# ABSTRACT

Title of dissertation: WAVELET AND FRAME THEORY: FRAME BOUND GAPS, GENERALIZED SHEARLETS, GRASSMANNIAN FUSION FRAMES, AND $P$-ADIC WAVELETS

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The first wavelet system was discovered by Alfréd Haar one hundred years ago. Since then the field has grown enormously. In 1952, Richard Duffin and Albert Schaeffer synthesized the earlier ideas of a number of illustrious mathematicians into a unified theory, the theory of frames. Interest in frames as intriguing objects in their own right arose when wavelet theory began to surge in popularity. Wavelet and frame analysis is found in such diverse fields as data compression, pseudo-differential operator theory and applied statistics.

We shall explore five areas of frame and wavelet theory: frame bound gaps, smooth Parseval wavelet frames, generalized shearlets, Grassmannian fusion frames, and $p$-adic wavlets. The phenomenon of a *frame bound gap* occurs when certain sequences of functions, converging in $L^2$ to a Parseval frame wavelet, generate systems with frame bounds that are uniformly bounded away from 1. In the 90's, Bin Han proved the existence of Parseval wavelet frames which are smooth and compactly

supported on the frequency domain and also approximate wavelet set wavelets. We discuss problems that arise when one attempts to generalize his results to higher dimensions.

A shearlet system is formed using certain classes of dilations over $\mathbb{R}^2$ that yield directional information about functions in addition to information about scale and position. We employ representations of the extended metaplectic group to create shearlet-like transforms in dimensions higher than 2. Grassmannian frames are in some sense optimal representations of data which will be transmitted over a noisy channel that may lose some of the transmitted coefficients. Fusion frame theory is an incredibly new area that has potential to be applied to problems in distributed sensing and parallel processing. A novel construction of Grassmannian fusion frames shall be presented. Finally, $p$-adic analysis is a growing field, and $p$-adic wavelets are eigenfunctions of certain pseudo-differential operators. A construction of a 2-adic wavelet basis using dilations that have not yet been used in $p$-adic analysis is given.

# WAVELET AND FRAME THEORY: FRAME BOUND GAPS, GENERALIZED SHEARLETS, GRASSMANNIAN FUSION FRAMES, AND $P$-ADIC WAVELETS

by

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# List of Abbreviations

| | |
|---|---|
| MRA | multiresolution analysis |
| ONB | orthonormal basis |
| MSF | minimally supported frequency |
| Eqn | equation |

# Chapter 1

# Introduction

## 1.1 Background

This dissertation contains five distinct components, which are all unified under the umbrella of frame and wavelet theory.

Alfréd Haar probably did not foresee the impact that the first wavelet system, which was a seemingly innocuous example presented in an appendix of his 1909 dissertation, would have on the mathematical and scientific communities ([59], [60]). This set of functions existed many years with out a name or a greater context to be viewed in. About 70 years later, Jean Morlet and Alex Grossman resurrected this mathematical concept to analyze geophysical measurements and other physical phenomena (see, for example [49], [55], and [56]). They named the objects *ondelettes*, *little waves*, which was later translated to *wavelets*, and started building the foundation of wavelet theory. Meyer and Mallat then developed the multiresolution analysis scheme ([85] and [83]). Since then the field has grown enormously. Wavelet analysis is used for data compression, pattern recognition, noise reduction and transient recognition, and wavelet algorithms work in such varied areas as applied statistics, numerical PDEs and image processing. An excellent resource for the study of wavelet theory is Daubechies' book [36]. Heil and Walnut also wrote an expository paper about wavelet theory that caught a snapshot of the field as it was beginning to expand at lightening speed, [67]. For a thorough collection of fundamental papers (or their translations, if necessary) in the field of wavelet theory, see [66].

In their seminal paper "A class of nonharmonic Fourier series" [45], Richard Duffin and Albert Schaeffer synthesized the earlier ideas of a number of illustrious mathematicians, including Ralph Boas Jr ([17], [18]), Raymond Paley and Norbert Wiener ([88]) into a unified theory, the theory of frames. Interest in frames as intriguing objects in their own right, apart from their connection to nonharmonic Fourier series, remained dormant for many years. *Frame theory* became a subject of interest when wavelet theory began to surge in popularity. Frames are intricately connected to sampling theory ([45]) and operator theory ([65]) and have applications in many fields, including wavelet theory ([36]), pseudodifferential operators ([54]), signal processing ([75]) and wireless communication ([96]).

We shall explore five areas of frame and wavelet theory: frame bound gaps, smooth Parseval wavelet frames, generalized shearlets, Grassmannian fusion frames, and $p$-adic wavlets. In Chapter 2, we introduce the following: a new method to improve frame bound estimation; a shrinking technique to construct frames; and a nascent theory concerning frame bound gaps. The phenomenon of a *frame bound gap* occurs when certain sequences of functions, converging in $L^2$ to a Parseval frame wavelet, generate systems with frame bounds that are uniformly bounded away from 1. In [62] and [63], Bin Han proved the existence of Parseval wavelet frames which are smooth and compactly supported on the frequency domain and also approximate wavelet set wavelets. In Chapter 3, we discuss problems that arise when one attempts to generalize his results to higher dimensions. Chapters 2 and 3 solely concern dyadic wavelet systems. A shearlet system is formed using certain classes of non-dyadic dilations over $\mathbb{R}^2$ that yield directional information about functions in addition to information about scale and position. In Chapter 4, we employ representations of the extended metaplectic group to create shearlet-like transforms in dimensions higher than 2. Grassmannian frames are in some sense optimal representations of data which will be transmitted over a noisy channel that may lose some of the transmitted coefficients. Fusion frame theory is an incredibly new area that has potential to be applied to problems in distributed sensing and parallel processing. A novel construction of Grassmannian fusion frames shall be presented in Chapter 5. Finally, $p$-adic analysis is a growing field, with applications in such areas as quantum physics ([73]) and DNA sequencing ([44]). As eigenfunctions of certain pseudo-differential operators, $p$-adic wavelets play an important role in these applications. A construction of a 2-adic wavelet basis using dilations that have not yet been used in $p$-adic analysis is in Chapter 6.

## 1.2 Preliminaries

We now document certain notation, definitions, and conventions that will be used throughout the thesis.

**Definition 1.** For

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix} \in \mathbb{C}^d \text{ and } y = \begin{pmatrix} y_1 \\ \vdots \\ y_d \end{pmatrix} \in \mathbb{C}^d,$$

$$x \cdot y = \langle x, y\rangle x_1\overline{y_1} + \ldots + x_d\overline{y_d};$$

that is, the dot product is conjugate linear in the second entry.

**Definition 2.** For a function $f \in L^1(\mathbb{R}^d)$, the *Fourier transform of* $f$ is defined to be

$$\mathcal{F}(f)(\gamma) = \hat{f}(\gamma) = \int f(x)e^{-2\pi i x\cdot\gamma}dx.$$

By Plancherel's Theorem, $\mathcal{F}$ extends from $L^1 \cap L^2$ to a unitary operator $L^2 \to L^2$. We denote the inverse Fourier transform of a function $g \in L^2(\widehat{\mathbb{R}}^d)$ as $\mathcal{F}^{-1}g = \check{g}$.

**Definition 3.** For $f : \mathbb{R}^d \to \mathbb{C}$, $y \in \mathbb{R}^d$, $\xi \in \widehat{\mathbb{R}}^d$, and $A \in \mathrm{GL}(\mathbb{R}, d)\backslash\mathbb{R}^*I$ define the following operators

$$\begin{aligned} T_y f(x) &= f(x-y), \\ M_\xi f(x) &= e^{2\pi i\xi\cdot x} f(x), \text{ and} \\ D_A f(x) &= |\det A|^{1/2} f(Ax). \end{aligned}$$

In Chapters 2 and 3, for $t \in \mathbb{R}^*$, we shall define

$$D_t f(x) = 2^{td/2} f(2^t x) \tag{1.1}$$

since dyadic dilations are very commonly used.

These operators are unitaries which satisfy the following commutation relations, which are all easily verified (see, for example [9], [53]) :

$$\begin{aligned}
M_\xi T_y &= e^{2\pi i \xi \cdot y} T_y M_\xi \\
M_\xi D_A &= D_A M_{A^{-1}\xi} \\
D_A T_y &= T_{A^{-1}y} D_A \\
\mathcal{F} T_y &= M_{-y} \mathcal{F} \\
\mathcal{F} M_\xi &= T_\xi \mathcal{F} \text{ and} \\
\mathcal{F} D_A &= D_{{}^tA^{-1}} \mathcal{F},
\end{aligned}$$

where ${}^tA$ denotes the transpose of $A$. We are now able to define the term *wavelet*.

**Definition 4.** Let $\psi \in L^2\left(\mathbb{R}^d\right)$ and define the *(dyadic) wavelet system* (using the notation in (1.1),

$$\mathcal{W}(\psi) = \{D_n T_k \psi(x) : n \in \mathbb{Z}, k \in \mathbb{Z}^d\} = \{2^{nd/2}\psi(2^n x - k) : n \in \mathbb{Z}, k \in \mathbb{Z}^d\}.$$

If $\mathcal{W}(\psi)$ is an orthonormal basis for $L^2\left(\mathbb{R}^d\right)$, then $\psi$ is an *orthonormal dyadic wavelet* or simply a *wavelet* for $L^2(\mathbb{R}^d)$.

We can extend some of these definitions to general fields and dilations.

**Definition 5.** Let $\mathbb{F}$ be a field with valuation $|\cdot|$. For $f : \mathbb{F}^d \to \mathbb{C}$, $y \in \mathbb{F}^d$, and $A \in \mathrm{GL}(\mathbb{F}, d)$ define the following operators

$$\begin{aligned}
T_y f(x) &= f(x - y) \text{ and} \\
D_A f(x) &= |\det A|^{1/2} f(Ax),
\end{aligned}$$

where in Chapters 2 and 3, the dilation is defined as in (1.1). We will also call

$$\{D_A T_y \psi(x) : A \in \mathcal{A} \subset \mathrm{GL}(\mathbb{F}, d), y \in \mathcal{Z} \subset \mathbb{F}^d\}$$

a *wavelet system* and $\psi$ a *wavelet.*

Next, we define the term *frame.*

**Definition 6.** A sequence $\{e_j\}_{j\in J}$ in a Hilbert space $\mathcal{H}$ is a *frame* for $\mathcal{H}$ if there exist constants $0 < A \leq B < \infty$ such that

$$\forall f \in \mathcal{H}, \qquad A\|f\|^2 \leq \sum_{j\in J} |\langle f, e_j\rangle|^2 \leq B\|f\|^2. \tag{1.2}$$

The maximal such $A$ and minimal such $B$ are the *optimal frame bounds.* In this thesis, the phrase *frame bound* will always mean the optimal frame bound, where $A$ is the *lower frame bound* and $B$ is the *upper frame bound.* A frame is *tight* if $A = B$, and it is *Parseval* if $A = B = 1$. If a frame $\{e_j\}_{j\in J}$ for $\mathcal{H}$ has the property that for all $k \in J$, $\{e_j\}_{j\neq k}$ is not a frame for $\mathcal{H}$, then $\{e_j\}_{j\in J}$ is a *Riesz basis* for $\mathcal{H}$. If the second inequality of (1.2) is true, but possibly not the first, then $\{e_j\}_{j\in J}$ is a *Bessel sequence.* In this case, we shall still refer to $B$ as the upper frame bound to simplify statements of certain theorem. We note that it is usually called the *Bessel bound.* A frame is *normalized* if $\|e_j\| = 1$ for $j \in \mathcal{J}$. A frame is *equiangular* if for some $\alpha$, $|\langle e_j, e_i\rangle| = \alpha$ for all $i \neq j$.

Every orthonormal basis is a frame. One may view frames as generalizations of orthonormal bases which mimic the reconstruction properties (i.e.: $\forall x, x = \sum\langle x, e_j\rangle e_j$) of orthonormal bases but may have some redundancy. We remark that $\{e_j\}$ is a tight frame with frame bound $A$ if and only if

$$\forall f \in \mathcal{H}, \qquad Af = \sum_{j\in J}\langle f, e_j\rangle e_j. \tag{1.3}$$

In Definition 4, we deal with wavelet systems that are orthonormal bases. However, there is no reason that we should not consider systems $\mathcal{W}(\psi)$ which form frames (respectively, Bessel sequences) for $L^2(\mathbb{R}^d)$. In this case, $\psi$ is a *frame wavelet* (respectively, *Bessel wavelet*).

**Definition 7.** Let $X$ be a measure space. For any measurable set $S \subseteq X$, the *characteristic function of* $S$, $\mathbb{1}_S$, is

$$\mathbb{1}_S(x) = \begin{cases} 1 & ; \quad x \in S \\ 0 & ; \quad \text{else} \end{cases} .$$

Finally, we note that our definition of support will not be the traditional one.

**Definition 8.** Let $(X, \mu)$ be a measure space and $f$ a complex-valued function defined on $X$. The *support of* $f$, $\operatorname{supp} f$ is the following equivalence class of measurable sets

$$\left\{ S \subseteq X : \int_{X \setminus S} |f(x)| d\mu(x) = 0, \text{ and if } R \subset S \text{ and } \int_{X \setminus R} |f(x)| d\mu(x) = 0 \text{ then } \mu(S \setminus R) = 0 \right\}.$$

We shall still speak of *the* support of a function, just as we refer to *a* function in an $L^p$ space. So, $\operatorname{supp} f \subseteq S$ means that at least one element in the equivalence class is a subset of $S$ and $f$ *is compactly supported* means that $\operatorname{supp} f \subseteq K$, where $K$ is a compact set.

Chapter 2

# Smooth Functions Associated with Wavelet Sets on $\mathbb{R}^d$ and Frame Bound Gaps

## 2.1 Introduction

### 2.1.1 Problem

Wavelet theory for $\mathbb{R}^d$, $d > 1$, was historically associated with multiresolution analysis (MRA), *e.g.*, [86]. In particular, for dyadic wavelets, it is well-known that $2^d - 1$ wavelets are required to provide a wavelet orthonormal basis (ONB) with an MRA for $L^2(\mathbb{R}^d)$, cf., [82], [4], and [95]. In fact, until the mid-1990s, it was assumed that it would be impossible to construct a single dyadic wavelet $\psi$ generating an ONB for $L^2(\mathbb{R}^d)$. This changed with the groundbreaking work of Dai and Larson [33] and Dai, Larson, and Speegle [34], [35]. The earliest known examples of such single dyadic wavelets for $d > 1$ had complicated spectral properties, see [6], [12], [8], [13], [33], [34], [35], [69], [70], [93], [98]. Further, such wavelets have discontinuous Fourier transforms. As such it is a natural problem to construct single wavelets with better temporal decay. Further, even on $\mathbb{R}$, in order to improve the temporal decay, one must consider systems of frames rather than orthonormal bases [5], [25], [62], [63] or wavelets which have an MRA structure [69], [70]. We shall address the problem of smoothing $\widehat{\psi}$ by convolution, where $\psi$ is derived by the so-called neighborhood mapping method; see Section 2.1.3. This method has the advantage of being general and constructive. Although there are other smoothing techniques that have been introduced in the area of wavelet theory, e.g., [62] and [63], we choose to smooth by convolution because of its theoretical simplicity and computational effectiveness. However, as will be shown later in the thesis, convolutional smoothing on the frequency domain yields counterintuitive results.

### 2.1.2 Preliminaries

Recall that in this chapter, $D_t f(x) = 2^{td/2} f(2^t x)$. The *Haar wavelet* is the function $\psi = \mathbb{1}_{[0,1/2)} - \mathbb{1}_{[1/2,1)}$. The Haar wavelet is well localized in the time domain but not in the frequency domain. There are wavelets that are characteristic functions in the frequency domain and thus are not localized in the time domain. A classical example of a wavelet which is the inverse Fourier transform of a characteristic function is the *Shannon* or *Littlewood-Paley wavelet*, $\check{\mathbb{1}}_{[-1,-1/2)\cup[1/2,1)}$. Another example is the Journé wavelet,

$$\check{\mathbb{1}}_{[-\frac{16}{7},-2)\cup[-\frac{1}{2},-\frac{2}{7})\cup[\frac{2}{7},\frac{1}{2})\cup[2,\frac{16}{7})}.$$

At an AMS special session in 1992, Dai and Larson introduced the term *wavelet set*, which generalizes this phenomenon. Their original publications concerning wavelet sets are [33] and also [34] and [35], which were written with Speegle. Hernàndez, Wang, and Weiss developed a similar theory in [69] and [70], using the terminology *minimally supported frequency* (*MSF*) *wavelets*.

**Definition 9.** If $K$ is a measurable subset of $\widehat{\mathbb{R}}^d$ and $\check{\mathbb{1}}_K$ is a wavelet for $L^2(\mathbb{R}^d)$, then $K$ is a *wavelet set*.

We can extend this definition to frames.

**Definition 10.** If $L$ is a measurable subset of $\widehat{\mathbb{R}}^d$ and $\mathcal{W}(\check{\mathbb{1}}_L)$ is a frame (respectively, tight frame or Parseval frame) for $L^2(\mathbb{R}^d)$, then $L$ is a *frame* (respectively, *tight frame* or *Parseval frame*) *wavelet set*.

We need the following definition in order to characterize wavelet sets and Parseval frame wavelet sets.

