ABSTRACT

Title of dissertation: SPECTRUM AUCTIONS FOR DYNAMIC SPECTRUM ACCESS NETWORKS

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We examine the problem of designing an auction-based market mechanism for dynamic spectrum sharing when there are multiple sellers and multiple buyers. We assume that the sellers are selfish players and focus on an optimal auction mechanism that maximizes the expected payoff or profit of the seller. First, we study the interaction among homogeneous buyers of the spectrum as a noncooperative game and show the existence of a symmetric mixed strategy Nash equilibrium (SMSNE). We investigate the uniqueness of the SMSNE in some special cases and discuss the convergence to the unique SMSNE. Second, we prove that there exists an incentive for risk neutral sellers of the spectrum to cooperate in order to maximize their expected profits at the SMSNEs of buyers' noncooperative game. This is done by modeling the interaction among the sellers as a cooperative game and demonstrating that the core of the cooperative game is nonempty. We show that there exists a way for the sellers to share the profits in a such manner that no subset of sellers will have an incentive to deviate or power to increase their expected profits by deviating. We also introduce the algorithms for achieving any profit sharing in the core. Finally, we

introduce an optimal auction mechanism in which the spectrum bands in multiple regions are sold simultaneously and the buyers are simple-minded in the sense that each buyer wants to buy the same number of frequency bands only the regions where they operate.

SPECTRUM AUCTIONS FOR DYNAMIC SPECTRUM ACCESS NETWORKS

by

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BM	Branco's Mechanism
CAB	Coordinated Access Band
CDMA	Code Division Multiple Access
CR	Cognitive Radio
FCC	Federal Communications Commission
FDD	Frequency Division Duplex
FDMA	Frequency Division Multiple Access
GBM	Generalized Branco's Mechanism
GSM	Global System for Mobile communications
MAC	Medium Access Control
MNO	Mobile Network Operator
MSNE	Mixed Strategy Nash Equilibrium
MVNO	Mobile Virtual Network Operator
PCS	Personal Communication Services
PSP	Primary Service Provider
QoS	Quality of Service
RA	Radio Access
SINR	Signal-to-Interference plus Noise Ratio
SMSNE	Symmetric Mixed Strategy Nash Equilibrium
SSP	Secondary Service Provider
UMTS	Universal Mobile Telecommunications System
UWB	Ultra-WideBand
VCG	Vickrey-Clarke-Groves
WCDMA	Wideband Code Division Multiple Access

Chapter 1

Introduction

A conventional way of managing available frequency spectrum is a static allocation to a set of users, where each user receives dedicated spectrum. In many countries, a government agency (e.g., the Federal Communications Commission (FCC) in the U.S. [1]) bears the responsibility to plan, allocate, and manage the spectrum. Unfortunately, this static assignment of available spectrum leads to several drawbacks. First, it hampers the entrance of a new service provider. Secondly, recent studies [2, 23, 24, 46, 63] suggest that much of the assigned spectrum is under-utilized in many places. Thus, a natural question that arises is: "How can we increase the frequency usage efficiency?"

There are several new approaches put forth to address this issue. One approach to increasing the spectrum utilization in cellular frequency bands introduces a new class of service providers called Mobile Virtual Network Operators (MVNOs). An MVNO is an operator that provides mobile communication services without its own *licensed* spectrum and necessary infrastructure. In order to provide the services, they have business agreements with Mobile Network Operators (MNOs) to use the frequency spectrum and some of infrastructure owned by the MNOs. In the U.S., Virgin Mobile has successfully launched its service with Sprint Nextel as its MNO.

Another approach to more flexible use of spectrum is based on Cognitive Radio

(CR) [49], which is being considered as a candidate for a new frequency management scheme by the FCC [3]. The CR is based on software-defined radio technology; it allows a CR user to switch its radio access (RA) technology based on the availability and/or performance of available networks. As a result, in principle a CR user can utilize *any* frequency band by adopting a suitable RA technology. CR users, however, should not interfere with *licensed* users, also called *primary* users, that paid for the spectrum.

Several solutions are proposed for ensuring that CR users do not interfere with licensed users: Under a spectrum rental protocol [50] the owner advertises the frequency bands for rent, and a renter (i.e., a CR user) may express interest. Another solution is spectrum sensing; CR users continually scan the spectrum to find an idle frequency band, called *spectrum hole*. The CR users can utilize the idle frequency band until an activity by a primary user is detected, at which point the CR users must relinquish the band. In [14], a framework is proposed to use unlicensed band under the assumption that the physical layer has the capability to detect primary users' activities. Mishra et al. [48] showed that a cooperative sensing could reduce the sensitivity requirements on an individual CR. A third approach is based on an interference metric called *interference temperature* [2]. Under the proposed solution, a CR user can make use of a frequency band as long as the interference level at *every* primary user's receiver remains below a certain threshold. In [16], new physical and MAC layer protocols are proposed for CR using the interference temperature model. Instead of the interference temperature model, the maximum rate of collisions can also be used as a constraint [60].

There are several existing studies on dynamic spectrum sharing between primary users and unlicensed secondary users: Mutlu et al. [51] investigated an efficient pricing policy of an MNO for secondary spectrum usage of MVNOs in the presence of both primary and secondary users. Wang et al. [64] proposed a novel joint power/channel allocation scheme to improve the network's performance by modeling the spectrum allocation problem as a noncooperative game among the CR users. Etkin et al. suggested a repeated game approach to enforce an efficient and fair outcome and incentive compatible spectrum sharing [25]. In [53] the channel allocation problem in a CR network was formulated as a potential game that has provable convergence to a Nash equilibrium. The interference temperature model is applied in an auction-based spectrum sharing mechanism in [34]. Bae et al. [9] proposed a sequential auction mechanism for sharing spectrum and power among competing transmitters.

These recently proposed solutions have the potential to improve the spectrum usage by filling spectrum holes without interfering with the services of primary users. However, they also suffer from several drawbacks that have not been addressed effectively. First, since the MVNOs share the infrastructure with the MNOs, MVNOs are often forced to employ the same RA technologies. This subordinate relationship limits the set of services the MVNOs can provide to their customers.[1]

Second, most of existing studies on CR focus primarily on the resource allocation among the secondary users and often assume that the secondary users can use

[1]For instance, MVNOs may not provide a different set of data rates since it is mandatory to follow a specifications the MNO employs.

the spectrum free of charge. This may be reasonable if the owner or licensee is a government agency that is interested in maximizing social welfare or if the spectrum is set aside for research purposes. However, in many cases, the frequency spectrum is allocated for commercial use and primary service providers (PSPs) have paid for the exclusive right to the spectrum. In such a scenario, it may be unrealistic to assume that the PSPs will share their spectrum without charging for the use, even when the secondary users do not interfere with the services to their customers. Hence, it is more reasonable to assume that the unlicensed users will have to pay for access to licensed spectrum in this case.

Third, when there is no centralized authority, individual unlicensed users may access under-utilized frequency bands in a distributed, unorganized manner. The gain in spectrum utilization from such unorganized access, however, may be limited. We suspect that introducing *secondary service providers* (SSPs) that can grant access to under-utilized spectrum in a more organized manner, by leveraging, for instance, CR users, may present a better recourse.

A well designed spectrum sharing and pricing scheme between PSPs and SSPs will encourage and facilitate sharing of spectrum in a more dynamic and flexible fashion. This is the scenario we consider in this dissertation. We assume that there are (i) SSPs whose infrastructure and customers' equipments have the capability for dynamic spectrum access (e.g., CR) and (ii) PSPs that wish to lend their surplus frequency spectrum according to a contract with the SSPs. This is shown in Figure 1.1. Our setting is also applicable to the spectrum trading between PSPs (e.g., [12]).

Realizing dynamic sharing of under-utilized spectrum between PSPs and SSPs

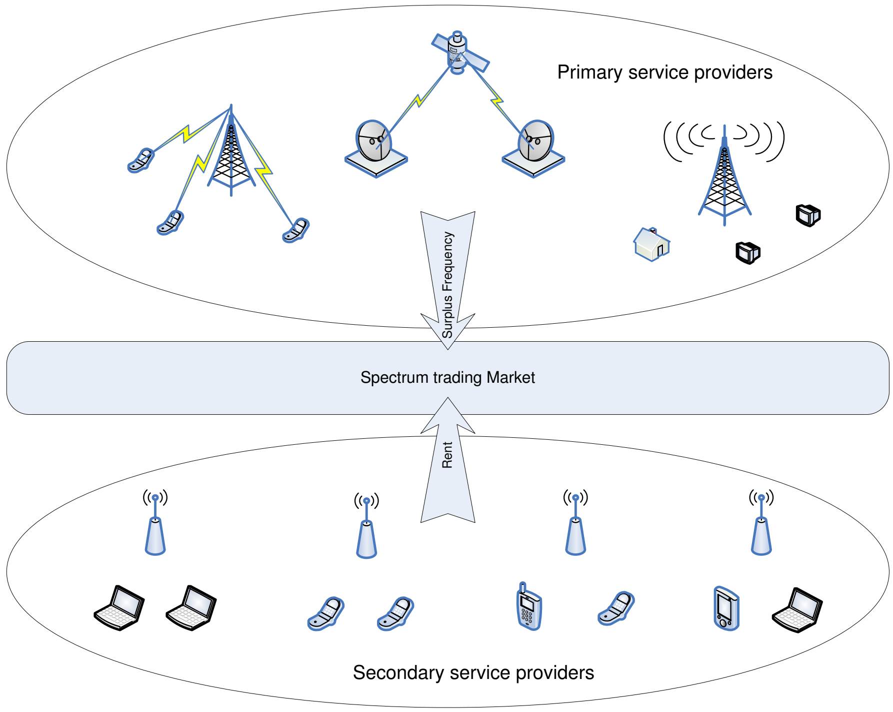

Figure 1.1: Dynamic spectrum sharing market.

calls for a new spectrum trading mechanism. To share or trade the goods, i.e., the under-utilized spectrum, a (*virtual*) market is formed and transactions take place in the market.

The mechanism design theory provides a coherent framework for analyzing various types of *allocation mechanisms*, with a focus on the problems associated with incentives and private information [4]. Thus, the mechanism design deals with how the participants' information is presented and how decisions are made with consideration of individual preferences so that the outcome is acceptable to all self-interested participants. Even though the application of the mechanism design theory is not limited to auction mechanisms, we focus on auction mechanisms in this dissertation.[2]

[2]Reference [4] gives a few possible applications such as regulation, auditing, and social choice problem.

There are many examples in which mechanism design is used in practice, especially in allocation of radio spectrum. For instance, the FCC auctioned thousands of the Personal Communication Services (PCS) licenses and other licenses each of which grants the exclusive right to use particular radio spectrum over a geographic area from July 1994 to May 1996. The employed mechanism was designed between August 1993 and March 1994, and the final mechanism adopted was a simultaneous multiple-round auction. The mechanism was similar to an ascending-bid *English* auction. However, in each round of the auction, buyers could bid on any of the offered licenses simultaneously. A detailed analysis of the six auctions conducted by the FCC is given in [19]. Similarly, the licenses for radio spectrum in many countries have been sold by auction mechanisms. For example, in April 2000, the licenses for Turkish GSM 1800 MHz bands were awarded by using a sequential first price sealed bid auction.[3] In many European countries, e.g., UK, Netherlands, Italy, Germany, Switzerland, and Austria, a spectrum for the third generation system (Universal Mobile Telecommunications System, UMTS) was auctioned in 2000. Except for Germany and Austria, they employed the same simultaneous multiple-round auction mechanism which was used in the U.S. [30].

In the system we consider, both PSPs who have goods for sale, i.e., surplus frequency bands, and SSPs who want to buy the goods and are willing to pay for them participate in a trading market. In order to trade the surplus spectrum between PSPs and SSPs, they need a means of exchanging the information and defining allocation and payment schemes. To ensure desirable outcomes, we need to analyze

[3] The auction mechanism and results are analyzed in [22].

the strategies of participating PSPs and SSPs. The mechanism design provides a suitable tool. In this dissertation, we propose an auction-based framework for devising such a mechanism with PSPs and SSPs as sellers and buyers, respectively. PSPs hold auction(s) to lend their exclusive rights for surplus frequency bands to SSPs for specified time, e.g., 30 minutes. The SSPs pay for this period and the PSPs do not use the bands during the period. The allocation and payment mechanism is chosen or defined by PSPs. PSPs are free to form any coalition among themselves and the members of a coalition share their information and hold a single auction. Each SSP participates in one of the auctions and submits its bids. The items, i.e., frequency bands, are allocated based on the submitted bids, and payments are determined according to the predefined rule. More details will be provided in Chapter 4.

There exists much work available in the literature on auction theory (a brief summary is provided in Chapter 2). However, most of it focuses on efficient allocation of items, i.e., maximization of social welfare. Throughout the dissertation, we take the viewpoint that PSPs are private entities that are interested in their own profits or revenue (rather than social welfare). Hence, we focus on *optimal* auction mechanisms that maximize sellers' profits.

The rest of the dissertation is organized as follows: Chapter 2 presents the basic auction mechanisms and some important concepts in the auction mechanism design. Several existing auction-based dynamic spectrum sharing schemes are discussed in Chapter 3. Chapter 4 introduces the model and the proposed problems with the summary of results. The optimal mechanism we assume the sellers adopt for allocating and pricing the spectrum bands is developed in Chapter 5. The noncooperative game among the buyers is studied in Chapter 6. Chapter 7 demonstrates the existence of an incentive for cooperation among the sellers (section 7.1) and the nonempty *core* (section 7.2), followed by our proposed profit sharing mechanisms. Chapter 8 introduces an optimal auction mechanism that maximizes the auctioneer's expected payoff when the frequency bands in multiple areas are sold in the same auction. Finally, concluding remarks are provided in Chapter 9.

Note on notations: Throughout the dissertation, we denote the expectation with respect to a random variable X by $\mathbf{E}_X[\cdot]$. Similarly, the expectation with respect to a random vector $\mathbf{X}$ is denoted by $\mathbf{E}_{\mathbf{X}}[\cdot]$. We denote the set of real numbers (resp. nonnegative real numbers) by $\mathbb{R}$ (resp. $\mathbb{R}_+$). Similarly, the set of nonnegative integer is given by $\mathbb{Z}_+ := \{0, 1, 2, \ldots\}$. In this dissertation, since PSPs may form any coalition among themselves and hold a single auction, we need to distinguish between seller(s) of the frequency bands and seller of the auction. We will use *seller(s)* to denote PSP(s), and *auctioneer* refers to the seller of an auction. Hence, when a PSP holds a separate auction it is a seller and also an auctioneer. When some PSPs form a coalition and hold a single auction, however, they have one auctioneer that represents all the sellers in the coalition and conducts the auction.

Chapter 2

Literature survey of auction mechanisms

In this chapter, we briefly introduce some of basic concepts and existing auction mechanisms in the literature and explain their limitations. For this chapter we define $\mathcal{S} = \{1, 2, \ldots, N\}$ to be the set of buyers and assume that there are $m,\ m \geq 1$, items for sale. For example, in a single unit auction, $m = 1$.

2.1 Values and bids

In general the auctioneer or the seller at an auction may not know how much the buyers are willing to pay for the item(s), which are decided by what are called the *values* of the buyers. In an auction mechanism, there is a process through which each buyer communicates its willingness to pay for the item(s), called *bids*. From the received bids, the auctioneer attempts to estimate the buyers' value(s) for the item(s).

In the mechanisms we will deal with, (assuming multiple items for sale) the values of the buyers are determined by buyers' *types*, denoted by $\{t_j;\ j \in \mathcal{S}\}$. In general cases, buyer j's values may depend not only on its type t_j, but also on those of other buyers. However, we assume that the values of buyer j is determined only by t_j throughout this dissertation, unless stated otherwise.[1]

[1]Inter-dependent values are considered only in section 2.3.2.

Each buyer j's type is modeled using a continuous random variable T_j with some support $\mathcal{T}_j$.[2] In general, the type of each buyer is assumed to be private information. Hence, each buyer's values for the items are known only to the buyer at the beginning.

Since each buyer's willingness to pay depends on how much it values the items, we describe the willingness to pay or bids of each buyer as a function of its type. For each $j \in \mathcal{S}$, define $\hat{\mathcal{B}}_j$ to be a set of possible bids of buyer j. Let $\hat{\mathcal{B}} = (\hat{\mathcal{B}}_j; j \in \mathcal{S})$. Then, we can define the bidding function $\hat{\beta}_j : \mathcal{T}_j \to \hat{\mathcal{B}}_j$ for each buyer j.[3] Here, $\hat{\beta}_j$ represents a strategy of buyer j in the auction and determines its bids given its type. Auction mechanism design deals with the strategies of the buyers, i.e., bidding strategies, and the allocation and pricing schemes.

2.2 Direct mechanism and revelation principle

Definition 1. A mechanism is called a *direct mechanism* if the bid of every buyer is its type.

It is clear from the definition that the only action required of each buyer in a direct mechanism is to report its type. However, the reported type is not necessarily its true type. If there is an equilibrium in which every buyer reveals its true type, then

[2] In a multiple item auction, t_j may be either a scalar or a vector. In the first case, the buyer's value for each item will be given by some function of t_j. In the latter case, the vector may contain the values themselves. We adopt the first case in this dissertation.

[3] Again, in more general cases where the values of buyer j depend on the types of other buyers as well, the bidding function $\hat{\beta}_j$ will be a mapping from $\mathcal{T}$ to $\hat{\mathcal{B}}_j$.

the direct mechanism is said to have a truthful equilibrium. Let $U_j(t_j^*; t_j)$ be the (*expected*) payoff of buyer j when its reported type is t_j^* and its true type is t_j.

Definition 2. *(Incentive compatibility)* A direct mechanism is said to be *incentive compatible* if

$$U_j(t_j; t_j) \geq U_j(t_j^*; t_j) \quad \textit{for all } j \in \mathcal{S} \textit{ and } t_j, t_j^* \in \mathcal{T}_j.$$

By employing an incentive compatible mechanism in an auction, an auctioneer can encourage all buyers to report their true types.

The following theorem shows that the allocation and payments from any equilibrium of any mechanism can be replicated by a truthful equilibrium of some direct mechanism.

Theorem 2.1. *[38, p.63] (Revelation principle) Given a mechanism and an equilibrium for that mechanism, there exists a direct mechanism in which (i) it is an equilibrium for each buyer to report its value truthfully and (ii) the outcomes are the same as in the given equilibrium of the original mechanism.*

Even though the direct mechanisms comprise only a small subclass of mechanisms in real world, due to the revelation principle, *any* mechanism can be translated to a direct mechanism that has simple structure and is easier to design and analyze. For this reason, we restrict our attention only to direct mechanisms.

Since a buyer can guarantee itself zero payoff by not participating in an auction, the expected payoff of a buyer participating in an auction should not be smaller zero.

Definition 3. *(Individual rationality)* A direct mechanism is said to be *individually rational* if

$$U_j(t_j; t_j) \geq 0 \quad \textit{for all } j \in \mathcal{S} \textit{ and } t_j \in \mathcal{T}_j.$$

2.3 Efficient and optimal mechanisms

Auction mechanisms can be categorized in various ways according to characteristics of goods or buyers, process of the auction, payment rule of the auction, goal of the auction, and so on. For instance, an auction can be classified into either a single item auction or a multiple item auction, depending on the number of items for sale. Similarly, an auction can be conducted in an open manner (e.g., oral bids) or through sealed bids. In the first case, the bids are known to all participants, whereas in the latter case the bids remain unknown unless the auctioneer announces them. First price auction, second price auction, discriminatory auction, and uniform price auction are well known auction mechanisms classified based on the payment rule. Here, we introduce two classes of auction mechanisms based on the goal of the auctioneer – efficient mechanisms and optimal mechanisms.

2.3.1 Efficient mechanisms

In mechanism design, an auction mechanism that assigns items or objects to buyers with the highest values is said to be an *efficient* mechanism [38]. In other words, efficient auction mechanisms maximize 'social welfare'. If we denote auctioneer's payoff and buyer j's payoff by u_0 and u_j, respectively, the social welfare is given by $u_0+\sum_{j\in\mathcal{S}} u_j$: Given that buyer j's type is t_j, let $v_{j,k}(t_j)$ be buyer j's value for the k-th item it receives and $c_j(t_j)$ buyer j's payment. Here, we assume that the buyer j's values for the items depend only on its type t_j and $v_{j,k}(t_j) \geq v_{j,k+1}(t_j)$ for all $t_j \in \mathcal{T}_j$ and $k = 1, 2, \ldots, m-1$. Define the indicator function

$$I_{j,k} = \begin{cases} 1 & \text{if the buyer } j \text{ wins at least } k \text{ items} \quad , \\ 0 & \text{otherwise} \quad . \end{cases}$$

Note that $\sum_{j\in\mathcal{S}} \sum_{k=1}^{m} I_{j,k} \leq m$. Then, assuming the auctioneer has zero values for the items, the buyer's and auctioneer's payoffs are given by

$$\begin{aligned} u_j &= \sum_{k=1}^{m} v_{j,k}(t_j) \times I_{j,k} - c_j(t_j) \text{ , and} \\ u_0 &= \sum_{j\in\mathcal{S}} c_j(t_j) \ . \end{aligned}$$

Since the social welfare $u_0 + \sum_{j\in\mathcal{S}} u_j = \sum_{j\in\mathcal{S}} \sum_{k=1}^{m} v_{j,k}(t_j) \times I_{j,k}$, the auctioneer should allocate m items to the buyers who have the m highest positive values in an efficient auction mechanism.

An efficient auction mechanism is often selected when the object is a public asset and the auctioneer wants to assign the object to the buyer with the highest value, even though the revenue from the auction may be less than that from some other auction mechanisms. Efficient auction mechanisms may be adopted, for instance, in government auctions.

2.3.1.1 Single unit efficient mechanism

An efficient allocation in an auction with a single item to sell assigns the item to the buyer who values the item the most. Some of well-known single-item auctions, such as Dutch auction, English auction, first price auction, and second price auction, are efficient [38].

The Dutch auction is an open descending auction. In this auction the first price is called high enough so that no buyer will buy at that price. Then, the price is gradually lowered until a buyer accepts the price. The item is sold at the given price to this buyer. In the English auction, in an opposite way to Dutch auction, the auctioneer initially calls a low price and buyers indicate their interests. The price is then gradually increased until only one of the buyers shows interest. This buyer wins the item and pays the price at which the second to last buyer drops out.

First price auction and second price auction are the sealed bid counterparts of the Dutch and English auctions. In the first price auction, each buyer submits a bid in a sealed envelope so that the bid is not known to each other. The item goes to the buyer submitting the highest bid and payment is set to the winner's bid. In the second price auction, buyers' sealed bids are collected and the item is sold to the buyer who submitted the highest bid in the same way as in the first price auction. However, the winner pays the second highest bid. The second price auction is also called Vickey auction [61].

2.3.1.2 Multiple unit efficient mechanism

When more than one homogeneous item is available for sale, a discriminatory auction, a uniform price auction, or a Vickrey-Clarke-Groves (VCG) mechanism can be used. In these auctions, buyers are allowed to put in a bid for more than one item. When m units are available for sale, the m highest bids will win the m items.

The discriminatory auction is a multi-unit extension of the first price auction; in the discriminatory auction each buyer submits m bids and auctioneer chooses the m highest bids. Each buyer pays the sum of its winning bids. The uniform price auction is the multi-unit extension of the second price auction; in the uniform price auction all units are sold at a market-clearing price which is the highest losing bid. A buyer that wins m_w units in a VCG mechanism pays the sum of the m_w highest losing bids of *the other* buyers. When there is only one unit, the VCG mechanism is also equivalent to the second price auction. An example is provided in Tables 2.1 and 2.2.

Table 2.1: Example with three buyers and 5 units for sale.

Buyer	submitted bids
Buyer 1	60, 57, 42, 35, 10
Buyer 2	51, 43, 32, 23, 15
Buyer 3	53, 39, 28, 24, 9
Winning bids	60, 57, 53, 51, 43

The multi-item auctions described above are examples of *simultaneous* auctions; the assignment is made simultaneously for all the items. An auctioneer with multiple items for sale can also employ a *sequential* auction in which the items are sold in multiple rounds of auctions. In each round a single item is made available for auction [38]. The auction used in each round is allowed to be different. For

Table 2.2: Prices paid by the buyers

Buyer	Discriminatory	Uniform price	VCG
Buyer 1	60+57 = 117	42+42 = 84	32+39 = 71
Buyer 2	51+43 = 94	42+42 = 84	39+42 = 81
Buyer 3	53	42	42

example, when two items are available, a first price auction may be used in the first round, and a second price auction may be employed in the second round.

The resulting allocations, the buyers' strategies, and the prices paid by the winners in these efficient auctions are well studied in earlier years (1960s and 1970s). However, efficient mechanisms do not necessarily maximize the revenues or profits of the sellers.

2.3.2 Optimal mechanisms

An optimal auction mechanism maximizes the seller's expected revenue or profit, subject to the incentive compatibility and individual rationality constraints. In this subsection we discuss two optimal mechanisms related to our work – one by Myerson [52] and the other by Branco [10]. As we will see in Myerson's and Branco's auction mechanisms, in an optimal auction mechanism, the buyer who values the item most may not be the winner for the item.

2.3.2.1 Myerson's optimal mechanism

In [52], Myerson proposed a general framework for studying auction mechanisms for a single item. Recall that each buyer j's type is modeled using a continuous random variable T_j. The distribution (resp. density) of T_j is denoted by $\mathcal{G}_j$ (resp. g_j). In Myerson's framework, the distribution and density functions are common knowledge. In addition, the type of buyer j, T_j, represents buyer j's value for the item and is assumed to lie in a compact interval $\mathcal{T}_j := [t_{j,\min},\ t_{j,\max}]$. Let $\mathbf{T} = (T_j;\ j \in \mathcal{S})$ be the random vector of the types of the buyers and

$$\mathcal{T} := \mathcal{T}_1 \times \mathcal{T}_2 \times \cdots \times \mathcal{T}_N.$$

For each $j \in \mathcal{S}$, define $\mathcal{T}_{-j} = \prod_{j^* \in \mathcal{S}, j^* \neq j} \mathcal{T}_{j^*}$. The auctioneer's type, which is its value of the item, is given by a continuous random variable T_0 and lies in $\mathcal{T}_0 := [t_{0,\min},\ t_{0,\max}]$.

An auction mechanism is described by a pair of functions (p, c) such that, if $\mathbf{t} \in \mathcal{T}$ is the vector of the types of the buyers,

- $p_j(\mathbf{t})$ denotes the probability that buyer j wins the item, and
- $c_j(\mathbf{t})$ is the payment of buyer j.

A broad set of mechanisms can be considered using his framework, including the case of asymmetric buyers with different distributions $\mathcal{G}_j$, $j \in \mathcal{S}$. Making use of these extensions, Myerson investigated the problem of designing an optimal auction mechanism.[4]

[4]He restricted his attention to the direct mechanisms in which each buyer reports its type.

Suppose that the auctioneer's type $T_0 = t_0$ and buyer j's type $T_j = t_j$ for all $j \in \mathcal{S}$. If the types or values of the buyers are mutually independent, the joint density function evaluated at $\mathbf{t} \in \mathcal{T}$ is given by $g(\mathbf{t}) = \prod_{j \in \mathcal{S}} g_j(t_j)$. Similarly, the joint density function of the types of all other buyers at $\mathbf{t}_{-j} = (t_{j^*}; j^* \in \mathcal{S}, j^* \neq j)$ is given by $g_{-j}(\mathbf{t}_{-j}) = \prod_{j^* \in \mathcal{S}, j^* \neq j} g_{j^*}(t_{j^*})$. The problem is said to be *regular* if the function, which is called the *virtual value* and is given by

$$\theta_j(t_j) = t_j - \frac{1 - \mathcal{G}_j(t_j)}{g_j(t_j)} ,$$

is a monotone strictly increasing function of t_j for every $j \in \mathcal{S}$.

With the regularity assumption in place, Myerson proposed a deterministic optimal mechanism: For any vector $\mathbf{t}_{-j}$, let

$$z_j(\mathbf{t}_{-j}) = \inf\{\tilde{t_j} \mid \theta_j(\tilde{t_j}) \geq t_0 \text{ and } \theta_j(\tilde{t_j}) \geq \theta_{j^*}(t_{j^*}),\ \forall j^* \neq j\} . \tag{2.1}$$

Myerson's proposed optimal allocation rule is given by

$$p_j(t_j, \mathbf{t}_{-j}) = \begin{cases} 1 & \text{if } t_j > z_j(\mathbf{t}_{-j}) , \\ 0 & \text{otherwise} , \end{cases} \tag{2.2}$$

and the payment is

$$c_j(\mathbf{t}) = \begin{cases} z_j(\mathbf{t}_{-j}) & \text{if } p_j(\mathbf{t}) = 1 , \\ 0 & \text{otherwise} . \end{cases} \tag{2.3}$$

In the Myerson's mechanism, a buyer pays only if it gets the item and the price is the smallest value that would make it a winner. The auctioneer keeps the item if the highest virtual value is smaller than auctioneer's value of the item, t_0, i.e., $t_j < \theta_j^{-1}(t_0)$ (equivalently $\theta_j(t_j) < t_0$) for all $j \in \mathcal{S}$.

In the case of independent and identically distributed values of the buyers, Myerson's mechanism becomes a modified Vickey auction [61] in which the auctioneer submits a bid equal to $\theta_j^{-1}(t_0)$ (or sets the reserve price of $\theta_j^{-1}(t_0)$)[5] and then sells the item to the highest buyer at the second highest bid (or the reserve price if the second highest bid is below the reserve price). Since Myerson considered only single item cases, his work is *not* directly applicable to our problem with multiple items.

Following Myerson's work, there have been many studies on the design of optimal auction mechanisms (including multi-unit cases). A nice summary of these studies is provided in a survey paper by Zhan [67]. Unfortunately, most of these studies deal only with the case in which each buyer has a unit demand, i.e., wishes to purchase only one item.

2.3.2.2 Branco's optimal mechanism

Branco generalized Myerson's optimal auction mechanism to multi-unit auctions with multi-unit demands [10]. He used the generalized framework due to Myerson, and considered asymmetric buyers as well. In Branco's work, the auction mechanism is given by a pair (p, c), where

- $p_{j,k}(\mathbf{t})$ is the probability that buyer j wins at least k items, and
- $c_j(t_j)$ is the expected payment of buyer j.[6]

[5] In this symmetric case, all θ_j, $j \in \mathcal{S}$, are the same and the regularity assumption guarantees that θ_j is invertible.

[6] Unlike the Myerson's payment rule, $c_j(t_j)$ denotes the *expected* payment with respect to both

Suppose that the auctioneer has m units of homogeneous item. The values of each buyer j is determined by its *type*, denoted by T_j, which is private information. As defined in subsection 2.3.2.1, the distribution (resp. density) of T_j is denoted by $\mathcal{G}_j$ (resp. g_j). The type of buyer j, T_j, is assumed to lie in a compact interval $\mathcal{T}_j := [t_{j,\min},\ t_{j,\max}]$. Let $\mathbf{T} = (T_j;\ j \in \mathcal{S})$ be the random vector of the types of the buyers and

$$\mathcal{T} := \prod_{j \in \mathcal{S}} \mathcal{T}_j \ .$$

For each $j \in \mathcal{S}$, define $\mathcal{T}_{-j} := \prod_{j^* \in \mathcal{S}, j^* \neq j} \mathcal{T}_{j^*}$.

For each $k \in \{1, 2, \ldots, m\}$, let $V_{j,k} : \mathcal{T} \to \mathbb{R}_+$ be the function that determines buyer j's value for the k-th item it wins (i.e., $V_{j,k}(\mathbf{t})$ is the value buyer j has for the k-th item it receives when buyers' type vector is $\mathbf{t} \in \mathcal{T}$). The functions $V_{j,k}$ are increasing and differentiable with respect to T_j.

As a counterpart to virtual values in the Myerson's mechanism, so-called *contributions* are computed for the buyers: The contribution of buyer j for the k-th unit is given by

$$\pi_{j,k}(\mathbf{t}) = V_{j,k}(\mathbf{t}) - \frac{\partial V_{j,k}(\mathbf{t})}{\partial T_j} \frac{1 - \mathcal{G}_j(t_j)}{g_j(t_j)} \ .$$

After receiving the buyers' types, the auctioneer computes the contributions, orders them, and allocates the m units to the buyers who have the m highest positive contributions.[7] If fewer than m contributions are positive, the auctioneer keeps the remaining units.

the other buyers' types and the probability of winning a varying number of items.

[7]Although Branco assumed $t_0 = 0$ in his paper, one can easily apply non-zero auctioneer's values as will be shown later in Chapter 5.

The problem is said to be *regular* when following two conditions are satisfied. For all $j \in \mathcal{S}$, $k = 1, 2, \ldots, m$, and any given $\mathbf{t}_{-j} \in \mathcal{T}_{-j}$,

- (i) $(t_j - \tilde{t}_j)(\pi_{j,k}(t_j, \mathbf{t}_{-j}) - \pi_{j,k}(\tilde{t}_j, \mathbf{t}_{-j})) \geq 0$ for all $t_j, \tilde{t}_j \in \mathcal{T}_j$, and

- (ii) if $\pi_{j,k+1}(t_j, \mathbf{t}_{-j}) \geq 0$, then $\pi_{j,k}(t_j, \mathbf{t}_{-j}) \geq \pi_{j,k+1}(t_j, \mathbf{t}_{-j})$ for all $t_j \in \mathcal{T}_j$.

The *regularity* assumption implies that (i) the contribution is non-decreasing in its type and (ii) the nonnegative contribution is non-increasing in the number of units it receives. Thus, if we order the contributions of a certain buyer by decreasing value, the second condition guarantees that the k-th contribution precedes the $(k+1)$-th contribution. Since the auctioneer allocates the items by going down the list of ordered contributions, in the case with symmetric buyers,[8] the buyer with larger type has a higher probability of winning at least k items than another buyer with smaller type.

Denote the l-th highest contribution among all contributions from all buyers by $\pi_{(l)}(\mathbf{t})$. In the *regular* case, Branco's mechanism can be summarized as follows: For any type vector $\mathbf{t}_{-j} \in \mathcal{T}_{-j}$, let

$$T^*_{j,k}(\mathbf{t}_{-j}) = \inf\{\tilde{t}_j \in T_j \mid \pi_{j,k}(\tilde{t}_j, \mathbf{t}_{-j}) \geq \max\{0, \pi_{(m+1)}(\tilde{t}_j, \mathbf{t}_{-j})\}\} \ .$$

Branco's optimal allocation rule is

$$p_{j,k}(t_j, \mathbf{t}_{-j}) = \begin{cases} 1 & \text{if } t_j > T^*_{j,k}(\mathbf{t}_{-j}) \ , \\ 0 & \text{otherwise} \ , \end{cases} \tag{2.4}$$

[8]Buyers with the identical distribution $\mathcal{G}_j$ and the same valuation functions are said to be symmetric or homogeneous.

and the expected payment is

$$c_j(t_j) = \mathrm{E}_{T_{-j}} \left[\sum_{k=1}^{m} V_{j,k}(T^*_{j,k}(T_{-j}), T_{-j})\ p_{j,k}(t_j, T_{-j}) \right] . \tag{2.5}$$

Let $\hat{c}_{j,k}(\mathbf{t})$ be buyer j's payment for the k-th unit given the type vector $\mathbf{t}$. Although Branco showed only the expected payment given by (2.5), the payment $\hat{c}_{j,k}(\mathbf{t})$ can be shown to be

$$\hat{c}_{j,k}(\mathbf{t}) = \begin{cases} V_{j,k}(T^*_{j,k}(\mathbf{t}_{-j}), \mathbf{t}_{-j}) & \text{if } p_{j,k}(\mathbf{t}) = 1 \ , \\ 0 & \text{otherwise} \ . \end{cases} \tag{2.6}$$

In other words, for the k-th unit, the winner pays the smallest value that would win the k-th unit.

We compare the revenue[9] of the VCG mechanism and Branco's mechanism for an example. The expected revenues are given in Table 2.3, and the parameters used in the example are given in Table 2.4. As we expect, the optimal auction mechanism generates a higher expected revenue than the VCG mechanism.

Table 2.3: Expected revenue

Setting	VCG	BM
1	2.4651	2.7667
2	483.7636	524.7744

Even though Branco's work can provide a framework for designing new multi-unit auction mechanisms, it still considers only single seller cases. Since our problem may deal with multiple sellers, Branco's optimal mechanism is not directly applicable

[9]The revenue equals the sum of received payments from the buyers.

Table 2.4: Setting for average revenue computation

Setting 1		Setting 2	
Parameter	Value	Parameter	Value
# of units	5	# of units	5
Auctioneer's value for item	0 for all units	Auctioneer's value for item	0 for all units
# of buyers	10	# of buyers	10
$\mathcal{T}_j \ \ (\forall j)$	[0,1]	$\mathcal{T}_j \ \ (\forall j)$	$[0,\infty)$
$\mathcal{G}_j(t_j) \ \ (\forall j)$	t_j	$\mathcal{G}_j(t_j) \ \ (\forall j)$	$1 - \frac{1}{\lambda} e^{-\lambda t_j}$
λ	N/A	λ	$\frac{1}{100}$
$V_{j,k}(t_j) \ \ (\forall j)$	$\frac{1}{k} t_j$	$V_{j,k}(t_j) \ \ (\forall j)$	$\frac{1}{k} t_j$
# of iterations	10000	# of iterations	10000

without assuming that the sellers agree to cooperate and hold a single auction to sell all their items. Moreover, in Branco's optimal mechanism, sellers are assumed to have zero values of all units for sale. However, since the PSPs paid for their spectrum, even when some of their spectrum is underutilized, they may not be willing to lend it to SSPs without a compensation in order to recover a part of their price. Secondly, the underutilized spectrum may be needed in the future when the demand from their customers increases. Hence, when they lease out their spectrum, they run the risk of not being able to serve some of their customers resulting in a loss of revenue. For these reasons, the sellers may have positive values for the frequency bands made available for sale.

Therefore, in order to analyze proposed frequency spectrum trading system, an optimal multi-unit auction mechanism that can handle multi-unit demands of the buyers and positive reserve prices of the seller is needed. This new optimal mechanism can be designed by generalizing/extending the original Branco's mechanism, and we will describe it in Chapter 5.

Chapter 3

Related work

Recently many auction-based mechanisms for dynamic spectrum access have been proposed. In this chapter, we introduce some of them and discuss their limitations and differences with the system we propose in this dissertation.

3.1 Power allocation

When spread spectrum radio access technologies, e.g., Code Division Multiple Access (CDMA), Wideband Code Division Multiple Access (W-CDMA), or Ultra-WideBand (UWB), are employed in a system, users share the same frequency band simultaneously. In this case, a user's Quality of Service (QoS) is affected not only by received signal power and noise at its receiver but also by the interference from other users. Thus, in this system, each user may try to achieve higher Signal-to-Interference plus Noise Ratio (SINR) to receive high QoS as much as possible and, hence, each user's SINR can be viewed as its utility. As we mentioned in Chapter 1, one of solutions for ensuring that CR users do not interfere with licensed users is applying an *interference temperature* constraint so that CR users can use certain spectrum bands as long as the interference level remains under a specified threshold at the primary users. When the interference temperature constraint is employed in a dynamic spectrum sharing system, power needs to be adjusted at CR users in such a way that the total interference level at the primary users remains under the threshold. Here, the power allocation can be performed using an auction-based mechanism.

Huang et al. [34] proposed and analyzed an auction-based spectrum sharing mechanism with the interference temperature as the constraint. In order to deal with frequent arrivals and departures of users, this power allocation auction needs to be performed often. Thus, periods between auctions may be very short in this system. In their work, the primary goal of the mechanism design is to allocate power to users in a way that maximizes the sum of users' SINRs with the constraint that total interference at measured point remains below a threshold. In order to compute the SINR of each user, the seller needs to receive information of all channel gains between users, which can introduce large signaling overhead. To deal with this issue of large overhead, they developed an auction mechanism that requires less information exchange and computation. Although the proposed power allocation auction does not lead to the optimal allocation, it provides the seller with a means of achieving sub-optimal result.

Bae et al. [9] analyzed a sequential auction mechanism for sharing power among competing transmitters. Even though they did not explicitly mention, their work can be applied to power sharing among CR users. In a sequential mechanism, at each round, one unit of power, which is predefined, is sold by a second price auction. The winning buyer then increases its transmission power by one unit. Thus, if a seller has m units of power, the second price auction is executed m times. Each buyer's value for each unit available for sale is determined by a function that depends on the (current) SINR. Since a buyer's SINR depends not only on its power but also on other buyers' power, the values are interdependent. Under the assumption that every buyer's utility is known to each other, Bae et al. analyzed the auction and showed that the sequential auction may lose efficiency, i.e., a buyer who values some unit the most may not win it. It can happen, for example, in a two rounds sequential auction where a buyer who cannot win the unit in the second round may bid more aggressively in the first round than a buyer who can win the unit in the second round.

Since the buyers are end-users (usually mobile users) in the proposed power allocation mechanism for dynamic spectrum sharing, when the SINR of the users varies quickly, the auction may need to be performed very frequently (at the same timescale as the variation in the SINRs). In addition, the inter-dependency between buyers' utilities[1] tends to increase the complexity of the algorithm. Since this line of research focuses on the efficient mechanisms that maximize the social welfare, the power sharing model may not be suitable for a private and selfish seller interested in maximizing its revenue (or payoff).[2]

3.2 Spectrum allocation

In a dynamic spectrum sharing system, if underutilized spectrum bands are allocated to unlicensed users exclusively, each unlicensed user can use the allocated

[1]SINR depends on the other users' power as well.

[2]The power sharing mechanisms may be more applicable to power control in a spread spectrum system.

band(s) for an agreed time period. Thus, if the allocation is conducted by an auction, unlike power allocation, the auction can be performed less often. Additionally, the value for an item (i.e., frequency band) is not expected to depend on the other users' value. Therefore, the users' utilities are relatively easier to compute.

In [12], Buddhikot et al. introduced a conceptual model we adopt as a basic model in this dissertation. They proposed that a contiguous block of spectrum which is designated for dynamic use, called Coordinated Access Band (CAB), is managed by a *spectrum broker*. The spectrum broker divides the CAB into several sub-bands and allocates the sub-bands to the service providers who do not own any spectrum during an agreed time duration. These networks, i.e., service providers and customers, are capable of using leased spectrum by employing CR.

Based on Buddhikot's model, several studies examined how *spectrum broker* should allocate its spectrum to service providers. Here, we introduce some of them that are auction-based mechanisms: Gandhi et al. proposed a framework for spectrum auction with an interference constraint that a same frequency band is not allocated in adjacent areas to avoid conflict [27]. In the framework, a seller divides its spectrum into a large number of homogeneous channels with equal bandwidth and holds a multi-unit auction. Since the interference constraint is considered, the auction mechanism is designed for frequency bands in multiple areas. In the auction, however, each buyer can bid for only one area and a buyer who requests the frequency bands in multiple area is regarded as a different buyer in the auction. Let m_j be the number of units buyer j obtains. The per-unit price charged by the seller, denoted by $p_j(m_j)$, is given by buyer j's willingness to pay (per-unit) for m_j

units (i.e., the average per-unit value of buyer j). This implies that buyer j's total payment for received units is equal to $m_j \times p_j(m_j)$. In the study, $p_j(m_j)$ is assumed to be a continuous, concave demand curve, e.g., $p_j(m_j) = -a_j m_j + b_j,\ a_j \geq 0, b_j > 0$ for $0 \leq m_j \leq \frac{b_j}{a_j}$. Assuming that each buyer reports its demand curve *truthfully*, the seller computes the *optimal* price and the number of winning units for each buyer such that the total revenue of the seller is maximized subject to the interference constraint.

Similar research has also been done in [59]. Rather than dividing spectrum equally, in their model a seller can divide the available spectrum into some finite number of bands for each type of buyers' networks, e.g., 1.25 MHz for CDMA, 200 KHz for GSM, and 5 MHz for W-CDMA. The division is made after receiving buyers' requests, i.e., each buyer reports the type of its network and the price it is willing to pay, in a way that maximizes the total revenue under the interference constraint. In other words, the seller manages available spectrum as a whole and divides it according to the buyers' demands.

