significantly from the expected trend. The cause of this error lies in finding the nature of finding the exact solution. For the variance to be positive the following equation must hold true.

$$k\beta^2\Gamma\left(1+\frac{2}{\alpha}\right)\sum_i \binom{k-1}{i}\frac{(-1)^i}{(i+1)^{(1+2/\alpha)}} \geq {\mu_{M_k}}^2 \qquad (2\text{-}17)$$

Round off errors and precision loss in the binomial expansion can result in Equation 2-17 becoming false. Errors in this analysis appear around the 40 to 45 year mark. After these years the loss of precision removes any confidence in the results. Any proposed method must account for the possibility of loss of precision in the binomial expansion calculations in order to be justified.

Numerical integration gave results that seemed more appropriate for the 50th year. The numeric approach also does not require the expansion of the $(x-\mu)^2$ term in order to calculate. Not calculating the $(x-\mu)^2$ term means the variance will not become negative. As the numerical integration method does not diverge it was more robust as an estimation method, though more time consuming.

The sagging moment results of the exact method versus the numeric method did not have the same precision loss. However due to the nature of precision loss there is the possibility round off errors will cancel out. Despite appearing to be correct values above 40 years were considered unjustifiable to be used. Any other differences between the two results stemmed from the nature of the numeric method. Since 1% relative error was deemed acceptable in the analysis no further iterations were done once an error less than that level was reached. Considering for the 50-year standard deviation for the sagging moment the relative error between the final step of size chosen and the previously used step size was roughly 0.46% the accuracy of the integral could be expected to have a maximum error of that value. To compare the exact values, the approximated value and the error between successive step sizes is found in Table 2.7.

Table 2.7. Sagging Exact Method versus Numeric Integration Results

	Mean			Standard Deviation		
Year	Exact (ft-lton)	Numeric (ft-lton)	Step Size Relative Error (%)	Exact (ft-lton)	Numeric (ft-lton)	Step Size Relative Error (%)
15	19325.93	19325.93	4.04165E-05	13482.66	13482.6	0.00070354
20	21959.55	21953.8	-0.026109257	14185.83	14147.65	-0.267892208
25	24120.52	24113.34	-0.029712432	14721.58	14676.53	-0.304562458
30	25961.17	25952.55	-0.033126913	15153.22	15101.63	-0.338872899
35	27569.14	27559.09	-0.03639381	15514.03	15456.14	-0.371321236
40	28999.8	28988.3	-0.039540898	15823.59	15759.64	-0.402248965
45	30290.68	30277.55	-0.042587936	16094.04	16024.58	-0.431900708
50	31473.06	31453.27	-0.045549619	16324.55	16259.4	-0.460459332

Table 2.7 displays that the differences between the two methods are within the range of the maximum possible error due to the step sizes used in calculating the moments numerically. Therefore the exact method could be a very efficient tool so long as the “k” range is valid.

2.6.3. Estimation Method Comparison

Each method had to be analyzed for its suitability as an estimation process. Based on the results the approximation method was not viable as a means of forecasting. The approximation was the simplest to implement and required little time or computational power but it was not accurate enough to be justifiably used.

The exact method was examined for its potential as an estimation process. It was accurate within the range in which it was valid. Programming the method was not difficult. Two main problems needed to be addressed before it could be used. One was the range of validity of the integration result. In order to use the exact method with confidence the range over which the equation could provide accurate results needed to be determined. Doing so required finding a range in terms of the shape factor and year. The other issue was the factorial nature of the permutations in the binomial expansion terms. Factorial calculations can become large quickly, reaching the overflow limit of a computer if the design life is large. Compounding the precision loss was the “α” term in the summation. A ship’s design life is generally short enough that any precision loss will only occur in later years. Then there was also the loss of precision as the number of year’s increases.