**Definition 11.** Let $K$ and $L$ be two measurable subsets of $\widehat{\mathbb{R}}^d$. A *partition* of $K$ is a collection $\{K_l : l \in \mathbb{Z}\}$ of subsets of $K$ such that $\bigcup_l K_l$ and $K$ differ by a set of measure 0 and, for all $l \neq j$, $K_l \cap K_j$ is a set of measure 0. If there exist a partition $\{K_l : l \in \mathbb{Z}\}$ of $K$ and a sequence $\{k_l : l \in \mathbb{Z}\} \subseteq \mathbb{Z}^d$ such that $\{K_l + k_l : l \in \mathbb{Z}\}$ is a partition of $L$, then $K$ and $L$ are $\mathbb{Z}^d$*-translation congruent*. Similarly, if there exist a partition $\{K_l : l \in \mathbb{Z}\}$ of $K$ and a sequence $\{n_l : l \in \mathbb{Z}\} \subseteq \mathbb{Z}$, where $\{2^{n_l} K_l : l \in \mathbb{Z}\}$ is a partition of $L$, then $K$ and $L$ are *dyadic-dilation congruent*.

The following proposition appears in [35].

**Proposition 12.** *Let $K \subseteq \widehat{\mathbb{R}}^d$ be measurable. The following are equivalent:*

- *$K$ is a (Parseval frame) wavelet set.*
- *$K$ is $\mathbb{Z}^d$-translation congruent to (a subset of) $[0,1)^d$, and $K$ is dyadic-dilation congruent to $[-1,1)^d \backslash [-\frac{1}{2}, \frac{1}{2})^d$.*
- *$\{K + k : k \in \mathbb{Z}^d\}$ is a partition of (a subset of) $\widehat{\mathbb{R}}^d$ and $\{2^n K : n \in \mathbb{Z}\}$ is a partition of $\widehat{\mathbb{R}}^d$.*

### 2.1.3 Neighborhood mapping construction

An infinite iterative construction of wavelet sets, called the *neighborhood mapping construction*, is given by Leon, Sumetkijakan, and Benedetto in [14], [12], and [8]. See also [98], [6], and [93]. In dimensions $d \geq 2$, the example wavelet sets $K$ formed by this process are fractal-like but not fractals. Following a question by E. Weber, the authors proved that the sets $(K_m \backslash A_m)$ they defined, formed after a finite number of steps of the neighborhood mapping construction, are actually Parseval frame wavelet sets.

We shall require the following definition and theorem from [14].

**Definition 13.** Let $K_0$ be a bounded neighborhood of the origin in $\widehat{\mathbb{R}}^d$. Assume that $K_0$ is $\mathbb{Z}^d$-translation congruent to $[0,1]^d$. Let $S$ be a measurable map $S : \widehat{\mathbb{R}}^d \to \widehat{\mathbb{R}}^d$ satisfying the following properties:

- $S$ is a $\mathbb{Z}^d$-translated map, i.e.,

$$\forall \gamma \in \mathbb{R}^d, \qquad \exists k_\gamma \in \mathbb{Z}^d \qquad \text{such that } S(\gamma) = \gamma + k_\gamma;$$

- $S$ is injective;
- The range of $S - I$ is bounded, where $I$ is the identity map on $\mathbb{R}^d$;
- $\left[\cup_{k=1}^\infty S^k(K_0)\right] \cap \left[\cup_{n=0}^\infty 2^{-n} K_0\right] = \emptyset$, where $S^0 = I$ and $S^k \equiv \underbrace{S \circ \cdots \circ S}_{k\text{-fold}}$.

For each $m \in \mathbb{N} \cup \{0\}$ define

$$\begin{aligned} A_m &= K_m \cap \left[\textstyle\bigcup_{n=1}^\infty 2^{-n} K_m\right], \\ K_{m+1} &= (K_m \backslash A_m) \cup S(A_m), \\ \text{and} \qquad K &= \left[K_0 \backslash \textstyle\bigcup_{m=0}^\infty A_m\right] \cup \left[\textstyle\bigcup_{m=0}^\infty \left(S(A_m) \backslash \bigcup_{n>m} A_n\right)\right]. \end{aligned}$$

This process is the *neighborhood mapping construction*. Loosely speaking, $K$ is the limit of the $K_m$.

**Theorem 14.** *Let $K$ be defined by the neighborhood mapping construction. $K$ is a wavelet set. Further, for each $m \geq 0$, $K_m \backslash A_m$ is a Parseval frame wavelet set.*

These frame wavelet sets are finite unions of convex sets. The delicate, complicated shape of an orthonormal wavelet set $K$ constructed in [14] makes it difficult to use natural methods with which to smooth it. It is for this reason that we shall deal with frame wavelets and with the smoothing of $\check{\mathbb{1}}_L$, where $L$ is a $K_m \backslash A_m$. We shall use the following collection of sets in Section 2.2.

**Example 15.** Let

$$K_0 = \left[-\frac{1}{2}, \frac{1}{2}\right)^d \text{ and } S(\gamma_1, \cdots, \gamma_d) = (\gamma_1 + 2\,\mathrm{sign}(\gamma_1), \cdots, \gamma_d + 2\,\mathrm{sign}(\gamma_d)).$$

When $d = 1$, the resulting $K$ is the Journé wavelet set.

It should be mentioned that Merrill [84] has recently found examples of orthonormal wavelet sets for $d = 2$ which may be represented as finite unions of 5 or more convex sets. She uses the generalized scaling set technique from [6]. It is unknown if the construction can be used for $d > 2$. Moreover, the question of existence of orthonormal wavelet sets in $\widehat{\mathbb{R}}^d$ for $d > 2$, which are the finite union of convex sets, is still an open problem. Furthermore, in [14], it is shown that a wavelet set in $\widehat{\mathbb{R}}^d$ can not be decomposed into a union of $d$ or fewer convex sets. It is possible that this bound is not sharp for $d = 2$; that is, it is still not known if there exists a wavelet set in $\widehat{\mathbb{R}}^2$ which may be written as the union of 3 or 4 convex sets.

### 2.1.4 Outline and results

We shall smooth Parseval wavelet sets $L$ by convolving $\mathbb{1}_L$ with auxiliary functions to obtain $\hat{\psi}$ and consider the properties of $\mathcal{W}(\psi)$. In many cases, the resulting $\mathcal{W}(\psi)$ is a frame. In Section 2.2, we develop methods to estimate the resulting frame bounds. We apply those methods to a canonical example in Section 2.3. However, we see in Section 2.4 that there exists a Parseval wavelet set $L$ such that $\mathcal{W}((\mathbb{1}_L * \frac{m}{2}\mathbb{1}_{[-\frac{1}{m},\frac{1}{m}]})^\vee)$ is not a frame for any $m > 0$. Later in Section 2.4, we introduce the *shrinking method*, with which we modify the preceding example to obtain a frame. This method may be used to modify Parseval frame wavelets sets in such a way that they may be smoothed using our techniques or other methods, like those in [63]. Section 2.5 contains Theorems 44 and 48, which show that *frame bound gaps* occur with many wavelet sets. In fact, for certain Parseval frame wavelet sets $L$ and approximate identities $\{k_\lambda\}$, the system $\mathcal{W}((\mathbb{1}_L * k_\lambda)^\vee)$ does not have frame bounds that converge to 1 as $\lambda \to \infty$, even though, for all $1 \leq p < \infty$,

$$\lim_{\lambda\to\infty} \|\mathbb{1}_L * k_\lambda - \mathbb{1}_L\|_{L^p(\widehat{\mathbb{R}}^d)} = 0.$$

Furthermore, when we smooth a specific class of Parseval frame wavelet sets $L_d \subseteq \widehat{\mathbb{R}}^d$ with certain approximate identities $k_{\lambda,d} = \otimes_{i=1}^d k_\lambda$, the corresponding upper frame bounds increase and converge to 2 as $d \to \infty$.

## 2.2 Frame bounds and approximate identities

### 2.2.1 Approximating frame bounds

In this section we give several methods, mostly well-known, to evaluate frame bounds. Our goal is to manipulate Parseval frame wavelet set wavelets on the frequency domain in order to construct frames with faster temporal decay than the original Parseval frames.

**Remark 16.** The following calculation and ones similar to it are commonly used to prove facts about frame wavelet bounds. Define $Q_n = [0, 2^{-n}]^d$ and $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Using the Parseval-Plancherel theorem on both $\mathbb{R}^d$ and $\mathbb{T}^d$ as well as a standard $L^1$ periodization technique, we let $\psi \in L^2(\mathbb{R}^d)$ and have the following calculation:

$$\forall f \in L^2(\mathbb{R}), \quad \sum_{n\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^d} |\langle f, D_n T_k \psi\rangle|^2 \;=\; \sum_{n\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^d} \left|\langle \hat{f}, D_{-n} M_{-k} \hat{\psi}\rangle\right|^2$$

$$= \sum_{n\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^d} \left| \langle \hat{f}, D_n M_k \hat{\psi} \rangle \right|^2$$

$$= \sum_{n\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^d} \left| \int \hat{f}(\gamma) 2^{dn/2} e^{2\pi i k\cdot 2^n\gamma} \overline{\hat{\psi}(2^n\gamma)} d\gamma \right|^2$$

$$= \sum_n 2^{dn} \sum_k \left| \int_{Q_n} \sum_{l\in\mathbb{Z}^d} \hat{f}(\gamma + 2^{-n} l) e^{2\pi i k\cdot 2^n(\gamma+2^{-n}l)} \overline{\hat{\psi}(2^n\gamma + l)} d\gamma \right|^2$$

$$= \sum_n \int_{Q_n} \left| \sum_l \hat{f}(\gamma + 2^{-n} l) \overline{\hat{\psi}(2^n\gamma + l)} \right|^2 d\gamma$$

$$= \sum_n \int_{Q_n} \sum_l \sum_{k\in\mathbb{Z}^d} \hat{f}(\gamma + 2^{-n} l) \overline{\hat{\psi}(2^n\gamma + l) \hat{f}(\gamma + 2^{-n}k)} \hat{\psi}(2^n\gamma + k) d\gamma$$

$$= \sum_n \int \sum_k \hat{f}(\gamma) \overline{\hat{f}(\gamma + 2^{-n}k) \hat{\psi}(2^n\gamma)} \hat{\psi}(2^n\gamma + k) d\gamma \tag{2.1}$$

$$= \int \left| \hat{f}(\gamma) \right|^2 \sum_n \left| \hat{\psi}(2^n\gamma) \right|^2 d\gamma + \int \sum_n \sum_{k\neq 0} \hat{f}(\gamma) \overline{\hat{f}(\gamma + 2^{-n}k) \hat{\psi}(2^n\gamma)} \hat{\psi}(2^n\gamma + k) d\gamma. \tag{2.2}$$

Here, (2.1) and (2.2) are formally computed, but the calculations will be justified when they are used later in the thesis. To simplify notation, we define

$$F(f) = \int \left| \hat{f}(\gamma) \right|^2 \sum_n \left| \hat{\psi}(2^n\gamma) \right|^2 d\gamma + \int \sum_n \sum_{k\neq 0} \hat{f}(\gamma) \overline{\hat{f}(\gamma + 2^{-n}k) \hat{\psi}(2^n\gamma)} \hat{\psi}(2^n\gamma + k) d\gamma. \tag{2.3}$$

We would like to find explicit upper and lower bounds of $F(f)$ in terms of $\|f\|^2$. Clearly, these bounds correspond to frame bounds for the system $\mathcal{W}(\psi)$. Specifically, if $\mathcal{W}(\psi)$ has frame bounds $A$, $B$, then

$$A = \inf_{\|f\|_2=1} F(f) \quad \text{and} \quad B = \sup_{\|f\|_2=1} F(f).$$

Consequently, if $f \in L^2\left(\mathbb{R}^d\right)$ has unit norm, then $A \leq F(f) \leq B$.

Calculations such as these play a basic role in proving the following well-known theorem ([36], [24]) and its variants.

**Theorem 17.** *Let* $\psi \in L^2(\widehat{\mathbb{R}}^d)$*, and let* $a > 0$ *be arbitrary. Define*

$$\begin{aligned} \mu_\psi(\gamma) &= \sum_{k\in\mathbb{Z}^d}\sum_{n\in\mathbb{Z}} \left|\hat{\psi}\left(2^n\gamma\right)\hat{\psi}\left(2^n\gamma+k\right)\right| \ \textit{and} \\ M_\psi &= \operatorname{esssup}_{\gamma\in\widehat{\mathbb{R}}^d}\mu_\psi(\gamma) = \operatorname{esssup}_{a\le\|\gamma\|\le 2a}\mu_\psi(\gamma). \end{aligned}$$

*If* $M_\psi < \infty$*, then* $\mathcal{W}(\psi)$ *is a Bessel sequence with upper frame bound* $B$*, and* $M_\psi \geq B$*. Similarly, define*

$$\begin{aligned} \nu_\psi(\gamma) &= \sum_{n\in\mathbb{Z}} \left|\hat{\psi}\left(2^n\gamma\right)\right|^2 - \sum_{k\neq 0}\sum_{n\in\mathbb{Z}} \left|\hat{\psi}\left(2^n\gamma\right)\hat{\psi}\left(2^n\gamma+k\right)\right| \ \textit{and} \\ N_\psi &= \operatorname{essinf}_{\gamma\in\widehat{\mathbb{R}}^d}\nu_\psi(\gamma) = \operatorname{essinf}_{a\le\|\gamma\|\le 2a}\nu_\psi(\gamma). \end{aligned}$$

*If* $N_\psi > 0$*, then* $\mathcal{W}(\psi)$ *is a frame with lower frame bound* $A \geq N_\psi$*.*

We refer to $M_\psi$ and $N_\psi$ as the *Daubechies-Christensen bounds*. Christensen proved Theorem 17 for functions $\psi \in L^2(\mathbb{R})$, but his proof extends to $L^2(\mathbb{R}^d)$ with only minor modifications. Chui and Shi proved necessary conditions for a wavelet system in $L^2(\mathbb{R})$ to have certain frame bounds, [27]. Jing extended this result to $L^2(\mathbb{R}^d)$ for $d \geq 1$, [72].

**Proposition 18.** *Define* $\kappa_\psi(\gamma) = \sum_{n\in\mathbb{Z}} \left|\hat{\psi}\left(2^n\gamma\right)\right|^2$*. If* $\mathcal{W}(\psi)$ *is a wavelet frame for* $L^2(\mathbb{R}^d)$ *with bounds* $A$ *and* $B$*, then, for almost all* $\gamma \in \widehat{\mathbb{R}}^d$*,*

$$A \leq \kappa_\psi(\gamma) \leq B.$$

*Define* $\overline{K}_\psi = \operatorname{esssup}_{\gamma\in\widehat{\mathbb{R}}^d}\kappa_\psi(\gamma)$ *and* $\underline{K}_\psi = \operatorname{essinf}_{\gamma\in\widehat{\mathbb{R}}^d}\kappa_\psi(\gamma)$

We may combine the previous two results to obtain the following corollary.

**Corollary 19.** *Let* $\psi \in L^2(\mathbb{R}^d)$*. Let* $a > 0$ *be arbitrary. If* $M_\psi < \infty$*, then* $\mathcal{W}(\psi)$ *is a Bessel sequence with bound* $B$ *satisfying* $\overline{K}_\psi \leq B \leq M_\psi$*. If, further,* $N_\psi > 0$*, then* $\mathcal{W}(\psi)$ *is a frame with lower frame bound* $A$ *satisfying* $N_\psi \leq A \leq \underline{K}_\psi$*.*

Many of the $\psi$ that we mention in this thesis are continuous. In these cases, we shall simply calculate the supremum and infimum of $\kappa_\psi$, rather than the essential supremum and essential infimum.

### 2.2.2 Approximate Identities

**Definition 20.** An *approximate identity* is a family $\{k_{(\lambda)} : \lambda > 0\} \subseteq L^1(\mathbb{R}^d)$ of functions with the following properties:

*i.* $\forall \lambda > 0$, $\int k_{(\lambda)}(x)dx = 1$;

*ii.* $\exists K$ such that $\forall \lambda > 0$, $\|k_{(\lambda)}\|_{L^1(\mathbb{R}^d)} \leq K$;

*iii.* $\forall \eta > 0$, $\lim_{\lambda\to\infty} \int_{\|x\|\geq\eta} |k_{(\lambda)}(x)|dx = 0$.

The following result is well-known, e.g., [9], [48], [94].

**Proposition 21.** *Suppose $k \in L^1(\mathbb{R}^d)$ satisfies $\int k(x)dx = 1$. Define the family,*

$$\{k_\lambda : k_\lambda(x) = \lambda^d k(\lambda x), \lambda > 0\},$$

*of dilations. Then, the following assertions hold.*

*a.* $\{k_\lambda\}$ *is an* approximate identity*;*

*b. If $f \in L^p(\mathbb{R}^d)$ for some $1 \leq p < \infty$, then $\lim_{\lambda\to\infty} \|f * k_\lambda - f\|_{L^p(\mathbb{R}^d)} = 0$;*

*c. If $k$ is an even function, there exists a subsequence $\{\lambda_m\}$ of $\{\lambda\}$ such that*

$$\lim_{m\to\infty} \int f(u) T_x k_{\lambda_m}(u)du = f(x) \;\; a.e. \;\; x \in \mathbb{R}^d.$$

*Proof.* *a.* To verify the condition of Definition 20.$i$, we compute

$$\int k_\lambda(x)dx = \lambda^d \int k(\lambda x)dx = \int k(u)du = 1.$$

For part $ii$ we compute

$$\int |k_\lambda(x)|dx = \lambda^d \int |k(\lambda x)|dx = \int |k(u)|du = K < \infty,$$

where $K$ is finite since $k \in L^1(\mathbb{R}^d)$. For part $iii$, take $\eta > 0$ and compute

$$\int_{\|x\|\geq\eta} |k_\lambda(x)|dx = \lambda^d \int_{\|x\|\geq\eta} |k(\lambda x)|dx = \int_{\|u\|\geq\lambda\eta} |k(u)|du;$$

this last term tends to 0 as $\lambda$ tends to $\infty$ since $\eta > 0$ and because of the definition of the integral.