Although the VCG mechanism guarantees incentive compatibility, there is known shortcoming [7]. For instance, consider an auction of two identical items to three buyers [7]. Buyer 1 has a value of 2 for the pair of items and no value for a single item. Each buyer 2 and 3 has zero value for the pair and a value of 0.5 for a single item. In the VCG mechanism, buyer 1 should win both items, for a payment of 1, and buyer 2 and 3 should win nothing. However, suppose that buyer 2 and 3 raise their bids to 2. Then, each of buyers 2 and 3 becomes a winner and pays zero price. In [66], Wu et al. developed an auction-based mechanism that addresses the vulnerability of the VCG mechanism (e.g., the collusion between buyers 2 and 3 in the example) by sacrificing the incentive compatibility. The proposed mechanism allocates the items to the buyers who value the items most, i.e., efficient mechanism, with an interference constraint. Even though they considered multiple sellers and multiple buyers, they limited their study to a scenario where each buyer demands only one unit of frequency band and each seller has at most one frequency band for sale. They also assumed that all frequency bands of the sellers are auctioned altogether.

A mechanism that allows buyers to request frequency bands in multiple geographical areas was introduced in [36] with the constraint that the same frequency band cannot be allocated in neighboring regions. The mechanism is designed to cover the case that there is one seller who has unused frequency bands and each buyer is interested in purchasing a certain number of bands in each of a fixed set of regions. However, unless a buyer obtains the same number of frequency bands it requested in its desired areas, it receives no value for the allocated items. In the mechanism, since a partial allocation is not permitted, each buyer has a scalar value for a bundle of items, i.e., the required number of frequency bands in each of the desired regions. With the assumption that the seller knows the distribution of each buyer's value, an optimal auction mechanism that maximizes the expected revenue is devised. The proposed mechanism is based on Myerson's optimal mechanism and, as a result, *incentive compatibility* is guaranteed.

When multiple units of spectrum are available for sale and at most one unit is granted to each buyer, Sengupta and Chatterjee analyzed the bidding strategies in a sequential auction and a simultaneous auction in [58]. In each round of the sequential auction they studied, a seller employs the first price sealed bid auction and announces the lowest bid after awarding a unit of frequency band to the buyer who submitted the highest bid. The simultaneous auction is conducted by a first price sealed bid auction. With the assumption that the buyers' *bids* are uniformly distributed, they showed that the sequential auction provides more revenue for the seller than the simultaneous auction.

In [58], a new simultaneous auction mechanism was also proposed for multi-unit demand case. In the proposed auction, each buyer requests a bundle of items and a partial allocation is not permitted.[3] Authors considered two scenarios, an asynchronous auction and a synchronous auction. In the asynchronous auction, buyers can have different leasing periods from each other and request the spectrum at any time. In the synchronous auction, however, a seller restricts the buyers to have the same leasing periods and executes the auction at the beginning of each period. Through simulations, it was found that the synchronous auction is more beneficial to the seller.

In the auction mechanism design, we implicitly assume that buyers know their own value(s) for the item(s). Based on this assumption, we can analyze buyers' strategies in any auction mechanisms, e.g., first price auction and second price auction. However, one may wonder how a buyer knows its true value(s) for the item(s). Especially, when the buyers are also wireless service providers and are seeking extra frequency bands, the buyers need to compute or estimate the benefits that they can

[3] This situation is directly mapped to a well known 0-1 knapsack problem [18].

earn from providing more services to their own customers using the frequency bands that they wish to buy in an auction. Rodriguez et al. [56] studied this problem for the buyers who operate CDMA based-systems in which terminals, e.g., end users or customers, are serviced at different data rates.

3.3 Discussion

As we illustrated, most of previous studies that are based on auction mechanisms for dynamic spectrum access focused on single seller cases.[4] However, when there are multiple sellers in the market, the sellers have two options: (i) Each seller holds a separate auction or (ii) a subset of sellers form a coalition and sell their items together. As a special case of the second case, all sellers may cooperate to form a grand coalition and hold a single auction. Thus, when designing the spectrum sharing market mechanism, we need to investigate buyers' and sellers' behaviors.

As explained earlier, when sellers are private entities, it is likely that they would be interested in maximizing the (expected) profit (or payoff). Regardless of coalitions that emerge, we assume that each coalition employs an optimal auction mechanism in order to maximize the overall expected profit (or payoff). For the purpose of analyzing more realistic scenarios, unlike the previous studies, we investigate the cases where each seller may have positive values for the items and each buyer may acquire any number of units i.e., the buyer does not request a particular

[4]Multiple sellers cases are covered in [66]. However, authors assumed that each seller has a single unit and all sellers sell their units together in a single auction. Thus, there is no difference from the cases where a single seller has multiple units.

number of units.

In most studies with multiple regions, the constraint that the same frequency band (or channel) should not be allocated in neighboring areas was adopted.[5] However, when a buyer employing a spread spectrum technology, e.g, CDMA, W-CDMA, has the same frequency band(s) in neighboring areas, it can provide soft-handover feature to its customers. For this reason, we allow the mechanism to allocate the same frequency band in every region. However, since we do not assume that the same frequency band in adjacent areas must be allocated to the same buyer who demands the frequency band(s) in both areas, this can introduce interference from neighboring regions. We do not explicitly address this issue; instead we assume that the buyers can handle it using, for example, interference cancellation techniques.

[5]Except for [36], [27, 59, 66] assumed that each buyer requests the frequency band(s) in a single region. Even in [36], the buyer cannot obtains the same frequency band(s) in adjacent region.

Chapter 4

Setup and research problems

4.1 Overview

In this dissertation, we are interested in designing a new spectrum trading mechanism for PSPs and SSPs. We study the setting where there are multiple PSPs with surplus frequency bands and multiple SSPs interested in leasing them. We assume that spectrum trading is performed periodically, for instance, by an electronic system with participating service providers. The PSPs are the *sellers* and the SSPs are the *buyers* or *bidders*. Sellers are interested in lending their surplus frequency bands which are the goods or items to be sold. In order to make progress, we assume that the frequency spectrum is traded in an agreed unit (e.g., 100 kHz). For example, as shown in Figure 4.1, a seller that wants to sell 3 MHz spectrum will have 30 units of 100 kHz frequency bands. In general, a buyer may prefer to win a block of contiguous frequency bands. However, we assume that the buyers do not differentiate the frequency bands, i.e, frequency bands are homogeneous, and the total value a buyer receives from winning one or more frequency bands depends only on the total number of frequency bands it receives.

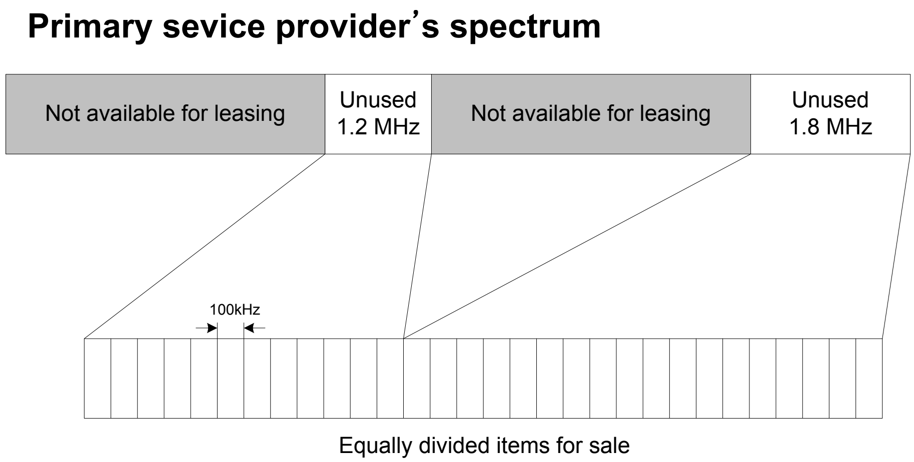

Figure 4.1: Example: Frequency bands for sale.

4.2 Model

Let $\mathcal{P} = \{1, 2, \ldots, M\}$ be the set of sellers and $\mathcal{S} = \{1, 2, \ldots, N\}$ the set of buyers. The sellers are assumed risk neutral and interested in maximizing their expected payoffs or profits. In a general setting, the area over which a seller operates (e.g., the United States) is partitioned into regions or markets (e.g., Washington D.C. metropolitan area). This partition is given by $\mathcal{R}$. However, except for in Chapter 8, we consider a simpler setting with only one region. The setting we explain here is for a single region case. We will describe the setting for multiple regions case in Chapter 8. For a single region case, the spectrum is divided into a set of frequency bands, denoted by $\mathcal{F}$. We assume that buyers are also risk neutral and interested in maximizing their expected payoff.[1]

[1]The buyer's payoff is defined to be the total value of the items the buyer wins minus the payment.

1) Sellers: Each seller owns a set of frequency bands. We denote the set of frequency bands owned by a seller $i \in \mathcal{P}$ by $\mathcal{F}^i$, and the set of frequency bands assigned to the sellers is given by $\cup_{i \in \mathcal{P}} \mathcal{F}^i \subset \mathcal{F}$. Moreover, we assume that a frequency band $f \in \mathcal{F}$ is owned by at most one seller, i.e., $\mathcal{F}^i \cap \mathcal{F}^{\tilde{i}} = \emptyset$ for all $i, \tilde{i} \in \mathcal{P}$ $(i \neq \tilde{i})$.

Sellers with under-utilized or extra frequency band(s) may participate in spectrum trading. When a seller partakes in the trading, it provides a list of frequency bands it wishes to lend to buyers (over an agreed period). Let K^i be the number of frequency bands seller i wants to sell, and $K_T = \sum_{i \in \mathcal{P}} K^i$ the total number of frequency bands available for lease.

We denote seller i's value for the ℓ-th item it wants to sell by V^i_ℓ, $\ell = 1, 2, \ldots, K^i$. In other words, seller i would prefer not to sell the ℓ-th frequency band if it cannot receive at least V^i_ℓ for it. The set of all sellers' values is given by $\mathcal{V} := \{V^i_\ell;\ i \in \mathcal{P} \text{ and } \ell \in \{1, 2, \ldots, K^i\}\}$. Without loss of generality, we assume that the seller's items are ordered by increasing value, i.e., $V^i_1 \leq \cdots \leq V^i_{K^i}$.

We assume that sellers can form arbitrary coalitions among themselves. The members of each coalition are assumed to share their information, e.g., the frequency bands for sale and received bids from buyers, and hold one auction to sell their spectrum together.

2) Buyers: Each buyer $j \in \mathcal{S}$ has private information, namely its *type*, which is denoted by T_j. We assume that T_j, $j \in \mathcal{S}$, are mutually independent continuous random variables. The distribution of T_j is $\mathcal{G}_j$ with support $\mathcal{T}_j := [t_{j,\min},\ t_{j,\max}]$. Moreover, we assume that $\mathcal{G}_j$ yields a density function g_j. The value of T_j is revealed only to buyer j at the beginning. Let $\mathbf{T} = (T_j;\ j \in \mathcal{S})$ be the random vector of the types of the buyers and $\mathcal{T} := \prod_{j\in\mathcal{S}} \mathcal{T}_j$.

The type of a buyer determines its values for the items it wins: For each $k \in \{1, 2, \ldots, K_T\}$, let $V_{j,k} : \mathcal{T}_j \to \mathbb{R}_+$ be the function that determines buyer j's value for the k-th item it wins, i.e., $V_{j,k}(t_j)$ is the value buyer j has for the k-th item it receives when its type is t_j. In general, the values of a buyer may depend on the types of other buyers as well. However, in this dissertation, we assume that the values of a buyer depend only on its own type, but not on those of other buyers. The functions $V_{j,k}$ are increasing and differentiable. We define the maximum number of frequency bands buyer j would like to lease from the sellers to be buyer j's demand and denote it by D_j. When D_j is strictly less than K_T, $V_{j,k}(t_j) = 0$ for all $t_j \in \mathcal{T}_j$ and $k = D_j + 1, \ldots, K_T$. However, we assume that $V_{j,k}(t_j) > 0$ for all $t_j \in \mathcal{T}_j$ and $k = 1, 2, \ldots, K_T$, although they can be arbitrarily close to zero. This implies that the demand of buyer j is at least K_T regardless of its type. As we will see, in an optimal auction mechanism sellers employ, each buyer may acquire the items less than D_j. In order to reflect the law of diminishing return, we also assume that $V_{j,1}(t_j) \geq V_{j,2}(t_j) \geq \cdots \geq V_{j,K_T}(t_j) \geq 0$ for all $t_j \in \mathcal{T}_j$.[2]

[2]This valuation model is originally introduced in [47]. In [10], Branco proposes an optimal multi-unit auction with this model. Given these valuation function, the auctioneer can compute the values the buyers receive from the items they wins, using buyers' types. This simplifies modeling of buyers' values and reduces the complexity of the auction mechanism. To the best of our knowledge, there is no known mechanism that deals with multiple units for sale, multi-unit demands of buyers and multi-dimensional values reported by the buyers.

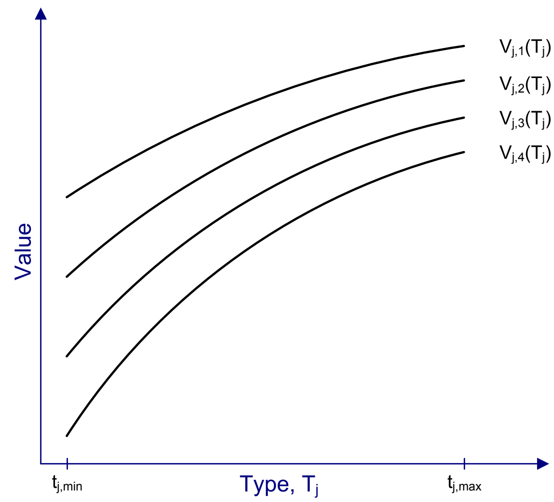

Figure 4.2: Valuation model.

4.3 Problems and the summary of results

In this dissertation, under the settings we described, we are interested in answering the following questions.

1. How do we model the interaction among the buyers that compete for the frequency bands available for lease?

2. Given the behavior of the buyers, is there an incentive for the sellers to cooperate to raise their expected profit?

3. If the answer to the second question is positive, how should they share the revenue or profit so that no subset of sellers will have an incentive to deviate from cooperation?

In addition to answering the above questions, we are also interested in designing an optimal auction mechanism for a seller who has surplus frequency bands in multiple regions.

Branco's mechanism can be used when a seller or a coalition (i.e., subsets of sellers) with multiple units of item pursues revenue maximization and buyers have multi-unit demands. However, it assumes that the seller has zero values for the items. As mentioned earlier, it is likely that the sellers, i.e., PSPs, have nonzero values for the items. Thus, we generalize Branco's optimal mechanism so that it can be applied to the cases where the sellers have positive values for the items while maintaining desired properties, such as incentive compatibility and individual rationality. We call the proposed mechanism generalized Branco's mechanism (GBM), which is explained in Chapter 5.

The GBM also follows the same valuation model of Branco's mechanism and the values a buyer has for the frequency bands it wins are determined by a function of the *type* of the buyer. We assume that the buyers are homogeneous and independent (i.e., their types are independent and identically distributed (i.i.d.) and the valuation functions are the same).

Since sellers are free to form any coalition, it is necessary to investigate how the sellers form or join the coalitions.[3] In order to examine the sellers behaviors, we have to compare the (expected) payoff (or profit) between possible coalitions the sellers can form. However, the coalitions formed by the sellers may depend on buyers' strategies in the spectrum trading market. As a simple example, suppose that there are only two sellers 1 and 2. If each of them holds a separate auction and all buyers choose to participate in seller 1's auction with probability one, then

[3]Each seller may prefer a coalition under which it can earn a higher payoff (or profit) than the other coalitions.

seller 1 has no reason to form a coalition with seller 2. Therefore, we study the buyers' strategies first. In order to make progress, we assume that the probability that a given coalition will form is known to the buyers (i.e., the SSPs) initially. This probability can be interpreted as an initial belief buyers have when they do not know for sure which coalitions emerge.

We begin by modeling the interaction among the buyers as a noncooperative game under the assumption that each coalition of sellers employs the GBM. A bidding strategy for a buyer can be divided into two steps: First, each buyer, at the beginning, selects a seller whose auction it will participate in. After choosing a seller, each buyer reports its type to the seller. However, as we will show, because of the incentive compatibility of the GBM, the optimal strategy of a buyer is to report its true type. Consequently, the only decision a buyer needs to make is the selection of a seller.

We show that there exists a *symmetric* mixed strategy Nash equilibrium (SMSNE). The Nash equilibrium is, however, not necessarily unique in general, except for in some special cases. For example, if (i) each seller holds its separate auction with probability 1 or (ii) there are at most five sellers and the probability that each seller holds its own auction is strictly positive, there is a unique SMSNE. In such cases, we also investigate whether or not the buyers' mixed strategies converge to the unique SMSNE. In general cases with more than one SMSNE, we assume that the buyers can agree on one of the SMSNEs.

In order to answer the second question stated at the beginning of the section, we demonstrate that, if C_1 and C_2 are two disjoint coalitions of the sellers, the sum of the expected payoffs of these two coalitions is not larger than the expected payoff of the coalition $C_1 \cup C_2$. This implies that, *assuming* that they can find a suitable way of sharing the payoff, risk neutral sellers will have an incentive to cooperate and form a single coalition that includes all the sellers, in order to maximize their expected payoffs.

To determine whether equitable sharing of payoff is feasible or not, we model the interaction among the sellers as a cooperative game. We prove that its *core*[4] is not empty. This tells us that there exists a way for the sellers to share the payoff so that no subset of sellers would deviate from the grand coalition to increase its expected payoff. We propose a payoff sharing scheme that can achieve *any* payoff sharing vector in the core.[5] Thus, from this finding, we expect that the sellers would form the grand coalition and hold a single auction in the spectrum trading market.

Finally, we turn our attention to more general cases where a seller wishes to sell frequency bands in several service regions (e.g., Washington D.C., Baltimore, and Philadelphia) and buyers operate in multiple regions and want to lease spectrum in those regions. We introduce an auction mechanism in which the spectrum bands in different service areas are sold simultaneously to the buyers who request the frequency bands in different (or multiple) areas. We show that the proposed auction mechanism is optimal in that it maximizes the seller's expected payoff.

[4]The core of our cooperative game is a set of the expected payoff sharing vectors among the sellers such that no subset of the sellers can increase its expected payoff by deviating from the cooperation. A formal definition and a characterization of the core are given in section 7.2.

[5]As we will show, since there is only constant difference between the payoff and profit, a desired expected payoff vector can be attained by the profit sharing scheme.

4.4 Challenges and contributions

In the case where there is one unit available for sale at the auction or each buyer requires only one unit of item in a multi-unit auction, each buyer has one dimensional information, i.e., value or type. Hence, if the distributions of buyers' types are known, by looking at the order statistics, we can compute the distribution of other quantities of interest, such as the winning bids, prices paid by the winners and the revenue of the auctioneer. This is the conventional way of computing these quantities. However, when each buyer requires multiple units and may acquire any number of units, computation of such quantities becomes more challenging. As a result, calculation of the expected payoff or profit of the auctioneer also becomes difficult. In order to skirt these difficulties and to analyze the cooperative games among the sellers, we take an alternative approach based on the framework used by Branco.

In order to investigate the buyers' strategies in our setting, it is necessary to formulate and calculate a buyer's expected payoff. However, the allocation rule in an optimal mechanism for multi-unit supply and multi-unit demands is rather complicated. For instance, as we can see the allocation rule (2.4) in Chapter 2, the contribution of each buyer and unit is computed and compared given buyers' types. This complexity in computing a buyer's expected payoff arises also in the GBM that we will describe in Chapter 5. Thus, this difficulty in computing a buyer's expected payoff limits analyzable cases. Under this limitation, we investigate the properties of the buyer's expected payoff and examine the buyers' strategy in some cases.

We expect that the GBM we introduce can be used for more realistic multi-unit auctions in which a seller has its own values for the items and can allocate the items in a more flexible manner. The approach taken in this dissertation and the findings may provide a guideline for designing a practical mechanism for a spectrum trading market where there are multiple sellers. Also, the proposed profit sharing mechanism can help sellers maintain cooperation. Finally, we hope that the developed optimal auction mechanism for multiple-region cases will servce as a reference and promote design/development of other sub-optimal mechanism with lower computational complexity.

Chapter 5

Generalized Branco's mechanism (GBM)

In this chapter, we introduce the GBM. Since sellers may have nonzero values for their items, as we mentioned in Chapter 2, Branco's original mechanism cannot be applied without any modification. We follow the same steps used in the development of Branco's original mechanism to show that the GBM is optimal in the sense that it maximizes the auctioneer's expected payoff and satisfies incentive compatibility and individual rationality.

5.1 Setup

Assume that a total of m items are available for sale from a seller or a coalition of sellers that are interested in selling their frequency bands together. Without loss of generality, we assume that the m items are ordered by increasing value of the item, i.e., $0 \leq V_0^{(1)} \leq V_0^{(2)} \leq \cdots \leq V_0^{(m)}$, where $V_0^{(k)}$ is the value of the auctioneer for the k-th item in an auction. Here, we denote the ordered value of the auctioneer (either a single seller or a coalition of sellers of frequency bands) by $V_0^{(k)}$ instead of $V_l^i (i \in \mathcal{P}, l \in K^i)$ introduced in section 4.2 for notational convenience; in the auction mechanism, items in a coalition are not distinguished according to their owners. For each buyer $j \in \mathcal{S}$, the valuation functions $V_{j,k}$, $k \in \{1, 2, \ldots, m\}$, are as defined in section 4.2, and $\mathcal{T} = \prod_{j \in \mathcal{S}} \mathcal{T}_j$. The auctioneer is assumed to know the valuation functions $V_{j,k}$, $j \in \mathcal{S}$ and $k \in \{1, 2, \dots, m\}$, the distributions $\mathcal{G}_j$, $j \in \mathcal{S}$, and the density functions g_j, $j \in \mathcal{S}$, of buyers' types, but not their realizations. Recall that, in Branco's framework, the auction mechanism is given by a pair (p, c), where

- $p_{j,k} : \mathcal{T} \to [0, 1]$, $j \in \mathcal{S}$ and $k \in \{1, 2, \dots, m\}$, where $p_{j,k}(\mathbf{t})$ is the probability that buyer j wins at least k items given that the buyers' type vector $\mathbf{T}$ is equal to $\mathbf{t}$, and

- $c_j : \mathcal{T}_j \to \mathbb{R}_+$, $j \in \mathcal{S}$, where $c_j(t_j)$ is the expected payment of buyer j of type t_j.[1]

Following Branco's framework, we are interested in a mechanism with the allocation rule with the property $p_{j,k}(\mathbf{t}) \in \{0, 1\}$ for all $j \in \mathcal{S}$, $k \in \{1, 2, \dots, m\}$, and $\mathbf{t} \in \mathcal{T}$. Given the buyers' type vector $\mathbf{t} \in \mathcal{T}$, denote the number of sold units by $m^\star(\mathbf{t}) = \sum_{j \in \mathcal{S}} \sum_{k=1}^{m} p_{j,k}(\mathbf{t})$. Then, the auctioneer's expected payoff, denoted by U_0, is defined as the expected payment it receives for the items sold plus the expected value of the unsold items:

$$\begin{aligned} U_0 &= \sum_{j \in \mathcal{S}} \mathbf{E}_{T_j}\left[c_j(T_j)\right] + \mathbf{E}_{\mathbf{T}}\left[\sum_{k=m^\star(\mathbf{T})+1}^{m} V_0^{(k)}\right] && (5.1) \\ &= \sum_{j \in \mathcal{S}} \mathbf{E}_{T_j}\left[c_j(T_j)\right] + \sum_{k=1}^{m} V_0^{(k)} - \mathbf{E}_{\mathbf{T}}\left[\sum_{k=1}^{m^\star(\mathbf{T})} V_0^{(k)}\right] && (5.2) \end{aligned}$$

In the GBM, each buyer reports its type $t_j^* \in \mathcal{T}_j$ to the auctioneer. The reported type t_j^* is not necessarily its true type t_j.

[1] This is the expected payment with respect to both the other buyers' types and the probability of winning a varying number of items. The payment rule will be provided in section 5.3.

5.2 Conditions for the GBM

We are interested in an allocation rule $p_{j,k}(\mathbf{t})$ that satisfies following conditions:

$$\textstyle\sum_{j\in\mathcal{S}}\sum_{k=1}^{m} p_{j,k}(\mathbf{t}) \le m \ , \tag{5.3}$$

$$p_{j,k}(\mathbf{t}) \ge p_{j,k+1}(\mathbf{t}) \ , \ \text{and} \tag{5.4}$$

$$p_{j,k}(\mathbf{t}) \in \{0,1\} \ . \tag{5.5}$$

The first condition (5.3) ensures that the total number of allocated items does not exceed the number of available items for sale. Conditions (5.4) and (5.5) follow from the definition of p and our restriction on p, respectively.

When the buyer j's reported type is t_j^* and its true type is t_j, assuming every other buyer report its true type, since the buyer is assumed risk neutral, its utility can be written as

$$U_j(t_j^*;t_j) = \mathbf{E}_{\mathbf{T}_{-j}}\left[\sum_{k=1}^{m} V_{j,k}(t_j)p_{j,k}(t_j^*,\mathbf{T}_{-j})\right] - c_j(t_j^*) \ . \tag{5.6}$$

Since our goal is to design an optimal auction mechanism which is both incentive compatible and individually rational, we impose the following requirements for all $j \in \mathcal{S}$ and $t_j, t_j^* \in \mathcal{T}_j$:

$$U_j(t_j;t_j) \ \ge \ U_j(t_j^*;t_j) \ \ \text{and} \tag{5.7}$$

$$U_j(t_j;t_j) \ \ge \ 0 \ . \tag{5.8}$$

Substituting the right hand side of (5.6) for $U_j(t_j^*;t_j)$ in (5.7), we obtain

$$\begin{aligned} U_j(t_j;t_j) &\ge U_j(t_j^*;t_j^*) + \big(U_j(t_j^*;t_j) - U_j(t_j^*;t_j^*)\big) \\ &= U_j(t_j^*;t_j^*) + \mathbf{E}_{\mathbf{T}_{-j}}\left[\sum_{k=1}^{m}\big(V_{j,k}(t_j) - V_{j,k}(t_j^*)\big)\, p_{j,k}(t_j^*,\mathbf{T}_{-j})\right] \ . \end{aligned} \tag{5.9}$$

Using (5.9) twice, we get

$$\mathbf{E}_{\mathbf{T}_{-j}}\left[\sum_{k=1}^{m}\left(V_{j,k}(t_j)-V_{j,k}(t_j^*)\right)p_{j,k}(t_j^*,\mathbf{T}_{-j})\right]$$
$$\leq U_j(t_j;t_j)-U_j(t_j^*;t_j^*)\leq \mathbf{E}_{\mathbf{T}_{-j}}\left[\sum_{k=1}^{m}\left(V_{j,k}(t_j)-V_{j,k}(t_j^*)\right)p_{j,k}(t_j,\mathbf{T}_{-j})\right] \quad (5.10)$$

By dividing all terms by $t_j - t_j^*$ and taking the limit as $t_j^* \to t_j$, we see that both bounds converge to the same limit. Therefore, $U_j(t_j;t_j) - U_j(t_j^*;t_j^*)$ can be obtained by integrating the limit of the upper bound in (5.10) (from t_j^* to t_j).

$$U_j(t_j;t_j) = U_j(t_j^*;t_j^*) + \mathbf{E}_{\mathbf{T}_{-j}}\left[\sum_{k=1}^{m}\int_{t_j^*}^{t_j}\frac{dV_{j,k}(x)}{dx}p_{j,k}(x,\mathbf{T}_{-j})dx\right] . \quad (5.11)$$

From (5.11) and the first part of (5.10),

$$\mathbf{E}_{\mathbf{T}_{-j}}\left[\sum_{k=1}^{m}\int_{t_j^*}^{t_j}\frac{dV_{j,k}(x)}{dx}p_{j,k}(x,\mathbf{T}_{-j})dx\right]$$
$$\geq \mathbf{E}_{\mathbf{T}_{-j}}\left[\sum_{k=1}^{m}\left(V_{j,k}(t_j)-V_{j,k}(t_j^*)\right)p_{j,k}(t_j^*,\mathbf{T}_{-j})\right] . \quad (5.12)$$

Since (5.9) follows from (5.11) and (5.12), we can replace condition (5.7) with (5.11) and (5.12).

Suppose that condition (5.8) holds.[2] Then,

$$U_j(t_{j,\min};t_{j,\min}) \geq 0 . \quad (5.13)$$

If (5.11) and (5.13) hold, since $V_{j,k}$ is an increasing function, condition (5.8) also holds. Therefore, we can drop conditions (5.7) and (5.8) and use conditions (5.11), (5.12), and (5.13) in their place.

[2]Incentive compatibility holds.

5.3 Allocation and payment schemes

In this section, we design the allocation and payment rules of the GBM so that the expected payoff U_0 is maximized under conditions (5.3), (5.4), (5.5), (5.11), (5.12), and (5.13). Since the GBM is designed to be incentive compatible, from now on, we assume that buyers report their types truthfully. In the following theorem, we first formulate an optimization problem with object function (5.1) and derive the expected payment of the buyers.

Theorem 5.1. *Suppose that the allocation rule $p^\star$ solves the following optimization problem:*

$$\begin{aligned}\text{maximize}_{p(\cdot)} \quad & \mathbf{E}_{\mathbf{T}}\left[\sum_{j\in\mathcal{S}}\sum_{k=1}^{m}\left(V_{j,k}(T_j)-\frac{dV_{j,k}(T_j)}{dT_j}\frac{1-\mathcal{G}_j(T_j)}{g_j(T_j)}\right)p_{j,k}(\mathbf{T})\right.\\ & \qquad\left.+\sum_{k=m^\star(\mathbf{T})+1}^{m}V_0^{(k)}\right] \qquad (5.14)\\ \text{subject to} \quad & (5.3),(5.4),(5.5),(5.11),(5.12),\ \text{and}\ (5.13)\ ,\end{aligned}$$

and that the expected payment $c^\star$ is given by

$$c_j^\star(t_j)=\mathbf{E}_{\mathbf{T}_{-j}}\left[\sum_{k=1}^{m}\left(V_{j,k}(t_j)p_{j,k}^\star(t_j,\mathbf{T}_{-j})-\int_{t_{j\,\min}}^{t_j}\frac{dV_{j,k}(x)}{dx}p_{j,k}^\star(x,\mathbf{T}_{-j})dx\right)\right] .(5.15)$$

Then $(p^\star, c^\star)$ is an optimal mechanism.

Proof. The first term in (5.1), i.e., expected payment the auctioneer receives, can be manipulated as

$$
\begin{aligned}
&\sum_{j\in\mathcal{S}} \mathbf{E}_{T_j}\left[c_j(T_j)\right] \\
&= \sum_{j\in\mathcal{S}} \mathbf{E}_{T_j}\left[\mathbf{E}_{\mathbf{T}_{-j}}\left[\sum_{k=1}^{m} V_{j,k}(T_j)p_{j,k}(T_j,\mathbf{T}_{-j})\right]\right] \\
&\quad - \sum_{j\in\mathcal{S}} \mathbf{E}_{T_j}\left[\mathbf{E}_{\mathbf{T}_{-j}}\left[\sum_{k=1}^{m} V_{j,k}(T_j)p_{j,k}(T_j,\mathbf{T}_{-j})\right] - c_j(T_j)\right] \\
&= \sum_{j\in\mathcal{S}} \mathbf{E}_{T_j}\left[\mathbf{E}_{\mathbf{T}_{-j}}\left[\sum_{k=1}^{m} V_{j,k}(T_j)p_{j,k}(T_j,\mathbf{T}_{-j})\right]\right] - \sum_{j\in\mathcal{S}} \mathbf{E}_{T_j}\left[U_j(T_j;T_j)\right] \\
&= \sum_{j\in\mathcal{S}} \mathbf{E}_{T_j}\left[\mathbf{E}_{\mathbf{T}_{-j}}\left[\sum_{k=1}^{m} V_{j,k}(T_j)p_{j,k}(T_j,\mathbf{T}_{-j})\right]\right] \\
&\quad - \sum_{j\in\mathcal{S}} \mathbf{E}_{T_j}\left[U_j(t_{j,\min};t_{j,\min}) + \mathbf{E}_{\mathbf{T}_{-j}}\left[\sum_{k=1}^{m}\int_{t_{j,\min}}^{T_j} \frac{dV_{j,k}(x)}{dx} p_{j,k}(x,\mathbf{T}_{-j})dx\right]\right] \ ,
\end{aligned}
\tag{5.16}
$$

where the second equality follows from (5.6) and the last equality is a consequence of (5.11). Since

$$
\begin{aligned}
&\mathbf{E}_{T_j}\left[\int_{t_{j,\min}}^{T_j} \frac{dV_{j,k}(x)}{dx} p_{j,k}(x,\mathbf{T}_{-j})dx\right] \\
&= \int_{t_{j,\min}}^{t_{j,\max}} \left(\int_{t_{j,\min}}^{y} \frac{dV_{j,k}(x)}{dx} p_{j,k}(x,\mathbf{T}_{-j})dx\right) g_j(y)dy \\
&= \int_{t_{j,\min}}^{t_{j,\max}} \left(\int_{x}^{t_{j,\max}} g_j(y)dy\right) \frac{dV_{j,k}(x)}{dx} p_{j,k}(x,\mathbf{T}_{-j})dx \\
&= \int_{t_{j,\min}}^{t_{j,\max}} \left(1-\mathcal{G}_j(x)\right) \frac{dV_{j,k}(x)}{dx} p_{j,k}(x,\mathbf{T}_{-j})dx \\
&= \mathbf{E}_{T_j}\left[\frac{dV_{j,k}(T_j)}{dT_j}\frac{1-\mathcal{G}_j(T_j)}{g_j(T_j)} p_{j,k}(T_j,\mathbf{T}_{-j})\right] \ ,
\end{aligned}
\tag{5.17}
$$

using (5.16) and (5.17), we can rewrite (5.1) as

$$\begin{aligned}
U_0 &= \sum_{j\in\mathcal{S}} \mathbf{E}_{T_j}\left[\mathbf{E}_{\mathbf{T}_{-j}}\left[\sum_{k=1}^{m}\left(V_{j,k}(T_j) - \frac{dV_{j,k}(T_j)}{dT_j}\frac{1-\mathcal{G}_j(T_j)}{g_j(T_j)}\right)p_{j,k}(T_j,\mathbf{T}_{-j})\right]\right] \\
&\quad - \sum_{j\in\mathcal{S}} U_j(t_{j,\min};t_{j,\min}) + \mathbf{E}_{\mathbf{T}}\left[\sum_{k=m^\star(\mathbf{T})+1}^{m} V_0^{(k)}\right] \\
&= \mathbf{E}_{\mathbf{T}}\left[\sum_{j\in\mathcal{S}}\sum_{k=1}^{m}\left(V_{j,k}(T_j) - \frac{dV_{j,k}(T_j)}{dT_j}\frac{1-\mathcal{G}_j(T_j)}{g_j(T_j)}\right)p_{j,k}(\mathbf{T}) + \sum_{k=m^\star(\mathbf{T})+1}^{m} V_0^{(k)}\right] \\
&\quad - \sum_{j\in\mathcal{S}} U_j(t_{j,\min};t_{j,\min}) \ . \qquad (5.18)
\end{aligned}$$

In (5.18), buyer j's expected payment, c_j, appears only in $U_j(t_{j,\min};t_{j,\min})$. Thus, in order to maximize the auctioneer's expected payoff, c_j needs to be selected such that $U_j(t_{j,\min};t_{j,\min})$ is minimized. From (5.6) and (5.11),

$$\begin{aligned}
&U_j(t_{j,\min};t_{j,\min}) \\
&= U_j(t_j;t_j) - \mathbf{E}_{\mathbf{T}_{-j}}\left[\sum_{k=1}^{m}\int_{t_{j,\min}}^{t_j}\frac{dV_{j,k}(x)}{dx}p_{j,k}(x,\mathbf{T}_{-j})dx\right] \\
&= \mathbf{E}_{\mathbf{T}_{-j}}\left[\sum_{k=1}^{m}\left(V_{j,k}(t_j)p_{j,k}(t_j,\mathbf{T}_{-j}) - \int_{t_{j\,\min}}^{t_j}\frac{dV_{j,k}(x)}{dx}p_{j,k}(x,\mathbf{T}_{-j})dx\right)\right] \\
&\quad - c_j(t_j) \ . \qquad (5.19)
\end{aligned}$$

Since $U_j(t_{j,\min};t_{j,\min}) \geq 0$, $c_j(t_j)$ should be selected so that $U_j(t_{j,\min};t_{j,\min}) = 0$. Therefore, the optimal expected payment should be (5.15)[3] and, from (5.18), the allocation rule should maximize (5.14). □

Given the types of the buyers, the contribution of buyer j for the k-th item ($k = 1, 2, \ldots, m$) is a mapping $\pi_{j,k} : \mathcal{T}_j \to \mathbb{R}$,[4] where

$$\pi_{j,k}(t_j) = V_{j,k}(t_j) - \left.\frac{dV_{j,k}(T_j)}{dT_j}\right|_{T_j=t_j}\frac{1-\mathcal{G}_j(t_j)}{g_j(t_j)} \ .$$

[3]This expected payment satisfies (5.11) and (5.13).

[4]Note that, in the original Branco's mechanism, the contribution is the mapping $\pi_{j,k} : \mathcal{T} \to \mathbb{R}$.

We order the contributions of all buyers by decreasing value and denote the ℓ-th highest contribution ($\ell = 1, 2, \ldots, N \cdot m$) by $\pi_{(\ell)}(\mathbf{t})$.[5]

Throughout the dissertation, we assume that the following regularity conditions hold: For all $j \in \mathcal{S}$ and $k = 1, 2, \ldots, m$,

- (i) $(t_j - \tilde{t}_j)(\pi_{j,k}(t_j) - \pi_{j,k}(\tilde{t}_j)) \geq 0$ for all $t_j, \tilde{t}_j \in \mathcal{T}_j$, and
- (ii) if $\pi_{j,k+1}(t_j) \geq 0$, then $\pi_{j,k}(t_j) \geq \pi_{j,k+1}(t_j)$ for all $t_j \in \mathcal{T}_j$.

When these conditions are satisfied, the problem is said to be *regular*.[6] The *regularity* assumption implies that (i) the contribution is non-decreasing in its type and (ii) the nonnegative contribution is non-increasing in the number of units it receives. Thus, if we order the contributions of a certain buyer by decreasing value, the second condition guarantees that the k-th contribution precedes the $(k+1)$-th contribution.[7] Since the auctioneer allocates the items by going down the list of the ordered contributions, in the case with symmetric buyers, the buyer with larger type has a higher probability of winning at least k items than another buyer with smaller type.

The following theorem provides a sufficient condition for regularity conditions.

Theorem 5.2. *[10, p.87] For all $j \in \mathcal{S}$ and $k \in \{1, 2, \ldots, m\}$, suppose that $\frac{1-\mathcal{G}_j(T_j)}{g_j(T_j)}$ is an non-increasing function in T_j, $V_{j,k}(T_j)$ is concave, and*

$$\frac{dV_{j,k}(T_j)/dT_j|_{T_j=t_j}}{V_{j,k}(t_j)} \leq \frac{dV_{j,k+1}(T_j)/dT_j|_{T_j=t_j}}{V_{j,k+1}(t_j)} \quad \textit{for all } t_j \in \mathcal{T}_j \ .$$

[5] In the event of measure zero that there are ties in the contributions, we break the ties randomly.

[6] Again, note that the contribution $\pi_{j,k}(\cdot)$ depends only on its type t_j in the GBM.

[7] This ensures that (5.4) holds.

Then, the regularity conditions (i) and (ii) are satisfied.

Now we define two functions:

(1) For each $\ell = 1, 2, \ldots, m$,

$$\eta_\ell(\mathbf{t}) := \max\{V_0^{(\ell)}, \pi_{(\ell+1)}(\mathbf{t})\} \ .$$

(2) For each $j \in \mathcal{S}$ and $k = 1, 2, \ldots, m$,

$$\varsigma_{j,k}(\mathbf{t}_{-j}) := \inf\{\hat{t}_j \in \mathcal{T}_j \mid \pi_{j,k}(\hat{t}_j) > \min\{\eta_\ell(\hat{t}_j, \mathbf{t}_{-j});\ \ell = 1, 2, \ldots, m\}\}, \quad (5.20)$$

where $\mathbf{t}_{-j} = \{t_{j^*};\ j^* \in \mathcal{S} \setminus \{j\}\}$.

Theorem 5.3. *If the problem is regular, an optimal mechanism* $(p^\star, c^\star)$ *satisfies*

$$p^\star_{j,k}(\mathbf{t}) = \begin{cases} 1 & \text{if } t_j > \varsigma_{j,k}(\mathbf{t}_{-j}) \ , \\ 0 & \text{otherwise,} \end{cases} \quad (5.21)$$

and

$$c^\star_j(t_j) = \mathbf{E}_{\mathbf{T}_{-j}}\left[\sum_{k=1}^{m} V_{j,k}(\varsigma_{j,k}(\mathbf{T}_{-j}))\ p^\star_{j,k}(t_j, \mathbf{T}_{-j})\right] \ . \quad (5.22)$$

Proof. The objective function in problem (5.14) can be written as

$$\mathbf{E}_{\mathbf{T}}\left[\sum_{j \in \mathcal{S}} \sum_{k=1}^{m} \pi_{j,k}(T_j) p_{j,k}(\mathbf{T}) - \sum_{k=1}^{m^\star(\mathbf{T})} V_0^{(k)}\right] + \sum_{k=1}^{m} V_0^{(k)} \ .$$

In order to maximize the payoff, an auctioneer would collect the m-highest *contributions*, i.e., $\pi_{(1)}(\mathbf{t})$, $\pi_{(2)}(\mathbf{t})$, …, $\pi_{(m)}(\mathbf{t})$. However, at the same time, the auctioneer loses the values for sold items. Therefore, the auctioneer should allocate k-th unit only if $\pi_{(k)}(\mathbf{t}) > V_0^{(k)}$. By the regularity assumption, $\varsigma_{j,k}(\mathbf{t}_{-j})$ returns the highest losing type given $\mathbf{t}_{-j}$. Thus, the item should be sold only if t_j is greater than $\varsigma_{j,k}(\mathbf{t}_{-j})$, and the optimal allocation rule $p^\star$ is given by (5.21). Since $p^\star_{j,k}(\mathbf{t})$ is nondecreasing in t_j and the value $V_{j,k}(t_j)$, for any $j \in \mathcal{S}$ and $k \in \{1, 2, \ldots, m\}$, is increasing in t_j, the condition (5.12) is satisfied. The other conditions (5.3), (5.4), and (5.5) follow from the allocation rule $p^\star$.[8] By substituting (5.21) in (5.15) we obtain the expected payment is (5.22). $\square$

From the expected payment given by (5.22), the following payment rule can be used.

$$\hat{c}_{j,k}(\mathbf{t}) = \begin{cases} V_{j,k}(\varsigma_{j,k}(\mathbf{t}_{-j})) & \text{if } p_{j,k}(\mathbf{t}) = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{5.23}$$

In other words, the price buyer j pays for the k-th unit it wins is equal to the smallest value for the k-th unit that would win the unit.

From the allocation and payment rules of our GBM in (5.21) and (5.23), it is clear that $m^\star(\mathbf{t})$ items are awarded to the buyers with the $m^\star(\mathbf{t})$ highest contributions, where

$$m^\star(\mathbf{t}) := \max\{\ell \in \{1, 2, \ldots, m\} \mid \pi_{(\ell)}(\mathbf{t}) > V_0^{(\ell)}\} \; . \tag{5.24}$$

When the set on the right-hand side is empty, the maximum is defined to be zero.

For example, suppose that an auctioneer has three items for sale. The values of the auctioneer are $V_0^{(1)} = 0.2$, $V_0^{(2)} = 0.24$, and $V_0^{(3)} = 0.26$. We assume that there are three homogeneous buyers. The buyers' types are uniformly distributed over the interval [0,1], and the valuation function $V_{j,k}(t_j) = \frac{1}{k} t_j$, $j \in \{1, 2, 3\}$ and

[8] Note that while we find the optimal expected payment (5.15), conditions (5.11) and (5.13) are satisfied.