Implementing the exact method in a programming language like C++ or MATLAB is simple and does not require as many calculations as the other methods examined. The issue was whether or not a specific year can be determined where precision loss becomes problematic. MATLAB's binomial expansion tool warns when the binomial expansion values will exceed the precision limit so a similar option could be programmed for the calculations here. These binomial expansion values are also multiplied by a term involving the shape factor "α", potentially adding to the precision loss. The multipliers for the mean, denoted M, and standard deviation, denoted S, both for the i^{th} year in the summation terms of Equations 2-15 and 2-16 are given in Equations 2-18 and 2-19.

$$M = \frac{(-1)^i}{(i+1)^{(1+1/\alpha)}} \tag{2-18}$$

$$S = \frac{(-1)^i}{(i+1)^{(1+2/\alpha)}} \tag{2-19}$$

In all cases Equations 2-18 and 2-19 will be less than one, as "α" is always positive. The additional precision loss due to the "α" term will be shifted towards the smallest order of magnitude of any binomial coefficient that can be accurately represented in a program. Depending on the desired level of accuracy Equations 2-18 and 2-19 could guide when to start using the numeric method. If the moments are only required to be found to the a 10^0 order of magnitude then at the point that a code is only capable of giving results of that accuracy that value could be checked with numeric integration. After that point a numeric method could be used.

A check of the numeric approach for accuracy and efficiency was done. Upfront calculations were required in order to find the PDF and generate the values for each step of the integrals. No curve fitting was required, nor any specialized software to do the integration. One source of complication was finding a step size that would result in the desired accuracy. For this data set there was a step size of 1000 ft-lton was generally sufficient but two complete integrations were required to verify the results. An advantage of the numeric approach was that was valid even if the lower bound was shifted away from the origin. Overall the approach proved robust, and easy to implement. The method was calculation heavy but did not require any distribution fitting, and it could produce results as accurately as desired.

The distribution fitting method also needed to be looked at for its capabilities and to see if any improvement was actually made. Two primary faults with it were that it introduced errors in regression, and it required point generation and curve fitting. The former problem emanated from the repeated multiplication of small numbers together. Calculations of this type could cause a program to reach its precision limit quickly; resulting in any fitted curve produced providing incorrect parameters. Similar to the exact method the error appears for longer design lives and could cause the results to diverge from expected trends. The point generation was also problematic. Depending on the desired accuracy, and the nature of the distribution, a large number of points may be required in order to get an appropriate fit. Compounding the point generation problem is the curve fitting process, which took extra computational time. For the case examined here neither problem proved too problematic. The @Risk software fit the distribution quickly and the calculations to generate the necessary points were easily handled. One benefit of the distribution fitting technique is that it is valid regardless of the lower bound of the Weibull meaning no shift or alteration to the given data. Comparing methods, distribution fitting was not as robust as the numeric integration and not as efficient as the exact method.

It is the conclusion of this report that the mean and standard deviation be calculated by the exact method for k up to the point the loss of precision is unacceptable. Unless the forecasting goes beyond 40 years precision loss should not be problematic. The exact method requires fewer calculations and is accurate to the level of precision in any program used. A numeric approach is more robust and there are several algorithms optimized for this purpose and may also be used up to the 40 year mark. After 40 years all calculations should be done using a numerical approach. Numeric integration is more calculation heavy than the other two but resolves the issue of precision loss.

2.6.3. Validation of Methodology

A final check of if the fatigue load results could be used to estimate an extreme loading distribution was done. The mean and standard deviation of the hogging fatigue load histogram with the still-water bending moment uncoupled were calculated. A parent distribution was generated and an estimated distribution was found for the 15th year. The coefficient of variation was calculated and compared to the 0.25 prescribed as per Ayyub 2011b. Table 2.8 displays these results

Table 2.8. Results of 15 Year Estimation for Shifted Data

1 year Histogram, Hogging		
Mean	=	6396.017
Standard Deviation	=	4848.457

Parent Distribution	
Shape =	1.332183
Scale =	6958.167

15 Year Estimated Moments		
Mean	=	16907.38
Standard Deviation	=	4733.13
COV	=	0.279945

15 Year Estimated Parameters	
Shape =	4.005104
Scale =	18651.93

Symmetry of the hogging and sagging moments meant only the one of the bending moment results needed to be done. The exact method was used for calculating moments as 15 years was within the limit where precision errors were negligible.

Table 2.8 shows that the coefficient of variation is fairly close to the 0.25 given. Considering that other sources accept up to 0.3 it can be stated that the fatigue data can be justifiably used in the manner that it was during the analysis for this data set.

3. Observations and Recommendations

3.1. SPECTRA Values and a Proposed Forecasting Method

3.1.1. SPECTRA Values

Analyzing the SPECTRA output revealed that the given Weibull parameters were not fit to be used for an extreme value distribution. The mean and standard deviations presented are used for the reliability section of the output and not the statistical moments for the histogram data. Any given parameters or moments from SPECTRA cannot be used for the purposes of any extreme load estimation.