*b.* Setting $w = \lambda u$, we have

$$\begin{aligned} f * k_\lambda(x) - f(x) &= \int [f(x-u) - f(x)]\, k_\lambda(u)du \\ &= \int \left[f(x - \frac{w}{\lambda}) - f(x)\right] k(w)dw \\ &= \int \left[T_{\frac{w}{\lambda}} f(x) - f(x)\right] k(w)dw. \end{aligned}$$

Apply Minkowsi's inequality for integrals:

$$\|f * k_\lambda - f\|_p \leq \int \|T_{\frac{w}{\lambda}} f - f\|_p |k(w)|dw.$$

As $\|T_{\frac{w}{\lambda}} f - f\|_p$ is bounded by $2\|f\|_p$ and tends to 0 as $\lambda \to \infty$ for each $w$, the assertion follows from the dominated convergence theorem.

*c.* The last part follows from the evenness of $k$.

$$\int f(u) T_x k_{\lambda_m}(u)du = \int f(u) k_{\lambda_m}(u-x)du = \int f(u) k_{\lambda_m}(x-u)du = f * k_{\lambda_m}(x).$$

$\square$

We shall use approximate identities on $\widehat{\mathbb{R}}^d$. The following notation will streamline our arguments.

**Proposition 22.** *Fix a non-negative, compactly supported, bounded, even function $k : \widehat{\mathbb{R}}^d \to \mathbb{C}$ with the property that $\int k(\gamma)d\gamma = 1$. Then, $k \in L^1 \cap L^2(\widehat{\mathbb{R}}^d)$ and the results of Proposition 21 hold. For $\omega \in \widehat{\mathbb{R}}^d$ and $\alpha > 0$, define $g_{\lambda,\alpha,\omega} \in L^2(\widehat{\mathbb{R}}^d)$ by $g_{\lambda,\alpha,\omega} = \sqrt{\alpha T_\omega k_\lambda}$. If $\alpha = 1$, we write $g_{\lambda,\omega} = g_{\lambda,1,\omega}$. Note that $\|g_{\lambda,\omega}\|_2 = 1$ for all $\lambda, \omega$.*

The following 2 propositions may be seen as special cases of Proposition 18. Our results in this subsection require more hypotheses than the results just referenced, but the proofs are decidedly less technical, requiring fewer analytic estimates, and they also give greater insight as to why these bounds are valid. Furthermore, methods used later in the thesis which improve the bound estimates provided by Corollary 19 are inspired by these proofs. Recall the function $\kappa_\psi(\gamma) = \sum_{n\in\mathbb{Z}} \left|\hat{\psi}(2^n\gamma)\right|^2$.

**Proposition 23.** *Let $\psi \in L^2(\mathbb{R}^d)$ be a function with non-negative Fourier transform. Further, assume that $\kappa_\psi(\gamma) \in L^p(\widehat{\mathbb{R}}^d)$ for some $1 \le p \le \infty$. If $\mathcal{W}(\psi)$ is a Bessel sequence with upper frame bound $B$, then $\kappa_\psi(\gamma) \in L^\infty(\mathbb{R}^d)$ and $B \ge \overline{K}_\psi$.*

*Proof.* We have assumed $\hat{\psi}(\gamma) \ge 0$ for all $\gamma \in \widehat{\mathbb{R}}^d$. For any $f \in L^2(\mathbb{R}^d)$ with non-negative Fourier transform, lines (2.1) and (2.2) hold by the Tonelli theorem, and we have

$$F(f) \ge \int \left|\hat{f}(\gamma)\right|^2 \kappa_\psi(\gamma)d\gamma.$$

Thus, for a fixed $\omega \in \widehat{\mathbb{R}}^d$,

$$F(\check{g}_{\lambda,\omega}) \geq \int T_\omega k_\lambda(\gamma)\kappa_\psi(\gamma)d\gamma. \tag{2.4}$$

By Proposition 21, there exists a subsequence $\{\lambda_m\}$ of $\{\lambda\}$ such that the right hand side of (2.4) approaches $\kappa_\psi(\omega)$ as $m \to \infty$ for almost every $\omega$. Since $B \geq F(\check{g}_{\lambda,\omega})$ for all $\lambda$ and $\omega$, $B \geq \operatorname{esssup}_{\omega\in\widehat{\mathbb{R}}^d} \kappa_\psi(\omega)$. $\square$

**Proposition 24.** *Let $\psi \in L^2(\mathbb{R}^d)$ be a function for which $\operatorname{supp}\hat{\psi}$ is compact and $\operatorname{dist}(0, \operatorname{supp}\hat{\psi}) > 0$. Further, assume that $\hat{\psi} \in L^\infty(\widehat{\mathbb{R}}^d)$. If $\mathcal{W}(\psi)$ is a Bessel sequence with upper frame bound $B$, then $B \geq \overline{K}_\psi$.*

*Proof.* Since $\hat{\psi} \in L^\infty(\widehat{\mathbb{R}}^d)$ and the support of $\hat{\psi}$ is bounded and of positive distance from the origin, we have $\kappa_\psi(\gamma) \in L^\infty(\widehat{\mathbb{R}}^d)$. Thus, we may use Proposition 21. Furthermore, the sums in lines (2.1) and (2.2) are finite due to the support hypothesis and thus the calculations are justified. Fix a point $\omega \in \widehat{\mathbb{R}}^d$. As in the preceding proof, we would like to ignore the cross terms of $F(\check{g}_{\lambda,\omega})$ in order to obtain the desired result. We shall prove that the cross terms disappear for certain $\lambda$. If $\omega \neq 0$, since $\operatorname{supp}\hat{\psi}$ is bounded, there exists an $N \in \mathbb{Z}$ and a neighborhood $\mathcal{N}$ of $\omega$ such that $\hat{\psi}(2^n\gamma) = 0$ for all $n > N$ and all $\gamma \in \mathcal{N}$.

As $\lambda$ increases, the support of $g_{\lambda,\omega}$ decreases. Hence, let $L_1$ have the property that $\operatorname{supp}(g_{L_1,\omega}) \subseteq \mathcal{N}$. For all $\lambda > L_1$, $n < N$, and $\gamma \in \widehat{\mathbb{R}}^d$, we have

$$g_{\lambda,\omega}(\gamma)\overline{\hat{\psi}(2^n\gamma)} = 0. \tag{2.5}$$

On the other hand, choose an $L_2 > 0$ large enough so that for all $-n \geq N$, $\lambda \geq L_2$, and $l \in \mathbb{Z}^d$, we have

$$\operatorname{supp} g_{\lambda,\omega} \bigcap \operatorname{supp} T_{-2^{-n}l}\, g_{\lambda,\omega} = \emptyset.$$

Set $\tilde{L} = \max\{L_1, L_2\}$. Then, for any $\lambda > \tilde{L}$, $n \in \mathbb{Z}$, and $\gamma \in \widehat{\mathbb{R}}^d$,

$$g_{\lambda,\omega}(\gamma)\overline{g_{\lambda,\omega}(\gamma + 2^{-n}l)}\overline{\hat{\psi}(2^n\gamma)}\hat{\psi}(2^n\gamma + l) = 0.$$

Thus for $\lambda > \tilde{L}$,

$$F(\check{g}_{\lambda,\omega}) = \int T_\omega k_\lambda(\gamma)\kappa_\psi(\gamma)d\gamma.$$

Letting a certain subsequence of $\lambda$ get larger, we obtain $B \geq \kappa_\psi(\omega)$ for almost every $\omega$. Thus, $B \geq \operatorname{esssup}_{\omega\in\widehat{\mathbb{R}}^d} \kappa_\psi(\omega)$. $\square$

## 2.3 A canonical example

For this section, let $L = \left[-\frac{1}{2}, -\frac{1}{4}\right) \cup \left[\frac{1}{4}, \frac{1}{2}\right)$, which is $K_0 \backslash A_0$ from the 1-$d$ Journé construction; see Example 15.

**Example 25.** We shall compute some Bessel bounds.

*a.* $\mathcal{W}(\mathbb{1}_L^\vee)$ is a Parseval frame. Smooth $\mathbb{1}_L$ by defining $\hat{\psi} = \mathbb{1}_L * 8\mathbb{1}_{[-\frac{1}{16},\frac{1}{16}]}$. We would like to determine if $\mathcal{W}(\psi)$ is a Bessel sequence and, if so, to determine its upper frame bound. We compute $\overline{K}_\psi = \frac{17}{16}$. Within the dyadic interval $\left[\frac{9}{32}, \frac{9}{16}\right)$ this supremum occurs at $\frac{7}{16}$. Also, $M_\psi = \frac{17}{16}$, where the supremum occurs at the same point. Thus, by Corollary 19, the upper frame bound of $\mathcal{W}(\psi)$ is $\frac{17}{16}$.

*b.* Similarly, if $\hat{\psi} = \mathbb{1}_{[-\frac{1}{2},\frac{1}{2})^2\backslash[-\frac{1}{4},\frac{1}{4})^2} * 64\mathbb{1}_{[-\frac{1}{16},\frac{1}{16}]^2}$, then the upper frame bound of $\mathcal{W}(\psi)$ is $\frac{305}{256}$.

**Example 26.** Once again, let $\hat{\psi} = \mathbb{1}_L * 8\mathbb{1}_{[-\frac{1}{16},\frac{1}{16}]}$.

*a.* We have that $\underline{K}_\psi = \frac{9}{20}$ and $N_\psi = \frac{2}{9}$. It now follows from Corollary 19 that $\mathcal{W}$ is a frame with lower frame bound $A$, satisfying $\frac{2}{9} \le A \le \frac{9}{20}$. We would like to tighten these bounds around $A$. This is a delicate operation. For this estimate, we shall use functions consisting of multiple spikes, scaled by positive and negative numbers. We have that $\underline{K}_\psi$ occurs at $\frac{21}{40}$ within the dyadic interval $\left[\frac{9}{32}, \frac{9}{16}\right)$. By symmetry, this infimum is also achieved at $-\frac{21}{40}$. Further,

$$\sup_\gamma \sum_n \sum_{l \neq 0} \hat{\psi}(2^n\gamma)\overline{\hat{\psi}(2^n\gamma + l)} = \frac{1}{4}.$$

This supremum occurs at $\pm\frac{1}{2}$. In order to compute the lower frame bound, we need to minimize $F(f)$, defined in (2.3), over all $f \in L^2(\mathbb{R}^d)$. We shall refer to the summands,

$$\hat{f}(\gamma)\overline{\hat{f}(\gamma + 2^{-n}k)\hat{\psi}(2^n\gamma)}\hat{\psi}(2^n\gamma + k),$$

in $F(f)$ as *cross terms*. We would like to find an $\hat{f} \in L^2(\widehat{\mathbb{R}}^d)$ that allows us to use the cross terms to mitigate the other terms as much as possible. Since $\pm\frac{21}{40}$ is close to $\pm\frac{1}{2}$, one possibility is to set $\hat{f}_\lambda = g_{\lambda,\frac{1}{2},\frac{1}{2}} - g_{\lambda,\frac{1}{2},-\frac{1}{2}}$. The centers of the bumps are chosen to be a distance 1 apart from each other so that the cross terms do not disappear as $\lambda$ gets larger, while the negative coefficient is chosen so that the cross terms cancel out some of the other terms. For large enough $\lambda$, $\mathrm{supp}(g_{\lambda,\frac{1}{2},\frac{1}{2}}) \cap \mathrm{supp}(g_{\lambda,\frac{1}{2},-\frac{1}{2}}) = \emptyset$. We may always rescale the $k$ which generates the $g_{\lambda,\frac{1}{2},\pm\frac{1}{2}}$ so that these supports are disjoint for all $\lambda$. Thus, without loss of generality, assume that the supports are disjoint for all $\lambda$. We have $\left|\hat{f}_\lambda\right|^2 = \frac{1}{2}T_{\frac{1}{2}}k_\lambda + \frac{1}{2}T_{-\frac{1}{2}}k_\lambda$. Also,

$$\begin{aligned} \hat{f}_\lambda(\gamma)\hat{f}_\lambda(\gamma+1) &= -\frac{1}{2}T_{\frac{1}{2}}k_\lambda(\gamma) \\ \text{and} \quad \hat{f}_\lambda(\gamma)\hat{f}_\lambda(\gamma-1) &= -\frac{1}{2}T_{-\frac{1}{2}}k_\lambda(\gamma). \end{aligned}$$

These equalities rely on the evenness of the $k_\lambda$. For an appropriate subsequence $\lambda_\ell$, it is true that

$$\begin{aligned} F(f_{\lambda_\ell}) &\to \frac{1}{2}\Bigg\{ \sum_n \left[ \left|\hat{\psi}(2^n\frac{1}{2})\right|^2 + \left|\hat{\psi}(2^n\left(-\frac{1}{2}\right))\right|^2 \right] \\ &- \sum_n \sum_{l\neq 0} \left[ \hat{\psi}(2^n\frac{1}{2})\hat{\psi}(2^n\frac{1}{2}+l) + \hat{\psi}(2^n\left(-\frac{1}{2}\right))\hat{\psi}(2^n\left(-\frac{1}{2}\right)+l) \right] \Bigg\} = \frac{1}{4} \end{aligned}$$

as $\ell \to \infty$. Thus, the lower frame bound $A$ of $\psi$ is bounded above by $\frac{1}{4}$.

*b.* Can we use similar methods to tighten this lower frame bound estimate? For example, although the maximum of the cross terms occurs at $\frac{1}{2}$, the minimum of the remaining terms occurs at $\frac{21}{40}$. Perhaps it would be better to consider $\hat{f}_\lambda = g_{\lambda,\frac{1}{2},\frac{21}{40}} - g_{\lambda,\frac{1}{2},-\frac{19}{40}}$. Further, values of $\alpha$ different from $\frac{1}{2}$ might yield better results. Actually, neither of these options changes the results. If we choose $0 < \alpha < 1$ and $\omega \in [\frac{7}{16}, \frac{9}{16})$ and set $\hat{f}_\lambda = g_{\lambda,\alpha,\omega} - g_{\lambda,1-\alpha,1-\omega}$, then the minimum bound obtained for $A$ using the same method as in part *a* is $\frac{1}{4}$. We note that $\omega$ must be chosen from the interval $\left[\frac{7}{16}, \frac{9}{16}\right)$ (or the reflection of the interval to the negative $\mathbb{R}$ axis) because that is the only region in the support of $\hat{\psi}$ where, for $\gamma$ lying in that region, $\hat{\psi}(\gamma)\hat{\psi}(\gamma+l)$ is non-zero for any $l \in \mathbb{Z}\backslash\{0\}$.

*c.* Recalling that the Daubechies-Christensen bound is $\frac{2}{9}$, we conclude that the lower frame bound satisfies $\frac{2}{9} \le A \le \frac{1}{4}$.

This method of fine tuning lower frame bounds is difficult to generalize.

A natural idea that arises when attempting to obtain Parseval frames with frequency smoothness is to use elements of an approximate identity to convolve with $\mathbb{1}_L$ in order to obtain $\mathcal{W}(\psi)$ with frame bounds $A$ and $B$ which are arbitrarily close to 1, specifically using an approximate identity, $\{\phi_m\}$, that consists of the dilations of a non-negative function $\phi$ with $L^1$-norm 1. We know that $\mathbb{1}_L * \phi_m$ converges to $\mathbb{1}_L$ in $L^p$, $1 \le p < \infty$. Thus, there is a subsequence which converges almost everywhere to $\mathbb{1}_L$. However, one may think that the corresponding frame bounds converge to 1, but this does not happen.

**Proposition 27.** *Consider the approximate identity* $\{\phi_m = \frac{m}{2}\mathbb{1}_{[-\frac{1}{m},\frac{1}{m}]} : m > 12\}$. *Although* $\mathbb{1}_L * \phi_m \to \mathbb{1}_L$ *in* $L^p$, $1 \le p < \infty$, *the upper frame bounds of* $\mathcal{W}((\mathbb{1}_L * \phi_m)^\vee)$ *are all* $\frac{17}{16}$, *and the lower frame bounds are bounded between* $\frac{2}{9}$ *and* $\frac{1}{4}$.