$k \in \{1,2,3\}$, for all $t_j \in \mathcal{T}_j = [0,1]$. Suppose that the reported buyers' types are $\mathbf{t}^* = (0.85, 0.75, 0.65)$. Then, buyer's contributions are given by $\pi_{j,k}(t_j^*) = \frac{2}{k}t_j^* - \frac{1}{k}$ for $j \in \{1,2,3\}$ and $k \in \{1,2,3\}$. Hence, we have

- $\pi_{1,1}(0.85) = \frac{7}{10}$, $\pi_{1,2}(0.85) = \frac{7}{20}$, and $\pi_{1,3}(0.85) = \frac{7}{30}$.
- $\pi_{2,1}(0.75) = \frac{5}{10}$, $\pi_{2,2}(0.75) = \frac{5}{20}$, and $\pi_{2,3}(0.75) = \frac{5}{30}$.
- $\pi_{3,1}(0.65) = \frac{3}{10}$, $\pi_{3,2}(0.65) = \frac{3}{20}$, and $\pi_{3,3}(0.65) = \frac{3}{30}$.

The ordered contributions are $\pi_{(1)}(\mathbf{t}^*) = \pi_{1,1}(t_1^*)$, $\pi_{(2)}(\mathbf{t}^*) = \pi_{2,1}(t_2^*)$, $\pi_{(3)}(\mathbf{t}^*) = \pi_{1,2}(t_1^*)$, $\pi_{(4)}(\mathbf{t}^*) = \pi_{3,1}(t_3^*)$, $\pi_{(5)}(\mathbf{t}^*) = \pi_{2,2}(t_2^*)$, $\pi_{(6)}(\mathbf{t}^*) = \pi_{1,3}(t_1^*)$, $\pi_{(7)}(\mathbf{t}^*) = \pi_{2,3}(t_2^*)$, $\pi_{(8)}(\mathbf{t}^*) = \pi_{3,2}(t_3^*)$, and $\pi_{(9)}(\mathbf{t}^*) = \pi_{3,3}(t_3^*)$. From (5.24), we can see that buyer 1 wins two items and buyer 2 wins one item. From (5.20), $\varsigma_{1,1}(\mathbf{t}_{-1}^*) = 0.63$. Hence, for the first item buyer 1 wins, buyer 1 pays $V_{1,1}(\varsigma_{1,1}(\mathbf{t}_{-1}^*)) = 0.63$. In the same way, since $\varsigma_{1,2}(\mathbf{t}_{-1}^*) = 0.53$ and $\varsigma_{2,1}(\mathbf{t}_{-2}^*) = 0.65$, buyer 1 pays $V_{1,2}(\varsigma_{1,2}(\mathbf{t}_{-1}^*)) = 0.265$ for the second item it wins and buyer 2 pays $V_{2,1}(\varsigma_{2,1}(\mathbf{t}_{-2}^*)) = 0.65$ for the item it wins.

5.4 Properties of the GBM

From the previous sections, we can state following lemma.

Lemma 1. *The generalized Branco's mechanism is both incentive compatible and individually rational.*

We define the expected profit of an auctioneer as the expected payment it receives from the buyers minus the expected values of the sold items;

$$\breve{U}_0 = \sum_{j \in \mathcal{S}} \mathbf{E}_{T_j}\left[c_j(T_j)\right] - \mathbf{E}_{\mathbf{T}}\left[\sum_{k=1}^{m^\star(\mathbf{T})} V_0^{(k)}\right]. \tag{5.25}$$

Lemma 2. *The generalized Branco's mechanism is optimal in the sense that it maximizes the expected profit of the auctioneer.*

Proof. From (5.2) and (5.25), it is obvious,

$$\breve{U}_0 = U_0 - \sum_{k=1}^{m} V_0^{(k)} \ . \tag{5.26}$$

In other words, since the values of items $\sum_{k=1}^{m} V_0^{(k)}$ is constant, the expected payoff and expected profit differ only by a fixed constant $\sum_{k=1}^{m} V_0^{(k)}$. Thus, since the GBM maximizes the expected payoff of the auctioneer,[9] the GBM also maximizes the expected profit of the auctioneer. □

[9] See Theorems 5.1 and 5.3.

Chapter 6

Noncooperative game among the buyers

6.1 Bidding strategy

There are many different ways in which the sellers can sell their available frequency bands to the buyers. For example, individual sellers can hold separate individual auctions, or a group of sellers can form a *coalition* to sell their available frequency bands together. In the latter case, each coalition will hold one auction by sharing their information (e.g., the received bids, the number of frequency bands, and the reserved value for each frequency band) and the profit according to an agreement between its members. As we showed in Chapter 5, the payoff and the profit of a seller differ only by a fixed amount that equals the total value of the items it has for sale. Hence, we can compute one from the other. For this reason, maximizing the payoff is equivalent to maximizing the profit.

In order for a coalition of sellers to emerge, the sellers in the coalition must find it advantageous to cooperate and a proper profit sharing scheme must be in place. In general, it would require that (i) the expected profit of the coalition from a single auction be no smaller than the total expected profit the members can achieve by forming a set of smaller coalitions and (ii) there exist a suitable profit sharing scheme that allocates the profits in a way no subset of members finds it beneficial to leave the coalition. It is obvious that the expected profit of every seller i should be at least its expected profit from holding an individual auction.

Before we can understand how the sellers would behave, we must first examine buyers' behavior. To this end we model the interaction among the buyers as a noncooperative game [29]: At the beginning of the game each buyer first chooses a seller whose auction it will participate in[1] and then reports its type to the selected seller. We assume that either a buyer's selection of the seller takes place before the type is revealed to the buyer or the selection does not depend on the revealed type.

Sellers are free to form any coalition(s) among themselves. They do not announce the coalitions they form to the buyers before the buyers select sellers. In other words, buyers choose the sellers without the knowledge of the coalitions formed by the sellers; instead they only know the *probabilities* that different coalitions will emerge. Sellers in a coalition share the reported types of the buyers that choose a member of the coalition and decide on the set of frequency bands to be allocated and the prices to be charged according to the GBM. The buyers are then informed of the number of frequency bands they have won and the prices to pay. In Figure 6.1, an example is given: Sellers 1 and 2 form a coalition and sell their items together and seller 3 holds a separate auction. Each buyer selects a seller and reports its type before sellers announce the coalitions they form.

Let us first examine the actions to be taken by the buyers. As mentioned earlier, each buyer must first choose one of the M sellers and report its type to the seller. However, since the GBM is incentive compatible, once a buyer chooses a

[1]Here, we assume that each buyer joins only one auction. However, we will show later that this does not impose any restrictions on our findings.

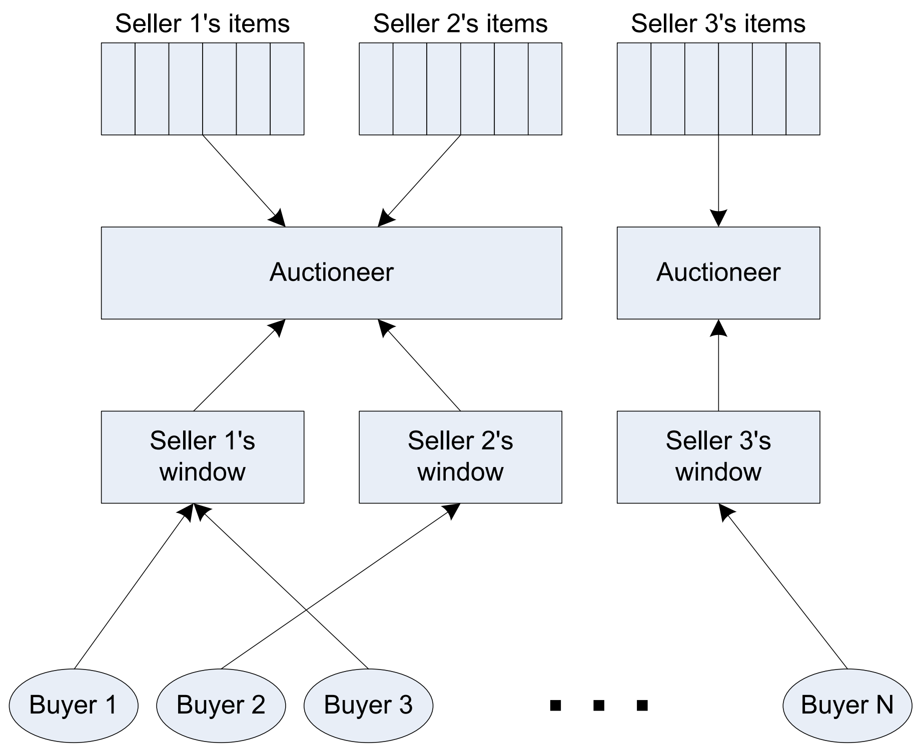

Figure 6.1: Example of coalition.

seller, the optimal strategy of a buyer in the GBM is to report its true type. Hence, the only action required of a buyer is the selection of a seller. We formulate this problem as a noncooperative game among the buyers.

Let $\Omega_{\mathcal{P}}$ be the set of all possible partitions of the set of sellers $\mathcal{P}$ and μ a distribution over the set $\Omega_{\mathcal{P}}$. The probability that coalitions in a partition $\omega \in \Omega_{\mathcal{P}}$ will form is given by $\mu(\omega)$. For example, suppose that $\mathcal{P} = \{1, 2\}$ and $\Omega_{\mathcal{P}} = \{\omega_1, \omega_2\} = \{\{\{1\}, \{2\}\}, \{\{1, 2\}\}\}$. Then, $\mu(\omega_1)$ is the probability that the coalitions $\{1\}$ and $\{2\}$ will form (i.e., two sellers do not cooperate) and $\mu(\omega_2)$ is the probability that coalition $\{1, 2\}$ will form (i.e., they will cooperate with each other). We assume that the distribution μ is common knowledge, and buyers know the probability a coalition $C \subset \mathcal{P}$ will form, which is given by

$$\mathbf{Pr}\left[\text{coalition } C \text{ forms}\right] = \sum_{\omega \in \Omega_{\mathcal{P}} : C \in \omega} \mu(\omega).$$

Since each buyer must choose a seller, the pure strategy space $\mathcal{B}_j$ of buyer $j \in \mathcal{S}$ is given by the set of sellers $\mathcal{P}$. The (expected) payoff of buyer j given a strategy profile $\mathbf{b} := (b_1, b_2, \ldots, b_N)$, where $b_j \in \mathcal{B}_j$ for all $j \in \mathcal{S}$, is given by $u_j(\mathbf{b})$. Then, the noncooperative game among the buyers is presented by $\Gamma = \{\mathcal{S}, (\mathcal{B}_j;\ j \in \mathcal{S}), (u_j;\ j \in \mathcal{S})\}$. The goal of each buyer is to maximize its expected payoff.

A mixed strategy of a buyer j is simply a distribution $\bar{\xi} := (\xi_i; i \in \mathcal{P})$ over $\mathcal{B}_j = \mathcal{P}$, where $\xi_i, i \in \mathcal{P}$, is the probability that buyer j will choose seller i. A mixed strategy Nash equilibrium (MSNE), $\Xi = (\bar{\xi}^1, \bar{\xi}^2, \ldots, \bar{\xi}^N)$, is a set of mixed strategies, one for each buyer, such that no buyer can increase its expected payoff by unilaterally deviating from the equilibrium strategy. An MSNE, Ξ, is called a *symmetric* MSNE if $\bar{\xi}^1 = \bar{\xi}^2 = \cdots = \bar{\xi}^N$.

In the rest of the dissertation we consider independent homogeneous buyers: The types of the buyers $t_j, j \in \mathcal{S}$, are independent and identically distributed, and the valuation functions $V_{j,k}$ are identical for all $j \in \mathcal{S}$.

Definition 4. *A game Γ is symmetric if the players have identical strategy spaces, i.e., $\mathcal{B}_1 = \mathcal{B}_2 = \cdots = \mathcal{B}_N$, and $u_j(b_j, \mathbf{b}_{-j}) = u_l(b_l, \mathbf{b}_{-l})$ if $b_j = b_l$ and $\mathbf{b}_{-j} = \mathbf{b}_{-l}$ for all $j, l \in \mathcal{S}$, where $\mathbf{b}_{-j} = (b_m; m \neq j)$ is the strategy profile of the other players.*

Theorem 6.1. *[15, p.25] A finite symmetric game has a symmetric mixed strategy Nash equilibrium.*

Theorem 6.2. *There always exists a symmetric mixed strategy Nash equilibrium in our game* Γ.

Proof. In our game Γ, the buyers have a common finite strategy space $\mathcal{P}$ (i.e., $\mathcal{B}_j = \mathcal{P}$, for all $j \in \mathcal{S}$) and have the identical payoff function because of the homogeneity assumption. Thus, Γ is a finite symmetric game, and by Theorem 6.1, there exists a symmetric mixed strategy Nash equilibrium. $\square$

In general a symmetric MSNE is not guaranteed to be unique. However, when no seller cooperates with any other seller(s) with probability 1 (w.p. 1), i.e., $\mu(\{\{1\}, \{2\}, \{3\}, \ldots, \{M\}\}) = 1$, the symmetric MSNE is unique if the valuation functions $V_{j,k}$ satisfy the following condition:

$$\pi_{j,\lceil (K^i+1)/2 \rceil}(t_{j,\max}) > V_1^i \quad \text{for all } i \in \mathcal{P} \tag{6.1}$$

The condition (6.1) guarantees that (i) the possible highest contribution, i.e., $\pi_{j,1}(t_{j,\max})$, is larger than the minimum value of every seller V_1^i, $i \in \mathcal{P}$, and (ii) assuming each seller has the identical value for every item, i.e., $V_1^i = V_2^i = \cdots = V_{K^i}^i$, when there are more than two buyers participating in seller i's auction, the number of contributions larger than the value of the items exceeds the supply of the seller with strictly positive probability. This implies that the expected payoff of a buyer partaking in seller i's auction will be strictly decreasing in the number of other buyers that join the auction. Note that the condition in (6.1) is the same for all buyers from the homogeneity assumption. Throughout the dissertation, we assume that the condition (6.1) holds.

Theorem 6.3. *Suppose that condition (6.1) holds. If every seller holds a separate auction with probability 1, there is a unique symmetric mixed strategy Nash equilibrium.*

Before we prove the theorem, we introduce a lemma that will be used in the proof. We denote the set of buyers that choose seller i by $\mathcal{S}_i$. Let $\tilde{U}_j^{(i)}(n)$ be the conditional expected payoff of buyer j given that (i) buyer j chooses seller $i \in \mathcal{P}$ and (ii) $|\mathcal{S}_i| = n+1$, i.e., exactly n other buyers choose seller i as well.

Lemma 3. *Suppose that buyer j chooses seller i. Then, $\tilde{U}_j^{(i)}(n-1) > \tilde{U}_j^{(i)}(n)$ for all $n \in \{1, 2, ..., N-1\}$.*

Proof. Since the buyers are homogeneous, without loss of generality, assume that $j = 1$ and $\mathcal{S}_i = \{1, 2, \ldots, n+1\}$. Define $\mathbf{T}_{-1}^{(n)} = (T_l; l = 2, \ldots, n+1)$ and $\mathbf{t}_{-1}^{(n)} = (t_l; l = 2, \ldots, n+1)$. From the allocation rule in (5.21), for fixed $\mathbf{t}_{-1}^{(n)}$, the probability $p_{1,k}(t_1, \mathbf{t}_{-1}^{(n)})$ is nondecreasing in t_1.[2] In particular, $p_{1,k}(t_1, \mathbf{t}_{-1}^{(n)}) = 0$ if $t_{1,\min} \leq t_1 \leq \varsigma_{1,k}(\mathbf{t}_{-1}^{(n)})$ and $p_{1,k}(t_1, \mathbf{t}_{-1}^{(n)}) = 1$ if $\varsigma_{1,k}(\mathbf{t}_{-1}^{(n)}) < t_1$.

From the payment rule in (5.23), we can show that the expected payoff of buyer 1 that participates in seller i's auction when there are n other buyers is given

[2]Here, for different values of n, we have different auctions. However, with some abuse of notation, we do not explicitly indicate the dependence of $p_{1,k}(t_1, \mathbf{t}_{-1}^{(n)})$, $\varsigma_{1,k}(\mathbf{t}_{-1}^{(n)})$, and $\hat{c}_{1,k}(T_1, \mathbf{T}_{-1}^{(n)})$ on n (or more precisely, on the set $\mathcal{S}_i$).

by

$$\tilde{U}_1^{(i)}(n) = \mathbf{E}_{\mathbf{T}(i)}\left[\sum_{k=1}^{K^i}\Big(V_{1,k}(T_1)\ p_{1,k}(T_1, \mathbf{T}_{-1}^{(n)}) - \hat{c}_{1,k}(T_1, \mathbf{T}_{-1}^{(n)})\Big)\right]$$
$$= \mathbf{E}_{\mathbf{T}(i)}\left[\sum_{k=1}^{K^i}\left(\int_{t_{1,\min}}^{T_1} \frac{dV_{1,k}(x)}{dx}\ p_{1,k}(x, \mathbf{T}_{-1}^{(n)})\ dx\right)\right], \tag{6.2}$$

where $\mathbf{T}(i) = \{T_j; j \in \mathcal{S}_i\}$, and the expectation is taken over the types $\mathbf{T}(i)$. From the allocation rule (5.21), for any t_1 and $\mathbf{t}_{-1}$, we have

$$p_{1,k}(t_1, \mathbf{t}_{-1}^{(n-1)}) \geq p_{1,k}(t_1, \mathbf{t}_{-1}^{(n)}). \tag{6.3}$$

The lemma now follows from (6.1) - (6.3). □

Proof of Theorem 6.3. As mentioned earlier, the existence of a symmetric MSNE follows from Theorem 6.1. Suppose that there are two symmetric MSNEs, $\Xi^1 = (\bar{\xi}^1, \ldots, \bar{\xi}^1)$ and $\Xi^2 = (\bar{\xi}^2, \ldots, \bar{\xi}^2)$, where $\bar{\xi}^k = (\xi_1^k, \ldots, \xi_M^k)$, $k = 1, 2$, such that $\bar{\xi}^1 \neq \bar{\xi}^2$. We will show that this leads to a contradiction, thus proving the uniqueness of a symmetric MSNE.

Let $U_j^{(i)}(\bar{\xi})$ denote the conditional expected payoff of buyer j, given that buyer j selects seller i, when all buyers employ the same mixed strategy $\bar{\xi}$. The buyer j's expected payoff is equal to

$$U_j(\bar{\xi}) = \sum_{i \in \mathcal{P}} \xi_i \cdot U_j^{(i)}(\bar{\xi})\ . \tag{6.4}$$

One can easily show that, at any symmetric MSNE $\Xi^\star = (\bar{\xi}^\star, \ldots, \bar{\xi}^\star)$, we must have

$$U_j^{(1)}(\bar{\xi}^\star) = \cdots = U_j^{(M)}(\bar{\xi}^\star) \qquad \text{for all } j \in \mathcal{S}. \tag{6.5}$$

Since the buyers are assumed to select sellers independently of each other, for each $i \in \mathcal{P}$,

$$U_j^{(i)}(\bar{\xi}^\star) = \sum_{n=0}^{N-1} \binom{N-1}{n} \; (\xi_i^\star)^n \;\; (1-\xi_i^\star)^{N-1-n} \; \tilde{U}_j^{(i)}(n).$$

We can compute the derivative of $U_j^{(i)}(\bar{\xi})$.

$$\begin{aligned}
&\frac{\partial U_j^{(i)}(\bar{\xi})}{\partial \xi_i} \\
&= \sum_{n=0}^{N-1} \binom{N-1}{n} \tilde{U}_j^{(i)}(n) \left[n(\xi_i)^{n-1}(1-\xi_i)^{(N-1-n)} - (N-1-n)(\xi_i)^n (1-\xi_i)^{(N-2-n)} \right] \\
&= -\binom{N-1}{1}(1-\xi_i)^{(N-2)}(\tilde{U}_j^{(i)}(0) - \tilde{U}_j^{(i)}(1)) \\
&\quad -\binom{N-1}{1}(N-2)(\xi_i)(1-\xi_i)^{(N-3)}(\tilde{U}_j^{(i)}(1) - \tilde{U}_j^{(i)}(2)) \\
&\quad -\binom{N-1}{2}(N-3)(\xi_i)^2(1-\xi_i)^{(N-4)}(\tilde{U}_j^{(i)}(2) - \tilde{U}_j^{(i)}(3)) \\
&\quad -\cdots \\
&\quad -\binom{N-1}{N-2}(\xi_i)^{N-2}(\tilde{U}_j^{(i)}(N-2) - \tilde{U}_j^{(i)}(N-1))
\end{aligned} \tag{6.6}$$

and, from Lemma 3, $\partial U_j^{(i)}(\bar{\xi})/\partial \xi_i < 0$.

If $\bar{\xi}^1 \neq \bar{\xi}^2$, there must exist i^+ and i^* such that (i) $\xi_{i^+}^1 < \xi_{i^+}^2$ and (ii) $\xi_{i^*}^1 > \xi_{i^*}^2$. Our finding that $U_j^{(i)}(\bar{\xi})$ is strictly decreasing in ξ_i implies that $U_j^{(i^+)}(\bar{\xi}^1) > U_j^{(i^+)}(\bar{\xi}^2) = U_j^{(i^*)}(\bar{\xi}^2) > U_j^{(i^*)}(\bar{\xi}^1)$, which contradicts (6.5). □

Theorem 6.4. *Suppose that there are at most five sellers and the probability that each seller holds its own auction is strictly positive. Then, there is a unique symmetric mixed strategy Nash equilibrium.*

Proof. A proof is given in Appendix A. □

In the proofs of Theorems 6.3 and 6.4 (in Appendix A), we make use of only the properties of the buyer's conditional expected payoff, e.g., $U_j^{(i)}(\bar{\xi})$ is strictly decreasing in ξ_i for all $i \in \mathcal{P}$. In order to verify whether or not there is a unique symmetric MSNE when there are *more than five sellers* and the probability that each seller holds its own auction is strictly positive, we may need to calculate each buyer's expected payoff. As can be seen from equations (5.6) and (5.22),[3] in order to compute buyers' expected payoffs, we need to know the allocation for every realization, which is difficult for general cases. In addition, when there are more than five sellers, the approach employed in the proof of Theorem 6.4 becomes impractical as the number of cases we need to consider gets large. As a result, proving the uniqueness of the symmetric MSNE by contradiction becomes harder. For this reason, we leave the question of the uniqueness of the symmetric MSNE in general cases as an open problem.

When the probability that each seller holds its own auction is *not* strictly positive, one can easily find examples where symmetric MSNE is not unique. For example, suppose that $M \geq 3$ and $\mu(\{\{1,2\},\{3\},\{4\},\ldots,\{M\}\}) = 1$. Since sellers 1 and 2 always cooperate, this does not satisfy the assumption that each seller holds its own auction with strictly positive probability in Theorem 6.4 (when $M \leq 5$). Note that this implies that the assumption in Theorem 6.3 is not satisfied, either. Consider a new game $\Gamma^\star$ with $M-1$ sellers, where the new seller 1 combines both sellers 1 and 2 in the original game Γ. Then, from Theorem 6.3, there is a unique symmetric MSNE, $\Xi^\star = (\bar{\xi}^\star, \bar{\xi}^\star, \ldots, \bar{\xi}^\star)$, in this new game $\Gamma^\star$. One can easily verify

[3] Buyer's expected payoff and expected payment.

that any strategy profile $\Xi^\dagger = (\bar{\xi}^\dagger, \ldots, \bar{\xi}^\dagger)$, where $\xi_1^\dagger + \xi_2^\dagger = \xi_1^\star$ and $\xi_l^\dagger = \xi_l^\star$ for all $l = 3, \ldots, M$, is a symmetric MSNE of Γ. Thus, if $\xi_1^\star > 0$, there are uncountably many symmetric MSNEs of Γ.

When there are more than one symmetric MSNEs, we assume that the buyers can agree on or reach one of the symmetric MSNEs and behave according to the chosen symmetric MSNE. For example, the buyers may choose the symmetric MSNE that yields the largest expected payoff. In addition, we suspect that, although there may be many symmetric MSNEs, the expected payoffs of the buyer are the same under all symmetric MSNEs.

Conjecture 1. *Suppose that $\Xi^\star = (\bar{\xi}^\star, \bar{\xi}^\star, \ldots, \bar{\xi}^\star)$ and $\Xi^\dagger = (\bar{\xi}^\dagger, \ldots, \bar{\xi}^\dagger)$ are two symmetric MSNEs. Then, $U_j(\xi^\star) = U_j(\xi^\dagger)$ for all $j \in \mathcal{S}$.*

A sufficient condition for Conjecture 1 is given in Lemma 4.

Lemma 4. *Suppose $\Xi^\star = (\bar{\xi}^\star, \ldots, \bar{\xi}^\star)$ and $\Xi^\dagger = (\bar{\xi}^\dagger, \ldots, \bar{\xi}^\dagger)$ are two symmetric MSNEs and $\sum_{i \in C} \xi_i^\star = \sum_{i \in C} \xi_i^\dagger$ for every $C \subset \mathcal{P}$ for which $\mathbf{Pr}\,[\text{coalition } C \text{ forms}] > 0$. Then, $U_j(\xi^\star) = U_j(\xi^\dagger)$ for all $j \in \mathcal{S}$.*

Proof. Let $\tilde{\mu}(C)$ be the probability that a coalition $C \subset \mathcal{P}$ forms and $\hat{U}_j^{(i)}(C, \bar{\xi})$ be the conditional expected payoff of buyer j choosing seller i in the coalition C, given that the coalition C forms and all buyers employ the same mixed strategy $\bar{\xi}$.[4] Recall that $U_j^{(i)}(\bar{\xi})$ denotes the conditional expected payoff of buyer j, given that buyer j selects seller i and all buyers adopt the same mixed strategy $\bar{\xi}$. Then,

$$U_j^{(i)}(\bar{\xi}) = \sum_{C \subset \mathcal{P} \setminus \{i\}} \tilde{\mu}(C \cup \{i\}) \times \hat{U}_j^{(i)}(C \cup \{i\}, \bar{\xi}) \; . \tag{6.7}$$

[4]These are the same definitions introduced in Appendix A.

From the proof of Theorem 6.3, we can see that $\hat{U}_j^{(i)}(C, \bar{\xi})$ depends on $\bar{\xi}$ only through $\sum_{l \in C} \xi_l$. Thus, if $\sum_{i \in C} \xi_i^\star = \sum_{i \in C} \xi_i^\dagger$ for every $C \subset \mathcal{P}$ with $\tilde{\mu}(C) > 0$, we have $U_j^{(i)}(\bar{\xi}^\star) = U_j^{(i)}(\bar{\xi}^\dagger)$ for every $i \in \mathcal{P}$. Therefore, from (6.4), we have $U_j(\bar{\xi}^\star) = U_j(\bar{\xi}^\dagger)$. □

A main obstacle to proving Conjecture 1 is the difficulty in computing the buyers' expected payoffs. It also hampers the studies of other interesting questions such as the convergence and stability of symmetric MSNEs. Even though some of our analysis on buyers' symmetric MSNE is carried out for limited cases in this dissertation, in practice, we expect that the number of sellers in the same region to be small and exceeds five only infrequently.

6.2 Convergence of buyers' strategies

In this section, we investigate whether or not the buyers' symmetric mixed strategy converges to a symmetric MSNE under a certain strategy update scheme. As we mentioned in the previous section, due to the difficulty in calculation of the expected payoff of buyers, the analyzable cases are limited. We illustrate the difficulty and discuss the case in which the convergence of the buyers' symmetric mixed strategy can be established.

Since we assume that the buyers are selfish and do not communicate/cooperate among themselves, i.e., noncooperative game among buyers, the information each buyer can use is limited to the received payoff in the past auctions. Thus, we assume that a buyer's strategy updating scheme depends only on the past payoffs. To make progress, we make several simplifying assumptions:

1. A buyer's mixed strategy is updated periodically, e.g, every 50 auctions (discrete time update). Between two consecutive updates, the buyers can estimate the expected payoffs (reasonably accurately).

2. Every buyer updates the strategy at the same time (synchronous update).

3. All buyers use the same initial mixed strategy.

4. No buyer stops participating in the auctions.

5. Buyers' valuation functions $V_{j,k}$, $j \in \mathcal{S}, k \in \{1, 2, \ldots, K_T\}$, remain the same.

6. The number of items available from each seller remains the same.

Let $\bar{\xi}(0)$ be the initial mixed strategy of the buyers. Denote, by $\bar{\xi}(n) := (\xi_1(n), \xi_2(n), \ldots, \xi_M(n))$, the buyers' mixed strategy at update step $n \in \mathbb{Z}_+$. Recall that $U_j^{(i)}(\bar{\xi}(n))$ denotes the conditional expected payoff of buyer j, given that buyer j selects seller i, when all buyers employ the same mixed strategy $\bar{\xi}(n)$, and the buyer j's expected payoff is denoted by $U_j(\bar{\xi}(n))$. Note that since we assume homogeneous buyers and a synchronous update scheme, if all buyers apply the same update scheme, all buyers have same mixed strategy $\bar{\xi}(n)$ for all n. This implies that all buyers have the same expected payoff $U_1(\bar{\xi}(n)) = U_2(\bar{\xi}(n)) = \cdots = U_M(\bar{\xi}(n))$. Thus, for notational convenience, we omit the subscript j in the expected payoff.

Assume that buyers adopt the following update rule for every $i \in \mathcal{P}$.[5]

$$\xi_i(n+1) = \xi_i(n) + \alpha(n)\ \xi_i(n) \left(U^{(i)}(\bar{\xi}(n)) - U(\bar{\xi}(n)) \right) , \tag{6.8}$$

where $U(\bar{\xi}(n)) = \sum_{i \in \mathcal{P}} \xi_i(n) \cdot U^{(i)}(\bar{\xi}(n))$, and $\alpha(n) > 0$ for all $n \in \mathbb{Z}_+$. Since $\sum_{i \in \mathcal{P}} \xi_i(n) = 1$ and $\sum_{i \in \mathcal{P}} \xi_i(n) \left(U^{(i)}(\bar{\xi}(n)) - U(\bar{\xi}(n)) \right) = 0$, it is clear that $\sum_{i \in \mathcal{P}} \xi_i(n+1) = 1$.

Suppose that there is a mapping $\hat{T} : \mathcal{X} \mapsto \mathcal{X}$, where $\mathcal{X} = \{\bar{\xi} \in \mathbb{R}^M \mid 0 < \xi_i < 1 \text{ for all } i \in \mathcal{P} \text{ and } \sum_{i \in \mathcal{P}} \xi_i = 1\} \subset \mathbb{R}^M$.

Definition 5. *The vector $x^* \subset \mathcal{X}$ is called a fixed point of $\hat{T}$ if $x^* = \hat{T}(x^*)$.*

Definition 6. *A mapping $\hat{T} : \mathcal{X} \mapsto \mathcal{X}$ is called a pseudocontraction if the mapping $\hat{T}$ has a fixed point x^* and the following property holds:*

$$\|\hat{T}(x) - x^*\| \leq \beta \|x - x^*\|, \ \forall\ x \in \mathcal{X} , \tag{6.9}$$

where β, called the modulus of $\hat{T}$, is a constant in $[0, 1)$.

Theorem 6.5. *[11, p.183] Suppose that the mapping $\hat{T} : \mathcal{X} \mapsto \mathcal{X}$ is a pseudocontraction with a fixed point $x^* \in \mathcal{X}$ and modulus $\beta \in [0, 1)$. Then, $\hat{T}$ has no other fixed points and the sequence $\{x(n)\}$ generated by $x(n+1) = \hat{T}(x(n))$ converges to x^* as $n \to \infty$.*

Suppose that there is a unique symmetric MSNE $\Xi^\star = (\bar{\xi}^\star, \bar{\xi}^\star, \ldots, \bar{\xi}^\star)$. At equilibrium $\bar{\xi}^\star$, we have $U^{(1)}(\bar{\xi}^\star) = U^{(2)}(\bar{\xi}^\star) = \cdots = U^{(M)}(\bar{\xi}^\star) = U(\bar{\xi}^\star)$. If $\bar{\xi}(n) = \bar{\xi}^\star$,

[5] Updating rule (6.8) is motivated by the replicator dynamic model which is introduced in evolutionary game theory [65].

from (6.8), $\bar{\xi}(n+1) = \bar{\xi}(n) = \bar{\xi}^\star$. Thus, if the update rule (6.8) gives rise to a pseudocontraction with some modulus $\beta \in [0,1)$, the mixed strategy sequence $\bar{\xi}(n)$ converges to $\bar{\xi}^\star$ by Theorem 6.5. Note that, since a pseudocontraction with modulus $\beta = 0$ is unlikely, we assume that $\beta \in (0,1)$. In order to examine whether or not the update rule (6.8) yields a pseudocontraction with some modulus $\beta \in (0,1)$, we will investigate the conditions the update rule (6.8) has to satisfy.

Let $\|\cdot\|$ be the L_1 norm, i.e., $\|x\| = \sum_{i\in\mathcal{P}} |x_i|$ for $x \in \mathbb{R}^M$. For a given $n \in \mathbb{Z}_+$, define the following index sets:

$$\Theta_1(n) := \{k \in \mathcal{P} \mid \xi_k(n) > \xi_k^\star\}$$

$$\Theta_2(n) := \mathcal{P} \setminus \Theta_1(n)$$

$$\Theta_3(n) := \Theta_1(n) \cap \Theta_1(n+1)$$

$$\Theta_4(n) := \Theta_1(n) \cap \Theta_2(n+1)$$

$$\Theta_5(n) := \Theta_2(n) \cap \Theta_2(n+1)$$

$$\Theta_6(n) := \Theta_2(n) \cap \Theta_1(n+1)$$

$$\Theta_7(n) := \{k \in \mathcal{P} | U^{(k)}(\bar{\xi}(n)) > U(\bar{\xi}(n))\}$$

$$\Theta_8(n) := \{k \in \mathcal{P} | U^{(k)}(\bar{\xi}(n)) = U(\bar{\xi}(n))\}$$

$$\Theta_9(n) := \{k \in \mathcal{P} | U^{(k)}(\bar{\xi}(n)) < U(\bar{\xi}(n))\}$$

Suppose that $\bar{\xi}(n) \in \mathcal{X}$. In order for the update rule (6.8) to be a mapping from $\mathcal{X}$ to $\mathcal{X}$, we must ensure that $\bar{\xi}(n+1)$ lies in $\mathcal{X}$. In other words, for every $i \in \mathcal{P}$,

$$0 < \xi_i(n) + \alpha(n)\ \xi_i(n) \left(U^{(i)}(\bar{\xi}(n)) - U(\bar{\xi}(n))\right) < 1\ . \tag{6.10}$$

Let

$$\begin{aligned}
\alpha_1^*(n) &= \min\left\{\frac{1-\xi_k(n)}{\xi_k(n)\left(U^{(k)}(\bar{\xi}(n)) - U(\bar{\xi}(n))\right)};\ k \in \Theta_7(n)\right\} \\
\alpha_2^*(n) &= \min\left\{\frac{1}{\left(U(\bar{\xi}(n)) - U^{(k)}(\bar{\xi}(n))\right)};\ k \in \Theta_9(n)\right\}\ .
\end{aligned}$$

Then, in order to ensure that (6.10) holds, the step size in the update rule must satisfy

$$0 < \alpha(n) < \min\{\alpha_1^*(n), \alpha_2^*(n)\}\ . \tag{6.11}$$

Now we examine the conditions for the update rule (6.8) to yield a pseudo-contraction. Since

$$\begin{aligned}
\|\bar{\xi}(n+1) - \bar{\xi}^\star\| &= \sum_{i\in\Theta_3(n)} \xi_i(n) + \alpha(n)\xi_i(n)\left(U^{(i)}(\bar{\xi}(n)) - U(\bar{\xi}(n))\right) - \xi_i^\star \\
&+ \sum_{i\in\Theta_6(n)} \xi_i(n) + \alpha(n)\xi_i(n)\left(U^{(i)}(\bar{\xi}(n)) - U(\bar{\xi}(n))\right) - \xi_i^\star \\
&+ \sum_{i\in\Theta_4(n)} \xi_i^\star - \xi_i(n) - \alpha(n)\xi_i(n)\left(U^{(i)}(\bar{\xi}(n)) - U(\bar{\xi}(n))\right) \\
&+ \sum_{i\in\Theta_5(n)} \xi_i^\star - \xi_i(n) - \alpha(n)\xi_i(n)\left(U^{(i)}(\bar{\xi}(n)) - U(\bar{\xi}(n))\right)
\end{aligned}$$

and

$$\begin{aligned}
\|\bar{\xi}(n) - \bar{\xi}^\star\| &= \sum_{i\in\Theta_3(n)} \xi_i(n) - \xi_i^\star + \sum_{i\in\Theta_4(n)} \xi_i(n) - \xi_i^\star \\
&+ \sum_{i\in\Theta_5(n)} \xi_i^\star - \xi_i(n) + \sum_{i\in\Theta_6(n)} \xi_i^\star - \xi_i(n)\ ,
\end{aligned}$$

the condition (6.9) can be rewritten as

$$\begin{aligned}
&\sum_{i\in\Theta_3(n)\cup\Theta_6(n)} \alpha(n)\xi_i(n)\left(U^{(i)}(\bar{\xi}(n)) - U(\bar{\xi}(n))\right) \\
&\quad - \sum_{i\in\Theta_4(n)\cup\Theta_5(n)} \alpha(n)\xi_i(n)\left(U^{(i)}(\bar{\xi}(n)) - U(\bar{\xi}(n))\right) \\
&\leq (\beta-1)\left(\sum_{i\in\Theta_3(n)} \xi_i(n) - \xi_i^\star + \sum_{i\in\Theta_5(n)} \xi_i^\star - \xi_i(n)\right) \\
&\quad +(\beta+1)\left(\sum_{i\in\Theta_4(n)} \xi_i(n) - \xi_i^\star + \sum_{i\in\Theta_6(n)} \xi_i^\star - \xi_i(n)\right) .
\end{aligned} \tag{6.12}$$

Note that $\sum_{i\in\Theta_3(n)\cup\Theta_4(n)\cup\Theta_5(n)\cup\Theta_6(n)} \alpha(n)\xi_i(n)\left(U^{(i)}(\bar{\xi}(n)) - U(\bar{\xi}(n))\right) = 0$. Thus, the left hand side of (6.12) is equal to

$$\begin{aligned}
&\sum_{i\in\Theta_3(n)\cup\Theta_6(n)} \alpha(n)\xi_i(n)\left(U^{(i)}(\bar{\xi}(n)) - U(\bar{\xi}(n))\right) \\
&\quad - \sum_{i\in\Theta_4(n)\cup\Theta_5(n)} \alpha(n)\xi_i(n)\left(U^{(i)}(\bar{\xi}(n)) - U(\bar{\xi}(n))\right) \\
&= \sum_{i\in\Theta_3(n)\cup\Theta_6(n)} 2\alpha(n)\xi_i(n)\left(U^{(i)}(\bar{\xi}(n)) - U(\bar{\xi}(n))\right) \\
&= \sum_{i\in\Theta_4(n)\cup\Theta_5(n)} -2\alpha(n)\xi_i(n)\left(U^{(i)}(\bar{\xi}(n)) - U(\bar{\xi}(n))\right) .
\end{aligned} \tag{6.13}$$

Define

$$\begin{aligned}
A(n) &:= \left(\sum_{i\in\Theta_3(n)} \xi_i(n) - \xi_i^\star + \sum_{i\in\Theta_5(n)} \xi_i^\star - \xi_i(n)\right) , \text{ and} \\
B(n) &:= \left(\sum_{i\in\Theta_4(n)} \xi_i(n) - \xi_i^\star + \sum_{i\in\Theta_6(n)} \xi_i^\star - \xi_i(n)\right) .
\end{aligned}$$

Note that $A(n) \geq 0$ and $B(n) \geq 0$ for every $n \in \mathbb{Z}_+$.

Case (i): Given $n \in \mathbb{Z}_+$, if right hand side of (6.12) is positive, i.e., $(\beta-1)A(n) + (\beta+1)B(n) > 0$, and $-\sum_{i\in\Theta_4(n)\cup\Theta_5(n)} 2\xi_i(n)\left(U^{(i)}(\bar{\xi}(n)) - U(\bar{\xi}(n))\right) \leq 0$, the property (6.12) holds for all positive $\alpha(n) > 0$. Case (ii): If $(\beta-1)A(n) + (\beta+$

$1)B(n) > 0$ and $-\sum_{i\in\Theta_4(n)\cup\Theta_5(n)} 2\xi_i(n)\left(U^{(i)}(\bar{\xi}(n)) - U(\bar{\xi}(n))\right) > 0$, the property (6.12) is satisfied with every

$$\alpha(n) \in \left(0, \frac{((\beta-1)A(n)+(\beta+1)B(n))}{-\sum_{i\in\Theta_4(n)\cup\Theta_5(n)} 2\xi_i(n)\left(U^{(i)}(\bar{\xi}(n)) - U(\bar{\xi}(n))\right)}\right].$$

Case (iii): If $(\beta-1)A(n)+(\beta+1)B(n) \leq 0$, however, the left hand side of (6.12) must be non-positive.

Consider the case that $B(n) > 0$. If $A(n)-B(n) \leq 0$, then $(\beta-1)A(n)+(\beta+1)B(n) > 0$ for any $\beta \in (0,1)$.[6] If $A(n)-B(n) > 0$, then $(\beta-1)A(n)+(\beta+1)B(n) > 0$ for any $\beta \in (\frac{A(n)-B(n)}{A(n)+B(n)}, 1)$. We assume that the modulus β is selected so that $(\beta-1)A(n)+(\beta+1)B(n) > 0$ for $n \in \mathbb{Z}_+$ with $B(n) > 0$. Define

$$\alpha_3^*(n) := \frac{(\beta-1)A(n)+(\beta+1)B(n)}{-\sum_{i\in\Theta_4(n)\cup\Theta_5(n)} 2\xi_i(n)\left(U^{(i)}(\bar{\xi}(n)) - U(\bar{\xi}(n))\right)}.$$

Then, choosing $\alpha(n)$ so that

$$\alpha(n) < \begin{cases} \min\{\alpha_1^*(n), \alpha_2^*(n), \alpha_3^*(n)\} \\ \qquad \text{if } -\sum_{i\in\Theta_4(n)\cup\Theta_5(n)} \xi_i(n)\left(U^{(i)}(\bar{\xi}(n)) - U(\bar{\xi}(n))\right) > 0 \\ \min\{\alpha_1^*(n), \alpha_2^*(n)\} \qquad \text{otherwise} \end{cases} \quad (6.14)$$

guarantees that the property (6.12) holds when $B(n) > 0$.

Note that $B(n) = 0$ and $A(n) = 0$ if and only if $\bar{\xi}(n) = \bar{\xi}^\star$. Thus, we focus on the case with $B(n) = 0$ and $A(n) > 0$. In this case, the right hand side of (6.12), i.e., $(\beta-1)A(n)+(\beta+1)B(n)$, is negative. Here, we must show two things. First, we need to show that the left hand side of (6.12) is negative when $B(n) = 0$ and $A(n) > 0$. Suppose that we can show that the left hand side of (6.12) is negative.