3.1.2. Proposed Estimation Method

The analysis lead to the development of a method to use the fatigue load data to generate extreme load distributions. An adaptive methodology was chosen to achieve a desired level of accuracy in an optimal manner. By utilizing both the exact method where it provides valid results and numeric integration when the precision limit of a program is starting to reach its limit the limitations of both can be overcome. Any program should us double precision in implementation and avoid a long double format due to the lack of consistency in how long double precision numbers are defined across compilers and programs.

1) Calculate the mean and standard deviation from the histogram data treating the number of cycles at a given moment as weight factors. These values should have the still-water bending moment, if any, uncoupled so that they represent only the wave loading, this can be done based on the work of

Appendices A and B. Alternatively the still water bending moment may be set as zero for the input into SPECTRA, thus giving only the wave induced moments (Nikolaidas 1993).

2) Using any valid parameter estimation technique like the method of moments (Al-Fawzan 2000) or a root finding algorithm, estimate the parent distribution parameters using the statistical moments from Step 1 and zero as the shift factor.

3) Estimate the mean and standard deviation for any year in question using an adaptive method. For times below 40 years the exact method provided here should be used, however beyond that it any values obtained from the exact method should be checked using a numerical method. When using numeric methods the moments should be recalculated until the relative error between two successive calculations achieves the desired level of accuracy.

4) Estimate the parameters of a Weibull using the mean and standard deviation calculated in Step 3.

5) Use these distributions in any reliability calculations, such as Equation 1-1.

Only checking the coefficient for a single data set does not provide sufficient grounds to state that the process is valid. Therefore Validation using other data sets is required.

3.2. Applications

The primary application of the proposed method is for extending the usefulness of the SPECTRA program or other similar programs. The need to predict extreme loadings is by not limited to naval engineering. Thus the use of this technique could be extended to several other areas in physics and engineering. Any structure involving a cyclic loading capable of producing extreme loading can benefit from using extreme value analysis on fatigue load data. Wind loading on bridges is one example of a situation where this procedure could be used. The loading in this analysis followed a Weibull but other distributions could be used.

3.3. Further Research

The analysis done here could be used as a basis for other studies. A full examination of the effects of the shape factor on the divergence of the integrals could be beneficial in increasing the efficiency of the any program that utilizes the proposed method. A general means of finding a valid k range would also increase confidence in the results. Other approximation methods similar to Equations 2-11 and 2-12 could be developed. Doing so would increase the efficiency of the methodology by removing the need for a numeric integration process or an adaptive technique. Sensitivity analysis to check error propagation through the process would help identify where the most accuracy is lost. Knowing where the error is accumulated would allow recommended tolerances to be determined and which areas require further refinement. Other studies could potentially be done to check applicability to other engineering fields and systems.

4. Conclusions

Histograms and probability distributions based on lifetime fatigue load data are needed for ship design. Several programs, like SPECTRA, have been coded specifically to estimate their characteristics. However the extreme loading events that a ship may encounter are also of importance. An analysis was done to see if it was possible to extend the functionality of lifetime fatigue loading output for use in estimating extreme loading distributions. Doing so entailed an analysis of the hypothetical SPECTRA output for its validity in the use of an extreme value estimation. Analysis showed that only the fatigue load histograms were usable and parent distributions were generated from it and compared. Two predominant issues in the resulting distributions were found, the means of the fitted distribution did not agree with the calculated histogram mean, and the symmetry of the histogram loading data was lost. The analysis continued using only the @Risk distributions and statistical moments to allow a baseline for the comparison of the moment estimation methods. Errors in the validity of the method would be adjusted in the final recommendations after a method was chosen and verified. The comparison consisted of a proposed approximation method which was wholly unsuitable, a distribution fitting method using @Risk, numeric integration, and an exact method involving the closed form solution for the mean and variance of the k^{th} year. Comparing results showed the numeric method proved the most robust, while the exact method was most efficient. Precision loss in calculating the exact moments became problematic after 40 years due to precision loss. A combination of the exact method up to 40 years, combined with a numeric integration method proved to be most efficient. The proposed method is as follows

1) Calculate the mean and standard deviation from the histogram data treating the number of cycles at a given moment as weight factors. These values should have the still-water bending moment, if any, uncoupled so that they represent only the wave loading, this can be done based on the work of Appendices A and B. Alternatively the still water bending moment may be set as zero for the input into SPECTRA, thus giving only the wave induced moments (Nikolaidas 1993).