*Proof.* For $m > 12$, we initially calculate

$$\mathbb{1}_L * \phi_m(\gamma) = \begin{cases} 0 & \text{for } \gamma < -\frac{1}{2} - \frac{1}{m} \\ -\frac{m}{2}(-\gamma - \frac{1}{2} - \frac{1}{m}) & \text{for } -\frac{1}{2} - \frac{1}{m} \le \gamma < -\frac{1}{2} + \frac{1}{m} \\ 1 & \text{for } -\frac{1}{2} + \frac{1}{m} \le \gamma < -\frac{1}{4} - \frac{1}{m} \\ \frac{n}{2}(-\gamma - \frac{1}{4} + \frac{1}{m}) & \text{for } -\frac{1}{4} - \frac{1}{m} \le \gamma < -\frac{1}{4} + \frac{1}{m} \\ 0 & \text{for } -\frac{1}{4} + \frac{1}{m} \le \gamma < \frac{1}{4} - \frac{1}{m} \\ \frac{m}{2}(\gamma - \frac{1}{4} + \frac{1}{m}) & \text{for } \frac{1}{4} - \frac{1}{m} \le \gamma < \frac{1}{4} + \frac{1}{m} \\ 1 & \text{for } \frac{1}{4} + \frac{1}{m} \le \gamma < \frac{1}{2} - \frac{1}{m} \\ -\frac{m}{2}(\gamma - \frac{1}{2} - \frac{1}{m}) & \text{for } \frac{1}{2} - \frac{1}{m} \le \gamma < \frac{1}{2} + \frac{1}{m} \\ 0 & \text{for } \frac{1}{2} + \frac{1}{m} \le \gamma \end{cases}$$

Let $\hat{\psi}_m = \mathbb{1}_L * \phi_m$. Just as above, we then calculate $\kappa_{\psi_m}(\gamma)$. Because of symmetry, we only need to calculate $\kappa_{\psi_m}$ over the positive dyadic interval $\left[\frac{1}{4} + \frac{1}{2m}, \frac{1}{2} + \frac{1}{m}\right]$.

$$\kappa_{\psi_m}(\gamma) = \begin{cases} \left(\frac{m^2}{4}\right)\left(\gamma^2 + \left(\frac{2}{m} - \frac{1}{2}\right)\gamma + \left(\frac{1}{16} - \frac{1}{2m} + \frac{1}{m^2}\right)\right) & \text{for } \frac{1}{4} + \frac{1}{2m} \leq \gamma < \frac{1}{4} + \frac{1}{m} \\ 1 & \text{for } \frac{1}{4} + \frac{1}{m} \leq \gamma < \frac{1}{2} - \frac{2}{m} \\ \left(\frac{m^2}{4}\right)\left(\frac{1}{4}\gamma^2 + \left(\frac{1}{m} - \frac{1}{4}\right)\gamma + \left(\frac{1}{16} - \frac{1}{2m} + \frac{5}{m^2}\right)\right) & \text{for } \frac{1}{2} - \frac{2}{m} \leq \gamma < \frac{1}{2} - \frac{1}{m} \\ \left(\frac{m^2}{4}\right)\left(\frac{5}{4}\gamma^2 - \left(\frac{5}{4} + \frac{1}{m}\right)\gamma + \left(\frac{5}{16} + \frac{1}{2m} + \frac{2}{m^2}\right)\right) & \text{for } \frac{1}{2} - \frac{1}{m} \leq \gamma < \frac{1}{2} + \frac{1}{m} \end{cases}$$

The maximum value of $g$ is $\frac{17}{16}$ and occurs at $\frac{1}{2} - \frac{1}{m}$. So the upper frame bound of $\mathcal{W}(\psi_m)$ is at least $\frac{17}{16}$. We now calculate $(\mu_{\psi_m} - \kappa_{\psi_m})(\gamma) = \sum_n \sum_{l \neq 0} \left| \hat{\psi}_m(2^n\gamma)\hat{\psi}_m(2^n\gamma + l) \right|$ over the same interval and obtain

$$(\mu_{\psi_m} - \kappa_{\psi_m})(\gamma) = \begin{cases} 0 & \text{for } \frac{1}{4} + \frac{1}{2m} \leq \gamma < \frac{1}{2} - \frac{1}{m} \\ \left(\frac{m^2}{4}\right)\left(-\gamma^2 + \gamma + \left(-\frac{1}{4} + \frac{1}{m^2}\right)\right) & \text{for } \frac{1}{4} + \frac{1}{2m} \leq \gamma < \frac{1}{4} + \frac{1}{m} \end{cases}$$

The upper frame bound of $\mathcal{W}(\psi_m)$ is bounded above by $M_{\psi_m} = \sup_\gamma(\kappa_{\psi_m}(\gamma) + (\mu_{\psi_m} - \kappa_{\psi_m})(\gamma))$, which is also $\frac{17}{16}$. Hence $\mathcal{W}(\psi_m)$ has upper frame bound $\frac{17}{16}$ for every $m \geq 12$. Now consider

$$N_{\psi_m} = \inf_\gamma \left(\kappa_{\psi_m}(\gamma) - (\mu_{\psi_m} - \kappa_{\psi_m})(\gamma)\right) = \frac{2}{9}.$$

By Theorem 17, $\mathcal{W}(\psi_m)$ is a frame with lower frame bound $A \geq \frac{2}{9}$. If we now calculate $F((g_{\lambda,\frac{1}{2},\frac{1}{2}} - g_{\lambda,\frac{1}{2},-\frac{1}{2}})^\vee)$, as in Example 26, we obtain $A \leq \frac{1}{4}$. $\square$

One may hope to improve the frame bounds of the smooth frame wavelets, e.g., by bringing both of the bounds closer to 1, by convolving with a linear spline. The following proposition shows that, in this case, the resulting upper frame bound is closer to 1, than for the case of Proposition 27, but that it also constant for large enough $m$. Further, in the limit, there is a positive gap between upper and lower frame bounds.

**Proposition 28.** *Consider the approximate identity* $\{\phi_m : m > 12\}$, *where* $\phi_m(\gamma) = \max(m(1 - m|\gamma|), 0)$, $\gamma \in \widehat{\mathbb{R}}$. *Although* $\mathbb{1}_L * \phi_m \to \mathbb{1}_L$ *pointwise a.e. and in* $L^p$, $1 \le p < \infty$, *the upper frame bounds of* $\mathcal{W}((\mathbb{1}_L * \phi_m)^\vee)$ *are all* $\frac{65}{64}$, *and the lower frame bounds are bounded between* $\frac{2}{9}$ *and* $\frac{1}{4}$.

*Proof.* Let $\hat{\psi}_m = \mathbb{1}_L * \phi_m$. By utilizing basic methods of optimization from calculus, we evaluate

$$\overline{K}_{\psi_m} = M_{\psi_m} = \frac{65}{64}.$$

It follows from Corollary 19 that the upper frame bound of $\mathcal{W}(\psi_m)$ is equal to $\frac{65}{64}$, independent of which $m > 12$ is used.

As in Example 26, set $\hat{f}_\lambda = g_{\lambda,\frac{1}{2},\frac{1}{2}} - g_{\lambda,\frac{1}{2},-\frac{1}{2}}$. Then, we can verify that there exists a subsequence $\lambda_\ell$ such that

$$\begin{aligned} F(f_{\lambda_\ell}) \;\; &\to \;\; \frac{1}{2}\Bigg\{ \sum_n \left[ \left|\hat{\psi}(2^n\left(\frac{1}{2}\right))\right|^2 + \left|\hat{\psi}(2^n\left(-\frac{1}{2}\right))\right|^2 \right] \\ &- \;\; \sum_n \sum_{l\neq 0} \left[ \hat{\psi}(2^n\left(\frac{1}{2}\right))\hat{\psi}(2^n\left(\frac{1}{2}\right) - l) + \hat{\psi}(2^n\left(-\frac{1}{2}\right))\hat{\psi}(2^n\left(-\frac{1}{2}\right) - k) \right] \Bigg\} = \frac{1}{4}, \end{aligned}$$

as $\ell \to \infty$. Also, the lower Daubechies-Christensen bound is $\frac{2}{9}$, yielding the desired bounds on the lower frame bound. $\square$

We shall call the phenomenon which occurs in Propositions 27 and 28 a *frame bound gap*. The results presented in this section prompt the following questions, which we address in Sections 2.4 and 2.5.

- Do we obtain a frame when we try to smooth $K_1\backslash A_1$ from the 1-$d$ Journé neighborhood mapping construction?
- Can we ever precisely determine the lower frame bound?
- What happens when we smooth $K_0\backslash A_0$ from higher dimensional Journé constructions?
- Does a frame bound gap occur for other wavelet sets and other approximate identities?

## 2.4 A shrinking method to obtain frames

### 2.4.1 The shrinking method

When we try to smooth $\mathbb{1}_L$ for other sets $L$ obtained using the neighborhood mapping constrution, we do not necessarily obtain a frame.

**Example 29.** Let

$$L = \left[-\frac{9}{4}, -2\right) \cup \left[-\frac{1}{2}, -\frac{9}{32}\right) \cup \left[\frac{9}{32}, \frac{1}{2}\right) \cup \left[2, \frac{9}{4}\right),$$

which is $K_1\backslash A_1$ from the neighborhood mapping construction of the 1-$d$ Journé set (Example 15). For $m \in \mathbb{N}$, define $\hat{\psi}_m = \mathbb{1}_L * \frac{m}{2}\mathbb{1}_{[-\frac{1}{m},\frac{1}{m}]}$. Then $\mathcal{W}(\psi_m)$ is not a frame for any $m$. This can be shown by considering $F((g_{\lambda,\frac{1}{2},\frac{1}{2}} - g_{\lambda,\frac{1}{2},-\frac{1}{2}})^\vee)$ for arbitrarily large $\lambda$, just as in Example 26. Specifically, a subsequence of $F((g_{\lambda,\frac{1}{2},\frac{1}{2}} - g_{\lambda,\frac{1}{2},-\frac{1}{2}})^\vee)$ converges to 0, while each $g_{\lambda,\frac{1}{2},\frac{1}{2}} - g_{\lambda,\frac{1}{2},-\frac{1}{2}}$ has unit norm. However, for arbitrary $m$, $\mathcal{W}(\psi_m)$ is a Bessel sequence, and for any $m > 64$, the Bessel bound is bounded between $\frac{305}{256}$ and $\frac{11}{8}$. Again, we see that the upper frame bound does not converge to 1.

It seems reasonable to assume that smoothing $\mathbb{1}_L$ in Example 29 with a linear spline may yield a frame; however, the following example shows that this does not happen.

**Example 30.** Let

$$L = \left[-\frac{9}{4}, -2\right) \cup \left[-\frac{1}{2}, -\frac{9}{32}\right) \cup \left[\frac{9}{32}, \frac{1}{2}\right) \cup \left[2, \frac{9}{4}\right),$$

and for $m > 64$, let $\phi_m$ be the linear spline $\phi_m(\gamma) = \max(m(1 - m|\gamma|), 0)$. Set $\hat{\psi} = \mathbb{1}_L * \phi_m$. Then, using Mathematica we obtain

$$\begin{aligned} M_\psi &= \frac{41}{32} \approx 1.28125 \\ \overline{K}_\psi &\approx 1.14833 \\ \underline{K}_\psi &\approx 0.38092 \\ N_\psi &= 0. \end{aligned}$$

In fact, for some subsequence $\{\lambda_\ell\}$,

$$F((g_{\lambda_\ell, \frac{1}{2}, \frac{1}{2}} - g_{\lambda_\ell, \frac{1}{2}, -\frac{1}{2}})^\vee) \to \nu_\psi(2) = 0$$

If $\mathcal{W}(\psi)$ formed a frame, then it would have a lower frame bound $0 = N_\psi \leq A \leq \nu_\psi(2) = 0$. Thus $\mathcal{W}(\psi)$ is not a frame, but it is a Bessel sequence with upper frame bound $1.14833 \leq B \leq 1.28125$.

We would not only like to construct frames, but also to determine the exact lower frame bound of such a frame rather than a range of possible values. The following definitions and theorem will help us do that.

**Definition 31.** For any measurable subset $L \subseteq \widehat{\mathbb{R}}^d$ define

$$\Delta(L) = \operatorname{dist}\big(L, \bigcup_{k \in \mathbb{Z}^d \setminus \{0\}} (L + k)\big).$$

**Definition 32.** If $f$ is a function $\widehat{\mathbb{R}}^d \to \mathbb{R}$, define $f^+ = \frac{|f|+f}{2}$. For $\epsilon \geq 0$, define

$$\operatorname{supp}_\epsilon f = \operatorname{supp}(f(\cdot) - \epsilon)^+.$$

Essentially, $\operatorname{supp}_\epsilon f$ returns the regions over which $f$ takes values greater than $\epsilon$. Notice that $\operatorname{supp} f = \operatorname{supp}_0 |f|$.

**Theorem 33.** *Let $\hat{\psi} \in L_c^\infty(\widehat{\mathbb{R}}^d)$ be a non-negative function. If there exists an $\epsilon > 0$ such that for $L = \operatorname{supp}_\epsilon \hat{\psi}$, $\bigcup_{n \in \mathbb{Z}} 2^n L = \widehat{\mathbb{R}}^d$ up to a set of measure 0, and for $\tilde{L} = \operatorname{supp} \hat{\psi}$, $\Delta(\tilde{L}) > 0$, and $\operatorname{dist}(0, \tilde{L}) > 0$. Then, $\mathcal{W}(\psi)$ is a frame for $L^2(\mathbb{R}^d)$. The frame bounds are $\operatorname{essinf}_\gamma \kappa_\psi(\gamma)$ and $\operatorname{esssup}_\gamma \kappa_\psi(\gamma)$.*

**Remark 34.** If the $L \subseteq \widehat{\mathbb{R}}^d$ is a Parseval frame wavelet set and the closure $\overline{L} \subseteq (-\frac{1}{2}, \frac{1}{2})^d$, then $\hat{\psi} = \mathbb{1}_L$ and $0 < \epsilon < 1$ satisfy the hypotheses with $L = \tilde{L}$.

*Proof.* We first note that since $\hat{\psi}$ is compactly supported and bounded, it lies in $L^2(\widehat{\mathbb{R}}^d)$. Thus $\psi \in L^2(\mathbb{R}^d)$. We now prove that $\mathcal{W}(\psi)$ is a frame. Since $\Delta(\tilde{L}) > 0$,

$$\forall \gamma \in \widehat{\mathbb{R}}^d \qquad \sum_{n \in \mathbb{Z}} \sum_{k \neq 0} \left| \hat{\psi}(2^n \gamma) \hat{\psi}(2^n \gamma + k) \right| = 0. \tag{2.6}$$

So

$$M_\psi = \overline{K}_\psi.$$

By assumption, $\hat{\psi}$ is bounded. Furthermore, since $\operatorname{dist}(0, \tilde{L}) > 0$ and $\tilde{L}$ is bounded, for any $\gamma \in \widehat{\mathbb{R}}^d$, $\hat{\psi}(2^n \gamma)$ is non-zero for only finitely many $n \in \mathbb{Z}$. Putting these two facts together, we conclude that

$$\overline{K}_\psi < \infty.$$

Similarly,

$$N_\psi = \underline{K}_\psi.$$

Since dyadic dilations of $L$ cover $\widehat{\mathbb{R}}^d$, for almost every $\gamma \in \widehat{\mathbb{R}}^d$, there exists $n \in \mathbb{Z}$ such that $2^n\gamma \in L$, which implies that $\hat{\psi}(2^n\gamma) > \epsilon$. Thus $\operatorname{essinf}_{\gamma\in\widehat{\mathbb{R}}^d} \kappa_\psi(\gamma) > 0$. Thus, by Corollary 19, $\mathcal{W}(\psi)$ is a frame with bounds $A$ and $B$ which satisfy

$$\begin{aligned} A &= \underline{K}_\psi \\ \text{and} \quad B &= \overline{K}_\psi. \end{aligned}$$

□

A statement very similar to the preceding theorem appears camouflaged (through a number of auxiliary functions) as Theorem 8 in [26].

**Remark 35.** Let $\psi \in L^2(\mathbb{R})$ satisfy the hypotheses of Theorem 33. Then for $C = \max\{A^{-1}, B\}$ and almost all $\gamma \in \widehat{\mathbb{R}}$

$$0 < C^{-1} \le \kappa_\psi(\gamma) \le C < \infty.$$

Furthermore, it follows from line (2.6) that for almost all $\gamma \in \widehat{\mathbb{R}}$,

$$\overline{\hat{\psi}\,(2^n\gamma)}\hat{\psi}\,(2^n\gamma + 2^n k) = 0 \quad \forall k \in \mathbb{Z}\backslash 2\mathbb{Z}\ ,\ k \in \mathbb{N} \cup \{0\}.$$

Thus, by Proposition 2.2 of [37], if $S : L^2(\widehat{\mathbb{R}}) \to L^2(\widehat{\mathbb{R}})$ is the frame operator defined as

$$Sf = \sum_{n\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^d} \langle f, D_n T_k \psi \rangle D_n T_k \psi,$$

then $S$ is translation invariant. That is, for all $x \in \mathbb{R}$, $ST_x = T_x S$ as operators.