[6] $(\beta-1)A(n)+(\beta+1)B(n) = \beta(A(n)+B(n)) - A(n) + B(n)$

Then, in order for the update rule (6.8) to satisfy condition (6.12),

$$\alpha(n) \geq \frac{(\beta-1)A(n)}{\sum_{i\in\Theta_3(n)\cup\Theta_6(n)} 2\xi_i(n)\left(U^{(i)}(\bar{\xi}(n)) - U(\bar{\xi}(n))\right)} \tag{6.15}$$

or

$$\alpha(n) \geq \frac{(\beta-1)A(n)}{-\sum_{i\in\Theta_4(n)\cup\Theta_5(n)} 2\xi_i(n)\left(U^{(i)}(\bar{\xi}(n)) - U(\bar{\xi}(n))\right)}\ . \tag{6.16}$$

In addition, we must satisfy the condition (6.11). Thus, the second requirement we need to meet is that either

$$\frac{(\beta-1)A(n)}{\sum_{i\in\Theta_3(n)\cup\Theta_6(n)} 2\xi_i(n)\left(U^{(i)}(\bar{\xi}(n)) - U(\bar{\xi}(n))\right)} < \min\{\alpha_1^*(n), \alpha_2^*(n)\} \tag{6.17}$$

or

$$\frac{(\beta-1)A(n)}{-\sum_{i\in\Theta_4(n)\cup\Theta_5(n)} 2\xi_i(n)\left(U^{(i)}(\bar{\xi}(n)) - U(\bar{\xi}(n))\right)} < \min\{\alpha_1^*(n), \alpha_2^*(n)\} \tag{6.18}$$

when $B(n) = 0$ and $A(n) > 0$. However, due to the aforementioned difficulty in calculating the conditional expected payoff $U^{(i)}(\bar{\xi}(n))$ for any $i \in \mathcal{P}$, proving that these two conditions hold in general settings is challenging, if possible at all.

Theorem 6.6. *Suppose that there is a unique symmetric mixed strategy Nash equilibrium in noncooperative game among buyers and there are two sellers in the market. Then, the mixed strategy $\bar{\xi}(n)$ converges to the unique symmetric mixed strategy under the update rule in (6.8).*

Proof. For any $n \in \mathbb{Z}_+$, since there exists $\alpha(n) > 0$ such that (6.12) holds if $B(n) > 0$,[7] we need to show only that, when $B(n) = 0$ and $A(n) > 0$,[8] (i) the left hand side of (6.12) is negative and (ii) either (6.17) or (6.18) holds.

[7] We can choose $\alpha(n)$ so that (6.14) holds.

[8] Recall that $A(n) = 0$ and $B(n) = 0$ if and only if $\bar{\xi}(n) = \bar{\xi}^\star$.

Recall that $\tilde{\mu}(C)$ is the probability that a coalition $C \subset \mathcal{P}$ forms and $\hat{U}^{(i)}(C, \bar{\xi}(n))$ is the conditional expected payoff of a buyer choosing seller i in the coalition C, assuming the coalition C forms and all buyers employ the same mixed strategy $\bar{\xi}(n)$. Then, for a mixed strategy $\bar{\xi}(n)$, $n \in \mathbb{Z}_+$, we have

$$\begin{aligned} U^{(1)}(\bar{\xi}(n)) &= \tilde{\mu}(\{1\})\hat{U}^{(1)}(\{1\}, \bar{\xi}(n)) + \tilde{\mu}(\{1,2\})\hat{U}^{(1)}(\{1,2\}, \bar{\xi}(n)) \text{ and} \\ U^{(2)}(\bar{\xi}(n)) &= \tilde{\mu}(\{2\})\hat{U}^{(2)}(\{2\}, \bar{\xi}(n)) + \tilde{\mu}(\{1,2\})\hat{U}^{(2)}(\{1,2\}, \bar{\xi}(n)) \ . \end{aligned}$$

Suppose $\Xi^\star = (\bar{\xi}^\star, \bar{\xi}^\star, \ldots, \bar{\xi}^\star)$, where $\bar{\xi}^\star = (\xi_1^\star, \xi_2^\star)$, is the unique symmetric MSNE. Since $\hat{U}^{(i)}(C, \bar{\xi}(n))$ is determined by $\sum_{k \in C} \xi_k(n)$,

$$\hat{U}^{(1)}(\{1,2\}, \bar{\xi}(n)) = \hat{U}^{(2)}(\{1,2\}, \bar{\xi}(n)) = \hat{U}^{(1)}(\{1,2\}, \bar{\xi}^\star) = \hat{U}^{(2)}(\{1,2\}, \bar{\xi}^\star) \ . \tag{6.19}$$

When $\xi_1(n) > \xi_1^\star$ and $\xi_2(n) < \xi_2^\star$,

$$\begin{aligned} \hat{U}^{(1)}(\{1\}, \bar{\xi}(n)) &< \hat{U}^{(1)}(\{1\}, \bar{\xi}^\star) \text{ and} \\ \hat{U}^{(2)}(\{2\}, \bar{\xi}(n)) &> \hat{U}^{(2)}(\{2\}, \bar{\xi}^\star) \ . \end{aligned}$$

Thus, this yields

$$\hat{U}^{(1)}(\{1\}, \bar{\xi}(n)) < \hat{U}^{(1)}(\{1\}, \bar{\xi}^\star) = \hat{U}^{(2)}(\{2\}, \bar{\xi}^\star) < \hat{U}^{(2)}(\{2\}, \bar{\xi}(n)) \ . \tag{6.20}$$

Similarly, when $\xi_1(n) < \xi_1^\star$ and $\xi_2(n) > \xi_2^\star$, we can show that

$$\hat{U}^{(1)}(\{1\}, \bar{\xi}(n)) > \hat{U}^{(1)}(\{1\}, \bar{\xi}^\star) = \hat{U}^{(2)}(\{2\}, \bar{\xi}^\star) > \hat{U}^{(2)}(\{2\}, \bar{\xi}(n)) \ . \tag{6.21}$$

Note that $B(n) = 0$ if $\Theta_4(n) = \emptyset$ and $\Theta_6(n) = \emptyset$.[9] From (6.19), (6.20), and (6.21), we can see that, for every $l \in \Theta_3(n)$ and $k \in \Theta_5(n)$,

$$U^{(k)}(\bar{\xi}(n)) > U^{(k)}(\bar{\xi}^\star) = U^{(l)}(\bar{\xi}^\star) > U^{(l)}(\bar{\xi}(n)) \ .$$

[9] In this case, if $\xi_1(n) > \xi_1^\star$ and $\xi_2(n) < \xi_2^\star$, $\Theta_3(n) = \{1\}$ and $\Theta_5(n) = \{2\}$. Similarly, if $\xi_1(n) < \xi_1^\star$ and $\xi_2(n) > \xi_2^\star$, $\Theta_3(n) = \{2\}$ and $\Theta_5(n) = \{1\}$.

Thus, $\sum_{i\in\Theta_5(n)} \xi_i(n) \left(U^{(i)}(\bar{\xi}(n)) - U(\bar{\xi}(n))\right) > 0$. By (6.13), this implies that the left hand side of (6.12) is negative and there exists $\alpha(n) > 0$ such that (6.12) holds. Now we need to ensure that either (6.17) or (6.18) holds

From (6.20) and (6.21), we can see that $\Theta_7(n) = \Theta_5(n)$ and $\Theta_9(n) = \Theta_3(n)$. Since there are only two sellers,

$$\begin{aligned}
\alpha_1^*(n) &= \frac{1 - \xi_k(n)}{\xi_k(n)\left(U^{(k)}(\bar{\xi}(n)) - U(\bar{\xi}(n))\right)} \quad \text{and} \\
\alpha_2^*(n) &= \frac{1}{\left(U(\bar{\xi}(n)) - U^{(l)}(\bar{\xi}(n))\right)} \ ,
\end{aligned}$$

where $l \in \Theta_3(n)$ and $k \in \Theta_5(n)$. From the definition of $\Theta_3(n)$ and $\Theta_5(n)$,

$$\begin{aligned}
A(n) &= \sum_{i\in\Theta_3(n)} 2 \ \left(\xi_i(n) - \xi_i^\star\right) \\
&= \sum_{i\in\Theta_5(n)} 2 \ \left(\xi_i^\star - \xi_i(n)\right) \ .
\end{aligned}$$

Suppose that $\min\{\alpha_1^*(n), \alpha_2^*(n)\} = \alpha_1^*(n)$. The condition (6.18) can be rewritten as

$$\begin{aligned}
\frac{(\beta-1)A(n)}{-\sum_{i\in\Theta_5(n)} 2\xi_i(n)\left(U^{(i)}(\bar{\xi}(n)) - U(\bar{\xi}(n))\right)} &= \frac{(\beta-1)\left(\xi_k(n) - \xi_k^\star\right)}{\xi_k(n)\left(U^{(k)}(\bar{\xi}(n)) - U(\bar{\xi}(n))\right)} \\
&< \frac{1 - \xi_k(n)}{\xi_k(n)\left(U^{(k)}(\bar{\xi}(n)) - U(\bar{\xi}(n))\right)} \ , \qquad (6.22)
\end{aligned}$$

where $k \in \Theta_5(n)$. Since $\beta \in (0, 1)$ and $\xi_k(n) \leq \xi_k^\star < 1$ for $k \in \Theta_5(n)$, the condition (6.22) holds.

Suppose that $\min\{\alpha_1^*(n), \alpha_2^*(n)\} = \alpha_2^*(n)$. The condition (6.17) can be rewritten as

$$\begin{aligned}
\frac{(\beta-1)A(n)}{\sum_{i\in\Theta_3(n)} 2\xi_i(n)\left(U^{(i)}(\bar{\xi}(n)) - U(\bar{\xi}(n))\right)} &= \frac{(\beta-1)\left(\xi_l(n) - \xi_l^\star\right)}{\xi_l(n)\left(U^{(l)}(\bar{\xi}(n)) - U(\bar{\xi}(n))\right)} \\
&< \frac{-1}{\left(U^{(l)}(\bar{\xi}(n)) - U(\bar{\xi}(n))\right)} \ , \qquad (6.23)
\end{aligned}$$

where $l \in \Theta_3(n)$. Since $\xi_l(n) > \xi_l^\star$ for $l \in \Theta_3(n)$, the condition (6.23) holds. $\square$

When there are three or more sellers, as we did in the proof of Theorem 6.6, we need to show that (6.17) or (6.18) holds. However, without explicitly computing the buyers' expected payoffs, it becomes hard to prove in general.

Simulation results: Figures 6.2 and 6.3 show the simulation results when buyers' types are uniformly and exponentially distributed, respectively. The parameters used in simulation are given in Table 6.1. In the simulation, each buyer estimates the expected payoff $U^{(i)}(\xi(n))$ for all $i \in \mathcal{P}$ and compute $U(\xi(n))$ after 500 rounds of the auction. From the figures, it is clear that the buyers' strategy converges under the update rule (6.8). But, due to the difficulty of calculating the buyer's expected payoff, we leave the analysis on the convergence of the buyers' strategy in general cases as an open problem.

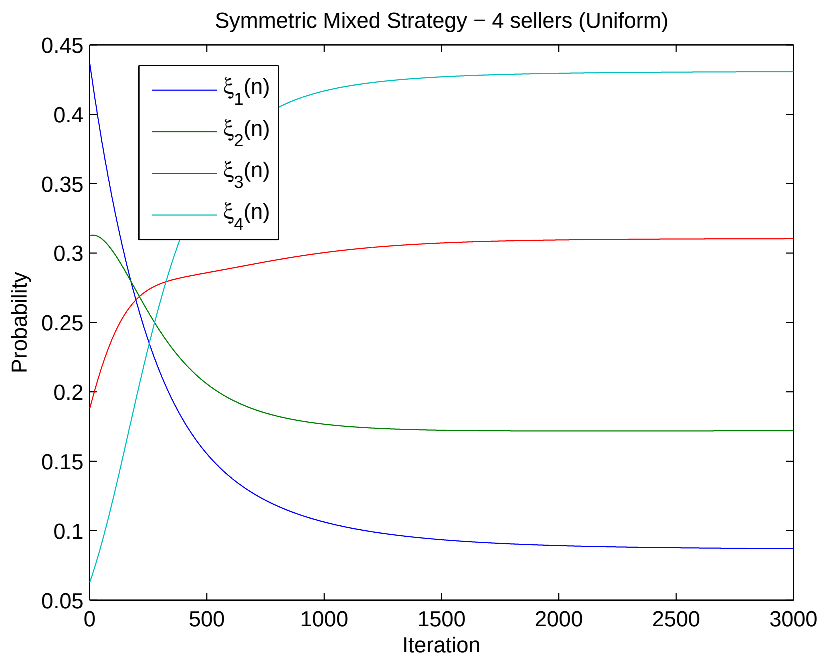

Figure 6.2: The buyer's mixed strategy with setting 1.

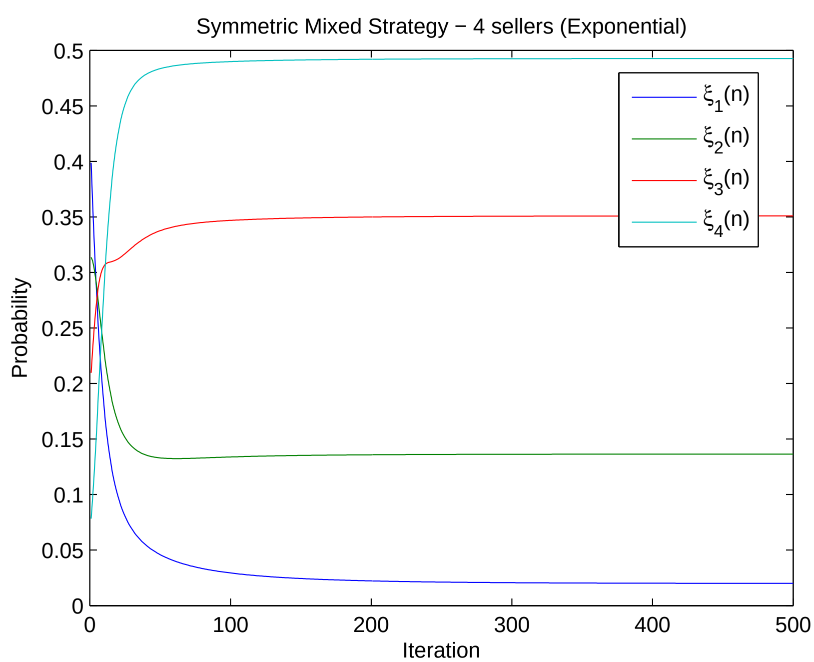

Figure 6.3: The buyer's mixed strategy with setting 2.

Table 6.1: Convergence check simulation setting

Setting 1		Setting 2	
Parameter	Value	Parameter	Value
# of sellers	4	# of sellers	4
# of units (seller 1)	2	# of units (seller 1)	2
# of units (seller 2)	3	# of units (seller 2)	3
# of units (seller 3)	5	# of units (seller 3)	5
# of units (seller 4)	7	# of units (seller 4)	7
Seller's value for item	0 for all units	Seller's value for item	0 for all units
# of buyers	10	# of buyers	10
$\mathcal{T}_j \quad (\forall j)$	[0,1]	$\mathcal{T}_j \quad (\forall j)$	$[0,\infty)$
$\mathcal{G}_j(t_j) \quad (\forall j)$	t_j	$\mathcal{G}_j(t_j) \quad (\forall j)$	$1 - \frac{1}{\lambda} e^{-\lambda t_j}$
λ	N/A	λ	$\frac{1}{100}$
$V_{j,k}(t_j) \quad (\forall j)$	$\frac{1}{k} t_j$	$V_{j,k}(t_j) \quad (\forall j)$	$\frac{1}{k} t_j$
$\mu(\omega)$	$1/\lvert\Omega_{\mathcal{P}}\rvert$	$\mu(\omega)$	$1/\lvert\Omega_{\mathcal{P}}\rvert$
$\alpha(n)$	0.1	$\alpha(n)$	0.01
# of iterations	3000	# of iterations	500

Chapter 7

Cooperative game among the sellers

7.1 Existence of an incentive for cooperation among the sellers

As mentioned earlier, a coalition of sellers will emerge only if its members find it beneficial to cooperate in that they can earn higher expected payoffs (or profits). In order to examine the existence of such an incentive for some or all of the sellers to cooperate, we compare the expected payoffs of different coalitions at a symmetric MSNE of the noncooperative game Γ. As we mentioned in the proof of Lemma 2, since there is only a fixed constant difference between the profit and the payoff of the seller, even though we examine the existence of incentive using the expected payoff, the result is also applicable to the expected profit term.

In the GBM with n buyers and m items to be sold, we can see that the expected payoff of an auctioneer is equal to[1]

$$\begin{aligned} U_0 &= \sum_{j=1}^{n} \mathrm{E}_{T_j}\left[c_j(T_j)\right] + \mathrm{E}_{\mathbf{T}}\left[\sum_{k=m^\star(\mathbf{T})+1}^{m} V_0^{(k)}\right] \\ &= \sum_{j=1}^{n} \mathrm{E}_{\mathbf{T}}\left[\sum_{k=1}^{m} \pi_{j,k}(T_j)\ p_{j,k}(\mathbf{T})\right] + \mathrm{E}_{\mathbf{T}}\left[\sum_{k=m^\star(\mathbf{T})+1}^{m} V_0^{(k)}\right], \end{aligned} \tag{7.1}$$

where $\mathbf{T}$ is the random vector of buyers' types, and $m^\star(\mathbf{T})$ is the number of items sold. From (7.1) and the allocation rule (5.21) under the GBM, we can see that the expected payoff of the seller is equal to the expected values of the contributions that

[1] Details are provided in Chapter 5.

are selected by the allocation rule plus the unsold items. This implies that when we compute the expected payoff of the seller we can use the contributions instead of the payments.

Recall that the total number of frequency bands for sale in the market is K_T. Define a random vector $\mathbf{B} := (B^j; j \in \mathcal{S})$, where B^j is the seller chosen by buyer j (using the selected symmetric MSNE strategy), and $\mathcal{S}_i(\mathbf{B}) = \{j \in \mathcal{S} \mid B^j = i\} \subset \mathcal{S}$. Given fixed types of the buyers, $\mathbf{t} \in \mathcal{T}$,

- $\pi_{j\cdot}(t_j) = \{\pi_{j,k}(t_j); k = 1, 2, \ldots, K_T\}$ is the set of the contributions of buyer j,
- $\tilde{\Pi}_{\mathbf{t}} := \{\pi_{j\cdot}(t_j); j \in \mathcal{S}\}$,
- $\Pi_{\mathbf{t}} := (\pi_{(k)}(\mathbf{t}); k = 1, 2, \ldots, D_T)$ is the vector of the contributions in $\tilde{\Pi}_{\mathbf{t}}$ ordered by decreasing value, where $D_T := N \cdot K_T$,
- for every $i \in \mathcal{P}$, $\tilde{\Pi}^i_{\mathbf{t}}(\mathbf{B}) := \{\pi_{j\cdot}(t_j); j \in \mathcal{S}_i(\mathbf{B})\}$, and
- $\Pi^i_{\mathbf{t}}(\mathbf{B})$, $i \in \mathcal{P}$, is the order statistics of $\tilde{\Pi}^i_{\mathbf{t}}(\mathbf{B})$.

The variables and functions which are used in this chapter are listed in Appendix B.

For every $\mathbf{t} \in \mathcal{T}$, define a mapping $\overline{\Pi}_{\mathbf{t}} : \mathcal{B} \to \mathcal{H}(\mathbf{t})$, where $\mathcal{B} = \mathcal{P}^N$, $\overline{\Pi}_{\mathbf{t}}(\mathbf{b}) = \{\Pi^i_{\mathbf{t}}(\mathbf{b}); i \in \mathcal{P}\}$, and $\mathcal{H}(\mathbf{t}) := \{\overline{\Pi}_{\mathbf{t}}(\mathbf{b});\ \mathbf{b} \in \mathcal{B}\}$. For each $\overline{\pi} \in \mathcal{H}(\mathbf{t})$, π^i denotes the ordered contributions of the buyers that choose seller i when the types of the buyers are given by $\mathbf{t}$.

Let $\mathbf{b}_{\mathbf{t}} : \mathcal{H}(\mathbf{t}) \to \mathcal{B}$, where $\mathbf{b}_{\mathbf{t}}(\overline{\pi})$, $\overline{\pi} \in \mathcal{H}(\mathbf{t})$, is the vector that tells us the selected sellers of the buyers under $\overline{\pi}$. Suppose that $\nu_{\mathbf{t}}$ is a distribution over the set

$\mathcal{H}(\mathbf{t})$, where $\nu_{\mathbf{t}}(\overline{\pi})$, $\overline{\pi} \in \mathcal{H}(\mathbf{t})$, is the probability $\mathbf{Pr}\left[\mathbf{B} = \mathbf{b}_{\mathbf{t}}(\overline{\pi})\right]$ determined by the symmetric MSNE.

As we mentioned in Chapter 5, from (5.24), for given buyers' types $\mathbf{t} \in \mathcal{T}$, the GBM allocates $m^{\star}(\mathbf{t})$ items, i.e., frequency bands, to the buyers with the $m^{\star}(\mathbf{t})$ highest contributions such that $m^{\star}(\mathbf{t})$ contributions are larger than $m^{\star}(\mathbf{t})$ smallest values of the seller(s). We call these selected contributions *winning* contributions. For a given $\overline{\pi} \in \mathcal{H}(\mathbf{t})$, denote the set of *winning contributions* in a coalition $C \subset \mathcal{P}$ by $\Psi_{\overline{\pi}}(C) \subset \tilde{\Pi}_{\mathbf{t}}$ and the sum of the winning contributions of coalition C by $\zeta(C, \overline{\pi})$. Similarly, define $\Phi_{\overline{\pi}}(C)$ to be the set of sellers' values of the unsold frequency bands in the coalition C, and $\lambda(C, \overline{\pi}) := \sum_{x \in \Phi_{\overline{\pi}}(C)} x$ the total value of the unsold items in coalition C.

For example, suppose that there are two sellers with a unit supply (i.e., one frequency band to sell) and that $\{\pi_{(1)}(\mathbf{t}), \pi_{(2)}(\mathbf{t}), \pi_{(3)}(\mathbf{t})\} \subset \Pi^1_{\mathbf{t}}(\mathbf{B})$ and $\{\pi_{(4)}(\mathbf{t}), \pi_{(5)}(\mathbf{t})\} \subset \Pi^2_{\mathbf{t}}(\mathbf{B})$, where $\pi_{(4)}(\mathbf{t}) < \pi_{(3)}(\mathbf{t})$. Also, sellers' values satisfy $V^1_1 \leq \pi_{(5)}(\mathbf{t})$ and $V^2_1 \leq \pi_{(5)}(\mathbf{t})$. Then, the winning *contributions* of the coalition $C = \{1, 2\}$ are $\pi_{(1)}(\mathbf{t})$ and $\pi_{(2)}(\mathbf{t})$, whereas the winning contribution of $C_1 = \{1\}$ and $C_2 = \{2\}$ is $\pi_{(1)}(\mathbf{t})$ and $\pi_{(4)}(\mathbf{t})$, respectively.

Let $m^{\star}_C = |\Psi_{\overline{\pi}}(C)|$ be the number of items sold by the coalition C and $p^{(C)}(t)$ the allocation rule of the coalition C according to the GBM. Then, we have

$$\begin{aligned} \zeta(C, \overline{\pi}) &= \sum_{k=1}^{K(C)} \Big(\sum_{j \in \mathcal{S}:\ b_{\mathbf{t},j}(\overline{\pi}) \in C} \pi_{j,k}(t_j)\ p^{(C)}_{j,k}(\mathbf{t}) \Big) \\ &= \sum_{\kappa \in\ \Psi_{\overline{\pi}}(C)} \kappa\ , \end{aligned} \tag{7.2}$$

where $K(C) = \sum_{i \in C} K^i$, and

$$p_{j,k}^{(C)}(\mathbf{t}) = \begin{cases} 1 & \text{if buyer } j \text{ is awarded at least } k \text{ items} \\ 0 & \text{otherwise.} \end{cases}$$

When each coalition holds a separate auction using the GBM, from the allocation rule (5.21), the coalition C awards $m_C^\star$ items to the buyers with the $m_C^\star$ highest *contributions* in $\bigcup_{i \in C} \pi^i$. Further, each unsold item's value is larger than or equal to that of every allocated item and every losing contribution. Therefore, it is clear that, for every disjoint coalitions C_1, $C_2 \subset \mathcal{P}$,

$$\begin{aligned} &\zeta(C_1, \overline{\pi}) + \lambda(C_1, \overline{\pi}) + \zeta(C_2, \overline{\pi}) + \lambda(C_2, \overline{\pi}) \\ &\leq \zeta(C_1 \cup C_2, \overline{\pi}) + \lambda(C_1 \cup C_2, \overline{\pi}). \end{aligned} \tag{7.3}$$

A strict inequality holds (i) if the smallest winning contribution in coalition C_1 is less than the largest losing contribution in coalition C_2 or vice versa or (ii) if the smallest value of unsold items in coalition C_1 is less than the largest losing contribution in coalition C_2 or vice versa.

For another example, as shown in Figure 7.1, suppose that there are three sellers; seller 1 has two frequency bands to sell and sellers 2 and 3 have a unit supply. Given $\mathbf{t} \in \mathcal{T}$, we assume that $\{\pi_{(1)}(\mathbf{t}), \pi_{(3)}(\mathbf{t}), \pi_{(4)}(\mathbf{t}), \pi_{(6)}(\mathbf{t})\} \subset \Pi_{\mathbf{t}}^1(\mathbf{B})$, $\{\pi_{(2)}(\mathbf{t}), \pi_{(5)}(\mathbf{t})\} \subset \Pi_{\mathbf{t}}^2(\mathbf{B})$, and $\{\pi_{(7)}(\mathbf{t}), \pi_{(8)}(\mathbf{t})\} \subset \Pi_{\mathbf{t}}^3(\mathbf{B})$, where $\pi_{(5)}(\mathbf{t}) < \pi_{(4)}(\mathbf{t})$. Also, sellers' values are assumed to satisfy $V_1^1 \leq V_2^1 < \pi_{(4)}(\mathbf{t})$, $V_1^2 < \pi_{(4)}(\mathbf{t})$, and $\pi_{(5)}(\mathbf{t}) \leq V_1^3 < \pi_{(4)}(\mathbf{t})$. Then, for every $C \subset \{1, 2, 3\}$, the sum of the winning contributions and the values of the unsold items in coalition C, $\zeta(C, \overline{\pi}) + \lambda(C, \overline{\pi})$, can be listed as follows:

- $\zeta(\{1\}, \overline{\pi}) + \lambda(\{1\}, \overline{\pi}) = \pi_{(1)}(\mathbf{t}) + \pi_{(3)}(\mathbf{t})$,
- $\zeta(\{2\}, \overline{\pi}) + \lambda(\{2\}, \overline{\pi}) = \pi_{(2)}(\mathbf{t})$,
- $\zeta(\{3\}, \overline{\pi}) + \lambda(\{3\}, \overline{\pi}) = V_1^3$,
- $\zeta(\{1,2\}, \overline{\pi}) + \lambda(\{1,2\}, \overline{\pi}) = \pi_{(1)}(\mathbf{t}) + \pi_{(2)}(\mathbf{t}) + \pi_{(3)}(\mathbf{t})$,
- $\zeta(\{1,3\}, \overline{\pi}) + \lambda(\{1,3\}, \overline{\pi}) = \pi_{(1)}(\mathbf{t}) + \pi_{(3)}(\mathbf{t}) + \pi_{(4)}(\mathbf{t})$,
- $\zeta(\{2,3\}, \overline{\pi}) + \lambda(\{2,3\}, \overline{\pi}) = \pi_{(2)}(\mathbf{t}) + V_1^3$,
- $\zeta(\{1,2,3\}, \overline{\pi}) + \lambda(\{1,2,3\}, \overline{\pi}) = \pi_{(1)}(\mathbf{t}) + \pi_{(2)}(\mathbf{t}) + \pi_{(3)}(\mathbf{t}) + \pi_{(4)}(\mathbf{t})$.

One can easily verify that (7.3) holds for every disjoint coalitions in the example.

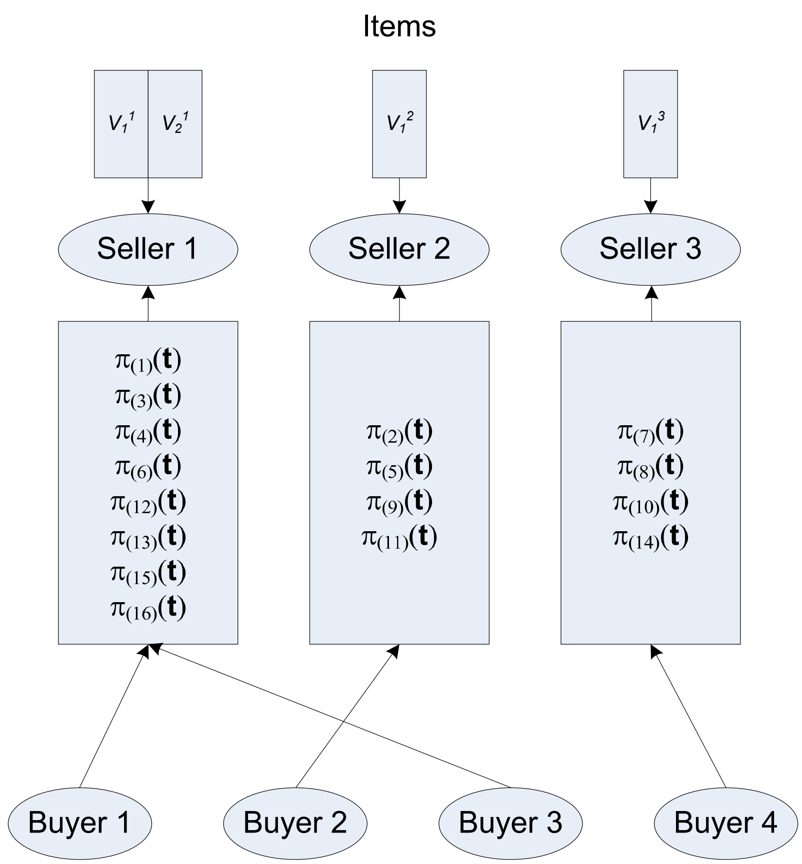

Figure 7.1: Example: Received contributions.

Let us first define, for each $\mathbf{t} \in \mathcal{T}$,

$$v(C;\mathbf{t}) := \sum_{\overline{\pi} \in \mathcal{H}(\mathbf{t})} \left(\zeta(C, \overline{\pi}) + \lambda(C, \overline{\pi})\right) \ \nu_{\mathbf{t}}(\overline{\pi}). \tag{7.4}$$

Then, the expected payoff of a coalition C is given by $\mathrm{E}_{\mathbf{T}}\left[v(C;\mathbf{T})\right]$. We can prove the following theorem from (7.3) and (7.4).

Theorem 7.1. *For every two disjoint coalitions C_1 and C_2,*

$$v(C_1) + v(C_2) \leq v(C_1 \cup C_2). \tag{7.5}$$

Theorem 7.1 tells us that the expected payoff function v satisfies the *super-additivity* property. In addition, it implies that risk neutral sellers will have an incentive to cooperate among themselves in order to increase their expected payoffs (resp. profits), assuming that they can find an equitable way of sharing the payoff (resp. profit).

7.2 Profit sharing and a cooperative game among the sellers

Our finding in the previous section indicates that the sellers will find it advantageous to cooperate with each other and form a grand coalition that includes all sellers if they want to maximize their expected payoffs (or profits). However, in order for the sellers to maintain such cooperation, they must be able to find an acceptable way of sharing the payoffs (or profits). In light of this, a natural question that arises is how the sellers should share the payoff (or profit) among themselves when they decide to cooperate. In order to answer this question we turn to cooperative game theory, and we model the interaction between the sellers as a cooperative game.

A cooperative game is often given by a *characteristic function* $v : 2^{\mathcal{P}} \to \mathbb{R}$. The characteristic function v assigns to each coalition $C \subset \mathcal{P}$ a value that is the total payoff of the members in the coalition they can guarantee themselves against the other players. The characteristic function of the cooperative game among the sellers in our problem is defined through the expected payoff of the coalitions at the assumed symmetric MSNE of the noncooperative game among the buyers. In other words, for every $C \subset \mathcal{P}$, $v(C)$ denotes the expected payoff the sellers in the coalition C can achieve without the help of the remaining sellers.

We first introduce following definitions [55].

Definition 7. *An imputation for an M-player cooperative game is a vector $x = (x_1, ..., x_M)$ that satisfies*

$$\begin{aligned} &(1) \quad \sum_{i \in \mathcal{P}} x_i = v(\mathcal{P}), \ \textit{and} \\ &(2) \quad x_i \geq v(\{i\}) \ \ \textit{for all} \ \ i \in \mathcal{P} \ . \end{aligned}$$

Definition 8. *Let x and y be two imputations. (i) Let $C \subset \mathcal{P}$ be a coalition. We say that x dominates y through C if*

$$\begin{aligned} &(1) \quad x_i > y_i \ \ \textit{for all} \ \ i \in C, \ \textit{and} \\ &(2) \quad \sum_{i \in C} x_i \leq v(C) \ . \end{aligned}$$

(ii) We say that x dominates y if there exists some coalition $C^\star \subset \mathcal{P}$ such that x dominates y through $C^\star$.

Definition 9. *The set of all undominated imputations is called the core of the cooperative game.*

The following theorem gives an alternate characterization of the core of a cooperative game and a means of finding it.

Theorem 7.2. *[55, p.219] The core is the set of all M-vectors x satisfying*

$$(1) \quad \sum_{i \in C} x_i \geq v(C) \ \ \textit{for all} \ \ C \subset \mathcal{P}, \quad \textit{and}$$
$$(2) \quad \sum_{i \in \mathcal{P}} x_i = v(\mathcal{P}) \ .$$

The conditions in Theorem 7.2 imply that no subset of the sellers (i.e., a coalition) has the power to increase its expected payoff by deviating from the grand coalition. Therefore, a payoff vector in the core can be viewed as a stable equilibrium and a candidate for fair sharing of the payoffs among the sellers.[2]

Unfortunately, the core of a cooperative game is in general not guaranteed to be nonempty, and proving the existence of a nonempty core can be nontrivial. However, we can show that the core of the cooperative game among the sellers under consideration is nonempty. This implies that indeed there exists a way for the sellers to share the payoffs (or profits) in such a way that no subset of the sellers will be able to leave the grand coalition and increase their expected payoffs (or profits).

Theorem 7.3. *The cooperative game v among the sellers has a nonempty core.*

In order to prove the theorem, we use the following well known result for the existence of a nonempty core: Let $\mathbf{y} = (y_C; C \subset \mathcal{P})$ be a nonnegative vector that

[2] Note that fair sharing of the payoffs represents fair sharing of the profits.

satisfies the condition

$$\sum_{C \subset \mathcal{P}: i \in C} y_C = 1 \;\; \text{for all} \;\; i \in \mathcal{P} \; . \tag{7.6}$$

Theorem 7.4. *[55, p.225] A necessary and sufficient condition for the game to have a nonempty core is that, for every nonnegative vector $(y_C; C \subset \mathcal{P})$ satisfying (7.6), we have*

$$\sum_{C \subset \mathcal{P}} y_C \cdot v(C) \leq v(\mathcal{P}) \; .$$

We first introduce following notation. Suppose that $\mathbf{y} = (y_C; C \subset \mathcal{P})$ is a nonnegative vector that satisfies (7.6). Then, for every $\mathbf{t} \in \mathcal{T}$ and $\mathbf{b} \in \mathcal{B}$, we have

$$\sum_{C \subset \mathcal{P}: \pi_{(k)}(\mathbf{t}) \in \Psi_{\overline{\pi}}(C)} \big(y_C \cdot \pi_{(k)}(\mathbf{t}) \big) \leq \pi_{(k)}(\mathbf{t}) \; , \tag{7.7}$$

where $\overline{\pi} = \overline{\Pi}_{\mathbf{t}}(\mathbf{b})$. Define $i_k(\mathbf{t}, \mathbf{b})$ to be the seller i whose $\Pi^i_{\mathbf{t}}(\mathbf{b})$ contains $\pi_{(k)}(\mathbf{t})$, i.e., $\pi_{(k)}(\mathbf{t}) \in \Pi^{i_k(\mathbf{t},\mathbf{b})}_{\mathbf{t}}(\mathbf{b})$. The equality in (7.7) holds if and only if $\pi_{(k)}(\mathbf{t})$ is a winning contribution in every coalition C that contains the seller $i_k(\mathbf{t}, \mathbf{b})$ and $y_C > 0$. With a little abuse of notation, for each $C \subset \mathcal{P}$, define

$$a_C^{(k)} = \begin{cases} y_C & \text{if } \pi_{(k)}(\mathbf{t}) \in \Psi_{\overline{\pi}}(C) \; , \\ 0 & \text{otherwise.} \end{cases}$$

Here, we denote the k-th smallest value of the item in the set of all sellers' values $\mathcal{V}$ by $V_0^{(k)}$. Recall that $\Phi_{\overline{\pi}}(C)$ is the set of sellers' values of the unsold frequency bands in the coalition C. It is clear

$$\sum_{C \subset \mathcal{P}: V_0^{(k)} \in \; \Phi_{\overline{\pi}}(C)} \left(y_C \cdot V_0^{(k)} \right) \leq V_0^{(k)}. \tag{7.8}$$

Let $\bar{i}_k^v$ be the seller that has the value $V_0^{(k)}$ for one of its frequency bands. The equality in (7.8) holds if and only if the frequency band with the k-th smallest value is unsold in every coalition C that includes the seller $\bar{i}_k^v$ and $y_C > 0$. The left hand side of (7.8) is equal to zero if the frequency band is allocated in every coalition C. For each $C \subset \mathcal{P}$, define

$$b_C^{(k)} = \begin{cases} y_C & \text{if } V_0^{(k)} \in \Phi_{\overline{\pi}}(C) \ , \\ 0 & \text{otherwise.} \end{cases}$$

We denote the number of items sold when all sellers cooperate by

$$k^\star := \max\{\ell \in \{1, 2, \ldots, K_T\} \mid \pi_{(\ell)}(\mathbf{t}) > V_0^{(\ell)}\}.$$

The maximum is equal to zero if the set on the right hand side is empty. Let $\mathcal{K}^\star := \{1, 2, \ldots, k^\star\}$.

We partition the set of items available for sale as follows:

$$\begin{aligned}
\Theta^1 &:= \{k \in \mathcal{K}^\star \mid \text{equality in (7.7) holds}\} \\
&= \{k \in \mathcal{K}^\star \mid \pi_{(k)}(\mathbf{t}) \in \Psi_{\overline{\pi}}(C) \text{ for all } C \subset \mathcal{P} \text{ such that } i_k(\mathbf{t}, \mathbf{b}) \in C\} \\
\Theta^2 &:= \mathcal{K}^\star \setminus \Theta^1 \\
\Theta^3 &:= \{k \in \{k^\star + 1, \ldots, D_T\} \mid \exists\, C \subset \mathcal{P} \text{ such that } \pi_{(k)}(\mathbf{t}) \in \Psi_{\overline{\pi}}(C)\} \\
\Theta^4 &:= \{k \in \Theta^2 \mid \pi_{(k)}(\mathbf{t}) \in \Psi_{\overline{\pi}}(\{i_k(\mathbf{t}, \mathbf{b})\})\} \\
\Theta^5 &:= \{k \in \Theta^3 \mid \pi_{(k)}(\mathbf{t}) \in \Psi_{\overline{\pi}}(\{i_k(\mathbf{t}, \mathbf{b})\})\} \\
\Theta^6 &:= \Theta^2 \setminus \Theta^4 \\
\Theta^7 &:= \Theta^3 \setminus \Theta^5 \\
\Theta^8 &:= \{k \in \{k^\star + 1, \ldots, K_T\} \mid \text{equality in (7.8) holds}\}
\end{aligned}$$

$$\Theta^9 := \{k \in \mathcal{K}^\star \mid \text{strict inequality in (7.8) holds }\}$$

$$\Theta^{10} := \{k \in \{k^\star + 1, \ldots, K_T\} \mid \exists\, C \subset \mathcal{P} \text{ such that } V_0^{(k)} \notin \Phi_{\overline{\pi}}(C)\}$$

$$\Theta^{11} := \{k \in \Theta^9 \mid V_0^{(k)} \in \Phi_{\overline{\pi}}(\{\bar{i}_k^v\})\}$$

$$\Theta^{12} := \{k \in \Theta^{10} \mid V_0^{(k)} \in \Phi_{\overline{\pi}}(\{\bar{i}_k^v\})\}$$

$$\Theta^{13} := \Theta^9 \setminus \Theta^{11}$$

$$\Theta^{14} := \Theta^{10} \setminus \Theta^{12}$$

Note from the definition of the sets

$$\Theta^2 = \Theta^4 \cup \Theta^6,\ \Theta^3 = \Theta^5 \cup \Theta^7, \tag{7.9}$$

$$\Theta^9 = \Theta^{11} \cup \Theta^{13},\ \text{and } \Theta^{10} = \Theta^{12} \cup \Theta^{14}.$$

Lemma 5.

$$\sum_{C \subset \mathcal{P}} \left(\sum_{k \in \Theta^2} a_C^{(k)} + \sum_{k \in \Theta^3} a_C^{(k)} \right) + \sum_{C \subset \mathcal{P}} \left(\sum_{k \in \Theta^9} b_C^{(k)} + \sum_{k \in \Theta^{10}} b_C^{(k)} \right)$$
$$= |\Theta^2| + |\Theta^{10}|. \tag{7.10}$$

Proof. For any given $C \subset \mathcal{P}$, define

$$\Theta_C^{L_c} := \{k \in \Theta^4 \cup \Theta^5 \mid i_k(\mathbf{t}, \mathbf{b}) \in C \text{ and } a_C^{(k)} = 0\},$$

$$\Theta_C^{W_c} := \{k \in \Theta^6 \cup \Theta^7 \mid i_k(\mathbf{t}, \mathbf{b}) \in C \text{ and } a_C^{(k)} = y_C\},$$

$$\Theta_C^{L_v} := \{k \in \Theta^{11} \cup \Theta^{12} \mid \bar{i}_k^v \in C \text{ and } b_C^{(k)} = 0\},\ \text{and}$$

$$\Theta_C^{W_v} := \{k \in \Theta^{13} \cup \Theta^{14} \mid \bar{i}_k^v \in C \text{ and } b_C^{(k)} = y_C\}.$$

From the definition of the sets Θ^n, $n = 1, 2, \ldots, 14$, the following observations can be made.

O1. Suppose that there exist $k_1 \in \Theta^6$ (resp. $k_1 \in \Theta^7$) and a coalition $C \subset \mathcal{P}$ such that $\pi_{(k_1)}(\mathbf{t})$ is a winning contribution in the coalition C. This implies that either (i) there exists $k_2 \in \Theta^4 \cup \Theta^5$ (resp. $k_2 \in \Theta^5$), where $k_2 > k_1$, such that the seller $i_{k_2}(\mathbf{t}, \mathbf{b}) \in C$ and $a_C^{(k_2)} = 0$, or (ii) there exists $k_2 \in \Theta^{11} \cup \Theta^{12}$ such that the seller $\bar{i}_{k_2}^v \in C$, $\pi_{(k_1)}(\mathbf{t}) > V_0^{(k_2)}$ and $b_C^{(k_2)} = 0$.

O2. Suppose that the item with seller's value $V_0^{(k_1)}$ is unsold in a coalition $C \subset \mathcal{P}$ for some $k_1 \in \Theta^{13}$ (resp. $k_1 \in \Theta^{14}$). Then, either (i) there exists $k_2 \in \Theta^5$ (resp. $k_2 \in \Theta^4 \cup \Theta^5$) such that the seller $i_{k_2}(\mathbf{t}, \mathbf{b}) \in C$, $V_0^{(k_1)} > \pi_{(k_2)}(\mathbf{t})$ and $a_C^{(k_2)} = 0$, or (ii) there exists $k_2 \in \Theta^{11}$ (resp. $k_2 \in \Theta^{11} \cup \Theta^{12}$), $k_1 > k_2$, such that the seller $\bar{i}_{k_2}^v \in C$ and $b_C^{(k_2)} = 0$.

O3. Suppose that there exist $k_1 \in \Theta^4$ (resp. $k_1 \in \Theta^5$) and a coalition $C \subset \mathcal{P}$ such that $\pi_{(k_1)}(\mathbf{t})$ is not a winning contribution in the coalition C. This implies that either (i) there exists $k_2 \in \Theta^6$ (resp. $k_2 \in \Theta^6 \cup \Theta^7$), where $k_1 > k_2$, such that the seller $i_{k_2}(\mathbf{t}, \mathbf{b}) \in C$ and $a_C^{(k_2)} = y_C$, or (ii) there exists $k_2 \in \Theta^{14}$ (resp. $\Theta^{13} \cup \Theta^{14}$) such that the seller $\bar{i}_{k_2}^v \in C$, $V_0^{(k_2)} > \pi_{(k_1)}(\mathbf{t})$ and $b_C^{(k_2)} = y_C$.