2) Using any valid parameter estimation technique like the method of moments (Al-Fawzan 2000) or a root finding algorithm, estimate the parent distribution parameters using the statistical moments from Step 1 and zero as the shift factor.

3) Estimate the mean and standard deviation for any year in question using an adaptive method. For times below 40 years the exact method provided here should be used, however beyond that it any values obtained from the exact method should be checked using a numerical method. When using numeric methods the moments should be recalculated until the relative error between two successive calculations achieves the desired level of accuracy.

4) Estimate the parameters of a Weibull using the mean and standard deviation calculated in Step 3.

This adaptive method utilizes both the efficiency of the exact method for years below 40, and the robustness of the numeric integration after that.

The method proposed in this analysis does have limitations. Unavailability of SPECTRA meant only hypothetical data was obtained, and only one set. Therefore the check of the coefficient of variation being near the acceptable value is possibly a statistical anomaly. Ideally more data sets would have been checked, allowing more confidence in the validity of the method. The effect of the shape factor on the divergence of the exact method is also not fully known. The cutoff for the use of the exact method must be carefully chosen as a result and may be ship dependent.

Despite the limitations, the method proposed here has the potential to extend the functionality of fatigue load data ships and possibly other structures. Any system subjected to a cyclic loading that can also produce extreme events may benefit from its usage. The procedure provided does not require any structure dependent calculations in determining the estimated distributions. All that is required is a check that the distributions produced are valid based on historical data or engineering judgment. Further validation of the procedure is required and the proposed method remains only a possible estimation method and not a fully reliable one.

Appendices

Appendix A. Weibull Shifted Distribution to Non-shifted Distribution

Assume the following set of data

$$X = \{x_1, x_2, x_3, \dots, x_n\}$$

follows a Weibull distribution shifted from the origin by some known quantity ω. The mean for this set is

$$\mu_S = \frac{\sum_{i=1}^{n} x_i}{n}$$

The mean and standard deviation in terms of the Weibull parameters are, respectively

$$\mu_S = \omega + (\beta_S - \omega)\left[\Gamma\left(1 + \frac{1}{\alpha_S}\right)\right]$$

Next assume that each value in X is shifted back to the origin, or

$$X_S = \{x_1 - \omega, x_2 - \omega, x_3 - \omega, \dots, x_n - \omega\}$$

With a mean in terms of the Weibull of

$$\mu_0 = (\beta_0)\left[\Gamma\left(1 + \frac{1}{\alpha_0}\right)\right] \tag{A-1}$$

This changes the mean equation to

$$\mu_0(x) = \frac{\sum_{i=1}^{n}(x_i - \omega)}{n} = \frac{\sum_{i=1}^{n}(x_i)}{n} - \frac{n\omega}{n} = \mu_S - \omega$$
$$\mu_S = \mu_0 + \omega$$

So

$$\mu_0 + \omega = \omega + (\beta_S - \omega)\left[\Gamma\left(1 + \frac{1}{\alpha_S}\right)\right]$$

$$\mu_0 = (\beta_S - \omega)\left[\Gamma\left(1 + \frac{1}{\alpha_S}\right)\right]$$

Inserting Equation A-1

$$(\beta_0)\left[\Gamma\left(1 + \frac{1}{\alpha_0}\right)\right] = (\beta_S - \omega)\left[\Gamma\left(1 + \frac{1}{\alpha_S}\right)\right]$$

In the event that the entire distribution is shifted a constant the shape factor remains unaffected, so $\alpha_S = \alpha_0$ canceling out the gamma function terms. As such

$$(\beta_0) = (\beta_S - \omega)$$

Therefore it can be stated that the scale factor for a shifted distribution is equal to that of a distribution with its lower bound at the origin with the shift added to the this value.

Appendix B. Weibull Parameters for Shifted Distribution

Table B-1 Presents the shape and scale factors for the case where the still-water bending moment is uncoupled, resulting in a symmetric case for the hogging and sagging loads.