**Corollary 36.** *Let $L$ be a Parseval frame wavelet set from the neighborhood mapping construction. Let $\delta = \operatorname{dist}(0, L) > 0$. Let $\alpha > 0$ be such that the closure $\alpha\overline{L} \subseteq (-\frac{1}{2}+\epsilon, \frac{1}{2}-\epsilon)^d$, for some $0 < \epsilon < \frac{1}{2}$. Further let $\phi$ be an essentially bounded non-negative function such that $\operatorname{supp}\phi \subseteq \min\{\frac{\alpha\delta}{2}, \epsilon\} \cdot (-1,1)^d$ and $\operatorname{supp}\phi$ contains a neighborhood about the origin. Then if $\hat{\psi} = \mathbb{1}_{\alpha L} * \phi$, $\mathcal{W}(\psi)$ is a frame for $L^2(\mathbb{R}^d)$.*

*Proof.* Define $\tilde{L} = \operatorname{supp}\hat{\psi}$. Since $\operatorname{supp}\phi$ contains a neighborhood about the origin, $\phi$ is non-negative, and $\hat{\psi}$ is continuous, there exists an $\epsilon > 0$ such that

$$\alpha L \subseteq \hat{\psi}^{-1}(\epsilon, \infty).$$

Thus, for this $\epsilon$,

$$\widehat{\mathbb{R}}^d = \alpha\widehat{\mathbb{R}}^d = \alpha \bigcup_{n\in\mathbb{Z}} 2^n L \subseteq \bigcup_{n\in\mathbb{Z}} 2^n \operatorname{supp}_\epsilon \hat{\psi},$$

up to a set of measure zero. As the convolution of two essentially bounded functions with compact support, $\hat{\psi} \in L^\infty$ immediately. It follows from Theorem 33 that $\mathcal{W}(\psi)$ is a frame for $L^2(\mathbb{R}^d)$. □

**Example 37.** Let

$$L = \left[-\frac{9}{32}, -\frac{1}{4}\right) \cup \left[-\frac{1}{16}, -\frac{9}{256}\right) \cup \left[\frac{9}{256}, \frac{1}{16}\right) \cup \left[\frac{1}{4}, \frac{9}{32}\right).$$

Then $L$ is $K_1 \backslash A_1$ from the 1-$d$ Journé construction, shrunk by a factor of 8. Further let $\hat{\psi}_m = \mathbb{1}_L * \frac{m}{2}\mathbb{1}_{[-\frac{1}{m},\frac{1}{m}]}$. Then for any $m \geq 384$, $\mathcal{W}(\psi_m)$ is frame with bounds $\frac{81}{260}$ and $\frac{305}{256}$. Note that $\mathcal{W}((\mathbb{1}_{8L} * \frac{m}{2}\mathbb{1}_{[-\frac{1}{m},\frac{1}{m}]})^\vee)$ is not a frame for any $m > 0$ (Example 29).

**Example 38.** Let $L_a = \left[-a, -\frac{a}{2}\right) \cup \left[\frac{a}{2}, a\right)$ for $0 < a < \frac{1}{2}$. Then $L_a$ is $\left[-\frac{1}{2}, -\frac{1}{4}\right) \cup \left[\frac{1}{4}, \frac{1}{2}\right)$ from the 1-$d$ Journé construction, dilated by a factor of $2a < 1$. Recall from

Proposition 27 that

$$\mathcal{W}((\mathbb{1}_{[-\frac{1}{2},-\frac{1}{4})\cup[\frac{1}{4},\frac{1}{2})} * \frac{m}{2}\mathbb{1}_{[-\frac{1}{m},\frac{1}{m})})^{\vee})$$

is a frame with upper frame bound $\frac{17}{16}$ and lower frame bound between $\frac{2}{9}$ and $\frac{1}{4}$. Define $\hat{\psi}_{m,a} = \mathbb{1}_{L_a} * \frac{m}{2}\mathbb{1}_{[-\frac{1}{m},\frac{1}{m}]}$. For $0 < a < \frac{1}{2}$ and $m \geq \max\{\frac{2}{1-2a}, \frac{6}{a}\}$, $\mathcal{W}(\psi_{m,a})$ is a frame with with frame bounds $\frac{9}{20}$ and $\frac{17}{16}$.

It follows from the calculations in Example 26 that the lower frame bound of $\mathcal{W}(\psi_{m,\frac{1}{2}})$ is bounded above by $\frac{1}{4}$, while the shrinking process brings the lower frame bound up to $\frac{9}{20}$, for $\mathcal{W}(\psi_{m,a})$, $0 < a < \frac{1}{2}$. Corollary 19, which is based on previously known results, only implies that the lower frame bound of $\mathcal{W}(\psi_{m,\frac{1}{2}})$ is bounded between $\frac{2}{9}$ and $\frac{9}{20}$. Thus without the methods introduced in Example 26, we would not know that the shrinking method actually improves the lower frame bound.

Further note that $\frac{17}{16} < \frac{305}{256}$ and $\frac{9}{20} > \frac{81}{260}$. Thus the frame bounds corresponding to shrinking $K_0 \backslash A_0$ from the 1-$d$ Journé construction are closer to 1 than the bounds obtained by shrinking $K_1 \backslash A_1$ in the Example 37.

### 2.4.2 Oversampling

Corollary 36 yields an easy method to obtain wavelet frames with certain decay properties from Parseval frame wavelet sets. It almost seems counterintuitive to believe that simply shrinking the support of the frequency domain can change a function which is not a frame generator into a function that is one. Although we have proven that this does indeed happen, we now give a heuristic argument that this method should work for dyadic-shrinking. If the collection $\{D_nT_k\psi : n \in \mathbb{Z}, k \in \mathbb{Z}^d\} \subseteq L^2(\mathbb{R}^d)$ is a Bessel sequence, then it is not a frame if and only if there exists a sequence $\{f_m : \|f_m\|_2 = 1, m \in \mathbb{Z}\} \subseteq L^2(\mathbb{R}^d)$ such that $\lim_{m\to\infty} \sum_{n\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^d} |\langle f_m, D_nT_k\psi\rangle|^2 = 0$. Having a positive lower frame bound is a stronger condition than being complete. However, if we add more elements to $\{D_nT_k\psi : n \in \mathbb{Z}, k \in \mathbb{Z}^d\}$, it is more likely that the system will be complete and thus also more likely that it will have a lower frame bound. We would like to show that shrinking the support of $\hat{\psi}$ will add more elements to the system. For $\alpha > 0$ and $\psi \in L^2(\mathbb{R}^d)$, let $\hat{\varphi}(\gamma) = \hat{\psi}(\alpha\gamma)$. Then $\mathcal{F}\varphi = \alpha^{-d/2} D_{\log_2 \alpha}\mathcal{F}\psi$,

$$\begin{aligned}
\Rightarrow \varphi &= \mathcal{F}^{-1}\mathcal{F}\varphi \\
&= \mathcal{F}^{-1}(\alpha^{-d/2} D_{\log_2 \alpha}\mathcal{F})\psi \\
&= \mathcal{F}^{-1}(\alpha^{-d/2} \mathcal{F} D_{-\log_2 \alpha})\psi \\
&= \alpha^{-d/2} D_{-\log_2 \alpha}\psi \\
\Rightarrow D_nT_k\varphi &= \alpha^{-d/2} D_nT_k D_{-\log_2 \alpha}\psi \\
&= \alpha^{-d/2} D_{n-\log_2 \alpha} T_{\frac{k}{\alpha}}\psi.
\end{aligned}$$

Hence, if $\alpha = 2^N$, for $N \in \mathbb{N}$,

$$\operatorname{span}\{D_nT_k\varphi : n \in \mathbb{Z}, k \in \mathbb{Z}^d\} = \operatorname{span}\{D_nT_{\frac{k}{2^N}}\psi : n \in \mathbb{Z}, k \in \mathbb{Z}^d\}.$$

Thus, dyadic shrinking on the Fourier domain has the effect of increasing the size of the system generated by dilations and translations by a power of 2. One may call this an *oversampling* of the continuous wavelet system $\{D_{\log_2 r}T_s\psi : r > 0, s \in \mathbb{R}\}$. If $L \subseteq \widehat{\mathbb{R}}^d$ is Parseval frame wavelet set and $\phi \in L_c^\infty(\widehat{\mathbb{R}}^d)$, $\mathcal{W}\left((\mathbb{1}_L * \phi)^\vee\right)$ is a

Bessel sequence but perhaps not a frame (see Example 37). Hence, dyadic shrinking increases the likelihood that $\mathcal{W}\left((\mathbb{1}_L * \phi)^\vee\right)$ is complete and thus also the likelihood that $\mathcal{W}\left((\mathbb{1}_L * \phi)^\vee\right)$ has a positive lower frame bound. In general, shrinking by any $\alpha > 1$ has the effect of increasing the number of translations in the original wavelet system and shifts each of the dilation operators by the same amount. We compare and contrast our results with the following two oversampling theorems found in [26].

**Theorem 39.** *Let $\mathcal{W}(\psi)$ be a frame for $L^2(\mathbb{R})$ with frame bounds $A$ and $B$. Then for every odd positive integer $N$, the family*

$$\{D_n T_{\frac{k}{N}} \psi : n, k \in \mathbb{Z}\}$$

*is a frame with bounds $\tilde{A}$ and $\tilde{B}$ which satisfy $\tilde{A} \geq NA$ and $\tilde{B} \leq NB$.*

**Theorem 40.** *Let $\psi \in L^2(\mathbb{R})$ decay sufficiently fast and satisfy $\int \psi(x)dx = 0$. If $\mathcal{W}(\psi)$ forms a frame, then for any positive integer $N$,*

$$\{D_n T_{\frac{k}{N}} \psi : n, k \in \mathbb{Z}\}$$

*is a frame also.*

**Remark 41.** The specific decay conditions in the hypothesis of Theorem 40 are described in [26], but are too lengthy to list here. The smoothed frame wavelets mentioned in this thesis all satisfy the decay conditions.

Only dyadic shrinking corresponds to oversampling in the Chui and Shi sense. Oversampling may potentially create a frame system from a pre-existing frame system, but we see in Example 37 that oversampling may change a non-frame system to a frame system. Furthermore, in Example 38 we see that oversampling can bring frame bounds closer to 1, rather than just scaling them as in Theorem 39.

## 2.5 Frame bound gaps

**Definition 42.** Let $\psi \in L^2(\widehat{\mathbb{R}}^d)$ be a Parseval frame wavelet and $\{\psi_m\}_{m\in\mathbb{N}} \subseteq L^2(\widehat{\mathbb{R}}^d)$ be a sequence of frame wavelets (or Bessel wavelets) with lower frame bounds $A_m$ and upper frame bounds $B_m$ (or just upper frame bounds $B_m$) for which

$$\lim_{m\to\infty} \|\psi - \psi_m\|_{L^2(\mathbb{R}^d)} = 0.$$

If $\overline{\lim}_{m\to\infty} A_m < 1$ or $\underline{\lim}_{m\to\infty} B_m > 1$, then there is a *frame bound gap.* By Parseval's equality, $\|\psi - \psi_m\|_{L^2(\mathbb{R}^d)} = \|\hat{\psi} - \hat{\psi}_m\|_{L^2(\widehat{\mathbb{R}}^d)}$, so it suffices to check for convergence on the frequency domain.

Many examples of frame bound gaps occur in the previous sections. We shall now prove that this phenomenon occurs in more general situations. First we make a quick comment.

**Remark 43.** Let $L \subseteq \widehat{\mathbb{R}}^d$ be bounded and measurable and $g \in L^1_{loc}(\widehat{\mathbb{R}}^d)$. For $m > 1$ define

$$\begin{aligned} g_{(m)}(\gamma) &= mg(m\gamma), \text{ and} \\ \hat{\psi}_m &= \mathbb{1}_L * g_{(m)}. \end{aligned}$$

Then

$$\begin{aligned}\hat{\psi}_m(u) &= \int \mathbb{1}_L(u-\gamma)g_{(m)}(\gamma)d\gamma \\ &= \int \mathbb{1}_L(u-\frac{\gamma}{m})g(\gamma)d\gamma \\ &= \int_{-mL+mu} g(\gamma)d\gamma.\end{aligned}$$

**Theorem 44.** *For $0 < a < 1/2$, let $L \subseteq \widehat{\mathbb{R}}^d$ be the Parseval frame wavelet set $[-a,a]^d \backslash [-\frac{a}{2}, \frac{a}{2}]^d$. Also let $g : \widehat{\mathbb{R}}^d \to \mathbb{R}$ satisfy the following conditions:*

*i.* $\operatorname{supp} g \subseteq \prod_{i=1}^d [-b_i, c_i]$*, where for all $i$, $b_i, c_i > 0$ and* $\operatorname{supp}_0 g$ *contains a neighborhood of* $0$*;*

*ii.* $\int g(\gamma)d\gamma = 1$*; and*

*iii.* $0 < \int_{\prod_{i=1}^d [\frac{c_i}{2}, c_i]} g(\gamma)d\gamma < 1$ *and* $0 < \int_{\prod_{i=1}^d [-\frac{b_i}{2}, c_i]} g(\gamma)d\gamma < 1.$

*Define $\hat{\psi}_m = \mathbb{1}_L * g_{(m)}$. For any*

$$m > \max_{1 \le i \le d} \{\max\left\{\frac{2(b_i+c_i)}{a}, \frac{b_i+c_i}{1-2a}, \frac{4b_i+c_i}{a}, \frac{4c_i+b_i}{a}\right\}\},$$

*$\mathcal{W}(\psi_m)$ is a frame with frame bounds $A_m$ and $B_m$, and there exist $\alpha < 1$ and $\beta > 1$, both independent of $m$, such that $A_m \le \alpha$ and $B_m \ge \beta$. In particular, there are frame bound gaps.*

**Remark 45.** Any non-negative function $g : \widehat{\mathbb{R}}^d \to \mathbb{R}$ which integrates to 1 and has support $\prod_{i=1}^d [-b_i, c_i] \supseteq \operatorname{supp} g \supseteq \prod_{i=1}^d (-b_i, c_i)$ satisfies the hypotheses.

**Remark 46.** This result holds true if $m \in \mathbb{N}$ or $m \in \mathbb{R}$.

*Proof.* Let $m > \max_{1 \le i \le d}\{\max\{\frac{2(b_i+c_i)}{a}, \frac{b_i+c_i}{1-2a}, \frac{4b_i+c_i}{a}, \frac{4c_i+b_i}{a}\}\}$.

Since $m > \frac{b_i+c_i}{1-2a}$, $\Delta(\operatorname{supp}\hat{\psi}_m) > 0$. Thus,

$$\mu_{\psi_m}(u) = \nu_{\psi_m}(u) = \kappa_{\psi_m}(u),$$

where $\kappa_{\psi_m}$ is compactly supported. Further, for all $1 \le i \le d$,

$$m > \frac{2(b_i + c_i)}{a} > \max\{\frac{2b_i}{a}, \frac{2c_i}{a}\},$$

so $\operatorname{dist}(0, \operatorname{supp}\hat{\psi}_m) > 0$. It follows from Theorem 33 and Corollary 19, that $\mathcal{W}(\psi_m)$ is a frame with bounds $A_m = \underline{K}_{\psi_m}$ and $B_m = \overline{K}_{\psi_m}$.

As $m > \max_{1\le i\le d}\{\max\{\frac{4b_i+c_i}{a}, \frac{4c_i+b_i}{a}\}\}$,

for $u \in \left(\prod_{i=1}^d[-a - \frac{b_i}{m}, a + \frac{c_i}{m}]\right) \setminus \left(\prod_{i=1}^d[-\frac{a}{2} - \frac{b_i}{2m}, \frac{a}{2} + \frac{c_i}{2m}]\right)$,

$$\begin{aligned} \kappa_{\psi_m}(u) &= (\hat{\psi}_m(u))^2 + (\hat{\psi}_m(\frac{u}{2}))^2, \text{ where} \\ \hat{\psi}_m(u) &= \int_{-mL+mu} g(\gamma)d\gamma. \end{aligned}$$

To bound $B_m$, we evaluate $\kappa_{\psi_m}(v)$ where $v = (a - \frac{b_1}{m}, a - \frac{b_2}{m}, \dots, a - \frac{b_d}{m})$. We first compute $\hat{\psi}_m(v)$. Since $[\frac{a}{2}, a]^d \subseteq L$,

$$\prod_{i=1}^d [-b_i, \frac{ma}{2} - b_i] \subseteq -mL + mv.$$

As $m > \frac{2(b_i+c_i)}{a}$ for all $1 \le i \le d$, $\prod_{i=1}^d[-b_i, c_i] \subseteq \prod_{i=1}^d[-b_i, \frac{ma}{2} - b_i]$. Hence,

$$\hat{\psi}_m(v) = \int_{-mL+mv} g(\gamma)d\gamma = \int_{\prod_{i=1}^d[-b_i,c_i]} g(\gamma)d\gamma = 1.$$

We now compute $\hat{\psi}_m(\frac{v}{2})$. Since $m$ is sufficiently large, for $1 \le i \le d$,

$$\begin{aligned} [-b_i, c_i] \cap (-m[-a, -\frac{a}{2}] + \frac{m}{2}(a - \frac{b_i}{m})) &= [-b_i, c_i] \cap ([-\frac{ma}{2} - \frac{b_i}{2}, -\frac{b_i}{2}]) \\ &= [-b_i, -\frac{b_i}{2}], \end{aligned}$$

$$\begin{aligned}[-b_i,c_i]\cap(-m[-\frac{a}{2},\frac{a}{2}]+\frac{m}{2}(a-\frac{b_i}{m})) &= [-b_i,c_i]\cap([-\frac{b_i}{2},ma-\frac{b_i}{2}]) \\ &= [-\frac{b_i}{2},c_i], \text{ and}\end{aligned}$$

$$\begin{aligned}[-b_i,c_i]\cap(-m[\frac{a}{2},a]+\frac{m}{2}(a-\frac{b_i}{m})) &= [-b_i,c_i]\cap([ma-\frac{b_i}{2},\frac{3ma}{2}-\frac{b_i}{2}]) \\ &= \emptyset.\end{aligned}$$

It follows that

$$\left(\prod_{i=1}^{d}[-b_i,c_i]\right)\cap\left(-mL+m(\frac{v}{2})\right)=\left(\prod_{i=1}^{d}[-b_i,c_i]\right)\setminus\left(\prod_{i=1}^{d}[-\frac{b_i}{2},c_i]\right),$$

and that

$$\hat{\psi}_m(\frac{v}{2})=\int_{-mL+m(\frac{v}{2})}g(\gamma)d\gamma=1-\int_{\prod_{i=1}^{d}[-\frac{b_i}{2},c_i]}g(\gamma)d\gamma.$$

Define $\beta=\kappa_{\psi_m}(v)=1+\left(1-\int_{\prod_{i=1}^{d}[-\frac{b_i}{2},c_i]}g(\gamma)d\gamma\right)^2$. Then, $B_m\geq\kappa_{\psi_m}(v)=\beta>1$, and $\beta$ is independent of $m$.

