ABSTRACT

Title of Dissertation: Microfunctions for Sheaves of Holomorphic Functions with Growth Conditions

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Mikio Sato devised microfunctions as a means of measuring the singularities of hyperfunctions. In 1970, Kawai and Sato introduced Fourier hyperfunctions in their study of partial differential operators. The class of Fourier hyperfunctions has been generalized by Saburi, Nagamachi, and Kaneko, among others, and most recently by Berenstein and Struppa.

Berenstein and Struppa introduced Fourier p-hyperfunctions, where p is a plurisubharmonic function satisfying certain smoothness and growth conditions. $p(z) = |z|^s, s \geq 1$ are the cases studied by Sato, Kawai, Nagamachi, and Kaneko.

Following the methods of Sato, Kawai and Kashiwara, this dissertation introduces Fourier p-microfunctions functorially, though under very severe conditions on p. These restrictions on p are satisfied when, for instance, $p(z) = \log^+ |f|$ where f is a product of 1 variable holomorphic functions with zeroes uniformly bounded away from the real axis. Kaneko has introduced Fourier microfunctions for $p(z) = |\mathfrak{Re}\, z|^s$, $s > 0$, using tubes. When $s < 1$ these p's are not plurisubharmonic. Thus the results here complement his.

MICROFUNCTIONS FOR SHEAVES OF HOLOMORPHIC FUNCTIONS WITH GROWTH CONDITIONS

BY
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PREFACE

In their work on Dirichlet series, Berenstein and Struppa [1988], introduced a new theory of interpolation for $A_{p,0}(\Gamma)$ (definition 1.2.1), the space holomorphic functions in an open cone $\Gamma \subseteq \mathbb{C}^n$ satisfying growth conditions depending upon a plurisubharmonic function p, and for its dual $\widehat{A_{p,0}(\Gamma)}'$. They noticed that the proofs of these interpolation theorems, and some theorems on mean periodic functions amounted to theorems on the vanishing of cohomology groups. In the spirit of Kawai [1970], they [preprint] thus introduced sheaves of holomorphic functions with growth conditions.

These sheaves are defined on the radial compactification $\mathbb{D}^n$ of $\mathbb{R}^n$, or its corresponding "complexification" $\widehat{\mathbb{C}^n} = \mathbb{D}^n + \sqrt{-1}\mathbb{R}^n$. Both compactifications were introduced by Sato and Kawai in Kawai [1970]. Sato and Kawai defined the sheaves, $\tilde{\mathscr{O}}$, of slowly increasing holomorphic functions and, $\underset{\sim}{\mathscr{O}}$, of rapidly decreasing holomorphic functions. Then they defined the Fourier hyperfunctions in the same manner that hyperfunctions are defined, namely as $\mathscr{R}_{\mathbb{D}^n} := \mathrm{R}^n\Gamma_{\mathbb{D}^n}(\tilde{\mathscr{O}})$. The sheaf of Fourier hyperfunctions $\mathscr{R}_{\mathbb{D}^n}$ on $\mathbb{D}^n$.

The sheaves Berenstein and Struppa [preprint] defined were the sheaf of holomorphic functions of minimal type p, ${}^{p}\mathscr{O}$, where the plurisubharmonic function p satisfies, among other things, Hörmander's condition (definition 1.2.1(4)(ii)[1]), and the sheaf of rapidly decreasing functions of type p, ${}_{p}\mathscr{O}$. As in Kawai [1970]

[1] Hörmander [1967].

they introduced the sheaf of Fourier p-hyperfunctions, here denoted ${}^p\mathscr{B}$. When $p(z) = |\cdot|$, these are the Fourier hyperfunctions of Kawai and Sato. Saburi [1978][2] introduced Fourier hyperfunctions using a radial compactification of $\mathbb{C}^n$, and Kaneko [1985] has introduced Fourier hyperfunctions when p is the (not necessarily plurisubharmonic) function $|\mathfrak{Re}z|^s$ $(s > 0)$.

As one of their interest lay in the singularities of Dirichlet series, Berenstein & Struppa asked what would correspond to microfunctions for Fourier p-hyperfunctions. Microfunctions (for ordinary hyperfunctions), it should be recalled, were introduced by Sato [1970], and defined functorially in Sato, Kawai & Kashiwara [1973]. They measure the extent to which hyperfunctions fail to be real analytic, thus measuring the singularities of hyperfunctions. By using tubes, Kaneko [1985] has introduced microfunctions for the Fourier hyperfunctions he defined.

Following Sato, Kawai & Kashiwara [1973], this paper introduces Fourier p-microfunctions. The results here complement Kaneko's. Eventhough the conditions to be imposed on the plurisubharmonic function p in chapter 2 will turn out to be rather severe, they allow for functions not considered by Kaneko. However, the results here do not include Kaneko's, since $p(z) = |\mathfrak{Re}z|^s$ is not plurisubharmonic when $s < 1$.

It is shown in chapter 4 that Fourier p-microfunctions defined here are concentrated in one degree. More specifically it is shown (theorem 4.2.18) that $\mathbb{S}^*\Omega$ is purely n-codimensional with respect to $\pi^{-1}\,{}^p\mathscr{O}$ in analogy to the case of (ordinary) microfunctions. It should be noted here that $\widehat{\mathbb{C}^n}$ and $\mathbb{D}^n$ are manifolds with boundary. Thus this result contrasts with microfunctions up to the boundary studied by Schapira[3], where the microfunctions are not in general

[2] For references to Saburi, the reader should also refer to Saburi [1982] and [1985].
[3] See for instance Schapira [1988] §2.

concentrated in one degree. On the other hand, Lieutenant [1986] showed that $\mathbb{S}^*\Omega$ is purely n-codimensional with respect to $\pi^{-1}i_*\mathcal{O}$, where $i : \mathbb{C}^n \hookrightarrow \widehat{\mathbb{C}^n}$ is the inclusion. Thus the result here is similar to his.

As a more tenuous justification for studying such microfunctions, one might note that Fourier hyperfunctions have appeared in quantum field theory as a means of enlarging the space of states[4].

[4] See Brüning & Nagamachi [1989] and references contained therein.

DEDICATION

Of Course

for

l. k. k. & c. g. c.

who have known for many
years what the
Old Masters
knew
well
and who are
Old Masters themselves

ACKNOWLEDGMENTS

This has been a long and oftentimes arduous journey. Along the way, I was aided by my advisor, Professor Carlos Berenstein, who suggested this problem, and Professor Daniele Struppa. They shared with me a preliminary version of their paper for which I am much obliged. They awaited my arrival with great patience, and made suggestions on avenues to take when I found myself at a dead end. Dr. Bao-Qin Li shared some of his thoughts with me. Professors Daniel Fivel, Mark Shayman, and Scott Wolpert served on my dissertation committee. I am thankful for everyone's help.

As the reader will notice, this work owes much to the English and french works of many authors on hyperfunctions and microfunctions. Those works provided the maps without which I would have been hopelessly lost.

It should be mentioned that this paper was typeset with $\mathcal{AMS}$-TEX, the TEX macro system of the American Mathematical Society, and XY-pic, K. H. Rose's TEX macro for typesetting diagrams.

Finally, I am, most of all, grateful (and much indebted) to Poh and Tom Willson for the shelter and sustenance they provided during thunderstorms. While they may find it hard to believe since I kept raiding their refrigerator, for a while they allowed me to think more about mathematics than food or fleas. This work would have been impossible without that assistance.

TABLE OF CONTENTS

CHAPTER I

INTRODUCTION AND THE BASIC TRIANGLES

David Harum says, "A reasonable amount of fleas is good for a dog. They keep him from broodin' on bein' a dog." A goodly supply of fleas might likewise keep man from brooding over anything deeper than the presence of these fleas, but in some cases this in itself is a rather serious thing to brood over.

—Asa C. Chandler, *Introduction to Parasitology* [1944].

§1.1 Introduction

To define the Fourier p microfunctions, I have basically followed the results of Sato, Kawai & Kashiwara [1973] for (ordinary) hyperfunctions. More specifically, in this chapter, we will note that all the basic triangles for hyperfunctions remain true without modification on open subsets of $\mathbb{D}^n$, and that most of the terms of these triangles can be computed in exactly the same manner. These results do not depend on any assumptions on p.

In the following chapters we proceed to compute the other terms of these triangles. Again as in Sato, Kawai & Kashiwara, these terms are the two vanishing theorems in chapter 4. The preliminaries needed to prove one of these vanishing theorems (4.2.18), are laid out in chapters 2 and 3. As in the case of hyperfunctions, proposition 1.4.12 below reduces one of the vanishing theorems (4.2.7) to a computation of $H^k_{V\cap G}(V;{}^p\mathscr{O})$, where G is a wedge and V an open set. Then following Kashiwara, Kawai & Kimura [1986] we show in chapter 3 and

§4.1 that $V \cap G$ can be written as $K' - K$ for suitable compact subsets of $\widehat{\mathbb{C}^n}$. The techniques involved are elementary if tedious. Theorem 2.4.8 in chapter 2, then shows $H^k_{K'-K}(\widehat{\mathbb{C}^n}; {}^{p}\mathcal{O}) = 0$ for $k \neq n$.

To prove theorem 2.4.8 is the goal of chapter 2. This theorem generalizes proposition 2.2.2 of Kashiwara, Kawai & Kimura [1986] to certain compact subsets of $\widehat{\mathbb{C}^n}$, and by remark 2.4.9, to compact subsets of $\mathbb{C}^n$ equal to their plurisubharmonic hull. This result is however essentially contained in Kawai [1970], and I have closely followed his ideas. Hörmander's L^2 methods provide the main tools. Crucial to this goal is Kawai's approximation theorem, which is noted to hold not only for subsets of $\mathbb{D}^n$, as stated in his paper, but also for compact subsets of $\widehat{\mathbb{C}^n}$ that are in some sense equal their plurisubharmonic hull.

It is in chapter 2 during the course of proving theorem 2.4.8 that restrictions are placed on the plurisubharmonic function p. These restrictions are the property (P_p) introduced by Berenstein & Struppa (in analogy with a condition in Meril [1983]), and the existence of holomorphic functions of "controlled growth", which is implicit in Kawai and Saburi's work.

As a philosophical point, one might note that such "controlled growth" functions will play only a catalytic role in the proofs. They are used locally only at points at infinity (of Ω, or, more precisely, of some $\widehat{\mathbb{C}^n}$ neighbourhood of Ω), and then only to bring functions from one space to another and then back again, by first multiplying and then dividing by the function of "controlled growth". Examples, although admittedly scant, of plurisubharmonic functions p that satisfy these conditions are given in chapter 2.

The other vanishing theorem (4.1.5) requires sufficiently many ${}^{p}\mathcal{O}$-pseudoconvex sets. Unlike the case of $\mathbb{C}^n$, where several characterizations of pseudoconvex sets are known, little more than the definition characterizes ${}^{p}\mathcal{O}$-pseudoconvex sets.[1] As a consequence to produce an ${}^{p}\mathcal{O}$-pseudoconvex set almost certainly requires the exhibition of an exhaustion function. This indeed was one of the problems that necessitated the explicit calculations to prove the previous vanishing theorem. In this case, the Grauert tubular theorem is used to exhibit the ${}^{p}\mathcal{O}$-pseudoconvex sets and exhaustion functions. This theorem is proven by Kawai [1970], and Berenstein & Struppa [preprint] for open subsets of $\mathbb{D}^n$, and in detail by Saburi [1985] for open subsets of $\mathbb{D}^n$ in a different compactification of $\mathbb{C}^n$. A proof following Harvey & Wells' [1972] proof of Grauert's original theorem (for real analytic submanifolds of complex manifolds) is supplied here for the reader's convenience.

After a proposition on smoothing plurisubharmonic functions, proposition 4.1.4 shows that points at infinity on $\sqrt{-1}\mathbb{S}\Omega$ have sufficiently many neighbourhoods whose projection on $\widehat{\mathbb{C}^n} - \mathbb{D}^n$ is ${}^{p}\mathcal{O}$-pseudoconvex. The classical proof of the vanishing theorem can then be used to show theorem 4.1.5 with no modification.

In summary, all the main ideas in this work are due to Sato, Kawai & Kashiwara [1973], and Kawai [1970]. Here, only a few calculations are added to their already extensive and formidable work.

§1.2 Review of Results

Listed below are some of the main definitions of Berenstein and Struppa [preprint]. As an important remark, the properties listed below that the plurisubharmonic function p are to satisfy form the ideal case. In actuality more severe restrictions shall have to be made; this is done in the following chapters. The results in §§3 and 4 hold regardless.

[1] For instance convex sets are pseudoconvex. This provides an abundance of albeit uninteresting pseudoconvex sets.

DEFINITION 1.2.1.

(1) $\mathscr{O}$ is the sheaf of holomorphic functions on $\mathbb{C}^n$;

(2) $\mathbb{D}^n$ is the radial compactification of $\mathbb{R}^n$, viz. $\mathbb{D}^n := \mathbb{R}^n \sqcup \mathbb{S}^\infty_{n-1}$, $\mathbb{S}^\infty_{n-1}$ being the $n-1$ sphere at infinity, which is identified with $\mathbb{R}^n - \{0\}/\mathbb{R}^+$;

(3) $\widehat{\mathbb{C}^n} := \mathbb{D}^n + \sqrt{-1}\mathbb{R}^n$;

(4) p is a smooth plurisubharmonic function on $\mathbb{C}^n$ satisfying:

(i) $p(z) \geq 0, \quad \log(1+|z|) = O(p(z))$,

(ii) there are constants K_1, K_2, K_3, K_4, such that

$|z_1 - z_2| \leq \exp(-K_1 p(z_1) - K_2)$ implies $p(z_1) \leq K_3 p(z_2) + K_4$; and

(iii) p is C^∞ and convex;

(5) For a pseudoconvex region, U, in $\mathbb{C}^n$, $A_p(U)$ is the set

$$\left\{ f \in \mathscr{O}(U) : \exists \text{ positive constants } A \text{ and } B \text{ s. t. } |f(z)| \leq A e^{Bp(z)} \right\}.$$

(6) For $U \subseteq \widehat{\mathbb{C}^n}$, open, ${}^p\mathscr{O}(U)$ is the set of all holomorphic functions $f \in \mathscr{O}(U \cap \mathbb{C}^n)$ such that, for any $\epsilon > 0$ and any compact set $K \subseteq U$,

$$\sup_{z \in K \cap \mathbb{C}^n} |f(z) e^{-\epsilon p(z)}| < \infty.$$

These ${}^p\mathscr{O}(U)$ form a sheaf, denoted ${}^p\mathscr{O}$;

(7) For $U \subseteq \widehat{\mathbb{C}^n}$, open, ${}_p\mathscr{O}(U)$ is the set of all holomorphic functions $f \in \mathscr{O}(U \cap \mathbb{C}^n)$ such that, for any compact set $K \subseteq U$, $\exists \delta > 0$ such that

$$\sup_{z \in K \cap \mathbb{C}^n} |f(z) e^{\delta p(z)}| < \infty.$$

These ${}^p\mathscr{O}(U)$ form a sheaf, denoted ${}^p\mathscr{O}$

(8) ${}^p\mathscr{R}_{\mathbb{D}^n}$, or simply ${}^p\mathscr{R}$, denotes the sheaf of Fourier hyperfunctions on $\mathbb{D}^n$; this by definition is the sheaf $\mathrm{R}^n\Gamma_{\mathbb{D}^n}({}^p\mathscr{O})$. $\Diamond$

REMARK 1.2.2. Kawai [1970] uses $\mathscr{R}$ to denote the Fourier hyperfunctions on $\mathbb{D}^n$; this corresponds to the case $p(z) = |z|$ in Berenstein and Struppa [preprint] Fourier p-hyperfunctions on $\mathbb{D}^n$, ${}^p\mathscr{R}$. $\triangleright$

Instead of the notation above for Fourier hyperfunctions, this paper will use

DEFINITION 1.2.3. Let Ω be an open subset of $\mathbb{D}^n$. Define the sheaf of Fourier p-hyperfunctions on Ω to be $\mathrm{R}^n\Gamma_\Omega({}^p\mathscr{O})$. This sheaf will be denoted by ${}^p\mathscr{B}_\Omega$ or simply ${}^p\mathscr{B}$. $\Diamond$

DEFINITION 1.2.4[2]. An open set $V \subseteq \mathbb{C}^n$ satisfies property (P_p) if

$$(P_p) \qquad \exists \phi \in \mathscr{O}(V) \text{ such that } \forall M > 0, \quad \sup_V(-\mathfrak{Re}\,\phi(z) + Mp(z)) < \infty.$$

REMARK 1.2.5. Clearly if $V' \supseteq V$ satisfies property (P_p) then so does V. $\triangleright$

REMARK 1.2.6. Any $V \subset\subset \mathbb{C}^n$ satisfies (P_q) for q plurisubharmonic and merely upper semicontinous on $\mathbb{C}^n$. Take $\phi \equiv 0$. $\triangleright$

DEFINITION 1.2.7[3]. An open set $U \subseteq \widehat{\mathbb{C}^n}$ is ${}^p\mathscr{O}$-pseudoconvex if

(1) $U \cap \mathbb{C}^n$ satisfies property (P_p);

(2) There is a C^2 plurisubharmonic function θ on $U \cap \mathbb{C}^n$ such that

  (i) $\forall c \in \mathbb{R}, \quad \{z : \theta(z) < c\} \subset\subset U$;

  (ii) $\forall K \subset U$, compact, $\exists M_K$ such that $\sup_{K\cap\mathbb{C}^n} \theta(z) < M_K$. $\Diamond$

THEOREM 1.2.8[4]. *Let $U \subseteq \widehat{\mathbb{C}^n}$ be ${}^p\mathscr{O}$-pseudoconvex. Then*

$$H^k(U; {}^p\mathscr{O}) = 0, \qquad \text{for } k \geq 1. \quad \square$$

[2]Meril [1983], Berenstein & Struppa [preprint].
[3]Kawai [1970], Saburi [1978], Nagamachi [1981], Berenstein & Struppa [preprint].
[4]Kawai, Meril, Saburi, Nagamachi, Kaneko, Berenstein & Struppa.

§1.3 The Basic Triangles

The general set-up will involve a convex set with what will be called "full trace" at infinity, and "thickenings" and closures of such sets in $\widehat{\mathbb{C}^n}$. Following Lieutenant [1986] these are assumed to "taper" linearly at the boundary.

DEFINITION 1.3.1. The trace at ∞ of a set $U \subseteq \widehat{\mathbb{C}^n}$, denoted $\mathrm{tr}_\infty U$, is the set of points in $\mathbb{S}_{n-1}\infty + i\mathbb{R}^n$ having $U \cup (\mathbb{S}_{n-1}\infty + i\mathbb{R}^n)$ as a neighbourhood. $\Diamond$

DEFINITION 1.3.2. An open subset $U \subset \widehat{\mathbb{C}^n}$ has full trace at infinity if $U = (U \cap \mathbb{C}^n) \cup \mathrm{tr}_\infty U$. $\Diamond$

Similarly

DEFINITION 1.3.3. The trace at ∞ of a set $\Omega \subset \mathbb{D}^n$, denoted $\mathrm{tr}_\infty \Omega$ is the set of points in $\mathbb{S}_{n-1}\infty$ having $\Omega \cup \mathbb{S}_{n-1}\infty$ as a neighbourhood. $\Diamond$

DEFINITION 1.3.4. An open subset $\Omega \subset \mathbb{R}^n$ has full trace at infinity if $\Omega = (\Omega \cap \mathbb{R}^n) \cup \mathrm{tr}_\infty \Omega$. $\Diamond$

REMARK 1.3.5. In some sense a set has full trace at infinity if it contains most of its interior frontier points at infinity.

There are clearly other definitions of traces at infinity such as closed traces, but only the ones above shall be used here. More generally the trace at "infinity" of a subset of a manifold with boundary can be defined. $\triangleright$.

NOTATION 1.3.6. Throughout the rest of this paper, Ω will denote an open subset of $\mathbb{D}^n$ with full trace at infinity such that $\Omega \cap \mathbb{C}^n$ is convex. $\triangle$

The following are modifications of Lieutenant's [1986] definitions and notations. Eventhough spaces involving the closure of Ω are defined, they will not be used in the rest of the paper.

Definition 1.3.7.

(1) For $\nu > 0$,

$$\Omega_\nu := \mathrm{cv}(\Omega \cup \{\pm\sqrt{-1}\nu' e_j : e_j \text{ is the } j\text{th unit vector in } \mathbb{R}^n,$$
$$j = 1, \ldots, n; 0 < \nu' < \nu\});$$

This is a complexification of Ω.

(2) $F := cl_{\mathbb{D}^n}\Omega$ and $F_\nu := cl_{\widehat{\mathbb{C}^n}}\Omega_\nu$.

(3) $\mathbb{S}\Omega := \Omega \times \mathbb{S}_{n-1}$; $\quad \mathbb{S}F := F \times \mathbb{S}_{n-1}$. These are the sphere bundles.

(4) $\mathbb{S}^*\Omega := \Omega \times \mathbb{S}^*_{n-1}$; $\quad \mathbb{S}^*F := F \times \mathbb{S}^*_{n-1}$. The dual sphere bundles.

(5) $\tilde{\Omega}_\nu := (\Omega_\nu - \Omega) \sqcup \mathbb{S}\Omega$; $\quad \tilde{F}_\nu := (F_\nu - F) \sqcup \mathbb{S}F$. The real monoidal transforms.

(6) $\tilde{\Omega}^*_\nu := (\Omega_\nu - \Omega) \sqcup \mathbb{S}^*\Omega$; $\quad \tilde{F}^*_\nu := (F_\nu - F) \sqcup \mathbb{S}^*F$. The real comonoidal transforms.

(7) $D\Omega := \{(x, \xi, \eta) \in \Omega \times \mathbb{S}_{n-1} \times \mathbb{S}^*_{n-1} : \langle \xi, \eta \rangle \leq 0\}$.

(8) $DF := \{(x, \xi, \eta) \in F \times \mathbb{S}_{n-1} \times \mathbb{S}^*_{n-1} : \langle \xi, \eta \rangle \leq 0\}$;

(9) $\widetilde{D\Omega^+_\nu} := (\Omega_\nu - \Omega) \sqcup D\Omega$; $\quad \widetilde{DF^+_\nu} := (F_\nu - F) \sqcup DF$;

(10) $i_\nu \colon \Omega_\nu \to F_\nu$; $\quad \alpha_\nu \colon \tilde{\Omega}_\nu \to \tilde{F}_\nu$;

(11) $\beta_\nu \colon \tilde{\Omega}^*_\nu \to \tilde{F}^*_\nu$; $\quad \epsilon_\nu \colon F_\nu - F \to F_\nu$. $\quad\Diamond$

Remark 1.3.8. To a map $f : X \to Y$ is associated the mapping cone triangle:

$$\mathscr{F}^\bullet \xrightarrow{\tau} f_* f^{-1}\mathscr{F}^\bullet \longrightarrow \mathrm{Co}(\tau) \xrightarrow{+1}$$

where $\mathscr{F}^\bullet \in \mathsf{K}^+(Y)$, and $\mathrm{Co}(\tau)$ is the mapping cone of the canonical adjunction τ.

Twisting and translating this triangle produces

$$\mathrm{Co}(\tau)[-1] \xrightarrow{\mathrm{pr}} \mathscr{F}^\bullet \xrightarrow{-\tau} f_* f^{-1}\mathscr{F}^\bullet \xrightarrow{+1}. \qquad \triangleright$$

DEFINITION 1.3.9. $\mathscr{D}\mathrm{ist}_\tau(\mathscr{F}^\bullet) = \mathrm{Co}(\tau)[-1]$. $\Diamond$

REMARK 1.3.10. $\mathscr{D}\mathrm{ist}_\tau$ can be considered a functor from $\mathsf{K}^+(Y)$ to $\mathsf{K}^+(Y)$. However $\mathrm{Co}(\mathscr{A}^\bullet \to \mathscr{B}^\bullet)$ is not a functor nor does it normally give rise to a derived functor.[5] $\triangleright$

As in Sato, Kawai & Kashiwara [1973], many of the triangles in the sequel will take the form

$$\mathscr{D}\mathrm{ist}_\tau(\mathscr{F}^\bullet) \longrightarrow \mathscr{F}^\bullet \xrightarrow{\tau} f_* f^{-1}\mathscr{F}^\bullet \xrightarrow{+1}. \tag{3-1}$$

We now use (3-1) in the following situation. Consider the inclusions and projections in the following diagram:

$$\begin{tikzcd}
 & \tilde{F}_\nu - \partial\tilde{\Omega}_\nu \arrow[d] & \\
\Omega_\nu - \Omega \arrow[r] \arrow[d] & \tilde{\Omega}_\nu \arrow[r, "\tau"] \arrow[d] & \Omega_\nu \arrow[d] \\
F_\nu - F \arrow[r] & \tilde{F}_\nu \arrow[r, "\tau"] & F_\nu
\end{tikzcd}$$

For $\mathscr{F}^\bullet \in \mathsf{K}^+(\Omega_\nu)$ or $\mathscr{F}^\bullet \in \mathsf{K}^+(F_\nu)$ there are triangles

$$\mathscr{F}^\bullet[-1] \longrightarrow \tau_*\tau^{-1}\mathscr{F}^\bullet[-1] \longrightarrow \mathscr{D}\mathrm{ist}_\tau(\mathscr{F}^\bullet) \xrightarrow{+1} \tag{3-5}$$

$$\mathscr{F}^\bullet[-1] \longrightarrow (\tau i)_*(\tau i)^{-1}\mathscr{F}^\bullet[-1] \longrightarrow \mathscr{D}\mathrm{ist}_{\tau i}(\mathscr{F}^\bullet) \xrightarrow{+1} \tag{3-6}$$

$$\tau_*\tau^{-1}\mathscr{F}^\bullet[-1] \longrightarrow (\tau i)_*(\tau i)^{-1}\mathscr{F}^\bullet[-1] \longrightarrow \tau_*\,\mathscr{D}\mathrm{ist}_i(\tau^{-1}\mathscr{F}^\bullet) \xrightarrow{+1} \tag{3-7}$$

These triangles form an octahedron, and the octahedral axiom provides the dashed arrows:

$$\begin{tikzcd}
\mathscr{F}^\bullet[-1] \arrow[r] \arrow[d, equal] & \tau_*\tau^{-1}\mathscr{F}^\bullet[-1] \arrow[r] \arrow[d] & \mathscr{D}\mathrm{ist}_\tau(\mathscr{F}^\bullet) \arrow[r, "+1"] \arrow[d, dashed] & {} \\
\mathscr{F}^\bullet[-1] \arrow[r] \arrow[d] & (\tau i)_*(\tau i)^{-1}\mathscr{F}^\bullet[-1] \arrow[r] \arrow[d, equal] & \mathscr{D}\mathrm{ist}_{\tau i}(\mathscr{F}^\bullet) \arrow[r, "+1"] \arrow[d, dashed] & {} \\
\tau_*\tau^{-1}\mathscr{F}^\bullet[-1] \arrow[r] \arrow[d] & (\tau i)_*(\tau i)^{-1}\mathscr{F}^\bullet[-1] \arrow[r] \arrow[d] & \tau_*\,\mathscr{D}\mathrm{ist}_i(\tau^{-1}\mathscr{F}^\bullet) \arrow[r, "+1"] \arrow[d, dashed, "+1"] & {} \\
{} & {} & {} &
\end{tikzcd}$$

[5] See Komatsu [1971] §8.

The triangle with dashed arrows gives rise to a triangle in the derived category $\mathbf{D}^+(Y)$:

$$\text{(3-8)} \qquad \mathbf{R}\,\mathscr{D}\mathrm{ist}_\tau(\mathscr{F}^\bullet) \to \mathbf{R}\,\mathscr{D}\mathrm{ist}_{\tau i}(\mathscr{F}^\bullet) \to \mathbf{R}\tau_*\mathbf{R}\,\mathscr{D}\mathrm{ist}_i(\tau^{-1}\mathscr{F}^\bullet) \xrightarrow{+1}$$

Note that

$$\mathbf{R}\,\mathscr{D}\mathrm{ist}_{\tau i}(\mathscr{F}^\bullet) = \mathbf{R}\Gamma_\Omega(\mathscr{F}^\bullet)$$

and

$$\mathbf{R}\tau_*\mathbf{R}\,\mathscr{D}\mathrm{ist}_i(\tau^{-1}\mathscr{F}^\bullet) = \mathbf{R}\tau_*\mathbf{R}\Gamma_{\mathbb{S}\Omega}\tau^{-1}\mathscr{F}^\bullet,$$

so (3-8) gives:

$$\text{(3-9)} \qquad \mathbf{R}\,\mathscr{D}\mathrm{ist}_\tau(\mathscr{F}^\bullet) \to \mathbf{R}\Gamma_\Omega(\mathscr{F}^\bullet) \to \mathbf{R}\tau_*\mathbf{R}\Gamma_{\mathbb{S}\Omega}\tau^{-1}\mathscr{F}^\bullet \xrightarrow{+1}$$

§1.4 Computation of Terms of the Triangles

The proofs given here are, with little or no modification, due to Sato, Kawai & Kashiwara [1971].

LEMMA 1.4.1. $\mathbf{R}\,\mathscr{D}\mathrm{ist}(\mathscr{F}) = (\mathscr{F})_\Omega[-n]$.

PROOF[6]. The long exact sequence from the triangle (3-5) gives

$$0 \longrightarrow \mathrm{R}^0\,\mathscr{D}\mathrm{ist}_\tau(\mathscr{F}) \longrightarrow \mathscr{F} \longrightarrow \tau_*\tau^{-1}\mathscr{F} \longrightarrow \mathrm{R}^1\,\mathscr{D}\mathrm{ist}_\tau(\mathscr{F}) \longrightarrow 0,$$

and the equality

$$\mathrm{R}^k\,\mathscr{D}\mathrm{ist}_\tau(\mathscr{F}) = \mathrm{R}^{k-1}\tau_*\tau^{-1}\mathscr{F}, \qquad \text{for } k \geqslant 2.$$

[6] *cf.* Prop. 2.3.3 of Kawai, Kashiwara & Kimura [1986].

Since τ is a closed map and Ω_ν is metrizable, one has, for $x \in \Omega_\nu$,[7]

$$\mathsf{R}^k\, \mathscr{D}\mathrm{ist}_\tau(\mathscr{F}) = H^k(\tau^{-1}\{x\} \to x; \mathscr{F}_x)$$

$$= \begin{cases} 0, & \text{if } x \notin \Omega \quad \text{or} \quad k \neq n-1, \\ \mathscr{F}_x, & \text{if } k = n-1. \end{cases}$$

This proves the lemma. □

LEMMA 1.4.2. *Let $\pi : \widetilde{D\Omega_\nu^+} \to \tilde{\Omega}_\nu$ be the canonical projection. Then there is an isomorphism*

$$\pi^{-1}\mathbf{R}\Gamma_{\mathbb{S}\Omega}(\tau^{-1}\mathscr{F}^\bullet) \xrightarrow{\sim} \mathbf{R}\Gamma_{D\Omega}\pi^{-1}\tau^{-1}\mathscr{F}^\bullet \tag{4-1}$$

PROOF. Let $\mathscr{G}^\bullet \in \mathsf{K}^+(\tilde{\Omega}_\nu)$. There is a composition of canonical maps

$$\begin{aligned} \pi^{-1}\Gamma_{\mathbb{S}\Omega}\mathscr{G}^\bullet &\longrightarrow \pi^{-1}\Gamma_{\mathbb{S}\Omega}\pi_*\pi^{-1}\mathscr{G}^\bullet = \pi^{-1}\pi_*\Gamma_{\pi^{-1}\mathbb{S}\Omega}\pi^{-1}\mathscr{G}^\bullet \\ &= \pi^{-1}\pi^*\Gamma_{D\Omega}\pi^{-1}\mathscr{G}^\bullet \longrightarrow \Gamma_{D\Omega}\pi^{-1}\mathscr{G}^\bullet. \end{aligned} \tag{4-2}$$

Consequently for $\mathscr{G}^\bullet \in \mathsf{D}^+(\tilde{\Omega}_\nu)$, this induces, in the derived category, a map

$$\pi^{-1}\mathbf{R}\Gamma_{\mathbb{S}\Omega}(\mathscr{G}^\bullet) \longrightarrow \mathbf{R}\Gamma_{D\Omega}\pi^{-1}\mathscr{G}^\bullet$$

When $\mathscr{G}^\bullet = \tau^{-1}\mathscr{F}^\bullet$ this is the map claimed in the lemma. That this map is an isomorphism is proven in Lieutenant [1986][8]. □

LEMMA 1.4.3. *Consider the following diagram and maps where the left arrows are inclusions*[9]*:*

$$\begin{tikzcd}
 & \widetilde{D\Omega_\nu^+} \arrow[dl, "\pi"'] \arrow[dr, "\tau"] & D\Omega \arrow[l] \arrow[dl] \arrow[dr] & \\
\tilde{\Omega}_\nu \arrow[dr, "\tau"'] & \mathbb{S}\Omega \arrow[l] \arrow[dr] & \tilde{\Omega}_\nu^* \arrow[dl, "\pi"'] & \mathbb{S}^*\Omega \arrow[l] \arrow[dl] \\
 & \Omega_\nu & \Omega \arrow[l] &
\end{tikzcd}$$

[7] Kawai, Kashiwara & Kimura [1986], proposition 2.3.6.
[8] Page 105 equation (1).
[9] After Sato, Kawai & Kashiwara [1973]

There is an isomorphism

$$\text{(4-3)} \qquad \mathbf{R}\tau_*\pi^{-1}\mathbf{R}\Gamma_{\mathbb{S}\Omega}(\tau^{-1}\mathscr{F}^\bullet) \xleftarrow{\sim} \mathbf{R}\Gamma_{\mathbb{S}^*\Omega}\pi^{-1}\mathscr{F}^\bullet.$$

PROOF. The triangle (3-1) gives rise to the following triangle in the derived category:

$$\mathbf{R}\,\mathscr{D}\mathrm{ist}_\tau(\pi^{-1}\mathscr{F}^\bullet) \longrightarrow \pi^{-1}\mathscr{F}^\bullet \longrightarrow \mathbf{R}\tau_*\tau^{-1}\pi^{-1}\mathscr{F}^\bullet \xrightarrow{+1}.$$

Applying $\mathbf{R}\Gamma_{\mathbb{S}^*\Omega}$ produces the triangle

$$\mathbf{R}\Gamma_{\mathbb{S}^*\Omega}\mathbf{R}\,\mathscr{D}\mathrm{ist}_\tau(\pi^{-1}\mathscr{F}^\bullet) \longrightarrow \mathbf{R}\Gamma_{\mathbb{S}^*\Omega}\pi^{-1}\mathscr{F}^\bullet \longrightarrow \mathbf{R}\Gamma_{\mathbb{S}^*\Omega}\mathbf{R}\tau_*\tau^{-1}\pi^{-1}\mathscr{F}^\bullet \xrightarrow{+1}.$$

By (4-1) the last term of this triangle is

$$\begin{aligned}\mathbf{R}\Gamma_{\mathbb{S}^*\Omega}\mathbf{R}\tau_*\tau^{-1}\pi^{-1}\mathscr{F}^\bullet &= \mathbf{R}\tau_*\mathbf{R}\Gamma_{D\Omega}\tau^{-1}\pi^{-1}\mathscr{F}^\bullet \\ &= \mathbf{R}\tau_*\mathbf{R}\Gamma_{D\Omega}\pi^{-1}\tau^{-1}\mathscr{F}^\bullet \\ &= \mathbf{R}\tau_*\pi^{-1}\mathbf{R}\Gamma_{\mathbb{S}\Omega}\tau^{-1}\mathscr{F}^\bullet.\end{aligned}$$

From the proof of Lemma 3.9 in Lieutenant [1986, 1988], $\mathbf{R}\,\mathscr{D}\mathrm{ist}_\tau(\pi^{-1}\mathscr{F}^\bullet) = 0$, so

$$\mathbf{R}\Gamma_{\mathbb{S}^*\Omega}\pi^{-1}\mathscr{F}^\bullet \xrightarrow{\sim} \mathbf{R}\tau_*\pi^{-1}\mathbf{R}\Gamma_{\mathbb{S}\Omega}(\tau^{-1}\mathscr{F}^\bullet). \quad \square$$

PROPOSITION 1.4.4.

$$\text{(4-4)} \qquad \mathbf{R}\tau_*\mathbf{R}\Gamma_{\mathbb{S}\Omega}(\tau^{-1}\mathscr{F}^\bullet) = \mathbf{R}\pi_*\mathbf{R}\Gamma_{\mathbb{S}^*\Omega}\pi^{-1}\mathscr{F}^\bullet.$$

PROOF. From the proof of lemma 1.4.2 we see that (4-2) gives the quasi-isomorphism

$$\pi_*\pi^{-1}\Gamma_{\mathbb{S}\Omega}\mathscr{G}^\bullet \longrightarrow \pi_*\Gamma_{D\Omega}\pi^{-1}\mathscr{G}^\bullet \simeq \Gamma_{\mathbb{S}\Omega}\pi_*\pi^{-1}G.$$

Taking derived functors gives and using the Vietoris-Begle isomorphism $\mathscr{G}^\bullet \to \mathbf{R}\pi_*\pi^{-1}\mathscr{G}^\bullet$ (which is possible by lemma 1.4.5 below) gives the isomorphism

$$\mathbf{R}\pi_*\pi^{-1}\mathbf{R}\Gamma_{\mathbb{S}\Omega}\mathscr{G}^\bullet \xrightarrow{\sim} \mathbf{R}\pi_*\mathbf{R}\Gamma_{D\Omega}\pi^{-1}\mathscr{G}^\bullet \simeq \mathbf{R}\Gamma_{\mathbb{S}\Omega}\mathbf{R}\pi_*\pi^{-1}\mathscr{G}^\bullet \simeq \mathbf{R}\Gamma_{\mathbb{S}\Omega}\mathscr{G}^\bullet.$$

For $\mathscr{G}^\bullet = \tau^{-1}\mathscr{F}^\bullet$ this isomorphism produces:

$$\begin{aligned}\mathbf{R}\tau_*\mathbf{R}\Gamma_{\mathbb{S}\Omega}(\tau^{-1}\mathscr{F}^\bullet) &= \mathbf{R}\tau_*\mathbf{R}\pi_*\pi^{-1}\mathbf{R}\Gamma_{\mathbb{S}\Omega}(\tau^{-1}\mathscr{F}^\bullet)\\ &= \mathbf{R}(\tau\pi)_*\pi^{-1}\mathbf{R}\Gamma_{\mathbb{S}\Omega}(\tau^{-1}\mathscr{F}^\bullet)\\ &= \mathbf{R}\pi_*\mathbf{R}\tau_*\pi^{-1}\mathbf{R}\Gamma_{\mathbb{S}\Omega}(\tau^{-1}\mathscr{F}^\bullet)\\ &= \mathbf{R}\pi_*\mathbf{R}\Gamma_{\mathbb{S}^*\Omega}\pi^{-1}\mathscr{F}^\bullet. \quad \square\end{aligned}$$

LEMMA 1.4.5. *$\pi : \widetilde{D\Omega^+} \longrightarrow \tilde{\Omega}_\nu$ is proper and has contractible fibres.* $\square$

THEOREM 1.4.6. *There is a triangle*

$$(\mathscr{F}^\bullet)_\Omega \longrightarrow \mathbf{R}\Gamma_\Omega(\mathscr{F}^\bullet)[n] \longrightarrow \mathbf{R}\pi_*\mathbf{R}\Gamma_{\mathbb{S}^*\Omega}(\pi^{-1}\mathscr{F}^\bullet)[n] \xrightarrow{+1}. \tag{4-4}$$

PROOF. Substitute the terms computed in lemma 1.4.1 and proposition 1.4.4 into the triangle obtained from the octahedral axiom (3-9) and translate by $[n]$. $\square$

DEFINITION 1.4.7.