O4. Suppose that the item with seller's value $V_0^{(k_1)}$ is sold in a coalition $C \subset \mathcal{P}$ for some $k_1 \in \Theta^{11}$ (resp. $k_1 \in \Theta^{12}$). Then, either (i) there exists $k_2 \in \Theta^6 \cup \Theta^7$ (resp. $k_2 \in \Theta^6$) such that the seller $i_{k_2}(\mathbf{t}, \mathbf{b}) \in C$, $\pi_{(k_2)}(\mathbf{t}) > V_0^{(k_1)}$ and $a_C^{(k_2)} = y_C$, or (ii) there exists $k_2 \in \Theta^{13} \cup \Theta^{14}$ (resp. $k_2 \in \Theta^{14}$), $k_2 > k_1$, such that the seller $\bar{i}_{k_2}^v \in C$ and $b_C^{(k_2)} = y_C$.

O5. From observations **O1**, **O2**, **O3**, and **O4**, $|\Theta_C^{L_c}| + |\Theta_C^{L_v}| = |\Theta_C^{W_c}| + |\Theta_C^{W_v}|$.

O6. One can show that $\Theta^1 \cup \Theta^4 \cup \Theta^5$ is the set of winning contributions and $\Theta^8 \cup \Theta^{11} \cup \Theta^{12}$ is the set of unsold items when sellers hold separate individual auctions.

Hence, the cardinality of their union is the number of available items K_T. Further, it is clear from their definitions that $\Theta^1 \cup \Theta^2 = \mathcal{K}^\star$ and $\Theta^8 \cup \Theta^{10} = \{k^\star + 1, \dots, K_T\}$. Thus, we have $|\Theta^4| + |\Theta^5| + |\Theta^{11}| + |\Theta^{12}| = |\Theta^2| + |\Theta^{10}|$.

From (7.9), we get

$$\begin{aligned} &\sum_{C \subset \mathcal{P}} \left(\sum_{k \in \Theta^2} a_C^{(k)} + \sum_{k \in \Theta^3} a_C^{(k)} + \sum_{k \in \Theta^9} b_C^{(k)} + \sum_{k \in \Theta^{10}} b_C^{(k)} \right) \\ &= \sum_{C \subset \mathcal{P}} \left(\sum_{k \in \Theta^4 \cup \Theta^5} a_C^{(k)} + \sum_{k \in \Theta^6 \cup \Theta^7} a_C^{(k)} \right) \\ &\quad + \sum_{C \subset \mathcal{P}} \left(\sum_{k \in \Theta^{11} \cup \Theta^{12}} b_C^{(k)} + \sum_{k \in \Theta^{13} \cup \Theta^{14}} b_C^{(k)} \right) \end{aligned} \tag{7.11}$$

Using the definitions of $a_C^{(k)}$ and $b_C^{(k)}$, we can rewrite terms in (7.11).

$$\begin{aligned} &(7.11) \\ &= \overbrace{\sum_{k \in \Theta^4 \cup \Theta^5} \Bigg(\sum_{C \subset \mathcal{P}:\ i_k(\mathbf{t}, \mathbf{b}) \in C} a_C^{(k)} \Bigg)}^{= \Upsilon_1} + \overbrace{\sum_{C \subset \mathcal{P}} \Bigg(\sum_{k \in \Theta^6 \cup \Theta^7} a_C^{(k)} \Bigg)}^{= \Upsilon_2} \\ &+ \underbrace{\sum_{k \in \Theta^{11} \cup \Theta^{12}} \Bigg(\sum_{C \subset \mathcal{P}:\ \bar{i}_k^v \in C} b_C^{(k)} \Bigg)}_{= \Upsilon_3} + \underbrace{\sum_{C \subset \mathcal{P}} \Bigg(\sum_{k \in \Theta^{13} \cup \Theta^{14}} b_C^{(k)} \Bigg)}_{= \Upsilon_4}. \end{aligned}$$

From observations **O1** through **O5**, for every $k \in \Theta^6 \cup \Theta^7$ and $C \subset \mathcal{P}$ such that $a_C^{(k)} = y_C$, we can find either (i) $k_1 \in \Theta^4 \cup \Theta^5$ such that $i_{k_1}(\mathbf{t}, \mathbf{b}) \in C$ and $a_C^{(k_1)} = 0$ or (ii) $k_2 \in \Theta^{11} \cup \Theta^{12}$ such that $\bar{i}_{k_2}^v \in C$ and $b_C^{(k_2)} = 0$. Similarly, for every $\tilde{k} \in \Theta^{13} \cup \Theta^{14}$ and $C \subset \mathcal{P}$ such that $b_C^{(\tilde{k})} = y_C$, we can find either (i) $k_3 \in \Theta^4 \cup \Theta^5$ such that $i_{k_3}(\mathbf{t}, \mathbf{b}) \in C$ and $a_C^{(k_3)} = 0$ or (ii) $k_4 \in \Theta^{11} \cup \Theta^{12}$ such that $\bar{i}_{k_4}^v \in C$ and $b_C^{(k_4)} = 0$. Therefore, we can swap the nonnegative $a_C^{(k)}$ or $b_C^{(k)}$ in Υ_2 and Υ_4, respectively, with the zero terms in Υ_1 and Υ_3. This swapping of the terms gives us

$$
\begin{aligned}
&\sum_{C\subset\mathcal{P}}\Big(\sum_{k\in\Theta^2} a_C^{(k)} + \sum_{k\in\Theta^3} a_C^{(k)} + \sum_{k\in\Theta^9} b_C^{(k)} + \sum_{k\in\Theta^{10}} b_C^{(k)}\Big)\\
&= \sum_{k\in\Theta^4}\Big(\sum_{C\subset\mathcal{P}:\ i_k(\mathbf{t},\mathbf{b})\in C} y_C\Big) + \sum_{k\in\Theta^5}\Big(\sum_{C\subset\mathcal{P}:\ i_k(\mathbf{t},\mathbf{b})\in C} y_C\Big)\\
&\quad + \sum_{k\in\Theta^{11}}\Big(\sum_{C\subset\mathcal{P}:\ \bar{i}_k^v\in C} y_C\Big) + \sum_{k\in\Theta^{12}}\Big(\sum_{C\subset\mathcal{P}:\ \bar{i}_k^v\in C} y_C\Big)\\
&= |\Theta^4| + |\Theta^5| + |\Theta^{11}| + |\Theta^{12}|\\
&= |\Theta^2| + |\Theta^{10}|, \qquad (7.12)
\end{aligned}
$$

where the last equality follows from observation **O6**. This proves the lemma. □

Proof of Theorem 7.3. First, define $\varphi_{(2)}^{(k)} := 1 - \sum_{C\subset\mathcal{P}} a_C^{(k)}$ for $k\in\Theta^2$ and $\varphi_{(10)}^{(k)} := 1 - \sum_{C\subset\mathcal{P}} b_C^{(k)}$ for $k\in\Theta^{10}$. From (7.12) we obtain

$$
\begin{aligned}
&\sum_{k\in\Theta^2}\varphi_{(2)}^{(k)} + \sum_{k\in\Theta^{10}}\varphi_{(10)}^{(k)}\\
&= |\Theta^2| - \sum_{C\subset\mathcal{P}}\Big(\sum_{k\in\Theta^2} a_C^{(k)}\Big) + |\Theta^{10}| - \sum_{C\subset\mathcal{P}}\Big(\sum_{k\in\Theta^{10}} b_C^{(k)}\Big)\\
&= \sum_{C\subset\mathcal{P}}\left(\sum_{k\in\Theta^3} a_C^{(k)} + \sum_{k\in\Theta^9} b_C^{(k)}\right) \qquad (7.13)
\end{aligned}
$$

Let $\pi_\star := \inf\{\pi_{(k)}(\mathbf{t});\ k\in\Theta^2\}$ and $V_\star := \inf\{V_0^{(k)};\ k\in\Theta^{10}\}$. Then, we have the following inequality.

$$
\begin{aligned}
&\sum_{k\in\Theta^2}\pi_{(k)}(\mathbf{t}) + \sum_{k\in\Theta^{10}} V_0^{(k)}\\
&= \sum_{k\in\Theta^2}\pi_{(k)}(\mathbf{t})\Big(\varphi_{(2)}^{(k)} + \sum_{C\subset\mathcal{P}} a_C^{(k)}\Big) + \sum_{k\in\Theta^{10}} V_0^{(k)}\Big(\varphi_{(10)}^{(k)} + \sum_{C\subset\mathcal{P}} b_C^{(k)}\Big)\\
&\geq \sum_{k\in\Theta^2}\Big(\pi_\star\ \varphi_{(2)}^{(k)} + \sum_{C\subset\mathcal{P}}\pi_{(k)}(\mathbf{t})\ a_C^{(k)}\Big) + \sum_{k\in\Theta^{10}}\Big(V_\star\ \varphi_{(10)}^{(k)} + \sum_{C\subset\mathcal{P}} V_0^{(k)} b_C^{(k)}\Big)\\
&\geq \sum_{k\in\Theta^2}\Big(\sum_{C\subset\mathcal{P}}\pi_{(k)}(\mathbf{t})\ a_C^{(k)}\Big) + \sum_{k\in\Theta^{10}}\Big(\sum_{C\subset\mathcal{P}} V_0^{(k)} b_C^{(k)}\Big)\\
&\qquad + \min\{\pi_\star, V_\star\}\Big(\sum_{k\in\Theta^2}\varphi_{(2)}^{(k)} + \sum_{k\in\Theta^{10}}\varphi_{(10)}^{(k)}\Big) \qquad (7.14)
\end{aligned}
$$

By interchanging the order of summations and from (7.13)

$$
\begin{aligned}
(7.14) = & \sum_{C \subset \mathcal{P}} \Big(\sum_{k \in \Theta^2} \pi_{(k)}(\mathbf{t}) \; a_C^{(k)} + \sum_{k \in \Theta^{10}} V_0^{(k)} \; b_C^{(k)} \Big) \\
& + \sum_{C \subset \mathcal{P}} \Big(\sum_{k \in \Theta^3} \min \{\pi_\star, V_\star\} a_C^{(k)} \Big) \\
& + \sum_{C \subset \mathcal{P}} \Big(\sum_{k \in \Theta^9} \min \{\pi_\star, V_\star\} b_C^{(k)} \Big) \; . \qquad (7.15)
\end{aligned}
$$

Note that $\pi_{(k)}(\mathbf{t}) \leq \min \{\pi_\star, V_\star\}$ for all $k \in \Theta^3$ and $V_0^{(k)} \leq \min\{\pi_\star, V_\star\}$ for all $k \in \Theta^9$. Thus, from (7.14) - (7.15) and these inequalities, we get

$$
\begin{aligned}
\sum_{k \in \Theta^2} \pi_{(k)}(\mathbf{t}) + \sum_{k \in \Theta^{10}} V_0^{(k)} \;\; \geq \;\; & \sum_{C \subset \mathcal{P}} \Big(\sum_{k \in \Theta^2} \pi_{(k)}(\mathbf{t}) \; a_C^{(k)} + \sum_{k \in \Theta^3} \pi_{(k)}(\mathbf{t}) \; a_C^{(k)} \Big) \\
& + \sum_{C \subset \mathcal{P}} \Big(\sum_{k \in \Theta^9} V_0^{(k)} \; b_C^{(k)} + \sum_{k \in \Theta^{10}} V_0^{(k)} \; b_C^{(k)} \Big). \quad (7.16)
\end{aligned}
$$

Finally, from (7.16) and the definition of Θ^1 and Θ^8,

$$
\begin{aligned}
& \sum_{C \subset \mathcal{P}} \Big(\sum_{k \in \Theta^1} a_C^{(k)} \; \pi_{(k)}(\mathbf{t}) + \sum_{k \in \Theta^2} a_C^{(k)} \; \pi_{(k)}(\mathbf{t}) + \sum_{k \in \Theta^3} a_C^{(k)} \; \pi_{(k)}(\mathbf{t}) \Big) \\
& \quad + \sum_{C \subset \mathcal{P}} \Big(\sum_{k \in \Theta^9} b_C^{(k)} \; V_0^{(k)} + \sum_{k \in \Theta^{10}} b_C^{(k)} \; V_0^{(k)} + \sum_{k \in \Theta^8} b_C^{(k)} \; V_0^{(k)} \Big) \\
& = \sum_{C \subset \mathcal{P}} \big(y_C \;\; (\zeta(C, \overline{\pi}) + \lambda(C, \overline{\pi})) \big) \\
& \leq \sum_{k \in \Theta^1 \cup \Theta^2} \pi_{(k)}(\mathbf{t}) + \sum_{k \in \Theta^{10}} V_0^{(k)} + \sum_{k \in \Theta^8} V_0^{(k)}.
\end{aligned}
$$

Since $\sum_{k \in \Theta^1 \cup \Theta^2} \pi_{(k)}(\mathbf{t}) + \sum_{k \in \Theta^8 \cup \Theta^{10}} V_0^{(k)} = \zeta(\mathcal{P}, \overline{\pi}) + \lambda(\mathcal{P}, \overline{\pi})$, we can conclude

$$
\sum_{C \subset \mathcal{P}} y_C \; v(C) \leq v(\mathcal{P}) \; . \qquad (7.17)
$$

□

7.3 Profit sharing mechanisms

Since the *core* exists, i.e., is nonempty, in the cooperative game among the sellers, assuming that sellers can achieve some imputation, i.e., a vector of sellers' expected payoffs, in the core, they are likely to cooperate. Hence, the next natural question is how the sellers should reach such an imputation in the core. In this section we introduce a family of profit sharing schemes that can realize *any* imputation in the core.

Assume that we know the *core*. Let $\mathbf{x}^\star = (x_1^\star, ..., x_M^\star)$ be an imputation in the core the sellers agree on. Recall (i) $\mathcal{T} = \prod_{j \in \mathcal{S}} \mathcal{T}_j$, where $\mathcal{T}_j = [t_{j,\min},\ t_{j,\max}]$, (ii) the random vector $\mathbf{B} = (B^j; j \in \mathcal{S})$, where B^j is the seller chosen by buyer j (using the selected symmetric MSNE strategy), and (iii) $\mathcal{B} = \mathcal{P}^N$. Define $\mathcal{W} := \mathcal{T} \times \mathcal{B}$. Suppose that $\nu^{\mathcal{W}}$ is a distribution over the set $\mathcal{W}$.

For each realization $\mathbf{w} \in \mathcal{W}$, let $r_t^{(g)}(\mathbf{w})$ be the total profit of the grand coalition, $r_i^{(g)}(\mathbf{w})$ the received profit of seller $i \in \mathcal{P}$ in the grand coalition, and $r_i^{(s)}(\mathbf{w})$ the profit seller i can make in a separate auction by itself. Denote by $\bar{v}^i(\mathbf{w})$ seller i's total value of the sold items under the grand coalition, and $\bar{V}^i$ the seller i's total value of all items it has. Then, the revenue (i.e., total received payment) and the payoff of seller i in the grand coalition are $r_i^{(g)}(\mathbf{w}) + \bar{v}^i(\mathbf{w})$ and $r_i^{(g)}(\mathbf{w}) + \bar{V}^i$, respectively. Also, we can find the vector of sellers' expected profits, $\tilde{\mathbf{x}}^\star = (\tilde{x}_1^\star, \ldots, \tilde{x}_M^\star)$ from the relation $\tilde{x}_i^\star = x_i^\star - \bar{V}^i$ for all $i \in \mathcal{P}$. We assume that the sellers have different positive value for each unit of item. The allocation is determined by the GBM. Hence, if $k^\star$ units are sold, $k^\star$ units with the smallest values are allocated.

We first introduce two simple profit sharing mechanisms under which sellers' expected payoff vector lies in the core. As we will explain later, while they are simple, they have some undesirable properties.

7.3.1 Simple profit sharing schemes

Proportional sharing: One of the simplest way is dividing the profit proportionally according to the selected expected profit vector $\tilde{\mathbf{x}}^\star$ in any realization $\mathbf{w} \in \mathcal{W}$.

Mechanism 1. *(Proportional sharing)*

$$r_i^{(g)}(\mathbf{w}) = \frac{\tilde{x}_i^\star}{\sum_{l \in \mathcal{P}} \tilde{x}_l^\star} \times r_t^{(g)}(\mathbf{w}) \ .$$

Define random vector $\mathbf{W} := (\mathbf{T}, \mathbf{B})$. Then, the total expected profit of the grand coalition

$$\begin{aligned} \mathbf{E}_{\mathbf{W}}\left[r_t^{(g)}(\mathbf{W})\right] &= \int_{\mathbf{w} \in \mathcal{W}} r_t^{(g)}(\mathbf{w}) \ d\nu^{\mathcal{W}}(\mathbf{w}) \\ &= \sum_{l \in \mathcal{P}} \tilde{x}_l^\star \ . \end{aligned}$$

Hence, the expected profit of seller i, $\mathbf{E}_{\mathbf{W}}\left[r_i^{(g)}(\mathbf{W})\right]$, equals $\tilde{x}_i^\star$. Note that, in the proportional sharing mechanism, each seller receives some profit even when it does not provide any item that is sold or any winning contributions in the grand coalition.

Surplus sharing: If the sellers' expected payoffs are in the core, by Theorem 7.2 [55, p.219], each seller's expected payoff is larger than or equal to the expected payoff the seller can obtain in a separate auction. Bearing this in mind, one may consider employing a mechanism which reflects the payoff that each seller can receive in separate individual auctions.

Mechanism 2. *(Surplus sharing)*

$$r_i^{(g)}(\mathbf{w}) = r_i^{(s)}(\mathbf{w}) + \alpha_i \times \left(r_t^{(g)}(\mathbf{w}) - \sum_{l \in \mathcal{P}} r_l^{(s)}(\mathbf{w}) \right) \ ,$$

where $\sum_{i \in \mathcal{P}} \alpha_i = 1$ *and* $\alpha_i \geq 0$ *for all* $i \in \mathcal{P}$.

From Theorem 7.2, if the expected payoff vector $\mathbf{x}$ is in the core, for all $i \in \mathcal{P}$,

$$\sum_{i\in\mathcal{P}} x_i^\star = \mathbf{E_W}\left[r_t^{(g)}(\mathbf{W})\right] + \sum_{i\in\mathcal{P}} \bar{V}^i \text{ and}$$
$$x_i^\star \geq \mathbf{E_W}\left[r_i^{(s)}(\mathbf{W}) + \bar{V}^i\right] = \mathbf{E_W}\left[r_i^{(s)}(\mathbf{W})\right] + \bar{V}^i \ .$$

Similarly, for all $i \in \mathcal{P}$,

$$\sum_{i\in\mathcal{P}} \tilde{x}_i^\star = \mathbf{E_W}\left[r_t^{(g)}(\mathbf{W})\right] \text{ and } \tilde{x}_i^\star \geq \mathbf{E_W}\left[r_i^{(s)}(\mathbf{W})\right] \ .$$

Then, one can find $\alpha_i > 0$ such that

$$\tilde{x}_i^\star = \mathbf{E_W}\left[r_i^{(s)}(\mathbf{W})\right] + \alpha_i \times \left(\mathbf{E_W}\left[r_t^{(g)}(\mathbf{W})\right] - \sum_{l\in\mathcal{P}} \mathbf{E_W}\left[r_l^{(s)}(\mathbf{W})\right]\right)$$

for all $i \in \mathcal{P}$ and $\sum_{i\in\mathcal{P}} \alpha_i = 1$.

It is clear that this mechanism reflects the profit of each seller in separate individual auctions. However, when $\sum_{l\in\mathcal{P}} r_l^{(s)}(\mathbf{w}) > r_t^{(g)}(\mathbf{w})$ for some realization $\mathbf{w} \in \mathcal{W}$, some sellers may have negative profit ($r_i^{(g)}(\mathbf{w}) < 0$) which may not be acceptable to some sellers.[3]

7.3.2 Proposed profit sharing scheme

Even though the shared expected payoffs of the sellers under Mechanisms 1 and 2 lie in the core, they may not be attractive because either some sellers should share their payoffs or profits with other sellers who do not have any winning contributions or allocated items (mechanism 1) or some sellers are asked to give up their items for a payment less than their values of the items or even pay for joining the grand

[3]Those sellers should either receive the revenue less than the values of sold items or pay some 'fee'.

coalition (mechanism 2). In order to find a more attractive mechanism, we introduce following two constraints:

1. A seller who does not contribute anything, i.e., neither winning contributions nor items, receives no profit.

2. Sellers shall have non-negative profit for every realization, i.e., each seller receives at least the total value of sold items.

First, we divide the set $\mathcal{W}$ into 2^M subsets according to the set of sellers providing winning contributions or allocated items. For example, in a two seller case ($M = 2$), there are four sub-cases: (1) Both sellers 1 and 2 provide items or winning contributions, (2) only seller 1 has winning contributions and allocated items, (3) only seller 2 brings winning contributions and allocated items, and (4) no item is sold in the auction. We number these sets (from 1 to 2^M). Here, we always number the subset that none of items is sold 2^M.

We denote the subset of $\mathcal{W}$ that is numbered k by $\mathcal{W}^{(k)}$ so that $\bigcup_{k=1}^{2^M} \mathcal{W}^{(k)} = \mathcal{W}$. Define Λ_k to be the set of sellers who receive a share of the revenue because they provide either winning contributions or allocated items in $\mathcal{W}^{(k)}$. Note that, since $\mathcal{W}^{(2^M)}$ is the subset where none of items is sold, $\Lambda_{2^M} = \emptyset$. For each $i \in \mathcal{P}$, let

$$\begin{aligned}
\Psi_1^{(i)} &:= \{k \in \{1, 2, \ldots, 2^M\} \mid i \notin \Lambda_k\} \ , \\
\Psi_2^{(i)} &:= \{k \in \{1, 2, \ldots, 2^M\} \mid \Lambda_k = \{i\}\} \ , \text{ and} \\
\Psi_3^{(i)} &:= \{1, 2, \ldots, 2^M\} \setminus \left(\Psi_1^{(i)} \cup \Psi_2^{(i)}\right) \ .
\end{aligned}$$

From the definitions, when $k \in \Psi_1^{(i)}$ and $\mathbf{w} \in \mathcal{W}^{(k)}$, seller i shall not receive any revenue. If $k \in \Psi_2^{(i)}$ and $\mathbf{w} \in \mathcal{W}^{(k)}$, seller i takes all the revenue from the auction.

We propose the following profit sharing mechanism,

Mechanism 3. *For every $\mathbf{w} \in \mathcal{W}$, seller i's profit is given by*

$$r_i^{(g)}(\mathbf{w}) = \alpha_i(\mathbf{w}) \times r_t^{(g)}(\mathbf{w}) \ ,$$

where $\sum_{i \in \mathcal{P}} \alpha_i(\mathbf{w}) = 1$ and $\alpha_i(\mathbf{w}) \geq 0$. When $k \in \Psi_1^{(i)}$ and $\mathbf{w} \in \mathcal{W}^{(k)}$, $\alpha_i(\mathbf{w}) = 0$.

In order to complete the proposed mechanism, we need to specify how the coefficients $\alpha_i(\mathbf{w})$, $i \in \mathcal{P}$ and $\mathbf{w} \in \mathcal{W}$, are computed. We focus on the case where $\alpha_i(\mathbf{w})$ is the same for all $\mathbf{w} \in \mathcal{W}^{(k)}$, i.e., it does not depend on $\mathbf{w}$ in $\mathcal{W}^{(k)}$. In order to see whether or not there exist such sharing coefficients, we introduce following notation. Define a function $\hbar : 2^{\mathcal{P}} \to \{1, 2, \ldots, 2^M\}$ such that, given a subset of sellers S, $\Lambda_{\hbar(S)} = S$. In other words, $\hbar(S)$ refers to the case where only the sellers in S share the profit. Denote the expected profit over the subset $\mathcal{W}^{(k)}$ by $R_k := \int_{\mathbf{w} \in \mathcal{W}^{(k)}} r_t^{(g)}(\mathbf{w}) d\nu^{\mathcal{W}}(\mathbf{w})$.[4] For any given nonempty set of sellers $S \subsetneq \mathcal{P}$ and another seller $j \in \mathcal{P} \setminus S$, define following sets:

$\Theta_1 := \{k \in \{1, 2, \ldots, 2^M\} \mid \exists \text{ a nonempty set } S_s \subset S \text{ s.t. } S_s \cup \{j\} = \Lambda_k\}$

$\Theta_2 := \bigcup_{S_s \subset S; S_s \neq \emptyset} \{\hbar(S_s)\}$

$\Theta_3 := \Psi_2^{(j)}$

$\Theta_4 := \Theta_2 \cup \Theta_3$

$\Theta_5 := \bigcup_{i \in S} \Psi_3^{(i)} \setminus (\Theta_1 \cup \Theta_2)$

$\Theta_6 := \Psi_3^{(j)} \setminus (\Theta_1 \cup \Theta_3)$

[4] Note that $R_{2^M} = 0$.

$\Theta_7 := \Theta_5 \cup \Theta_6$

$\Theta_8 := \{1, 2, \ldots, 2^M\} \setminus (\Theta_1 \cup \Theta_4 \cup \Theta_7)$

For $S = \emptyset$, we set $\Theta_1 = \Theta_2 = \Theta_5 = \emptyset$. Note that $\Theta_1 \cup \Theta_4 \cup \Theta_7 \cup \Theta_8 = \{1, 2, \ldots, 2^M\}$. Under these definitions, the following holds.

1. If $k \in \Theta_1$, only some sellers in S and seller j share all the profit.

2. If $k \in \Theta_2$, only some sellers in S share all the profit.

3. If $k \in \Theta_3$, seller j takes all the profit.

4. If $k \in \Theta_5$, sellers in S share the profit with other sellers in $\mathcal{P} \setminus (S \cup \{j\})$. Seller j may receive some profit in this case.

5. If $k \in \Theta_6$, seller j share the profit with other sellers in $\mathcal{P} \setminus (S \cup \{j\})$. Sellers in S may receive some profit in this case.

6. If $k \in \Theta_8$, sellers in $S \cup \{j\}$ cannot receive any profit, i.e., only the seller in $\mathcal{P} \setminus (S \cup \{j\})$ share the profit.

Note that $\Theta_5 \cap \Theta_6$ may not be empty.

Proposition 1. *For any $S \subsetneq \mathcal{P}$ and $j \in \mathcal{P} \setminus S$,*

$$\sum_{i \in (S \cup \{j\})} \tilde{x}_i^\star \geq \sum_{k \in \Theta_1} R_k + \sum_{k \in \Theta_4} R_k.$$

Proof. Since $x^\star$ lies in the core, from the definition of $\tilde{x}^\star$, the sum of the expected profits of the sellers in $S \cup \{j\}$, $\sum_{i \in (S \cup \{j\})} \tilde{x}_i^\star$, is larger than or equal to the expected profit of the coalition $S \cup \{j\}$. For $k \in \Theta_1 \cup \Theta_4$, only the sellers in $S \cup \{j\}$ provide either the winning contributions or the items. Thus, if the sellers in $S \cup \{j\}$ form a coalition and hold their joint auction, they can guarantee the profit of at least $\sum_{k\in\Theta_1} R_k + \sum_{k\in\Theta_4} R_k$. $\square$

By the same argument in the proof of Proposition 1, we have

$$\sum_{i\in(\mathcal{P}\setminus(S\cup\{j\}))} \tilde{x}_i^\star \geq \sum_{k\in\Theta_8} R_k. \tag{7.18}$$

Since $\sum_{i\in\mathcal{P}} \tilde{x}_i^\star = \sum_{k\in\Theta_1\cup\Theta_4\cup\Theta_7\cup\Theta_8} R_k$, by subtracting (7.18), we can get the following proposition.

Proposition 2. *For any nonempty $S \subsetneq \mathcal{P}$ and $j \in \mathcal{P} \setminus S$,*

$$\sum_{i\in(S\cup\{j\})} \tilde{x}_i^\star \leq \sum_{k\in\Theta_1} R_k + \sum_{k\in\Theta_4} R_k + \sum_{k\in\Theta_7} R_k.$$

Theorem 7.5. *For any desired expected profits profile $\tilde{\mathbf{x}}^\star = (\tilde{x}_1^\star, \tilde{x}_2^\star, \ldots, \tilde{x}_M^\star)$, where the associated imputation $\mathbf{x}^\star = (x_1^\star, x_2^\star, \ldots, x_M^\star)$ lies in the core, there exist constants $\beta_k^{(i)}$, $k \in \{1, 2, \ldots, 2^M\}$ and $i \in \mathcal{P}$, for mechanism 3 such that the following conditions hold: For all $i \in \mathcal{P}$ and $k \in \{1, 2, \ldots, 2^M\}$,*

$$\beta_k^{(i)} \geq 0 \ , \tag{7.19}$$

$$\sum_{i\in\mathcal{P}} \beta_k^{(i)} = 1 \ , \tag{7.20}$$

$$\tilde{x}_i^\star = \sum_{k=1}^{2^M} \beta_k^{(i)} R_k \ , \tag{7.21}$$

$$\beta_k^{(i)} = 0, \ \textit{if } k \in \Psi_1^{(i)} \ . \tag{7.22}$$

The sharing coefficients in the mechanism 3 can now be set to $\alpha_i(\mathbf{w}) = \beta_k^{(i)}$ for all $\mathbf{w} \in \mathcal{W}^{(k)}$.

Proof. Our goal is to show the existence of $\beta_k^{(i)}$, $i \in \mathcal{P}$ and $k \in \{1, 2, \ldots, 2^M\}$, such that (7.19), (7.20), (7.21), and (7.22) hold. Note that if conditions (7.19), (7.20), (7.21), and (7.22) hold, $\beta_k^{(i)} = 1$ for $k \in \Psi_2^{(i)}$ and $\sum_{i\in\mathcal{P}} \tilde{x}_i^\star = \sum_{k=1}^{2^M} R_k$.

In order to prove the existence of such sharing coefficients, we will first show that there exists adequate sharing rule for sellers 1 and 2. Then, the sharing coefficients for seller 3 can be found without breaking validity of sharing coefficients of sellers 1 and 2.[5] Repeating this induction argument, our proof will illustrate that valid sharing coefficient can be found for sellers $1, 2, \ldots, M-1$. Finally, we will show that the sharing coefficients of seller M automatically satisfy the constraints.

$\triangleright$ *Step 1* : Without loss of generality, start with $S = \{1\}$ and $j = 2$. We denote by $k^\star$ the case where only sellers 1 and 2 provide either winning contributions or allocated items. In this case, we can set $\Theta_1 = \{k^\star\}$. Then, we need to show that there are $\beta_k^{(1)}$ and $\beta_k^{(2)}$, $k \in \{1, 2, \ldots, 2^M\}$, such that

1. $\beta_k^{(1)}, \beta_k^{(2)} \in [0, 1]$ and $\beta_k^{(1)} + \beta_k^{(2)} \leq 1$,

2. $\beta_{k^\star}^{(1)} + \beta_{k^\star}^{(2)} = 1$,

3. $\beta_k^{(1)} = 0$ for $k \in \Psi_1^{(1)}$ and $\beta_k^{(2)} = 0$ for $k \in \Psi_1^{(2)}$,

4. $\beta_k^{(1)} = 1$ for $k \in \Psi_2^{(1)}$ and $\beta_k^{(2)} = 1$ for $k \in \Psi_2^{(2)}$,

5. $\tilde{x}_1^\star = \sum_{k=1}^{2^M} \beta_k^{(1)} R_k$ and $\tilde{x}_2^\star = \sum_{k=1}^{2^M} \beta_k^{(2)} R_k$.

[5] The values of sharing coefficients of seller 1 and 2 can be changed. However, they still satisfy the constraints (7.19), (7.20), (7.21), and (7.22).

From Proposition 1, we have

$$\begin{aligned}\tilde{x}_1^\star &\geq R_{\hbar(\{1\})} ,\\ \tilde{x}_2^\star &\geq R_{\hbar(\{2\})} ,\\ \tilde{x}_1^\star + \tilde{x}_2^\star &\geq R_{k^\star} + R_{\hbar(\{1\})} + R_{\hbar(\{2\})} .\end{aligned}$$

From Proposition 2,

$$\tilde{x}_1^\star + \tilde{x}_2^\star \leq R_{k^\star} + R_{\hbar(\{1\})} + R_{\hbar(\{2\})} + \sum_{k\in\Theta_7} R_k . \tag{7.23}$$

Thus, we can always find $\beta_{k^\star}^{(1)}$ and $\beta_{k^\star}^{(2)}$ such that

$$\begin{aligned}&\tilde{x}_1^\star \geq \beta_{k^\star}^{(1)} R_{k^\star} + R_{\hbar(\{1\})} ,\\ &\tilde{x}_2^\star \geq \beta_{k^\star}^{(2)} R_{k^\star} + R_{\hbar(\{2\})} ,\\ &\beta_{k^\star}^{(1)} + \beta_{k^\star}^{(2)} = 1, \beta_{k^\star}^{(1)} \geq 0, \text{ and } \beta_{k^\star}^{(2)} \geq 0 .\end{aligned}$$

At the same time, from (7.23), there exist $\beta_k^{(1)}$ and $\beta_k^{(2)}$ such that

$$\begin{aligned}&\tilde{x}_1^\star = \sum_{k\in\Theta_5} \beta_k^{(1)} R_k + \beta_{k^\star}^{(1)} R_{k^\star} + R_{\hbar(\{1\})} ,\\ &\tilde{x}_2^\star = \sum_{k\in\Theta_6} \beta_k^{(2)} R_k + \beta_{k^\star}^{(2)} R_{k^\star} + R_{\hbar(\{2\})} ,\\ &\beta_k^{(1)} + \beta_k^{(2)} \leq 1, \beta_k^{(1)} \geq 0, \text{ and } \beta_k^{(2)} \geq 0 \text{ for } k \in \Theta_7 .\end{aligned}$$

We can verify the existence of such $\beta_k^{(1)}$ and $\beta_k^{(2)}$ as follows. Suppose that there is $\hat{k} \in \Theta_7$ such that $\beta_{\hat{k}}^{(1)} + \beta_{\hat{k}}^{(2)} > 1$. Then, we can adjust $\beta_{\hat{k}}^{(1)}$ and $\beta_{\hat{k}}^{(2)}$ to satisfy $\beta_{\hat{k}}^{(1)} + \beta_{\hat{k}}^{(2)} = 1$ and, in the process, increase other $\beta_k^{(1)}$ and $\beta_k^{(2)}$. In this operation, it is always possible to satisfy $\beta_k^{(1)} + \beta_k^{(2)} \leq 1$ for all $k \in \Theta_7$ due to (7.23).

$\triangleright$ *Step 2* : Assume that, for given $S = \{1, 2, \ldots, \eta\}$, $2 \leq \eta \leq M-2$, $\beta_k^{(i)} \geq 0$, $\sum_{i \in S} \beta_k^{(i)} \leq 1$, and $\tilde{x}_i^\star = \sum_{k=1}^{2^M} \beta_k^{(i)} R_k$ for all $i \in S$. Then, given S and $j = \eta + 1$, we need to show that there are $\beta_k^{(j)}$ and $\beta_k^{(i)}$, $k \in \{1, 2, \ldots, 2^M\}$, for all $i \in S$ such that

1. $\beta_k^{(j)}, \beta_k^{(i)} \in [0,1]$ and $\sum_{i \in S} \beta_k^{(i)} + \beta_k^{(j)} \leq 1$,
2. $\sum_{i \in S} \beta_{k^\star}^{(i)} + \beta_{k^\star}^{(j)} = 1$ for $k^\star \in \Theta_1$,
3. $\beta_k^{(i)} = 0$ for $k \in \Psi_1^{(i)}$, $i \in S$ and $\beta_k^{(j)} = 0$ for $k \in \Psi_1^{(j)}$,
4. $\beta_k^{(i)} = 1$ for $k \in \Psi_2^{(i)}$, $i \in S$ and $\beta_k^{(j)} = 1$ for $k \in \Psi_2^{(j)}$,
5. $\tilde{x}_i^\star = \sum_{k=1}^{2^M} \beta_k^{(i)} R_k$ for $i \in S$ and $\tilde{x}_j^\star = \sum_{k=1}^{2^M} \beta_k^{(j)} R_k$.

From Proposition 1,

$$\begin{aligned} \sum_{i \in S} \tilde{x}_i^\star &\geq \sum_{k \in \Theta_2} R_k \ , \\ \tilde{x}_j^\star &\geq \sum_{k \in \Theta_3} R_k \ , \\ \sum_{i \in S} \tilde{x}_i^\star + \tilde{x}_j^\star &\geq \sum_{k \in \Theta_1} R_k + \sum_{k \in \Theta_2} R_k + \sum_{k \in \Theta_3} R_k \ . \end{aligned} \tag{7.24}$$

From Proposition 2,

$$\begin{aligned} \sum_{i \in S} \tilde{x}_i^\star &\leq \sum_{k \in \Theta_1} R_k + \sum_{k \in \Theta_2} R_k + \sum_{k \in \Theta_5} R_k \ , \\ \sum_{i \in S} \tilde{x}_i^\star + \tilde{x}_j^\star &\leq \sum_{k \in \Theta_1} R_k + \sum_{k \in \Theta_2} R_k + \sum_{k \in \Theta_3} R_k + \sum_{k \in \Theta_7} R_k \ . \end{aligned} \tag{7.25}$$

Since we assumed that, in the previous round, $\beta_k^{(i)}$'s are selected in such a way that $0 \leq \beta_k^{(i)} \leq 1$ and $\tilde{x}_i^\star = \sum_{k=1}^{2^M} \beta_k^{(i)} R_k$ for all $i \in S$ and $\sum_{i \in S} \beta_k^{(i)} \leq 1$ for all $k \in \{1, 2, \ldots, 2^M\}$, the following condition is initially guaranteed in this round.

$$\sum_{i \in S} \tilde{x}_i^\star \geq \sum_{k \in \Theta_1} \left(\sum_{i \in S} \beta_k^{(i)} \right) R_k + \sum_{k \in \Theta_2} R_k \tag{7.26}$$

From the definition of Θ_1, because only the sellers in $S \cup \{j\}$ contribute,

$$\sum_{i \in S} \beta_k^{(i)} + \beta_k^{(j)} = 1 \text{ for } k \in \Theta_1 \; . \tag{7.27}$$

Thus, $\beta_k^{(j)}$ for $k \in \Theta_1$ can be determined by (7.27). However, the condition

$$\tilde{x}_j^\star \geq \sum_{k \in \Theta_1} \beta_k^{(j)} R_k + \sum_{k \in \Theta_3} R_k \; , \tag{7.28}$$

may not be satisfied at the same time.

Lemma 6. *Given* $S = \{1, 2, \ldots, \eta\}$*,* $2 \leq \eta \leq M - 2$*, suppose that* $\beta_k^{(i)} \geq 0$*,* $\sum_{i \in S} \beta_k^{(i)} \leq 1$*, and* $\tilde{x}_i^\star = \sum_{k=1}^{2M} \beta_k^{(i)} R_k$ *for all* $i \in S$*. Then, for* $j = \eta + 1$*,* $\sum_{k \in \Theta_5} \left(\sum_{i \in S} \beta_k^{(i)} \right) R_k \geq \sum_{k \in \Theta_1} \beta_k^{(j)} R_k + \sum_{k \in \Theta_3} R_k - \tilde{x}_j^\star$*.*

Proof. We have

$$\sum_{i \in S} \tilde{x}_i^\star = \sum_{k \in \Theta_1} \left(\sum_{i \in S} \beta_k^{(i)} \right) R_k + \sum_{k \in \Theta_2} R_k + \sum_{k \in \Theta_5} \left(\sum_{i \in S} \beta_k^{(i)} \right) R_k \; . \tag{7.29}$$

By subtracting (7.29) from (7.24) and substituting (7.27),

$$\tilde{x}_j^\star \geq \sum_{k \in \Theta_1} \beta_k^{(j)} R_k + \sum_{k \in \Theta_3} R_k - \sum_{k \in \Theta 5} \left(\sum_{i \in S} \beta_k^{(i)} \right) R_k \; .$$

□

Thus, when (7.28) is not satisfied, Lemma 6 tells us that we can adjust $\beta_k^{(i)}$ for $i \in S$ so that both (7.26) and (7.28) hold. In this adjustment, $\beta_k^{(i)}$, $i \in S$ and $k \in \Theta_1$, are increased and, as a result, $\beta_k^{(i)}$, $i \in S$ and $k \in \Theta_5$, are decreased because the condition $\tilde{x}_i^\star = \sum_{k \in \Theta_5} \beta_k^{(i)} R_k + \sum_{k \in \Theta_1} \beta_k^{(i)} R_k + \sum_{k \in \Theta_2} \beta_k^{(i)} R_k$ for $i \in S$ should be maintained. Hence, $\beta_k^{(i)}$, $i \in S$ and $k \in \Theta_1$, can be increased until each $\beta_k^{(i)}$, $i \in S$ and $k \in \Theta_5$, becomes zero. Then, $\beta_k^{(j)}$, $k \in \Theta_1$, that satisfy (7.28) can always be found because of Lemma 6. Note that decreasing $\beta_k^{(i)}$, $i \in S$ and $k \in \Theta_5$, may affect the $\beta_k^{(l)}$, $l \in \mathcal{P} \setminus (S \cup \{j\})$ that will be determined in a future round. However, by Lemma 6, there is always enough room to adjust $\beta_k^{(l)}$, $l \in \mathcal{P} \setminus (S \cup \{j\})$, if needed, so that $\tilde{x}_l^\star$, $l \in \mathcal{P} \setminus (S \cup \{j\})$, satisfies (7.28) when $\tilde{x}_l^\star$ is added.

Once the conditions (7.26) and (7.28) are satisfied, $\beta_k^{(j)}$ for $k \in \Theta_6$ can be found so that

$$\begin{aligned} \tilde{x}_i^\star &= \sum_{k\in\Theta_5} \beta_k^{(i)} R_k + \sum_{k\in\Theta_1} \beta_k^{(i)} R_k + \sum_{k\in\Theta_2} \beta_k^{(i)} R_k \quad \text{for all } i \in S \ , \\ \tilde{x}_j^\star &= \sum_{k\in\Theta_6} \beta_k^{(j)} R_k + \sum_{k\in\Theta_1} \beta_k^{(j)} R_k + \sum_{k\in\Theta_3} R_k \ , \\ \sum_{i\in S} & \beta_k^{(i)} + \beta_k^{(j)} \le 1 \text{ and } \beta_k^{(j)} \ge 0 \text{ for } k \in \Theta_6 \ . \end{aligned} \tag{7.30}$$

Even if $\beta_k^{(j)} + \sum_{i\in S} \beta_k^{(i)} = 1$, $i \in S$ and $k \in \Theta_6$, the condition (7.30) may not be satisfied, i.e., $\tilde{x}_j^\star > \sum_{k\in\Theta_6} \beta_k^{(j)} R_k + \sum_{k\in\Theta_1} \beta_k^{(j)} R_k + \sum_{k\in\Theta_3} R_k$. In this case, $\beta_k^{(i)}$, $i \in S$ and $k \in \Theta_1$, need to be decreased to meet (7.30) by increasing $\beta_k^{(j)}$, $k \in \Theta_1$. In this adjustment, as a result of decreasing $\beta_k^{(i)}$, $i \in S$ and $k \in \Theta_1$, $\beta_k^{(i)}$, $i \in S$ and $k \in \Theta_7 \setminus \Theta_6$, will be increased until they become 1. During this operation, we can always find suitable $\beta_k^{(i)}$, $i \in S$ and $k \in \Theta_7 \setminus \Theta_6$, because of (7.25).