Let $\omega=(a+\frac{c_1}{m},a+\frac{c_2}{m},\ldots,a+\frac{c_d}{m})$. We shall show that $\kappa_{\psi_m}(\omega)$ is strictly less than 1. We compute

$$\begin{aligned}-mL+m\omega &= \left(\prod_{i=1}^{d}[c_i,2ma+c_i]\right)\setminus\left(\prod_{i=1}^{d}[\frac{ma}{2}+c_i,\frac{3ma}{2}+c_i]\right) \\ &\Rightarrow \left(\prod_{i=1}^{d}[-b_i,c_i]\right)\cap(-mL+m\omega)=\emptyset.\end{aligned}$$

Thus, $\hat{\psi}_m(\omega)=0$. Furthermore,

$$-mL+m(\frac{\omega}{2})=\left(\prod_{i=1}^{d}[-\frac{ma}{2}+\frac{c_i}{2},\frac{3ma}{2}+\frac{c_i}{2}]\right)\setminus\left(\prod_{i=1}^{d}[\frac{c_i}{2},ma+\frac{c_i}{2}]\right).$$

It follows from our choice of $m$ that for all $1\leq i\leq d$, $-\frac{ma}{2}+\frac{c_i}{2}<-b_i$ and $c_i<ma+\frac{c_i}{2}<\frac{3ma}{2}+\frac{c_i}{2}$. Hence,

$$\left(\prod_{i=1}^{d}[-b_i,c_i]\right)\cap(-mL+m(\frac{\omega}{2}))=\left(\prod_{i=1}^{d}[-b_i,c_i]\right)\setminus\left(\prod_{i=1}^{d}[\frac{c_i}{2},c_i]\right), \text{ and}$$

$$\hat{\psi}_m(\frac{\omega}{2}) = 1 - \int_{\prod_{i=1}^d [\frac{c_i}{2}, c_i]} g(\gamma) d\gamma.$$

We define

$$\alpha = \kappa_{\psi_m}(\omega) = \left(1 - \int_{\prod_{i=1}^d [\frac{c_i}{2}, c_i]} g(\gamma) d\gamma\right)^2.$$

Consequently, $A_m \leq \alpha < 1$ for all sufficiently large $m$. $\square$

**Corollary 47.** *For $0 < a < \frac{1}{2}$, let $L_d \subseteq \widehat{\mathbb{R}}^d$ be the wavelet set $[-a, a]^d \backslash [-\frac{a}{2}, \frac{a}{2}]^d$. Also, let $g : \widehat{\mathbb{R}} \to \mathbb{R}$ satisfy the following conditions:*

*i.* $\operatorname{supp} g \subseteq [-b, c]$ *for some* $b, c > 0$ *and* $\operatorname{supp}_0 g$ *contains a neighborhood of* $0$*;*

*ii.* $\int g(\gamma) d\gamma = 1$*; and*

*iii.* $0 < \int_{\frac{c}{2}}^{c} g(\gamma) d\gamma < 1$ *and* $0 < \int_{-\frac{b}{2}}^{c} g(\gamma) d\gamma < 1$.

*Define $g_d = \bigotimes_{i=1}^d g : \widehat{\mathbb{R}}^d \to \mathbb{R}$. Further define $\hat{\psi}_{m,d} = \mathbb{1}_{L_d} * g_{d_{(m)}}$. Then, for each*

$$m > \max\{\frac{2(b+c)}{a}, \frac{b+c}{1-2a}, \frac{4b+c}{a}, \frac{4c+b}{a}\},$$

*and $d \geq 1$, $\mathcal{W}(\psi_{m,d})$ is a frame with bounds $A_{m,d}$ and $B_{m,d}$ which satisfy*

$$\begin{aligned} A_{m,d} &\leq \left(1 - \left(\int_{\frac{c}{2}}^{c} g(\gamma) d\gamma\right)^d\right)^2 < 1, \text{ and} \\ B_{m,d} &\geq \left(1 - \left(\int_{-\frac{b}{2}}^{c} g(\gamma) d\gamma\right)^d\right)^2 + 1 > 1. \end{aligned}$$

*Also for such $m$, $\lim_{d\to\infty} B_{m,d} = 2$.*

*Proof.* All of the hypotheses of Theorem 44 are satisfied, so

$$\begin{aligned} A_{m,d} &\leq \left(1 - \left(\int_{\frac{c}{2}}^{c} g(\gamma) d\gamma\right)^d\right)^2 < 1, \text{ and} \\ B_{m,d} &\geq \left(1 - \left(\int_{-\frac{b}{2}}^{c} g(\gamma) d\gamma\right)^d\right)^2 + 1 > 1, \end{aligned}$$

where

$$\lim_{d\to\infty}\left(1-\left(\int_{-\frac{b}{2}}^{c} g(\gamma)d\gamma\right)^{d}\right)^{2}+1=2,$$

since $0<\int_{-\frac{b}{2}}^{c} g(\gamma)d\gamma<1$. Furthermore,

$$\begin{aligned}
B_{m,d} &= \sup_{u\in[-a-\frac{b}{m},a+\frac{c}{m}]^d\setminus[-\frac{a}{2}-\frac{b}{2m},\frac{a}{2}+\frac{c}{2m}]^d} \kappa_{\psi_{m,d}}(u)\\
&= \sup_{u\in[-a-\frac{b}{m},a+\frac{c}{m}]^d\setminus[-\frac{a}{2}-\frac{b}{2m},\frac{a}{2}+\frac{c}{2m}]^d} (\hat{\psi}_{m,d}(u))^2+(\hat{\psi}_{m,d}(\frac{u}{2}))^2\\
&\leq 2.
\end{aligned}$$

Thus, $\lim_{d\to\infty} B_{m,d}=2$ for all large enough $m$. □

A similar result holds for a large class of wavelet sets in $\widehat{\mathbb{R}}$.

**Theorem 48.** *Let* $L=\bigcup_{j\in\mathcal{J}\subseteq\mathbb{Z}}[a_j,b_j]$, *with* $a_j<b_j$ *for all* $j\in\mathcal{J}$, *be a Parseval frame wavelet set. Let* $g:\widehat{\mathbb{R}}\to\mathbb{R}$ *be a non-negative function satisfying* $\int g(\gamma)d\gamma=1$ *and with support* $\operatorname{supp} g=[-c,d]$, *where* $c,d>0$ *and which contains a neighborhood of zero. Define* $\hat{\psi}_m=\mathbb{1}_L * g_{(m)}$ *for* $m>\frac{c+d}{b_j-a_j}$ *for all* $j\in\mathcal{J}$. *Then if* $\mathcal{W}(\psi_m)$ *forms a Bessel sequence, the upper frame bound satisfies* $B_m\geq\beta>1$, *where* $\beta$ *is independent of* $m$. *In particular, there is a frame bound gap.*

*Proof.* Set $a_k=\min\{a_j>0:j\in\mathcal{J}\}$, and let $b_i\in\{b_j\}_{j\in\mathcal{J}}$ be the unique $b_j>0$ such that there exists $N\in\mathbb{N}\cup\{0\}$ with $2^N a_k=b_i$. We wish to bound $\kappa_{\psi_m}(b_i-\frac{c}{m})$. Since $m>\frac{c+d}{b_i-a_i}$,

$$\begin{aligned}
-mL+m(b_i-\frac{c}{m}) &\supseteq m[-b_i,-a_i]+m(b_i-\frac{c}{m})\\
&= [-c,m(b_i-a_i)-c]\\
&\supseteq [-c,d],
\end{aligned}$$

implying that

$$\hat{\psi}_m(b_i - \frac{c}{m}) = \int_{-mL+m(b_i-\frac{c}{m})} g(\gamma)d\gamma = 1.$$

Similarly,

$$\begin{aligned} -mL + m(2^{-N}(b_i - \frac{c}{m})) &= -mL + ma_k - 2^{-N}c \\ &\supseteq m[-b_k, -a_k] + ma_k - 2^{-N}c \\ &= [m(a_k - b_k) - 2^{-N}c, -2^{-N}c] \\ &\supseteq [-c, -2^{-N}c]. \end{aligned}$$

So

$$\hat{\psi}_m(2^{-N}(b_i - \frac{c}{m})) \geq \int_{-c}^{-2^{-N}c} g(\gamma)d\gamma > 0.$$

Hence,

$$B_m \geq \kappa_{\psi_m}(b_i - \frac{c}{m}) \geq 1 + \left(\int_{-c}^{-2^{-N}c} g(\gamma)d\gamma\right)^2 > 1.$$

□

We note that by construction (*viii*) in Chapter 4 of [33], cantor-like wavelet sets exist. Thus, Theorem 48 does not apply to all wavelet sets in $\widehat{\mathbb{R}}$.

**Corollary 49.** *Let* $L = \bigcup_{j=1}^{J}[a_j, b_j] \subseteq (-\frac{1}{2}, \frac{1}{2})$, *with* $a_j < b_j$ *for all* $j \in \mathcal{J}$, *be a Parseval frame wavelet set. Let* $g : \widehat{\mathbb{R}} \to \mathbb{R}$ *be a non-negative function satisfying* $\int g(\gamma)d\gamma = 1$, *with support* $\operatorname{supp} g = [-c, d]$, *where* $c, d > 0$, *and which contains a neighborhood of zero. Define* $\hat{\psi}_m = \mathbb{1}_L * g_{(m)}$ *for all* $m$ *large enough that*

$$m > \max\left\{\frac{c+d}{(\min_j a_j) - (\max_j b_j) + 1}, \max_j\left\{\frac{c+d}{b_j - a_j}\right\}, \frac{d}{\operatorname{dist}(0, L)}, \frac{c}{\operatorname{dist}(0, L)}\right\}.$$

*Then* $\mathcal{W}(\psi_m)$ *forms a frame with upper frame bound* $B_m \geq \beta > 1$, *where* $\beta$ *is independent of* $m$.

*Proof.* Since $m > \frac{c+d}{(\min_j a_j)-(\max_j b_j)+1}$, $\mu_{\psi_m} = \nu_{\psi_m} = \kappa_{\psi_m}$. Because $\operatorname{supp}\hat{\psi}_m \supsetneq L$, $\inf \kappa_{\psi_m} > 0$. Finally, since $m > \max\{\frac{d}{\operatorname{dist}(0,L)}, \frac{c}{\operatorname{dist}(0,L)}\}$ and $\operatorname{supp}\hat{\psi}_m$ is compact, $\sup \kappa_{\psi_m} < \infty$. Hence $\mathcal{W}(\psi_m)$ is a frame.

The remainder of the claim follows from Theorem 48. □

In this chapter Parseval frame wavelets are smoothed on the frequency domain by elements of successive elements of approximate identities. However, the corresponding frame bounds do not converge to 1 even though $L$ is a Parseval frame wavelet set. We contrast these facts to the case of time domain smoothing. In [1], the Haar wavelet is smoothed using convolution on the time domain with members of particular approximate identities $\{k_\lambda\}$. The smoothed functions generate Riesz basis wavelets which have frame bounds which approach 1 as $\lambda \to \infty$. Thus, convolutional smoothing affects frame bounds dramatically differently depending on whether the smoothing is done on the temporal or frequency domains. Furthermore, Theorems 44 and 48 may be used to show that certain smooth functions are not the result of convolutional smoothing; see Section 3.5 in the next chapter. Finally, the shrinking method introduced in Section 2.4 may be used to simply modify non-complete systems in order to obtain frames.

# Chapter 3

# Smooth Parseval frames for $L^2(\mathbb{R})$ and generalizations to $L^2(\mathbb{R}^d)$

## 3.1 Introduction

### 3.1.1 Motivation

As in Chapter 2, we are concerned with finding frame wavelets which are smoothed approximations of Parseval wavelet set wavelets. We attempt to generalize Bin Han's non-constructive proof of the existence of Schwartz class functions which approximate Parseval wavelet set wavelets in $L^2(\mathbb{R})$ arbitrarily well. We show that the natural approaches to such a generalization fail. Furthermore, we show that a collection of well-known functions which also approximate wavelet set wavelets generate frames with upper frame bounds that converge to 1.

### 3.1.2 Background

Recall that in this chapter, $D_t f(x) = 2^{td/2} f(2^t x)$.

**Definition 50.**

- The space $C_c^\infty(\mathbb{R}^d)$ consists of functions $f : \mathbb{R}^d \to \mathbb{C}$ which are infinitely differentiable and compactly supported.
- Given a multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in (\mathbb{N} \cup \{0\})^d$, we write $|\alpha| = \sum_{i=1}^d \alpha_i$,

$x^\alpha = \prod_{i=1}^d x_i^{\alpha_i}$, and $D^\alpha = \frac{\partial_{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial_{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \frac{\partial_{\alpha_d}}{\partial x_d^{\alpha_d}}$. An infinitely differentiable function $f : \mathbb{R}^d \to \mathbb{C}$ is an element of the *Schwartz space* $\mathscr{S}(\mathbb{R}^d)$ if

$$\forall n = 0, 1, \ldots \qquad \sup_{|\alpha| \le n, \alpha \in (\mathbb{N} \cup \{0\})^d} \sup_{x \in \mathbb{R}^d} \left(1 + \|x\|^2\right)^\alpha |D^\alpha f(x)| < \infty.$$

Clearly $\mathscr{S} \subset L^1$, so the Fourier transform is well defined on $\mathscr{S}$ and is in fact a topological automorphism. Since $C_c^\infty \subseteq \mathscr{S}$, the (inverse) Fourier transform of a smooth compactly supported function is smooth.

- We will denote the space $\{f \in L^2(\mathbb{R}) : \operatorname{supp} \hat{f} \subseteq [0, \infty)\}$ as $H^2(\mathbb{R})$, as in [63].

We now make note of a now well-known result, which appeared in Bin Han's paper [63], as well as many other contemporary papers.

**Theorem 51.** *Let $\psi \in L^2(\mathbb{R}^d)$. Then $\mathcal{W}(\psi)$ is a Parseval frame if and only if*

$$\sum_{n \in \mathbb{Z}} |\hat{\psi}(2^n \gamma)|^2 = 1 \quad \textit{and} \quad \sum_{n=0}^{\infty} \hat{\psi}(2^n \gamma) \overline{\hat{\psi}(2^n(\gamma + m))} = 0 \tag{3.1}$$

*with absolute convergence for almost every $\gamma \in \widehat{\mathbb{R}}^d$ and for all $m \in \mathbb{Z}^d \backslash 2\mathbb{Z}^d$.*

### 3.1.3 Outline and Results

In Section 3.2.1, we present the results from [62] and [63] which concern the existence of smooth Parseval frames which approximate 1-dimensional Parseval frame wavelet sets. Bin Han's methods involve auxiliary smooth functions which we try to generalize to higher dimensions in Section 3.2.2. We show that forming tensor products or other similarly modified versions of the auxiliary functions from Section 3.2.1 either fails to yield a Parseval frame or fails to yield a smooth wavelet when used to smooth a certain type of wavelet set. However, some Parseval wavelet set wavelets in $\widehat{\mathbb{R}}^d$ can be smoothed using methods inspired by Han's work, see Section 3.3. In Section 3.4 we construct a class of $C_c^\infty$ functions which form frames with upper frame bounds converging to 1.

We conclude with Section 3.5 which contains a review of previously known methods to smooth frame wavelet set wavelets.

## 3.2 Schwartz class Parseval frames

### 3.2.1 Parseval frames for $L^2(\mathbb{R})$

In his Master's thesis, [62], as well as the paper [63], Bin Han proved the existence of $C^\infty$ Parseval frames for $H^2(\mathbb{R})$. The following two lemmas and definition appear in the paper [63].