(1) ${}^p\mathscr{A}_\Omega := ({}^p\mathscr{O})_\Omega$,

(2) ${}^p\mathscr{Q}_\Omega := \mathbf{R}\Gamma_{\mathbb{S}\Omega}(\tau^{-1}\,{}^p\mathscr{O})[+1]$,

(3) ${}^p\mathscr{C}_\Omega := \mathbf{R}\Gamma_{\mathbb{S}^*\Omega}(\pi^{-1}\,{}^p\mathscr{O})[n]^a$, where a is the antipodal map. $\Diamond$

(4-3) gives

PROPOSITION 1.4.8. $\mathbf{R}\tau_*\pi^{-1}\,{}^p\mathscr{Q} = {}^p\mathscr{C}[1-n]^a$. $\square$

Theorem 1.4.6 gives

PROPOSITION 1.4.9. *There is a triangle*

$$\text{(4-5)} \qquad {}^p\mathscr{A}_\Omega \longrightarrow \mathbf{R}\Gamma_\Omega({}^p\mathscr{O})[n] \longrightarrow \mathbf{R}\pi_*{}^p\mathscr{C}_\Omega \xrightarrow{+1} \qquad \square$$

PROPOSITION 1.4.10[10]. *Let $\mathscr{F}$ be a sheaf on Ω_ν, and let $x_0 + i\xi_0\infty \in \mathbb{S}^*\Omega$. Then there is an isomorphism*

$$\mathscr{H}^k_{\mathbb{S}^*\Omega}(\pi^{-1}\mathscr{F})_{x_0+i\xi_0\infty} \xleftarrow{\sim} \varinjlim_{\substack{V\ni x_0\\ G}} H^k_{V\cap G}(V;\mathscr{F}),$$

where V runs through neighbourhoods of x_0 in Ω_ν, and G through the following sets[11]*:*

$$\gamma_{m,\xi_0} := \{\xi \in \mathbb{S}^* : |\xi - \xi_0| \leq 1/m\};$$

$$G_{m,\xi_0} := \Omega + i\,\{y : \langle y|\xi\rangle \leq 0, \forall \xi \in \gamma_{m,\xi_0}\}.$$

PROOF. To be explicit, let V_ϵ $(\epsilon > 0)$ be the intersection of a basis of ${}^p\mathscr{O}$-pseudoconvex neighbourhoods of x_0 with Ω_ν that decreases to x_0 as ϵ decreases to 0. Let $U_{m,\epsilon}$ be the neighbourhoods of $x_0 + i\xi_0\infty$ in $\tilde{\Omega}^*_\nu$ defined by

$$\begin{aligned} U_{m,\epsilon} :=& \pi^{-1}V_\epsilon \cap \left((\Omega + i\gamma_{m,\xi_0}\infty) \cup G^c_{m,\xi_0}\right) \\ =& \pi^{-1}V_\epsilon \cap \left((\Omega + i\gamma_{m,\xi_0}\infty) \cup \left(\Omega + i\left\{y : y \neq 0, \frac{y}{\|y\|} \in \mathbb{S}_{n-1} - \gamma^{\perp}_{m,\xi_0}\right\}\right)\right). \end{aligned}$$

There is a morphism of triangles

(4-8)

$$\begin{array}{ccccccc} \Gamma_{G_{m,\xi_0}\cap V_\epsilon}(V_\epsilon;\mathscr{F}) & \longrightarrow & \Gamma(V_\epsilon;\mathscr{F}) & \longrightarrow & \Gamma(V_\epsilon - G_{m,\xi_0};\mathscr{F}) & \xrightarrow{+1} \\ \downarrow & & \downarrow & & \downarrow & \\ \Gamma_{\mathbb{S}^*\Omega\cap U_{m,\epsilon}}(U_{m,\epsilon};\pi^{-1}\mathscr{F}) & \rightarrow & \Gamma(U_{m,\epsilon};\pi^{-1}\mathscr{F}) & \rightarrow & \Gamma(U_{m,\epsilon} - \mathbb{S}^*\Omega;\pi^{-1}\mathscr{F}) & \xrightarrow{+1} \end{array}$$

---

[10] After Sato, Kawai & Kashiwara [1973].
[11] Notation follows Lieutenant [1986, 1988].

The vertical maps are essentially "restrictions": $\sigma \longmapsto \sigma\pi$, π being the canonical projection $\pi : \tilde{\Omega}^*_\nu \longrightarrow \Omega_\nu$.

The morphism of triangles, (4-8), gives rise to a commutative diagram of long exact sequences

(4-9)

$$\begin{array}{ccccccc}
0 \longrightarrow & H^0_{G_{m,\xi_0}\cap V_\epsilon}(V_\epsilon;\mathscr{F}) & \longrightarrow & H^0(V_\epsilon;\mathscr{F}) & \longrightarrow & H^0(V_\epsilon - G_{m,\xi_0};\mathscr{F}) & \to \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \to & H^0_{\mathbb{S}^*\Omega\cap U_{m,\epsilon}}(U_{m,\epsilon};\pi^{-1}\mathscr{F}) & \to & H^0(U_{m,\epsilon};\pi^{-1}\mathscr{F}) & \to & H^0(U_{m,\epsilon} - \mathbb{S}^*\Omega;\mathscr{F}) & \to
\end{array}$$

$$\begin{array}{ccccccc}
H^1_{G_{m,\xi_0}\cap V_\epsilon}(V_\epsilon;\mathscr{F}) & \longrightarrow & H^1(V_\epsilon;\mathscr{F}) & \longrightarrow & H^1(V_\epsilon - G_{m,\xi_0};\pi^{-1}\mathscr{F}) & \to \cdots \\
\downarrow & & \downarrow & & \downarrow & \\
H^1_{\mathbb{S}^*\Omega\cap U_{m,\epsilon}}(U_{m,\epsilon};\pi^{-1}\mathscr{F}) & \to & H^1(U_{m,\epsilon};\pi^{-1}\mathscr{F}) & \longrightarrow & H^1(U_{m,\epsilon} - \mathbb{S}^*\Omega;\mathscr{F}) & \longrightarrow \cdots
\end{array}$$

Next take the direct limit as ϵ tends to 0 and then the direct limit as m tends to ∞. Since $\widehat{\mathbb{C}^n}$ is paracompact and since, for Z closed,

$$\varinjlim_{U\supseteq Z} \Gamma(U;\mathscr{F}) \longrightarrow \Gamma(Z;\mathscr{F})$$

is an isomorphism, it follows that

$$\varinjlim_{m,\epsilon} H^k(V_\epsilon;\mathscr{F}|_{\Omega_\nu}) = \begin{cases} \mathscr{F}_{x_0}, & \text{for } k=0 \\ 0, & \text{for } k\neq 0, \end{cases}$$

and

$$\varinjlim_{m,\epsilon} H^k(U_{m,\epsilon};\pi^{-1}\mathscr{F}) = \begin{cases} (\pi^{-1}\mathscr{F})_{x_0+i\xi_0\infty}, & \text{for } k=0 \\ 0, & \text{for } k\neq 0. \end{cases}$$

Thus (4-9) provides

$$\text{(4-10)}\quad \begin{array}{ccc}
0 \longrightarrow \varinjlim_{m,\epsilon} H^0_{G_{m,\xi_0}\cap V_\epsilon}(V_\epsilon;\mathscr{F}) & \longrightarrow & \mathscr{F}_{x_0} \\
\downarrow & & \downarrow \\
0 \to \varinjlim_{m,\epsilon} H^0_{\mathbb{S}^*\Omega\cap U_{m,\epsilon}}(U_{m,\epsilon};\pi^{-1}\mathscr{F}) & \to & \mathscr{F}_{x_0}
\end{array}$$

$$\begin{array}{ccccc}
\longrightarrow \varinjlim_{m,\epsilon} H^0(V_\epsilon - G_{m,\xi_0};\mathscr{F}) & \longrightarrow & \varinjlim_{m,\epsilon} H^1_{G_{m,\xi_0}\cap V_\epsilon}(V_\epsilon;\mathscr{F}) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
\to \varinjlim_{m,\epsilon} H^0(U_{m,\epsilon} - \mathbb{S}^*\Omega;\pi^{-1}\mathscr{F}) & \to & \varinjlim_{m,\epsilon} H^1_{\mathbb{S}^*\Omega\cap U_{m,\epsilon}}(U_{m,\epsilon};\pi^{-1}\mathscr{F}) & \to & 0,
\end{array}$$

and for $k \geq 2$,

(4-11)

$$\begin{array}{ccccccc} 0 \longrightarrow & \varinjlim_{m,\epsilon} H^{k-1}(V_\epsilon - G_{m,\xi_0}; \mathscr{F}) & \longrightarrow & \varinjlim_{m,\epsilon} H^k_{G_{m,\xi_0} \cap V_\epsilon}(V_\epsilon; \mathscr{F}) & \longrightarrow 0 \\ & \downarrow & & \downarrow & \\ 0 \rightarrow & \varinjlim_{m,\epsilon} H^{k-1}(U_{m,\epsilon} - \mathbb{S}^*\Omega; \pi^{-1}\mathscr{F}) & \rightarrow & \varinjlim_{m,\epsilon} H^k_{\mathbb{S}^*\Omega \cap U_{m,\epsilon}}(U_{m,\epsilon}; \pi^{-1}\mathscr{F}) & \rightarrow 0 \end{array}$$

The vertical maps, being induced from isomorphisms, are themselves isomorphisms. It follows from the five lemma applied to (4-10), and from (4-11) that

$$\varinjlim_{m,\epsilon} H^k_{G_{m,\xi_0} \cap V_\epsilon}(V_\epsilon; \mathscr{F}) \simeq \varinjlim_{m,\epsilon} H^k_{\mathbb{S}^*\Omega \cap U_{m,\epsilon}}(U_{m,\epsilon}; \pi^{-1}\mathscr{F}) = \mathsf{R}^k\Gamma_{\mathbb{S}^*\Omega}(\pi^{-1}\mathscr{F})_{x_0+i\xi_0\infty}.$$

This proves the theorem. $\square$

REMARK 1.4.11. Clearly the proposition holds for $\mathscr{F}$ defined on F_ν *mutatis mutandis.* $\triangleright$

DEFINITION 1.4.12.

(1) ${}^p\mathscr{A}_F := (i_{\nu *}{}^p\mathscr{O})_\Omega$,

(2) ${}^p\mathscr{Q}_F := \mathsf{R}\Gamma_{\mathbb{S}F}(\tau^{-1} i_{\nu *}{}^p\mathscr{O})[+1]$,

(3) ${}^p\mathscr{C}_F := \mathsf{R}\Gamma_{\mathbb{S}^* F}(\pi^{-1} i_{\nu *}{}^p\mathscr{O})[n]$. $\lozenge$

The theory for these sheaves is however not complete at present.

CHAPTER II

THEOREMS OF KAWAI

Balances are delicate and easily tipped. The social status of a word, its force, its length, its history of use: anything can do it. Syntax sets up the scale, but semantics puts the weights in the pans. The following are out of balance: (1) "the bandit shot my son, stabbed me in the arm, and called me names," (2) "what bitter things both life and aspirin are!," (3) "I have boated everywhere—on the Po and on Pawtucket Creek," (4) "you say your marriage suffers from coital insufficiency and greasy fries?," (5) "yeah, my wife kisses her customers and brings their bad breath to bed."

—William H. Gass, *'And'* in *Habitations of the Word* [1985].

This chapter presents some restatements of Kawai's [1970] results, especially his theorems 2.2.1, 3.2.1 and 3.2.2. There is essentially nothing new here. The thrust of the effort has been to distill the essence of Kawai's results, to make sure that his results hold for these slightly more general plurisubharmonic functions. This has been carried out the way a janitor might go about making sure things are in order.

§2.1 Conditions on the Plurisubharmonic Function p

DEFINITION 2.1.1. Let $V \subseteq \mathbb{C}^n$ and $p(z)$ a plurisubharmonic function defined on $\mathbb{C}^n$. A holomorphic function $\psi \in \mathcal{O}(V)$ is controlled exponential type $(\kappa, p(\cdot))$ if

$$\exists\kappa', \quad 0 < \kappa' < \kappa, \quad \exists A_\kappa > 0, B_\kappa > 0 \quad \text{such that} \tag{1-1}$$

$$Be^{\kappa' p(z)} < |\psi(z)| < Ae^{\kappa p(z)} \qquad z \in V.$$

An open set $U \in \widehat{\mathbb{C}^n}$ is said to have a function of controlled exponential type (κ, p) if there is function of controlled exponential type (κ, p) on $U \cap \mathbb{C}^n$. $\Diamond$

REMARK 2.1.2. Suppose $U \supseteq U'$ has a function of controlled exponential type (κ, p) then clearly so does U'. $\rhd$

REMARK 2.1.3. Suppose $V \subset\subset \mathbb{C}^n$, and q is a continuous plurisubharmonic function. Then there are holomorphic functions of type (κ, q) on V for every $\kappa > 0$. Take $\psi \equiv 1$ in (1-1) and note that q attains its maximum and minimum on $cl_{\mathbb{C}^n} V$. $\rhd$

DEFINITION 2.1.4. In addition to the assumptions made in §1.1, we shall impose more restrictive conditions on the plurisubharmonic growth function p. Explicitly:

(1) $p \geq 0$, $p \in C^\infty$ is convex.

(2) For every compact $K \subseteq \widehat{\mathbb{C}^n}$, $\log(1 + |z|) = o(p(z))$ as $z \longrightarrow \infty$, $z \in K \cap \mathbb{C}^n$;

(3) $\exists A, B > 0$ such that $|z - \zeta| < 1 \implies p(\zeta) < Ap(z) + B$;[1]

(4) For sufficiently small ν every point of $\Omega_\nu - \mathbb{C}^n$ has a neighbourhood with functions of controlled type (κ, p) for every $\kappa > 0$. $\Diamond$

EXAMPLES 2.1.5.

(1) $p(z) = (1 + |z|^2)^{s/2}$ or $p(z) = |z|^s$, $s > 0$; when $s = 1$ this is the case considered by Kawai [1970], and Meril [1983].

(2) $p(z) = |\mathfrak{Re}\, z|^s$, $s \geq 1$; Kaneko [1985] considers the case $s > 0$. For $s < 1$ these p's are not plurisubharmonic, so the methods here are not applicable to his case.

[1] This condition goes back to Berenstein and Taylor. See references in Struppa [1983].

(3) $p(z) = \log^+ |f(z)|$ where $f(z) = \prod_1^n f_j(z^j)$, with f_j entire and uniformly bounded away from 0 in a $\dot{\hat{\mathbb{C}}}$ neighbourhood of $\mathbb{R}$. In this case (1-1) will be satisfied. $\triangleright$

Recall

DEFINITION 2.1.6[2]. An open set $V \subseteq \widehat{\mathbb{C}^n}$ is Saburi type (1) if for some $a > 0$

$$\text{(1-2)} \qquad \sup_{V \cap \mathbb{C}^n} \frac{|\Im\mathrm{m} z|}{|\Re\mathrm{e} z| + a} < 1.$$

Here $|\Im\mathrm{m} z| = \sqrt{\sum_j y^{j,2}}$ and $|\Re\mathrm{e} z| = \sqrt{\sum_j x^{j,2}}$. $\Diamond$

EXAMPLE 2.1.8. Suppose $V \subseteq \widehat{\mathbb{C}^n}$ is Saburi type (1). Let $p(z) = |z|$, the case considered by Kawai and Saburi. Then $\psi_\kappa(z) := \cosh\left(\kappa\sqrt{\frac{2}{3}}\sqrt{\sum_j z^{j,2}}\right)$ is a function of controlled exponential type $\kappa, |\cdot|$ for V.

PROOF. First note from the series expansion that ψ_κ is entire.

For computational purposes let $c := \sup_{V \cap \mathbb{C}^n} \dfrac{|\Im\mathrm{m} z|}{|\Re\mathrm{e} z| + a} < 1$, $\sigma(z) := \sqrt{\sum_j z^{j,2}}$, $\sigma_\mathfrak{r}(z) := \Re\mathrm{e}\,\sigma(z)$, and $\sigma_\mathfrak{i}(z) := \Im\mathrm{m}\,\sigma(z)$. Note that

$$\left|\cosh\sqrt{\tfrac{2}{3}}\kappa\sigma\right|^2 = \cosh 2\sqrt{\tfrac{2}{3}}\kappa\sigma_\mathfrak{r} + \cos 2\sqrt{\tfrac{2}{3}}\kappa\sigma_\mathfrak{i}.$$

Let $z^j = x^j + iy^j = r_j e^{i\theta_j}$. Then $\sigma(z) = \sqrt{\sum_j r_j^2 e^{2i\theta_j}}$. Define r and θ by $r^2 e^{2i\theta} := \sum_j r_j^2 e^{2i\theta_j}$. Then

$$r^2 = \sqrt{(\textstyle\sum_j r_j^2 \cos 2\theta_j)^2 + (\sum_j r_j^2 \sin 2\theta_j)^2},$$

$\sigma(z) = re^{i\theta} = r\cos\theta + i\sin\theta$, and $\cos 2\theta = \frac{\sum_j r_j^2 \cos 2\theta_j}{r^2}$. So up to sign

$$\begin{aligned}\sigma(z) &= \pm r\sqrt{\frac{1+\cos 2\theta}{2}} \pm ir\sqrt{\frac{1+\cos 2\theta}{2}} \\ &= \pm\sqrt{\frac{r^2 + \sum_j r_j^2 \cos 2\theta_j}{2}} \pm i\sqrt{\frac{r^2 - \sum_j r_j^2 \cos 2\theta_j}{2}}.\end{aligned}$$

[2]Saburi, Y. [1978].

Hence

$$(1\text{-}3) \qquad \sigma_{\mathfrak{r}} = \pm\sqrt{\frac{r^2 + \sum_j r_j^2 \cos 2\theta_j}{2}}; \qquad \sigma_{\mathfrak{i}} = \pm\sqrt{\frac{r^2 - \sum_j r_j^2 \cos 2\theta_j}{2}}$$

Similarly, using "cartesian" coordinates one gets

$$\sigma(z) = \sqrt{\sum_j (x^{j,2} - y^{j,2}) + 2i \sum_j x^j y^j} \quad = re^{i\theta}.$$

Then

$$(1\text{-}4) \qquad r^2 = \sqrt{(\sum_j x^{j,2} - y^{j,2})^2 + 4(\sum_j x^j y^j)^2}; \qquad \text{and}$$

$$\cos 2\theta = \frac{\sum_j x^{j,2} - y^{j,2}}{r^2}.$$

In terms of these coordinates

$$\sigma(z) = r\cos\theta + i\sin\theta = \pm\sqrt{\frac{r^2 + \sum_j (x^{j,2} - y^{j.2})}{2}} \pm i\sqrt{\frac{r^2 - \sum_j (x^{j,2} - y^{j,2})}{2}}.$$

Thus up to sign

$$(1\text{-}5) \qquad \sigma_{\mathfrak{r}} = \pm\sqrt{\frac{r^2 + \sum_j (x^{j,2} - y^{j,2})}{2}}; \qquad \sigma_{\mathfrak{i}} = \pm\sqrt{\frac{r^2 - \sum_j (x^{j,2} - y^{j,2})}{2}}.$$

1) Upper bound.

By

$$\left|\cosh\sqrt{\tfrac{2}{3}}\kappa\sigma\right|^2 \leq \cosh 2\sqrt{\tfrac{2}{3}}\kappa\sigma_{\mathfrak{r}} + 1 \leq e^{2\sqrt{\frac{2}{3}}\kappa|\sigma_{\mathfrak{r}}|} + 1.$$

Now estimate $\sigma_{\mathfrak{r}}$ using the "polar" coordinates (1-3).

$$|\sigma_{\mathfrak{r}}| = \sqrt{\frac{r^2 + \sum_j r_j^2 \cos 2\theta_j}{2}} \leq \sqrt{\frac{r^2 + \sum_j r_j^2}{2}}.$$

Since

$$r^2 \leq \left|\sum_j r_j^2 \cos 2\theta_j\right| + \left|\sum_j r_j^2 \sin 2\theta_j\right| \leq \sum_j r_j^2,$$

$$|\sigma_{\mathfrak{r}}| \leq \sqrt{\tfrac{3}{2}}|z| = \sqrt{\tfrac{3}{2}}p(z).$$

By choosing A sufficiently large, there is an upper bound

$$\left|\cosh\sqrt{\tfrac{2}{3}}\kappa\sigma\right| \leq Ae^{\kappa p}.$$

2) Lower bound.

For the lower bound the case when $\cosh\sqrt{\frac{2}{3}}\kappa\sigma = 0$ is dispensed with and then a asymptotic growth is obtained.

a) $\cosh\sqrt{\frac{2}{3}}\kappa\sigma = 0$ if and only if $2\sqrt{\frac{2}{3}}\kappa\sigma_{\mathfrak{r}} = 0$ and $2\sqrt{\frac{2}{3}}\kappa\sigma_{\mathfrak{i}} = (2k+1)\pi \quad k \in \mathbb{Z}$.

$$\sigma_{\mathfrak{r}} = \pm\sqrt{\frac{r^2 + \sum_j(x^{j,2} - y^{j,2})}{2}} = 0 \quad \Leftrightarrow \quad r^2 + \sum_j(x^{j,2} - y^{j,2}) = 0,$$

$$\sigma_{\mathfrak{i}} = \pm\sqrt{\frac{r^2 - \sum_j(x^{j,2} - y^{j,2})}{2}} = \frac{(2k+1)\pi}{2\sqrt{\frac{2}{3}}\kappa}$$

$$\Longrightarrow r^2 - \sum_j(x^{j,2} - y^{j,2}) = \frac{(2k+1)^2\pi^2}{\frac{4}{3}\kappa^2}.$$

Hence

$$\sum_j x^{j,2} - y^{j,2} = -\frac{(2k+1)^2\pi^2}{\frac{8}{3}\kappa^2}, \qquad \text{or}$$

$$\sum_j y^{j,2} = \sum_j x^{j,2} + \frac{(2k+1)^2\pi^2}{\frac{8}{3}\kappa^2}.$$

This would be impossible if it fails to satisfy Saburi's type (1) inequality (1-2).

So $\cosh\sqrt{\frac{2}{3}}\kappa\sigma \neq 0$ if

$$\sum_j y^{j,2} = \sum_j x^{j,2} + \frac{(2k+1)^2\pi^2}{\frac{8}{3}\kappa^2} > c^2\sum_j x^{j,2} + 2c^2 a\sum_j x^{j,2} + c^2a^2.$$

Simplifying and completing the square gives

$$
\begin{aligned}
0 &< \left(1-c^2\right)\textstyle\sum_j x^{j,2} - 2c^2 a\sqrt{\sum_j x^{j,2}} - c^2a^2 + \dfrac{(2k+1)^2\pi^2}{\frac{8}{3}\kappa^2} \\
&= \left(1-c^2\right)\left(\textstyle\sum_j x^{j,2} - 2\dfrac{c^2a}{1-c^2}\sqrt{\textstyle\sum_j x^{j,2}}\right) - c^2a^2 + \dfrac{(2k+1)^2\pi^2}{\frac{8}{3}\kappa^2} \\
&= \left(1-c^2\right)\left(\sqrt{\textstyle\sum_j x^{j,2}} - \dfrac{c^2a}{1-c^2}\right)^2 \\
&\qquad - \left(1-c^2\right)\left(\dfrac{c^2a}{1-c^2}\right)^2 - c^2a^2 + \dfrac{(2k+1)^2\pi^2}{\frac{8}{3}\kappa^2}.
\end{aligned}
$$

Since $(1-c^2)\left(\sqrt{\sum_j x^{j,2}} - \frac{c^2a}{1-c^2}\right)^2 \geq 0$, if κ can be chosen so that the other two terms of the last line are greater than zero, (1-2) will be false; i. e. κ has to be chosen so that

$$
-\left(1-c^2\right)\left(\frac{c^2a}{1-c^2}\right)^2 - c^2a^2 + \frac{(2k+1)^2\pi^2}{\frac{8}{3}\kappa^2} > 0, \qquad \text{for } k \in \mathbb{Z}.
$$

Equivalently

$$
\begin{aligned}
\frac{(2k+1)^2\pi^2}{\frac{8}{3}\kappa^2} &> \left(1-c^2\right)\left(\frac{c^2a}{1-c^2}\right)^2 + c^2a^2 \\
&= \frac{c^2a^2}{1-c^2}, \qquad \text{for } k \in \mathbb{Z}.
\end{aligned}
$$

This has to be true for all $k \in \mathbb{Z}$, so κ has to be chosen such that

$$
\frac{\pi^2}{\frac{8}{3}\kappa^2} > \frac{c^2a^2}{1-c^2}.
$$

This will clearly hold for $\kappa < \kappa_0$ for some small κ_0.

In summary when $0 < \kappa < \kappa_0$, $\left|\cosh\sqrt{\frac{2}{3}}\kappa\sigma\right| > 0$.

b) Now provide a lower bound for the asymptotic behaviour. Assume first of all that $|z| \gg 1$, and note that

$$
\text{(1-6)} \qquad \left|\cosh 2\sqrt{\tfrac{2}{3}}\kappa\sigma_{\mathfrak{r}}\right| = \frac{e^{2\sqrt{\frac{2}{3}}\kappa\sigma_{\mathfrak{r}}} + e^{-2\sqrt{\frac{2}{3}}\kappa\sigma_{\mathfrak{r}}}}{2} > \frac{1}{2}e^{2\sqrt{\frac{2}{3}}\kappa|\sigma_{\mathfrak{r}}|}.
$$

Now estimate $|\sigma_{\mathfrak{r}}|$ in (1-6). From (1-4)

$$r^2 \geq \left|\sum_j x^{j,2} - y^{j,2}\right|.$$

(1-2) implies that for sufficiently large $\left|\sqrt{\sum_j x^{j,2}}\right|$, $\sum_j y^{j,2} < \sum_j x^{j,2}$; so in this case (1-4) and (1-5) give

$$\begin{aligned}
|\sigma_{\mathfrak{r}}| &\geq \sqrt{\left(\left|\sum_j x^{j,2} - \sum_j y^{j,2}\right| + \sum_j x^{j,2} - \sum_j y^{j,2}\right)/2} \\
&= \sqrt{\sum_j x^{j,2} - \sum_j y^{j,2}} \\
&\geq \sqrt{\sum_j x^{j,2}} - \sqrt{\sum_j y^{j,2}} \\
&\geq \sqrt{\sum_j x^{j,2}} - \left(c\sqrt{\sum_j x^{j,2}} + ca\right) \\
&= (1-c)\sqrt{\sum_j x^{j,2}} - ca.
\end{aligned}$$

It follows that

$$\text{(1-7)} \qquad \frac{1}{2}e^{2\sqrt{\frac{2}{3}}\kappa|\sigma_{\mathfrak{r}}|} \geq \frac{1}{2}e^{-2\sqrt{\frac{2}{3}}\kappa ca}e^{2\sqrt{\frac{2}{3}}\kappa(1-c)\sqrt{\sum_j x^{j,2}}}.$$

On the other hand,

$$p(z) = \sqrt{\sum_j x^{j,2} + \sum_j y^{j,2}} \leq \sqrt{\sum_j x^{j,2}} + \sqrt{\sum_j y^{j,2}} \leq (1+c)\sqrt{\sum_j x^{j,2}} + ca,$$

and

$$\frac{p(z)}{1+c} - \frac{ca}{1+c} \leq \sqrt{\sum_j x^{j,2}}.$$

Together with (1-7), this yields

$$\frac{1}{2}e^{2\sqrt{\frac{2}{3}}\kappa|\sigma_{\mathfrak{r}}(z)|} \geq \frac{1}{2}e^{-2\sqrt{\frac{2}{3}}\kappa ca}e^{-2\sqrt{\frac{2}{3}}\kappa\frac{1-c}{1+c}ca}e^{2\sqrt{\frac{2}{3}}\kappa\frac{1-c}{1+c}p(z)}, \qquad \text{for large } \sqrt{\sum_j x^{j,2}}.$$

Since $\left|\cosh\sqrt{\frac{2}{3}}\kappa\sigma\right| > 0$ from step (2a), this lower asymptotic bound shows that

$$\left|\cosh\sqrt{\tfrac{2}{3}}\kappa\sigma\right| \geq m > 0.$$

Choosing B sufficiently small gives the lower bound

$$Be^{\sqrt{\frac{2}{3}}\kappa\frac{1-c}{1+c}p(z)} \leq \left|\cosh\sqrt{\tfrac{2}{3}}\kappa\sigma(z)\right|. \quad \square$$

§2.2 Spaces of Holomorphic Functions with Growth Conditions

The topologies of the spaces involved are first recalled from Saburi [1978], Nagamachi [1981], Meril [1983] and Berenstein & Struppa [preprint].

DEFINITION 2.2.1. Let $U \subseteq \widehat{\mathbb{C}^n}$ be open and $K'_j, K'_j \subset\subset int_{\widehat{\mathbb{C}^n}} K'_{j+1}$ be an exhaustion of U by compact subsets of $\widehat{\mathbb{C}^n}$.

$$\begin{aligned}
{}^p\tilde{X}^0(U) &:= \varprojlim_{j,\epsilon\searrow 0} L^2\left(int_{\widehat{\mathbb{C}^n}} K'_j \cap \mathbb{C}^n; \epsilon p(z)\right); \\
{}_p\tilde{X}^0_{\text{cpt}}(U) &:= \varinjlim_{j,\epsilon\searrow 0} L^2\left(int_{\widehat{\mathbb{C}^n}} K'_j \cap \mathbb{C}^n; -\epsilon p(z)\right); \\
{}^p\mathscr{O}(U) &:= \varprojlim_{j,\epsilon\searrow 0} L^\infty\left(int_{\widehat{\mathbb{C}^n}} K'_j \cap \mathbb{C}^n; \epsilon p(z)\right) \cap \mathscr{O}\left(int_{\widehat{\mathbb{C}^n}} K'_j \cap \mathbb{C}^n\right); \\
{}_p\mathscr{O}(U) &:= \varprojlim_{j} \varinjlim_{\delta\searrow 0} {}_p\mathscr{O}_{\text{Bdd}}\left(int_{\widehat{\mathbb{C}^n}} K'_j \cap \mathbb{C}^n; -\delta p(z)\right).
\end{aligned}$$

For K compact in $\widehat{\mathbb{C}^n}$ and $int_{\widehat{\mathbb{C}^n}} K_j$ a basis of compact neighbourhoods of K, let

$${}_p\mathscr{O}(K) := \varinjlim_{j,\delta\searrow 0} {}_p\mathscr{O}_{\text{Bdd}}\left(int_{\widehat{\mathbb{C}^n}} K_j \cap \mathbb{C}^n; -\delta p(z)\right),$$

where

$${}_p\mathscr{O}_{\text{Bdd}}(K;\phi) := \{f \in \mathscr{O}(K \cap \mathbb{C}^n) : \sup_{K\cap\mathbb{C}^n} |f|e^{-\phi} =: \|f\|_K < \infty\}, \qquad \text{and,}$$

$$\mathscr{O}^2(L;\phi) := \left\{f \in \mathscr{O}(L) : \sqrt{\int_L |f|^2 e^{-\phi} d\lambda} =: \|f\|_L^2 < \infty\right\}. \quad \Diamond$$

LEMMA 2.2.2[3]. *Let* $L'_j = int_{\widehat{\mathbb{C}^n}} K'_j \cap \mathbb{C}^n$, K'_j *increasing as in definition 2.2.1. Let* $m > 0$. *Then*

$$\varinjlim_j \mathcal{O}\left(L'_j; -\tfrac{1}{j}p(z) - 2m\log(1+|z|^2)\right) = \varinjlim_j \mathcal{O}\left(L'_j; -\tfrac{1}{j}p(z)\right) \qquad \text{as } TVS.$$

PROOF. Clearly

$$\mathcal{O}\left(L'_j; -\tfrac{1}{j}p(z) - 2m\log(1+|z|^2)\right) \longrightarrow \varinjlim_j \mathcal{O}\left(L'_j; -\tfrac{1}{j}p(z)\right)$$

is a continuous inclusion. Hence

$$\varinjlim_j \mathcal{O}\left(L'_j; -\tfrac{1}{j}p(z) - 2m\log(1+|z|^2)\right) \longrightarrow \varinjlim_j \mathcal{O}\left(L'_j; -\tfrac{1}{j}p(z)\right)$$

is a continuous injection.