Table B.1. Results for Weibull Parameters After Uncoupling the Still-Water Bending Moment

Hogging Case		Sagging Case	
Shifted Moment Values	Cumulative Distribution Function	Shifted Moment Values	Cumulative Distribution Function
0	0	62825	1
3078	0.514821	60685	0.999999
6155	0.704984	58551	0.999998
8617	0.820328	56423	0.999995
11080	0.891416	54242	0.999991
13545	0.934942	51890	0.999981
16010	0.96135	49322	0.999964
18475	0.977233	46539	0.999932
20940	0.986706	43600	0.999873
23408	0.992309	40617	0.999765
25882	0.995593	37643	0.99957
28363	0.9975	34676	0.999219
30855	0.998595	31775	0.998595
33359	0.999219	28954	0.9975
35876	0.99957	26227	0.995593
38404	0.999765	23588	0.992309
40945	0.999873	21020	0.986706
43491	0.999932	18502	0.977233
46029	0.999964	16015	0.96135
48512	0.999981	13547	0.934942
50935	0.999991	11082	0.891416
53340	0.999995	8618	0.820328
55748	0.999998	6159	0.704984
58164	0.999999	3083	0.514821
60584	1	0	0
Shape	0.90539	Shape	0.90525
Scale	4622	Scale	4626

Appendix C: Example Forecasting Calculations

Appendix C.1. Distribution Fitting Procedure Example

Table B-1 contains the series of calculations done to produce the results of the fitted distribution. An explanation of each heading and what it calculates follows.

Table C.1. Outline of Distribution Fitting Method

Parent Distribution
6363.353

Parent

X	Fx(x)
0	0
500	0.044133
1000	0.097874
1500	0.153711
2000	0.209467
2500	0.264022
3000	0.316723
3500	0.367173
4000	0.415139
4500	0.460497
5000	0.503197

15 Year

X	Fx(x)^15
0	0
500	4.69E-21
1000	7.24E-16
1500	6.32E-13
2000	6.56E-11
2500	2.11E-09
3000	3.24E-08
3500	2.97E-07
4000	1.87E-06
4500	8.88E-06
5000	3.36E-05

The Parent Distribution Cell at the top is actually an @Risk function which stores a Weibull distribution, “RiskWeibull(1.1903,6750.5)”, and can be used to generate points for a CDF using the function, “RiskTheoTarget(distribution cell,x value)”. This is how the first table was obtained. This value was then raised to the number of years, resulting in the second table. Both of these tables continued on until the X column reached 70000. This table could then have a distribution fit to it using @Risks distribution manager. All that was required was to highlight all the points and specify a lower bound, and the type of data, in this case CDF data. With that it returned several possible distributions, so the Weibull was chosen in the RiskWeibull function form.

Appendix C.2. Numeric Integration Example

Table C-2 contains the calculations used to determine the numeric integral. An explanation of each heading and what that column calculates.

Table C.2: Example of Numeric Integration method

50	Years

Shape Factor	1.1903
Scale	6750.5

X	$f_x(x)$	$F_x(x)$	$F_x(x)^{(k-1)}$	$f_m(x)$
0	0	0	0	0
500	1.03E-4	0.044	3.92E-67	2.01E-69
1000	1.11E-4	0.098	3.49E-50	1.93E-52
1500	1.12E-4	0.154	1.41E-40	7.89E-43
2000	1.11E-4	0.210	5.43E-34	3E-36
2500	1.07E-4	0.264	4.58E-29	2.46E-31
3000	1.03E-4	0.317	3.41E-25	1.76E-27

$x*f_m(x)$	Area	$(x-\mu)^2 f_m(x)$	Area
0		0	
1.01E-66	2.52E-64	5.03E-64	1.26E-61
1.93E-49	4.82E-47	1.93E-46	4.82E-44
1.18E-39	2.96E-37	1.78E-36	4.44E-34
6E-33	1.50E-30	1.2E-29	3E-27
6.14E-28	1.54E-25	1.54E-24	3.84E-22
5.29E-24	1.32E-21	1.59E-20	3.97E-18

The calculations are described in the headings, where “b” is the scale factor and “a” is the shape factor. Fx(x) and fx(x) are the CDF and PDF of the distribution respectively defined by Equations 2-4 and 2-3. Thus the first five column headings are self-explanatory. fm(x) is the PDF of the largest value distribution as defined in equation (3-1). X*fm(x) is the expression inside the definition of the mean, and the area is the area under the curve of x*fm(x) as calculated by the trapezoid rule. The last two columns are the definition of the variance and the area under the resulting curve. Not shown are the many other X values used. The sum of the first area column produces the mean, then this value is used in the variance calculation column, then the second area column is summed to get the variance. The square root of this then is the standard deviation.