This tell us that, starting with $S = \{1\}$ and $j = 2$, we can finally find $\beta_k^{(i)}$ for all $i \in \mathcal{P} \setminus \{M\}$ and $k \in \{1, 2, \ldots, 2^M\}$.[6]

$\triangleright$ *Step 3* : Finally, for $S = \{1, 2, \ldots, M-1\}$ and $j = M$, since $\sum_{i\in S} \tilde{x}_i^\star$ has all nonnegative $\beta_k^{(i)}$ satisfying $0 \le \sum_{i\in S} \beta_k^{(i)} \le 1$, from $\sum_{l\in\mathcal{P}} \tilde{x}_l^\star = \sum_{k=1}^{2^M} R_k$, we can

[6] Even though we added sellers, starting with seller 1, by increasing order, any arbitrary choice of j is possible in each stage.

obtain $\beta_k^{(j)}$ from

$$\beta_k^{(j)} = 1 - \sum_{i \in S} \beta_k^{(i)}.$$

This guarantees $0 \leq \beta_k^{(j)} \leq 1$ for all $k \in \Psi_3^{(j)}$. $\square$

Chapter 8

Optimal auction mechanism for multiple-region cases

8.1 Optimal multi-area auction

In the previous chapters, the auction conducted to trade the spectrum in one market or service area was introduced and analyzed. However, in general, the PSP may have obtained the frequency bands from the government in multiple geographic regions to provide its own services in a wide area, e.g., multiple cities, states, or nation-wide. In the general situation we consider here, the PSP, i.e., seller, has surplus frequency bands in multiple regions (markets) and the SSPs, i.e., buyers, also operate in multiple regions. In this chapter, we focus on designing an optimal multi-region auction mechanism with a single auctioneer.[1] When an auctioneer has frequency bands for sale in multiple regions, it can organize the auction in two different ways. In the first case, the auctioneer holds multiple auctions, one for each region or market (Figure 8.1). In this case, each buyer needs to participate in separate auctions for the regions in which it wants to lease spectrum bands. When a separate auction is held for each region, a buyer may win different numbers of frequency bands in different regions.

In general, the value of a buyer for a frequency band in a region may depend

[1] In the case where a coalition of sellers holds a single auction to sell all the frequency bands of coalition members together, we regard the coalition as a single auctioneer.

on the frequency bands it receives in other regions. In this case, the auctioneer may wish to sell the available frequency bands in different regions jointly in order to increase revenue, adding to the complexity in the design of a suitable auction mechanism. This may require the auctioneer to hold a single auction for all regions (Figure 8.2). In this type of auction, it is likely that the auctioneer should take into account dependency in buyers' values (or demands) in allocation and pricing schemes if it to maximize the profit.

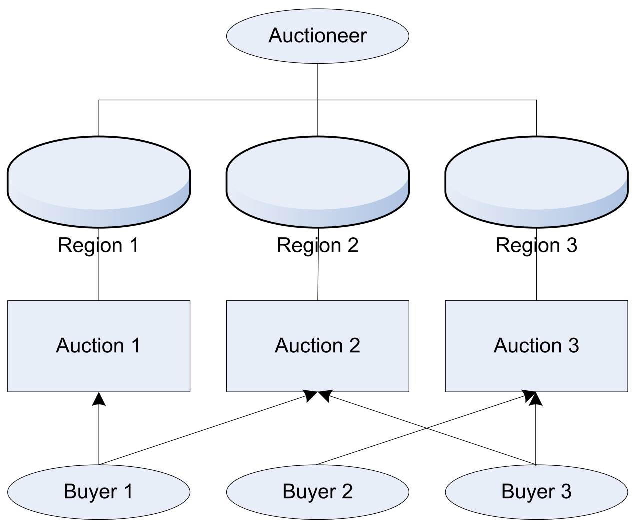

Figure 8.1: Separate auctions for multiple regions.

8.2 Model

In this chapter, we investigate the second case described above where an auctioneer holds one auction to lend all (surplus) frequency bands in a set of regions. As we already stated, we assume that there is only one risk neutral auctioneer. As we defined in Chapter 4, let $\mathcal{S} = \{1, 2, \ldots, N\}$ be the set of buyers and $\mathcal{R} = \{1, 2, \ldots, R\}$ the set of regions.

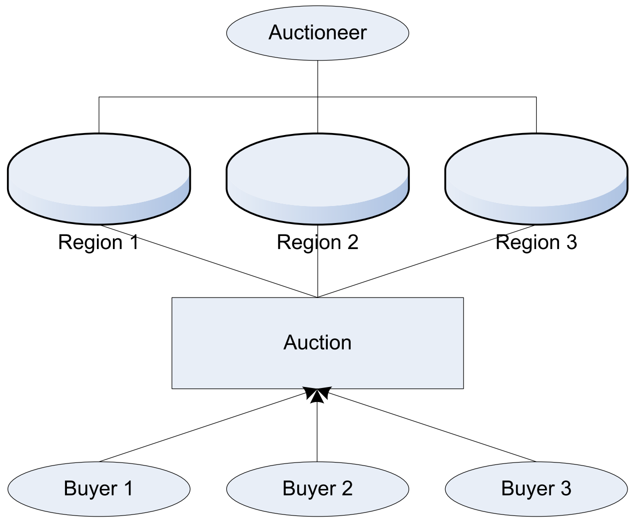

Figure 8.2: Combined auction for multiple regions.

1) Auctioneer: An auctioneer owns a set of frequency bands. We denote the number of available frequency bands for sale[2] in region $r \in \mathcal{R}$ by $\tilde{S}_r$. We define $\tilde{S}_{max} := \max\{\tilde{S}_1, \tilde{S}_2, \ldots, \tilde{S}_R\}$. We denote auctioneer's value for the ℓ-th item it wants to sell in region $r \in \mathcal{R}$ by $\hat{V}_r^{(\ell)}$, $\ell \in \{1, 2, \ldots, \tilde{S}_r\}$. In other words, the auctioneer would prefer not to sell the ℓ-th frequency band if it cannot receive at least $\hat{V}_r^{(\ell)}$ for it. Without loss of generality, we assume that the auctioneer's items are ordered by increasing value, i.e., $\hat{V}_r^{(1)} \leq \hat{V}_r^{(2)} \leq \cdots \leq \hat{V}_r^{(\tilde{S}_r)}$.

2) Buyers: Analogously to the single market cases, each buyer $j \in \mathcal{S}$ has private information, namely its *type* denoted by T_j. We assume that $T_j,\ j \in \mathcal{S}$, are given by mutually independent, continuous random variables. The distribution of T_j is $\mathcal{G}_j$ with support $\mathcal{T}_j := [t_{j,\min},\ t_{j,\max}]$. Moreover, we assume that $\mathcal{G}_j$ yields a density function g_j. Let $\mathbf{T} = (T_j;\ j \in \mathcal{S})$ be the random vector of the types of the

[2]Similarly to the single market case, we assume that frequency spectrum is divided into the same size units.

buyers and $\mathcal{T} := \prod_{j \in \mathcal{S}} \mathcal{T}_j$.

In order to make progress, we restrict our attention to the following case: Let $\mathcal{R}_j \subset \mathcal{R},\ j \in \mathcal{S}$, denote the set of regions in which buyer j operates. We assume that every buyer prefers to receive the same number of frequency bands in its operating regions.[3] In other words, given its type, the value buyer j acquires from a set of frequency bands it wins $\{a_r; r \in \mathcal{R}\}$, where a_r is the number of frequency bands it wins in region $r \in \mathcal{R}$, depends only on $\min_{r \in \mathcal{R}_j} a_r$. Note that this assumption implicitly implies that the buyer does not receive any value from winning frequency bands in the regions where it does *not* operate. We call such buyers *simple-minded buyers*.

An example is given in Table 8.1. Suppose that a buyer demands frequency bands in regions 1 and 2 only. When the buyer wins frequency band(s) in both regions 1 and 2, it earns a value which depends on the smallest number of units it receives in the two regions. As we can see, the buyer does not have any additional value for the frequency band(s) in region 3.

The type of a buyer determines its values for the *set* of frequency bands it wins:[4] For each $k \in \{1, 2, \ldots, \tilde{S}_{max}\}$, let $V_{j,k} : \mathcal{T}_j \to \mathbb{R}_+$ be the function that determines buyer j's value for the k-th set of frequency bands it wins (i.e., $V_{j,k}(t_j)$ is the value buyer j has for the k-th set of frequency bands it receives in its operating regions $\mathcal{R}_j$ when its type is t_j). The functions $V_{j,k}$ are increasing and differentiable.

[3]SSPs may need this to support handover to their customers. In [39], for multiple items with a single unit supply, the optimal combinatorial auction mechanism has been proposed.

[4]We still assume that the frequency bands in each region are homogeneous.

Table 8.1: Example of simple-minded buyer's value

The number of units			Value
Region 1	Region 2	Region 3	
1	1	0	\$1000
2	2	0	\$1800
2	2	1	\$1800
2	1	0	\$1000
1	2	0	\$1000
0	2	3	\$0
1	0	0	\$0
Buyer demands frequency bands in regions 1 and 2 only			

We also assume that $V_{j,1}(t_j) \geq V_{j,2}(t_j) \geq \cdots \geq V_{j,\tilde{S}_{max}}(t_j) \geq 0$ for all $t_j \in \mathcal{T}_j$. In order to make progress, we assume that the auctioneer knows the buyers' valuation functions $V_{j,k}$, $j \in \mathcal{S}$ and $k \in \{1, 2, \ldots, \tilde{S}_{max}\}$, and operating regions $\mathcal{R}_j$, $j \in \mathcal{S}$.

8.3 Optimal combinatorial auction mechanism for simple-minded buyers

In this section, we introduce an optimal combinatorial auction mechanism for multiple regions with simple-minded buyers. We follow the same framework which was used in Branco's mechanism and the GBM. An auction mechanism is given by a pair (p, c), where

- $p_{j,k} : \mathcal{T} \rightarrow [0,1]$, $j \in \mathcal{S}$ and $k \in \{1, 2, \ldots, \tilde{S}_{max}\}$, where $p_{j,k}(\mathbf{t})$ is the probability that buyer j wins at least k sets of frequency bands given that buyers' type vector $\mathbf{T} = \mathbf{t}$, and

- $c_j : \mathcal{T}_j \rightarrow \mathbb{R}_+$, $j \in \mathcal{S}$, where $c_j(t_j)$ is the expected payment of buyer j with type t_j.[5]

We are interested in a mechanism with the allocation rule with the property $p_{j,k}(\mathbf{t}) \in \{0, 1\}$ for all $j \in \mathcal{S}$, $k \in \{1, 2, \ldots, \tilde{S}_{max}\}$, and $\mathbf{t} \in \mathcal{T}$.

8.3.1 Conditions for the auction mechanism

Define indicator functions $I_r(j)$ where

$$I_r(j) = \begin{cases} 1 & \text{if } r \in \mathcal{R}_j \ , \\ 0 & \text{otherwise} \ . \end{cases}$$

Using these indicator functions, the conditions for an allocation rule can be stated as follows: For every $\mathbf{t} \in \mathcal{T}$,

$$\sum_{j \in \mathcal{S}} \sum_{k=1}^{\tilde{S}_{max}} p_{j,k}(\mathbf{t}) \times I_r(j) \leq \tilde{S}_r \text{ for all } r \in \mathcal{R} \ , \tag{8.1}$$

$$p_{j,k}(\mathbf{t}) \geq p_{j,k+1}(\mathbf{t}) \ , \text{ and} \tag{8.2}$$

$$p_{j,k}(\mathbf{t}) \in \{0, 1\} \ . \tag{8.3}$$

[5] Note that this is the expected payment with respect to both the other buyers' types and the probability of winning a varying number of items. The payment rule will be provided in section 8.3.2.

The condition (8.1) ensures that the auctioneer does not allocate more items than it has. The conditions (8.2) and (8.3) follow from the definition of p and our restriction on p, respectively.

Suppose that a buyer j reports its type $t_j^* \in \mathcal{T}_j$ to the auctioneer. The reported type t_j^* is not necessarily its true type t_j. Since the buyer is assumed risk neutral, its utility can be written as

$$U_j(t_j^*; t_j) = \mathbf{E}_{\mathbf{T}_{-j}}\left[\sum_{k=1}^{\tilde{S}_{max}} V_{j,k}(t_j) p_{j,k}(t_j^*, \mathbf{T}_{-j})\right] - c_j(t_j^*) \ , \tag{8.4}$$

where $\mathbf{T}_{-j} = (T_{j^*};\ j^* \in \mathcal{S} \setminus \{j\})$. Then, we can find the conditions that guarantee incentive compatibility and individual rationality.

In order for a mechanism to satisfy incentive compatibility, the following needs to hold for all $j \in \mathcal{S}$ and $t_j, t_j^* \in \mathcal{T}_j$:

$$U_j(t_j; t_j) \ \geq \ U_j(t_j^*; t_j) \ . \tag{8.5}$$

By following the same step used in developing the GBM in Chapter 5, we can find equivalent conditions to (8.5).

$$\begin{aligned} &\mathbf{E}_{\mathbf{T}_{-j}}\left[\sum_{k=1}^{\tilde{S}_{max}} \int_{t_j^*}^{t_j} \frac{dV_{j,k}(x)}{dx} p_{j,k}(x, \mathbf{T}_{-j}) dx\right] \\ &\geq \mathbf{E}_{\mathbf{T}_{-j}}\left[\sum_{k=1}^{\tilde{S}_{max}} \left(V_{j,k}(t_j) - V_{j,k}(t_j^*)\right) p_{j,k}(t_j^*, \mathbf{T}_{-j})\right] \text{ with} \end{aligned} \tag{8.6}$$

$$U_j(t_j; t_j) = U_j(t_j^*; t_j^*) + \mathbf{E}_{\mathbf{T}_{-j}}\left[\sum_{k=1}^{\tilde{S}_{max}} \int_{t_j^*}^{t_j} \frac{dV_{j,k}(x)}{dx} p_{j,k}(x, \mathbf{T}_{-j}) dx\right] \tag{8.7}$$

To guarantee individual rationality, we need

$$U_j(t_j; t_j) \ \geq \ 0 \tag{8.8}$$

for all $j \in \mathcal{S}$ and $t_j \in \mathcal{T}_j$. Thus, it is obvious that

$$U_j(t_{j,\min}; t_{j,\min}) \geq 0 \; . \tag{8.9}$$

Since we assume that $V_{j,k}$ is increasing, from (8.7) and (8.9), we can verify that (8.8) holds. Therefore, instead of (8.5) and (8.8), conditions (8.6), (8.7), and (8.9) can be used.

8.3.2 Allocation and payment schemes

Since the optimal combinatorial auction mechanism for simple-minded buyers is designed to be incentive compatible, from now on, we assume that buyers report their types truthfully. However, we will show that our proposed mechanism satisfies the incentive compatibility property. The number of allocated units in region $r \in \mathcal{R}$ is given by $\hat{m}_r^\star(\mathbf{t}) = \sum_{j\in\mathcal{S}} \sum_{k=1}^{\tilde{S}_r} p_{j,k}(\mathbf{t}) \times I_r(j)$ for $\mathbf{t} \in \mathcal{T}$. Then, the auctioneer's expected payoff is defined as

$$\begin{aligned} U_0 &= \sum_{j\in\mathcal{S}} \mathbf{E}_{T_j}\left[c_j(T_j)\right] + \mathbf{E}_{\mathbf{T}}\left[\sum_{r\in\mathcal{R}} \sum_{k=\hat{m}_r^\star(\mathbf{T})+1}^{\tilde{S}_r} \hat{V}_r^{(k)}\right] \\ &= \sum_{j\in\mathcal{S}} \mathbf{E}_{T_j}\left[c_j(T_j)\right] + \sum_{r\in\mathcal{R}} \sum_{k=1}^{\tilde{S}_r} \hat{V}_r^{(k)} - \mathbf{E}_{\mathbf{T}}\left[\sum_{r\in\mathcal{R}} \sum_{k=1}^{\hat{m}_r^\star(\mathbf{T})} \hat{V}_r^{(k)}\right] \; . \end{aligned} \tag{8.10}$$

Theorem 8.1. *Suppose that the allocation rule $p^\star$ solves the following problem*

$$\begin{aligned} \text{maximize}_{p(\cdot)} \quad & \mathbf{E}_{\mathbf{T}}\left[\sum_{j\in\mathcal{S}} \sum_{k=1}^{\tilde{S}_{max}} \left(V_{j,k}(T_j) - \frac{dV_{j,k}(T_j)}{dT_j} \frac{1-\mathcal{G}_j(T_j)}{g_j(T_j)}\right) p_{j,k}(\mathbf{T})\right] \\ & +\mathbf{E}_{\mathbf{T}}\left[\sum_{r\in\mathcal{R}} \sum_{k=\hat{m}_r{}^\star(\mathbf{T})+1}^{\tilde{S}_r} \hat{V}_r^{(k)}\right] \\ \text{subject to} \quad & (8.1), (8.2), (8.3), (8.6), (8.7), \text{ and } (8.9) \; , \end{aligned} \tag{8.11}$$

and that the expected payment $c^\star$ is given by

$$c^\star_j(t_j) = \mathbf{E}_{\mathbf{T}_{-j}}\left[\sum_{k=1}^{\tilde{S}_{max}}\left(V_{j,k}(t_j)p^\star_{j,k}(t_j,\mathbf{T}_{-j}) - \int_{t_{j\,\min}}^{t_j}\frac{dV_{j,k}(x)}{dx}p^\star_{j,k}(x,\mathbf{T}_{-j})\mathrm{d}x\right)\right]. \quad (8.12)$$

Then, $(p^\star, c^\star)$ is an optimal mechanism.

Proof. The expected payment the auctioneer receives can be written as [6]

$$\begin{aligned}
&\sum_{j\in\mathcal{S}}\mathbf{E}_{T_j}\left[c_j(T_j)\right]\\
&=\sum_{j\in\mathcal{S}}\mathbf{E}_{T_j}\left[\mathbf{E}_{\mathbf{T}_{-j}}\left[\sum_{k=1}^{\tilde{S}_{max}}V_{j,k}(T_j)p_{j,k}(T_j,\mathbf{T}_{-j})\right]\right]\\
&\qquad-\sum_{j\in\mathcal{S}}\mathbf{E}_{T_j}\left[\mathbf{E}_{\mathbf{T}_{-j}}\left[\sum_{k=1}^{\tilde{S}_{max}}V_{j,k}(T_j)p_{j,k}(T_j,\mathbf{T}_{-j})\right]-c_j(T_j)\right]\\
&=\sum_{j\in\mathcal{S}}\mathbf{E}_{T_j}\left[\mathbf{E}_{\mathbf{T}_{-j}}\left[\sum_{k=1}^{\tilde{S}_{max}}V_{j,k}(T_j)p_{j,k}(T_j,\mathbf{T}_{-j})\right]\right]-\sum_{j\in\mathcal{S}}\mathbf{E}_{T_j}\left[U_j(T_j;T_j)\right]\\
&=\sum_{j\in\mathcal{S}}\mathbf{E}_{T_j}\left[\mathbf{E}_{\mathbf{T}_{-j}}\left[\sum_{k=1}^{\tilde{S}_{max}}V_{j,k}(T_j)p_{j,k}(T_j,\mathbf{T}_{-j})\right]\right]\\
&\quad-\sum_{j\in\mathcal{S}}\mathbf{E}_{T_j}\left[U_j(t_{j,\min};t_{j,\min})+\mathbf{E}_{\mathbf{T}_{-j}}\left[\sum_{k=1}^{\tilde{S}_{max}}\int_{t_{j,\min}}^{T_j}\frac{dV_{j,k}(x)}{dx}p_{j,k}(x,\mathbf{T}_{-j})dx\right]\right]\\
&=\sum_{j\in\mathcal{S}}\mathbf{E}_{T_j}\left[\mathbf{E}_{\mathbf{T}_{-j}}\left[\sum_{k=1}^{\tilde{S}_{max}}\left(V_{j,k}(T_j)-\frac{dV_{j,k}(T_j)}{dT_j}\frac{1-\mathcal{G}_j(T_j)}{g_j(T_j)}\right)p_{j,k}(T_j,\mathbf{T}_{-j})\right]\right]\\
&\qquad-\sum_{j\in\mathcal{S}}U_j(t_{j,\min};t_{j,\min})\ .
\end{aligned}\quad (8.13)$$

The second equality follows from (8.4) and the third equality is a result of (8.7).

[6] It follows the same step used in Chapter 5.

Then, by using (8.13), the auctioneer's expected payoff (8.10) can be rewritten as

$$\begin{aligned}
U_0 &= \sum_{j\in\mathcal{S}} \mathbf{E}_{T_j}\left[\mathbf{E}_{\mathbf{T}_{-j}}\left[\sum_{k=1}^{\tilde{S}_{max}}\left(V_{j,k}(T_j) - \frac{dV_{j,k}(T_j)}{dT_j}\frac{1-\mathcal{G}_j(T_j)}{g_j(T_j)}\right)p_{j,k}(T_j,\mathbf{T}_{-j})\right]\right] \\
&\quad -\sum_{j\in\mathcal{S}} U_j(t_{j,\min};t_{j,\min}) + \mathbf{E}_{\mathbf{T}}\left[\sum_{r\in\mathcal{R}}\sum_{k=\hat{m}^\star_r(\mathbf{T})+1}^{\tilde{S}_r}\hat{V}_r^{(k)}\right] \\
&= \mathbf{E}_{\mathbf{T}}\left[\sum_{j\in\mathcal{S}}\sum_{k=1}^{\tilde{S}_{max}}\left(V_{j,k}(T_j) - \frac{dV_{j,k}(T_j)}{dT_j}\frac{1-\mathcal{G}_j(T_j)}{g_j(T_j)}\right)p_{j,k}(\mathbf{T})\right. \\
&\qquad\qquad \left. + \sum_{r\in\mathcal{R}}\sum_{k=\hat{m}^\star_r(\mathbf{T})+1}^{\tilde{S}_r}\hat{V}_r^{(k)}\right] \qquad (8.14) \\
&\quad -\sum_{j\in\mathcal{S}} U_j(t_{j,\min};t_{j,\min}) \ . \qquad (8.15)
\end{aligned}$$

Note that the buyer j's expected payment, c_j, appears only in $U_j(t_{j,\min};t_{j,\min})$. Since $p^\star$ is assumed to be an allocation rule that maximizes

$$\mathbf{E}_{\mathbf{T}}\left[\sum_{j\in\mathcal{S}}\sum_{k=1}^{\tilde{S}_{max}}\left(V_{j,k}(T_j) - \frac{dV_{j,k}(T_j)}{dT_j}\frac{1-\mathcal{G}_j(T_j)}{g_j(T_j)}\right)p^\star_{j,k}(\mathbf{T}) + \sum_{r\in\mathcal{R}}\sum_{k=\hat{m}^\star_r(\mathbf{T})+1}^{\tilde{S}_r}\hat{V}_r^{(k)}\right]$$

and $U_j(t_{j,\min};t_{j,\min}) \geq 0$ from individual rationality, in order to maximize the auctioneer's expected payoff U_0, c_j should be selected so that $U_j(t_{j,\min};t_{j,\min}) = 0$.[7] From (8.7),

$$\begin{aligned}
& U_j(t_{j,\min};t_{j,\min}) \\
&= U_j(t_j;t_j) - \mathbf{E}_{\mathbf{T}_{-j}}\left[\sum_{k=1}^{\tilde{S}_{max}}\int_{t_{j,\min}}^{t_j}\frac{dV_{j,k}(x)}{dx}p_{j,k}(x,\mathbf{T}_{-j})dx\right] \\
&= \mathbf{E}_{\mathbf{T}_{-j}}\left[\sum_{k=1}^{\tilde{S}_{max}}\left(V_{j,k}(t_j)p_{j,k}(t_j,\mathbf{T}_{-j}) - \int_{t_{j\,\min}}^{t_j}\frac{dV_{j,k}(x)}{dx}p_{j,k}(x,\mathbf{T}_{-j})dx\right)\right] \\
& -c_j(t_j) \ . \qquad (8.16)
\end{aligned}$$

Thus, the optimal expected payment is given by (8.12). □

[7] The optimal allocation rule should maximize (8.14). Given this, the optimal pricing scheme should minimize (8.15).

Note that (8.9) holds automatically with the optimal expected payment. By substituting (8.12) in (8.4), one can easily verify that (8.7) is also satisfied by the expected payment.

Given the type $\mathbf{t} = (t_j; j \in \mathcal{S})$, define the contribution of buyer j for the k-th set of frequency bands ($k = 1, 2, \ldots, \tilde{S}_{max}$) by

$$\pi_{j,k}(t_j) := V_{j,k}(t_j) - \left.\frac{dV_{j,k}(T_j)}{dT_j}\right|_{T_j=t_j} \frac{1-\mathcal{G}_j(t_j)}{g_j(t_j)} \ , \tag{8.17}$$

where $\mathcal{G}_j$ (resp. g_j) denotes the distribution (resp. density function) of T_j. The problem is *regular* if, for all $j \in \mathcal{S}$, $t_j, \hat{t}_j \in \mathcal{T}_j$, and $k = 1, 2, \ldots, \tilde{S}_{max}$,

$$(t_j - \hat{t}_j)\left(\pi_{j,k}(t_j) - \pi_{j,k}(\hat{t}_j)\right) \geq 0 \tag{8.18}$$

and, if $\pi_{j,k+1}(t_j) \geq 0$, then

$$\pi_{j,k}(t_j) \geq \pi_{j,k+1}(t_j) \ . \tag{8.19}$$

The regularity assumption implies that (i) the contribution is non-decreasing in its type and (ii) the nonnegative contribution is non-increasing in the number of the sets of frequency bands it receives.

Suppose that the expected payment is given by (8.12). Then, by substituting (8.13) with $U_j(t_{j,\min}; t_{j,\min}) = 0$ in (8.10) and using (8.17), we can rewrite U_0 as

$$\begin{aligned} U_0 \ &= \ \sum_{j\in\mathcal{S}} \sum_{k=1}^{\tilde{S}_{max}} \mathbf{E_T}\left[\pi_{j,k}(T_j) p_{j,k}(T_j, \mathbf{T}_{-j})\right] \\ &\qquad + \sum_{r\in\mathcal{R}} \sum_{k=1}^{\tilde{S}_r} \hat{V}_r^{(k)} - \mathbf{E_T}\left[\sum_{r\in\mathcal{R}} \sum_{k=1}^{\hat{m}_r^\star(\mathbf{T})} \hat{V}_r^{(k)}\right] \ . \end{aligned} \tag{8.20}$$

Thus, an optimal allocation rule needs to select the contributions so that (8.20) is maximized while the feasiblity conditions (8.1),(8.2), and (8.3) are satisfied. The condition (8.6) holds since $V_{j,k}$ is increasing in t_j and the regularity assumptions are in place. The conditions (8.7) and (8.9) hold because of the earlier selection of the expected payment while maximizing U_0.

Define the set of possible allocations as

$$\tilde{\mathcal{A}} = \{\tilde{\mathbf{a}} = (\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_N) \in \mathbb{Z}_+^N | \sum_{j \in \mathcal{S}} \tilde{a}_j \cdot I_r(j) \leq \tilde{S}_r \text{ for all } r \in \mathcal{R}\} \ .$$

For $\tilde{\mathbf{a}} \in \tilde{\mathcal{A}}$, denote the number of allocated items in region r by $\tilde{m}_r(\tilde{\mathbf{a}}) = \sum_{j \in \mathcal{S}} \tilde{a}_j \cdot I_r(j)$. Under the regularity assumptions, if $\pi_{j,k+1}(t_j)$ is a winning contribution, $\pi_{j,k}(t_j)$ is also a winning contribution. Then, for a given $\mathbf{t} \in \mathcal{T}$, the auctioneer's problem becomes

$$\text{maximize}_{\tilde{\mathbf{a}} \in \tilde{\mathcal{A}}} \sum_{j \in \mathcal{S}} \sum_{k=1}^{\tilde{a}_j} (\pi_{j,k}(t_j)) - \sum_{r \in \mathcal{R}} \sum_{k=1}^{\tilde{m}_r(\tilde{\mathbf{a}})} \hat{V}_r^{(k)} + \sum_{r \in \mathcal{R}} \sum_{k=1}^{\tilde{S}_r} \hat{V}_r^{(k)} \ . \tag{8.21}$$

Suppose that $\tilde{\mathbf{a}}^* \in \tilde{\mathcal{A}}$ is an allocation which solves (8.21). Then, $p_{j,k}(\mathbf{t}) = 1$ only for $k \leq \tilde{a}_j^*$. Here, it is clear that the constraints (8.1), (8.2), and (8.3) are satisfied. The chosen allocation rule $\tilde{\mathbf{a}}^\star$ is defined through the optimization in (8.21) such that, given $\mathbf{t} \in \mathcal{T}$,

$$\tilde{\mathbf{a}}^\star(\mathbf{t}) \in \text{argmax}_{\tilde{\mathbf{a}} \in \tilde{\mathcal{A}}} \sum_{j \in \mathcal{S}} \sum_{k=1}^{\tilde{a}_j} (\pi_{j,k}(t_j)) - \sum_{r \in \mathcal{R}} \sum_{k=1}^{\tilde{m}_r(\tilde{\mathbf{a}})} \hat{V}_r^{(k)} + \sum_{r \in \mathcal{R}} \sum_{k=1}^{\tilde{S}_r} \hat{V}_r^{(k)} \ .$$

Then, we rewrite the allocation rule,

$$p_{j,k}(\mathbf{t}) = \begin{cases} 1 & \text{if } k \leq \tilde{a}_j^\star(\mathbf{t}) \ , \\ 0 & \text{otherwise} \ . \end{cases} \tag{8.22}$$

Define, for each $j \in \mathcal{S}$ and $k = 1, 2, \ldots, \tilde{S}_{max}$,

$$\tilde{\varsigma}_{j,k}(\mathbf{t}_{-j}) := \inf\{\tilde{t}_j \in \mathcal{T}_j \mid p_{j,k}(\tilde{t}_j, \mathbf{t}_{-j}) = 1\},$$

where $\mathbf{t}_{-j} = \{t_{j^*};\ j^* \in \mathcal{S} \setminus \{j\}\}$. Then, we can state following theorem.

Theorem 8.2. *Suppose that the problem is regular. The following allocation rule $p^\star$ and payment scheme $c^\star$ give rise to an optimal mechanism.*

$$p^\star_{j,k}(\mathbf{t}) = \begin{cases} 1 & \text{if } t_j > \tilde{\varsigma}_{j,k}(\mathbf{t}_{-j}) \ , \\ 0 & \text{otherwise,} \end{cases} \tag{8.23}$$

$$c^\star_j(t_j) = \mathbf{E}_{\mathbf{T}_{-j}} \left[\sum_{k=1}^{\tilde{S}_{max}} V_{j,k}(\tilde{\varsigma}_{j,k}(\mathbf{T}_{-j}))\ p^\star_{j,k}(t_j, \mathbf{T}_{-j}) \right] \ . \tag{8.24}$$

From (8.24), for each realization $\mathbf{t} \in \mathcal{T}$, we can use the following payment rule.

$$\hat{c}_{j,k}(\mathbf{t}) := \begin{cases} V_{j,k}(\tilde{\varsigma}_{j,k}(\mathbf{t}_{-j})) & \text{if } p_{j,k}(\mathbf{t}) = 1 \ , \\ 0 & \text{otherwise} \ . \end{cases}$$

In other words, for the k-th set of frequency bands in the operating regions, the winner pays the smallest value that would win the k-th set of frequency bands.

In general, the combinatorial optimization problem in (8.21) is NP-complete. Thus, the computational complexity of the algorithm may be an issue in some cases. However, we believe that the number of frequency bands available for sale and the number of buyers are unlikely to be very large in the spectrum trading markets. Therefore, an electronic auction system should be able to handle this algorithm. We leave this complexity issue for future study.[8]

[8] One may be interested in designing an approximate algorithm which provides a sub-optimal solution in polynomial time.

Chapter 9

Conclusion and future work

9.1 Conclusion

We first investigated the problem of designing a suitable trading mechanism for dynamic spectrum sharing with multiple sellers and multiple buyers in a single market. We modeled the interaction between selfish buyers interested in leasing spectrum as a noncooperative game. We showed that, when the buyers are homogeneous, there exists a symmetric mixed strategy Nash equilibrium. When there are no more than five sellers in the market and they hold separate individual auctions with positive probability we showed that there is a unique mixed strategy Nash equilibrium. Moreover, we demonstrated that the buyers' strategy converges to a symmetric mixed strategy in two seller cases. Because of the difficulty involved with computing buyers' expected payoff, we leave the convergence analysis of more general cases as an open problem.

We demonstrated that risk neutral sellers have an incentive to cooperate with each other in order to maximize their expected profits when the buyers behave according to a symmetric mixed strategy Nash equilibrium. We formulated the interaction among the sellers as a cooperative game and proved that its core is nonempty. From this finding, we introduced the payoff/profit sharing schemes that allow the sellers to achieve any equitable sharing of payoff/profit in the core, hence encouraging them to cooperate.

Finally, we considered the scenario with multiple regions and simple-minded buyers, and provided an optimal auction mechanism for the auctioneer who has surplus frequency spectrum bands in several regions and wants to lend its bands using an auction. This mechanism can be used when the buyers have the same demand in every region they are interested in. The findings in this dissertation provide a general guidance for the sellers or auctioneers to design a spectrum sharing mechanism in dynamic spectrum access networks.

Based on the results of this dissertation, we expect that more interesting research will follow to enrich and complete the problem of designing an optimal mechanism for dynamic spectrum market.

9.2 Future work

Here we list some problems of interest for future work.

1. We modeled the interaction among the sellers as a cooperative game under the assumption that sellers would share their items and information, i.e., value of each item, *truthfully*. However, since sellers are selfish, they may have an incentive to lie about their private information. Thus, it would be interesting to study how such selfish nature of the sellers may change the picture and how one should design a mechanism that ensures the sellers to tell the truth.

2. Although we expect that the number of sellers and buyers in the spectrum trading market will not be very large, computational complexity may still

present an issue in the optimal auction mechanism for multiple-region cases. In order to reduce the complexity, one may wish to develop a sub-optimal mechanism which runs in polynomial time at the price of a lower expected payoff (or profit) of the seller.

3. We developed an optimal combinatorial mechanism for a seller who has surplus frequency bands in multiple regions. One may want to extend our investigations to the cases where multiple sellers have surplus frequency bands in multiple regions and are free to form any coalition among themselves. We believe that there are several interesting issues to be investigated related to the issue of complexity in the case of multiple regions, due to the combinatorial nature.

Appendix A

Proof of Theorem 6.4

Let $\tilde{\mu}(C)$ be the probability that a coalition $C \in \mathcal{P}$ forms and $\hat{U}_j^{(i)}(C, \bar{\xi})$ the conditional expected payoff of buyer j choosing seller i in the coalition C, assuming that the coalition C forms and all buyers employ the same mixed strategy $\bar{\xi} = (\xi_1, \xi_2, \ldots, \xi_M)$.[1] Since we assume homogeneous buyers, for notational convenience we omit the subscription j and simply use $\hat{U}^{(i)}(C, \bar{\xi})$. Note that $U^{(i)}(\bar{\xi})$ denotes the conditional expected payoff of a buyer, given that the buyer selects seller i and all buyers adopt the same mixed strategy $\bar{\xi}$.

A.1 Two sellers case ($M = 2$)

When there are only 2 sellers in the market, note that $\tilde{\mu}(\{1\}) = \tilde{\mu}(\{2\})$ and $\tilde{\mu}(\{1\}) + \tilde{\mu}(\{1,2\}) = \tilde{\mu}(\{2\}) + \tilde{\mu}(\{1,2\}) = 1$. In addition, $\hat{U}^{(1)}(\{1,2\}, \bar{\xi}) = \hat{U}^{(2)}(\{1,2\}, \bar{\xi})$. Given a mixed strategy $\bar{\xi}$, a buyer's expected payoff is equal to

$$\begin{aligned} U(\bar{\xi}) &= \xi_1 \left\{ \tilde{\mu}(\{1\}) \hat{U}^{(1)}(\{1\}, \bar{\xi}) + \tilde{\mu}(\{1,2\}) \hat{U}^{(1)}(\{1,2\}, \bar{\xi}) \right\} \\ &\quad + \xi_2 \left\{ \tilde{\mu}(\{2\}) \hat{U}^{(2)}(\{2\}, \bar{\xi}) + \tilde{\mu}(\{1,2\}) \hat{U}^{(2)}(\{1,2\}, \bar{\xi}) \right\} . \end{aligned}$$

[1] $\xi_i, i \in \mathcal{P}$, denotes the probability that buyer j chooses seller i.

Suppose that $\Xi^\star = (\bar{\xi}^\star, \bar{\xi}^\star, \ldots, \bar{\xi}^\star)$ is a symmetric MSNE. Then, at the equilibrium mixed strategy $\bar{\xi}^\star$, $\hat{U}^{(1)}(\bar{\xi}^\star) = \hat{U}^{(2)}(\bar{\xi}^\star)$. Hence,

$$\begin{aligned}&\tilde{\mu}(\{1\})\hat{U}^{(1)}(\{1\}, \bar{\xi}^\star) + \tilde{\mu}(\{1,2\})\hat{U}^{(1)}(\{1,2\}, \bar{\xi}^\star) \\ &= \tilde{\mu}(\{2\})\hat{U}^{(2)}(\{2\}, \bar{\xi}^\star) + \tilde{\mu}(\{1,2\})\hat{U}^{(2)}(\{1,2\}, \bar{\xi}^\star) \ .\end{aligned}$$

Since $\tilde{\mu}(\{1\}) = \tilde{\mu}(\{2\})$ and $\hat{U}^{(1)}(\{1,2\}, \bar{\xi}^\star) = \hat{U}^{(2)}(\{1,2\}, \bar{\xi}^\star)$, we must have

$$\hat{U}^{(1)}(\{1\}, \bar{\xi}^\star) = \hat{U}^{(2)}(\{2\}, \bar{\xi}^\star) \ . \tag{A.1}$$

Note that $\xi_1 + \xi_2 = 1$ and $\hat{U}^{(i)}(\{i\}, \bar{\xi})$ is a decreasing function of ξ_i.[2] Suppose that there exist two different symmetric MSNEs $\Xi^{\star^1} = (\bar{\xi}^{\star^1}, \bar{\xi}^{\star^1}, \ldots, \bar{\xi}^{\star^1})$ and $\Xi^{\star^2} = (\bar{\xi}^{\star^2}, \bar{\xi}^{\star^2}, \ldots, \bar{\xi}^{\star^2})$. Then, if $\xi_1^{\star^1} > \xi_1^{\star^2}$ and $\xi_2^{\star^1} < \xi_2^{\star^2}$, from (A.1) we have $\hat{U}^{(1)}(\{1\}, \bar{\xi}^{\star^1}) < \hat{U}^{(1)}(\{1\}, \bar{\xi}^{\star^2}) = \hat{U}^{(2)}(\{2\}, \bar{\xi}^{\star^2}) < \hat{U}^{(2)}(\{2\}, \bar{\xi}^{\star^1})$, which is a contradiction. We can draw similar contradiction when $\xi_1^{\star^1} < \xi_1^{\star^2}$ and $\xi_2^{\star^1} > \xi_2^{\star^2}$.

A.2 Three sellers case ($M = 3$)

In 3 seller cases, note that, for any mixed strategy $\bar{\xi}$, we have

$$\begin{aligned}\hat{U}^{(1)}(\{1,2\}, \bar{\xi}) &= \hat{U}^{(2)}(\{1,2\}, \bar{\xi}), \\ \hat{U}^{(2)}(\{2,3\}, \bar{\xi}) &= \hat{U}^{(3)}(\{2,3\}, \bar{\xi}), \\ \hat{U}^{(3)}(\{1,3\}, \bar{\xi}) &= \hat{U}^{(1)}(\{1,3\}, \bar{\xi}), \text{ and} \\ \hat{U}^{(1)}(\{1,2,3\}, \bar{\xi}) &= \hat{U}^{(2)}(\{1,2,3\}, \bar{\xi}) = \hat{U}^{(3)}(\{1,2,3\}, \bar{\xi}) \ .\end{aligned}$$

[2]See the equation (6.6).

The expected payoffs from choosing seller 1, 2, and 3 are given by

$$
\begin{aligned}
U^{(1)}(\bar{\xi}) &= \tilde{\mu}(\{1\})\hat{U}^{(1)}(\{1\},\bar{\xi}) + \tilde{\mu}(\{1,2\})\hat{U}^{(1)}(\{1,2\},\bar{\xi}) \\
&\quad + \tilde{\mu}(\{1,3\})\hat{U}^{(1)}(\{1,3\},\bar{\xi}) + \tilde{\mu}(\{1,2,3\})\hat{U}^{(1)}(\{1,2,3\},\bar{\xi}), \qquad \text{(A.2)} \\
U^{(2)}(\bar{\xi}) &= \tilde{\mu}(\{2\})\hat{U}^{(2)}(\{2\},\bar{\xi}) + \tilde{\mu}(\{2,3\})\hat{U}^{(2)}(\{2,3\},\bar{\xi}) \\
&\quad + \tilde{\mu}(\{1,2\})\hat{U}^{(2)}(\{1,2\},\bar{\xi}) + \tilde{\mu}(\{1,2,3\})\hat{U}^{(2)}(\{1,2,3\},\bar{\xi}), \text{ and} \qquad \text{(A.3)} \\
U^{(3)}(\bar{\xi}) &= \tilde{\mu}(\{3\})\hat{U}^{(3)}(\{3\},\bar{\xi}) + \tilde{\mu}(\{1,3\})\hat{U}^{(3)}(\{1,3\},\bar{\xi}) \\
&\quad + \tilde{\mu}(\{2,3\})\hat{U}^{(3)}(\{2,3\},\bar{\xi}) + \tilde{\mu}(\{1,2,3\})\hat{U}^{(3)}(\{1,2,3\},\bar{\xi}). \qquad \text{(A.4)}
\end{aligned}
$$

Suppose that $\Xi^\star = (\bar{\xi}^\star, \bar{\xi}^\star, \ldots, \bar{\xi}^\star)$ is a symmetric MSNE.

Recall that, at the equilibrium mixed strategy $\bar{\xi}^\star$, we have $U^{(1)}(\bar{\xi}^\star) = U^{(2)}(\bar{\xi}^\star) = U^{(3)}(\bar{\xi}^\star)$. Hence, by subtracting the common terms from (A.3) and (A.4), we get

$$
\begin{aligned}
&\tilde{\mu}(\{2\})\hat{U}^{(2)}(\{2\},\bar{\xi}^\star) + \tilde{\mu}(\{1,2\})\hat{U}^{(2)}(\{1,2\},\bar{\xi}^\star) \\
&\quad = \tilde{\mu}(\{3\})\hat{U}^{(3)}(\{3\},\bar{\xi}^\star) + \tilde{\mu}(\{1,3\})\hat{U}^{(3)}(\{1,3\},\bar{\xi}^\star)\ . \qquad \text{(A.5)}
\end{aligned}
$$

Assume that there are two different symmetric MSNEs $\Xi^{\star^1} = (\bar{\xi}^{\star^1}, \bar{\xi}^{\star^1}, \ldots, \bar{\xi}^{\star^1})$ and $\Xi^{\star^2} = (\bar{\xi}^{\star^2}, \bar{\xi}^{\star^2}, \ldots, \bar{\xi}^{\star^2})$. Without loss of generality, we can consider following case only.[3]

$$
\begin{aligned}
\xi_1^{\star^1} &< \xi_1^{\star^2} \\
\xi_2^{\star^1} &\leq \xi_2^{\star^2} \\
\xi_3^{\star^1} &> \xi_3^{\star^2}\ .
\end{aligned}
$$

In this case, since $\hat{U}^{(i)}(C,\bar{\xi})$, $i \in C$, depends on $\bar{\xi}$ only through $\sum_{l\in C}\xi_l$,

[3] Other cases can be obtained by permutating the indices in the subscript and can be handled in a similar way.

- $\tilde{\mu}(\{2\})\hat{U}^{(2)}(\{2\},\bar{\xi}^{\star^1}) \geq \tilde{\mu}(\{2\})\hat{U}^{(2)}(\{2\},\bar{\xi}^{\star^2})$,
- $\tilde{\mu}(\{1,2\})\hat{U}^{(2)}(\{1,2\},\bar{\xi}^{\star^1}) > \tilde{\mu}(\{1,2\})\hat{U}^{(2)}(\{1,2\},\bar{\xi}^{\star^2})$,
- $\tilde{\mu}(\{3\})\hat{U}^{(3)}(\{3\},\bar{\xi}^{\star^1}) < \tilde{\mu}(\{3\})\hat{U}^{(3)}(\{3\},\bar{\xi}^{\star^2})$, and
- $\tilde{\mu}(\{1,3\})\hat{U}^{(3)}(\{1,3\},\bar{\xi}^{\star^1}) \leq \tilde{\mu}(\{1,3\})\hat{U}^{(3)}(\{1,3\},\bar{\xi}^{\star^2})$.