On the other hand, let $f \in \mathcal{O}\left(L'_j; -\frac{1}{j}p(z)\right)$, and choose $\delta > 0$ such that $4m\delta - \frac{1}{j(j+1)} < 0$. Since $\log(1+|z|) = o(p(z))$ as $z \to \infty$ there is an R such that $\log(1+|z|) < \delta p(z)$ for $|z| > R$. Thus

$$e^{-\frac{1}{j+1}p(z)+4m\log(1+|z|)} < \delta p(z), \qquad |z| > R.$$

It follows that

$$\int_{L'_{j+1}} |f|^2 e^{\frac{1}{j+1}p(z)+2m\log(1+|z|^2)} d\lambda \leq \int_{L'_{j+1}} |f|^2 e^{\frac{1}{j}p(z)} e^{-\frac{1}{j(j+1)}p(z)+4m\log(1+|z|)} d\lambda$$

$$\leq M \int_{L'_{j+1}} |f|^2 e^{\frac{1}{j}p(z)} d\lambda.$$

So the map induced by restriction

$$\varinjlim_j \mathcal{O}\left(L'_j; -\tfrac{1}{j}p(z) - 2m\log(1+|z|^2)\right) \longleftarrow \varinjlim_j \mathcal{O}\left(L'_j; -\tfrac{1}{j}p(z)\right)$$

is continuous. This proves the lemma. $\square$

PROPOSITION 2.2.3[4]. *Let* $L_j = int_{\widehat{\mathbb{C}^n}} K_j \cap \mathbb{C}^n$, K_j *decreasing as in definition*

[3]Kawai [1970].
[4]Kawai [1970].

2.2.1. Then

$$\varinjlim_{int_{\widehat{\mathbb{C}^n}} K_j \supseteq K} {}_p\mathcal{O}\left(int_{\widehat{\mathbb{C}^n}} K_j; -\tfrac{1}{j}p(z)\right) = \varinjlim_{L_j} \mathcal{O}\left(L_j; -\tfrac{1}{j}p(z)\right) \qquad \text{as TVS.}$$

PROOF. First note that the map

$$\begin{aligned} {}_p\mathcal{O}\left(int_{\widehat{\mathbb{C}^n}} K_j; -\tfrac{1}{j}p(z)\right) &\longrightarrow \mathcal{O}\left(L_j; -\tfrac{1}{j}p(z)\right) \\ f &\longmapsto f \end{aligned}$$

is well-defined and continuous because if $f \in {}_p\mathcal{O}(int_{\widehat{\mathbb{C}^n}} K_j)$ then $\sup_{L_j} |f| e^{\frac{1}{j}p} < \infty$. Thus

$$\begin{aligned} \int_{L_j} |f|^2 e^{\frac{1}{j}p} d\lambda &= \int_{L_j} |f|^2 e^{\frac{2}{j}p} e^{-\frac{1}{j}p} d\lambda \\ &\leq \left(\sup_{L_j} |f| e^{\frac{1}{j}p}\right)^2 \int_{L_j} e^{-\frac{1}{j}p} d\lambda. \end{aligned}$$

However $\log(1+|z|) = o(p(z))$ as $z \to \infty$, so $\int_{L_j} e^{-\frac{1}{j}p} d\lambda < \infty$. Thus

$$\varinjlim_{j} {}_p\mathcal{O}\left(int_{\widehat{\mathbb{C}^n}} K_j; -\tfrac{1}{j}p(z)\right) \to \varinjlim_{j} \mathcal{O}\left(L_j; -\tfrac{1}{j}p(z)\right)$$

is continuous.

Next we show $\varinjlim_j {}_p\mathcal{O}\left(int_{\widehat{\mathbb{C}^n}} K_j; -\tfrac{1}{j}p(z)\right) \to \varinjlim_j \mathcal{O}\left(L_j; -\tfrac{1}{j}p(z)\right)$ is surjective. Consider the map given by restriction:

$${}_p\mathcal{O}\left(int_{\widehat{\mathbb{C}^n}} K_{[Aj+1]}; -\tfrac{1}{[Aj+1]}p(z)\right) \leftarrow \mathcal{O}\left(L_j; -\tfrac{1}{j}p(z)\right),$$

where the brackets $[\cdot]$ here denote the greatest integer, and the constant A comes from definition 2.1.4(3)

Choose r so that $B(z,r) \subseteq L_j$ for all $z \in L_{j+1}$. Then

$$|f(z)| \leq \frac{1}{\lambda(B(z,r))} \int_{B(z,r)} |f(\zeta)| d\lambda$$

$$\leq \frac{1}{\lambda(B(z,r))} \sqrt{\int_{B(z,r)} |f(\zeta)|^2 e^{\frac{1}{j}p(\zeta)} d\lambda} \sqrt{\int_{B(z,r)} e^{-\frac{1}{j}p(\zeta)} d\lambda}$$

But from definition 2.1.4(3) $p(\zeta) > (p(z) - B)/A$, so

$$|f(z)| \leq \frac{C}{r^{2n}} r^n e^{-p(z)/2jA} \leq C' e^{-p(z)/[2jA+1]}, \qquad z \in L_{j+1}.$$

Thus ${}_p\mathscr{O}\left(int_{\widehat{\mathbb{C}^n}} K_{[2jA+1]}; -\frac{1}{[2jA+1]}p(z)\right) \leftarrow \mathscr{O}\left(L_j; -\frac{1}{j}p(z)\right)$ is well-defined. This proves the surjectivity.

Since the preimages of barrels are barrels, $\varinjlim_j \mathscr{O}\left(L_j; -\frac{1}{j}p(z)\right)$ is barreled. Moreover as the direct limit of injective[5] weakly compact[6] maps it is a DFS* space, and thus Hausdorff.

$\varinjlim_j {}_p\mathscr{O}\left(int_{\widehat{\mathbb{C}^n}} K_j; -\frac{1}{j}p(z)\right) = {}_p\mathscr{O}(K)$ is a DFS space, and the strong dual of a Fréchet space, thus it is fully complete[7].

Thus $\varinjlim_j {}_p\mathscr{O}\left(int_{\widehat{\mathbb{C}^n}} K_j; -\frac{1}{j}p(z)\right) \to \varinjlim_j \mathscr{O}\left(L_j; -\frac{1}{j}p(z)\right)$ is open[8]. It is clearly 1-1. □

REMARK 2.2.4.

$$\varinjlim_j {}_p\mathscr{O}\left(L_j; -\tfrac{1}{j}p(z)\right) = \varinjlim_j \varinjlim_\delta \mathscr{O}\left(L_j; -\delta p(z)\right)$$

Similarly for $\mathscr{O}$ replaced with ${}_p\mathscr{O}$ and the weight $-\frac{1}{j}p$ replaced with $-\frac{1}{j}p - 2m\log(1+|z|^2)$.

This follows because limits commute and from "diagram chasing". ▷

---

[5]Each component of $int_{\widehat{\mathbb{C}^n}} K_j$ intersect K by assumption.

[6]The spaces $\mathscr{O}\left(L_j; -\frac{1}{j}p(z)\right)$ are Hilbert spaces; Aloaglu- Bourbaki theorem.

[7]Page, W. [1988] theorem 21.3(ii).

[8]Page, W. [1988] corollary 21.9.

Definition 2.2.5[9]. Let $X\left(L_j; -\delta p(z)\right)$ denote the closure of $\mathscr{O}\left(L_j; -2\delta p(z)\right)$ in $L^2\left(L_j; -\delta p(z)\right)$. Similarly for $X\left(L_j; -\delta p(z) - 2m\log(1+|z|^2)\right)$. $\Diamond$

Remark 2.2.6. $X\left(L_j; -\delta p(z)\right) \subseteq \mathscr{O}\left(L_j; -\delta p(z)\right)$, since $\mathscr{O}\left(L_j; -\delta p(z)\right)$, being the kernel of $-\bar{\partial}$, is closed in $L^2\left(L_j; -\delta p(z)\right)$. $\triangleright$

Lemma 2.2.7[10].

$$\varinjlim_{\delta\searrow 0} X\left(L_j; -\delta p(z) - 2m\log(1+|z|^2)\right) = \varinjlim_{\delta\searrow 0} X\left(L_j; -\delta p(z)\right).$$

Proof. The proof is essentially the same as in lemma 2.2.2. First consider the map

$$\begin{aligned} X\left(L_j; -\delta p(z) - 2m\log(1+|z|^2)\right) &\longrightarrow X\left(L_j; -\delta p(z)\right); \\ f &\longmapsto f. \end{aligned}$$

This is well-defined and continuous since

$$\int_{L_j} |f|^2 e^{\delta p(z) + 2m\log(1+|z|^2)}\, d\lambda \geq \int_{L_j} |f|^2 e^{\delta p(z)}\, d\lambda.$$

If $f_k \in \mathscr{O}\left(L_j; -2\delta p(z) - 2m\log(1+|z|^2)\right)$, and $f_k \to f$ in $L^2\left(L_j; -\delta p(z) - 2m\log(1+|z|^2)\right)$, then $f_k \in \mathscr{O}\left(L_j; -2\delta p(z)\right)$ and $f_k \to f$ in $L^2\left(L_j; -\delta p(z)\right)$.

On the other hand, for $\delta < \delta'$,

$$\begin{aligned} X\left(L_j; -\delta p(z) - 2m\log(1+|z|^2)\right) &\longleftarrow X\left(L_j; -\delta' p(z)\right) \\ f &\longleftarrow\!\!\shortmid f \end{aligned}$$

[9] Kawai [1970].
[10] Kawai [1970].

is well-defined since

$$\int_{L_j} |f|^2 e^{\delta p(z)+2m\log(1+|z|^2)} d\lambda \leq \int_{L_j} |f|^2 e^{\delta' p(z)} e^{-(\delta'-\delta)p(z)+2m\log(1+|z|^2)} d\lambda$$

$$\leq M \int_{L_j} |f|^2 e^{\delta' p(z)} d\lambda.$$

Again if $f_k \in \mathscr{O}(L_j; -2\delta' p(z))$, $f_k \to f \in L^2(L_j; -\delta' p(z))$, then $f_k \in \mathscr{O}(L_j; -2\delta p(z) - 2m\log(1+|z|^2))$ and $f_k \to f$ in $L^2(L_j; -\delta p(z) - 2m\log(1+|z|^2))$. $\square$

LEMMA 2.2.8[11]. *Let $K_j \subseteq U$ be compact neighbourhoods of K that decrease to K. Suppose U satisfies property (P_p). Then*

$$X\left(L_j; -\delta p(z) - 2m\log(1+|z|^2)\right) = \mathscr{O}\left(L_j; -\delta p(z) - 2m\log(1+|z|^2)\right).$$

PROOF. There is an injection

$$cl_{L^2(L_j;-\delta p(z)-2m\log(1+|z|^2))}\mathscr{O}\left(L_j; -2\delta p(z) - 2m\log(1+|z|^2)\right)$$

$$= X\left(L_j; -\delta p(z) - 2m\log(1+|z|^2)\right) \hookrightarrow \mathscr{O}\left(L_j; -\delta p(z) - 2m\log(1+|z|^2)\right).$$

Let

$$\mu \in L^2\left(L_j; -\delta p(z) - 2m\log(1+|z|^2)\right)'$$

such that

$$\mu\left(\mathscr{O}\left(L_j; -2\delta p(z) - 2m\log(1+|z|^2)\right)\right) = 0.$$

Then

$$\exists u \in L^2\left(L_j; -\delta p(z) - 2m\log(1+|z|^2)\right)$$

[11] Kawai [1970].

such that

$$\mu(v) = \int_{L_j} v\bar{u}e^{\delta p(z)+2m\log(1+|z|^2)}d\lambda.$$

Suppose $f \in \mathscr{O}\left(L_j; -\delta p(z) - 2m\log(1+|z|^2)\right)$. Let ϕ be the holomorphic function given by property (P_p). Then $fe^{-\frac{1}{k}\phi} \in \mathscr{O}\left(L_j; -2\delta p(z) - 2m\log(1+|z|^2)\right)$ for all $k \in \mathbb{Z}^+$ because

$$\begin{aligned}
&\int_{L_j} |fe^{-\frac{1}{k}\phi}|^2 e^{\delta p(z)+2m\log(1+|z|^2)}d\lambda \\
&\qquad = \int_{L_j} |f|^2 e^{-\frac{2}{k}\phi+\delta p(z)} e^{\delta p(z)+2m\log(1+|z|^2)}d\lambda \\
&\qquad \le \int_{L_j} |fe^{\delta p(z)+2m\log(1+|z|^2)}d\lambda e^{\sup_{L_j}\left(-\frac{2}{k}\Re\mathfrak{e}\phi+\delta p(z)\right)} \\
&\qquad < \infty.
\end{aligned}$$

So

$$0 = \mu(fe^{-\frac{1}{k}\phi}) = \int_{L_j} fe^{-\frac{1}{k}\phi}\bar{u}e^{\delta p(z)+2m\log(1+|z|^2)}d\lambda.$$

Now, for some $\xi > 0$,

$$\left|e^{-\frac{1}{k}\phi}\right| = e^{-\Re\mathfrak{e}\phi/k} \le (e^{\sup_\Omega -\Re\mathfrak{e}\phi})^{1/k} \le (e^{\sup_\Omega -\Re\mathfrak{e}\phi+\xi p})^{1/k} < \infty, \qquad \forall k.$$

So Lebesgue's dominated convergence theorem gives

$$\mu(f) = 0 = \int_{L_j} f\bar{u}e^{\delta p(z)+2m\log(1+|z|^2)}d\lambda, \qquad \forall f \in \mathscr{O}\left(L_j; -\delta p(z) - 2m\log(1+|z|^2)\right).$$

By the Hahn-Banach theorem, $\mathscr{O}\left(L_j; -2\delta p(z) - 2m\log(1+|z|^2)\right)$ is dense in $\mathscr{O}\left(L_j; -\delta p(z) - 2m\log(1+|z|^2)\right)$. $\square$

DEFINITION 2.2.9. For an open set $U \subseteq \widehat{\mathbb{C}^n}$ and a family of increasing compact sets $K_c, c \in \mathbb{R}$, $K_c \uparrow U$, $K_c \subset\subset int_{\widehat{\mathbb{C}^n}} K_{c'}$ for $c < c'$, define

$$\mathcal{O}^2(U \cap \mathbb{C}^n) := \varprojlim_{c \nearrow \infty} \varinjlim_{\delta' \searrow 0} \mathcal{O}\left(K_c \cap \mathbb{C}^n; -\delta' p(z) - 2m log(1+|z|^2)\right),$$

$${}_p\mathcal{O}(U) := \varprojlim_{c \nearrow \infty} \varinjlim_{\delta' \searrow 0} \mathcal{O}\left(K_c; -\delta' p(z) - 2m log(1+|z|^2)\right). \quad \Diamond$$

LEMMA 2.2.10. *For U and K_c as in definition 2.2.9,*

$$\mathcal{O}^2(U \cap \mathbb{C}^n) = {}_p\mathcal{O}(U) \qquad \text{as sets.}$$

PROOF. The proof follows that of Proposition 2.2.3.

Let ${}_p\mathcal{O}(U) \longrightarrow \mathcal{O}^2(U \cap \mathbb{C}^n)$ be the "identity": $f \mapsto f$. To show this is well-defined, let K be a compact subset of U. Without loss of generality, K can be taken to be K_c for some c. By definition

$$\exists \delta > 0 \quad \text{such that} \quad \sup_{K_c \cap \mathbb{C}^n} |f| e^{\delta p(z) + 2m \log(1+|z|^2)} < \infty.$$

Thus

$$\int_{K_c \cap \mathbb{C}^n} |f|^2 e^{\delta p(z) + 2m \log(1+|z|^2)} d\lambda$$

$$= \int_{K_c \cap \mathbb{C}^n} |f|^2 e^{2\delta p(z) + 4m \log(1+|z|^2)} e^{-\delta p(z) - 2m \log(1+|z|^2)} d\lambda$$

$$\leq \left(\sup_{K_c \cap \mathbb{C}^n} |f| e^{\delta p(z) + 2m \log(1+|z|^2)} \right)^2 \int_{K_c \cap \mathbb{C}^n} e^{-\delta p(z) - 2m \log(1+|z|^2)} d\lambda$$

$$< \infty.$$

To show that the inverse ${}_p\mathcal{O}(U) \longleftarrow \mathcal{O}^2(U \cap \mathbb{C}^n)$ is well defined, let K be a compact subset of U, and $f \in \mathcal{O}^2(U \cap \mathbb{C}^n)$. Then $K \subset\subset int_{\widehat{\mathbb{C}^n}} K_c$ for some c.

Choose $r > 0$ so that $B(z,r) \subset K_c$ for all $z \in K \cap \mathbb{C}^n$. By definition there is a $\delta > 0$ such that

$$\int_{K_c \cap \mathbb{C}^n} |f|^2 e^{\delta p(\zeta) + 2m \log(1+|z|^2)} d\lambda < M < \infty.$$

Following the argument in proposition 2.2.3, we have for $z \in K$,

$$\begin{aligned} |f(z)| &\leq \frac{1}{\lambda(B(z,r))} \int_{B(z,r)} |f(\zeta)| d\lambda \\ &\leq \frac{1}{\lambda(B(z,r))} \sqrt{\int_{B(z,r)} |f(\zeta)|^2 e^{\delta p(\zeta)} d\lambda} \sqrt{\int_{B(z,r)} e^{-\delta p(\zeta)} d\lambda} \end{aligned}$$

Thus

$$|f(z)| \leq \frac{C}{r^{2n}} r^n e^{-\delta' p(z)} \leq C' e^{-\delta'' p(z) + 2m \log(1+|z|^2)}, \qquad z \in L_{j+1}.$$

So $\sup_{K \cap \mathbb{C}^n} |f(z)| e^{\delta'' p(z) + 2m log(1+|z|^2)} < \infty$. This proves the lemma. □

LEMMA 2.2.11[12]. *Suppose K_j is a decreasing sequence of compact neighbourhoods of a compact set $K \subseteq U$ and that U satisfies (P_p). Then for $\delta < \delta'$ there is a dense inclusion*

$$\mathcal{O}\left(L_j; -\delta' p(z) - 2m \log(1+|z|^2)\right) \hookrightarrow \mathcal{O}\left(L_j; -\delta p(z) - 2m \log(1+|z|^2)\right).$$

In fact the closure of the image in $L^2\left(L_j; -\delta p(z) - 2m \log(1+|z|^2)\right)$ is $\mathcal{O}\left(L_j; -\delta' p(z) - 2m \log(1+|z|^2)\right)$.

PROOF. Recall that $\mathcal{O}\left(L_j; -\delta p(z) - 2m \log(1+|z|^2)\right)$ is a closed subspace of $L^2\left(L_j; -\delta' p - 2m \log(1+|z|^2)\right)$. Follow the proof of lemma 2.2.8. □

[12] Kawai [1970].

Corollary 2.2.12.

$$\varinjlim_j \mathcal{O}\left(L_j; -\delta' p - 2m \log(1+|z|^2)\right) \hookrightarrow \varinjlim_j \mathcal{O}\left(L_j; -\delta p(z) - 2m \log(1+|z|^2)\right)$$

has dense image.

Proof. This follows from general definitions of direct limits. Let μ be a continuous linear functional and suppose each f_j has dense image:

$$\begin{tikzcd}
A_i \arrow[r, "f_j"] \arrow[d, "\rho_j"] & A_i' \arrow[d, "\rho_j'"] & \\
\varinjlim_j A_j \arrow[r, "f"] & \varinjlim_j A_j' \arrow[r, "\mu"] & \mathbb{C}
\end{tikzcd}$$

Suppose $\mu f = 0$. Then $\mu f \rho_j = 0 = \mu \rho_j' f_j$, and hence $\mu \rho_j = 0 \quad \forall j$. This implies that $\mu = 0$. $\square$

§2.3 Kawai's Approximation Theorem

In this section we note that Kawai's approximation theorem remains true for sets not necessarily in $\mathbb{D}^n$ [13].

Lemma 2.3.1. *Consider the inductive system $\{A_\epsilon\}$ in an abelian category. (For simplicity assume this category is concrete.)*

$$\begin{tikzcd}
& \rho_\epsilon^* L' \arrow[d] \arrow[dr] & & \\
\rho_\epsilon^* L \arrow[r, "\rho_\epsilon^* f"] \arrow[dd] & A_\epsilon \arrow[rr] \arrow[ddr] & & A_{\epsilon'} \arrow[dddl, dash] \\
& & L' \arrow[d, "f'"] & \\
L \arrow[rr, "f"'] & & \varinjlim_\epsilon A_\epsilon =: A \arrow[d, "\mu"] & \\
& & B &
\end{tikzcd}$$

[13] See also Saburi [1985] §2.3.

Given morphisms f and f' consider the pull-backs $\rho_\epsilon^* f$ and $\rho_\epsilon^* f'$.

The following are equivalent:

(1) $\mu : A \to B$; $\mu f = 0 \implies \mu f' = 0$;

(2) $\mu_\epsilon : A_\epsilon \to B$; $\forall \epsilon \quad \mu_\epsilon \rho_\epsilon f = 0 \implies \mu_\epsilon \rho_\epsilon f' = 0$.

PROOF. [14] Given μ_ϵ, μ exists from the definition of direct limits. Let $a \in A$. Then there are an ϵ and an $a' \in A$ such that $\rho_\epsilon(a') = a$. If moreover $a = f(l)$, then $(l, a') \in \rho_\epsilon^* L$.

So $\mu f(a) = \mu \rho_\epsilon(\rho_\epsilon^* f)(l, a') = 0$. By hyphothesis this implies that $\mu f' = 0$. So $\mu_\epsilon(\rho_\epsilon^* f) = 0 \quad \forall \epsilon$.

Suppose μ is given such that $\mu f = 0$. Let $\mu_\epsilon := \mu \rho_\epsilon$. Then $\mu f = 0 \implies \mu_\epsilon(\rho_\epsilon^* f) = 0 \quad \forall \epsilon$. By hypothesis this implies $\mu_\epsilon(\rho_\epsilon^* f') = 0 \quad \forall \epsilon$. Let $l' \in L'$. $\exists a' \in A_\epsilon$, for some ϵ, such that $f'(l') = \rho_\epsilon(a')$. So $(l', a') \in \rho_\epsilon^* L'$. But then $\mu f'(l') = \mu \rho_\epsilon(\rho_\epsilon^* f')(l', a') = 0$; i. e. $\mu f' = 0$. $\square$

LEMMA 2.3.2. *Suppose $\omega \in L^2\left(U; \delta p(z) + 2m \log(1 + |z|^2)\right)$ and $|\psi| > Be^{\kappa' p}$ on U. Then*

$$\frac{\omega}{\psi} \in L^2\left(U; -\epsilon p(z) - 2m \log(1 + |z|^2)\right), \qquad \text{for } \epsilon < 2\kappa' - \delta.$$

PROOF.

$$\begin{aligned}
&\left|\frac{\omega}{\psi}\right|^2 e^{\epsilon p(z) + 2m \log\left(1 + |z|^2\right)} \\
&\qquad \le |\omega|^2 e^{-\delta p(z) - 2m \log(1+|z|^2)} \frac{e^{(\epsilon+\delta)p + 4m \log(1+|z|^2)}}{|\psi|^2} \\
&\qquad \le \frac{1}{B^2} |\omega|^2 e^{-\delta p(z) - 2m \log(1+|z|^2)} e^{(\epsilon+\delta-2\kappa')p(z) + 4m \log(1+|z|^2)}.
\end{aligned}$$

Now note that $e^{(\epsilon+\delta-2\kappa')p(z)+4m\log(1+|z|^2)} \in L^\infty$ when $\epsilon < 2\kappa' - \delta$, and $|\omega|^2 e^{-\delta p(z) - 2m \log(1+|z|^2)} \in L^1$. $\square$

[14] This should be true without assuming that the objects are sets.

LEMMA 2.3.3. *Suppose* $v \in L^2\left(U; -\delta p(z) - 2m\log(1+|z|^2)\right)$ *and* $|\psi| < Ae^{\kappa p}$ *on* U. *Then*

$$v\psi \in L^2\left(U; \epsilon p(z) + 2m\log(1+|z|^2)\right), \qquad \text{for } 2\kappa - \delta \leq \epsilon.$$

PROOF.

$$\begin{aligned} |v\psi|^2 e^{-\epsilon p - 2m\log(1+|z|^2)} &\leq |v|^2 e^{\delta p + 2m\log(1+|z|^2)} |\psi|^2 e^{-(\epsilon+\delta)p - 4m\log(1+|z|^2)} \\ &\leq A^2 |v|^2 e^{\delta p + 2m\log(1+|z|^2)} e^{(2\kappa-\epsilon-\delta)p - 4m\log(1+|z|^2)}. \end{aligned}$$

Note that $e^{(2\kappa-\epsilon-\delta)p-4m\log(1+|z|^2)} \in L^\infty$ when $2\kappa - \delta \leq \epsilon$. $\square$

REMARK 2.3.4. Note that ${}_p\mathcal{O}(K_0)$ injects into $\varinjlim_{\epsilon \searrow 0} L^2(L_\epsilon; -\epsilon p(z) - 2m\log(1+|z|^2))$. By lemma 2.2.3, the induced topology is the same as the original topology on ${}_p\mathcal{O}(K_0)$.

PROPOSITION 2.3.5 (KAWAI[15]). *Let* $U \subseteq \widehat{\mathbb{C}^n}$ *be* ${}^p\mathcal{O}$*-pseudoconvex with a* C^2 *strictly plurisubharmonic exhaustion function* θ. *Define*

$$L_c := \{\theta < c\}; \quad \text{and} \quad K_c := cl_{\widehat{\mathbb{C}^n}} L_c; \qquad c \in \mathbb{R}.$$

Suppose U *has a holomorphic function with controlled exponential type* (κ, p) *for some* $\kappa > 0$.

Then ${}_p\mathcal{O}(U) \longrightarrow {}_p\mathcal{O}(K_0)$ *has dense image in the topology induced by* $\varinjlim_{\epsilon \searrow 0} L^2\left(L_\epsilon; -\epsilon p(z) - 2m\log(1+|z|^2)\right)$.

PROOF. Note first that ${}_p\mathcal{O}(U)$ injects into ${}_p\mathcal{O}(K_0)$. Its image will again be denoted ${}_p\mathcal{O}(U)$. The Hahn-Banach theorem will be applied to show

$$\mu \in \left(\varinjlim_{\epsilon \searrow 0} L^2\left(L_\epsilon; -\epsilon p(z) - 2m\log(1+|z|^2)\right)\right)' \qquad \text{and} \qquad \mu\left({}_p\mathcal{O}(U)\right) = 0$$

$$\text{implies} \qquad \mu\left({}_p\mathcal{O}(K_0)\right) = 0.$$

[15] *cf.* Hörmander [1990] lemma 4.3.1.

By lemma 2.3.1 this is equivalent to showing

$$\mu \in L^2\left(L_\epsilon; -\epsilon p(z) - 2m\log(1+|z|^2)\right)' \qquad \text{and} \qquad \mu\left(\rho_\epsilon^{-1}{}_p\mathscr{O}(U)\right) = 0$$

$$\text{implies} \qquad \mu\left(\rho_\epsilon^{-1}{}_p\mathscr{O}(K_0)\right) = 0,$$

where

$$\rho_\epsilon : L^2(L_\epsilon; -\epsilon p(z) - 2m\log(1+|z|^2)) \longrightarrow \varinjlim_{\epsilon' \searrow 0} L^2(L'_\epsilon; -\epsilon' p(z) - 2m\log(1+|z|^2)).$$

By the Riesz representation theorem $\exists u \in L^2\left(L_\epsilon; -\epsilon p(z) - 2m\log(1+|z|^2)\right)$ such that

$$\mu(v) = \int_{K_\epsilon \cap \mathbb{C}^n} v\bar{u}e^{\epsilon p(z)+2m\log(1+|z|^2)}\,d\lambda, \qquad v \in L^2\left(L_\epsilon; -\epsilon p(z) - 2m\log(1+|z|^2)\right).$$

Extend u by 0 to $U \cap \mathbb{C}^n$. Then μ can be defined for $v \in L^2(U \cap \mathbb{C}^n; -\epsilon p(z) - 2m\log(1+|z|^2))$ by the same integral:

$$\mu(v) = \int_U v\bar{u}e^{\epsilon p(z)+2m\log(1+|z|^2)}\,d\lambda.$$

Let ψ_κ be the assumed holomorphic function of controlled exponential type κ, p. Then since $\bar{u}e^{\epsilon p(z)+2m\log(1+|z|^2)} \in L^2\left(U \cap \mathbb{C}^n; \epsilon p(z) + 2m\log(1+|z|^2)\right)$ lemma 2.3.2 gives

$$\text{(2-7)} \qquad \frac{\bar{u}}{\psi_\kappa}e^{\epsilon p(z)+2m\log(1+|z|^2)} \in L^2\left(U \cap \mathbb{C}^n; -\xi p(z) - 2m\log(1+|z|^2)\right), \qquad \text{for } \xi < 2\kappa' - \epsilon.$$

Assume $\epsilon < 2\kappa'$. Let $w \in L^2\left(U \cap \mathbb{C}^n; \xi p(z) + 2m\log(1+|z|^2)\right)$. Define

$$\tilde{\mu} \in L^2\left(U \cap \mathbb{C}^n; \xi p(z) + 2m\log(1+|z|^2)\right)'$$

by

$$\tilde{\mu}(w) := \int_{U \cap \mathbb{C}^n} w \frac{\bar{u}}{\psi_\kappa} e^{\epsilon p(z) + 2m\log(1+|z|^2)} d\lambda. \tag{2-8}$$

Let $\theta^+ := \max(0, \theta - \epsilon)$, θ being the exhaustion function of U. Let

$$\Lambda := \cup_{\lambda > 0} L^2\left(U \cap \mathbb{C}^n; \xi p(z) + 2m\log(1+|z|^2) + \lambda\theta^+\right).$$

Note that $\Lambda \subseteq L^2\left(U \cap \mathbb{C}^n; \xi p(z) - 2m\log(1+|z|^2)\right)$.

Claim: Let $L^2_{\text{loc}}\left(U \cap \mathbb{C}^n; -\epsilon p(z) - 2m\log(1+|z|^2)\right)$ denote the set of functions square integrable over compact subsets of $\widehat{\mathbb{C}^n}$ with respect to the given weight; i. e.

$$\begin{aligned}
&L^2_{\text{loc}}\left(U \cap \mathbb{C}^n; -\epsilon p(z) - 2m\log(1+|z|^2)\right) \\
&\qquad = \varprojlim_j L^2\left(L_c; -\epsilon p(z) - 2m\log(1+|z|^2)\right) \\
&\qquad = \left\{ f : \forall \text{ compact } K \subseteq U, \quad \int_{K \cap \mathbb{C}^n} |f|^2 e^{\epsilon p(z) + 2m\log(1+|z|^2)} d\lambda < \infty \right\}.
\end{aligned}$$

If $w \in \Lambda$ then $\frac{w}{\psi_\kappa} \in L^2_{\text{loc}}\left(U \cap \mathbb{C}^n; -\epsilon p(z) - 2m\log(1+|z|^2)\right)$.

Proof. Suppose $w \in L^2\left(U \cap \mathbb{C}^n; \xi p(z) + 2m\log(1+|z|^2) + \lambda\theta^+\right)$. Then

$$\begin{aligned}
&\left|\frac{w}{\psi_\kappa}\right|^2 e^{\delta p(z) + 2m\log(1+|z|^2) - \lambda\theta^+} \\
&\qquad = |w|^2 e^{-\xi p(z) - 2m\log(1+|z|^2) - \lambda\theta^+} \frac{e^{(\xi+\delta)p(z) + 4m\log(1+|z|^2) - \lambda\theta^+}}{|\psi_\kappa|^2} \\
&\qquad < B' e^{(-2\kappa' + \xi + \delta)p(z) + 4m\log(1+|z|^2)}.
\end{aligned}$$

Consequently $\frac{w}{\psi_\kappa} \in L^2\left(U \cap \mathbb{C}^n; -\delta p(z) - 2m\log(1+|z|^2) + \lambda\theta^+\right)$ when $\delta < 2\kappa' - \xi$. By (2-7) $\xi < 2\kappa' - \epsilon$ or $\epsilon < 2\kappa' - \xi$. So

$$\frac{w}{\psi_\kappa} \in L^2\left(U \cap \mathbb{C}^n; -\epsilon p(z) - 2m\log(1+|z|^2) + \lambda\theta^+\right).$$

Let $K \subseteq U$ be compact. By definition $\exists M < \infty$ such that $\sup_{K\cap\mathbb{C}^n} e^{\lambda\theta^+} < M$. But then

$$\begin{aligned}\infty &> \int_{K\cap\mathbb{C}^n} \left|\frac{w}{\psi_\kappa}\right|^2 e^{\epsilon p(z)+2m\log(1+|z|^2)} e^{-\lambda\theta^+} d\lambda \\ &\geq \frac{1}{M}\int_{K\cap\mathbb{C}^n} \left|\frac{w}{\psi_\kappa}\right|^2 e^{\epsilon p(z)+2m\log(1+|z|^2)} d\lambda.\end{aligned}$$

This proves the claim.

Now the hypothesis of Hörmander's proposition 2.3.2 [1965] are shown to hold with his ψ as the θ here, and his $\phi = \xi p(z) + 2m\log(1+|z|^2)$. Note that $\xi p(z) + 2m\log(1+|z|^2)$ is strictly plurisubharmonic, and

$$\frac{\bar{u}}{\psi_\kappa} e^{\epsilon p(z)+2m\log(1+|z|^2)} \in L^2\left(U \cap \mathbb{C}^n; -\xi p(z) - 2m\log(1+|z|^2)\right).$$

Suppose $w \in \Lambda$ and $\bar\partial w = 0$, so that w is analytic. Then $\bar\partial \frac{w}{\psi_\kappa} = 0$. Moreover $\tilde\mu(w) = \mu\left(\frac{w}{\psi_\kappa}\right)$, (2-8). By the claim above

$$\frac{w}{\psi_\kappa} \in L^2_{\text{loc}}\left(U \cap \mathbb{C}^n; -\epsilon p(z) - 2m\log(1+|z|^2)\right) \cap \mathcal{O}(U \cap \mathbb{C}^n).$$

From lemma 2.2.10

$$\begin{aligned}{}_p\mathcal{O}^2(U) :&= \varprojlim_{c\nearrow\infty} \varinjlim_{\epsilon'\searrow 0} \mathcal{O}\left(K_c \cap \mathbb{C}^n; -\epsilon p(z) - 2m\log(1+|z|^2)\right) \\ &= {}_p\mathcal{O}(U), \qquad \text{as sets,} \\ &=: \varprojlim_{c\nearrow\infty} \varinjlim_{\epsilon'\searrow 0} \mathcal{O}\left(K_c; -\epsilon p(z) - 2m\log(1+|z|^2)\right).\end{aligned}$$

Note that

$$\rho_\epsilon^{-1}\left({}_p\mathscr{O}^2(U)\right) = L^2_{\text{loc}}\left(U \cap \mathbb{C}^n; -\epsilon p(z) - 2m\log(1+|z|^2)\right) \cap \mathscr{O}(U \cap \mathbb{C}^n)|_{L_\epsilon}.$$

So $\frac{w}{\psi_\kappa} \in \rho_\epsilon^{-1}\left({}_p\mathscr{O}^2(U)\right)$. Thus $\tilde{\mu}(w) = \mu\left(\frac{w}{\psi_\kappa}\right) = 0$.