**Lemma 52.** There exists a function $\theta \in C^\infty(\mathbb{R})$ satisfying $\theta(x) = 0$ when $x \leq -1$ and $\theta(x) = 1$ when $x \geq 1$ and

$$\theta(x)^2 + \theta(-x)^2 = 1, \qquad x \in \mathbb{R}.$$

**Definition 53.** Given a closed interval $I = [a, b]$ and two positive numbers $\delta_1$, $\delta_2$ such that $\delta_1 + \delta_2 \leq b - a$, we define

$$f_{(I;\delta_1,\delta_2)}(x) = \begin{cases} \theta\left(\frac{x-a}{\delta_1}\right) & \text{when } x < a + \delta_1 \\ 1 & \text{when } a + \delta_1 \leq x \leq b - \delta_2 \\ \theta\left(\frac{b-x}{\delta_2}\right) & \text{when } x > b - \delta_2 \end{cases}$$

Note that $\text{supp}(f_{(I;\delta_1,\delta_2)}) \subseteq [a - \delta_1, b + \delta_2]$.

**Lemma 54.** For any positive numbers $\delta_1$, $\delta_2$, $\delta_3$ and $0 < a < b < c$,

$$f_{(I;\delta_1,\delta_2)}(2^n x) = f_{(2^{-k}I;2^{-k}\delta_1,2^{-k}\delta_2)}(x)$$

and

$$f^2_{([a,b];\delta_1,\delta_2)}(x) + f^2_{([b,c];\delta_2,\delta_3)}(x) = f^2_{([a,c];\delta_1,\delta_3)}(x).$$

The preceding lemmas are used to prove

**Proposition 55** ([63]). *Suppose that a family of disjoint closed intervals $I_i = [a_i, b_i]$, $1 \le i \le l$ in $(0, \infty)$ is arranged in a decreasing order, i.e., $0 < b_l < b_{l-1} < \ldots < b_1$ and $\cup_{i=1}^{l} I_i$ is dyadic dilation congruent to $[1/2, 1) \subseteq \widehat{\mathbb{R}}$. If $\Delta(\cup_{i=1}^{l} I_i) > 0$. Then for any*

$$0 < \delta < \frac{1}{2}\min\{\Delta(\cup_{i=1}^{l} I_i), \min_{1\le i\le l}\{b_i - a_i\}, \min_{1\le i<l}\operatorname{dist}(I_i, I_{i+1})\},$$

*let*

$$\hat{\psi}_\delta = f_{(I_1;\frac{\delta}{2},\delta)} + \sum_{i=2}^{l} f_{(I_i;2^{-k_i-1}\delta,2^{-k_i-1}\delta)}$$

*where $k_i$ is the unique non-negative integer such that $2^{k_i} I_i \subseteq [\frac{1}{2}b_1, b_1]$. We have $\psi_\delta \in \mathscr{S}(\mathbb{R})$ and $\mathcal{W}(\psi_\delta)$ is a Parseval frame in $H^2(\mathbb{R})$.*

Bin Han states that a similar proposition holds for $L^2(\mathbb{R})$, but does not explicitly state it nor prove it. However, it is easy to extend the result using similar methods to his proof of the preceding proposition.

**Proposition 56.** *Suppose that a family of disjoint closed intervals $I_i = [a_i, b_i]$, $1 \le i \le l$ in $\widehat{\mathbb{R}}$ is arranged in a decreasing order, i.e., $b_l < b_{l-1} < \ldots < b_1$ where $b_j < 0 < a_{j-1}$ and $\cup_{i=1}^{l} I_i$ is dyadic dilation congruent to $[-1, -1/2) \cup [1/2, 1) \subseteq \widehat{\mathbb{R}}$.*

*If* $\Delta(\cup_{i=1}^{l} I_i) > 0$*, then for any*

$$0 < \delta < \frac{1}{2}\min\{\Delta(\cup_{i=1}^{l} I_i), \min_{1\le i\le l}\{b_i - a_i\}, \min_{1\le i<l}\operatorname{dist}(I_i, I_{i+1})\},$$

*let*

$$\hat{\psi}_\delta = f_{(I_1;\frac{\delta}{2},\delta)} + \left[\sum_{i=2}^{l-1} f_{(I_i;2^{-k_i-1}\delta,2^{-k_i-1}\delta)}\right] + f_{(I_l;\delta,\frac{\delta}{2})}$$

*where for* $2 \le i \le j-1$*,* $k_i$ *is the unique non-negative integer such that* $2^{k_i} I_i \subseteq [\frac{1}{2}b_1, b_1]$ *and for* $j \le i \le l-1$*,* $k_i$ *is the unique non-negative integer such that* $2^{k_i} I_i \subseteq [a_l, \frac{1}{2}a_l]$*. We have* $\psi_\delta \in \mathscr{S}(\mathbb{R})$ *and* $\mathcal{W}(\psi_\delta)$ *is a Parseval frame in* $L^2(\mathbb{R})$*.*

*Proof.* We'd like to make use of Theorem 51. To this end, let

$$\begin{aligned} f_1 &= f_{(I_1;\frac{\delta}{2},\delta)} \\ f_i &= f_{(I_i;2^{-k_i-1}\delta,2^{-k_i-1}\delta)} \qquad \text{for } 2 \le i \le l-1 \\ f_l &= f_{(I_l;\delta,\frac{\delta}{2})} \end{aligned}$$

In order for the $f_i$ to be well defined, we need

$$\begin{aligned} \frac{\delta}{2} + \delta &\le b_1 - a_1 \\ 2^{-k_i-1}\delta + 2^{-k_i-1}\delta &\le b_i - a_i \qquad \text{for } 2 \le i \le l-1 \\ \frac{\delta}{2} + \delta &\le b_l - a_l \end{aligned}$$

By our choice of $\delta$, for all $1 \le i \le l$,

$$\begin{aligned} \frac{\delta}{2} + \delta &< \frac{\frac{1}{2}(b_i - a_i)}{2} + \frac{1}{2}(b_i - a_i) \\ &= \frac{3}{4}(b_i - a_i) \\ &< b_i - a_i. \end{aligned}$$

So for $i = 1$ and $i = l$ the desired inequalities hold. Also because $k_i \geq 0$ for each $2 \leq i \leq l-1$,

$$\begin{aligned} 2^{-k_i-1}\delta + 2^{-k_i-1}\delta &= 2^{-k_i}\delta \\ &< 2^{-k_i}\left(\frac{b_i - a_i}{2}\right) \\ &= 2^{-k_i-1}(b_i - a_i) \\ &< b_i - a_i \end{aligned}$$

So the $f_i$ are all well-defined and

$$\begin{aligned} \operatorname{supp} f_1 &\subseteq [a_1 - \frac{\delta}{2}, b_1 + \delta], \\ \operatorname{supp} f_i &\subseteq [a_i - 2^{-k_i-1}\delta, b_i + 2^{-k_i-1}\delta] \qquad \text{for any } 2 \leq i \leq l-1 \\ \operatorname{supp} f_l &\subseteq [a_l - \frac{\delta}{2}, b_l + \delta]. \end{aligned}$$

Hence for any $1 \leq i \leq l$, $\operatorname{supp} f_i \subset [a_i - \delta, b_i + \delta]$. Since $0 < \delta < \frac{1}{2}\Delta(\cup_{i=1}^{l})$,

$$\Delta(\cup_{i=1}^{l} \operatorname{supp} f_i) \geq \Delta(\cup_{i=1}^{l}[a_i - \delta, b_i + \delta]) \geq \Delta(\cup_{i=1}^{l} I_i) - 2\delta > 0.$$

Thus for all $1 \leq i, k \leq l$, $n \in \mathbb{N} \cup \{0\}$, $m \in \mathbb{Z}\backslash 0$,

$$f_i(2^n\gamma) f_k(2^n(\gamma + m)) = 0,$$

since $2^n m \in \mathbb{Z}\backslash\{0\}$. Hence for any $n \in \mathbb{N} \cup \{0\}$ and $m \in \mathbb{Z}\backslash\{0\}$,

$$\overline{\hat{\psi}}_\delta(2^n\gamma)\hat{\psi}_\delta(2^n(\gamma + m)) = 0 \text{ a.e.}$$

By construction,

$$\mathbb{1}_{I_1} + \left(\sum_{i=2}^{l-1} \mathbb{1}_{2^{k_i} I_i}\right) + \mathbb{1}_{I_l} = \mathbb{1}_{[a_l, \frac{1}{2}a_l] \cup [\frac{1}{2}b_1, b_1]}.$$

Using the Lemma 54, we obtain

$$
\begin{aligned}
f_1^2(\gamma)+\sum_{i=2}^{j-1} f_i^2(2^{-k_i}\gamma) &= f^2_{(I_1;\frac{\delta}{2},\delta)}(\gamma)+\sum_{i=2}^{j-1} f^2_{(I_i;2^{-k_i-1}\delta,2^{-k_i-1}\delta)}(2^{-k_i}\gamma)\\
&= f^2_{(I_1;\frac{\delta}{2},\delta)}(\gamma)+\sum_{i=2}^{j-1} f^2_{(2^{k_i}I_i;\frac{1}{2}\delta,\frac{1}{2}\delta)}(\gamma)\\
&= f^2_{([\frac{1}{2}b_1,b_1];\frac{1}{2}\delta,\delta)}(\gamma).
\end{aligned}
$$

Similarly,

$$\sum_{i=j}^{l-1} f_i^2(2^{-k_i}\gamma)+f_l^2(\gamma)=f^2_{([a_l,\frac{1}{2}a_l];\delta,\frac{1}{2}\delta)}(\gamma).$$

So we know that

$$
\begin{aligned}
\sum_{n\in\mathbb{Z}}\sum_{i=1}^{l} f_i^2(2^n\gamma) &= \sum_{n\in\mathbb{Z}}\left(f^2_{([a_l,\frac{1}{2}a_l];\delta,\frac{1}{2}\delta)}(2^n\gamma)+f^2_{([\frac{1}{2}b_1,b_1];\frac{1}{2}\delta,\delta)}(2^n\gamma)\right)\\
&= \sum_{n\in\mathbb{Z}}\left(f^2_{(2^{-n}[a_l,\frac{1}{2}a_l];2^{-n}\delta,2^{-n-1}\delta)}(\gamma)+f^2_{(2^{-n}[\frac{1}{2}b_1,b_1];2^{-n-1}\delta,2^{-n}\delta)}(\gamma)\right).
\end{aligned}
$$

For any $n\in\mathbb{Z}$, we again apply Lemma 54 to get

$$
\begin{aligned}
&\left(f^2_{(2^{-n}[a_l,\frac{1}{2}a_l];2^{-n}\delta,2^{-n-1}\delta)}(\gamma)+f^2_{(2^{-n}[\frac{1}{2}b_1,b_1];2^{-n-1}\delta,2^{-n}\delta)}(\gamma)\right)\\
+&\left(f^2_{(2^{-n-1}[a_l,\frac{1}{2}a_l];2^{-n-1}\delta,2^{-n-2}\delta)}(\gamma)+f^2_{(2^{-n-1}[\frac{1}{2}b_1,b_1];2^{-n-2}\delta,2^{-n-1}\delta)}(\gamma)\right)\\
=\;& f^2_{([2^{-n}a_l,2^{-n-2}a_l];2^{-n}\delta,2^{-n-2}\delta)}(\gamma)+f^2_{([2^{-n-2}b_1,2^{-n}b_1];2^{-n-2}\delta,2^{-n}\delta)}(\gamma)
\end{aligned}
$$

As a result of these 2 equalities, $\sum_{i=1}^{l}\sum_{n\in\mathbb{Z}} f_i(2^n\gamma)=\mathbb{1}_{(-\infty,\infty)}(\gamma)$, where the convergence is almost everywhere and absolute for $1\le i\le l$. Since $0<\delta<\frac{1}{2}\min_{1\le i\le l}\{\mathrm{dist}(I_i,I_{i+1})\}$, $\operatorname{supp} f_i\subseteq[a_i-\delta,b_i+\delta]$,

$$|\hat{\psi}_\delta(\gamma)|^2=\sum_{i=1}^{l} f_i^2(\gamma).$$

Thus

$$\sum_{n\in\mathbb{Z}}|\hat{\psi}_\delta(2^n\gamma)|^2=\sum_{n\in\mathbb{Z}}\sum_{i=1}^{l} f_i^2(2^n\gamma)=1$$

for almost all $\gamma$. Hence $\psi_\delta$ generates a Parseval wavelet frame in $L^2(\mathbb{R})$. Clearly $\psi_\delta \in \mathscr{S}(\mathbb{R})$ since $f_i \in \mathscr{S}(\mathbb{R})$, $1 \leq i \leq l$. $\square$

In Corollary 36, we shrunk Parseval frame wavelet sets and obtained frame wavelets with better decay than the original Parseval frame wavelet. We again employ that idea in order to modify any bounded Parseval frame wavelet set in $\widehat{\mathbb{R}}$ so that we may apply Proposition 56 in order to obtain a smooth Parseval frame wavelet set.

**Corollary 57.** *Let $L \subseteq \widehat{\mathbb{R}}$ be a Parseval frame wavelet set. Let $N \in \mathbb{Z}$ have the trait that $\overline{2^N L} \subseteq (-\frac{1}{2}, \frac{1}{2})$. Then there exists a $\psi \in \mathscr{S}(\mathbb{R})$ such that $\hat{\psi} \in C_c^\infty(\mathbb{R})$ and $\mathcal{W}(\psi)$ is a Parseval frame and the measure of* $\mathrm{supp}(\psi)\backslash 2^N L$ *is arbitrarily small.*

### 3.2.2 Extensions of Han's construction

We would like to extend Han's results even further in order to create Schwartz class Parseval frames over $L^2(\mathbb{R}^d)$ for $d > 1$. The basic idea of Han's construction is to replace each $\mathbb{1}_{[a_i,b_i]}(x)$ with an appropriate $C_c^\infty$ bump function $f_{([a_i,b_i];\delta_i,\tilde{\delta}_i)}(x)$, where $\cup_i[a_i,b_i]$ is a Parseval frame wavelet set with $\Delta(\cup_i[a_i,b_i]) > 0$. We will attempt to generalize the smoothing techniques on the class of Parseval frame wavelet sets

$$\{L_a\} = \{[-2a, 2a]^2 \backslash [-a, a]^2 : 0 < a < \frac{1}{4}\}.$$

It is easy to see that any such $L_a$ is indeed a Parseval frame wavelet set since each tiles the plane under dyadic dilation and $\Delta(L_a) > 0$ for $0 < a < \frac{1}{4}$ (Proposition 12). These sets are natural ones to start with because of their simplicity. When

$a = 2^N$ for $N \leq -3$, then $L_a$ may be obtained by dyadically shrinking the $K_0 \backslash A_0$ set obtained from the neighborhood mapping construction of the 2-$d$ Journé set, as in Example 15. We need to define an appropriate family of bump functions to replace each $\mathbb{1}_{L_a}(x, y)$. We try the following functions:

$$h_{(L_a;\delta,\frac{\delta}{2})}(x,y) = f_{([-2a,2a];\delta,\delta)}(x) f_{([-2a,2a];\delta,\delta)}(y) - f_{([-a,a];\frac{\delta}{2},\frac{\delta}{2})}(x) f_{([-a,a];\frac{\delta}{2},\frac{\delta}{2})}(y), \text{ and} \tag{3.2}$$

$$g_{(L_a;\delta,\frac{\delta}{2})}(x,y) = \begin{cases} \theta\left(\frac{2a-|y|}{\delta}\right) & \text{when } |x| \in [0, 2a-\delta] \text{ and } |y| \in [2a-\delta, 2a+\delta] \\ \theta\left(\frac{2a-|y|}{\delta}\right)\theta\left(\frac{2a-|x|}{\delta}\right) & \text{when } |x|, |y| \in [2a-\delta, 2a+\delta] \\ \theta\left(\frac{2a-|x|}{\delta}\right) & \text{when } |y| \in [0, 2a-\delta] \text{ and } |x| \in [2a-\delta, 2a+\delta] \\ 1 & \text{when } (|x|, |y|)^T \in [0, 2a-\delta]^2 \backslash [0, a+\frac{\delta}{2}]^2 \\ \theta\left(\frac{|y|-a}{\delta/2}\right) & \text{when } |x| \in [0, a-\frac{\delta}{2}] \text{ and } |y| \in [a-\frac{\delta}{2}, a+\frac{\delta}{2}] \\ \theta\left(\frac{|y|-a}{\delta/2}\right)\theta\left(\frac{|x|-a}{\delta/2}\right) & \text{when } |x|, |y| \in [a-\frac{\delta}{2}, a+\frac{\delta}{2}] \\ \theta\left(\frac{|x|-a}{\delta/2}\right) & \text{when } |y| \in [0, a-\frac{\delta}{2}] \text{ and } |x| \in [a-\frac{\delta}{2}, a+\frac{\delta}{2}] \\ 0 & \text{otherwise,} \end{cases} \tag{3.3}$$

where $\theta$ is as in Lemma 52. We first note that $g$ is well defined even though the piecewise domains overlap. In order to form $h$, we tensor the 1-dimensional interval bump functions to create 2-dimensional rectangle bump functions and then subtract such functions corresponding to $[-2a, 2a]^2$ and $[-a, a]^2$. The function $g$ may be seen as a piecewise tensor product. In fact,

$$h_{(L_a;\delta,\frac{\delta}{2})}(x,y) = g_{(L_a;\delta,\frac{\delta}{2})}(x,y) \text{ for } (x,y)^T \notin [-a-\frac{\delta}{2}, a+\frac{\delta}{2}]^2 \backslash [-a+\frac{\delta}{2}, a-\frac{\delta}{2}]^2$$

and $\operatorname{supp} h_{(L_a;\delta,\frac{\delta}{2})} = \operatorname{supp} g_{(L_a;\delta,\frac{\delta}{2})}$. Although both of these functions seem promising, neither $\sum_{n\in\mathbb{Z}} h^2_{(L_a;\delta,\frac{\delta}{2})}(2^n x, 2^n y)$ nor $\sum_{n\in\mathbb{Z}} g^2_{(L_a;\delta,\frac{\delta}{2})}(2^n x, 2^n y)$ are equal to 1 almost everywhere. It follows from Theorem 51 that neither $\mathcal{W}(\check{h})$ nor $\mathcal{W}(\check{g})$ are Parseval frames.