Therefore, we have

$$\begin{aligned}
&\tilde{\mu}(\{2\})\hat{U}^{(2)}(\{2\},\bar{\xi}^{\star^1}) + \tilde{\mu}(\{1,2\})\hat{U}^{(2)}(\{1,2\},\bar{\xi}^{\star^1}) \\
&\quad > \tilde{\mu}(\{2\})\hat{U}^{(2)}(\{2\},\bar{\xi}^{\star^2}) + \tilde{\mu}(\{1,2\})\hat{U}^{(2)}(\{1,2\},\bar{\xi}^{\star^2}) \text{ and} \\
&\tilde{\mu}(\{3\})\hat{U}^{(3)}(\{3\},\bar{\xi}^{\star^1}) + \tilde{\mu}(\{1,3\})\hat{U}^{(3)}(\{1,3\},\bar{\xi}^{\star^1}) \\
&\quad < \tilde{\mu}(\{3\})\hat{U}^{(3)}(\{3\},\bar{\xi}^{\star^2}) + \tilde{\mu}(\{1,3\})\hat{U}^{(3)}(\{1,3\},\bar{\xi}^{\star^2}) \ .
\end{aligned}$$

By (A.5), this yields the contradiction $U^{(2)}(\bar{\xi}^{\star^1}) > U^{(2)}(\bar{\xi}^{\star^2}) = U^{(3)}(\bar{\xi}^{\star^2}) > U^{(3)}(\bar{\xi}^{\star^1})$.

A.3 Four sellers case ($M = 4$)

The expected payoffs from choosing seller 1, 2, 3, and 4 are equal to

$$\begin{aligned}
U^{(1)}(\bar{\xi}) \;=\; & \tilde{\mu}(\{1\})\hat{U}^{(1)}(\{1\},\bar{\xi}) + \tilde{\mu}(\{1,2\})\hat{U}^{(1)}(\{1,2\},\bar{\xi}) + \tilde{\mu}(\{1,3\})\hat{U}^{(1)}(\{1,3\},\bar{\xi}) \\
& +\tilde{\mu}(\{1,4\})\hat{U}^{(1)}(\{1,4\},\bar{\xi}) + \tilde{\mu}(\{1,2,3\})\hat{U}^{(1)}(\{1,2,3\},\bar{\xi}) \\
& +\tilde{\mu}(\{1,2,4\})\hat{U}^{(1)}(\{1,2,4\},\bar{\xi}) + \tilde{\mu}(\{1,3,4\})\hat{U}^{(1)}(\{1,3,4\},\bar{\xi}) \\
& +\tilde{\mu}(\{1,2,3,4\})\hat{U}^{(1)}(\{1,2,3,4\},\bar{\xi}) \ , \qquad \text{(A.6)}
\end{aligned}$$

$$
\begin{aligned}
U^{(2)}(\bar{\xi}) \;\; = \;\; & \tilde{\mu}(\{2\})\hat{U}^{(2)}(\{2\},\bar{\xi}) + \tilde{\mu}(\{1,2\})\hat{U}^{(2)}(\{1,2\},\bar{\xi}) + \tilde{\mu}(\{2,3\})\hat{U}^{(2)}(\{2,3\},\bar{\xi}) \\
& +\tilde{\mu}(\{2,4\})\hat{U}^{(2)}(\{2,4\},\bar{\xi}) + \tilde{\mu}(\{1,2,3\})\hat{U}^{(2)}(\{1,2,3\},\bar{\xi}) \\
& +\tilde{\mu}(\{1,2,4\})\hat{U}^{(2)}(\{1,2,4\},\bar{\xi}) + \tilde{\mu}(\{2,3,4\})\hat{U}^{(2)}(\{2,3,4\},\bar{\xi}) \\
& +\tilde{\mu}(\{1,2,3,4\})\hat{U}^{(2)}(\{1,2,3,4\},\bar{\xi}) \ ,
\end{aligned}
$$

$$
\begin{aligned}
U^{(3)}(\bar{\xi}) \;\; = \;\; & \tilde{\mu}(\{3\})\hat{U}^{(3)}(\{3\},\bar{\xi}) + \tilde{\mu}(\{1,3\})\hat{U}^{(3)}(\{1,3\},\bar{\xi}) + \tilde{\mu}(\{2,3\})\hat{U}^{(3)}(\{2,3\},\bar{\xi}) \\
& +\tilde{\mu}(\{3,4\})\hat{U}^{(3)}(\{3,4\},\bar{\xi}) + \tilde{\mu}(\{1,2,3\})\hat{U}^{(3)}(\{1,2,3\},\bar{\xi}) \\
& +\tilde{\mu}(\{1,3,4\})\hat{U}^{(3)}(\{1,3,4\},\bar{\xi}) + \tilde{\mu}(\{2,3,4\})\hat{U}^{(3)}(\{2,3,4\},\bar{\xi}) \\
& +\tilde{\mu}(\{1,2,3,4\})\hat{U}^{(3)}(\{1,2,3,4\},\bar{\xi}) \ , \text{ and}
\end{aligned}
$$

$$
\begin{aligned}
U^{(4)}(\bar{\xi}) \;\; = \;\; & \tilde{\mu}(\{4\})\hat{U}^{(4)}(\{4\},\bar{\xi}) + \tilde{\mu}(\{1,4\})\hat{U}^{(4)}(\{1,4\},\bar{\xi}) + \tilde{\mu}(\{2,4\})\hat{U}^{(4)}(\{2,4\},\bar{\xi}) \\
& +\tilde{\mu}(\{3,4\})\hat{U}^{(4)}(\{3,4\},\bar{\xi}) + \tilde{\mu}(\{1,2,4\})\hat{U}^{(4)}(\{1,2,4\},\bar{\xi}) \\
& +\tilde{\mu}(\{1,3,4\})\hat{U}^{(4)}(\{1,3,4\},\bar{\xi}) + \tilde{\mu}(\{2,3,4\})\hat{U}^{(4)}(\{2,3,4\},\bar{\xi}) \\
& +\tilde{\mu}(\{1,2,3,4\})\hat{U}^{(4)}(\{1,2,3,4\},\bar{\xi}) \ .
\end{aligned}
$$

Suppose that $\Xi^\star = (\bar{\xi}^\star, \bar{\xi}^\star, \ldots, \bar{\xi}^\star)$ is a symmetric MSNE. Then, at the equilibrium mixed strategy $\bar{\xi}^\star$, $U^{(1)}(\bar{\xi}^\star) = U^{(2)}(\bar{\xi}^\star) = U^{(3)}(\bar{\xi}^\star) = U^{(4)}(\bar{\xi}^\star)$. Hence, by subtracting the common terms from (A.7) and (A.7), we get

$$
\begin{aligned}
& \tilde{\mu}(\{3\})\hat{U}^{(3)}(\{3\},\bar{\xi}^\star) + \tilde{\mu}(\{1,3\})\hat{U}^{(3)}(\{1,3\},\bar{\xi}^\star) + \tilde{\mu}(\{2,3\})\hat{U}^{(3)}(\{2,3\},\bar{\xi}^\star) \\
& \quad +\tilde{\mu}(\{1,2,3\})\hat{U}^{(3)}(\{1,2,3\},\bar{\xi}^\star) \\
& = \tilde{\mu}(\{4\})\hat{U}^{(4)}(\{4\},\bar{\xi}^\star) + \tilde{\mu}(\{1,4\})\hat{U}^{(4)}(\{1,4\},\bar{\xi}^\star) + \tilde{\mu}(\{2,4\})\hat{U}^{(4)}(\{2,4\},\bar{\xi}^\star) \\
& \quad +\tilde{\mu}(\{1,2,4\})\hat{U}^{(4)}(\{1,2,4\},\bar{\xi}^\star) \ . \qquad \text{(A.7)}
\end{aligned}
$$

Similarly, from $U^{(1)}(\bar{\xi}^\star) = U^{(3)}(\bar{\xi}^\star)$,

$$\begin{aligned}
&\tilde{\mu}(\{1\})\hat{U}^{(1)}(\{1\},\bar{\xi}^\star) + \tilde{\mu}(\{1,2\})\hat{U}^{(1)}(\{1,2\},\bar{\xi}^\star) + \tilde{\mu}(\{1,4\})\hat{U}^{(1)}(\{1,4\},\bar{\xi}^\star) \\
&\quad +\tilde{\mu}(\{1,2,4\})\hat{U}^{(1)}(\{1,2,4\},\bar{\xi}^\star) \\
&= \tilde{\mu}(\{3\})\hat{U}^{(3)}(\{3\},\bar{\xi}^\star) + \tilde{\mu}(\{2,3\})\hat{U}^{(3)}(\{2,3\},\bar{\xi}^\star) + \tilde{\mu}(\{3,4\})\hat{U}^{(3)}(\{3,4\},\bar{\xi}^\star) \\
&\quad +\tilde{\mu}(\{2,3,4\})\hat{U}^{(3)}(\{2,3,4\},\bar{\xi}^\star)\ , \qquad \text{(A.8)}
\end{aligned}$$

and using $U^{(2)}(\bar{\xi}^\star) = U^{(4)}(\bar{\xi}^\star)$,

$$\begin{aligned}
&\tilde{\mu}(\{2\})\hat{U}^{(2)}(\{2\},\bar{\xi}^\star) + \tilde{\mu}(\{1,2\})\hat{U}^{(2)}(\{1,2\},\bar{\xi}^\star) + \tilde{\mu}(\{2,3\})\hat{U}^{(2)}(\{2,3\},\bar{\xi}^\star) \\
&\quad +\tilde{\mu}(\{1,2,3\})\hat{U}^{(2)}(\{1,2,3\},\bar{\xi}^\star) \\
&= \tilde{\mu}(\{4\})\hat{U}^{(4)}(\{4\},\bar{\xi}^\star) + \tilde{\mu}(\{1,4\})\hat{U}^{(4)}(\{1,4\},\bar{\xi}^\star) + \tilde{\mu}(\{3,4\})\hat{U}^{(4)}(\{3,4\},\bar{\xi}^\star) \\
&\quad +\tilde{\mu}(\{1,3,4\})\hat{U}^{(4)}(\{1,3,4\},\bar{\xi}^\star)\ . \qquad \text{(A.9)}
\end{aligned}$$

Suppose that there are two different symmetric MSNEs $\Xi^{\star^1} = (\bar{\xi}^{\star^1}, \bar{\xi}^{\star^1}, \ldots, \bar{\xi}^{\star^1})$ and $\Xi^{\star^2} = (\bar{\xi}^{\star^2}, \bar{\xi}^{\star^2}, \ldots, \bar{\xi}^{\star^2})$. Without loss of generality, we consider following two cases.[4]

Case 1:

$$\begin{aligned}
\xi_1^{\star^1} &< \xi_1^{\star^2}\ , \\
\xi_2^{\star^1} &\leq \xi_2^{\star^2}\ , \\
\xi_3^{\star^1} &\leq \xi_3^{\star^2}\ , \\
\xi_4^{\star^1} &> \xi_4^{\star^2}\ .
\end{aligned}$$

[4] Other cases can be obtained by permutating the indices in the subscript and can be handled in a similar way.

Case 2:

$$\begin{aligned}
\xi_1^{\star^1} &< \xi_1^{\star^2} \ , \\
\xi_2^{\star^1} &\leq \xi_2^{\star^2} \ , \\
\xi_3^{\star^1} &\geq \xi_3^{\star^2} \ , \\
\xi_4^{\star^1} &> \xi_4^{\star^2} \ .
\end{aligned}$$

In case 1, we have the following inequalities:

$$\begin{aligned}
\xi_1^{\star^1} + \xi_4^{\star^1} &\geq \xi_1^{\star^2} + \xi_4^{\star^2}, \\
\xi_2^{\star^1} + \xi_4^{\star^1} &> \xi_2^{\star^2} + \xi_4^{\star^2}, \\
\xi_3^{\star^1} + \xi_4^{\star^1} &> \xi_3^{\star^2} + \xi_4^{\star^2}, \\
\xi_1^{\star^1} + \xi_2^{\star^1} + \xi_4^{\star^1} &\geq \xi_1^{\star^2} + \xi_2^{\star^2} + \xi_4^{\star^2}, \\
\xi_1^{\star^1} + \xi_3^{\star^1} + \xi_4^{\star^1} &\geq \xi_1^{\star^2} + \xi_3^{\star^2} + \xi_4^{\star^2}, \text{ and} \\
\xi_2^{\star^1} + \xi_3^{\star^1} + \xi_4^{\star^1} &> \xi_2^{\star^2} + \xi_3^{\star^2} + \xi_4^{\star^2}.
\end{aligned}$$

Since $\hat{U}^{(i)}(C, \bar{\xi})$, $i \in C$, depends on $\bar{\xi}$ only through $\sum_{l \in C} \xi_l$ and is strictly decreasing in $\sum_{l \in C} \xi_l$, we have

- $\tilde{\mu}(\{3\})\hat{U}^{(3)}(\{3\}, \bar{\xi}^{\star^1}) \geq \tilde{\mu}(\{3\})\hat{U}^{(3)}(\{3\}, \bar{\xi}^{\star^2})$,
- $\tilde{\mu}(\{1,3\})\hat{U}^{(3)}(\{1,3\}, \bar{\xi}^{\star^1}) > \tilde{\mu}(\{1,3\})\hat{U}^{(3)}(\{1,3\}, \bar{\xi}^{\star^2})$,
- $\tilde{\mu}(\{2,3\})\hat{U}^{(3)}(\{2,3\}, \bar{\xi}^{\star^1}) \geq \tilde{\mu}(\{2,3\})\hat{U}^{(3)}(\{2,3\}, \bar{\xi}^{\star^2})$,
- $\tilde{\mu}(\{1,2,3\})\hat{U}^{(3)}(\{1,2,3\}, \bar{\xi}^{\star^1}) > \tilde{\mu}(\{1,2,3\})\hat{U}^{(3)}(\{1,2,3\}, \bar{\xi}^{\star^2})$,
- $\tilde{\mu}(\{4\})\hat{U}^{(4)}(\{4\}, \bar{\xi}^{\star^1}) < \tilde{\mu}(\{4\})\hat{U}^{(4)}(\{4\}, \bar{\xi}^{\star^2})$,

- $\tilde{\mu}(\{1,4\})\hat{U}^{(4)}(\{1,4\},\bar{\xi}^{\star^1}) \leq \tilde{\mu}(\{1,4\})\hat{U}^{(4)}(\{1,4\},\bar{\xi}^{\star^2})$,

- $\tilde{\mu}(\{2,4\})\hat{U}^{(4)}(\{2,4\},\bar{\xi}^{\star^1}) < \tilde{\mu}(\{2,4\})\hat{U}^{(4)}(\{2,4\},\bar{\xi}^{\star^2})$, and

- $\tilde{\mu}(\{1,2,4\})\hat{U}^{(4)}(\{1,2,4\},\bar{\xi}^{\star^1}) \leq \tilde{\mu}(\{1,2,4\})\hat{U}^{(4)}(\{1,2,4\},\bar{\xi}^{\star^2})$.

Using the equality $U^{(3)}(\bar{\xi}^{\star}) = U^{(4)}(\bar{\xi}^{\star})$ at the equilibrium and the above inequalities in the expression for $U^{(3)}(\bar{\xi}^{\star^1})$, $U^{(3)}(\bar{\xi}^{\star^2})$, $U^{(4)}(\bar{\xi}^{\star^1})$, and $U^{(4)}(\bar{\xi}^{\star^2})$, we can construct the contradiction $U^{(3)}(\bar{\xi}^{\star^1}) > U^{(3)}(\bar{\xi}^{\star^2}) = U^{(4)}(\bar{\xi}^{\star^2}) > U^{(4)}(\bar{\xi}^{\star^1})$.

In case 2, the following inequalities hold.

$$\begin{aligned}
\xi_1^{\star^1} + \xi_2^{\star^1} &< \xi_1^{\star^2} + \xi_2^{\star^2}, \\
\xi_3^{\star^1} + \xi_4^{\star^1} &> \xi_3^{\star^2} + \xi_4^{\star^2}, \\
\xi_1^{\star^1} + \xi_2^{\star^1} + \xi_3^{\star^1} &< \xi_1^{\star^2} + \xi_2^{\star^2} + \xi_3^{\star^2}, \\
\xi_1^{\star^1} + \xi_2^{\star^1} + \xi_4^{\star^1} &\leq \xi_1^{\star^2} + \xi_2^{\star^2} + \xi_4^{\star^2}, \\
\xi_1^{\star^1} + \xi_3^{\star^1} + \xi_4^{\star^1} &\geq \xi_1^{\star^2} + \xi_3^{\star^2} + \xi_4^{\star^2}, \text{ and} \\
\xi_2^{\star^1} + \xi_3^{\star^1} + \xi_4^{\star^1} &> \xi_2^{\star^2} + \xi_3^{\star^2} + \xi_4^{\star^2}.
\end{aligned}$$

Here, the relation between $\xi_2^{\star^1} + \xi_3^{\star^1}$ and $\xi_2^{\star^2} + \xi_3^{\star^2}$ is unknown. Hence, we consider two different cases.

Case 2-(i): Suppose that $\xi_2^{\star^1} + \xi_3^{\star^1} \geq \xi_2^{\star^2} + \xi_3^{\star^2}$. Then, $\xi_1^{\star^1} + \xi_4^{\star^1} \leq \xi_1^{\star^2} + \xi_4^{\star^2}$, and the following inequalities can be shown:

- $\mu(\{1\})\hat{U}^{(1)}(\{1\},\bar{\xi}^{\star^1}) > \mu(\{1\})\hat{U}^{(1)}(\{1\},\bar{\xi}^{\star^2})$,

- $\mu(\{1,2\})\hat{U}^{(1)}(\{1,2\},\bar{\xi}^{\star^1}) > \mu(\{1,2\})\hat{U}^{(1)}(\{1,2\},\bar{\xi}^{\star^2})$,

- $\mu(\{1,4\})\hat{U}^{(1)}(\{1,4\},\bar{\xi}^{\star^1}) \geq \mu(\{1,4\})\hat{U}^{(1)}(\{1,4\},\bar{\xi}^{\star^2})$,

- $\mu(\{1,2,4\})\hat{U}^{(1)}(\{1,2,4\},\bar{\xi}^{\star^1}) \geq \mu(\{1,2,4\})\hat{U}^{(1)}(\{1,2,4\},\bar{\xi}^{\star^2})$,

- $\mu(\{3\})\hat{U}^{(3)}(\{3\},\bar{\xi}^{\star^1}) \leq \mu(\{3\})\hat{U}^{(3)}(\{3\},\bar{\xi}^{\star^2})$,

- $\mu(\{2,3\})\hat{U}^{(3)}(\{2,3\},\bar{\xi}^{\star^1}) \leq \mu(\{2,3\})\hat{U}^{(3)}(\{2,3\},\bar{\xi}^{\star^2})$,

- $\mu(\{3,4\})\hat{U}^{(3)}(\{3,4\},\bar{\xi}^{\star^1}) < \mu(\{3,4\})\hat{U}^{(3)}(\{3,4\},\bar{\xi}^{\star^2})$, and

- $\mu(\{2,3,4\})\hat{U}^{(3)}(\{2,3,4\},\bar{\xi}^{\star^1}) < \mu(\{2,3,4\})\hat{U}^{(3)}(\{2,3,4\},\bar{\xi}^{\star^2})$.

Using the equalities $U^{(1)}(\bar{\xi}^{\star^1}) = U^{(3)}(\bar{\xi}^{\star^1})$ and $U^{(1)}(\bar{\xi}^{\star^2}) = U^{(3)}(\bar{\xi}^{\star^2})$ and the above inequalities, one can draw a contradiction that $U^{(1)}(\bar{\xi}^{\star^1}) > U^{(1)}(\bar{\xi}^{\star^2}) = U^{(3)}(\bar{\xi}^{\star^2}) > U^{(3)}(\bar{\xi}^{\star^1})$.

Case 2-(ii): Suppose that $\xi_2^{\star^1} + \xi_3^{\star^1} < \xi_2^{\star^2} + \xi_3^{\star^2}$. Then, $\xi_1^{\star^1} + \xi_4^{\star^1} > \xi_1^{\star^2} + \xi_4^{\star^2}$ and we have

- $\tilde{\mu}(\{2\})\hat{U}^{(2)}(\{2\},\bar{\xi}^{\star^1}) \geq \tilde{\mu}(\{2\})\hat{U}^{(2)}(\{2\},\bar{\xi}^{\star^2})$,

- $\tilde{\mu}(\{1,2\})\hat{U}^{(2)}(\{1,2\},\bar{\xi}^{\star^1}) > \tilde{\mu}(\{1,2\})\hat{U}^{(2)}(\{1,2\},\bar{\xi}^{\star^2})$,

- $\tilde{\mu}(\{2,3\})\hat{U}^{(2)}(\{2,3\},\bar{\xi}^{\star^1}) > \tilde{\mu}(\{2,3\})\hat{U}^{(2)}(\{2,3\},\bar{\xi}^{\star^2})$,

- $\tilde{\mu}(\{1,2,3\})\hat{U}^{(2)}(\{1,2,3\},\bar{\xi}^{\star^1}) > \tilde{\mu}(\{1,2,3\})\hat{U}^{(2)}(\{1,2,3\},\bar{\xi}^{\star^2})$,

- $\tilde{\mu}(\{4\})\hat{U}^{(4)}(\{4\},\bar{\xi}^{\star^1}) < \tilde{\mu}(\{4\})\hat{U}^{(4)}(\{4\},\bar{\xi}^{\star^2})$,

- $\tilde{\mu}(\{1,4\})\hat{U}^{(4)}(\{1,4\},\bar{\xi}^{\star^1}) < \tilde{\mu}(\{1,4\})\hat{U}^{(4)}(\{1,4\},\bar{\xi}^{\star^2})$,

- $\tilde{\mu}(\{3,4\})\hat{U}^{(4)}(\{3,4\},\bar{\xi}^{\star^1}) < \tilde{\mu}(\{3,4\})\hat{U}^{(4)}(\{3,4\},\bar{\xi}^{\star^2})$, and

- $\tilde{\mu}(\{1,3,4\})\hat{U}^{(4)}(\{1,3,4\},\bar{\xi}^{\star^1}) \leq \tilde{\mu}(\{1,3,4\})\hat{U}^{(4)}(\{1,3,4\},\bar{\xi}^{\star^2})$.

By the equalities $U^{(2)}(\bar{\xi}^{\star^1}) = U^{(4)}(\bar{\xi}^{\star^1})$ and $U^{(2)}(\bar{\xi}^{\star^2}) = U^{(4)}(\bar{\xi}^{\star^2})$, the above inequalities yield the contradiction that $U^{(2)}(\bar{\xi}^{\star^1}) > U^{(2)}(\bar{\xi}^{\star^2}) = U^{(4)}(\bar{\xi}^{\star^2}) > U^{(4)}(\bar{\xi}^{\star^1})$.

A.4 Five sellers case ($M = 5$)

Given the mixed strategy $\bar{\xi}$, the expected payoffs from choosing seller 1, 2, 3, 4, and 5, are equal to

$$\begin{aligned}
U^{(1)}(\bar{\xi}) \quad = \quad & \tilde{\mu}(\{1\})\hat{U}^{(1)}(\{1\},\bar{\xi}) + \tilde{\mu}(\{1,2\})\hat{U}^{(1)}(\{1,2\},\bar{\xi}) + \tilde{\mu}(\{1,3\})\hat{U}^{(1)}(\{1,3\},\bar{\xi}) \\
& +\tilde{\mu}(\{1,4\})\hat{U}^{(1)}(\{1,4\},\bar{\xi}) + \tilde{\mu}(\{1,5\})\hat{U}^{(1)}(\{1,5\},\bar{\xi}) \\
& +\tilde{\mu}(\{1,2,3\})\hat{U}^{(1)}(\{1,2,3\},\bar{\xi}) + \tilde{\mu}(\{1,2,4\})\hat{U}^{(1)}(\{1,2,4\},\bar{\xi}) \\
& +\tilde{\mu}(\{1,2,5\})\hat{U}^{(1)}(\{1,2,5\},\bar{\xi}) + \tilde{\mu}(\{1,3,4\})\hat{U}^{(1)}(\{1,3,4\},\bar{\xi}) \\
& +\tilde{\mu}(\{1,3,5\})\hat{U}^{(1)}(\{1,3,5\},\bar{\xi}) + \tilde{\mu}(\{1,4,5\})\hat{U}^{(1)}(\{1,4,5\},\bar{\xi}) \\
& +\tilde{\mu}(\{1,2,3,4\})\hat{U}^{(1)}(\{1,2,3,4\},\bar{\xi}) + \tilde{\mu}(\{1,2,3,5\})\hat{U}^{(1)}(\{1,2,3,5\},\bar{\xi}) \\
& +\tilde{\mu}(\{1,2,4,5\})\hat{U}^{(1)}(\{1,2,4,5\},\bar{\xi}) + \tilde{\mu}(\{1,3,4,5\})\hat{U}^{(1)}(\{1,3,4,5\},\bar{\xi}) \\
& +\tilde{\mu}(\{1,2,3,4,5\})\hat{U}^{(1)}(\{1,2,3,4,5\},\bar{\xi}) \ ,
\end{aligned}$$

$$
\begin{aligned}
U^{(2)}(\bar{\xi}) \quad = \quad & \tilde{\mu}\{2\})\hat{U}^{(2)}(\{2\},\bar{\xi}) + \tilde{\mu}(\{1,2\})\hat{U}^{(2)}(\{1,2\},\bar{\xi}) + \tilde{\mu}(\{2,3\})\hat{U}^{(2)}(\{2,3\},\bar{\xi}) \\
& + \tilde{\mu}(\{2,4\})\hat{U}^{(2)}(\{2,4\},\bar{\xi}) + \tilde{\mu}(\{2,5\})\hat{U}^{(2)}(\{2,5\},\bar{\xi}) \\
& + \tilde{\mu}(\{1,2,3\})\hat{U}^{(2)}(\{1,2,3\},\bar{\xi}) + \tilde{\mu}(\{1,2,4\})\hat{U}^{(2)}(\{1,2,4\},\bar{\xi}) \\
& + \tilde{\mu}(\{1,2,5\})\hat{U}^{(2)}(\{1,2,5\},\bar{\xi}) + \tilde{\mu}(\{2,3,4\})\hat{U}^{(2)}(\{2,3,4\},\bar{\xi}) \\
& + \tilde{\mu}(\{2,3,5\})\hat{U}^{(2)}(\{2,3,5\},\bar{\xi}) + \tilde{\mu}(\{2,4,5\})\hat{U}^{(2)}(\{2,4,5\},\bar{\xi}) \\
& + \tilde{\mu}(\{1,2,3,4\})\hat{U}^{(2)}(\{1,2,3,4\},\bar{\xi}) + \tilde{\mu}(\{1,2,3,5\})\hat{U}^{(2)}(\{1,2,3,5\},\bar{\xi}) \\
& + \tilde{\mu}(\{1,2,4,5\})\hat{U}^{(2)}(\{1,2,4,5\},\bar{\xi}) + \tilde{\mu}(\{2,3,4,5\})\hat{U}^{(2)}(\{2,3,4,5\},\bar{\xi}) \\
& + \tilde{\mu}(\{1,2,3,4,5\})\hat{U}^{(2)}(\{1,2,3,4,5\},\bar{\xi}) \ ,
\end{aligned}
$$

$$
\begin{aligned}
U^{(3)}(\bar{\xi}) \quad = \quad & \tilde{\mu}(\{3\})\hat{U}^{(3)}(\{3\},\bar{\xi}) + \tilde{\mu}(\{1,3\})\hat{U}^{(3)}(\{1,3\},\bar{\xi}) + \tilde{\mu}(\{2,3\})\hat{U}^{(3)}(\{2,3\},\bar{\xi}) \\
& + \tilde{\mu}(\{3,4\})\hat{U}^{(3)}(\{3,4\},\bar{\xi}) + \tilde{\mu}(\{3,5\})\hat{U}^{(3)}(\{3,5\},\bar{\xi}) \\
& + \tilde{\mu}(\{1,2,3\})\hat{U}^{(3)}(\{1,2,3\},\bar{\xi}) + \tilde{\mu}(\{1,3,4\})\hat{U}^{(3)}(\{1,3,4\},\bar{\xi}) \\
& + \tilde{\mu}(\{1,3,5\})\hat{U}^{(3)}(\{1,3,5\},\bar{\xi}) + \tilde{\mu}(\{2,3,4\})\hat{U}^{(3)}\{2,3,4\},\bar{\xi}) \\
& + \tilde{\mu}(\{2,3,5\})\hat{U}^{(3)}(\{2,3,5\},\bar{\xi}) + \tilde{\mu}(\{3,4,5\})\hat{U}^{(3)}(\{3,4,5\},\bar{\xi}) \\
& + \tilde{\mu}(\{1,2,3,4\})\hat{U}^{(3)}(\{1,2,3,4\},\bar{\xi}) + \tilde{\mu}(\{1,2,3,5\})\hat{U}^{(3)}(\{1,2,3,5\},\bar{\xi}) \\
& + \tilde{\mu}(\{1,3,4,5\})\hat{U}^{(3)}(\{1,3,4,5\},\bar{\xi}) + \tilde{\mu}(\{2,3,4,5\})\hat{U}^{(3)}(\{2,3,4,5\},\bar{\xi}) \\
& + \tilde{\mu}(\{1,2,3,4,5\})\hat{U}^{(3)}(\{1,2,3,4,5\},\bar{\xi})
\end{aligned}
$$

$$
\begin{aligned}
U^{(4)}(\bar{\xi}) \;\; = \;\; & \tilde{\mu}(\{4\})\hat{U}^{(4)}(\{4\},\bar{\xi}) + \tilde{\mu}(\{1,4\})\hat{U}^{(4)}(\{1,4\},\bar{\xi}) + \tilde{\mu}(\{2,4\})\hat{U}^{(4)}(\{2,4\},\bar{\xi}) \\
& +\tilde{\mu}(\{3,4\})\hat{U}^{(4)}(\{3,4\},\bar{\xi}) + \tilde{\mu}(\{4,5\})\hat{U}^{(4)}(\{4,5\},\bar{\xi}) \\
& +\tilde{\mu}(\{1,2,4\})\hat{U}^{(4)}(\{1,2,4\},\bar{\xi}) + \tilde{\mu}(\{1,3,4\})\hat{U}^{(4)}(\{1,3,4\},\bar{\xi}) \\
& +\tilde{\mu}(\{1,4,5\})\hat{U}^{(4)}(\{1,4,5\},\bar{\xi}) + \tilde{\mu}(\{2,3,4\})\hat{U}^{(4)}(\{2,3,4\},\bar{\xi}) \\
& +\tilde{\mu}(\{2,4,5\})\hat{U}^{(4)}(\{2,4,5\},\bar{\xi}) + \tilde{\mu}(\{3,4,5\})\hat{U}^{(4)}(\{3,4,5\},\bar{\xi}) \\
& +\tilde{\mu}(\{1,2,3,4\})\hat{U}^{(4)}(\{1,2,3,4\},\bar{\xi}) + \tilde{\mu}(\{1,2,4,5\})\hat{U}^{(4)}(\{1,2,4,5\},\bar{\xi}) \\
& +\tilde{\mu}(\{1,3,4,5\})\hat{U}^{(4)}(\{1,3,4,5\},\bar{\xi}) + \tilde{\mu}(\{2,3,4,5\})\hat{U}^{(4)}(\{2,3,4,5\},\bar{\xi}) \\
& +\tilde{\mu}(\{1,2,3,4,5\})\hat{U}^{(4)}(\{1,2,3,4,5\},\bar{\xi}) \text{ , and}
\end{aligned}
$$

$$
\begin{aligned}
U^{(5)}(\bar{\xi}) \;\; = \;\; & \tilde{\mu}(\{5\})\hat{U}^{(5)}(\{5\},\bar{\xi}) + \tilde{\mu}(\{1,5\})\hat{U}^{(5)}(\{1,5\},\bar{\xi}) + \tilde{\mu}(\{2,5\})\hat{U}^{(5)}(\{2,5\},\bar{\xi}) \\
& +\tilde{\mu}(\{3,5\})\hat{U}^{(5)}(\{3,5\},\bar{\xi}) + \tilde{\mu}(\{4,5\})\hat{U}^{(5)}(\{4,5\},\bar{\xi}) \\
& +\tilde{\mu}(\{1,2,5\})\hat{U}^{(5)}(\{1,2,5\},\bar{\xi}) + \tilde{\mu}(\{1,3,5\})\hat{U}^{(5)}(\{1,3,5\},\bar{\xi}) \\
& +\tilde{\mu}(\{1,4,5\})\hat{U}^{(5)}(\{1,4,5\},\bar{\xi}) + \tilde{\mu}(\{2,3,5\})\hat{U}^{(5)}(\{2,3,5\},\bar{\xi}) \\
& +\tilde{\mu}(\{2,4,5\})\hat{U}^{(5)}(\{2,4,5\},\bar{\xi}) + \tilde{\mu}(\{3,4,5\})\hat{U}^{(5)}(\{3,4,5\},\bar{\xi}) \\
& +\tilde{\mu}(\{1,2,3,5\})\hat{U}^{(5)}(\{1,2,3,5\},\bar{\xi}) + \tilde{\mu}(\{1,2,4,5\})\hat{U}^{(5)}(\{1,2,4,5\},\bar{\xi}) \\
& +\tilde{\mu}(\{1,3,4,5\})\hat{U}^{(5)}(\{1,3,4,5\},\bar{\xi}) + \tilde{\mu}(\{2,3,4,5\})\hat{U}^{(5)}(\{2,3,4,5\},\bar{\xi}) \\
& +\tilde{\mu}(\{1,2,3,4,5\})\hat{U}^{(5)}(\{1,2,3,4,5\},\bar{\xi}) \text{ .}
\end{aligned}
$$

Suppose that $\Xi^\star = (\bar{\xi}^\star, \bar{\xi}^\star, \ldots, \bar{\xi}^\star)$ is a symmetric MSNE. Then, at the equilibrium mixed strategy $\bar{\xi}^\star$, $U^{(1)}(\bar{\xi}^\star) = U^{(2)}(\bar{\xi}^\star) = U^{(3)}(\bar{\xi}^\star) = U^{(4)}(\bar{\xi}^\star) = U^{(5)}(\bar{\xi}^\star)$. From

$U^{(1)}(\bar{\xi}^\star) = U^{(5)}(\bar{\xi}^\star)$ and their expressions, we can show

$$
\begin{aligned}
&\tilde{\mu}(\{1\})\hat{U}^{(1)}(\{1\},\bar{\xi}^\star) + \tilde{\mu}(\{1,2\})\hat{U}^{(1)}(\{1,2\},\bar{\xi}^\star) + \tilde{\mu}(\{1,3\})\hat{U}^{(1)}(\{1,3\},\bar{\xi}^\star) \\
&+\tilde{\mu}(\{1,4\})\hat{U}^{(1)}(\{1,4\},\bar{\xi}^\star) + \tilde{\mu}(\{1,2,3\})\hat{U}^{(1)}(\{1,2,3\},\bar{\xi}^\star) \\
&+\tilde{\mu}(\{1,2,4\})\hat{U}^{(1)}(\{1,2,4\},\bar{\xi}^\star) + \tilde{\mu}(\{1,3,4\})\hat{U}^{(1)}(\{1,3,4\},\bar{\xi}^\star) \\
&+\tilde{\mu}(\{1,2,3,4\})\hat{U}^{(1)}(\{1,2,3,4\},\bar{\xi}^\star) \\
&= \tilde{\mu}(\{5\})\hat{U}^{(5)}(\{5\},\bar{\xi}^\star) + \tilde{\mu}(\{2,5\})\hat{U}^{(5)}(\{2,5\},\bar{\xi}^\star) + \tilde{\mu}(\{3,5\})\hat{U}^{(5)}(\{3,5\},\bar{\xi}^\star) \\
&+\tilde{\mu}(\{4,5\})\hat{U}^{(5)}(\{4,5\},\bar{\xi}^\star) + \tilde{\mu}(\{2,3,5\})\hat{U}^{(5)}(\{2,3,5\},\bar{\xi}^\star) \\
&+\tilde{\mu}(\{2,4,5\})\hat{U}^{(5)}(\{2,4,5\},\bar{\xi}^\star) + \tilde{\mu}(\{3,4,5\})\hat{U}^{(5)}(\{3,4,5\},\bar{\xi}^\star) \\
&+\tilde{\mu}(\{2,3,4,5\})\hat{U}^{(5)}(\{2,3,4,5\},\bar{\xi}^\star)\ . \qquad \text{(A.10)}
\end{aligned}
$$

From $U^{(2)}(\bar{\xi}^\star) = U^{(4)}(\bar{\xi}^\star)$,

$$
\begin{aligned}
&\tilde{\mu}(\{2\})\hat{U}^{(2)}(\{2\},\bar{\xi}^\star) + \tilde{\mu}(\{1,2\})\hat{U}^{(2)}(\{1,2\},\bar{\xi}^\star) + \tilde{\mu}(\{2,3\})\hat{U}^{(2)}(\{2,3\},\bar{\xi}^\star) \\
&+\tilde{\mu}(\{2,5\})\hat{U}^{(2)}(\{2,5\},\bar{\xi}^\star) + \tilde{\mu}(\{1,2,3\})\hat{U}^{(2)}(\{1,2,3\},\bar{\xi}^\star) \\
&+\tilde{\mu}(\{1,2,5\})\hat{U}^{(2)}(\{1,2,5\},\bar{\xi}^\star) + \tilde{\mu}(\{2,3,5\})\hat{U}^{(2)}(\{2,3,5\},\bar{\xi}^\star) \\
&+\tilde{\mu}(\{1,2,3,5\})\hat{U}^{(2)}(\{1,2,3,5\},\bar{\xi}^\star) \\
&= \tilde{\mu}(\{4\})\hat{U}^{(4)}(\{4\},\bar{\xi}^\star) + \tilde{\mu}(\{1,4\})\hat{U}^{(4)}(\{1,4\},\bar{\xi}^\star) + \tilde{\mu}(\{3,4\})\hat{U}^{(4)}(\{3,4\},\bar{\xi}^\star) \\
&+\tilde{\mu}(\{4,5\})\hat{U}^{(4)}(\{4,5\},\bar{\xi}^\star) + \tilde{\mu}(\{1,3,4\})\hat{U}^{(4)}(\{1,3,4\},\bar{\xi}^\star) \\
&+\tilde{\mu}(\{1,4,5\})\hat{U}^{(4)}(\{1,4,5\},\bar{\xi}^\star) + \tilde{\mu}(\{3,4,5\})\hat{U}^{(4)}(\{3,4,5\},\bar{\xi}^\star) \\
&+\tilde{\mu}(\{1,3,4,5\})\hat{U}^{(4)}(\{1,3,4,5\},\bar{\xi}^\star)\ , \qquad \text{(A.11)}
\end{aligned}
$$

and using $U^{(3)}(\bar{\xi}^{\star}) = U^{(4)}(\bar{\xi}^{\star})$,

$$
\begin{aligned}
&\tilde{\mu}(\{3\})\hat{U}^{(3)}(\{3\},\bar{\xi}^{\star}) + \tilde{\mu}(\{1,3\})\hat{U}^{(3)}(\{1,3\},\bar{\xi}^{\star}) + \tilde{\mu}(\{2,3\})\hat{U}^{(3)}(\{2,3\},\bar{\xi}^{\star}) \\
&+\tilde{\mu}(\{3,5\})\hat{U}^{(3)}(\{3,5\},\bar{\xi}^{\star}) + \tilde{\mu}(\{1,2,3\})\hat{U}^{(3)}(\{1,2,3\},\bar{\xi}^{\star}) \\
&+\tilde{\mu}(\{1,3,5\})\hat{U}^{(3)}(\{1,3,5\},\bar{\xi}^{\star}) + \tilde{\mu}(\{2,3,5\})\hat{U}^{(3)}(\{2,3,5\},\bar{\xi}^{\star}) \\
&+\tilde{\mu}(\{1,2,3,5\})\hat{U}^{(3)}(\{1,2,3,5\},\bar{\xi}^{\star}) \\
&= \tilde{\mu}(\{4\})\hat{U}^{(4)}(\{4\},\bar{\xi}^{\star}) + \tilde{\mu}(\{1,4\})\hat{U}^{(4)}(\{1,4\},\bar{\xi}^{\star}) + \tilde{\mu}(\{2,4\})\hat{U}^{(4)}(\{2,4\},\bar{\xi}^{\star}) \\
&+\tilde{\mu}(\{4,5\})\hat{U}^{(4)}(\{4,5\},\bar{\xi}^{\star}) + \tilde{\mu}(\{1,2,4\})\hat{U}^{(4)}(\{1,2,4\},\bar{\xi}^{\star}) \\
&+\tilde{\mu}(\{1,4,5\})\hat{U}^{(4)}(\{1,4,5\},\bar{\xi}^{\star}) + \tilde{\mu}(\{2,4,5\})\hat{U}^{(4)}(\{2,4,5\},\bar{\xi}^{\star}) \\
&+\tilde{\mu}(\{1,2,4,5\})\hat{U}^{(4)}(\{1,2,4,5\},\bar{\xi}^{\star}) \ . \qquad \text{(A.12)}
\end{aligned}
$$

Assume that there are two different symmetric MSNEs $\Xi^{\star^1} = (\bar{\xi}^{\star^1}, \bar{\xi}^{\star^1}, \ldots, \bar{\xi}^{\star^1})$ and $\Xi^{\star^2} = (\bar{\xi}^{\star^2}, \bar{\xi}^{\star^2}, \ldots, \bar{\xi}^{\star^2})$. Without loss of generality, we can consider only two cases.[5]

Case 1:

$$
\begin{aligned}
\xi_1^{\star^1} &< \xi_1^{\star^2} \\
\xi_2^{\star^1} &\leq \xi_2^{\star^2} \\
\xi_3^{\star^1} &\leq \xi_3^{\star^2} \\
\xi_4^{\star^1} &\leq \xi_4^{\star^2} \\
\xi_5^{\star^1} &> \xi_5^{\star^2} \ .
\end{aligned}
$$

[5]Again, other cases are obtained by permutating the indices and can be handled in a similar way.