Hence proposition 2.3.2 of Hörmander [1965] shows that

$$\exists F \in L^2_{(0,1)}\left(U \cap \mathbb{C}^n; -\xi p(z) - 2m\log(1+|z|^2)\right)$$

such that

$$\Theta F = \frac{u}{\bar{\psi}_\kappa} e^{\epsilon p(z) + 2m\log(1+|z|^2)} \qquad \text{in the sense of distributions.}$$

Here

$$\Theta g = -\sum\nolimits_j \frac{\partial g_j}{\partial z^j}; \qquad g = \sum\nolimits_j g_j d\bar{z}^j.$$

Moreover F vanishes when $\theta > \epsilon$; i. e.$F = 0$ on $K_\epsilon^c \cap \mathbb{C}^n$.

Let

$$T : L^2_{(p,q)}\left(U; \xi p(z) + 2m\log(1+|z|^2)\right) \longrightarrow L^2_{(p,q)}\left(U; \xi p(z) + 2m\log(1+|z|^2)\right)$$

be the densely defined operator $T = \bar{\partial}$. According to proposition 2.2.1 of Hörmander [1965], $\mathscr{D}_{(p,q)}$ is dense in graph norm in $Dom(T)$. It follows that

$$\begin{aligned}\tilde{\mu}(w) &= \int_U w\overline{\Theta F} d\lambda \\ &= \int_U \sum\nolimits_j \frac{\partial w}{\partial \bar{z}^j} \bar{F}_j d\lambda, \qquad \text{for} \quad w \in Dom(T).\end{aligned}$$

In paticular this is true for $w \in \mathcal{O}\left(L_\epsilon; \xi p(z) + 2m\log(1+|z|^2)\right)$ since w can be extended by 0 to all of U, and since F and u both vanish outside L_ϵ. Such w are thus in $Dom(T)$. The formula above shows that

$$\tilde{\mu}\left(\mathcal{O}\left(L_\epsilon; \xi p(z) + 2m\log(1+|z|^2)\right)\right) = 0.$$

Recall that (2-8)

$$\tilde{\mu}(g) = \int_{L_\epsilon} \frac{g}{\psi_\kappa} \bar{u} e^{\epsilon p(z) + 2m\log(1+|z|^2)} = \mu\left(\frac{g}{\psi_\kappa}\right).$$

Hence μ vanishes on $\frac{1}{\psi_\kappa}\mathcal{O}\left(L_\epsilon; \xi p(z) + 2m\log(1+|z|^2)\right)$. By lemma 2.3.3

$$\psi_\kappa \mathcal{O}\left(L_\epsilon; -\delta p(z) - 2m\log(1+|z|^2)\right) \subseteq \mathcal{O}\left(L_\epsilon; \xi p(z) + 2m\log(1+|z|^2)\right), \qquad \text{for} \quad 2\kappa - \xi \le \delta.$$

Since $\epsilon < 2\kappa' - \xi \le 2\kappa - \xi =: \epsilon''$,

$$\mathcal{O}\left(L_\epsilon; -\epsilon'' p(z) - 2m\log(1+|z|^2)\right) \subseteq \frac{1}{\psi_\kappa}\mathcal{O}\left(L_\epsilon; \xi p(z) + 2m\log(1+|z|^2)\right).$$

Thus μ vanishes on $\mathcal{O}\left(L_\epsilon; -\epsilon'' p(z) - 2m\log(1+|z|^2)\right)$. By lemma 2.2.11 $\mathcal{O}\left(L_\epsilon; -\epsilon'' p(z) - 2m\log(1+|z|^2)\right)$ is dense in $\mathcal{O}\left(L_\epsilon; -\epsilon p(z) - 2m\log(1+|z|^2)\right)$. Thus μ vanishes on $\mathcal{O}\left(L_\epsilon; -\epsilon p(z) - 2m\log(1+|z|^2)\right)$. But note that

$$\rho_\epsilon^{-1}\left({}_p\mathcal{O}(K_0)\right) = \mathcal{O}\left(L_\epsilon; -\epsilon p(z) - 2m\log(1+|z|^2)\right).$$

So the proposition is proven. $\square$

Lemma 2.3.6. *Let I be a directed index set. Then $\varinjlim_j A_j = \varinjlim_k \varinjlim_{j>k} A_j$.*

Proof. (Here $j > k$ means $j \geq k$ and $j \neq k$. Suppose given $f_j : A_j \to B$. Consider the diagram

$$\begin{tikzcd}
A_j \arrow[rrrr] \arrow[dr] \arrow[dddrr] & & & & A_{j'} \arrow[dl] \arrow[dddll] \\
& \varinjlim_{l>j''} A_l \arrow[rr] \arrow[dr] \arrow[ddr] & & \varinjlim_{l>j'''} A_l \arrow[dl] \arrow[ddl] & \\
& & \varinjlim_k \varinjlim_{l>k} A_l \arrow[d, dotted] & & \\
& & B & &
\end{tikzcd}$$

The maps into $\varinjlim_{l>j''} A_l$ and $\varinjlim_{l>j'''} A_l$ are well defined, and a unique dashed arrow exists. $\square$

Remark 2.3.7. The lemma above is applied to the theorem below in step 2 with index set $I = \{K_V : K \subset\subset K_V \subset\subset V\}$. $\triangleright$

The lemma and proposition 2.3.5 give the following approximation theorem essentially due to Kawai[16].

Theorem 2.3.8 (Kawai). *Suppose K is a compact subset of an ${}^{p}\mathscr{O}$-pseudoconvex set $U \subseteq \widehat{\mathbb{C}^n}$. Suppose that for every $\widehat{\mathbb{C}^n}$ neighbourhood V of $K, V \subseteq U$, there is a C^2 strictly plurisubharmonic function θ_V, depending on V, such that*

(1) $\{\theta < c\} \subset\subset U, \qquad$ *for* $c \in \mathbb{R}$;

(2) $K \cap \mathbb{C}^n \subseteq \{\theta < 0\} \subseteq cl_{\widehat{\mathbb{C}^n}} \{\theta < 0\} =: K_0 \subseteq V$;

(3) $\sup_{K' \cap \widehat{\mathbb{C}^n}} \theta < \infty$ *for every compact subset* $K' \subseteq U$.

Moreover suppose that U has a function of controlled exponential type (κ, p) for some $\kappa > 0$. Then ${}_p\mathscr{O}(U) \to {}_p\mathscr{O}(K)$ has dense image.

[16] Kawai [1970]. *cf.* Hörmander [1990] theorem 4.3.2. Kawai states his result only for subsets in $\mathbb{D}^n$, eventhough it is applicable without this restriction.

PROOF.

$$\begin{aligned}
{}_p\mathscr{O}(K) &= \varinjlim_{U\supset\supset V\supseteq K} \varinjlim_{\delta} {}_p\mathscr{O}_{\mathrm{Bdd}}(V; -\delta p(z)) \\
&= \varinjlim_{K_V\supset K} \varinjlim_{W\supseteq K_V} \varinjlim_{\delta} {}_p\mathscr{O}_{\mathrm{Bdd}}(W; -\delta p(z)), \qquad (K_V := cl_{\widehat{\mathbb{C}^n}}\{\theta_V < 0\}), \\
&= \varinjlim_{K_V\supset K} {}_p\mathscr{O}(K_V).
\end{aligned}$$

By proposition 2.3.5, ${}_p\mathscr{O}(U)$ is dense in ${}_p\mathscr{O}(K_V)$. The proof of corollary 2.2.12 shows that ${}_p\mathscr{O}(U)$ is dense in $\varinjlim_V {}_p\mathscr{O}(K_V) = {}_p\mathscr{O}(K)$. $\square$

We shall Kawai's approximation theorem in the following form.

COROLLARY 2.3.9 (KAWAI). *Let U and K be as in the theorem, and let $K \subseteq K'$ be compact in U. Then ${}_p\mathscr{O}(K') \to {}_p\mathscr{O}(K)$ has dense image.*

PROOF. There is a commutative diagram

$$\begin{tikzcd}
{}_p\mathscr{O}(U) \arrow[r, "f_K"] \arrow[d, "f_{K'}"] & {}_p\mathscr{O}(K) \\
{}_p\mathscr{O}(K') \arrow[ur, "in"'] &
\end{tikzcd}$$

f_K has dense image and $\mathrm{im} f_K = \mathrm{im}\, in \circ f_K$. So im in is dense. $\square$

§2.4 A Vanishing Theorem

DEFINITION 2.4.1[17]. Let U be an open subset of $\widehat{\mathbb{C}^n}$.

$${}^p\mathcal{X}(U) := \left\{ f \in L^2_{\mathrm{loc}}(U) : \forall K \subset\subset U, \quad \forall\epsilon \quad \int_{K\cap\mathbb{C}^n} |f|^2 e^{-\epsilon p(z)} d\lambda(z) < \infty \right\}.$$

Let ${}^p\mathcal{X}_{(p,q)}(U)$ denote the corresponding (p,q) forms. $\Diamond$

[17] Berenstein & Struppa [preprint].

DEFINITION 2.4.2[18]. Let U be an open subset of $\widehat{\mathbb{C}^n}$.

$$_p\mathcal{X}(U) := \left\{ f \in L^2_{\text{loc}}(U) : \forall K \subset\subset U, \quad \exists \delta_K, \quad \int_{K \cap \mathbb{C}^n} |f|^2 e^{\delta_K p(z)} d\lambda(z) < \infty \right\}.$$

Let $_p\mathcal{X}_{(p,q)}(U)$ denote the corresponding (p,q) forms. $\Diamond$

Recall the following propositions

PROPOSITION 2.4.3[19]. *Suppose $U \subseteq \widehat{\mathbb{C}^n}$ is $^p\mathcal{O}$-pseudoconvex. Then the sequence*

$$^p\mathcal{X}_{(p,0)}(U) \xrightarrow{\bar\partial} {^p\mathcal{X}_{(p,1)}(U)} \xrightarrow{\bar\partial} {^p\mathcal{X}_{(p,2)}(U)} \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} {^p\mathcal{X}_{(p,n)}(U)} \xrightarrow{\bar\partial} 0$$

is exact. $\square$

PROPOSITION 2.4.4[20]. *Suppose $K \subseteq \Omega_\nu$ is compact and has a fundamental system of $^p\mathcal{O}$-pseudoconvex neighbourhoods. Moreover suppose that for every $\kappa > 0$ one of these neighbourhoods has a function of controlled exponential type (κ, p). Then the sequence*

$$_p\mathcal{X}_{(p,0)}(K) \xrightarrow{\bar\partial} {_p\mathcal{X}_{(p,1)}(K)} \xrightarrow{\bar\partial} {_p\mathcal{X}_{(p,2)}(K)} \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} {_p\mathcal{X}_{(p,n)}(K)} \xrightarrow{\bar\partial} 0$$

is exact.

PROOF[21]. Let $f \in {_p\mathcal{X}(K)}$ satisfy $\bar\partial f = 0$. Since K is compact and $\widehat{\mathbb{C}^n}$ is Hausdorff, $_p\mathcal{X}(K) = \varinjlim_{V \supseteq K} {_p\mathcal{X}(V)}$ where V may be assumed to be relatively compact $^p\mathcal{O}$-pseudoconvex neighbourhoods of K. The representative of f in $_p\mathcal{X}_{(p,q)}(V)$ for some V satisfies

$$\int_V |f|^2 e^{\delta p(z)} d\lambda < \infty.$$

[18] Berenstein & Struppa [preprint].

[19] Kawai [1970], Saburi [1978], Nagamachi [1981], Meril [1983], Berenstein & Struppa [preprint].

[20] Kawai [1970], Saburi [1978], Nagamachi [1981], Berenstein & Struppa [preprint].

[21] Saburi [1978].

By choosing κ sufficiently small, and restricting f to a smaller ${}^p\mathscr{O}$-pseudoconvex neighbourhood if necessary, we may suppose that $\psi_\kappa f \in {}^p\mathcal{X}_{(p,q)}(V)$, where ψ_κ is a function of controlled exponential type (κ, p). Since $\bar\partial(\psi_\kappa f) = 0$ there is a $g \in {}^p\mathcal{X}_{(p,q-1)}(V)$ such that $\bar\partial g = \psi_\kappa f$ by proposition 2.4.3 and lemma 2.3.2. Then $\bar\partial \frac{g}{\psi_\kappa} = f$ and $\frac{g}{\psi_\kappa} \in {}_p\mathcal{X}_{(p,q-1)}(V)$. $\square$

COROLLARY 2.4.5[22]. *There is an exact sequence*

$$0 \to {}_p\mathscr{O} \to {}_p\mathcal{X}_{(p,0)} \xrightarrow{\bar\partial} {}_p\mathcal{X}_{(p,1)} \xrightarrow{\bar\partial} \cdots {}_p\mathcal{X}_{(p,n)} \xrightarrow{\bar\partial} 0$$

PROOF. This follows from the assumption that points at infinity $(\Omega_\nu - \mathbb{C}^n)$ have a basis of neighbourhoods having functions of controlled exponential growth p for every κ. $\square$

COROLLARY 2.4.6. *Let K be a compact subset of $\widehat{\mathbb{C}^n}$ satisfying the conditions of proposition 2.4.4. Then*

$$H^k(K; {}_p\mathscr{O}) = 0, \qquad \textit{for } k > 0. \quad \square$$

Recall the following theorem from Kawai [1970], Nagamachi [1981], Berenstein & Struppa [preprint].

THEOREM 2.4.7. *Let $K \subseteq U \subseteq \widehat{\mathbb{C}^n}$, where K is compact and U is ${}^p\mathscr{O}$-pseudoconvex. Suppose $H^k(K; {}_p\mathscr{O}) = 0$ for $k > 0$. Then*

$$H^k_K(U; {}^p\mathscr{O}) = 0, \qquad \textit{for } k \neq n,$$

$$\textit{and} \qquad H^n_K(U; {}^p\mathscr{O}) \simeq {}_p\mathscr{O}(K)'. \quad \square$$

These results together yield the main theorem of the chapter.

[22] Kawai, Saburi, Nagamachi, Berenstein & Struppa.

THEOREM 2.4.8[23]. *Let $K \subseteq K' \subseteq \widehat{\mathbb{C}^n}$ be two compact subsets of $\widehat{\mathbb{C}^n}$ satisfying*

(1) *K' and K have fundamental systems of ${}^{p}\mathscr{O}$-pseudoconvex neighbourhoods;*

(2) *there is an open ${}^{p}\mathscr{O}$-pseudoconvex neighbourhood U of K' having a holomorphic function of controlled exponential type (κ, p) for any $\kappa > 0$;*

(3) *there is a function θ_V for every $\widehat{\mathbb{C}^n}$ neighbourhood V of K satisfying the conditions of theorem 2.3.8.*

Then $H^k_{K'-K}(\widehat{\mathbb{C}^n}; {}^{p}\mathscr{O}) = 0$ for $k \neq n$.

PROOF. Recall that $\Gamma_Z(X; \mathscr{F}) = \Gamma_Z(V; \mathscr{F})$, where Z is locally closed and V is an open set containing Z as a closed subset. Thus for the situation here $H^k_K(U; {}^{p}\mathscr{O}) = H^k_K(\widehat{\mathbb{C}^n}; {}^{p}\mathscr{O}) =$ and similarly for K'.

Now consider the long exact sequence

$$
\begin{aligned}
0 &\longrightarrow H^0_K(\widehat{\mathbb{C}^n}; {}^{p}\mathscr{O}) \longrightarrow H^0_{K'}(\widehat{\mathbb{C}^n}; {}^{p}\mathscr{O}) \longrightarrow H^0_{K'-K}(\widehat{\mathbb{C}^n}; {}^{p}\mathscr{O}) \\
&\longrightarrow H^1_K(\widehat{\mathbb{C}^n}; {}^{p}\mathscr{O}) \longrightarrow \cdots \\
&\longrightarrow H^{n-1}_K(\widehat{\mathbb{C}^n}; {}^{p}\mathscr{O}) \longrightarrow H^{n-1}_{K'}(\widehat{\mathbb{C}^n}; {}^{p}\mathscr{O}) \longrightarrow H^{n-1}_{K'-K}(\widehat{\mathbb{C}^n}; {}^{p}\mathscr{O}) \\
&\longrightarrow H^n_K(\widehat{\mathbb{C}^n}; {}^{p}\mathscr{O}) \longrightarrow H^n_{K'}(\widehat{\mathbb{C}^n}; {}^{p}\mathscr{O}) \longrightarrow H^n_{K'-K}(\widehat{\mathbb{C}^n}; {}^{p}\mathscr{O}) \\
&\longrightarrow H^{n+1}_K(\widehat{\mathbb{C}^n}; {}^{p}\mathscr{O}) \longrightarrow \cdots .
\end{aligned}
$$

Since $H^k_{K'}(\widehat{\mathbb{C}^n}; {}^{p}\mathscr{O}) = H^k_K(\widehat{\mathbb{C}^n}; {}^{p}\mathscr{O}) = 0$, for $k \neq n$ by corollary 2.4.6 and theorem 2.4.7,

$$H^k_{K'-K}(\widehat{\mathbb{C}^n}; {}^{p}\mathscr{O}) = 0, \qquad \text{for } k \neq n-1, n.$$

[23] *cf.* Kawai, Kashiwara & Kimura [1986] proposition 2.2.2.

For $k = n-1, n$ there is the exact sequence

$$\begin{array}{ccccccccc} 0 \to H^{n-1}_{K'-K}(\widehat{\mathbb{C}^n}; {}^p\mathscr{O}) \to & H^n_K(\widehat{\mathbb{C}^n}; {}^p\mathscr{O}) & \to & H^n_{K'}(\widehat{\mathbb{C}^n}; {}^p\mathscr{O}) & \to H^n_{K'-K}(\widehat{\mathbb{C}^n}; {}^p\mathscr{O}) \to 0 \\ & \| & & \| & \\ & {}_p\mathscr{O}(K)' & \longrightarrow & {}_p\mathscr{O}(K')' & \end{array}$$

By corollary 2.3.9 ${}_p\mathscr{O}(K)' \longrightarrow {}_p\mathscr{O}(K')'$ is injective. Hence $H^{n-1}_{K'-K}(\widehat{\mathbb{C}^n}; {}^p\mathscr{O}) = 0$; i. e.

$$H^k_{K'-K}(\widehat{\mathbb{C}^n}; {}^p\mathscr{O}) = 0, \qquad \text{for } k \neq n. \quad \square$$

REMARK 2.4.9. If K and K' are compact in $\mathbb{C}^n$ satisfying $K = \widehat{K}^P_U$, the plurisubharmonic hull of K, and $K' = \widehat{K'}^P_U$, then the conditions of the theorem are automatically satisfied by remarks 1.2.6 and 2.1.3 above, and theorem 2.6.11 in Hörmander [1990]. (See also scholium 4.3.1 below.) Thus the theorem generalizes proposition 2.2.2 of Kawai, Kashiwara & Kimura [1990], which states that $H^k_{K'-K}(\mathbb{C}^n; \mathscr{O}) = 0$ for $k \neq n$ when K and K' are compact analytic polyhedra. $\triangleright$

CHAPTER III

TOPOLOGICAL LEMMATA

> Who, if I cried out, would hear me among the angels'
> hierarchies? and even if one of them pressed me
> suddenly against his heart: I would be consumed
> in that overwhelming existence. For beauty is nothing
> but the beginning of terror, which we still are just able to endure,
> and we are so awed because it serenely disdains
> to annihilate us. Every angel is terrifying.
>
> —Rainer M. Rilke, *Duino Elegies* [1923].[1]

The purpose of this chapter is to show that the traces at infinity (definition 1.3.1) of certain neighbourhoods are well behaved. The method used is simply to look at the asymptotic expansions of the functions that define these neighbourhoods. These calculations are simple and terrifying, but, unfortunately, not beautiful.

§3.1 Exhaustion functions

The following functions will be crucial in this and the next chapter. While they play an important role, their importance is merely technical in that they serve only to make the machinery work.

CONVENTION 3.1.1. In this and the following chapter, sums over k run from $2, \dots, n$, while sums over j run from $1, \dots, n$ (n being as usual the n in $\mathbb{C}^n$). $\triangle$

[1] Translated by Stephen Mitchell.

DEFINITION 3.1.2[3].

(1) $\rho^\alpha(z) := \dfrac{\sum_k |z^k|^2 + y^{1,2}}{\chi(x^1 - 1/\alpha)} + \sum_j |y^j|^2 + \dfrac{1}{\chi(x^1 - 1/\alpha)}$, where

$$\chi(t) = \begin{cases} 0, & t \leq 0, \\ t^2, & t > 0. \end{cases}$$

For simplicity χ will not be explicitly written in most cases. Instead ρ^α shall be written as

$$\text{(1-1)} \qquad \rho^\alpha(z) := \frac{\sum_k |z^k|^2 + y^{1,2}}{(x^1 - 1/\alpha)^2} + \sum_j |y^j|^2 + \frac{1}{(x^1 - 1/\alpha)^2};$$

(2) $\Psi_a(z) := ia + \sum_j P^j z - \dfrac{\lambda i \sum_k z^{k,2} - \sum_k P^k z - (P^1 z - 1/\alpha)}{(z^1 - 1/\alpha)^2 + 2i(z^1 - 1/\alpha)} - \dfrac{i}{z^1 - 1/\alpha}$;

(3) $\psi_a(z) := \Im\mathrm{m}\Psi_a(z)$. $\Diamond$

NOTATION 3.1.3. To simplify notation let $x := x^1 - 1/\alpha$ when dealing with ρ^α, and $x := x^1 - 1/\epsilon$ when dealing with ρ^ϵ. No confusion should arise from this imprecision. Superscripts are used to denote coordinates, and this necessitated the more perverse notation $x^{,2}$ (etc.) for $(x^1 - 1/\alpha)^2$. $\triangle$

LEMMA 3.1.4.

$$\text{(1-2)} \quad \psi_a(z) := a - \frac{x}{x^{,2} + y^{1,2}} + \sum_j P^j y$$
$$- \frac{\left\{ \begin{array}{l} (\lambda \sum_k (x^{k,2} - y^{k,2}) - \sum_j P^j y)(x^{,2} - y^{1,2} - 2y^1) \\ \qquad + (2\lambda \sum_k x^k y^k + \sum_k P^k x + P x) 2x(y^1 + 1) \end{array} \right\}}{(x^{,2} - y^{1,2} - 2y^1)^2 + 4x^{,2}(y^1 + 1)^2}.$$

PROOF.

$$\Psi_a = ia + \sum_j (P^j x + iP^j y) - \frac{i}{x + iy^1} - \frac{\lambda i \sum_k z^{k,2} - \sum_k P^k z - (P^1 z - 1/\alpha)}{x^{,2} - y^{1,2} + 2ixy^1 + 2ix - 2y^1}$$

[3]The definition of ρ^α is essentially due to Nagamachi [1981]. The idea for the function Ψ comes from a similar function in Kawai Kashiwara, & Kimura [1986].

$$= ia + \sum_j (P^j x + iP^j y) - \frac{i(x - iy^1)}{x^{,2} + y^{1,2}}$$

$$- \frac{\left(\lambda i \sum_k z^{k,2} - \sum_k P^k z - (z^1 - 1/\alpha)\right)\left(x^{,2} - y^{1,2} - 2y^1 - 2xi(y^1 + 1)\right)}{(x^{,2} - y^{1,2} - 2y^1)^2 + 4x^{,2}(y^1 + 1)^2}$$

$$= ia + \sum_j (P^j x + iP^j y) - \frac{y^1 + ix}{x^{,2} + y^{1,2}}$$

$$- \frac{\left[\begin{array}{c} \Big(\lambda i \sum_k (x^{k,2} - y^{k,2} + 2ix^k y^k) - \sum_j (P^k x + iP^k y) - Px - iP^1 y\Big) \\ \times \left(x^{,2} - y^{1,2} - 2y^1 - 2ix(y^1 + 1)\right) \end{array}\right]}{(x^{,2} - y^{1,2} - 2y^1)^2 + 4x^{,2}(y^1 + 1)^2}$$

$$= ia + \sum_j (P^j x + iP^j y) - \frac{y^1 + ix}{x^{,2} + y^{1,2}}$$

$$- \frac{\left[\begin{array}{c} \left(\begin{array}{l} -Px - \sum_k P^k x - 2\lambda \sum_k x^k y^k \\ \qquad + i\left(\sum_k \left(\lambda x^{k,2} - \lambda y^{k,2} - P^k y\right) - P^1 y\right) \end{array}\right) \\ \times \left(x^{,2} - y^{1,2} - 2y^1 - 2ix(y^1 + 1)\right) \end{array}\right]}{(x^{,2} - y^{1,2} - 2y^1)^2 + 4x^{,2}(y^1 + 1)^2}.$$

Thus

$$\psi_a(z^1, \dots, z^n) = a + \sum_j P^j y - \frac{x}{x^{,2} + y^{1,2}}$$

$$- \frac{\left\{\begin{array}{l} \left(\lambda \sum_k (x^{k,2} - y^{k,2}) - \sum_k P^k y - P^1 y\right)(x^{,2} - y^{1,2} - 2y^1) \\ \qquad + (2\lambda \sum_k x^k y^k + \sum_k P^k x + Px) 2x(y^1 + 1) \end{array}\right\}}{(x^{,2} - y^{1,2} - 2y^1)^2 + 4x^{,2}(y^1 + 1)^2}. \quad \square$$

REMARK 3.1.5.

(1) $\psi_a > 0$ if and only if

$$\text{(1-3)} \qquad a > \frac{x}{x^{,2} + y^{1,2}} - \sum_j P^j y$$

$$+ \frac{\left\{\begin{array}{l} \left(\lambda \sum_k (x^{k,2} - y^{k,2}) - \sum_k P^k y - P^1 y\right)(x^{,2} - y^{1,2} - 2y^1) \\ \qquad + (2\lambda \sum_k x^k y^k + \sum_k P^k x + Px) 2x(y^1 + 1) \end{array}\right\}}{(x^{,2} - y^{1,2} - 2y^1)^2 + 4x^{,2}(y^1 + 1)^2};$$

(2) $\psi_a(z) = a + \psi_0(z)$;

(3) $\psi_{a+b}(z) = a + \psi_b(z)$. $\triangleright$

NOTATION 3.1.6. For the rest of this chapter let $x_0 := (1, 0, \ldots, 0) \in \mathbb{D}^n - \mathbb{R}^n$. $\triangle$

LEMMA 3.1.7 (NAGAMACHI).

(1) *ρ^α is C^∞ strictly plurisubharmonic where it is defined;*

(2) *Let $S_\epsilon := \{z \in \mathbb{C}^n : \rho^\epsilon(z) < \epsilon\}$, and let $\tilde{S}_\epsilon = int_{\widehat{\mathbb{C}^n}}\, cl_{\widehat{\mathbb{C}^n}}\, S_\epsilon$. Then $\{\tilde{S}_\epsilon\}_{0<\epsilon<\frac{1}{4}}$ is a fundamental system of neighbourhoods of $x_0 + i0$.*

(3) *$\tilde{S}_\epsilon$ is $\mathscr{PO}$-pseudoconvex, having*

$$q^\epsilon(z) := \frac{1}{\epsilon - \rho^\epsilon(z)}; \qquad int_{\widehat{\mathbb{C}^n}}\, cl_{\widehat{\mathbb{C}^n}}\{z : \rho^\epsilon(z) < \beta\}, \quad 0 < \beta < \epsilon$$

as exhaustion function and exhaustion sets.

PROOF.

(1) Since the last two terms of ρ^α, $\sum_j y^{j,2}$ and $1/x^{,2}$ are C^∞ plurisubharmonic where ever ρ^α is defined, it is sufficient to show likewise for the first term of ρ^α,

$$T_1 := \frac{\sum_k z^k \bar{z}^k + y^{1,2}}{x^{,2}} = \frac{\sum_k z^k \bar{z}^k + (\frac{z^1 - \bar{z}^k}{2i})^2}{(\frac{z^1 + \bar{z}^1}{2} - 1/\alpha)^2}.$$

Compute the Levi form:

$$\frac{\partial T_1}{\partial z^1} = \frac{\frac{1}{i}(\frac{z^1 - \bar{z}^1}{2i})}{(\frac{z^1 - \bar{z}^1}{2} - \frac{1}{\epsilon})^2} - \frac{\sum_k z^k \bar{z}^k + (\frac{z^1 - \bar{z}^1}{2i})^2}{\left(\frac{z^1 + \bar{z}^1}{2} - \epsilon^{-1}\right)^3}.$$

$$\frac{\partial^2 T_1}{\partial z^1 \partial \bar{z}^1} = \frac{1}{2x^{,2}} - \frac{y^1}{ix^{,3}} + \frac{y^1}{x^{,3}} + \frac{3}{2}\frac{\sum_k |z^k|^2 + y^{1,2}}{x^{,4}}$$

$$= \frac{1}{2x^{,2}} + \frac{3}{2}\frac{\sum_k |z^k|^2 + y^{1,2}}{x^{,4}}.$$

$$\frac{\partial^2 T_1}{\partial z^1 \partial \bar{z}^k} = -\frac{z^k}{x^{,3}}.$$

$$\frac{\partial^2 T_1}{\partial z^k \partial \bar{z}^k} = \frac{1}{x^{,2}}.$$

Thus the matrix of the Levi form is

$$\frac{1}{x^{,2}} \begin{pmatrix} \frac{1}{2} + \frac{3}{2}\frac{\sum_k |z^k|^2 + y^{1,2}}{x^{,2}} & -\frac{z^2}{x} & -\frac{z^3}{x} & \dots & -\frac{z^n}{x} \\ -\frac{\bar{z}^2}{x} & 1 & 0 & \dots & 0 \\ -\frac{\bar{z}^3}{x} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{\bar{z}^n}{x} & 0 & 0 & \dots & 1 \end{pmatrix}.$$

This is positive definite.

(2) Let

$$N' := \left\{ z \in \mathbb{C}^n : \frac{\sum_k x^{k,2}}{(x^1 - 3/\epsilon)^2} < \frac{\epsilon}{3}, \quad \sum_j |y^j|^2 < \frac{\epsilon}{4} \right\},$$

and let $N := int_{\widehat{\mathbb{C}^n}}\, cl_{\widehat{\mathbb{C}^n}}\, N' \subseteq \widehat{\mathbb{C}^n}$. Then N is a neighbourhood of $x_0\infty + i0$. Let $z \in N'$. Then $x^1 > \frac{3}{\epsilon}$ implies that

$$\frac{1}{(x^1 - \frac{1}{\epsilon})^2} < \frac{\epsilon^2}{4} < \frac{\epsilon}{4}$$

since $\epsilon < 1$. Thus

$$\frac{1}{(x^1 - \frac{1}{\epsilon})^2} + \sum_j |y^j|^2 + \frac{\sum_k (x^{k,2} + y^{k,2}) + y^{1,2}}{(x^1 - \frac{1}{\epsilon})^2}$$

$$< \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\sum_k x^{k,2}}{(x^1 - \frac{1}{\epsilon})^2} + \frac{\frac{\epsilon}{4}}{(x^1 - \frac{1}{\epsilon})^2}$$

$$< \frac{\epsilon}{2} + \frac{\sum_k x^{k,2}}{(x^1 - 3/\epsilon)^2} + \frac{\epsilon}{4}\frac{\epsilon}{4}$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{3} + \frac{\epsilon}{16}$$

$$< \epsilon.$$

So $z \in S_\epsilon$. Thus $N \subseteq \tilde{S}_\epsilon$.

Clearly given any "conic" basic neighbourhood of $x_0 + i0$, there is an ϵ such that $\tilde{S}_\epsilon$ is contained in that neighbourhood.

(3) $t \mapsto 1/(\epsilon - t)$ is a convex increasing function for $t < \epsilon$. Hence $q^\epsilon(z) := 1/(\epsilon - \rho^\epsilon(z))$ is C^∞ strictly plurisubharmonic. The corresponding exhaustion sets are $\{q^\epsilon < c\}$, for $\frac{1}{\epsilon} < c < \infty$, or since $q^\epsilon < c$ if and only if $\rho^\epsilon < \epsilon - \frac{1}{c}$, these sets can be rewritten as $\{\rho^\epsilon < \beta\}$, for $0 < \beta < \epsilon$. $\square$

§3.2 General Lemmas

LEMMA 3.2.1. *If A is open in a topological space X then $cl_X\, int_X\, cl_X A = cl_X A$.*

PROOF. $A \subseteq int_X\, cl_X A$ so $clA \subseteq cl_X\, int_X\, cl_X A$. If C is closed and $C \supseteq A$ then $C \supseteq int_X\, cl_X A$. So $C \supseteq cl_X\, int_X\, cl_X A$. Thus $clA \supseteq cl_X\, int_X\, cl_X A$. $\square$

LEMMA 3.2.2. *Let X be a topological space and let U be an open subset of X. For $A \subseteq X$, $(cl_X A) \cap U = cl_U(A \cap U)$.*

PROOF. $(cl_X A) \cap U$ is a closed subset of U containing $A \cap U$. So $(cl_X A) \cap U \supseteq cl_U(A \cap U)$. Now let $x \in (cl_X A) \cap U$. Then $x \in U$ and every neighbourhood N of x meets A. Since U is open, $N \cap U$ is a neighbourhood of x. So $N \cap A \cap U \neq \varnothing$. This implies that $(N \cap U) \cap (A \cap U) \neq \varnothing$. Thus $x \in cl_U(A \cap U)$. $\square$

LEMMA 3.2.3. *Let X, U, A be as in the previous lemma. Then $(int_X A) \cap U = int_U(A \cap U)$.*

PROOF. $(int_X A) \cap U$ is open in U and is contained in $A \cap U$. So $(int_X A) \cap U \subseteq int_U(A \cap U)$. On the other hand, $int_U(A \cap U)$ is open in X since U is open. Thus $(int_X A) \cap U \supseteq int_U(A \cup U)$. $\square$

LEMMA 3.2.4. *Let A be an open convex subset of a TVS X. Then $int_X\, cl_X A = A$.*

PROOF. Clearly $A \subseteq int_X\, cl_X A$. Suppose now that $z_0 \in int_X\, cl_X A$. By translation, z_0 can be taken to be 0. Since $0 \in int_X\, cl_X A$, there is a neighbourhood, N, of 0 such that $N \subseteq cl_X A$. By considering $N \cap -N$ we may suppose $N = -N$.