Appendix C.3. Derivation of Exact Mean and Standard Deviation

Starting from Equation 2-9

$$\mu_{M_k} = \frac{k\alpha}{\beta^\alpha}\int_0^\infty x^\alpha e^{-\left(\frac{x}{\beta}\right)^\alpha}\left[1 - e^{-\left(\frac{x}{\beta}\right)^\alpha}\right]^{k-1} dx$$

Expanding the binomial

$$\mu_{M_k} = \frac{k\alpha}{\beta^\alpha}\int_0^\infty x^\alpha e^{-\left(\frac{x}{\beta}\right)^\alpha}\left[a_0 - a_1 e^{-\left(\frac{x}{\beta}\right)^\alpha} + a_2 e^{-2\left(\frac{x}{\beta}\right)^\alpha} - \cdots (-1)^{k-1} a_{k-1} e^{-(k-1)\left(\frac{x}{\beta}\right)^\alpha}\right] dx$$

$$\mu_{M_k} = \frac{k\alpha}{\beta^\alpha}\int_0^\infty \left[a_0 x^\alpha e^{-\left(\frac{x}{\beta}\right)^\alpha} - a_1 x^\alpha e^{-2\left(\frac{x}{\beta}\right)^\alpha} + a_2 x^\alpha e^{-3\left(\frac{x}{\beta}\right)^\alpha} - \cdots (-1)^{k-1} a_{k-1} x^\alpha e^{-k\left(\frac{x}{\beta}\right)^\alpha}\right] dx$$

Distributing the integral

$$\mu_{M_k} = \frac{k\alpha}{\beta^\alpha}\left\{\int_0^\infty \left[a_0 x^\alpha e^{-\left(\frac{x}{\beta}\right)^\alpha}\right] dx - \int_0^\infty \left[a_1 x^\alpha e^{-2\left(\frac{x}{\beta}\right)^\alpha}\right] dx + \cdots \int_0^\infty \left[(-1)^{k-1} a_{k-1} x^\alpha e^{-k\left(\frac{x}{\beta}\right)^\alpha}\right] dx\right\}$$

From integration tables

$$\int_0^\infty \left[x^n e^{-ax^p}\right] dx = \frac{\Gamma(\frac{n+1}{p})}{pa^{(\frac{n+1}{p})}} \tag{C-1}$$

So

$$\mu_{M_k} = \frac{k\alpha}{\beta^\alpha}\left\{a_0 \frac{\Gamma(1+\frac{1}{\alpha})}{\alpha\left(\frac{1}{\beta^\alpha}\right)^{(1+\frac{1}{\alpha})}} - a_1 \frac{\Gamma(1+\frac{1}{\alpha})}{\alpha\left(\frac{2}{\beta^\alpha}\right)^{(1+\frac{1}{\alpha})}} + \cdots a_{k-1}(-1)^{k-1} \frac{\Gamma(1+\frac{1}{\alpha})}{\alpha\left(\frac{2}{\beta^\alpha}\right)^{(1+\frac{1}{\alpha})}}\right\}$$

The constants can be factored out

$$\mu_{M_k} = \frac{k\Gamma(1+\frac{1}{\alpha})}{\beta^\alpha}\frac{1}{\left(\frac{1}{\beta^\alpha}\right)^{(1+\frac{1}{\alpha})}}\left\{a_0 - a_1 \frac{1}{2^{(1+\frac{1}{\alpha})}} + \cdots a_{k-1}(-1)^{k-1}\frac{1}{k^{(1+\frac{1}{\alpha})}}\right\}$$

Applying a binomial expansion series and simplifying the β terms results in the following series form for the mean for the kth year.

$$\mu_{M_k} = k\beta\Gamma\left(1+\frac{1}{\alpha}\right)\sum_i \binom{k-1}{i}\frac{(-1)^i}{(i+1)^{(1+1/\alpha)}} \qquad \text{(C-2)}$$