Case 2:

$$\begin{aligned} \xi_1^{\star^1} &< \xi_1^{\star^2} \\ \xi_2^{\star^1} &\leq \xi_2^{\star^2} \\ \xi_3^{\star^1} &\leq \xi_3^{\star^2} \\ \xi_4^{\star^1} &> \xi_4^{\star^2} \\ \xi_5^{\star^1} &> \xi_5^{\star^2} \ . \end{aligned} \tag{A.13}$$

In case 1, we can see that,

- $\tilde{\mu}(\{1\})\hat{U}^{(1)}(\{1\}, \bar{\xi}^{\star^1}) > \tilde{\mu}(\{1\})\hat{U}^{(1)}(\{1\}, \bar{\xi}^{\star^2})$,
- $\tilde{\mu}(\{1,2\})\hat{U}^{(1)}(\{1,2\}, \bar{\xi}^{\star^1}) > \tilde{\mu}(\{1,2\})\hat{U}^{(1)}(\{1,2\}, \bar{\xi}^{\star^2})$,
- $\tilde{\mu}(\{1,3\})\hat{U}^{(1)}(\{1,3\}, \bar{\xi}^{\star^1}) > \tilde{\mu}(\{1,3\})\hat{U}^{(1)}(\{1,3\}, \bar{\xi}^{\star^2})$,
- $\tilde{\mu}(\{1,4\})\hat{U}^{(1)}(\{1,4\}, \bar{\xi}^{\star^1}) > \tilde{\mu}(\{1,4\})\hat{U}^{(1)}(\{1,4\}, \bar{\xi}^{\star^2})$,
- $\tilde{\mu}(\{1,2,3\})\hat{U}^{(1)}(\{1,2,3\}, \bar{\xi}^{\star^1}) > \tilde{\mu}(\{1,2,3\})\hat{U}^{(1)}(\{1,2,3\}, \bar{\xi}^{\star^2})$,
- $\tilde{\mu}(\{1,2,4\})\hat{U}^{(1)}(\{1,2,4\}, \bar{\xi}^{\star^1}) > \tilde{\mu}(\{1,2,4\})\hat{U}^{(1)}(\{1,2,4\}, \bar{\xi}^{\star^2})$,
- $\tilde{\mu}(\{1,3,4\})\hat{U}^{(1)}(\{1,3,4\}, \bar{\xi}^{\star^1}) > \tilde{\mu}(\{1,3,4\})\hat{U}^{(1)}(\{1,3,4\}, \bar{\xi}^{\star^2})$,
- $\tilde{\mu}(\{1,2,3,4\})\hat{U}^{(1)}(\{1,2,3,4\}, \bar{\xi}^{\star^1}) > \tilde{\mu}(\{1,2,3,4\})\hat{U}^{(1)}(\{1,2,3,4\}, \bar{\xi}^{\star^2})$,
- $\tilde{\mu}(\{5\})\hat{U}^{(5)}(\{5\}, \bar{\xi}^{\star^1}) < \tilde{\mu}(\{5\})\hat{U}^{(5)}(\{5\}, \bar{\xi}^{\star^2})$,
- $\tilde{\mu}(\{2,5\})\hat{U}^{(5)}(\{2,5\}, \bar{\xi}^{\star^1}) < \tilde{\mu}(\{2,5\})\hat{U}^{(5)}(\{2,5\}, \bar{\xi}^{\star^2})$,
- $\tilde{\mu}(\{3,5\})\hat{U}^{(5)}(\{3,5\}, \bar{\xi}^{\star^1}) < \tilde{\mu}(\{3,5\})\hat{U}^{(5)}(\{3,5\}, \bar{\xi}^{\star^2})$,

- $\tilde{\mu}(\{4,5\})\hat{U}^{(5)}(\{4,5\},\bar{\xi}^{\star^1}) < \tilde{\mu}(\{4,5\})\hat{U}^{(5)}(\{4,5\},\bar{\xi}^{\star^2})$,

- $\tilde{\mu}(\{2,3,5\})\hat{U}^{(5)}(\{2,3,5\},\bar{\xi}^{\star^1}) < \tilde{\mu}(\{2,3,5\})\hat{U}^{(5)}(\{2,3,5\},\bar{\xi}^{\star^2})$,

- $\tilde{\mu}(\{2,4,5\})\hat{U}^{(5)}(\{2,4,5\},\bar{\xi}^{\star^1}) < \mu(\{2,4,5\})\hat{U}^{(5)}(\{2,4,5\},\bar{\xi}^{\star^2})$,

- $\tilde{\mu}(\{3,4,5\})\hat{U}^{(5)}(\{3,4,5\},\bar{\xi}^{\star^1}) < \tilde{\mu}(\{3,4,5\})\hat{U}^{(5)}(\{3,4,5\},\bar{\xi}^{\star^2})$, and

- $\tilde{\mu}(\{2,3,4,5\})\hat{U}^{(5)}(\{2,3,4,5\},\bar{\xi}^{\star^1}) < \tilde{\mu}(\{2,3,4,5\})\hat{U}^{(5)}(\{2,3,4,5\},\bar{\xi}^{\star^2})$.

Now, using these inequalities, we can get the contradiction $U^{(1)}(\bar{\xi}^{\star^1}) > U^{(1)}(\bar{\xi}^{\star^2}) = U^{(5)}(\bar{\xi}^{\star^2}) > U^{(5)}(\bar{\xi}^{\star^1})$.

In case 2, in (A.10), note that the relations between (i) $\xi_1^{\star^1}+\xi_4^{\star^1}$ and $\xi_1^{\star^2}+\xi_4^{\star^2}$, (ii) $\xi_1^{\star^1}+\xi_2^{\star^1}+\xi_4^{\star^1}$ and $\xi_1^{\star^2}+\xi_2^{\star^2}+\xi_4^{\star^2}$, and (iii) $\xi_1^{\star^1}+\xi_3^{\star^1}+\xi_4^{\star^1}$ and $\xi_1^{\star^2}+\xi_3^{\star^2}+\xi_4^{\star^2}$ are unknown. In order to deal with this, we will consider all possible subcases. Consider following conditions.

$$\begin{aligned}
\textbf{(S1)} &: \quad \xi_1^{\star^1}+\xi_4^{\star^1} > \xi_1^{\star^2}+\xi_4^{\star^2} \\
\textbf{(S2)} &: \quad \xi_1^{\star^1}+\xi_2^{\star^1}+\xi_4^{\star^1} > \xi_1^{\star^2}+\xi_2^{\star^2}+\xi_4^{\star^2} \\
\textbf{(S3)} &: \quad \xi_1^{\star^1}+\xi_3^{\star^1}+\xi_4^{\star^1} > \xi_1^{\star^2}+\xi_3^{\star^2}+\xi_4^{\star^2} \ .
\end{aligned}$$

If **S1** holds,

$$\xi_2^{\star^1}+\xi_3^{\star^1}+\xi_5^{\star^1} < \xi_2^{\star^2}+\xi_3^{\star^2}+\xi_5^{\star^2} \ .$$

If **S2** holds, combined with the earlier inequalities for case 2, the following inequalities hold.

$$\begin{aligned}
\xi_3^{\star^1} + \xi_5^{\star^1} &< \xi_3^{\star^2} + \xi_5^{\star^2}, \\
\xi_1^{\star^1} + \xi_4^{\star^1} &> \xi_1^{\star^2} + \xi_4^{\star^2}, \\
\xi_2^{\star^1} + \xi_4^{\star^1} &> \xi_2^{\star^2} + \xi_4^{\star^2}, \\
\xi_2^{\star^1} + \xi_3^{\star^1} + \xi_5^{\star^1} &< \xi_2^{\star^2} + \xi_3^{\star^2} + \xi_5^{\star^2}, \\
\xi_1^{\star^1} + \xi_3^{\star^1} + \xi_5^{\star^1} &< \xi_1^{\star^2} + \xi_3^{\star^2} + \xi_5^{\star^2} \ .
\end{aligned}$$

If **S3** holds, similarly,

$$\begin{aligned}
\xi_2^{\star^1} + \xi_5^{\star^1} &< \xi_2^{\star^2} + \xi_5^{\star^2}, \\
\xi_1^{\star^1} + \xi_4^{\star^1} &> \xi_1^{\star^2} + \xi_4^{\star^2}, \\
\xi_3^{\star^1} + \xi_4^{\star^1} &> \xi_3^{\star^2} + \xi_4^{\star^2}, \\
\xi_2^{\star^1} + \xi_3^{\star^1} + \xi_5^{\star^1} &< \xi_2^{\star^2} + \xi_3^{\star^2} + \xi_5^{\star^2}, \\
\xi_1^{\star^1} + \xi_2^{\star^1} + \xi_5^{\star^1} &< \xi_1^{\star^2} + \xi_2^{\star^2} + \xi_5^{\star^2} \ .
\end{aligned}$$

Note that if at least one of **S2** and **S3** holds, **S1** is automatically satisfied.[6]

If **S2** holds, the following inequalities hold.

- $\tilde{\mu}(\{3\})\hat{U}^{(3)}(\{3\}, \bar{\xi}^{\star^1}) \geq \tilde{\mu}(\{3\})\hat{U}^{(3)}(\{3\}, \bar{\xi}^{\star^2})$,
- $\tilde{\mu}(\{1,3\})\hat{U}^{(3)}(\{1,3\}, \bar{\xi}^{\star^1}) > \tilde{\mu}(\{1,3\})\hat{U}^{(3)}(\{1,3\}, \bar{\xi}^{\star^2})$,
- $\tilde{\mu}(\{2,3\})\hat{U}^{(3)}(\{2,3\}, \bar{\xi}^{\star^1}) \geq \tilde{\mu}(\{2,3\})\hat{U}^{(3)}(\{2,3\}, \bar{\xi}^{\star^2})$,
- $\tilde{\mu}(\{3,5\})\hat{U}^{(3)}(\{3,5\}, \bar{\xi}^{\star^1}) > \tilde{\mu}(\{3,5\})\hat{U}^{(3)}(\{3,5\}, \bar{\xi}^{\star^2})$,

[6] This is because $\xi_2^{\star^1} \leq \xi_2^{\star^2}$ and $\xi_3^{\star^1} \leq \xi_3^{\star^2}$.

- $\tilde{\mu}(\{1,2,3\})\hat{U}^{(3)}(\{1,2,3\},\bar{\xi}^{\star^1}) > \tilde{\mu}(\{1,2,3\})\hat{U}^{(3)}(\{1,2,3\},\bar{\xi}^{\star^2})$,

- $\tilde{\mu}(\{1,3,5\})\hat{U}^{(3)}(\{1,3,5\},\bar{\xi}^{\star^1}) > \tilde{\mu}(\{1,3,5\})\hat{U}^{(3)}(\{1,3,5\},\bar{\xi}^{\star^2})$,

- $\tilde{\mu}(\{2,3,5\})\hat{U}^{(3)}(\{2,3,5\},\bar{\xi}^{\star^1}) > \tilde{\mu}(\{2,3,5\})\hat{U}^{(3)}(\{2,3,5\},\bar{\xi}^{\star^2})$,

- $\tilde{\mu}(\{1,2,3,5\})\hat{U}^{(3)}(\{1,2,3,5\},\bar{\xi}^{\star^1}) > \tilde{\mu}(\{1,2,3,5\})\hat{U}^{(3)}(\{1,2,3,5\},\bar{\xi}^{\star^2})$,

- $\tilde{\mu}(\{4\})\hat{U}^{(4)}(\{4\},\bar{\xi}^{\star^1}) < \tilde{\mu}(\{4\})\hat{U}^{(4)}(\{4\},\bar{\xi}^{\star^2})$,

- $\tilde{\mu}(\{1,4\})\hat{U}^{(4)}(\{1,4\},\bar{\xi}^{\star^1}) < \tilde{\mu}(\{1,4\})\hat{U}^{(4)}(\{1,4\},\bar{\xi}^{\star^2})$,

- $\tilde{\mu}(\{2,4\})\hat{U}^{(4)}(\{2,4\},\bar{\xi}^{\star^1}) < \tilde{\mu}(\{2,4\})\hat{U}^{(4)}(\{2,4\},\bar{\xi}^{\star^2})$,

- $\tilde{\mu}(\{4,5\})\hat{U}^{(4)}(\{4,5\},\bar{\xi}^{\star^1}) < \tilde{\mu}(\{4,5\})\hat{U}^{(4)}(\{4,5\},\bar{\xi}^{\star^2})$,

- $\tilde{\mu}(\{1,2,4\})\hat{U}^{(4)}(\{1,2,4\},\bar{\xi}^{\star^1}) < \tilde{\mu}(\{1,2,4\})\hat{U}^{(4)}(\{1,2,4\},\bar{\xi}^{\star^2})$,

- $\tilde{\mu}(\{1,4,5\})\hat{U}^{(4)}(\{1,4,5\},\bar{\xi}^{\star^1}) \leq \tilde{\mu}(\{1,4,5\})\hat{U}^{(4)}(\{1,4,5\},\bar{\xi}^{\star^2})$,

- $\tilde{\mu}(\{2,4,5\})\hat{U}^{(4)}(\{2,4,5\},\bar{\xi}^{\star^1}) < \tilde{\mu}(\{2,4,5\})\hat{U}^{(4)}(\{2,4,5\},\bar{\xi}^{\star^2})$, and

- $\tilde{\mu}(\{1,2,4,5\})\hat{U}^{(4)}(\{1,2,4,5\},\bar{\xi}^{\star^1}) \leq \tilde{\mu}(\{1,2,4,5\})\hat{U}^{(4)}(\{1,2,4,5\},\bar{\xi}^{\star^2})$.

Using the equality $U^{(3)}(\bar{\xi}^{\star^1}) = U^{(4)}(\bar{\xi}^{\star^1})$ and $U^{(3)}(\bar{\xi}^{\star^2}) = U^{(4)}(\bar{\xi}^{\star^2})$ at the equilibrium and the above inequalities in the expression for $U^{(3)}(\bar{\xi}^{\star^1})$, $U^{(3)}(\bar{\xi}^{\star^2})$, $U^{(4)}(\bar{\xi}^{\star^1})$, and $U^{(4)}(\bar{\xi}^{\star^2})$, we can get the contradiction $U^{(3)}(\bar{\xi}^{\star^1}) > U^{(3)}(\bar{\xi}^{\star^2}) = U^{(4)}(\bar{\xi}^{\star^2}) > U^{(4)}(\bar{\xi}^{\star^1})$.

If **S3** holds, the following inequalities hold.

- $\tilde{\mu}(\{2\})\hat{U}^{(2)}(\{2\},\bar{\xi}^{\star^1}) \geq \tilde{\mu}(\{2\})\hat{U}^{(2)}(\{2\},\bar{\xi}^{\star^2})$,

- $\tilde{\mu}(\{1,2\})\hat{U}^{(2)}(\{1,2\},\bar{\xi}^{\star^1}) > \tilde{\mu}(\{1,2\})\hat{U}^{(2)}(\{1,2\},\bar{\xi}^{\star^2})$,

- $\tilde{\mu}(\{2,3\})\hat{U}^{(2)}(\{2,3\},\bar{\xi}^{\star^1}) \geq \tilde{\mu}(\{2,3\})\hat{U}^{(2)}(\{2,3\},\bar{\xi}^{\star^2})$,
- $\tilde{\mu}(\{2,5\})\hat{U}^{(2)}(\{2,5\},\bar{\xi}^{\star^1}) > \tilde{\mu}(\{2,5\})\hat{U}^{(2)}(\{2,5\},\bar{\xi}^{\star^2})$,
- $\tilde{\mu}(\{1,2,3\})\hat{U}^{(2)}(\{1,2,3\},\bar{\xi}^{\star^1}) > \tilde{\mu}(\{1,2,3\})\hat{U}^{(2)}(\{1,2,3\},\bar{\xi}^{\star^2})$,
- $\tilde{\mu}(\{1,2,5\})\hat{U}^{(2)}(\{1,2,5\},\bar{\xi}^{\star^1}) > \tilde{\mu}(\{1,2,5\})\hat{U}^{(2)}(\{1,2,5\},\bar{\xi}^{\star^2})$,
- $\tilde{\mu}(\{2,3,5\})\hat{U}^{(2)}(\{2,3,5\},\bar{\xi}^{\star^1}) > \tilde{\mu}(\{2,3,5\})\hat{U}^{(2)}(\{2,3,5\},\bar{\xi}^{\star^2})$,
- $\tilde{\mu}(\{1,2,3,5\})\hat{U}^{(2)}(\{1,2,3,5\},\bar{\xi}^{\star^1}) > \tilde{\mu}(\{1,2,3,5\})\hat{U}^{(2)}(\{1,2,3,5\},\bar{\xi}^{\star^2})$,
- $\tilde{\mu}(\{4\})\hat{U}^{(4)}(\{4\},\bar{\xi}^{\star^1}) < \tilde{\mu}(\{4\})\hat{U}^{(4)}(\{4\},\bar{\xi}^{\star^2})$,
- $\tilde{\mu}(\{1,4\})\hat{U}^{(4)}(\{1,4\},\bar{\xi}^{\star^1}) < \tilde{\mu}(\{1,4\})\hat{U}^{(4)}(\{1,4\},\bar{\xi}^{\star^2})$,
- $\tilde{\mu}(\{3,4\})\hat{U}^{(4)}(\{3,4\},\bar{\xi}^{\star^1}) < \tilde{\mu}(\{3,4\})\hat{U}^{(4)}(\{3,4\},\bar{\xi}^{\star^2})$,
- $\tilde{\mu}(\{4,5\})\hat{U}^{(4)}(\{4,5\},\bar{\xi}^{\star^1}) < \tilde{\mu}(\{4,5\})\hat{U}^{(4)}(\{4,5\},\bar{\xi}^{\star^2})$,
- $\tilde{\mu}(\{1,3,4\})\hat{U}^{(4)}(\{1,3,4\},\bar{\xi}^{\star^1}) < \tilde{\mu}(\{1,3,4\})\hat{U}^{(4)}(\{1,3,4\},\bar{\xi}^{\star^2})$,
- $\tilde{\mu}(\{1,4,5\})\hat{U}^{(4)}(\{1,4,5\},\bar{\xi}^{\star^1}) \leq \tilde{\mu}(\{1,4,5\})\hat{U}^{(4)}(\{1,4,5\},\bar{\xi}^{\star^2})$,
- $\tilde{\mu}(\{3,4,5\})\hat{U}^{(4)}(\{3,4,5\},\bar{\xi}^{\star^1}) < \tilde{\mu}(\{3,4,5\})\hat{U}^{(4)}(\{3,4,5\},\bar{\xi}^{\star^2})$, and
- $\tilde{\mu}(\{1,3,4,5\})\hat{U}^{(4)}(\{1,3,4,5\},\bar{\xi}^{\star^1}) \leq \tilde{\mu}(\{1,3,4,5\})\hat{U}^{(4)}(\{1,3,4,5\},\bar{\xi}^{\star^2})$.

Using the equality $U^{(2)}(\bar{\xi}^{\star^1}) = U^{(4)}(\bar{\xi}^{\star^1})$ and $U^{(2)}(\bar{\xi}^{\star^2}) = U^{(4)}(\bar{\xi}^{\star^2})$ at the equilibrium and the above inequalities in the expression for $U^{(2)}(\bar{\xi}^{\star^1})$, $U^{(2)}(\bar{\xi}^{\star^2})$, $U^{(4)}(\bar{\xi}^{\star^1})$, and $U^{(4)}(\bar{\xi}^{\star^2})$, this yields the contradiction $U^{(2)}(\bar{\xi}^{\star^1}) > U^{(2)}(\bar{\xi}^{\star^2}) = U^{(4)}(\bar{\xi}^{\star^2}) > U^{(4)}(\bar{\xi}^{\star^1})$.

If none of **S1**,**S2**, and **S3** holds, we have

- $\tilde{\mu}(\{1\})\hat{U}^{(1)}(\{1\},\bar{\xi}^{\star^1}) > \tilde{\mu}(\{1\})\hat{U}^{(1)}(\{1\},\bar{\xi}^{\star^2})$,
- $\tilde{\mu}(\{1,2\})\hat{U}^{(1)}(\{1,2\},\bar{\xi}^{\star^1}) > \tilde{\mu}(\{1,2\})\hat{U}^{(1)}(\{1,2\},\bar{\xi}^{\star^2})$,
- $\tilde{\mu}(\{1,3\})\hat{U}^{(1)}(\{1,3\},\bar{\xi}^{\star^1}) > \tilde{\mu}(\{1,3\})\hat{U}^{(1)}(\{1,3\},\bar{\xi}^{\star^2})$,
- $\tilde{\mu}(\{1,4\})\hat{U}^{(1)}(\{1,4\},\bar{\xi}^{\star^1}) \geq \tilde{\mu}(\{1,4\})\hat{U}^{(1)}(\{1,4\},\bar{\xi}^{\star^2})$,
- $\tilde{\mu}(\{1,2,3\})\hat{U}^{(1)}(\{1,2,3\},\bar{\xi}^{\star^1}) > \tilde{\mu}(\{1,2,3\})\hat{U}^{(1)}(\{1,2,3\},\bar{\xi}^{\star^2})$,b
- $\tilde{\mu}(\{1,2,4\})\hat{U}^{(1)}(\{1,2,4\},\bar{\xi}^{\star^1}) \geq \tilde{\mu}(\{1,2,4\})\hat{U}^{(1)}(\{1,2,4\},\bar{\xi}^{\star^2})$,
- $\tilde{\mu}(\{1,3,4\})\hat{U}^{(1)}(\{1,3,4\},\bar{\xi}^{\star^1}) \geq \tilde{\mu}(\{1,3,4\})\hat{U}^{(1)}(\{1,3,4\},\bar{\xi}^{\star^2})$,
- $\tilde{\mu}(\{1,2,3,4\})\hat{U}^{(1)}(\{1,2,3,4\},\bar{\xi}^{\star^1}) > \tilde{\mu}(\{1,2,3,4\})\hat{U}^{(1)}(\{1,2,3,4\},\bar{\xi}^{\star^2})$,
- $\tilde{\mu}(\{5\})\hat{U}^{(5)}(\{5\},\bar{\xi}^{\star^1}) < \tilde{\mu}(\{5\})\hat{U}^{(5)}(\{5\},\bar{\xi}^{\star^2})$,
- $\tilde{\mu}(\{2,5\})\hat{U}^{(5)}(\{2,5\},\bar{\xi}^{\star^1}) \leq \tilde{\mu}(\{2,5\})\hat{U}^{(5)}(\{2,5\},\bar{\xi}^{\star^2})$,
- $\tilde{\mu}(\{3,5\})\hat{U}^{(5)}(\{3,5\},\bar{\xi}^{\star^1}) \leq \tilde{\mu}(\{3,5\})\hat{U}^{(5)}(\{3,5\},\bar{\xi}^{\star^2})$,
- $\tilde{\mu}(\{4,5\})\hat{U}^{(5)}(\{4,5\},\bar{\xi}^{\star^1}) < \tilde{\mu}(\{4,5\})\hat{U}^{(5)}(\{4,5\},\bar{\xi}^{\star^2})$,
- $\tilde{\mu}(\{2,3,5\})\hat{U}^{(5)}(\{2,3,5\},\bar{\xi}^{\star^1}) \leq \tilde{\mu}(\{2,3,5\})\hat{U}^{(5)}(\{2,3,5\},\bar{\xi}^{\star^2})$,
- $\tilde{\mu}(\{2,4,5\})\hat{U}^{(5)}(\{2,4,5\},\bar{\xi}^{\star^1}) < \mu(\{2,4,5\})\hat{U}^{(5)}(\{2,4,5\},\bar{\xi}^{\star^2})$,
- $\tilde{\mu}(\{3,4,5\})\hat{U}^{(5)}(\{3,4,5\},\bar{\xi}^{\star^1}) < \tilde{\mu}(\{3,4,5\})\hat{U}^{(5)}(\{3,4,5\},\bar{\xi}^{\star^2})$, and
- $\tilde{\mu}(\{2,3,4,5\})\hat{U}^{(5)}(\{2,3,4,5\},\bar{\xi}^{\star^1}) < \tilde{\mu}(\{2,3,4,5\})\hat{U}^{(5)}(\{2,3,4,5\},\bar{\xi}^{\star^2})$.

By applying same approach, we can draw the contradiction $U^{(1)}(\bar{\xi}^{\star^1}) > U^{(1)}(\bar{\xi}^{\star^2}) = U^{(5)}(\bar{\xi}^{\star^2}) > U^{(5)}(\bar{\xi}^{\star^1})$.

One remaining subcase is when **S1** holds while **S2** and **S3** do not. Note that the conditions **S1**, **S2**, and **S3** are from unknown relations in (A.10). In the same way, there are many other unknown relations in (A.11), (A.12), and so on. For example, from (A.11), we can identify following similar conditions we work with.

$$\begin{aligned}
\textbf{(S4)} &: \ \xi_2^{\star^1} + \xi_5^{\star^1} > \xi_2^{\star^2} + \xi_5^{\star^2} \\
\textbf{(S5)} &: \ \xi_1^{\star^1} + \xi_2^{\star^1} + \xi_5^{\star^1} > \xi_1^{\star^2} + \xi_2^{\star^2} + \xi_5^{\star^2} \\
\textbf{(S6)} &: \ \xi_2^{\star^1} + \xi_3^{\star^1} + \xi_5^{\star^1} > \xi_3^{\star^2} + \xi_3^{\star^2} + \xi_5^{\star^2} .
\end{aligned}$$

In each case, by following similar steps, one can draw a contradiction except for one remaining subcase (e.g., **S4** holds while **S5** and **S6** do not). Now, we show that the intersection of those remaining subcases yields a contradiction. The intersection of subcases[7] is

$$\begin{aligned}
\xi_1^{\star^1} + \xi_4^{\star^1} &> \xi_1^{\star^2} + \xi_4^{\star^2} \\
\xi_1^{\star^1} + \xi_5^{\star^1} &> \xi_1^{\star^2} + \xi_5^{\star^2} \\
\xi_2^{\star^1} + \xi_4^{\star^1} &> \xi_2^{\star^2} + \xi_4^{\star^2} \\
\xi_2^{\star^1} + \xi_5^{\star^1} &> \xi_2^{\star^2} + \xi_5^{\star^2} \\
\xi_3^{\star^1} + \xi_4^{\star^1} &> \xi_3^{\star^2} + \xi_4^{\star^2} \\
\xi_3^{\star^1} + \xi_5^{\star^1} &> \xi_3^{\star^2} + \xi_5^{\star^2}
\end{aligned}$$

[7] The subcases are from $U^{(1)}(\bar{\xi}^\star) = U^{(5)}(\bar{\xi}^\star)$, $U^{(2)}(\bar{\xi}^\star) = U^{(4)}(\bar{\xi}^\star)$, $U^{(3)}(\bar{\xi}^\star) = U^{(4)}(\bar{\xi}^\star)$, $U^{(1)}(\bar{\xi}^\star) = U^{(4)}(\bar{\xi}^\star)$, $U^{(2)}(\bar{\xi}^\star) = U^{(5)}(\bar{\xi}^\star)$, and $U^{(3)}(\bar{\xi}^\star) = U^{(5)}(\bar{\xi}^\star)$, where $\bar{\xi}^\star$ is an equilibrium mixed strategy.

$$
\begin{aligned}
\xi_1^{\star^1} + \xi_2^{\star^1} + \xi_4^{\star^1} &\leq \xi_1^{\star^2} + \xi_2^{\star^2} + \xi_4^{\star^2} \\
\xi_1^{\star^1} + \xi_3^{\star^1} + \xi_4^{\star^1} &\leq \xi_1^{\star^2} + \xi_3^{\star^2} + \xi_4^{\star^2} \\
\xi_1^{\star^1} + \xi_2^{\star^1} + \xi_5^{\star^1} &\leq \xi_1^{\star^2} + \xi_2^{\star^2} + \xi_5^{\star^2} \\
\xi_1^{\star^1} + \xi_3^{\star^1} + \xi_5^{\star^1} &\leq \xi_1^{\star^2} + \xi_3^{\star^2} + \xi_5^{\star^2} \\
\xi_2^{\star^1} + \xi_3^{\star^1} + \xi_4^{\star^1} &\leq \xi_2^{\star^2} + \xi_3^{\star^2} + \xi_4^{\star^2} \\
\xi_2^{\star^1} + \xi_3^{\star^1} + \xi_5^{\star^1} &\leq \xi_2^{\star^2} + \xi_3^{\star^2} + \xi_5^{\star^2} \ .
\end{aligned}
\tag{A.14}
$$

Given two mixed strategies $\bar{\xi}^1$ and $\bar{\xi}^2$, define

$$
\Delta(C, \bar{\xi}^1, \bar{\xi}^2) = \tilde{\mu}(C) \ |\hat{U}^{(i)}(C, \bar{\xi}^1) - \hat{U}^{(i)}(C, \bar{\xi}^2)| \ , \tag{A.15}
$$

where $C \subset \mathcal{P}$ and $i \in C$. Note that the right hand side of (A.15) is the same for all $i \in C$.[8]

Rewriting (A.10) with $\bar{\xi}^{\star^1}$, we have

$$
\begin{aligned}
&\tilde{\mu}(\{1\})\hat{U}^{(1)}(\{1\}, \bar{\xi}^{\star^1}) + \tilde{\mu}(\{1,2\})\hat{U}^{(1)}(\{1,2\}, \bar{\xi}^{\star^1}) + \tilde{\mu}(\{1,3\})\hat{U}^{(1)}(\{1,3\}, \bar{\xi}^{\star^1}) \\
&+\tilde{\mu}(\{1,4\})\hat{U}^{(1)}(\{1,4\}, \bar{\xi}^{\star^1}) + \tilde{\mu}(\{1,2,3\})\hat{U}^{(1)}(\{1,2,3\}, \bar{\xi}^{\star^1}) \\
&+\tilde{\mu}(\{1,2,4\})\hat{U}^{(1)}(\{1,2,4\}, \bar{\xi}^{\star^1}) + \tilde{\mu}(\{1,3,4\})\hat{U}^{(1)}(\{1,3,4\}, \bar{\xi}^{\star^1}) \\
&+\tilde{\mu}(\{1,2,3,4\})\hat{U}^{(1)}(\{1,2,3,4\}, \bar{\xi}^{\star^1}) \\
&= \tilde{\mu}(\{5\})\hat{U}^{(5)}(\{5\}, \bar{\xi}^{\star^1}) + \tilde{\mu}(\{2,5\})\hat{U}^{(5)}(\{2,5\}, \bar{\xi}^{\star^1}) + \tilde{\mu}(\{3,5\})\hat{U}^{(5)}(\{3,5\}, \bar{\xi}^{\star^1}) \\
&+\tilde{\mu}(\{4,5\})\hat{U}^{(5)}(\{4,5\}, \bar{\xi}^{\star^1}) + \tilde{\mu}(\{2,3,5\})\hat{U}^{(5)}(\{2,3,5\}, \bar{\xi}^{\star^1}) \\
&+\tilde{\mu}(\{2,4,5\})\hat{U}^{(5)}(\{2,4,5\}, \bar{\xi}^{\star^1}) + \tilde{\mu}(\{3,4,5\})\hat{U}^{(5)}(\{3,4,5\}, \bar{\xi}^{\star^1} r) \\
&+\tilde{\mu}(\{2,3,4,5\})\hat{U}^{(5)}(\{2,3,4,5\}, \bar{\xi}^{\star^1}) \ .
\end{aligned}
\tag{A.16}
$$

[8]This is because $\hat{U}^{(i_1)}(C, \bar{\xi}) = \hat{U}^{(i_2)}(C, \bar{\xi})$ for any $i_1, i_2 \in C$.

and similarly with $\bar{\xi}^{\star^2}$,

$$\begin{aligned}
&\tilde{\mu}(\{1\})\hat{U}^{(1)}(\{1\},\bar{\xi}^{\star^2}) + \tilde{\mu}(\{1,2\})\hat{U}^{(1)}(\{1,2\},\bar{\xi}^{\star^2}) + \tilde{\mu}(\{1,3\})\hat{U}^{(1)}(\{1,3\},\bar{\xi}^{\star^2}) \\
&+\tilde{\mu}(\{1,4\})\hat{U}^{(1)}(\{1,4\},\bar{\xi}^{\star^2}) + \tilde{\mu}(\{1,2,3\})\hat{U}^{(1)}(\{1,2,3\},\bar{\xi}^{\star^2}) \\
&+\tilde{\mu}(\{1,2,4\})\hat{U}^{(1)}(\{1,2,4\},\bar{\xi}^{\star^2}) + \tilde{\mu}(\{1,3,4\})\hat{U}^{(1)}(\{1,3,4\},\bar{\xi}^{\star^2}) \\
&+\tilde{\mu}(\{1,2,3,4\})\hat{U}^{(1)}(\{1,2,3,4\},\bar{\xi}^{\star^2}) \\
&= \tilde{\mu}(\{5\})\hat{U}^{(5)}(\{5\},\bar{\xi}^{\star^2}) + \tilde{\mu}(\{2,5\})\hat{U}^{(5)}(\{2,5\},\bar{\xi}^{\star^2}) + \tilde{\mu}(\{3,5\})\hat{U}^{(5)}(\{3,5\},\bar{\xi}^{\star^2}) \\
&+\tilde{\mu}(\{4,5\})\hat{U}^{(5)}(\{4,5\},\bar{\xi}^{\star^1}) + \tilde{\mu}(\{2,3,5\})\hat{U}^{(5)}(\{2,3,5\},\bar{\xi}^{\star^2}) \\
&+\tilde{\mu}(\{2,4,5\})\hat{U}^{(5)}(\{2,4,5\},\bar{\xi}^{\star^1}) + \tilde{\mu}(\{3,4,5\})\hat{U}^{(5)}(\{3,4,5\},\bar{\xi}^{\star^2}r) \\
&+\tilde{\mu}(\{2,3,4,5\})\hat{U}^{(5)}(\{2,3,4,5\},\bar{\xi}^{\star^2}) \; . \qquad \text{(A.17)}
\end{aligned}$$

By subtracting the left hand side (resp. right hand side) of (A.16) from the left hand side (resp. right hand side) of (A.17) and using the inequalities in (A.14) and the definition in (A.15), we get

$$\begin{aligned}
&\Delta(\{1,4\},\bar{\xi}^{\star^1},\bar{\xi}^{\star^2}) - \Delta(\{1\},\bar{\xi}^{\star^1},\bar{\xi}^{\star^2}) - \Delta(\{1,2\},\bar{\xi}^{\star^1},\bar{\xi}^{\star^2}) - \Delta(\{1,3\},\bar{\xi}^{\star^1},\bar{\xi}^{\star^2}) \\
&-\Delta(\{1,2,3\},\bar{\xi}^{\star^1},\bar{\xi}^{\star^2}) - \Delta(\{1,2,4\},\bar{\xi}^{\star^1},\bar{\xi}^{\star^2}) - \Delta(\{1,3,4\},\bar{\xi}^{\star^1},\bar{\xi}^{\star^2}) \\
&-\Delta(\{1,2,3,4\},\bar{\xi}^{\star^1},\bar{\xi}^{\star^2}) \qquad \text{(A.18)} \\
&= \Delta(\{5\},\bar{\xi}^{\star^1},\bar{\xi}^{\star^2}) + \Delta(\{2,5\},\bar{\xi}^{\star^1},\bar{\xi}^{\star^2}) + \Delta(\{3,5\},\bar{\xi}^{\star^1},\bar{\xi}^{\star^2}) + \Delta(\{4,5\},\bar{\xi}^{\star^1},\bar{\xi}^{\star^2}) \\
&+\Delta(\{2,4,5\},\bar{\xi}^{\star^1},\bar{\xi}^{\star^2}) + \Delta(\{3,4,5\},\bar{\xi}^{\star^1},\bar{\xi}^{\star^2}) + \Delta(\{2,3,4,5\},\bar{\xi}^{\star^1},\bar{\xi}^{\star^2}) \\
&-\Delta(\{2,3,5\},\bar{\xi}^{\star^1},\bar{\xi}^{\star^2}) \; . \qquad \text{(A.19)}
\end{aligned}$$

Performing a similar procedure with (A.11) (instead of (A.10) above), we obtain

$$\begin{aligned}
&\Delta(\{2,5\},\bar{\xi}^{\star^1},\bar{\xi}^{\star^2}) - \Delta(\{2\},\bar{\xi}^{\star^1},\bar{\xi}^{\star^2}) - \Delta(\{1,2\},\bar{\xi}^{\star^1},\bar{\xi}^{\star^2}) - \Delta(\{2,3\},\bar{\xi}^{\star^1},\bar{\xi}^{\star^2}) \\
&-\Delta(\{1,2,3\},\bar{\xi}^{\star^1},\bar{\xi}^{\star^2}) - \Delta(\{1,2,5\},\bar{\xi}^{\star^1},\bar{\xi}^{\star^2}) - \Delta(\{2,3,5\},\bar{\xi}^{\star^1},\bar{\xi}^{\star^2}) \\
&-\Delta(\{1,2,3,5\},\bar{\xi}^{\star^1},\bar{\xi}^{\star^2}) \qquad\qquad \text{(A.20)} \\
&= \Delta(\{4\},\bar{\xi}^{\star^1},\bar{\xi}^{\star^2}) + \Delta(\{1,4\},\bar{\xi}^{\star^1},\bar{\xi}^{\star^2}) + \Delta(\{3,4\},\bar{\xi}^{\star^1},\bar{\xi}^{\star^2}) + \Delta(\{4,5\},\bar{\xi}^{\star^1},\bar{\xi}^{\star^2}) \\
&+\Delta(\{1,4,5\},\bar{\xi}^{\star^1},\bar{\xi}^{\star^2}) + \Delta(\{3,4,5\},\bar{\xi}^{\star^1},\bar{\xi}^{\star^2}) + \Delta(\{1,3,4,5\},\bar{\xi}^{\star^1},\bar{\xi}^{\star^2}) \\
&-\Delta(\{1,3,4\},\bar{\xi}^{\star^1},\bar{\xi}^{\star^2})\ . \qquad\qquad \text{(A.21)}
\end{aligned}$$

Denote the values of (A.18) and (A.20) by $\Delta_1^\star$ and $\Delta_2^\star$, respectively. Note that, since we assume that $\xi_1^{\star^1} < \xi_1^{\star^2}$, $\xi_5^{\star^1} > \xi_5^{\star^2}$, and $\tilde{\mu}(\{i\}) > 0$ for all $i \in \mathcal{P}$, we have

$$\Delta(\{1\},\bar{\xi}^{\star^1},\bar{\xi}^{\star^2}) > 0 \ \text{ and } \ \Delta(\{5\},\bar{\xi}^{\star^1},\bar{\xi}^{\star^2}) > 0\ . \qquad \text{(A.22)}$$

We will now proceed to show that if (A.18) is either strictly positive or negative, this leads to a contradiction.

Case (i): If $\Delta_1^\star \geq 0$, we see that $\Delta(\{1,4\},\bar{\xi}^{\star^1},\bar{\xi}^{\star^2}) > \Delta(\{1,3,4\},\bar{\xi}^{\star^1},\bar{\xi}^{\star^2})$ from (A.18) and $\Delta(\{1\},\bar{\xi}^{\star^1},\bar{\xi}^{\star^2}) > 0$ from (A.22). By comparing (A.18) and (A.21), we can get $\Delta_2^\star > \Delta_1^\star \geq 0$. In this case, since $\Delta_2^\star > 0$, from (A.20), $\Delta(\{2,5\},\bar{\xi}^{\star^1},\bar{\xi}^{\star^2}) > \Delta(\{2,3,5\},\bar{\xi}^{\star^1},\bar{\xi}^{\star^2})$ as remaining terms are less than or equal to 0. Then, by comparing (A.19) and (A.20), we can show that $\Delta_1^\star > \Delta_2^\star$ because $\Delta(\{5\},\bar{\xi}^{\star^1},\bar{\xi}^{\star^2}) > 0$ from (A.22). This contradicts the earlier finding that $\Delta_2^\star > \Delta_1^\star$.

Case (ii): If $\Delta_1^\star < 0$, we see that $\Delta(\{2,5\},\bar{\xi}^{\star^1},\bar{\xi}^{\star^2}) < \Delta(\{2,3,5\},\bar{\xi}^{\star^1},\bar{\xi}^{\star^2})$ from (A.19). Then, from (A.20) and the condition $\Delta(\{5\},\bar{\xi}^{\star^1},\bar{\xi}^{\star^2}) > 0$ in (A.19), we can get $0 > \Delta_1^\star > \Delta_2^\star$. In this case, since $\Delta_2^\star < 0$, from (A.21), $\Delta(\{1,4\},\bar{\xi}^{\star^1},\bar{\xi}^{\star^2}) <$

$\Delta(\{1,3,4\}, \bar{\xi}^{\star^1}, \bar{\xi}^{\star^2})$. Then, by comparing (A.18) and (A.21), we conclude $\Delta_2^\star > \Delta_1^\star$. This contradicts the earlier finding that $\Delta_1^\star > \Delta_2^\star$.

Appendix B

Variables and functions

Table B.1: Variables and functions defined in Section 7.1 (1)

Variable or Function	Definition
D_T	$N \cdot K_T$
$\mathcal{B}$	$\mathcal{P}^N$
$\mathbf{B}$	$(B^j; j \in \mathcal{S})$, where B^j the seller chosen by buyer j
$\mathcal{S}_i(\mathbf{B})$	$\{j \in \mathcal{S} \mid B^j = i\} \subset \mathcal{S}$
$\pi_{(l)}(\mathbf{t})$	l-th highest contributions ($l = 1, 2, \ldots, D_T$)
$\pi_{j\cdot}(t_j)$	$\{\pi_{j,k}(t_j); k = 1, 2, \ldots, K_T\}$
$\tilde{\Pi}_{\mathbf{t}}$	$\{\pi_{j\cdot}(t_j); j \in \mathcal{S}\}$
$\Pi_{\mathbf{t}}$	$(\pi_{(k)}(\mathbf{t}); k = 1, 2, \ldots, D_T)$
$\tilde{\Pi}^i_{\mathbf{t}}(\mathbf{B})$	$\{\pi_{j\cdot}(t_j); j \in \mathcal{S}_i(\mathbf{B})\}$
$\Pi^i_{\mathbf{t}}(\mathbf{B})$	order statistics of $\tilde{\Pi}^i_{\mathbf{t}}(\mathbf{B})$
$\overline{\Pi}_{\mathbf{t}}(\mathbf{B})$	$\{\Pi^i_{\mathbf{t}}(\mathbf{B}); i \in \mathcal{P}\}$
$\mathcal{H}(\mathbf{t})$	$\{\overline{\Pi}_{\mathbf{t}}(\mathbf{b});\ \mathbf{b} \in \mathcal{B}\}$

Table B.2: Variables and functions defined in Section 7.1 (2)

Variable or Function	Definition
$\overline{\pi} \in \mathcal{H}(\mathbf{t})$	$(\pi^1, \pi^2, \ldots, \pi^M)$
π^i	ordered contributions of the buyers that choose seller i
$\mathbf{b_t}(\overline{\pi})(\overline{\pi} \in \mathcal{H}(\mathbf{t}))$	vector that tells the selected sellers of the buyers
$\nu_{\mathbf{t}}$	distribution over the set $\mathcal{H}(\mathbf{t})$
$\Psi_{\overline{\pi}}(C)(\subset \tilde{\Pi}_{\mathbf{t}})$	set of *winning contributions* in a coalition $C \subset \mathcal{P}$
$\zeta(C, \overline{\pi})$	sum of the winning *contributions* of coalition C
$\Phi_{\overline{\pi}}(C)$	set of sellers' values of the unsold frequency bands
	in the coalition C
$\lambda(C, \overline{\pi})$	$\sum_{x \in \Phi_{\overline{\pi}}(C)} x$
$m_C^\star$	$\lvert\Psi_{\overline{\pi}}(C)\rvert$
$K(C)$	$\sum_{i \in C} K^i$

Table B.3: Variables and functions defined in Section 7.2 and 7.3

Variable or Function	Definition
$i_k(\mathbf{t}, \mathbf{b})$	seller i whose $\Pi^i_{\mathbf{t}}(\mathbf{b})$ contains $\pi_{(k)}(\mathbf{t})$
$\bar{i}^v_k$	seller who has the value $V_0^{(k)}$ for one of its frequency bands
$\mathcal{W}$	$\mathcal{T} \times \mathcal{B}$
$\nu^{\mathcal{W}}$	distribution over the set $\mathcal{W}$
$r_t^{(g)}(\mathbf{w})(\mathbf{w} \in \mathcal{W})$	total profit of the grand coalition
$r_i^{(g)}(\mathbf{w})(\mathbf{w} \in \mathcal{W})$	received profit of the seller $i \in \mathcal{P}$ in the grand coalition
$r_i^{(s)}(\mathbf{w})(\mathbf{w} \in \mathcal{W})$	profit the seller i can make in separate auction
$\bar{v}^i(\mathbf{w})(\mathbf{w} \in \mathcal{W})$	seller i's total value of the sold items
	under the grand coalition
$\bar{V}^i$	seller i's total value of all items
$\mathcal{W}^{(k)}$	subset of $\mathcal{W}$ assigned the number k
Λ_k	set of sellers who receive share of the revenue
$\Psi_1^{(i)}$	$\{k \in \{1, 2, \ldots, 2^M\} \mid i \notin \Lambda_k\}$
$\Psi_2^{(i)}$	$\{k \in \{1, 2, \ldots, 2^M\} \mid \Lambda_k = \{i\}\}$
$\Psi_3^{(i)}$	$\{1, 2, \ldots, 2^M\} \setminus \left(\Psi_1^{(i)} \cup \Psi_2^{(i)}\right)$
R_k	$\int_{\mathbf{w} \in \mathcal{W}^{(k)}} r_t^{(g)}(\mathbf{w}) d\nu^{\mathcal{W}}(\mathbf{w})$

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