Since 0 is a limit point of A, and A is open, there exist a and an open set V such that $a \in V \subseteq A \cap N$. Then $-V \subseteq N$. Now $-V$ must contain a point of A, for otherwise, $-a \in N$ is not in the closure of A, contradicting $N \subseteq A$. So there exists $b \in V \cap N \subseteq A$ such that $-b \in A \cap N$. Since A is convex, $0 = \frac{1}{2}b + \frac{1}{2}(-b) \in A$. Thus $int_X\, cl_X A \subseteq A$. $\square$

LEMMA 3.2.5. *Let $A \subseteq \mathbb{C}^n$. Suppose that $int_{\mathbb{C}^n}\, cl_{\mathbb{C}^n} A = A$. Then $int_{\widehat{\mathbb{C}^n}}\, cl_{\widehat{\mathbb{C}^n}} A = A \cup tr_\infty A$.*

PROOF. Clearly $int_{\widehat{\mathbb{C}^n}}\, cl_{\widehat{\mathbb{C}^n}} A \supseteq A \cup \mathrm{tr}_\infty A$.

Suppose $z_0 \in int_{\widehat{\mathbb{C}^n}}\, cl_{\widehat{\mathbb{C}^n}} A$. If $z_0 \in \mathbb{C}^n$, then $z_0 \in (int_{\widehat{\mathbb{C}^n}}\, cl_{\widehat{\mathbb{C}^n}} A) \cap \mathbb{C}^n = int_{\mathbb{C}^n}\, cl_{\mathbb{C}^n} A = A$.

If $z_0 \in \widehat{\mathbb{C}^n} - \mathbb{C}^n$, then there is a neighbourhood Γ of z_0 such that $\Gamma \subseteq cl_{\widehat{\mathbb{C}^n}} A$. Hence $\Gamma \cap \mathbb{C}^n \subseteq cl_{\widehat{\mathbb{C}^n}} A \cap \mathbb{C}^n = cl_{\mathbb{C}^n} A$. Since $\Gamma \cap \mathbb{C}^n$ is open in $\mathbb{C}^n$, $\Gamma \cap \mathbb{C}^n \subseteq int_{\mathbb{C}^n}\, cl_{\mathbb{C}^n} A = A$. By definition, $z_0 \in \mathrm{tr}_\infty A$. $\square$

Conversely there is

LEMMA 3.2.6. *Suppose $A \subseteq \mathbb{C}^n$ is open. Then $int_{\widehat{\mathbb{C}^n}}\, cl_{\widehat{\mathbb{C}^n}} A = A \cup tr_\infty A$ implies $int_{\mathbb{C}^n}\, cl_{\mathbb{C}^n} A = A$.*

PROOF. $int_{\mathbb{C}^n}\, cl_{\mathbb{C}^n} A = (int_{\widehat{\mathbb{C}^n}}\, cl_{\widehat{\mathbb{C}^n}} A) \cap \mathbb{C}^n = (A \cup \mathrm{tr}_\infty A) \cap \mathbb{C}^n = A$. $\square$

COROLLARY 3.2.7. *If $A \subseteq \mathbb{C}^n$ is convex then $int_{\widehat{\mathbb{C}^n}}\, cl_{\widehat{\mathbb{C}^n}} A = A \cup tr_\infty A$.* $\square$

LEMMA 3.2.8.

(1) $int_{\mathbb{C}^n}\, cl_{\mathbb{C}^n}\{\rho^\alpha < c\} = \{\rho^\alpha < c\}, \quad (c > 0);$

(2) $int_{\mathbb{C}^n}\, cl_{\mathbb{C}^n}\{\psi_a > 0\} = \{\psi_a > 0\}.$

PROOF.

(1) Since $\{\rho^\alpha < c\}$ is open, $int_{\mathbb{C}^n}\, cl_{\mathbb{C}^n}\{\rho^\alpha < c\} \supseteq \{\rho^\alpha < c\}$. Now let $z = (x^1, y^1, \ldots, x^n, y^n) \in int_{\mathbb{C}^n}\, cl_{\mathbb{C}^n}\{\rho^\alpha < c\} \subseteq int_{\mathbb{C}^n}\{\rho^\alpha \leq c\}$. Since a neighbourhood of z in $\mathbb{C}^n$ must project to a neighbourhood of $(y^1, \ldots, y^n)$, and since z is an interior point of $\{\rho^\alpha \leq c\}$ we cannot have

$$\rho^\alpha(z) = \frac{\sum_k x^{k,2} + \sum_j y^{j,2} + 1}{x^{,2}} + \sum_j y^{j,2} = c,$$

because increasing the values of y^j's will increase the value of ρ^α. Thus $z \in \{\rho^\alpha < c\}$.

(2) As above $int_{\mathbb{C}^n}\, cl_{\mathbb{C}^n}\{\psi_a > 0\} \supseteq \{\psi_a > 0\}$, and $int_{\mathbb{C}^n}\, cl_{\mathbb{C}^n}\{\psi_a > 0\} \subseteq int_{\mathbb{C}^n}\{\psi_a > 0\}$. Let $B \supseteq int_{\mathbb{C}^n}\, cl_{\mathbb{C}^n}\{\psi_a > 0\}$ be a neighbourhood of z and suppose that $\psi_a(z) = 0$. Since ψ_a is harmonic and $\psi_a(z) \geq 0$ for $z \in B$, the minimum principle implies that $\psi_a \equiv 0$ on B. But ψ_a is real analytic when $x > 0 (x = x^1 - 1/\alpha)$, so $\psi_a \equiv 0$. This is clearly impossible. Thus $\psi_a(z) > 0$; i. e. $int_{\mathbb{C}^n}\, cl_{\mathbb{C}^n}\{\psi_a > 0\} \subseteq \{\psi_a > 0\}$. $\square$

COROLLARY 3.2.9.

(1) $int_{\widehat{\mathbb{C}^n}}\, cl_{\widehat{\mathbb{C}^n}}\{\rho^\alpha < c\} = \{\rho^\alpha < c\} \cup tr_\infty\{\rho^\alpha < c\};$

(2) $int_{\widehat{\mathbb{C}^n}}\, cl_{\widehat{\mathbb{C}^n}}\{\psi_a > 0\} = \{\psi_a > 0\} \cup tr_\infty\{\psi_a > 0\}.$ $\square$

LEMMA 3.2.10.

(1) $cl_{\mathbb{C}^n}\{\rho^\alpha < c\} = \{\rho^\alpha \leq c\};$

(2) $cl_{\mathbb{C}^n}\{\psi_a > 0\} = \{\psi_a \geq 0\}, \qquad$ *for a outside a set of measure 0;*

(3) $int_{\mathbb{C}^n}\{\rho^\alpha \le c\} = \{\rho^\alpha < c\}$;

(4) $int_{\mathbb{C}^n}\{\psi_a \ge 0\} = \{\psi_a > 0\}$, *for a outside a set of measure* 0.

PROOF. (1) Suppose $z = (x^1 + iy^1, \ldots, x^n + iy^n)$ satisfies $\rho^\alpha(z) = c$. Then $z_t := (x^1 + ity^1, \ldots, x^n + ity^n)$ $\;0 \le t < 1$ satisfies

$$\rho^\alpha(z_t) = \frac{\sum_k x^{k,2} + t^2 \sum_j y^{j,2} + 1}{x^{,2}} + t^2 \sum_j y^{j,2} < c,$$

and $z_t \to z$ as $t \uparrow 1$.

(2) This is a consequence of Sard's theorem. Recall (remark 3.1.5) that $\psi_a = 0$ if and only if $\psi_0 = -a$. Since ψ_0 is C^∞ when $x > 0$, $\{\psi_0 = -a\}$ is a C^∞ hypersurface in $\mathbb{R}^{2n}$ when a is outside a set of measure 0. Suppose $z \in \mathbb{C}^n$ satisfy $\psi_a(z) = 0$. Since $\{\psi_a = 0\}$ is a (smooth) submanifold of $\mathbb{R}^{2n}$, there is a sequence $z_m \in \{\psi_a > 0\}$ that tends to $z_)0$. (Take for instance z_m to be a sequence along the normal.) So $z \in cl_{\mathbb{C}^n}\{\psi_a > 0\}$.

(3) is a corollary of (1) and lemma 3.2.8.

(4) is a corollary of (2) and lemma 3.2.8. □

§3.3 Lemmas on Traces

LEMMA 3.3.1. $tr_\infty\{\rho^\alpha < c\} = \bigcup_{0<d<c}\{\rho^\alpha < d\}$.

PROOF. It is clear that $\mathrm{tr}_\infty\{\rho^\alpha < c\} \supseteq \bigcup_{0<d<c}\{\rho^\alpha < d\}$, so it remains to show $\mathrm{tr}_\infty\{\rho^\alpha < c\} \subseteq \bigcup_{0<d<c}\{\rho^\alpha < d\}$.

To this end, let $z_* = x_*\infty + iy_* \in \mathrm{tr}_\infty\{\rho^\alpha < c\}$, where as usual $x_* \in \mathbb{S}_{n-1}$. Define $z_t := tx_* + iy_*$. By the definition of neighbourhoods at ∞, $\exists T > 0$ such that $t \ge T \implies z_t \in \{\rho^\alpha < c\}$.

Let

$$\tilde{\rho}(t) := \rho^\alpha(z_t) = \frac{t^2 \sum_k x_*^{k,2} + \sum_j y_*^{j,2} + 1}{(tx_*^1 - 1/\alpha)^2} + \sum_j y_*^{j,2}.$$

Then

$$\frac{d\tilde{\rho}}{dt}(t) = \frac{2t\sum_k x_*^{k,2}}{(tx_*^1 - 1/\alpha)^2} - 2x_*^1 \frac{t^2\sum_k x_*^{k,2} + \sum_j y_*^{j,2} + 1}{(tx_*^1 - 1/\alpha)^3}$$

$$= \frac{2t^2 x_*^1 \sum_k x_*^{k,2} - \frac{2}{\alpha} t \sum_k x_*^{k,2} - 2t^2 x_*^1 \sum_k x_*^{k,2} - 2x_*^1 \sum_j y_*^{j,2} - 2x_*^1}{(tx_*^1 - 1/\alpha)^3}$$

$$= -\frac{\frac{2}{\alpha} t \sum_k x_*^{k,2} + 2x_*^1 (\sum_j y_*^{j,2} + 1)}{(tx_*^1 - 1/\alpha)^3}$$

$$< 0, \qquad \text{since } x_*^1 > 0. \quad \square$$

So $\rho^\alpha(z_t) < \rho(z_T) < c$ for $t > T$. Thus $z_* \in cl_{\widehat{\mathbb{C}^n}}\{\rho^\alpha < \rho^\alpha(z_T) =: c'\}$.

Next it is shown that $z_* \in \mathrm{tr}_\infty\{\rho^\alpha < c'\}$.

Let $I :=]-1, 1[$. Let N_ϵ be the basic neighbourhood of z_* defined by

$$N_\epsilon :=$$

$$\{(Tx_*/\epsilon + sx') + iy' \in \mathbb{C}^n : s > 0, x' \in B_{\mathbb{R}^n}(x_*, \epsilon) \cap \mathbb{S}_{n-1}, y' \in y_* + \epsilon I^n\}$$
$$\bigcup \{x'\infty + iy' : x' \in B_{\mathbb{R}^n}(x_*, \epsilon) \cap \mathbb{S}_{n-1}, y' \in y_* + \epsilon I^n\}.$$

Claim: For sufficiently small ϵ, $N_\epsilon \cap \mathbb{C}^n \subseteq \{\rho^\alpha < c'\}$.

Proof. A sketch of the proof is given. Let $z'_s := Tx_0/\epsilon + sx' + iy' \in N_\epsilon \cap \mathbb{C}^n$. By drawing a picture, it is seen that $z'_s \in \{\rho^\alpha < c'\}$ for small ϵ: let $z'' = (x^{1,''} + iy^{1,''}, \ldots, x^{n,''} + iy^{n,''})$. Then

$$\text{(3-1)} \qquad \rho^\alpha(z'') < c' \iff 1 < \frac{(x^{1,''} - 1/\alpha)^2}{\left(\sqrt{\frac{1+\sum_j y^{j,'',2}}{c' - \sum_j y^{j,'',2}}}\right)^2} - \frac{\sum_k x^{k,'',2}}{\left(\sqrt{1+\sum_j y^{j,'',2}}\right)^2}.$$

Let $z'' = z_{T/\epsilon} = Tx_*/\epsilon + iy_*$. Then the inequality (3-1) will remain true for $y \in \epsilon I^n$ for all s when ϵ is small. Hence $z'_s \in \{\rho^\alpha < c'\}$. This proves the claim and the lemma. $\square$

LEMMA 3.3.2. *Let $V \subset\subset U \subseteq \widehat{\mathbb{C}^n}$. Suppose $U \subseteq int_{\widehat{\mathbb{C}^n}}\, cl_{\widehat{\mathbb{C}^n}}\{\rho^\alpha < c\}$. Then $\exists c', 0 < c' < c$ such that $V \subseteq int_{\widehat{\mathbb{C}^n}}\, cl_{\widehat{\mathbb{C}^n}}\{\rho^\alpha < c'\}$.*

PROOF.

$$\begin{aligned} int_{\widehat{\mathbb{C}^n}}\, cl_{\widehat{\mathbb{C}^n}}\{\rho^\alpha < c\} &= \{\rho^\alpha < c\} \cup \mathrm{tr}_\infty\{\rho^\alpha < c\} \\ &= \bigcup_{0<c<d} \{\rho^\alpha < d\} \cup \bigcup_{0<d<c} \mathrm{tr}_\infty\{\rho^\alpha < d\}, \quad \text{by lemma 3.3.1,} \\ &= \bigcup_{0<d<c} int_{\widehat{\mathbb{C}^n}}\, cl_{\widehat{\mathbb{C}^n}}\{\rho^\alpha < d\}. \end{aligned}$$

Since $cl_{\widehat{\mathbb{C}^n}} V$ is compact and $cl_{\widehat{\mathbb{C}^n}} V \subseteq \bigcup_{0<d<c} int_{\widehat{\mathbb{C}^n}}\, cl_{\widehat{\mathbb{C}^n}}\{\rho^\alpha < d\}$,

$$cl_{\widehat{\mathbb{C}^n}} V \subseteq int_{\widehat{\mathbb{C}^n}}\, cl_{\widehat{\mathbb{C}^n}}\{\rho^\alpha < c'\}, \qquad \text{for some } 0 < c' < c. \quad \square$$

LEMMA 3.3.3. *For $0 < c' < c$, $\quad int_{\widehat{\mathbb{C}^n}}\, cl_{\widehat{\mathbb{C}^n}}\{\rho^\alpha < c'\} \subset\subset int_{\widehat{\mathbb{C}^n}}\, cl_{\widehat{\mathbb{C}^n}}\{\rho^\alpha < c\}$.*

PROOF. A sketch of the proof is provided. It is sufficient (in fact equivalent) to prove that $\mathrm{tr}_\infty\{\rho^\alpha < c'\} \subset\subset \mathrm{tr}_\infty\{\rho^\alpha < c\}$ because it will then follow that

$$\begin{aligned} cl_{\widehat{\mathbb{C}^n}}\, int_{\widehat{\mathbb{C}^n}}\, cl_{\widehat{\mathbb{C}^n}}\{\rho^\alpha < c'\} &= cl_{\widehat{\mathbb{C}^n}}\{\rho^\alpha < c'\} \\ &= cl_{\widehat{\mathbb{C}^n}}\{\rho^\alpha < c'\} \cup cl_{\widehat{\mathbb{C}^n}}\mathrm{tr}_\infty\{\rho^\alpha < c'\} \\ &\subseteq \{\rho^\alpha < c\} \cup \mathrm{tr}_\infty\{\rho^\alpha < c\} \\ &= int_{\widehat{\mathbb{C}^n}}\, cl_{\widehat{\mathbb{C}^n}}\{\rho^\alpha < c\}, \end{aligned}$$

and moreover $cl_{\widehat{\mathbb{C}^n}}\, int_{\widehat{\mathbb{C}^n}}\{\rho^\alpha < c'\}$, having bounded imaginary parts, is compact.

But $\mathrm{tr}_\infty\{\rho^\alpha < c'\} \subset\subset \mathrm{tr}_\infty\{\rho^\alpha < c\}$ is clear from a picture. In fact let $z_* = x_*\infty + iy_*$, where $x_* = (x_*^1, x_*^k)$ and $y_* = (y_*^1, y_*^k)$. Then

$$\sqrt{\textstyle\sum_k x_*^{k,2}} \le \sqrt{c' - \textstyle\sum_j y_*^{j,2}}(x_*^1 - 1/\alpha) < \sqrt{c - \textstyle\sum_j y_*^{j,2}}(x_*^1 - 1/\alpha).$$

So for sufficiently large $s \in \mathbb{R}^+$, $sx_* + iy_* \in int_{\mathbb{C}^n}\{\rho^\alpha < c\}$. Hence $x_*\infty + iy_* \in \mathrm{tr}_\infty\{\rho^\alpha < c\}$. $\square$

LEMMA 3.3.4. *$tr_\infty\{\psi_a > c\} = \bigcup_{\infty>d>c} tr_\infty\{\psi_a > d\}$.*

PROOF. Clearly $\mathrm{tr}_\infty\{\psi_a > c\} \supseteq \bigcup_{\infty>d>c} \mathrm{tr}_\infty\{\psi_a > d\}$. So we show

$$\mathrm{tr}_\infty\{\psi_a > c\} \subseteq \bigcup_{\infty>d>c} \mathrm{tr}_\infty\{\psi_a > d\}.$$

Let $z_* := x_* + iy_* \in \mathrm{tr}_\infty\{\psi_a > c\}$ where $x_* \in \mathbb{S}_{n-1} = \mathbb{D}^n - \mathbb{R}^n$. Define $z_t := tx_* + iy_*$ for $t > 0$, and let N_ϵ be the basis of neighbourhoods of z_* defined by

$$N_\epsilon := \{x_*/\epsilon + sx + iy_* + i\epsilon^2 y \in \mathbb{C}^n : s > 0, x \in B_{\mathbb{R}^n}(x_*, \epsilon^2) \cap \mathbb{S}_{n-1}, y \in I^n\}$$
$$\bigcup \{x\infty + iy_* + \epsilon^2 iy : x \in B_{\mathbb{R}^n}(x_*, \epsilon^2) \cup \mathbb{S}_{n-1}, y \in I^n\},$$

where $I =]-1, 1[$. By definition of tr_∞, $\exists \epsilon_0 > 0$ such that $0 < \epsilon < \epsilon_0 \implies N_\epsilon \cup \mathbb{C}^n \subseteq \{\psi_a > c\}$.

Let $\hat{\psi}_\epsilon(s) := \psi_a(z_{\epsilon,s})$, where $z_{\epsilon,s} := x_*/\epsilon + sx + iy_* + i\epsilon^2 y \in N_\epsilon \mathbb{C}^n$. This of course depends on x and y. Explicitly

$$\hat{\psi}_\epsilon(s) = a - \frac{x_*^1/\epsilon + sx^1 - 1/\alpha}{(x_*^1/\epsilon + sx^1 - 1/\alpha)^2 + (y_*^1 + \epsilon^2 y^1)^2} + \sum_j (P^j y_* + \epsilon^2 P^j y) \tag{3-2}$$
$$- \frac{\left\{ \begin{array}{l} \left[\begin{array}{r} \left(\lambda \sum_k (x_*^k/\epsilon + sx^k)^2 - \lambda \sum_k (y_*^k + \epsilon^2 y^k)^2 - \sum_j (P^j y_* + \epsilon^2 P^j y)\right) \\ \times \left((x_*^1/\epsilon + sx^1 - 1/\alpha)^2 - (y_*^1 + \epsilon^2 y^1)^2 - 2(y_*^1 + \epsilon^2 y^1)\right) \end{array} \right] \\ + \\ \left[\begin{array}{r} 2 \left(\begin{array}{l} 2\lambda \sum_k (x_*^k/\epsilon + sx^k)(y_*^k + \epsilon^2 y^k) + \sum_k (P^k x_*/\epsilon + sP^k x) \\ \qquad + P^1 x_*/\epsilon + sP^1 x - 1/\alpha \end{array} \right) \\ \times (x_*^1/\epsilon + sx^1 - 1/\alpha)(y_*^1 + \epsilon^2 y^1 + 1) \end{array} \right] \end{array} \right\}}{\left[\begin{array}{r} \left((x_*^1/\epsilon + sx^1 - 1/\alpha)^2 - (y_*^1 + \epsilon^2 y^1)^2 - 2(y_*^1 + \epsilon^2 y^1)\right)^2 \\ + 4(x_*^1/\epsilon + sx^1 - 1/\alpha)^2 (y_*^1 + \epsilon^2 y^1 + 1)^2 \end{array} \right]}.$$

Now examine the asymptotics of this function when $0 < \epsilon \ll \epsilon_0$.

Assume first that $x_*^1 \neq 0$. Consider each of the terms above separately. For convenience set $X^j = x_*^j + s\epsilon x^j$, for $j = 1, \ldots, n$.

(1) 2nd term of (3-2):

$$
\begin{aligned}
-\frac{x_*^1/\epsilon + sx^1 - 1/\alpha}{(x_*^1/\epsilon + sx^1 - 1/\alpha)^2 + (y_*^1 + \epsilon^2 y^1)^2} &= -\frac{\epsilon(x_*^1 + s\epsilon x^1 - \epsilon/\alpha)}{(X^1 - \epsilon/\alpha)^2 + \epsilon^2(y_*^1 + \epsilon^2 y^1)^2} \\
&= -\frac{\epsilon(X^1 - \epsilon/\alpha)}{(X^1 - \epsilon/\alpha)^2 \left[1 + \epsilon^2 \frac{(y_*^1 + \epsilon^2 y^1)^2}{(X^1 - \epsilon/\alpha)^2}\right]} \\
&= -\frac{\epsilon}{X^1 - \epsilon/\alpha} + o(\epsilon^2).
\end{aligned}
$$

(2) 4th term numerator of (3-2):

$$
\frac{1}{\epsilon^4}\left\{
\begin{array}{l}
\left[\begin{array}{r}
\left(\lambda \sum_k X^{k,2} - \lambda\epsilon^2 \sum_k (y_*^k + \epsilon^2 y^k)^2 - \epsilon^2 \sum_j (P^j y_* + \epsilon^2 P^j y)\right) \\
\times \left((X^1 - \epsilon/\alpha)^2 - \epsilon^2 (y_*^1 + \epsilon^2 y^1)^2 - 2\epsilon^2 (y_*^1 + \epsilon^2 y^1)\right)
\end{array}\right] \\
+ \\
\left[\begin{array}{r}
2\epsilon^2 \left(2\lambda \sum_k X^k (y_*^k + \epsilon^2 y^k) + \sum_k P^k X + P^1 X - \epsilon/\alpha\right) \\
\times (X^1 - \epsilon/\alpha)(y_*^1 + \epsilon^2 y^1 + 1)
\end{array}\right]
\end{array}
\right\}
$$

$$
= \frac{1}{\epsilon^4}\left\{
\begin{array}{l}
\left(\lambda \sum_k X^{k,2} - \lambda\epsilon^2 \sum_k y_*^{k,2} - \epsilon^2 \sum_j P^j y_*\right)\left((X^1 - \epsilon/\alpha)^2 - \epsilon^2 y_*^{1,2} - 2\epsilon^2 y_*^1\right) \\
+ 2\epsilon^2 \left(2\lambda \sum_k X^k y_*^k + \sum_k P^k X + P^1 X - \epsilon/\alpha\right)(X^1 - \epsilon/\alpha)(y_*^1 + 1) \\
+ o(\epsilon^2)
\end{array}
\right\}
$$

$$
= \frac{1}{\epsilon^4}\left\{
\begin{array}{l}
\lambda \sum_k X^{k,2} (X^1 - \epsilon/\alpha)^2 - \lambda\epsilon^2 \sum_k y_*^{k,2} (X^1 - \epsilon/\alpha)^2 \\
\quad - \epsilon^2 \sum_j P^j y_* (X^1 - \epsilon/\alpha)^2 - \lambda\epsilon^2 y_*^{1,2} \sum_k X^{k,2} - 2\lambda\epsilon^2 y_*^1 \sum_k X^{k,2} \\
+ \quad 2\epsilon^2 (2\lambda \sum_k X^k y_*^k + \sum_k P^k X + P^1 X - \epsilon/\alpha)(X^1 - \epsilon/\alpha)(y_*^1 + 1) \\
\\
+ o(\epsilon^2)
\end{array}
\right\}.
$$

(3) 4th term denominator of (3-2):

$$
\frac{1}{\epsilon^4}\left[\begin{array}{r}
\left((X^1 - \epsilon/\alpha)^2 - \epsilon^2 (y_*^1 + \epsilon^2 y^1)^2 - 2\epsilon^2 (y_*^1 + \epsilon^2 y^1)\right)^2 \\
-4\epsilon^2 (X^1 - \epsilon/\alpha)^2 (y_*^1 + \epsilon^2 y^1 + 1)^2
\end{array}\right]
$$

$$= \frac{1}{\epsilon^4}\left[\begin{aligned}(X^1-\epsilon/\alpha)^4 - 2\epsilon^2(X^1-\epsilon/\alpha)^2 y_*^{1,2} - 4\epsilon^2(X^1-\epsilon/\alpha)^2 y_*^1 \\ -4\epsilon^2(X^1-\epsilon/\alpha)^2(y_*^1+1)^2 + o(\epsilon^2)\end{aligned}\right]$$

$$= \frac{1}{\epsilon^4}(X^1-\epsilon/\alpha)^4\left(1-\epsilon^2\left(\frac{2y_*^{1,2}+4y_*^1+4(y_*^1+1)^2}{(X^1-\epsilon/\alpha)^2}\right)+o(\epsilon^2)\right).$$

(4) 4th term of (3-2): Putting the numerator and denominator calculated above gives

$$-\frac{\lambda\sum_k X^{k,2}(X^1-\epsilon/\alpha)^2+o(\epsilon)}{(X^1-\epsilon/\alpha)^4}\left(1+\epsilon^2\frac{2y_*^{1,2}+4y_*^1+4(y_*^1+1)^2}{(X^1-\epsilon/\alpha)^2}+o(\epsilon^2)\right)$$

$$= -\frac{\lambda\sum_k X^{k,2}}{(X^1-\epsilon/\alpha)^2}+o(\epsilon).$$

(5) Hence

(3-3)

$$\hat{\psi}_\epsilon(s) = a - \frac{\epsilon}{(X^1-\epsilon/\alpha)} + \textstyle\sum_j y_*^j - \frac{\lambda\sum_k X^{k,2}}{(X^1-\epsilon/\alpha)^2} + o(\epsilon) > c, \qquad \text{for small} \quad \epsilon.$$

If ϵ (small) is decreased, $\hat{\psi}_\epsilon$ will decrease because of the second term. Now s occurs, if at all, only in the denominators of each term, including the $o(\epsilon)$–term. Moreover as s increases, the second term decreases in size, so that $\hat{\psi}_\epsilon$ has its minimum at finite s. Thus one sees that reducing ϵ to say ϵ' provides the estimate

$$\inf_{s,x\in B_{\mathbb{R}^n}(x_*;\epsilon^2)} \hat{\psi}_{\epsilon'}(s) > c.$$

Let $d = \inf_{s,x\in B_{\mathbb{R}^n}(x_*;\epsilon^2)} \hat{\psi}_{\epsilon'}(s)$, then $N_{\epsilon'} \cap \mathbb{C}^n \subseteq \mathrm{tr}_\infty\{\psi_a < d\}$; i. e. $z_* \in \mathrm{tr}_\infty\{\psi_a < d\}$.

(6) Consider now the case $x_*^1 = 0$. As before let $z_{\epsilon,s} := x_*/\epsilon + sx + iy_* + i\epsilon^2 y \in N_\epsilon \cap \mathbb{C}^n$. For notational simplicity, let $\tilde{y} = y_* + \epsilon^2 y$. When $x_*^1 = 0$, (3-2) reduces to

(3-4)

$$\hat{\psi}_\epsilon(s) = a - \frac{sx^1 - 1/\alpha}{(sx^1-1/\alpha)^2 + \tilde{y}^{1,2}} + \textstyle\sum_j P^j \tilde{y}$$

$$
-\frac{\left\{\begin{array}{l}\left[\begin{array}{r}\left(\lambda\sum_k(x_*^k/\epsilon+sx^k)^2-\lambda\sum_k\tilde{y}^{k,2}-\sum_j P^j\tilde{y}\right)\\ \times\left((sx^1-1/\alpha)^2-\tilde{y}^{1,2}-2\tilde{y}^1\right)\end{array}\right]\\ +2\left[\begin{array}{r}\left(\begin{array}{r}2\lambda\sum_k(x_*^k/\epsilon+sx^k)\tilde{y}^k+\sum_k(P^kx_*/\epsilon+sP^kx)\\ +P^1x_*/\epsilon+sP^1x-1/\alpha\end{array}\right)\\ \times(sx^1-1/\alpha)(\tilde{y}^1+1)\end{array}\right]\end{array}\right\}}{\left((sx^1-1/\alpha)^2-\tilde{y}^{1,2}-2\tilde{y}^1\right)^2+4(sx^1-1/\alpha)^2(\tilde{y}^1+1)^2}.
$$

By assumption $\exists\epsilon_0$ such that $0<\epsilon<\epsilon_0 \implies N_\epsilon\cap\mathbb{C}^n\subseteq\{\psi_a>c\}$; i. e.

$$
\hat{\psi}_\epsilon(s)>c \qquad \text{for} \quad z_{\epsilon,s}\in N_\epsilon\cap\mathbb{C}^n.
$$

This inequality must remain true for $x^1=0$. In this case only the 4th term depends on s. So consider its behaviour when s is large. As the denominator will in this case be independent of s, only the numerator will be significant.

(6i) 4th term numerator of (3-4):

$$
\begin{aligned}
&-\left(\lambda\sum_k(x_*^k/\epsilon+sx^k)^2-\lambda\sum_k\tilde{y}^{k,2}-\sum_j P^j\tilde{y}\right)(1/\alpha^2-\tilde{y}^{1,2}-2\tilde{y}^1)\\
&+\frac{2}{\alpha}\left(\begin{array}{r}2\lambda\sum_k(x_*^k/\epsilon+sx^k)\tilde{y}^k+\sum_k(P^kx_*/\epsilon+sP^kx)\\ +P^1x_*/\epsilon+sP^1x-1/\alpha\end{array}\right)(\tilde{y}^1+1).
\end{aligned}
$$

Let $\Delta x_*^j:=x^j-x_*^j$. Then the above numerator can be rewritten as

$$
\begin{aligned}
&-\left(\lambda\sum_k\left((s+\tfrac{1}{\epsilon})x_*^k+s\Delta x_*^k\right)^2-\lambda\sum_k\tilde{y}^{k,2}-\sum_j\tilde{P}^jy\right)(1/\alpha^2-\tilde{y}^{1,2}-2\tilde{y}^1)\\
&+\frac{2}{\alpha}\left(\begin{array}{r}2\lambda\sum_k\left((s+\tfrac{1}{\epsilon})x_*^k+s\Delta x_*^k\right)\tilde{y}^k\\ +\sum_j\left((s+\tfrac{1}{\epsilon})P^jx_*+sP^j\Delta x_*\right)-1/\alpha\end{array}\right)(\tilde{y}^1+1).
\end{aligned}
$$

For $s\gg1$ and $0<\epsilon\ll\epsilon_0$, this gives

$$
\begin{aligned}
-\lambda\Big(\sum_k\left((s+\tfrac{1}{\epsilon})^2x_*^{k,2}+2\left(s+\tfrac{1}{\epsilon}\right)sx_*^k\Delta x_*^k+s^2\Delta x_*^{k,2}\right)\Big)(1/\alpha^2-\tilde{y}^{1,2}-2\tilde{y}^1)\\
+o(s^2)
\end{aligned}
$$

$$=$$

$$\left(\begin{aligned} -\lambda \left(s+\tfrac{1}{\epsilon}\right)^2 \textstyle\sum_k x_*^{k,2} - 2\lambda \left(s+\tfrac{1}{\epsilon}\right) s \textstyle\sum_k x_*^k \Delta x_*^k \\ - \lambda s^2 \textstyle\sum_k \Delta x_*^{k,2} \end{aligned} \right)(1/\alpha^2 - \tilde{y}^{1,2} - 2\tilde{y}^1)$$

$$+ o(s^2).$$

This estimate provides an upperbound on $\hat{\psi}_\epsilon$ as follows. Recall that ϵ is small but fixed; that $1 = \sum_j x_*^{j,2} = \sum_k x_*^{k,2}$ since $x_*^1 = 0$; and that $x \in B_{\mathbb{R}^n}(x_*; \epsilon^2)$, so that $\sqrt{\sum_j \Delta x_*^{j,2}} < \epsilon^2$. Then

$$c < \left[\begin{aligned} \left(-\lambda \left(s+\tfrac{1}{\epsilon}\right)^2 \textstyle\sum_k x_*^{k,2} - 2\lambda \left(s+\tfrac{1}{\epsilon}\right) s \textstyle\sum_k x_*^k \Delta x_*^k - \lambda s^2 \textstyle\sum_k \Delta x_*^{k,2}\right) \\ \times(1/\alpha^2 - \tilde{y}^{1,2} - 2\tilde{y}^1) \end{aligned} \right] + o(s^2)$$

$$< \left(-\lambda \left(s+\tfrac{1}{\epsilon}\right)^2 + 2\lambda \left(s+\tfrac{1}{\epsilon}\right) s\epsilon^2 + \lambda s^2 \epsilon^4\right)(1/\alpha^2 - \tilde{y}^{1,2} - 2\tilde{y}^1) + o(s^2)$$

$$< \lambda s^2 \left(-\left(1+\tfrac{1}{s\epsilon}\right)^2 + 2\left(1+\tfrac{1}{s\epsilon}\right)\epsilon^2 + \epsilon^4\right)(1/\alpha^2 - \tilde{y}^{1,2} - 2\tilde{y}^1) + o(s^2).$$

Thus, since ϵ is small,

$$1/\alpha^2 - \tilde{y}^{1,2} - 2\tilde{y}^1 < 0. \tag{3-5}$$

(6*ii*) The next goal is to show

$$\frac{d\hat{\psi}_\epsilon}{ds}(s) > 0, \qquad \forall s \geq 0, \tag{3-6}$$

provided $0 < \epsilon \leq \epsilon'$ for some ϵ' which can be taken to be less than ϵ_0.

Let $0 < \epsilon'' < \epsilon'$. Once this is proven the proof of the lemma is completed by noting that

$$\begin{aligned} d' := \inf_{y \in I^n} \psi_a(x_*/\epsilon'' + iy_* + i\epsilon''^2 y) \\ < \psi_a\left(x_*/\epsilon' + \left(\frac{1}{\epsilon''} - \frac{1}{\epsilon'}\right) x_* + iy_* + i\epsilon'^{,2} y\right) \end{aligned} \tag{3-7}$$

$$\leq c.$$

Then choose d such that $c < d < d'$. For $z_{\epsilon',s} \in N_{\epsilon'} \cap \mathbb{C}^n$; $z_{\epsilon',s} := x_*/\epsilon' + sx + iy_* + i\epsilon'^2 y$,

$$\begin{aligned}\psi_a(z_{\epsilon',s}) &\geq \psi_a(z_{\epsilon',0})\\ &\geq \min_{y \in I^n} \psi_a(x_*/\epsilon' + iy_* + i\epsilon'^2 y)\\ &= d' > d.\end{aligned}$$

Thus $N_{\epsilon'} \cap \mathbb{C}^n \subseteq \{\psi_a > d\}$; i. e. $z_* = x_*\infty + iy_* \in \mathrm{tr}_\infty\{\psi_a > d\}$.