Finding the standard deviation starting from Equation 2-8

$$\sigma_{M_k}{}^2 = \int_0^\infty \left(x-\mu_{M_k}\right)^2 f_{M_k}(x)\,dx$$

Expanding

$$\sigma_{M_k}{}^2 = \int_0^\infty \left(x^2 - 2\mu_{M_k}x + \mu_{M_k}{}^2\right) f_{M_k}(x)\,dx$$

$$\sigma_{M_k}{}^2 = \int_0^\infty x^2 f_{M_k}(x)\,dx - 2\mu_{M_k}\int_0^\infty x f_{M_k}(x)\,dx + \mu_{M_k}{}^2\int_0^\infty f_{M_k}(x)dx$$

By definition

$$\int_0^\infty f_{M_k}(x)dx = 1$$

$$\int_0^\infty x f_{M_k}(x)\,dx = \mu_{M_k}$$

So

$$\sigma_{M_k}{}^2 = \int_0^\infty x^2 f_{M_k}(x)\,dx - \mu_{M_k}{}^2$$

Isolating the integral

$$\int_0^\infty x^2 f_X(x)\,dx = \frac{k\alpha}{\beta^\alpha}\int_0^\infty x^{\alpha+1} e^{-\left(\frac{x}{\beta}\right)^\alpha}\left[1-e^{-\left(\frac{x}{\beta}\right)^\alpha}\right]^{k-1}dx$$

The expansion of the binomial is the same process as before, as such

$$\frac{k\alpha}{\beta^\alpha}\int_0^\infty x^{\alpha+1} e^{-\left(\frac{x}{\beta}\right)^\alpha}\left[1-e^{-\left(\frac{x}{\beta}\right)^\alpha}\right]^{k-1}dx =$$

$$\frac{k\alpha}{\beta^\alpha}\left\{\int_0^\infty\left[a_0 x^{\alpha+1}e^{-\left(\frac{x}{\beta}\right)^\alpha}\right]dx - \int_0^\infty\left[a_1 x^{\alpha+1}e^{-2\left(\frac{x}{\beta}\right)^\alpha}\right]dx + \cdots \int_0^\infty\left[(-1)^{k-1}a_{k-1}x^{\alpha+1}e^{-k\left(\frac{x}{\beta}\right)^\alpha}\right]dx\right\}$$

Using C-1

$$\frac{k\alpha}{\beta^\alpha}\int_0^\infty x^{\alpha+1}e^{-\left(\frac{x}{\beta}\right)^\alpha}\left[1-e^{-\left(\frac{x}{\beta}\right)^\alpha}\right]^{k-1}\,dx\ =$$

$$\frac{k\alpha}{\beta^\alpha}\left\{a_0\frac{\Gamma(1+\frac{2}{\alpha})}{\alpha\left(\frac{1}{\beta^\alpha}\right)^{(1+\frac{2}{\alpha})}}-a_1\frac{\Gamma(1+\frac{2}{\alpha})}{\alpha\left(\frac{2}{\beta^\alpha}\right)^{(1+\frac{2}{\alpha})}}+\cdots a_{k-1}(-1)^{k-1}\frac{\Gamma(1+\frac{2}{\alpha})}{\alpha\left(\frac{2}{\beta^\alpha}\right)^{(1+\frac{2}{\alpha})}}\right\}$$

Pulling out constants and simplifying

$$\frac{k\alpha}{\beta^\alpha}\int_0^\infty x^{\alpha+1}e^{-\left(\frac{x}{\beta}\right)^\alpha}\left[1-e^{-\left(\frac{x}{\beta}\right)^\alpha}\right]^{k-1}\,dx$$
$$=k\beta^2\Gamma(1+\frac{2}{\alpha})\left\{a_0-a_1\frac{1}{2^{(1+\frac{2}{\alpha})}}+\cdots a_{k-1}(-1)^{k-1}\frac{1}{k^{(1+\frac{2}{\alpha})}}\right\}$$

Applying a binomial expansion series and reinserting into variance equation results in the equation for the variance in series form

$$\sigma_{M_k}{}^2=k\beta^2\Gamma\left(1+\frac{2}{\alpha}\right)\sum_i\binom{k-1}{i}\frac{(-1)^i}{(i+1)^{(1+2/\alpha)}}-\mu_{M_k}{}^2 \qquad \text{(B-3)}$$

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