(6*iii*) Differentiating (3-4) yields

(3-8)

$$\frac{d\hat{\psi}_\epsilon}{ds}(s) = -\frac{x^1}{(x_*^1/\epsilon + sx^1 - 1/\alpha)^2 + (y_*^1 + \epsilon^2 y^1)^2} + \frac{2(x_*^1/\epsilon + sx^1 - 1/\alpha)^2 x^1}{\left((x_*^1/\epsilon + sx^1 - 1/\alpha)^2 + (y_*^1 + \epsilon^2 y^1)^2\right)^2}$$

$$-\frac{\left\{\begin{array}{l} 2\lambda\sum_k (x_*^k/\epsilon + sx^k)x^k \left(\begin{array}{r}(x_*^1/\epsilon + sx^1 - 1/\alpha)^2 - (y_*^1 + \epsilon^2 y^1)^2\\ -2(y_*^1 + \epsilon^2 y^1)\end{array}\right)\\ \quad + \\ 2\left[\begin{array}{r}\left(\lambda\sum_k (x_*^k/\epsilon + sx^k)^2 - \lambda\sum_k (y_*^k + \epsilon^2 y^k)^2 - \sum_j (P^j y_* + \epsilon^2 P^j y)\right)\\ \times(x_*^1/\epsilon + sx^1 - 1/\alpha)x^1\end{array}\right]\\ \quad + \\ 2\left[\begin{array}{r}\left(2\lambda\sum_k x^k(y_*^k + \epsilon^2 y^k) + \sum_k P^k x + P^1 x\right)(x_*^1/\epsilon + sx^1 - 1/\alpha)\\ (y_*^1 + \epsilon^2 y^1 + 1)\end{array}\right]\\ \quad + \\ 2\left[\begin{array}{l}\left(\begin{array}{r}2\lambda\sum_k (x_*^k/\epsilon + sx^k)(y_*^k + \epsilon^2 y^k) + \sum_k (P^k x_*/\epsilon + sP^k x)\\ + P^1 x_*/\epsilon + sP^1 x - 1/\alpha\end{array}\right)\\ \quad \times x^1(y_*^1 + \epsilon^2 y^1 + 1)\end{array}\right]\end{array}\right\}}{\left[\begin{array}{r}\left((x_*^1/\epsilon + sx^1 - 1/\alpha)^2 - (y_*^1 + \epsilon^2 y^1)^2 - 2(y_*^1 + \epsilon^2 y^1)\right)^2\\ + 4(x_*^1/\epsilon + sx^1 - 1/\alpha)^2(y_*^1 + \epsilon^2 y^1)^2\end{array}\right]}$$

$$
+\left[\frac{\left\{\begin{array}{l}\left[\begin{array}{l}\left(\lambda\sum_k(x_*^k/\epsilon+sx^k)^2-\lambda\sum_k(y_*^k+\epsilon^2y^k)^2-\sum_j(y_*^j+\epsilon^2y^j)\right)\\ \quad\times((x_*^1/\epsilon+sx^1-1/\alpha)^2-(y_*^1+\epsilon^2y^1)^2-2(y_*^1+\epsilon^2y^1))\end{array}\right]\\ \quad+\\ 2\left[\begin{array}{l}\left(\begin{array}{l}2\lambda\sum_k(x_*^k/\epsilon+sx^k)(y_*^k+\epsilon^2y^k)+\sum_k(P^kx_*/\epsilon+sP^kx)\\ \qquad\qquad+P^1x_*/\epsilon+sP^1x-1/\alpha\end{array}\right)\\ \quad\times(x_*^1/\epsilon+sx^1-1/\alpha)(y_*^1+\epsilon^2y^1+1)\end{array}\right]\end{array}\right\}}{\left[\begin{array}{l}\left((x_*^1/\epsilon+sx^1-1/\alpha)^2-(y_*^1+\epsilon^2y^1)^2-2(y_*^1+\epsilon^2y^1)\right)^2\\ \qquad+4(x_*^1/\epsilon+sx^1-1/\alpha)^2(y_*^1+\epsilon^2y^1+1)^2\end{array}\right]^2}\right.
$$

$$
\left.\times\left\{\begin{array}{l}\left[\begin{array}{l}4\left((x_*^1/\epsilon+sx^1-1/\alpha)^2-(y_*^1+\epsilon^2y^1)^2-2(y_*^1+\epsilon^2y^1)\right)\\ \quad\times(x_*^1/\epsilon+sx^1-1/\alpha)x^1\end{array}\right]\\ \quad+8(x_*^1/\epsilon+sx^1-1/\alpha)x^1(y_*^1+\epsilon^2y^1+1)^2\end{array}\right\}\right].
$$

As in 6i) above, let $\tilde{y}=y_*+\epsilon^2y$. When $x_*^1=0$, (3-8) simplifies to

(3-9)

$$
\frac{d\hat{\psi}_\epsilon}{ds}(s)=-\frac{x^1}{(sx^1-1/\alpha)^2+\tilde{y}^{1,2}}+\frac{2(sx^1-1/\alpha)^2x^1}{\left((sx^1-1/\alpha)^2+\tilde{y}^{1,2}\right)^2}
$$

$$
-\frac{\left\{\begin{array}{l}2\lambda\sum_kX^kx^k\left((sx^1-1/\alpha)^2-\tilde{y}^{1,2}-2\tilde{y}^1\right)\\ \quad+2\left(\lambda\sum_kX^{k,2}-\lambda\epsilon^2\sum_k\tilde{y}^{k,2}-\epsilon^2\sum_jP^j\tilde{y}\right)(sx^1-1/\alpha)x^1\\ \quad+2\epsilon^2\left(2\lambda\sum_kx^k\tilde{y}^k+\sum_kP^kx+P^1x\right)(sx^1-1/\alpha)(\tilde{y}^1+1)\\ \quad+2\epsilon\left(2\lambda\sum_kX^k\tilde{y}^k+\sum_kP^kX+s\epsilon P^1x-\epsilon/\alpha\right)x^1(\tilde{y}^1+1)\end{array}\right\}}{\epsilon^2\left((sx^1-1/\alpha)^2-\tilde{y}^{1,2}-2\tilde{y}^1\right)^2+4\epsilon^2(sx^1-1/\alpha)^2(\tilde{y}^1+1)^2}
$$

$$
+\left[\frac{\left\{\begin{array}{l}\left(\lambda\sum_kX^{k,2}-\lambda\epsilon^2\sum_k\tilde{y}^{k,2}-\epsilon^2\sum_jP^j\tilde{y}\right)\left(\begin{array}{l}(sx^1-1/\alpha)^2\\ \qquad-\tilde{y}^{1,2}-2\tilde{y}^1\end{array}\right)\\ \qquad+2\left[\begin{array}{l}\left(2\lambda\sum_kX^k\tilde{y}^k+\sum_kP^kX+s\epsilon P^1x-\epsilon/\alpha\right)\\ \qquad\times(s\epsilon x^1-\epsilon/\alpha)(\tilde{y}^1+1)\end{array}\right]\end{array}\right\}}{\epsilon^2\left(\left((sx^1-1/\alpha)^2-\tilde{y}^{1,2}-2\tilde{y}^1\right)^2+4(sx^1-1/\alpha)^2(\tilde{y}^1+1)^2\right)^2}\right.
$$

$$
\left.\times\left(\begin{array}{l}4\left((sx^1-1/\alpha)^2-\tilde{y}^{1,2}-2\tilde{y}^1\right)(sx^1-1/\alpha)x^1\\ \qquad+8(sx^1-1/\alpha)x^1(\tilde{y}^1+1)^2\end{array}\right)\right].
$$

(6*iv*) First 2 terms of (3-8):

$$-\frac{x^1}{(sx^1-1/\alpha)^2+\tilde{y}^{1,2}}+\frac{2(sx^1-1/\alpha)^2x^1}{\left((sx^1-1/\alpha)^2+\tilde{y}^{1,2}\right)^2}$$

$$=\frac{-x^1\left((sx^1-1/\alpha)^2+\tilde{y}^{1,2}\right)+2(sx^1-1/\alpha)^2x^1}{\left((sx^1-1/\alpha)^2+\tilde{y}^{1,2}\right)^2}$$

$$=x^1\frac{(sx^1-1/\alpha)^2-\tilde{y}^{1,2}}{\left((sx^1-1/\alpha)^2+\tilde{y}^{1,2}\right)^2}.$$

But $x^1_* = 0$ and $x \in B_{\mathbb{R}^n}(x_*;\epsilon^2)$, implies $|x^1| < \epsilon^2$. Thus the first 2 terms is $o(\epsilon)$.

(6*v*) Numerator of the 3rd and 4th terms of (3-8).

First consider the numerator of the 3rd term of (3-8):

$$\begin{bmatrix}\text{(numerator of the 3rd term of (3-8))} \\ \qquad\times\left(\left((sx^1-1/\alpha)^2-\tilde{y}^{1,2}-2\tilde{y}^1\right)^2+4(sx^1-1/\alpha)^2(\tilde{y}^1+1)^2\right)\end{bmatrix}$$

$$=-2\lambda\textstyle\sum_k X^k x^k\left((sx^1-1/\alpha)^2-\tilde{y}^{1,2}-2\tilde{y}^1\right)^3$$

$$-8\lambda\textstyle\sum_k X^k x^k\left((sx^1-1/\alpha)^2-\tilde{y}^{1,2}-2\tilde{y}^1\right)(sx^1-1/\alpha)^2(\tilde{y}^1+1)^2$$

$$+\begin{bmatrix}2\left(-\lambda\sum_k X^{k,2}+\lambda\epsilon^2\sum_k\tilde{y}^{k,2}+\epsilon^2\sum_j P^j\tilde{y}\right)(sx^1-1/\alpha)x^1 \\ \qquad\times\left((sx^1-1/\alpha)^2-\tilde{y}^{1,2}-2\tilde{y}^1\right)^2\end{bmatrix}$$

$$+8\left(-\lambda\textstyle\sum_k X^{k,2}+\lambda\epsilon^2\sum_k\tilde{y}^{k,2}+\epsilon^2\sum_j P^j\tilde{y}\right)(sx^1-1/\alpha)^3x^1(\tilde{y}^1+1)^2$$

$$+o(\epsilon^0).$$

The numerator of the 4th term of (3-8) is

$$\begin{bmatrix}4\left(\lambda\sum_k X^{k,2}-\lambda\epsilon^2\sum_k\tilde{y}^{k,2}-\epsilon^2\sum_j P^j\tilde{y}\right) \\ \qquad\times\left((sx^1-1/\alpha)^2-\tilde{y}^{1,2}-2\tilde{y}^1\right)^2(sx^1-1/\alpha)^3x^1\end{bmatrix}$$

$$+\left[\begin{array}{r} 8\left(\lambda\sum_k X^{k,2} - \lambda\epsilon^2\sum_k \tilde{y}^{k,2} - \epsilon^2\sum_j P^j\tilde{y}\right) \\ \times\left((sx^1-1/\alpha)^2 - \tilde{y}^{1,2} - 2\tilde{y}^1\right)(sx^1-1/\alpha)x^1(\tilde{y}^1+1)^2 \end{array}\right]$$

$$+\, o(\epsilon^0).$$

With common denominator

$$\epsilon^2\left(\left((sx^1-1/\alpha)^2 - \tilde{y}^{1,2} - 2\tilde{y}^1\right)^2 + 4(sx^1-1/\alpha)^2(\tilde{y}^1+1)^2\right)^2, \tag{3-10}$$

the numerator of the 3rd and 4th terms of (3-9) combined is

(3-11)

$$-\,2\lambda\sum_k X^k x^k\left((sx^1-1/\alpha)^2 - \tilde{y}^{1,2} - 2\tilde{y}^1\right)^3$$

$$-\,8\lambda\sum_k X^k x^k\left((sx^1-1/\alpha)^2 - \tilde{y}^{1,2} - 2\tilde{y}^1\right)(sx^1-1/\alpha)^2(\tilde{y}^1+1)^2$$

$$+\,2\left[\begin{array}{c} \left(-\lambda\sum_k X^{k,2} + \lambda\epsilon^2\sum_k \tilde{y}^{k,2} + \epsilon^2\sum_j P^j\tilde{y}\right)(sx^1-1/\alpha) \\ \times x^1\left((sx^1-1/\alpha)^2 - \tilde{y}^{1,2} - 2\tilde{y}^1\right)^2 \end{array}\right]$$

$$+\,8\left(-\lambda\sum_k X^{k,2} + \lambda\epsilon^2\sum_k \tilde{y}^{k,2} + \epsilon^2\sum_j P^j\tilde{y}\right)(sx^1-1/\alpha)^3 x^1(\tilde{y}^1+1)^2$$

$$+\,4\left[\begin{array}{c} \left(\lambda\sum_k X^{k,2} - \lambda\epsilon^2\sum_k \tilde{y}^{k,2} - \epsilon^2\sum_j P^j\tilde{y}\right)\left((sx^1-1/\alpha)^2 - \tilde{y}^{1,2} - 2\tilde{y}^1\right)^2 \\ \times(sx^1-1/\alpha)^3 x^1 \end{array}\right]$$

$$+\,8\left[\begin{array}{c} \left(\lambda\sum_k X^{k,2} - \lambda\epsilon^2\sum_k \tilde{y}^{k,2} - \epsilon^2\sum_j P^j\tilde{y}\right)\left((sx^1-1/\alpha)^2 - \tilde{y}^{1,2} - 2\tilde{y}^1\right) \\ \times(sx^1-1/\alpha)x^1(\tilde{y}^1+1)^2 \end{array}\right]$$

$$+\, o(\epsilon^0)$$

Note that the $o(\epsilon^0)$ term cannot be disregarded because it contains s. This term will be studied separately in each of the two cases below.

($6vi$) Case 1: $x^1 = 0$.

In this case the denominator (3-10) is independent of s while the numerator (3-11) including the $o(\epsilon^0)$ term explicitly written out reduces to

(3-12)

$$-2\lambda\sum_k X^k x^k (1/\alpha^2 - \tilde{y}^{1,2} - \tilde{y}^1)^3 - \frac{8\lambda}{\alpha^2}\sum_k X^k x^k (1/\alpha^2 - \tilde{y}^{1,2} - \tilde{y}^1)(\tilde{y}^1 + 1)^2$$

$$+\epsilon^2 \frac{2}{\alpha}\left[\begin{array}{l}(2\lambda\sum_k x^k \tilde{y}^k + \sum_k P^k_{\cdot} x + P^1 x)(\tilde{y}^1 + 1) \\ \qquad\qquad \left((1/\alpha^2 - \tilde{y}^{1,2} - \tilde{y}^1)^2 + \frac{4}{\alpha}(\tilde{y}^1 + 1)^2\right)\end{array}\right].$$

In this case the $o(\epsilon^0)$ term is independent of s, so, for small ϵ, the dominating terms are

(3-13)

$$-2\lambda\sum_k X^k x^k (1/\alpha^2 - \tilde{y}^{1,2} - \tilde{y}^1)\left((1/\alpha^2 - \tilde{y}^{1,2} - \tilde{y}^1)^2 + \frac{4}{\alpha^2}(\tilde{y}^1 + 1)^2\right)$$

$$= -\left[\begin{array}{l}2\lambda\sum_k \left((1+s\epsilon)x^k_* + s\epsilon\Delta x^k_*\right)\left(x^k_* + \Delta x^k_*\right) \\ \qquad \times (1/\alpha^2 - \tilde{y}^{1,2} - \tilde{y}^1)\left((1/\alpha^2 - \tilde{y}^{1,2} - \tilde{y}^1)^2 + \frac{4}{\alpha^2}(\tilde{y}^1 + 1)^2\right)\end{array}\right].$$

Thus putting (3-10) and (3-13) together yields

$$\text{(3-14)} \qquad \frac{-2\lambda\sum_k(x^k_* + s\epsilon x^k)x^k(1/\alpha^2 - \tilde{y}^{1,2} - \tilde{y}^1)^2}{\epsilon^2\left((1/\alpha^2 - \tilde{y}^{1,2} - \tilde{y}^1)^2 + \frac{4}{\alpha^2}(\tilde{y}^1 + 1)^2\right)} + o(\epsilon^{-2}).$$

Now

$$\begin{aligned}
&\sum_k \left((1+s\epsilon)x^k_* + s\epsilon\Delta x^k_*\right)(x^k_* + \Delta x^k_*) \\
&\qquad =\sum_k(1+s\epsilon)x^{k,2}_* + \sum_k(1+2s\epsilon)\Delta x^k_* x^k_* + s\epsilon\sum_k \Delta x^{k,2}_* \\
&\qquad \geq(1+s\epsilon) - (1+2s\epsilon)\sqrt{\sum_k \Delta x^{k,2}_*}\sqrt{\sum_k x^{k,2}_*} - s\epsilon\epsilon^4 \\
&\qquad \geq(1+s\epsilon) - (1+2s\epsilon)\epsilon^2 - s\epsilon^5 \\
&\qquad =1 + s(\epsilon - 2\epsilon^3 - \epsilon^5) - \epsilon^2
\end{aligned}$$

>0 for ϵ sufficiently small. $(s \geq 0)$.

Together with (3-5), this shows that (3-14), the 3rd and 4th terms of (3-8) combined, is greater than 0 for small ϵ and all $s \geq 0$.

(6*vii*) Case 2: $x^1 \neq 0$.

In this case the largest power of s in the denominator (3-10) is 8 while it is at most 7 in the numerator (3-11)(this includes the $o(\epsilon^0)$ term). Thus the $o(\epsilon^0)$ term in (3-11) can be estimated by bounds independent of s, x^1 being estimated by ϵ^2.

Since $|x^1| < \epsilon^2$, the terms in the numerator involving x^1 are $o(\epsilon^2)$, and can be grouped with the $o(\epsilon^0)$ term; thus the sum of the 3rd and 4th terms of (3-8) is

(3-15)

$$\frac{\left\{\begin{array}{l} -2\lambda\sum_k X^k x^k \left((sx^1-1/\alpha)^2-\tilde{y}^{1,2}-\tilde{y}^1\right)^3 \\ \quad -8\lambda\sum_k X^k x^k \left((sx^1-1/\alpha)^2-\tilde{y}^{1,2}-\tilde{y}^1\right)(sx^1-1/\alpha)^2(\tilde{y}^1+1)^2+o(\epsilon^0) \end{array}\right\}}{\epsilon^2\left(\left((sx^1-1/\alpha)^2-\tilde{y}^{1,2}-\tilde{y}^1\right)^2+4(sx^1-1/\alpha)^2(\tilde{y}^1+1)^2\right)^2}.$$

As in the previous case

$$\sum_k (x_*^k + s\epsilon x^k)x^k = 1 + s\epsilon + (1+s\epsilon)\sum_k x_*^k \Delta x_*^k + s\epsilon \sum_k \Delta x_*^{k,2}.$$

Again the terms containing Δx_*^k are $o(\epsilon^1)$ and are thus $o(\epsilon^0)$. Thus (3-15) reduces to

$$-\frac{\left\{\begin{array}{l} 2\lambda(1+s\epsilon)\left((sx^1-1/\alpha)^2-\tilde{y}^{1,2}-\tilde{y}^1\right) \\ \quad \times\left(\left((sx^1-1/\alpha)^2-\tilde{y}^{1,2}-\tilde{y}^1\right)^2+4(sx^1-1/\alpha)^2(\tilde{y}^1+1)^2\right) \end{array}\right\}}{\epsilon^2\left(\left((sx^1-1/\alpha)^2-\tilde{y}^{1,2}-\tilde{y}^1\right)^2+4(sx^1-1/\alpha)^2(\tilde{y}^1+1)^2\right)^2}+o\left(\epsilon^{-2}\right)$$

$$=\frac{-2\lambda(1+s\epsilon)\left((sx^1-1/\alpha)^2-\tilde{y}^{1,2}-\tilde{y}^1\right)}{\epsilon^2\left((sx^1-1/\alpha)^2-\tilde{y}^{1,2}-\tilde{y}^1\right)^2+4\epsilon^2(sx^1-1/\alpha)^2(\tilde{y}^1+1)^2}+o\left(\epsilon^{-2}\right)$$

$> 0, \qquad$ for sufficiently small ϵ independent of s.

$(6ix)$ Together with the first two terms computed in $(6iv)$ it follows that

$$\frac{d\hat{\psi}_\epsilon}{ds}(s) > 0, \qquad \text{for sufficiently small } \epsilon \text{ and any } s \geq 0.$$

This proves the lemma. □

COROLLARY 3.3.5. *For a outside a set of measure 0,*

$$int_{\widehat{\mathbb{C}^n}}\, cl_{\widehat{\mathbb{C}^n}}\{\psi_a > c\} = \bigcup_{\infty > d > c} int_{\widehat{\mathbb{C}^n}}\, cl_{\widehat{\mathbb{C}^n}}\{\psi_a > d\}.$$

PROOF.

$$\begin{aligned} int_{\widehat{\mathbb{C}^n}}\, cl_{\widehat{\mathbb{C}^n}}\{\psi_a > c\} &= \{\psi_a > c\} \cup \mathrm{tr}_\infty\{\psi_a > c\} \\ &= \{\psi_a > c\} \cup \bigcup_{\infty > d > c} \mathrm{tr}_\infty\{\psi_a > d\} \\ &= \bigcup_{\infty > d > c} int_{\widehat{\mathbb{C}^n}}\, cl_{\widehat{\mathbb{C}^n}}\{\psi_a > d\}. \quad \square \end{aligned}$$

LEMMA 3.3.6. *Let $V \subset\subset U \subseteq \widehat{\mathbb{C}^n}$. Suppose $U \subset int_{\widehat{\mathbb{C}^n}}\, cl_{\widehat{\mathbb{C}^n}}\{\psi_a > c\}$. Then $\exists c', \infty > c' > c$ such that $V \subseteq int_{\widehat{\mathbb{C}^n}}\, cl_{\widehat{\mathbb{C}^n}}\{\psi_a > c'\}$.*

PROOF. $cl_{\widehat{\mathbb{C}^n}} V$ is compact and $cl_{\widehat{\mathbb{C}^n}} V \subseteq \bigcup_{\infty > d > c} int_{\widehat{\mathbb{C}^n}}\, cl_{\widehat{\mathbb{C}^n}}\{\psi_a > d\}$. □

LEMMA 3.3.7. *For $c' > c$, $\quad cl_{\widehat{\mathbb{C}^n}}\{\psi_a > c'\} \subseteq int_{\widehat{\mathbb{C}^n}}\, cl_{\widehat{\mathbb{C}^n}}\{\psi_a > c\}$, when a, c, c' are outside a set of measure* 0.

PROOF. As in lemma 3.3.3, this is equivalent to proving

$$cl_{\widehat{\mathbb{C}^n}}\, \mathrm{tr}_\infty\{\psi_a > c'\} \subseteq \mathrm{tr}_\infty\{\psi_a > c\}.$$

Let $z_* = x_*\infty + iy_* \in cl_{\widehat{\mathbb{C}^n}}\mathrm{tr}_\infty\{\psi_a > c'\}, x_* \in \mathbb{S}_{n-1}$. So there is a sequence $z_m = x_m\infty + iy_m \to z_*, z_m \in \mathrm{tr}_\infty\{\psi_a > c'\}$.

(1) Case 1: $x_*^1 \neq 0$. Since $x_m \to x_*$ it can be assumed that $x_m^1 \neq 0$. Then from (3-3) in the proof of lemma 3.3.4:

$$(3\text{-}16) \quad \psi_a(x_m/\epsilon_m + s_m x_m + iy_m)$$
$$= a - \frac{\epsilon_m}{(1+s_m\epsilon_m)x_m^1} + \sum\nolimits_j y_m^j - \frac{\lambda\sum_k(1+s_m\epsilon_m)^2 x_m^{k,2}}{(1+s_m\epsilon_m)^2 x_m^{1,2}} + o_m(\epsilon_m),$$

where the ϵ_m/α terms are collected with the $o_m(\epsilon_m)$ term. Since x in (3-3) is here chosen to be x_m, $\Delta x^j = 0$. Moreover, choose $\epsilon_m \to 0$ as $m \to \infty$.

By assumption $\psi_a(x_*/\epsilon_m + s_m x_m + iy_m) > c', \quad \forall s_m > 0$ when ϵ_m is sufficiently small.

On the other hand, for $x \in B_{\mathbb{R}^n}(x_*;\epsilon^2) \cap \mathbb{S}_{n-1}$ and $y \in I^n$,

$$\psi_a(x_*/\epsilon + sx + iy_* + i\epsilon^2 y)$$

$$= a - \frac{\epsilon}{(1+s\epsilon)x_*^1 + s\epsilon\Delta x_*^1} + \sum\nolimits_j(P^j y_* + \epsilon^2 P^j y)$$
$$- \frac{\lambda\sum_k\left((1+s\epsilon)x_*^k + s\epsilon\Delta x_*^k\right)^2}{\left((1+s\epsilon)x_*^1 + s\epsilon\Delta x_*^1\right)^2} + o(\epsilon)$$

$$= \psi_a(x_*/\epsilon_m + s_m x_m + iy_m) + \frac{\epsilon_m}{(1+s_m\epsilon_m)x_m^1} - \frac{\epsilon}{(1+s\epsilon)x_*^1}$$
$$+ \sum\nolimits_j(P^j y_* - P^j y_m) + \lambda\frac{\sum_k x_m^{k,2}}{x_m^{1,2}} - \lambda\frac{\sum_k\left((1+s\epsilon)x_*^k + s\epsilon\Delta x_*^k\right)^2}{\left((1+s\epsilon)x_*^1 + s\epsilon\Delta x_*^1\right)^2}$$
$$+ o_m(\epsilon_m) + o(\epsilon).$$

Clearly the 2nd, 3rd, and 4th terms can be made arbitrarily small for small ϵ and large m.

The 5th and 6th terms combined give:

$$\lambda \frac{\left\{ \begin{array}{l} \sum_k x_m^{k,2} \left((1+s\epsilon)^2 x_*^{1,2} + 2(1+s\epsilon)s\epsilon x_*^1 \Delta x_*^1 + s^2\epsilon^2 \Delta x_*^{1,2} \right) \\ \qquad - \sum_k \left((1+s\epsilon)^2 x_*^{k,2} + 2(1+s\epsilon)s\epsilon x_*^k \Delta x_*^k + s^2\epsilon^2 \Delta x_*^{k,2} \right) x_m^{1,2} \end{array} \right\}}{x_m^{1,2} \left((1+s\epsilon)x_*^1 + s\epsilon \Delta x_*^1 \right)^2}$$

$$= \lambda \frac{\left\{ \begin{array}{l} \sum_k (1+s\epsilon)^2 (x_m^{k,2} x_*^{1,2} - x_m^{1,2} x_*^{k,2}) \\ \qquad +2(1+s\epsilon)s\epsilon \sum_k (x_m^{k,2} x_*^1 \Delta x_*^1 - x_m^{1,2} x_*^k \Delta x_*^k) \\ \qquad\qquad +s^2\epsilon^2 \sum_k (x_m^{k,2} \Delta x_*^{1,2} - x_m^{1,2} \Delta x_*^{k,2}) \end{array} \right\}}{x_m^{1,2} \left((1+s\epsilon)x_*^1 + s\epsilon \Delta x_*^1 \right)^2}.$$

This can be made arbitrarily small since since $(x_m^{k,2} x_*^{1,2} - x_m^{1,2} x_*^{k,2}) \to 0$ as $m \to \infty$, and the other terms in the numerator contain Δx_*^j-terms which are small for $0 < \epsilon \ll 1$. As in lemma 3.3.4, the power of s in the numerator is no greater than that in the denominator. Thus by choosing all the terms except the first term to be less than δ in absolute value,

$$\begin{aligned} \psi_a(x_*/\epsilon + sx + iy_* + i\epsilon^2 y) &\geq \psi_a(x_m/\epsilon_m + s_m x_m + iy_m + iy_m) - \delta \\ &> c' - \delta \geq c, \qquad \text{for} \quad 0 < \delta \ll 1. \end{aligned}$$

That is $z_* \in N_\epsilon \subseteq int_{\widehat{\mathbb{C}^n}}\, cl_{\widehat{\mathbb{C}^n}} \{\psi_a > c\}$.

(2) Case 2: $x_*^1 = 0$.

In this case pick $z_m \to z_*$. We can assume $z_m \neq z_*$ and since $\mathrm{tr}_\infty\{\psi_a > c'\}$ is open choose z_m so that $x_m^1 \neq 0$ for all m.

Consider

$$\begin{aligned} &\psi_a(x_*/\epsilon + sx + iy_* + i\epsilon^2 y) \\ &\qquad = \psi_a(x_m/\epsilon_m + s_m x_m + iy_m) \\ &\qquad\qquad + \left(\psi_a(x_*/\epsilon + sx + iy_* + i\epsilon^2 y) - \psi_a(x_m/\epsilon_m + s_m x_m + iy_m) \right). \end{aligned}$$

By choosing ϵ_m sufficiently small, the first term on the right is greater than c' for all $s_m \geq 0$.

For simplicity, let

$$T_* := \psi_a(x_*/\epsilon + sx + iy_* + i\epsilon^2 y)$$

$$T_m := \psi_a(x_m/\epsilon_m + s_m x_m + iy_m).$$

The next step is to estimate $T_* - T_m$. From (3-4) and (3-16)

$$T_* - T_m =$$

$$a - \frac{sx^1 - 1/\alpha}{(sx^1 - 1/\alpha)^2 + \tilde{y}_*^{1,2}} + \textstyle\sum_j P^j \tilde{y}_*$$

$$+ \frac{\left\{ \begin{array}{l} \left[\begin{array}{l} (-\lambda\sum_k (x_*^k + s\epsilon x^k)^2 + \lambda\epsilon^2 \sum_k \tilde{y}_*^{k,2} + \epsilon^2 \sum_k P^k \tilde{y}_*) \\ \qquad \times \left((sx^1 - 1/\alpha)^2 - \tilde{y}_*^{1,2} - 2\tilde{y}_*^1\right) \end{array} \right] \\ \\ -2\epsilon \left[\left(\begin{array}{l} 2\lambda \sum_k (x_*^k + s\epsilon x^k)\tilde{y}_*^k \\ \quad + \sum_k (P^k x_* + s\epsilon P^k x) \\ \quad + P^1 x_* + s\epsilon P^1 x - \epsilon/\alpha \end{array} \right) (sx^1 - 1/\alpha)(\tilde{y}_*^1 + 1) \right] \end{array} \right\}}{\epsilon^2 \left((sx^1 - 1/\alpha)^2 - \tilde{y}_*^{1,2} - 2\tilde{y}_*^1 \right)^2 + 4\epsilon^2 (sx^1 - 1/\alpha)^2 (\tilde{y}_*^1 + 1)^2}$$

$$- a + \frac{\epsilon_m}{(1 + s_m\epsilon_m)x_m^1} - \textstyle\sum_j P^j y_m + \frac{\lambda \sum_k (1 + s_m \epsilon_m)^2 x_m^{k,2}}{(1 + s_m\epsilon_m)^2 x_m^{1,2}} + o(\epsilon)$$

$$= \frac{\epsilon_m}{(1 + s_m\epsilon_m)x_m^1} - \frac{sx^1 - 1/\alpha}{(sx^1 - 1/\alpha)^2 + \tilde{y}_*^{1,2}} + \textstyle\sum_j (P^j y_* - P^j y_m) + \frac{\lambda \sum_k x_m^{k,2}}{x_m^{1,2}}$$

$$
+\frac{\left\{\begin{array}{l}\left[\begin{array}{r}\left(-\lambda\sum_k(x_*^k+s\epsilon x^k)^2+\lambda\epsilon^2\sum_k\tilde{y}_*^{k,2}+\epsilon^2\sum_k\tilde{P}^k y_*\right)\\ \times\left((sx^1-1/\alpha)^2-\tilde{y}_*^{1,2}-2\tilde{y}_*^1\right)\end{array}\right]\\ \\ -2\epsilon\left[\begin{array}{r}\left(\begin{array}{l}2\lambda\sum_k(x_*^k+s\epsilon x^k)\tilde{y}_*^k+\sum_k(P^k x_*+s\epsilon P^k x)\\ \qquad +P^1x_*+s\epsilon P^1 x-\epsilon/\alpha\end{array}\right)\\ \times(sx^1-1/\alpha)(\tilde{y}_*^1+1)\end{array}\right]\end{array}\right\}}{\epsilon^2\left((sx^1-1/\alpha)^2-\tilde{y}_*^{1,2}-2\tilde{y}_*^1\right)^2+4\epsilon^2(sx^1-1/\alpha)^2(\tilde{y}_*^1+1)^2}
$$

$$
+\,o(\epsilon).
$$

The first term is bounded. The 2nd term can be made arbitrarily small by choosing m sufficiently large so that ϵ_m is small, and then fixing m and choosing s_m large. Since $y_m \to y_*$, $\sum_j(P^j y_* - P^j y_m) > -\delta$ for an arbitrary $\delta > 0$ by choosing m large.

To estimate the 5th term, consider the following two cases.

(3) Case 2a: ($x_*^1 = 0$ and) $x^1 = 0$.

In this case the 5th term reduces to

$$
\frac{\left[\begin{array}{l}-\lambda\sum_k(x_*^k+s\epsilon x^k)^2(1/\alpha^2-\tilde{y}_*^{1,2}-2\tilde{y}_*^1)\\ +2\dfrac{\epsilon}{\alpha}\left(2\lambda\sum_k(x_*^k+s\epsilon x^k)\tilde{y}_*^k+\sum_j(P^jx_*+s\epsilon P^jx)\right)(\tilde{y}_*^1+1)\end{array}\right]}{\epsilon^2(1/\alpha^2-\tilde{y}_*^{1,2}-2\tilde{y}_*^1)^2+4\epsilon^2/\alpha^2(\tilde{y}_*^1+1)^2}+o(\epsilon^{-1}).
$$

Estimate the numerator

$$
\begin{aligned}
&-\lambda\sum_k(x_*^k+s\epsilon x^k)^2(1/\alpha^2-\tilde{y}_*^{1,2}-2\tilde{y}_*^1)\\
&\qquad+2\frac{\epsilon}{\alpha}\left(2\lambda\sum_k(x_*^k+s\epsilon x^k)\tilde{y}_*^k+\sum_k(P^jx_*+s\epsilon P^jx)\right)(\tilde{y}_*^1+1)\\
&\geq-\lambda\sum_k(x_*^k+s\epsilon x^k)^2(1/\alpha^2-\tilde{y}_*^{1,2}-2\tilde{y}_*^1)\\
&\qquad-2\frac{\epsilon}{\alpha}\sqrt{\sum_k(x_*^k+s\epsilon x^k)^2}\left(2\lambda\sqrt{\sum_k\tilde{y}_*^{k,2}}+K_{\|P^j\|}\right)|\tilde{y}_*^1+1|
\end{aligned}
$$

$$= \|x_* + s\epsilon x\| \left[\begin{array}{r} -\lambda \|x_* + s\epsilon x\| (1/\alpha^2 - \tilde{y}_*^{1,2} - 2\tilde{y}_*^1) \\ -2\frac{\epsilon}{\alpha}\left(2\lambda\sqrt{\sum_k \tilde{y}_*^{k,2}} + K_{\|P^j\|}\right)|\tilde{y}_*^1 + 1| \end{array} \right].$$

The first term in square brackets is greater than 0 by (3-5). Thus choosing $\epsilon \ll 1$ makes the 5th term as large as needed.

(4) Case 2b: ($x_*^1 = 0$ and) $x^1 \neq 0$.

Because the power of s in the numerator is less than that in the denominator, the dominating term in the numerator, for small ϵ, will be

$$-\lambda \sum_k (x_*^k + s\epsilon x^k)^2 (1/\alpha^2 - \tilde{y}_*^{1,2} - 2\tilde{y}_*^1),$$

since $|x^1| = |x^1 - x_0^1| < \epsilon^2$. Again by (3-5), this is greater than 0. Thus the 5th term can be made as large as needed.

In conclusion $T_* - T_m > -\delta$ for $\delta > 0$ by suitably choosing $\epsilon \ll 1$. Thus $T_* = T_m + (T_* - T_m) > c' - (c' - c) = c$ on N_ϵ. So $z_* \in \mathrm{tr}_\infty\{\psi_a > c\}$. This proves the lemma. $\square$

CHAPTER IV

THEOREMS ON PURE CODIMENSIONALITY AND FUNDAMENTAL EXACT SEQUENCES

> Whoso has sixpence is sovereign (to the length of sixpence) over all men; commands cooks to feed him, philosophers to teach him, kings to mount guard over him,—to the length of sixpence.
>
> —Thomas Carlyle, *Sartor Resartus* [1833].

In this chapter we show that $\mathbb{S}\Omega$ is pure 1-codimensional with respect to $\tau^{-1}{}^{p}\mathscr{O}$ (i. e. $\mathscr{H}^{k}_{\mathbb{S}\Omega}(\tau^{-1}{}^{p}\mathscr{O}) = 0$ for $k \neq 1$); and $\mathbb{S}^{*}\Omega$ is pure n-codimensional with respect to $\pi^{-1}{}^{p}\mathscr{O}$.

Since the Fourier-Sato transform works just as well on $\Omega \subseteq \mathbb{D}^{n}$ as on a real analytic manifold, many of the usual results for microfunctions on a real analytic manifold are seen to remain true for Fourier p-microfunctions. Specifically one has the usual short exact sequences on the sphere and cosphere bundles, $\mathbb{S}\Omega$ and $\mathbb{S}^{*}\Omega$ respectively. These are stated in §4.3.

§4.1 Computation of $\mathscr{H}^{k}_{\mathbb{S}\Omega}(\tau^{-1}{}^{p}\mathscr{O})$

Some preliminaries are needed to begin. Proposition 4.1.2 allows one to smooth plurisubharmonic exhaustion. It is modelled after a classical result. Next we recall the Grauert[1] tubular neighbourhood theorem in the form Kawai

[1] Grauert [1958], §3.

used for open subsets of $\mathbb{D}^n$. The proof given here is almost exactly Harvey and Wells' [1972] proof that dispenses with Grauert's original cone construction.

Finally essentially by intersecting the Grauert tubular neighbourhood with a wedge, we show that every point of $\sqrt{-1}\mathbb{S}\Omega$ has a basis of neighbourhoods whose projection on $\widehat{\mathbb{C}^n} - \mathbb{D}^n$ is ${}^{\mathcal{P}}\mathcal{O}$-pseudoconvex. $\mathscr{H}^k_{\mathbb{S}\Omega}(\tau^{-1}\,{}^{\mathcal{P}}\mathcal{O})$ can then be calculated using the classical proof.

Lemma 4.1.1. *Let $(X,\mathcal{U})$ be a uniform space, and let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be uniformly continuous. Then $f \vee g := max(f,g)$ is uniformly continuous.* $\square$

Proposition 4.1.2[2]. *Let U be an open subset of $\widehat{\mathbb{C}^n}$, and K_0 a compact subset of U. Suppose q is a continuous plurisubharmonic function such that*

(1) $\{q < c\} \subset\subset U, \qquad c \in \mathbb{R}$;

(2) $\sup_{K_0 \cap \mathbb{C}^n} q < 0$;

(3) *for every compact subset $K \subseteq U$, $\sup_{K \cap \mathbb{C}^n} q < \infty$; and*

(4) *for every compact subset $K \subseteq U$, q is uniformly continuously on $K \cap \mathbb{C}^n$.*

Then $\exists \hat{q} \in C^\infty(U \cap \mathbb{C}^n)$ strictly plurisubharmonic, $\hat{q} \geq q$, satisfying (1), (2), (3) and (4).

Proof. Let $V_j := \{q < j\}$, and

$$v_j(z) := \int_{V_{j+1}} q(\zeta)\, \varphi\!\left(\frac{z-\zeta}{\delta_j}\right) \delta_j^{-2n} d\lambda + \delta_j |\mathfrak{Im} z|^2,$$

where φ is a Friedrich mollifier, and δ_j is chosen so small that $\sup_{K_0 \cap \mathbb{C}^n} v_0 < 0$, and $\sup_{K_0 \cap \mathbb{C}^n} v_1 < 0$. This is possible because of condition (2) in the statement and because for a compact set K_0, $\sup_{K_0} |\mathfrak{Im} z|^2 < \infty$. Moreover uniform continuity of q, condition (4), shows that for $j = 2, 3, \ldots$ the δ_j's can be chosen so

[2] This is essentially the second part of Hörmander [1990] theorem 2.6.11. The same proof goes through with these new hypotheses.

that $v_j < q+1$ on V_j since

$$
\begin{aligned}
v_j(z) - q(z) &= \int_{V_{j+1}\cap B(z,\delta_j)} (q(\zeta) - q(z))\, \varphi\!\left(\frac{z-\zeta}{\delta_j}\right) \delta_j^{-2n} d\lambda + \delta_j |\mathfrak{Im} z|^2, \\
&\le \int_{V_{j+1}\cap B(z,\delta_j)} |q(\zeta) - q(z)|\, \varphi\!\left(\frac{z-\zeta}{\delta_j}\right) \delta_j^{-2n} d\lambda + \delta_j M_{V_j} \\
&< 1, \qquad \text{for small } \delta_j.
\end{aligned}
$$

Thus there is a $\widehat{\mathbb{C}^n}$-neighbourhood, $\tilde{V}_j$ of $cl_{\mathbb{C}^n} V_j$, such that on $\tilde{V}_j \cap \mathbb{C}^n$, v_j is strictly plurisubharmonic (because of the $|\mathfrak{Im} z|^2$ term) and is $> q$. Moreover note that v_j vanishes outside a δ_j-neighbourhood of V_{j+1}. Let $\chi(t)$ be a convex C^∞ function that is 0 when $t \le 0$, and > 0 when $t > 0$, such that $\chi' > 0$ when $t > 0$.

Then $\chi\left(v_j + \frac{3}{2} - j\right)$ is strictly plurisubharmonic in a $\widehat{\mathbb{C}^n}$ neighbourhood of $cl_{\mathbb{C}^n} V_j - V_{j-1}$ (intersected with $\mathbb{C}^n$) since

(1-1)

$$
\begin{aligned}
\bar{w}^t \frac{\partial^2}{\partial \bar{z} \partial z} \chi\left(v_j + \tfrac{3}{2} - j\right) w &= \chi''(v_j + \tfrac{3}{2} - j) \frac{\partial v_j}{\partial \bar{z}} \bar{w} \frac{\partial v_j}{\partial z} w \\
&\quad + \chi'\left(v_j + \tfrac{3}{2} - j\right) \bar{w}^t \frac{\partial^2 v_j}{\partial \bar{z} \partial z} w \\
&\ge \chi'(\tfrac{1}{4}) \bar{w}^t \frac{\partial^2 v_j}{\partial \bar{z} \partial z} w, \qquad \text{for } z \text{ outside } V_{j-\frac{5}{4}}.
\end{aligned}
$$

Next inductively choose constants a_j and define u_m by

$$
u_m = v_0 + \textstyle\sum_1^m a_j \chi\left(v_j + \frac{3}{2} - j\right),
$$

so that u_m is strictly plurisubharmonic on a $\widehat{\mathbb{C}^n}$ neighbourhood of $cl_{\mathbb{C}^n} V_m$, and $u_m > q$.

u_m can be chosen strictly plurisubharmonic since v_{m-1} vanishes outside a

δ_{m-1} neighbourhood of V_m giving

$$\begin{aligned}
\bar{w}^t \frac{\partial^2 v_{m-1}}{\partial \bar{z} \partial z}\left(\frac{z-\zeta}{\delta_{m-1}}\right) w &= \int_{V_{m-1}} q(\zeta) \bar{w}^t \frac{\partial^2 \varphi}{\partial \bar{z} \partial z}\left(\frac{z-\zeta}{\delta_{m-1}}\right) w \frac{d\lambda}{\delta_{m-1}^{2n}} \\
&\geq -(m+1) \int_{V_{m-1}} \left| \frac{\partial^2 \varphi}{\partial \bar{z} \partial z}\left(\frac{z-\zeta}{\delta_{m-1}}\right) \right| \delta_{m-1}^{-2n} d\lambda |w|^2 \\
&\geq -(m+1) M_{\delta_{m-1}} |w|^2, \qquad \text{for some constant } M_{\delta_{m-1}}.
\end{aligned}$$

Thus a similar calculation as in (1-1) shows that

$$\bar{w}^t \frac{\partial^2 \chi\left(v_{m-1} + \frac{3}{2} - j\right)}{\partial \bar{z} \partial z} w \geq -C_{m-1} |w|^2, \qquad \text{outside } V_{m-1}.$$

Choosing a_m sufficiently large thus makes u_m strictly plurisubharmonic.

u_m can be chosen $> q$ since on $V_m - V_{m-1}$

$$a_m \chi\left(v_m + \tfrac{3}{2} - m\right) \geq a_m \chi\left(\tfrac{1}{2}\right)$$

can be chosen greater than $m+1$, the maximum of q there.

Let $\hat{q} := \lim_{m \to \infty} u_m$. $\hat{q}$ is C^∞ and strictly plurisubharmonic on $U \cap \mathbb{C}^n$. This is uniformly continuous on $K \cap \mathbb{C}^n$ since each v_j is. $\hat{q}$ satisfies the other requirements of the proposition. $\square$

Recall the following Grauert tubular neighbourhood theorem from Kawai [1970][3].

THEOREM 4.1.3. *Let O be an open subset of $\mathbb{D}^n$, and U a complex neighbourhood of O such that $U \cap \mathbb{D}^n = O$. There is an $\mathscr{W}$-pseudoconvex neighbourhood W of O such that $O \subseteq W \subseteq U$ and $W \cap \mathbb{D}^n = O$.*

Moreover a strictly plurisubharmonic exhaustion function, q, of V can be chosen to satisfy

(1) $q \geq 0$;

[3]See also Saburi [1985]. The proof given here follows Harvey & Wells [1972].

(2) *q is C^∞ on W (W considered as a manifold with boundary);*

(3) *For every compact subset $K \subseteq W$ there is a constant λ_K such that the Levi form of q, L_q, satisfies $L_q(z)(w,w) \geq \lambda_K |w|^2$, for $z \in K$.*

PROOF. As in Saburi [1985], let ϖ be the C^∞ diffeomorphism of $\widehat{\mathbb{C}^n}$ onto $\bar{B}(0;1) + i\mathbb{R}^n$ given by

$$\varpi(x+iy) = \begin{cases} \dfrac{x'}{|x'|} + iy, & \text{if} \quad x = x'\infty \in \mathbb{S}_{n-1}\infty = \mathbb{D}^n - \mathbb{R}^n, \\ \dfrac{x}{\sqrt{1+|x|^2}} + iy, & \text{if} \quad x + iy \in \mathbb{C}^n. \end{cases}$$

Let K_k, $k = 0, 1, 2, \ldots$ be an exhaustion of $\varpi(U)$; i. e.

$$K_k \subset int_{\widehat{\mathbb{C}^n}} K_{k+1}; \qquad K_k \subset\subset \varpi(U); \qquad \varpi(U) = \cup_k K_k.$$

Let

$$U_0 = int_{\widehat{\mathbb{C}^n}} K_1; \qquad U_k := int_{\widehat{\mathbb{C}^n}} K_{k+1} - K_{k-1}, \qquad k \geq 1.$$

This is a locally finite cover of $\varpi(U)$. Take a partition of unity, ψ_k, subordinate to this cover. We may suppose that only finitely many ψ_j are non-zero on K_k.

Let $\varphi(z) := |\Im\mathrm{m} z|^2$. Then φ is C^∞ on $\widehat{\mathbb{C}^n}$ and strictly plurisubharmonic on $\mathbb{C}^n$. We shall consider C^∞ functions $\epsilon(\varpi(z))$ that vanish at $\mathrm{fr}_{\widehat{\mathbb{C}^n}}(U)$ for which $\varphi - \epsilon \circ \varpi$ is strictly plurisubharmonic on $U \cap \mathbb{C}^n$.

For instance, take $0 < \epsilon(\tilde{z}) := \sum_k a_k \psi_k(\tilde{z})$, $\tilde{z} \in \varpi(U) \subseteq \widehat{\mathbb{C}^n}$. For small $a_k > 0$, $\varphi(z) - \epsilon(\varpi(z))$ will be strictly plurisubharmonic on $U \cap \mathbb{C}^n$. This follows by directly computing the Levi form on each $\varpi^{-1}(K_k) \cap \mathbb{C}^n$ and using Lemma

2.2.1 of Saburi [1985] (with his notation):

$$\begin{aligned}
&\textstyle\sum_{l,m} \frac{\partial^2}{\partial \bar{z}_l \partial z_m} \left(\varphi(z) - \epsilon(\varpi(z))\right) \bar{w}_l w_m \\
&\quad \geq 2|w|^2 - B \textstyle\sum_{k=0}^{q} \frac{a_k}{1+|x|^2} \left(|\nabla^2 \psi_k|(\varpi(z)) + |\nabla \psi_k|(\varpi(z))\right) |w|^2 \\
&\quad \geq \left[2 - B \textstyle\sum_{j=0}^{q} \frac{a_j}{1+|x|^2} \sup_{K_k \cap \mathbb{C}^n} \left(|\nabla^2 \psi_j|(\varpi(z)) + |\nabla \psi_j|(\varpi(z))\right) \right] |w|^2, \\
&\qquad\qquad\qquad \text{for } z \in K_k - K_{k-1}.
\end{aligned}$$

By choosing a_k sufficiently small for large k, ϵ vanishes at the boundary $\mathrm{fr}_{\widehat{\mathbb{C}^n}}(U)$. Such ϵ are C^∞ on U.

Let $U_\epsilon := \{z \in U : \varphi(z) < \epsilon(\varpi(z))\}$. Then as in Harvey & Wells [1972], $\{U_\epsilon\}_\epsilon$ form a basis of neighbourhoods of O; and $(\epsilon \circ \varpi - \varphi)^{-1}|_{\mathbb{C}^n}$ is an $\mathscr{W}$ exhaustion function satisfying the conditions of the theorem. □

PROPOSITION 4.1.4[4]. *Every point of $\sqrt{-1}\mathbb{S}\Omega$ has a basis of neighbourhoods, $\tilde{U}$, such that $\tilde{U} - \sqrt{-1}\mathbb{S}\Omega$ is $\mathscr{W}$-pseudoconvex.*

PROOF. If the point is not in $\mathbb{S}_{n-1}\infty \times \sqrt{-1}\mathbb{S}_{n-1}0$, then the usual convex, relatively compact basis of neighbourhoods suffices.

So let $\Gamma' := \Gamma \cup \mathrm{tr}_\infty \Gamma$ be a $\widehat{\mathbb{C}^n}$ neighbourhood of $x_0\infty$, $\|x_0\| = 1$. For simplicity suppose Γ is convex so that $\Gamma' = int_{\widehat{\mathbb{C}^n}}\, cl_{\widehat{\mathbb{C}^n}} \Gamma$. Let W and q be obtained from the Grauert theorem 4.1.3, with W an $\mathscr{W}$-pseudoconvex neighbourhood of Γ' contained in Ω_ν.

Let

$$\begin{aligned}
V'_\epsilon &:= \Gamma' + \left\{ y \in \mathbb{R}^n - 0 : \left| \frac{y}{\|y\|} - v_0 \right| < \epsilon \right\} \\
S'_\epsilon &:= \Gamma' + \left\{ iv0 \in \sqrt{-1}\mathbb{S}_{n-1} : |v - v_0| < \epsilon \right\} \\
\tilde{U}'_\epsilon &:= (V'_\epsilon \cup S_\epsilon) \cap ((W - \Gamma') \cup \sqrt{-1}\mathbb{S}\Gamma')
\end{aligned}$$

[4]This should really be a corollary of a $\widehat{\mathbb{C}^n}$ version of theorem due to Bros and Iagolnitzer [1976] essentially stating that every tuboid with convex base contains a smaller pseudoconvex tuboid with the same profile.

Then the $\tilde{U}'_\epsilon$ from a basis of neighbourhoods of $x_0\infty + iv_00$, with W and ϵ varying. We show that

$$U_\epsilon := \tilde{U}'_\epsilon - \sqrt{-1}\mathbb{S}\Omega = V'_\epsilon \cap W$$

is $\mathscr{P}\mathscr{O}$-pseudoconvex.

Note that $V_\epsilon := V'_\epsilon \cap \mathbb{C}^n$ is convex hence pseudoconvex. Thus $-\log d(z, V_\epsilon^c)$ is a continuous plurisubharmonic exhaustion function.

Let $\theta(z) := \max(-\log d(z, V_\epsilon^c), q(z))$. Then θ is a continuous non-negative plurisubharmonic function on $U_\epsilon := U'_\epsilon \cap \mathbb{C}^n$.

U_ϵ satisfies (P_p) since W does. Moreover

$$\{\theta < c\} = \{-\log d(z, V_\epsilon^c) < c\} \cap \{q < c\} \subset\subset U_\epsilon.$$

Now let K be a compact subset of U'_ϵ. Then

$$\sup_{K\cap\mathbb{C}^n} \theta \leq \max\left(\sup_{K\cap\mathbb{C}^n} -\log d(z, V_\epsilon^c),\ \sup_{K\cap\mathbb{C}^n} q\right).$$

Suppose that $\sup_{K\cap\mathbb{C}^n} -\log d(z, V_\epsilon^c) = \infty$; i. e. $\exists z_k = x_k + iy_k \in K \cap \mathbb{C}^n$ such that $d(z_k, V_\epsilon^c) \to 0$. Since K is compact in U'_ϵ, by taking a subsequence if necessary, we can assume $z_k \to z_*$ for some $z_* \in K$. Clearly $z_* \notin \mathbb{C}^n$ for otherwise $d(z_*, V_\epsilon^c) = 0$ contradicting $z_* \in K$.

So $z_* = x_*\infty + iy_*$. Now, there is a neighbourhood $(\Gamma_1 \cup \Gamma_1\infty) + \sqrt{-1}\,(y_0 +]-\delta, \delta[^{\,n})$ of z_* such that $\Gamma_1 \subset\subset \Gamma$ and z_k's are contained in this neighbourhood for large k.

For simplicity let

$$B := \left\{y \in \mathbb{R}^n - 0 : \left|\frac{y}{\|y\|} - v_0\right| < \epsilon\right\}$$

Since $d(x_k, \Gamma) > c > 0$ for some constant c, we must have $d(y_k, B^c) \to 0$. Thus $y_k \to 0$ or $\left|\frac{y_k}{\|y_k\|} - v_0\right| \to \epsilon$. But since $y_k \to y_0 \neq 0$ and $y_0 \in B$, this is a contradiction. Thus $\sup_{K\cap\mathbb{C}^n} -\log d(z, V_\epsilon^c) < \infty$

Thus if K is compact in U'_ϵ,

$$\sup_{K\cap\mathbb{C}^n} \theta < \infty.$$

Moreover q is uniformly continuous on $K \cap \mathbb{C}^n$. Next we show the same is true for $-\log d(z, V_\epsilon^c)$, and hence for θ.

Let $z, z' \in K \cap \mathbb{C}^n$. Then

$$\begin{aligned}
|-\log d(z, V_\epsilon^c) + \log d(z', V_\epsilon^c)| &= \left|\log \frac{d(z', V_\epsilon^c)}{d(z, V_\epsilon^c)}\right| \\
&= \left|\log\left(1 + \frac{d(z', V_\epsilon^c)}{d(z, V_\epsilon^c)} - 1\right)\right| \\
&\leq \frac{3}{2}\left|\frac{d(z', V_\epsilon^c)}{d(z, V_\epsilon^c)} - 1\right|, \quad \text{when } \left|\frac{d(z', V_\epsilon^c)}{d(z, V_\epsilon^c)} - 1\right| < \frac{1}{2}, \\
&= \frac{1}{d(z, V_\epsilon^c)}|d(z', V_\epsilon^c) - d(z, V_\epsilon^c)|.
\end{aligned}$$

But by what was proven earlier, $1/d(z, V_\epsilon^c) \leq M_0$ on $K \cap \mathbb{C}^n$. Since $d(\cdot, V_\epsilon^c)$ is uniformly continuous on $\mathbb{C}^n$, it follows that $-\log d(\cdot, V_\epsilon^c)$ is uniformly continuous on $K \cap \mathbb{C}^n$.

The proposition now follows from proposition 4.1.2. $\square$

THEOREM 4.1.5. $\mathsf{R}^k\Gamma_{\mathbb{S}\Omega}(\tau^{-1}\mathscr{O}) = 0, \qquad$ *for* $k \neq 1$.

PROOF[5]. Let $\mathscr{F}$ be a sheaf on Ω_ν. Recall the following maps:

$$(\Omega_\nu - \Omega) \xrightarrow{j} \tilde{\Omega}_\nu \xrightarrow{\tau} \Omega_\nu$$

[5] After Kawai, Kashiwara & Kimura [1986].

This gives the triangle

$$\mathbf{R}\Gamma_{\mathbb{S}\Omega}\mathscr{F} \to \mathbf{R}\Gamma_{\tilde{\Omega}_\nu}(\mathscr{F}) \to \mathbf{R}\Gamma_{\Omega_\nu - \Omega}(\mathscr{F}) \xrightarrow{+1}$$

Since $\Omega_\nu - \Omega$ is open in $\widehat{\mathbb{C}^n}$, and the functor $\Gamma_{\Omega_\nu - \Omega} = j_* j^{-1}$, this triangle with $\mathscr{F} = \tau^{-1}{}^p\mathscr{O}$ is

$$\mathbf{R}\Gamma_{\mathbb{S}\Omega}\tau^{-1}{}^p\mathscr{O} \to \mathbf{R}\Gamma_{\tilde{\Omega}_\nu}(\tau^{-1}{}^p\mathscr{O}) \to \mathbf{R}\Gamma_{\Omega_\nu - \Omega}(\tau^{-1}{}^p\mathscr{O}) \xrightarrow{+1} .$$

This gives the long exact sequence

$$\begin{aligned} 0 &\to \mathscr{H}^0_{\mathbb{S}\Omega}(\tau^{-1}{}^p\mathscr{O}) \to \tau^{-1}{}^p\mathscr{O} \to j_* j^{-1}\tau^{-1}{}^p\mathscr{O} \\ &\to \mathscr{H}^1_{\mathbb{S}\Omega}(\tau^{-1}{}^p\mathscr{O}) \longrightarrow 0 \longrightarrow \mathrm{R}^1 j_* j^{-1}{}^p\mathscr{O} \\ &\to \mathscr{H}^2_{\mathbb{S}\Omega}(\tau^{-1}{}^p\mathscr{O}) \longrightarrow 0 \longrightarrow \cdots \end{aligned}$$

Thus there is a sequence

$$0 \longrightarrow \mathscr{H}^0_{\mathbb{S}\Omega}(\tau^{-1}{}^p\mathscr{O}) \longrightarrow \tau^{-1}{}^p\mathscr{O} \longrightarrow j_* j^{-1}\tau^{-1}{}^p\mathscr{O} \longrightarrow \mathscr{H}^1_{\mathbb{S}\Omega}(\tau^{-1}{}^p\mathscr{O}) \longrightarrow 0, \tag{1-2}$$

and isomorphisms

$$\mathrm{R}^k j_* j^{-1}\tau^{-1}{}^p\mathscr{O} \simeq \mathscr{H}^{k+1}_{\mathbb{S}\Omega}(\tau^{-1}{}^p\mathscr{O}), \qquad \text{for } k \geq 1. \tag{1-3}$$

Consider first the morphism $\tau^{-1}{}^p\mathscr{O} \longrightarrow j_* j^{-1}\tau^{-1}{}^p\mathscr{O} = j_*{}^p\mathscr{O}|_{\Omega_\nu - \Omega}$ in (1-2). It shall be shown that this map is injective. This map is obtained as follows. Let $\tilde{U} \subset \tilde{\Omega}_\nu$ be a neighbourhood of $z_0 \in \tilde{\Omega}_\nu$. The map above is the direct limit as $\tilde{U}$ runs through a basis of neighbourhoods z_0 of

$$\tau^{-1}{}^p\mathscr{O}(\tilde{U}) \longrightarrow j_*{}^p\mathscr{O}|_{\Omega_\nu - \Omega}\left(\tilde{U}\right) = {}^p\mathscr{O}|_{\Omega_\nu - \Omega}\left(\tilde{U} \cap (\Omega_\nu - \Omega)\right).$$

Recall that $\tau^{-1}{}^{\nu}\mathcal{O}(\tilde{U})$ consists of sections $\sigma' \circ \tau$, where σ' is continuous and $\pi\sigma' = \mathrm{id}$ as in

$$\begin{tikzcd} & {}^{\nu}\mathcal{O} \arrow[d, shift left, "pr"] \\ \tilde{U} \arrow[ur, "\sigma"] \arrow[r, "\tau"'] & \tau\tilde{U} \subseteq \Omega_\nu \arrow[u, shift left, "\sigma'"] \end{tikzcd}$$

Thus since $\tilde{U} \cap (\Omega_\nu - \Omega) = \tilde{U} - \mathbb{S}\Omega$ the map above is $\sigma' \circ \tau \longmapsto \sigma'|_{\tilde{U} - \mathbb{S}\Omega}$.

Suppose that $\sigma'|_{\tilde{U}-\mathbb{S}\Omega} = 0$. If $z_0 \in \Omega_\nu - \Omega = \tilde{\Omega}_\nu - \mathbb{S}\Omega$, then for sufficiently small $\tau\tilde{U} = \tilde{U} \subseteq \Omega_\nu - \Omega$, then $\sigma' = 0$ and hence $\sigma = 0$.

On the other hand, if $z_0 = x_0 + iv_0 0 \in \mathbb{S}\Omega$ where $v_0 = 1$ then we can take $\tilde{U}$ to be the sets $\tilde{U}_\epsilon$ defined as follows.

If $x_0 \in \mathbb{R}^n$ define

$$A_\epsilon := \{x + iv0 : |x - x_0| < \epsilon, |v - v_0| < \epsilon, \|v\| = 1\} \quad \subseteq \quad \mathbb{S}\Omega$$

If $x_0 \in \mathbb{D}^n - \mathbb{R}^n$ say $x_0 = x'\infty$ where $\|x'\| = 1$, define

$$\Gamma_\epsilon := \left\{ x \in \mathbb{R}^n - \{0\} : \frac{x}{\|x\|} \in B_{\mathbb{R}^n}(x', \epsilon) \right\} \cup \{x\infty \in \mathbb{S}_{n-1}\infty : x \in B_{\mathbb{R}^n}(x', \epsilon)\}$$
$$A_\epsilon := (\Gamma_\epsilon + x_0/\epsilon) + i\{v : |v - v_0| < \epsilon, \quad \|v\| = 1\}0 \quad \subseteq \quad \mathbb{S}\Omega.$$

In both cases define

$$B_\epsilon := \{x + itv : 0 < t < \epsilon, x + iv0 \in A_\epsilon, \quad \|v\| = 1\}$$
$$\tilde{U}_\epsilon := (A_\epsilon \cup B_\epsilon) \cap \tilde{\Omega}_\nu$$

Then note that in either case $\tau\tilde{U}_\epsilon$ is open in Ω_ν. Hence σ' is a section of ${}^{\nu}\mathcal{O}$, $\sigma' \in {}^{\nu}\mathcal{O}(\tau\tilde{U}_\epsilon)$. Thus $\sigma' \in \mathcal{O}(\tau\tilde{U}_\epsilon \cap \mathbb{C}^n)$ is an analytic function and by the uniqueness of analytic continuation, $\sigma'|_{\tilde{U}_\epsilon - \mathbb{S}\Omega} = 0$ implies $\sigma' \equiv 0$ on $\tau\tilde{U}_\epsilon$.

This proves that $\tau^{-1}{}^{\nu}\mathcal{O} \longrightarrow j_* j^{-1}\tau^{-1}{}^{\nu}\mathcal{O} = j_*{}^{\nu}\mathcal{O}|_{\Omega_\nu - \Omega}$ is injective. It follows from (1-2) that $\mathcal{H}^0_{\mathbb{S}\Omega}(\tau^{-1}{}^{\nu}\mathcal{O}) = 0$.

Now consider the isomorphisms (1-3). As before $\mathsf{R}^k j_* j^{-1}\tau^{-1}{}^p\mathscr{O} = \mathsf{R}^k j_*({}^p\mathscr{O}|_{\Omega_\nu - \Omega})$. This is the sheaf associated to the presheaf

$$\tilde{U} \longmapsto H^k\left(\tilde{U} \cap (\Omega_\nu - \Omega); {}^p\mathscr{O}|_{\Omega_\nu - \Omega}\right).$$

For $z \in \tilde{\Omega}_\nu$,

$$\mathsf{R}^k j_*({}^p\mathscr{O}|_{\Omega_\nu - \Omega})_z = \varinjlim_{\tilde{U} \ni z} H^k(\tilde{U} \cap (\Omega_\nu - \Omega); {}^p\mathscr{O}). \tag{1-4}$$

If $z \in \tilde{\Omega}_\nu - \mathbb{S}\Omega = \Omega_\nu - \Omega$, (1-4) becomes the direct limit over neighbourhoods of z in $\widehat{\mathbb{C}^n}$.

$$\varinjlim_{U \ni z} H^k(U; {}^p\mathscr{O}) = 0, \qquad \text{for } k \geq 1.$$

If on the other hand $z \in \mathbb{S}\Omega$, $z = x_0 + iv_0 0$, with $x_0 \in \Omega$, and $\|v_0\| = 1$, take $\tilde{U} = \tilde{U}_\epsilon$ as in proposition 4.1.4, with $\tilde{U}_\epsilon$ forming a basis of neighbourhoods of z. Then since $\tilde{U}_\epsilon \cap (\Omega_\nu - \Omega)$ is ${}^p\mathscr{O}$-pseudoconvex,

$$H^k\left(\tilde{U}_\epsilon \cap (\Omega_\nu - \Omega); {}^p\mathscr{O}\right) = 0.$$

So in either case the direct limit vanishes. This proves the claim and the theorem. $\square$

From the proof of the lemma one has

COROLLARY 4.1.6. *There is short exact sequence*

$$0 \longrightarrow \tau^{-1}{}^p\mathscr{O} \longrightarrow \widetilde{{}^p\mathscr{O}} \longrightarrow {}^p\mathscr{Q} \longrightarrow 0. \quad \square$$

§4.2 Computation of $\mathscr{H}^k_{\mathbb{S}^*\Omega}(\pi^{-1}\mathscr{O})$

DEFINITION 4.2.1. $W_a := int_{\widehat{\mathbb{C}^n}} cl_{\widehat{\mathbb{C}^n}}(\{z : \psi_a(z) > 0\} \cap \{z : \rho^\alpha(z) < 2a\})$. $\Diamond$

REMARK 4.2.2. As in chapter 3, we set $x_0 := (1, 0, \ldots, 0)\infty \in \mathbb{D}^n - \mathbb{R}^n$. These sets W_a will be a basis of neighbourhoods of x_0, as is seen below. We will then write these sets as a difference of compact sets, K^1 and K^2_a. $\rhd$

LEMMA 4.2.3. *$\{W_a\}$ form a basis of neighbourhoods of $x_0 + i0 \in \widehat{\mathbb{C}^n}$ for $a > 0$.*

PROOF. By definition $W_a \subseteq int_{\widehat{\mathbb{C}^n}} cl_{\widehat{\mathbb{C}^n}}\{\rho^\alpha < 2a\}$. It remains to show that for sufficiently small ϵ, $\{\rho^\alpha < \epsilon\} \subseteq \{\psi_a > 0\}$.

Suppose $\rho^\alpha(z) < \epsilon$, where as usual $z = (x^1 + iy^1, \ldots, x^n + iy^n)$. Next each of the terms of ψ_a is estimated. Write $\psi_a = a - T_1 - T_2 - T_3$ in (III.1-2). Since $\rho^\alpha < \epsilon$, it follows from (III.1-1) that

$$T_1 := \frac{x}{x^{,2} + y^{1,2}} \leq \frac{x}{x^{,2}} = \frac{1}{x} < \sqrt{\epsilon};$$

$$T_2 := -\textstyle\sum_j P^j y < \sum_j |P^j y| \leq K_{P^j} \|y\| \leq K\sqrt{\epsilon}.$$

Now examine the third term:

$$T_3 := \frac{\left\{\begin{aligned}&(\lambda\textstyle\sum_k(x^{k,2} - y^{k,2}) - \sum_j P^j y)(x^{,2} - y^{1,2} - 2y^1)\\ &\qquad + 2x\left(2\lambda\textstyle\sum_k x^k y^k + \sum_k P^k x + Px\right)(y^1 + 1)\end{aligned}\right\}}{(x^{,2} - y^{1,2} - 2y^1)^2 + 4x^{,2}(y^1 + 2)^2}$$

$$= \frac{\left\{\begin{aligned}&(\lambda\textstyle\sum_k(x^{k,2} - y^{k,2}) - \sum_j P^j y)(x^{,2} - y^{1,2} - 2y^1)\\ &\qquad + 2x(y^1 + 1)\left(2\lambda\textstyle\sum_k x^k(y^k + 1) + \sum_k P^k x\right)\end{aligned}\right\}}{x^{,4} - 2x^{,2}y^1(y^1 + 2) + y^{1,2}(y^1 + 2)^2 + 4x^{,2}(y^{1,2} + 2y^1 + 1)}$$

$$+ \frac{2xPx(y^1 + 1)}{x^{,4} - 2x^{,2}y^1(y^1 + 2) + y^{1,2}(y^1 + 2)^2 + 4x^{,2}(y^{1,2} + 2y^1 + 1)}.$$

The denominator of these two terms simplifies:

$$
\begin{aligned}
x'^4 &- 2x'^2y^1(y^1+2) + y^{1,2}(y^1+2)^2 + 4x'^2(y^{1,2}+2y^1+1) \\
&= x'^4 - 2x'^2y^1(y^1+2) + y^{1,2}(y^1+2)^2 + 4x'^2 + 4x'^2y^1(y^1+2) \\
&= x'^4 + 4x'^2 + y^{1,2}(y^1+2)^2 + 2x'^2y^1(y^1+2) \\
&= x'^4 + y^{1,2}(y^1+2)^2 + 2x'^2(y^{1,2}+2y^1+2)
\end{aligned}
$$

Note that $y^{1,2} + 2y^1 + 2 > 0$, so the 2nd summand of T_3 can be estimated as follows

$$
\begin{aligned}
&\frac{2x(P^1x - 1/\alpha)(y^1+1)}{x'^4 + y^{1,2}(y^1+2)^2 + 2x'^2(y^{1,2}+2y^1+2)} \\
&\qquad < \frac{2(P^1x - 1/\alpha)(y^1+1)}{x'^2x} \\
&\qquad = \frac{2(y^1+1)(K_{\|P^1\|}\|x\| + 1/\alpha)}{x'^2x} \\
&\qquad < 2(1+\sqrt{\epsilon})\epsilon \cdot \left(K_{\|P^1\|}\sqrt{M+\epsilon} + \frac{1}{\alpha}\sqrt{\epsilon}\right).
\end{aligned}
$$

Similarly for the first summand of T_3,

$$
\frac{\left[\begin{array}{r}\left(\lambda\sum_k(x^{k,2}-y^{k,2}) - \sum_j P^j y\right)(x'^2 - y^{1,2} - 2y^1) \\ + 2x(y^1+1)(2\lambda\sum_k x^k y^k + \sum_k P^k x)\end{array}\right]}{x'^4 + y^{1,2}(y^1+2)^2 + 2x'^2(y^{1,2}+2y^1+2)}
$$

$$
< \frac{\left[\begin{array}{l}\left(\lambda\sum_k|z^k|^2 + \sum_j|P^jy|\right)(|x'^2 - y^{1,2} - 2y^1|) \\ + 2x|y^1+1|(2\lambda\sum_k|x^k||y^k| + \sum_k|P^kx|)\end{array}\right]}{x'^4}
$$

$$
< \left\{\begin{array}{l}\dfrac{(\lambda\epsilon|x|^2 + K_{\|P^j\|}\sqrt{\epsilon})(x'^2 + \sqrt{\epsilon}(2-\sqrt{\epsilon}))}{x'^4} \\ \qquad + \dfrac{2x\left(2\lambda\sqrt{\sum_k|x^k|^2}\sqrt{\sum_k|y^k|^2} + K_{\|P^k\|}\sqrt{M+\epsilon}(1+\sqrt{\epsilon})x'^2\right)}{x'^4}\end{array}\right\}
$$

$$
< o(1), \qquad \text{as } \epsilon \to 0.
$$

Thus $T_1 + T_2 + T_3 < a$, for $\epsilon > 0$ sufficiently small. That is, $\psi_a(z) > 0$ when ϵ is sufficiently small. This proves the lemma. $\square$

LEMMA 4.2.4. $W_a \cap \mathbb{C}^n = \{\psi_a > 0\} \cap \{\rho^\alpha < 2a\}$.

PROOF. $int_{\widehat{\mathbb{C}^n}}(cl_{\widehat{\mathbb{C}^n}}\{\psi_a > 0\} \cap \{\rho^\alpha < 2a\}) \cap \mathbb{C}^n = int_{\mathbb{C}^n}((cl_{\mathbb{C}^n}\{\psi_a > 0\} \cap \{\rho^\alpha < 2a\}) \cap \mathbb{C}^n) = int_{\mathbb{C}^n}\, cl_{\mathbb{C}^n}\{\psi_a > 0\} \cap int_{\mathbb{C}^n}\, cl_{\mathbb{C}^n}\{\rho^\alpha < 2a\} = \{\psi_a > 0\} \cap \{\rho^\alpha < 2a\}$. $\square$

LEMMA 4.2.5. *For* $\epsilon > 0 \quad cl_{\widehat{\mathbb{C}^n}} W_{a-\epsilon} \subseteq W_a$.

PROOF. Recall that $W_a := int_{\widehat{\mathbb{C}^n}}\, cl_{\widehat{\mathbb{C}^n}}(\{\psi_a > 0\} \cap \{\rho^\alpha < 2a\})$. Thus the lemma is a corollary of corollary 3.2.9, and lemmas 3.3.2, 3.3.6, 3.3.7, since $\psi_{a-\epsilon} > 0$ if and only if $\psi_a > \epsilon$. $\square$

ASSUMPTION 4.2.6. From now on assume that α is very small, and in particular smaller than

(1) $0 < \alpha \leq \frac{4}{9}$;
(2) $\frac{1}{2} - \frac{3}{4}\alpha - 3\sqrt{\alpha} \geq 0$;

This is used in lemma 4.2.9 below.

DEFINITION 4.2.7.

(1) $G := \{(z^1, \ldots, z^n) : P^1 y \leq 0, \ldots, P^n y \leq 0.\}$;
(2) $K^1 := cl_{\widehat{\mathbb{C}^n}}\left(G \cap \{\rho^\alpha \leq \frac{\alpha}{2}\}\right)$;
(3) $K_a^2 := cl_{\widehat{\mathbb{C}^n}}\left(K^1 \cap \{\psi_a \leq 0\}\right)$.

LEMMA 4.2.8. $K^1 - K_a^2 \supseteq W_a \cap G$.

PROOF. From corollary 3.2.9, $W_a \cap G \cap \mathbb{C}^n = \{\psi_a > 0\} \cap \{\rho^\alpha < 2a\} \cap G$; i. e. 'the "$int_{\widehat{\mathbb{C}^n}}\, cl_{\widehat{\mathbb{C}^n}}$" operator does not add points of $\mathbb{C}^n$ to $\{\psi_a > 0\} \cap \{\rho^\alpha < 2a\}$.'

Similarly $(K^1 - K_a^2) \cap \mathbb{C}^n = G \cap \{\rho^\alpha \leq \frac{\alpha}{2}\} \cap \{\psi_a > 0\}$. Since $0 < a < \frac{\alpha}{4}$, $2a < \frac{\alpha}{2}$. So $W_a \cap G \cap \mathbb{C}^n \subseteq K^1 - K_a^2$. Let $z \in (W_a \cap G) - \mathbb{C}^n$. Then by the lemmas 3.3.2 and 3.3.7 (since $z \in W_a - \mathbb{C}^n$) there is a (conic) neighbourhood of z, say Γ, such that $\Gamma \cap \mathbb{C}^n \subseteq W_a$. So $\psi_a(\Gamma \cap \mathbb{C}^n) \geq \delta > 0$. Hence $z \notin cl_{\widehat{\mathbb{C}^n}}\{\psi_a \leq 0\}$. Moreover $z \in K^1$ since $\{\psi_a > 0\} \cap \{\rho^\alpha < 2a\} \cap G \subseteq K^1$. So $W_a \cap G \subseteq K^1 - K_a^2$. $\square$

LEMMA 4.2.9. $K^1 - K_a^2 \subseteq W_a \cap G$.

PROOF. First we show $K^1 - K_a^2 \cap \mathbb{C}^n \subseteq \{\psi_a > 0\} \cap \{\rho^\alpha < 2a\}$. From (III.1-3)

(2-1)

$$\psi_a > 0 \Leftrightarrow a > \frac{x}{x^{,2} + y^{1,2}} - \textstyle\sum_j P^j y + (\lambda\rho^\alpha - \lambda\rho^\alpha)$$

$$+ \frac{\left\{ \begin{array}{r} (\lambda\sum_k (x^{k,2} - y^{k,2}) - \sum_k P^k y - P^1 y)(x^{,2} - y^{1,2} - 2y^1) \\ +(2\lambda\sum_k x^k y^k + \sum_k P^k x + Px) 2x(y^1 + 1) \end{array} \right\}}{(x^{,2} - y^{1,2} - 2y^1)^2 + 4x^{,2}(y^1 + 1)^2}$$

$$= \left(-\textstyle\sum_j P^j y - \lambda \sum_j y^{j,2}\right) + \left(\frac{x}{x^{,2} + y^{1,2}} - \frac{\lambda}{x^{,2}}\right)$$

$$+ \left\{ \begin{array}{c} \dfrac{\left[\begin{array}{r} (\lambda\sum_k (x^{k,2} - y^{k,2}) - \sum_k P^k y - P^1 y)(x^{,2} - y^{1,2} - 2y^1) \\ +(2\lambda\sum_k x^k y^k + \sum_k P^k x + Px) 2x(y^1 + 1) \end{array} \right]}{(x^{,2} - y^{1,2} - 2y^1)^2 + 4x^{,2}(y^1 + 1)^2} \\ - \lambda \dfrac{\sum_k |z^k|^2 + y^{1,2}}{x^{,2}} \end{array} \right\}$$

$$+ \lambda\rho^\alpha.$$

As in the proof of the previous lemma $(K^1 - K_a^2) \cap \mathbb{C}^n = G \cap \{\rho^\alpha \leq \frac{\alpha}{2}\} \cap \{\psi_a > 0\} \subseteq \{\rho^\alpha \leq \frac{\alpha}{2}\} \cap \{\psi_a > 0\}$. As usual let $z = (x^1 + iy^1, \ldots, x^n + iy^n)$. We shall show that if $P^j y \leq 0$ then $\psi_a(z) > 0 \Rightarrow \rho^\alpha(z) < 2a$ when $z \in \{\rho^\alpha \leq \frac{\alpha}{2}\}$. To this end, we shall show that the first three summands together are nonnegative when $P^j y \leq 0$

1) The first summand of (2-1)

$$
\begin{aligned}
-\sum_j P^j y - \lambda \sum_j y^{j,2} &= \sum_j |P^j y| - \lambda \|y\|_2^2 \\
&\geq \|Py\|_1 - \lambda \|y\|_2, \qquad \text{since } \|y\|_2 < 1, \\
&\geq K\|y\|_2 - \lambda\|y\|_2 \\
&\geq 0, \qquad \text{for } \lambda < K.
\end{aligned}
$$

2) The second summand of (2-1). (III.3-1) gives

$$
\rho^\alpha(z) \leq \frac{\alpha}{2} \iff 1 \leq \frac{\frac{\alpha}{2} - \sum_j y^{j,2}}{1 + \sum_j y^{j,2}} (x^{,1} - 1/\alpha)^2 - \frac{\sum_k x^{k,2}}{1 + \sum_j y^{j,2}}.
$$

So

$$
x \geq \sqrt{\frac{1 + \sum_j y^{j,2}}{\frac{\alpha}{2} - \sum_j y^{j,2}}} \geq \sqrt{\frac{2}{\alpha}}, \qquad (\alpha \leq 2).
$$

Moreover by assumption 4.2.6

$$
\frac{y^{1,2}}{x^2} \leq \frac{3\alpha}{4} \leq 1.
$$

Hence

$$
\frac{x}{x^{,2} + y^{1,2}} - \frac{\lambda}{x^{,2}} \geq \frac{x}{x^{,2} + x^{,2}} - \frac{\lambda}{x^{,2}} \geq \frac{1}{2x}\left(1 - \frac{2\lambda}{x}\right) \geq 0.
$$

3) Now estimate the third summand of (2-1)

(2-2)

$$
\frac{\left\{ \begin{array}{l} (\lambda \sum_k (x^{k,2} - y^{k,2}) - \sum_k P^k y - P^1 y)(x^{,2} - y^{1,2} - 2y^1) \\ \qquad + (2\lambda \sum_k x^k y^k + \sum_k P^k x + Px) 2x(y^1 + 1) \end{array} \right\}}{(x^{,2} - y^{1,2} - 2y^1)^2 + 4x^{,2}(y^1 + 1)^2}
$$

$$
- \lambda \frac{\sum_k |z^k|^2 + y^{1,2}}{x^{,2}}
$$

$$= \frac{\left\{\begin{array}{l} (\lambda\sum_k(x^{k,2} - y^{k,2}) - \sum_k P^k y - P^1 y)(x^{,2} - y^{1,2} - 2y^1)x^{,2} \\ \quad + (2\lambda\sum_k x^k y^k + \sum_k P^k x + Px)2x^{,3}(y^1+1) \\ \quad - \left((x^{,2} - y^{1,2} - 2y^1)^2 + 4x^{,2}(y^1+1)^2\right)\left(\lambda\sum_k |z^k|^2 + \lambda y^{1,2}\right) \end{array}\right\}}{\left((x^{,2} - y^{1,2} - 2y^1)^2 + 4x^{,2}(y^1+1)^2\right)x^{,2}}.$$

We shall examine the numerator of (2-2) by collecting powers of x.

(2-3)
$$\begin{array}{l} (\lambda\sum_k(x^{k,2} - y^{k,2}) - \sum_k P^k y - P^1 y)(x^{,2} - y^1(y^1+2))x^{,2} \\ + (2\lambda\sum_k x^k y^k + \sum_k P^k x + Px)2x^{,3}(y^1+1) \\ - \left[\begin{array}{r} \left(x^{,4} - 2x^{,2}y^1(y^1+2) + y^{1,2}(y^1+2)^2 + 4x^{,2}(y^1+1)^2\right) \\ \times\left(\lambda\sum_k |z^k|^2 + \lambda y^{1,2}\right) \end{array}\right] \end{array}$$

$$= \quad x^{,4}\left(\begin{array}{l} -\lambda\sum_k y^{k,2} - \sum_j P^j y - \frac{\lambda y^1(y^1+2)}{x^{,2}}\sum_k x^{k,2} \\ + \frac{2(y^1+1)}{x}(2\lambda\sum_k x^k y^k + \sum_k P^k x + Px) \\ - \lambda\sum_j y^{j,2} + \frac{2\lambda y^1(y^1+2)}{x^{,2}}(\sum_k x^{k,2} + \sum_j y^{j,2}) \\ - \frac{4\lambda(y^1+1)^2}{x^{,2}}(\sum_k x^{k,2} + \sum_j y^{j,2}) \end{array}\right)$$

$$+$$

$$x^{,2}\left(y^1(y^1+2)(\lambda\sum_k y^{k,2} + \sum_j P^j y) - \frac{\lambda y^{1,2}(y^1+2)^2}{x^{,2}}(\sum_k x^{k,2} + \sum_j y^{j,2})\right)$$

$$
= \quad x'^4 \begin{pmatrix} -2\lambda\sum_k y^{k,2} - \sum_k P^k y - \lambda y^{1,2} - P^1 y \\ \quad - \dfrac{\lambda y^1(y^1+2)}{x'^2}(\sum_k x^{k,2} + \sum_j y^{j,2} + 1) \\ \quad + \dfrac{2(y^1+1)}{x}(2\lambda\sum_k x^k y^k + \sum_k P^k x + Px) \\ \quad - 2\lambda\dfrac{y^{1,2}+2y^1+2}{x'^2}(\sum_k x^{k,2} + \sum_j y^{j,2} + 1) \end{pmatrix}
$$

$$
+ x'^2 \begin{pmatrix} y^1(y^1+2)(\lambda\sum_k y^{k,2} + \lambda\sum_j y^{j,2} + \sum_j P^j y) \\ \quad - \lambda\dfrac{y^{1,2}(y^1+2)^2}{x'^2}(\sum_k x^{k,2} + \sum_j y^{j,2} + 1) \\ \quad + \lambda y^1(y^1+2) + 2\lambda(y^{1,2}+2y^1+2) \end{pmatrix}
$$

$$
+ \frac{y^{1,2}(y^1+2)^2}{2} .
$$

$$
= \quad x'^4 \begin{pmatrix} -2\lambda\sum_k y^{k,2} - \sum_j P^j y - \lambda y^{1,2} \\ \quad - 3\lambda(y^{1,2}+2y^1+\tfrac{4}{3})(\rho^\alpha(z) - \sum_j y^{j,2}) \\ \quad + \dfrac{2(y^1+1)}{x}(\sum_k x^k y^k + \sum_k P^k x + Px) \end{pmatrix}
$$

$$
+ x'^2 \begin{pmatrix} y^1(y^1+2)(\lambda\sum_k y^{k,2} + \lambda\sum_j y^{j,2} + \sum_j P^j y) \\ \quad - \lambda y^{1,2}(y^1+2)^2(\rho^\alpha(z) - \sum_j y^{j,2}) \\ \quad + 2\lambda(y^{1,2}+3y^1+2) \end{pmatrix}
$$

$$
+ \lambda y^{1,2}(y^1+2)^2
$$

(2-4)

$$= \quad x'^4 \begin{pmatrix} 3\lambda(y^{1,2}+2y^1+\frac{4}{3})\sum_j y^{j,2} + \dfrac{2(y^1+1)}{x}(\sum_k x^k y^k + \sum_k P^k x + Px) \\ -3\lambda(y^{1,2}+2y^1+\frac{4}{3})\rho^\alpha(z) - 2\lambda\sum_k y^{k,2} - \sum_j P^j y - \lambda y^{1,2} \end{pmatrix}$$

$$+$$

$$x'^2 \begin{pmatrix} y^1(y^1+2)(\lambda\sum_k y^{k,2} + \lambda\sum_j y^{j,2} + \sum_j P^j y) + \lambda y^{1,2}(y^1+2)^2\sum_j y^{j,2} \\ -\lambda y^{1,2}(y^1+2)^2\rho^\alpha(z) + 2\lambda(\dfrac{3}{2}y^{1,2} + 3y^1 + 2) \end{pmatrix}$$

$$+ \lambda y^{1,2}(y^1+2)^2$$

Let c_4 denote the coefficient of x'^4, c_0 the coefficient of x'^2, and let c_0 be the last term of (2-4). By assuming α is sufficiently small, a computation shows

a) $c_0 \geq 0$

b) $c_2 > 0$ since $|y^1| < \sqrt{\alpha}$ when $\rho^\alpha(z) < \alpha/2$.

c) On the other hand c_4 may be less than 0. However together with the term computed in °(2) above, we see that for small α this term is $o(1/x)$. Thus the 2nd and 3rd summands of (2-1) together are greater than 0 for small α.

Hence if $\psi_a(z) > 0$ then

$$a > \tfrac{1}{2}\rho^\alpha(z), \quad \text{when } z \in \{\rho^\alpha \leq \tfrac{\alpha}{2}\}.$$

This shows

$$\text{(2-5)} \qquad (K^1 - K^2_a) \cap \mathbb{C}^n \subseteq \{\psi_a > 0\} \cap \{\rho^\alpha < 2a\} \subseteq W_a.$$

Now let $z_* \in (K^1 - K^2_a) - \mathbb{C}^n$. $\exists z_n \in G \cap \{\rho^\alpha \leq \frac{\alpha}{2}\} - K^2_a$ such that $z_n \to z_*$ in $\widehat{\mathbb{C}^n}$. By corollary 3.2.9, $z_* \notin K^2_a$ implies there is a (conic) neighbourhood

$\Gamma \ni z_*$ such that $\psi_a(\Gamma \cap \mathbb{C}^n) > 0$. Then for $z_* \in \Gamma' \subset\subset \Gamma$, $\exists \epsilon > 0$ such that $\psi_{a-\epsilon}(\Gamma' \cap \mathbb{C}^n) > 0$ (lemma 3.3.6).

Now $z_n \in \Gamma'$ for large n, and by (2-5), lemmas 3.3.3 and 3.3.6, is contained in

$$\begin{aligned} cl_{\mathbb{C}^n}\{\psi_{a-\epsilon} > 0\} \cap \{\rho^\alpha < 2(a-\epsilon)\} \\ &= cl_{\widehat{\mathbb{C}^n}}\, int_{\widehat{\mathbb{C}^n}}\, cl_{\widehat{\mathbb{C}^n}}\{\psi_{a-\epsilon} > 0\} \cap \{\rho^\alpha < 2(a-\epsilon)\} \\ &= cl_{\widehat{\mathbb{C}^n}} W_{a-\epsilon} \\ &\subset W_a \qquad \text{(lemmas 3.3.3, 3.3.6)} \end{aligned}$$

So $K^1 - K^2_a \subseteq W_a$; and since $K^1 - K^2_a \subseteq G$, $K^1 - K^2_a \subseteq W_a \cap G$. This proves the lemma. $\square$

The two previous lemmas show

COROLLARY 4.2.10. $K^1 - K^2_a = W_a \cap G$, *for small* $a > 0$. $\square$

REMARK 4.2.11. (III.1-3) shows that if $b < 0$ then $int_{\widehat{\mathbb{C}^n}}\, cl_{\widehat{\mathbb{C}^n}}\{\psi_b < 0\}$ is a $\widehat{\mathbb{C}^n}$ neighbourhood of K^1. $\triangleright$

DEFINITION 4.2.12. Suppose α and a are given.

Let $\theta' := max(\beta\psi_a, \rho^\alpha - \frac{\alpha}{2}, P^1y, \ldots, P^ny)$, where $\beta > 0$ is chosen so small that $a - \frac{\alpha}{2\beta} < 0$.

Let $\tilde{\theta} := max(\rho^\alpha - \frac{\alpha}{2}, P^1y, \ldots, P^ny)$.

Let $U_0 := int_{\widehat{\mathbb{C}^n}}\, cl_{\widehat{\mathbb{C}^n}}\{\theta' < \frac{\alpha}{2}\}$.

Let $G := int_{\widehat{\mathbb{C}^n}}\, cl_{\widehat{\mathbb{C}^n}}\{z \in \mathbb{C}^n : P^jy \leq 0, j = 1, \ldots, n\}$. $\diamond$

REMARK 4.2.13. By remark 4.2.11, U_0 is a neighbourhood of K^1. $\triangleright$

REMARK 4.2.14. θ' is plurisubharmonic, and $\{\theta' < \frac{\alpha}{2}\} \subseteq \{\rho^\alpha < \alpha\}$. The same is true for $\tilde{\theta}$. $\triangleright$

LEMMA 4.2.15. *θ' and $\tilde{\theta}$ are uniformly continuous on $\{\rho^\alpha < r\alpha\}$.*

PROOF. Clearly each $y \mapsto P^j y$ is uniformly continuous on $\{\rho^\alpha < r\alpha\}$. By taking the derivatives of $\rho^\alpha - \frac{\alpha}{2}$, and showing that each of the partials is bounded on $\{\rho^\alpha < r\alpha\}$, one concludes that $\rho^\alpha - \frac{\alpha}{2}$ is uniformly continuous on $\{\rho^\alpha < r\alpha\}$.

Similarly the techniques of the previous chapter and those of lemma 3.3.4 in particular show that $2\psi_a$ has bounded partials on $\{\rho^\alpha < r\alpha\}$. $\square$

LEMMA 4.2.16. *$int_{\widehat{\mathbb{C}^n}}\, cl_{\widehat{\mathbb{C}^n}}\{\theta' < \epsilon\}_{0<\epsilon<\frac{a}{2}}$ is a basis of $\mathcal{P}\mathcal{O}$-pseudoconvex neighbourhoods of K_a^2.*

PROOF. First note that

$$cl_{\widehat{\mathbb{C}^n}}\{\psi_a \le 0\} \subseteq int_{\widehat{\mathbb{C}^n}}\, cl_{\widehat{\mathbb{C}^n}}\left\{\psi_a < \tfrac{\epsilon}{2}\right\}, \qquad \text{for } \epsilon > 0$$

since

$$\begin{aligned}
cl_{\widehat{\mathbb{C}^n}}\{\psi_a \le 0\} &= int_{\widehat{\mathbb{C}^n}}\left(\{\psi_a \le 0\}^c\right)^c \\
&= \left(int_{\widehat{\mathbb{C}^n}}\, cl_{\widehat{\mathbb{C}^n}}\{\psi_a > 0\}\right)^c \\
&= \left(\bigcup_{d>0} int_{\widehat{\mathbb{C}^n}}\, cl_{\widehat{\mathbb{C}^n}}\{\psi_a > d\}\right)^c, \qquad \text{by lemma 3.3.5} \\
&= \bigcap_{d>0} cl_{\widehat{\mathbb{C}^n}}\, int_{\widehat{\mathbb{C}^n}}\left(\{\psi_a \le d\} \cup \mathbb{S}_{n-1}\infty + i\mathbb{R}^n\right) \\
&= \bigcap_{d>0} cl_{\widehat{\mathbb{C}^n}}\, int_{\widehat{\mathbb{C}^n}}\, cl_{\widehat{\mathbb{C}^n}}\{\psi_a < d\} \\
&= \bigcap_{d>0} cl_{\widehat{\mathbb{C}^n}}\{\psi_a < d\} \\
&= \bigcap_{d>0} int_{\widehat{\mathbb{C}^n}}\, cl_{\widehat{\mathbb{C}^n}}\{\psi_a < d\}.
\end{aligned}$$

In particular one has

$$cl_{\widehat{\mathbb{C}^n}}\{\psi_a \le 0\} = \bigcap_{\epsilon>0} int_{\widehat{\mathbb{C}^n}}\, cl_{\widehat{\mathbb{C}^n}}\left\{\psi_a < \tfrac{\epsilon}{2}\right\}.$$

Similarly

$$cl_{\widehat{\mathbb{C}^n}}\{y^j \le 0\} = \bigcap_{\epsilon>0} int_{\widehat{\mathbb{C}^n}}\, cl_{\widehat{\mathbb{C}^n}}\{y^j < \epsilon\};$$
$$cl_{\widehat{\mathbb{C}^n}}\left\{\rho^\alpha \le \tfrac{\alpha}{2}\right\} = \bigcap_{\epsilon>0} int_{\widehat{\mathbb{C}^n}}\, cl_{\widehat{\mathbb{C}^n}}\left\{\rho^\alpha < \tfrac{\alpha}{2} + \epsilon\right\}.$$

Moreover each of the sets in the intersection on the right hand side is a neighbourhood of the corresponding set on the left.

Note that

$$\begin{aligned} K_a^2 &= cl_{\widehat{\mathbb{C}^n}}\{\theta' \le 0\} \\ &= cl_{\widehat{\mathbb{C}^n}}\{\psi_a \le 0\} \cap cl_{\widehat{\mathbb{C}^n}}\left\{\rho^\alpha \le \tfrac{\alpha}{2}\right\} \cap \bigcap_j cl_{\widehat{\mathbb{C}^n}}\{y^j \le 0\}. \end{aligned}$$

So

$$\begin{aligned} int_{\widehat{\mathbb{C}^n}}\, cl_{\widehat{\mathbb{C}^n}}\left\{\psi_a < \tfrac{\epsilon}{2}\right\} \cap int_{\widehat{\mathbb{C}^n}}\, cl_{\widehat{\mathbb{C}^n}}\left\{\rho^\alpha < \tfrac{\alpha}{2} + \epsilon\right\} \cap \bigcap_j int_{\widehat{\mathbb{C}^n}}\, cl_{\widehat{\mathbb{C}^n}}\{y^j < \epsilon\} \\ = int_{\widehat{\mathbb{C}^n}}\, cl_{\widehat{\mathbb{C}^n}}\{\theta' < \epsilon\} \end{aligned}$$

is a neighbourhood of K_a^2. In fact these form a fundamental system of neighbourhoods of K_a^2 since K_a^2 is compact and $\widehat{\mathbb{C}^n}$ is metrizable. These sets are relatively compact for each ϵ and tend to K_a^2 as ϵ tends to 0. Moreover these sets are ${}^{n}\mathcal{O}$-pseudoconvex: by lemma 4.2.15, θ' is uniformly continuous on compact subsets; then consider $(\epsilon - \theta')^{-1}$; finally smooth these according to proposition 4.1.2.

This proves the lemma. □

Similarly one has

LEMMA 4.2.17. *$int_{\widehat{\mathbb{C}^n}}\, cl_{\widehat{\mathbb{C}^n}}\{\tilde{\theta} < \epsilon\}_{0<\epsilon\ll 1}$ is a basis of ${}^{n}\mathcal{O}$-pseudoconvex neighbourhoods of K^1.* □

THEOREM 4.2.18. *$\mathscr{H}^k_{\mathbb{S}^*\Omega}(\pi^{-1}\mathscr{P}\mathscr{O}) = 0$, for $k \neq n$.*

PROOF. By proposition 1.4.12 this is equivalent to showing

$$\varinjlim_{\substack{V \ni x_0 \\ G'}} H^k_{V \cap G'}(V; \mathscr{P}\mathscr{O}) = 0, \qquad \text{for } k \neq n.$$

If $x_0 \in \mathbb{C}^n$ then this reduces the usual result about microfunctions on $\mathbb{C}^n$ (scholium 4.3.2 below).

Suppose first that $x_0 = (1, 0, \ldots, 0)\infty \in \mathbb{D}^n - \mathbb{R}^n$, and let $G' = G$ (definition 4.2.12). Since $\{W_a\}_{0<a\ll 1}$ (a outside a set of measure 0) form a basis of $\widehat{\mathbb{C}^n}$-neighbourhoods of x_0 by lemma 4.2.3, V can be taken to be the W_a. But then corollary 4.2.10 gives

$$\varinjlim_{\substack{W_a \ni x_0 \\ G}} H^k_{W_a \cap G}(V; \mathscr{P}\mathscr{O}) = \varinjlim_{a} H^k_{K^1 - K^2_a}(\widehat{\mathbb{C}^n}; \mathscr{P}\mathscr{O}).$$

Thus the theorem will follow if the conditions of theorem 2.4.8 hold. We proceed to show this next.

Let θ' and U_0 be as in definition 4.2.12. Then recall that

$$K^2_a = cl_{\widehat{\mathbb{C}^n}}\{\theta' \leq 0\}; \qquad K^2_a \cap \mathbb{C}^n = \{\theta' \leq 0\}.$$

By lemma 4.2.16, given a $\widehat{\mathbb{C}^n}$ neighbourhood V of K^2_a

$$\begin{aligned} \exists \epsilon_V > 0, \tfrac{\alpha}{2} > \epsilon_V \text{ such that } K^2_a &\subseteq int_{\widehat{\mathbb{C}^n}}\, cl_{\widehat{\mathbb{C}^n}}\{\theta' - \epsilon_V < 0\} \\ &\subset\subset int_{\widehat{\mathbb{C}^n}}\, cl_{\widehat{\mathbb{C}^n}}\left\{\theta' - \tfrac{\epsilon_V}{2} < 0\right\} \\ &\subseteq V. \end{aligned}$$

Now let $\chi_V(\cdot)$ be a convex increasing function such that

(1) χ_V is uniformly continuous on $\{t : t \leq d\}$ for every $d \in \mathbb{R}$;

(2) $\lim_{t\to\frac{\alpha}{2}} \chi_V(t) = \infty$;

(3) $\chi_V\left(\frac{\epsilon_V}{2}\right) < 0$; and

(4) $\chi_V \circ \theta' > 0$ on V^c.

Then $\chi_V \circ \theta'$ is a plurisubharmonic exhaustion function of U_0 satisfying the conditions of proposition 4.1.2. Thus it can be smoothed to produce an $\mathscr{H}$-pseudoconvex exhaustion function, θ_V, of U_0. θ_V satisfies the requirements of theorem 2.3.8.

For general $x_0 \in \mathbb{D}^n - \mathbb{R}^n$, take a unitary transformation R mapping x_0 to $(1, 0, \ldots, 0)\infty$. Modify the functions ρ^α and ψ_a as follows:

$$\rho^\alpha(z) = \frac{\sum_k |(Rz)^k|^2 + (Ry)^{1,2}}{\left((Rx)^1 - 1/\alpha\right)^2} + \sum_j |y^j|^2 + \frac{1}{\left((Rx)^1 - 1/\alpha\right)^2};$$

$$\Psi_a(z) = ia + \sum_j P^j z - \frac{\lambda i \sum_k (Rz)^{k,2} - \sum_k P^k(Rz) - \left(P^1(Rz) - 1/\alpha\right)}{\left((Rz)^1 - 1/\alpha\right)^2 + 2i((Rz)^1 - 1/\alpha)} - \frac{i}{(Rz)^1 - 1/\alpha};$$

$$\begin{aligned}\psi_a(z) &= \mathfrak{Im}\Psi_a(z) \\ &= a - \frac{(Rx)^1 - 1/\alpha}{\left((Rx)^1 - 1/\alpha\right)^2 + (Ry)^{1,2}} + \sum_j P^j y \\ &\quad - \frac{\left\{\begin{array}{l}\left[\begin{array}{l}\left(\lambda\sum_k \left((Rx)^{k,2} - (Ry)^{k,2}\right) - \sum_j P^j(Ry)\right) \\ \qquad \times \left(\left((Rx)^1 - 1/\alpha\right)^2 - (Ry)^{1,2} - 2(Ry)^1\right)\end{array}\right] \\ + \\ \left[\begin{array}{l}(2\lambda\sum_k (Rx)^k (Ry)^k + \sum_k P^k(Rx) + \left(P^1(Rx) - 1/\alpha\right)) \\ \qquad \times 2x(y^1 + 1)\end{array}\right]\end{array}\right\}}{\left[\begin{array}{l}\left(\left((Rx)^1 - 1/\alpha\right)^2 - (Ry)^{1,2} - 2(Ry)^1\right)^2 \\ \qquad + 4\left((Rx)^1 - 1/\alpha\right)^2 \left((Ry)^1 + 1\right)^2\end{array}\right]}.\end{aligned}$$

Calculations similar to those in chapter 3 and in lemma 4.2.9 show that lemma 4.2.10 still holds.

This proves the theorem. □

§4.3 Fundamental Exact Sequences

SCHOLIUM 4.3.1. *Let U be an open subset of a topological space X. Let Z be a locally closed subset of a topological space U and V an open subset of U containing Z as a closed set. For a sheaf $\mathscr{F}$ on X, $H^k_Z(V;\mathscr{F}) = H^k_Z(V;\mathscr{F}|_U)$.*

PROOF. In fact take a flabby resolution $\mathscr{L}^\bullet$ of $\mathscr{F}$. Then since U is open $\mathscr{L}^\bullet|_U$ is a flabby resolution of $\mathscr{F}|_U$. Moreover $\Gamma_Z(V;\mathscr{F}) = \Gamma_Z(V;F|_U)$ and similarly $\Gamma_Z(V;\mathscr{L}^\bullet) = \Gamma_Z(V;\mathscr{L}^\bullet|_U)$. □

SCHOLIUM 4.3.2. ${}^p\mathscr{B}_\Omega|_{\mathbb{R}^n} = {}^p\mathscr{B}_{\Omega\cap\mathbb{R}^n}$.

PROOF. This follows from the previous scholium since ${}^p\mathscr{B}_\Omega(U) = H^n_{\Omega\cap U}(U;{}^p\mathscr{O})$, and ${}^p\mathscr{O}|_{\mathbb{C}^n} = \mathscr{O}$. □

SCHOLIUM 4.3.3. ${}^p\mathscr{C}_\Omega|_{(\Omega\cap\mathbb{R}^n)+\sqrt{-1}\mathbb{S}_{n-1}\infty} = \mathscr{C}_{\Omega\cap\mathbb{R}^n}$.

PROOF. Let $x_* + i\xi_*\infty \in (\Omega\cap\mathbb{R}^n) + \sqrt{-1}\mathbb{S}_{n-1}\infty$. Then

$$\mathscr{H}^k_{\mathbb{S}^*\Omega}(\pi^{-1}\,{}^p\mathscr{O})_{x_*+i\xi_*\infty} \xleftarrow{\sim} \varinjlim_{\substack{V\ni x_0 \\ G}} H^k_{V\cap G}(V;{}^p\mathscr{O}), \tag{4-1}$$

But since $x_* \in \mathbb{R}^n$, V runs through bounded of $\mathbb{C}^n$. Thus again by scholium 4.3.1, (4-1) becomes the usual limit for microfunctions on $\Omega\cup\mathbb{R}^n$. □

We have now computed all the terms in the triangle (I.3-5) of proposition 1.3.9. We suppose as always the conditions on the plurisubharmonic function p stated in Chapter II.

Recall that $\mathbf{R}\Gamma_\Omega({}^p\mathscr{O})[n]$ vanishes except in degree 0; it is the sheaf of Fourier p-hyperfunctions, ${}^p\mathscr{B}_\Omega$, and ${}^p\mathscr{A}$ vanishes except in degree 0. From the long exact sequence associated to the triangle in proposition 1.4.9 it follows that $\mathsf{R}^j\pi_*{}^p\mathscr{C}_\Omega = 0$, for $j \neq 0$. From theorem 4.2.7 we now have the stronger result that ${}^p\mathscr{C}$ is concentrated in degree 0. Thus there is

THEOREM 4.3.4. *Let Ω be an open subset of $\mathbb{D}^n$. There there is a short exact sequence*

$$0 \longrightarrow {}^{p}\mathscr{A}_{\Omega} \longrightarrow {}^{p}\mathscr{B}_{\Omega} \longrightarrow \pi_{*}{}^{p}\mathscr{C}_{\Omega} \longrightarrow 0$$

PROOF. Take the long exact sequence of triangle (I.3-5), and use theorem 4.2.7. □

Similarly one can produce the other short exact sequences involving ${}^{p}\mathscr{C}_{\Omega}$ as in Sato, Kawai & Kashiwara [1973].

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