**Proposition 58.** *Let $0 < a < \frac{1}{4}$, set $L = [-2a, 2a]^2 \backslash [-a, a]^2$, and pick a $\delta$ such that $0 < \delta < \frac{1}{2}\min\{1-4a, a\}$. Let $h$ be as in Equation 3.2. Then $\sum_{n\in\mathbb{Z}} h^2_{(L_a;\delta,\frac{\delta}{2})}(2^n x, 2^n y)$ is not equal to 1 a.e.*

*Proof.* We first rewrite $h_{(L_a;\delta,\frac{\delta}{2})}$ in terms of $\theta$ (from Lemma 52).

$$h_{(L_a;\delta,\frac{\delta}{2})}(x,y) = \begin{cases} \theta\left(\frac{2a-|y|}{\delta}\right) & \text{when } |x| \in [0, 2a-\delta] \text{ and } |y| \in [2a-\delta, 2a+\delta] \\ \theta\left(\frac{2a-|y|}{\delta}\right)\theta\left(\frac{2a-|x|}{\delta}\right) & \text{when } |x|, |y| \in [2a-\delta, 2a+\delta] \\ \theta\left(\frac{2a-|x|}{\delta}\right) & \text{when } |y| \in [0, 2a-\delta] \text{ and } |x| \in [2a-\delta, 2a+\delta] \\ 1 & \text{when } (|x|,|y|)^T \in [0, 2a-\delta]^2 \backslash [0, a+\frac{\delta}{2}]^2 \\ 1-\theta\left(\frac{a-|y|}{\delta/2}\right) & \text{when } |x| \in [0, a-\frac{\delta}{2}] \text{ and } |y| \in [a-\frac{\delta}{2}, a+\frac{\delta}{2}] \\ 1-\theta\left(\frac{a-|y|}{\delta/2}\right)\theta\left(\frac{a-|x|}{\delta/2}\right) & \text{when } |x|, |y| \in [a-\frac{\delta}{2}, a+\frac{\delta}{2}] \\ 1-\theta\left(\frac{a-|x|}{\delta/2}\right) & \text{when } |y| \in [0, a-\frac{\delta}{2}] \text{ and } |x| \in [a-\frac{\delta}{2}, a+\frac{\delta}{2}] \\ 0 & \text{otherwise.} \end{cases}$$

We will prove the claim if we show that $\sum_{n\in\mathbb{Z}} h^2_{(L_a;\delta,\frac{\delta}{2})}(2^n x, 2^n y) < 1$ on a set of positive measure. Note that for $(x,y)^T \in [0, a-\frac{\delta}{2}] \times (a-\frac{\delta}{2}, a+\frac{\delta}{2}]$,

$$\begin{aligned} \sum_{n\in\mathbb{Z}} h^2_{(L_a;\delta,\frac{\delta}{2})}(2^n x, 2^n y) &= h^2_{(L_a;\delta,\frac{\delta}{2})}(x,y) + h^2_{(L_a;\delta,\frac{\delta}{2})}(2x,2y) \\ &= \left(1-\theta\left(\frac{a-y}{\delta/2}\right)\right)^2 + \theta^2\left(\frac{a-y}{\delta/2}\right) \\ &= 1 + 2\theta\left(\frac{a-|y|}{\delta/2}\right)\left[\theta\left(\frac{a-|y|}{\delta/2}\right) - 1\right] \\ &< 1 \end{aligned}$$

since $0 < \theta\left(\frac{a-|y|}{\delta/2}\right) < 1$ for $y > a - \frac{\delta}{2}$. □

**Proposition 59.** *Let* $0 < a < \frac{1}{4}$, *set* $L = [-2a, 2a]^2 \backslash [-a, a]^2$, *and pick a* $\delta$ *such that* $0 < \delta < \frac{1}{2}\min\{1-4a, a\}$. *Let* $g$ *be as in Equation 3.3. Then* $\sum_{n\in\mathbb{Z}} g^2_{(L_a;\delta,\frac{\delta}{2})}(2^n x, 2^n y)$ *is not equal to* 1 *a.e.*

*Proof.* We first compute the following for $x, y > 0$, making use of Lemma 54:

$$g^2_{(L_a;\delta,\frac{\delta}{2})}(x, y) + g^2_{(L_a;\delta,\frac{\delta}{2})}(2x, 2y)$$

$$
= \begin{cases}
\theta^2\left(\frac{2a-y}{\delta}\right) & \text{when } (x,y)^T \in [0,2a-\delta]\times[2a-\delta,2a+\delta] \\
\theta^2\left(\frac{2a-y}{\delta}\right)\theta^2\left(\frac{2a-x}{\delta}\right) & \text{when } (x,y)^T \in [2a-\delta,2a+\delta]^2 \\
\theta^2\left(\frac{2a-x}{\delta}\right) & \text{when } (x,y)^T \in [2a-\delta,2a+\delta]\times[0,2a-\delta] \\
1 & \text{when } (x,y)^T \in [0,2a-\delta]^2\backslash[0,a+\frac{\delta}{2}]^2 \\
\theta^2\left(\frac{a-y}{\delta/2}\right)+\theta^2\left(\frac{y-a}{\delta/2}\right) & \text{when } (x,y)^T \in [0,a-\frac{\delta}{2}]\times[a-\frac{\delta}{2},a+\frac{\delta}{2}] \\
\theta^2\left(\frac{a-y}{\delta/2}\right)\theta^2\left(\frac{a-x}{\delta/2}\right)+\theta^2\left(\frac{y-a}{\delta/2}\right)\theta^2\left(\frac{x-a}{\delta/2}\right) & \text{when } (x,y)^T \in [a-\frac{\delta}{2},a+\frac{\delta}{2}]^2 \\
\theta^2\left(\frac{a-x}{\delta/2}\right)+\theta^2\left(\frac{x-a}{\delta/2}\right) & \text{when } (x,y)^T \in [a-\frac{\delta}{2},a+\frac{\delta}{2}]\times[0,a-\frac{\delta}{2}] \\
1 & \text{when } (x,y)^T \in [0,a-\frac{\delta}{2}]^2\backslash[0,\frac{a}{2}+\frac{\delta}{4}]^2 \\
\theta^2\left(\frac{y-a/2}{\delta/4}\right) & \text{when } (x,y)^T \in [0,\frac{a}{2}-\frac{\delta}{4}]\times[\frac{a}{2}-\frac{\delta}{4},\frac{a}{2}+\frac{\delta}{4}] \\
\theta^2\left(\frac{y-a/2}{\delta/4}\right)\theta^2\left(\frac{x-a/2}{\delta/4}\right) & \text{when } (x,y)^T \in [\frac{a}{2}-\frac{\delta}{4},\frac{a}{2}+\frac{\delta}{4}]^2 \\
\theta^2\left(\frac{x-a/2}{\delta/4}\right) & \text{when } (x,y)^T \in [\frac{a}{2}-\frac{\delta}{4},\frac{a}{2}+\frac{\delta}{4}]\times[0,\frac{a}{2}-\frac{\delta}{4}] \\
0 & \text{otherwise}
\end{cases}
$$

$$
= \begin{cases}
\theta^2\left(\frac{2a-y}{\delta}\right) & \text{when } (x,y)^T \in [0,2a-\delta]\times[2a-\delta,2a+\delta] \\
\theta^2\left(\frac{2a-y}{\delta}\right)\theta^2\left(\frac{2a-x}{\delta}\right) & \text{when } (x,y)^T \in [2a-\delta,2a+\delta]^2 \\
\theta^2\left(\frac{2a-x}{\delta}\right) & \text{when } (x,y)^T \in [2a-\delta,2a+\delta]\times[0,2a-\delta] \\
1 & \text{when } (x,y)^T \in [0,2a-\delta]^2\backslash[0,a+\frac{\delta}{2}]^2 \\
1 & \text{when } (x,y)^T \in [0,a-\frac{\delta}{2}]\times[a-\frac{\delta}{2},a+\frac{\delta}{2}] \\
\theta^2\left(\frac{a-y}{\delta/2}\right)\theta^2\left(\frac{a-x}{\delta/2}\right)+\theta^2\left(\frac{y-a}{\delta/2}\right)\theta^2\left(\frac{x-a}{\delta/2}\right) & \text{when } (x,y)^T \in [a-\frac{\delta}{2},a+\frac{\delta}{2}]^2 \\
1 & \text{when } (x,y)^T \in [a-\frac{\delta}{2},a+\frac{\delta}{2}]\times[0,a-\frac{\delta}{2}] \\
1 & \text{when } (x,y)^T \in [0,a-\frac{\delta}{2}]^2\backslash[0,\frac{a}{2}+\frac{\delta}{4}]^2 \\
\theta^2\left(\frac{y-a/2}{\delta/4}\right) & \text{when } (x,y)^T \in [0,\frac{a}{2}-\frac{\delta}{4}]\times[\frac{a}{2}-\frac{\delta}{4},\frac{a}{2}+\frac{\delta}{4}] \\
\theta^2\left(\frac{y-a/2}{\delta/4}\right)\theta^2\left(\frac{x-a/2}{\delta/4}\right) & \text{when } (x,y)^T \in [\frac{a}{2}-\frac{\delta}{4},\frac{a}{2}+\frac{\delta}{4}]^2 \\
\theta^2\left(\frac{x-a/2}{\delta/4}\right) & \text{when } (x,y)^T \in [\frac{a}{2}-\frac{\delta}{4},\frac{a}{2}+\frac{\delta}{4}]\times[0,\frac{a}{2}-\frac{\delta}{4}] \\
0 & \text{otherwise.}
\end{cases}
$$

Continuing inductively we obtain $\sum_{n\in\mathbb{Z}} g^2_{(L_a;\delta,\frac{\delta}{2})}(2^n x, 2^n y)$

$$
= \begin{cases} 0 & \text{when } x = y = 0 \\ \theta^2\left(\frac{a-2^m y}{\delta/2}\right)\theta^2\left(\frac{a-2^m x}{\delta/2}\right) & \text{when } (2^m x, 2^m y)^T \in [a-\frac{\delta}{2}, a+\frac{\delta}{2}]^2 \text{ for } m \in \mathbb{Z} \\ \quad +\theta^2\left(\frac{2^m y-a}{\delta/2}\right)\theta^2\left(\frac{2^m x-a}{\delta/2}\right) & \\ 1 & \text{otherwise.} \end{cases}
$$

We would like to show that $\theta^2\left(\frac{a-y}{\delta/2}\right)\theta^2\left(\frac{a-x}{\delta/2}\right)+\theta^2\left(\frac{y-a}{\delta/2}\right)\theta^2\left(\frac{x-a}{\delta/2}\right)$ does not take the value 1 for almost all $(x,y)^T \in [a-\frac{\delta}{2}, a+\frac{\delta}{2}]^2$. Assume that $\frac{1}{2} < \beta < 1$ and $0 < \alpha < 1$. Then $\frac{\beta}{2\beta-1} > 1$

$$
\begin{aligned}
\text{and so} \quad & \alpha \neq \frac{\beta}{2\beta-1} \\
\Rightarrow \quad & 2\alpha\beta - \alpha \neq \beta \\
\Rightarrow \quad & 1 + 2\alpha\beta - \alpha - \beta \neq 1 \\
\Rightarrow \quad & \alpha\beta + (1-\alpha)(1-\beta) \neq 1
\end{aligned}
$$

It follows from the intermediate value theorem and continuity that the measure of $E = \{(x,y)^T \in [a-\frac{\delta}{2}, a+\frac{\delta}{2}]^2 : \frac{1}{2} < \theta^2\left(\frac{a-y}{\delta/2}\right) < 1, 0 < \theta^2\left(\frac{a-x}{\delta/2}\right) < 1\}$ is positive. So for $(x,y)^T \in E$,

$$
\begin{aligned}
& \theta^2\left(\frac{a-y}{\delta/2}\right)\theta^2\left(\frac{a-x}{\delta/2}\right)+\theta^2\left(\frac{y-a}{\delta/2}\right)\theta^2\left(\frac{x-a}{\delta/2}\right) \\
= \; & \theta^2\left(\frac{a-y}{\delta/2}\right)\theta^2\left(\frac{a-x}{\delta/2}\right)+\left(1-\theta^2\left(\frac{a-y}{\delta/2}\right)\right)\left(1-\theta^2\left(\frac{a-x}{\delta/2}\right)\right) \\
\neq \; & 1
\end{aligned}
$$

□

However, as the calculations above show, $\sum_{n\in\mathbb{Z}} h^2_{(L_a;\delta,\frac{\delta}{2})}(2^m x, 2^n y) = 1$ for all

$$(|x|,|y|)^T \notin \{0\} \cup \left( \bigcup_{m\in\mathbb{Z}} 2^m \left[a - \frac{\delta}{2}, a + \frac{\delta}{2}\right]^2 \right).$$

So we adjust $h_{(L_a;\delta,\frac{\delta}{2})}(x,y)$ on $C = [a-\frac{\delta}{2}, a+\frac{\delta}{2}]^2 \cup [2a-\delta, 2a+\delta]^2$ in hopes of obtaining a Parseval frame. We do this by setting

$$f_{(L_a;\delta,\frac{\delta}{2})}(x,y) = h_{(L_a;\delta,\frac{\delta}{2})}(x,y) \text{ for } (|x|,|y|)^T \notin C$$

and

$$f_{(L_a;\delta,\frac{\delta}{2})}(x,y) = f_{(L_a;\delta,\frac{\delta}{2})}(\tilde{x},\tilde{y}),$$

for $(|x|,|y|)^T, (|\tilde{x}|,|\tilde{y}|)^T \in C$, $|x|+|y| = |\tilde{x}|+|\tilde{y}|$, and $|x|+|y|$ small enough. Explicitly, $f_{(L_a;\delta,\frac{\delta}{2})}(x,y) =$

$$\begin{cases} \theta\left(\frac{2a-|y|}{\delta}\right) & \text{when } |x| \in [0, 2a-\delta] \text{ and } |y| \in [2a-\delta, 2a+\delta] \\ \theta\left(\frac{4a-|x|-|y|-\delta}{\delta}\right) & \text{when } |x|,|y| \in [2a-\delta, 2a+\delta], \text{ where } 4a-2\delta \le |x|+|y| \le 4a \\ \theta\left(\frac{2a-|x|}{\delta}\right) & \text{when } |y| \in [0, 2a-\delta] \text{ and } |x| \in [2a-\delta, 2a+\delta] \\ 1 & \text{when } (|x|,|y|)^T \in [0, 2a-\delta]^2 \backslash [0, a+\frac{\delta}{2}]^2 \\ 1 & \text{when } |x|,|y| \in [a-\frac{\delta}{2}, a+\frac{\delta}{2}] \text{ where } 2a \le |x|+|y| \le 2a+\delta \\ \theta\left(\frac{|y|-a}{\delta/2}\right) & \text{when } |x| \in [0, a-\frac{\delta}{2}] \text{ and } |y| \in [a-\frac{\delta}{2}, a+\frac{\delta}{2}] \\ \theta\left(\frac{|x|+|y|-2a+\delta/2}{\delta/2}\right) & \text{when } |x|,|y| \in [a-\frac{\delta}{2}, a+\frac{\delta}{2}] \text{ where } 2a-\delta \le |x|+|y| \le 2a \\ \theta\left(\frac{|x|-a}{\delta/2}\right) & \text{when } |y| \in [0, a-\frac{\delta}{2}] \text{ and } |x| \in [a-\frac{\delta}{2}, a+\frac{\delta}{2}] \\ 0 & \text{otherwise.} \end{cases}$$

**Proposition 60.** *Let $0 < a < \frac{1}{4}$, set $L = [-2a, 2a]^2 \backslash [-a, a]^2$, and pick a $0 < \delta < \frac{1}{2}\min\{1-4a, a\}$. Let $\hat{\psi}_\delta = f_{(L_a;\delta,\frac{\delta}{2})}$. For $x, y > 0$,*

$$\hat{\psi}^2_\delta(\vec{x}) + \hat{\psi}^2_\delta(2\vec{x}) = f^2_{([-2a,2a]^2\backslash[-\frac{a}{2},\frac{a}{2}]^2;\delta,\frac{\delta}{4})}(\vec{x}).$$

*Proof.*

$$\hat{\psi}_\delta^2(\vec{x}) + \hat{\psi}_\delta^2(2\vec{x})$$

$$= \begin{cases} \theta^2\left(\frac{2a-y}{\delta}\right) & \text{when } (x,y)^T \in [0, 2a-\delta] \times [2a-\delta, 2a+\delta] \\ \theta^2\left(\frac{4a-x-y-\delta}{\delta}\right) & \text{when } (x,y)^T \in [2a-\delta, 2a+\delta]^2 \\ \theta^2\left(\frac{2a-x}{\delta}\right) & \text{when } (x,y)^T \in [2a-\delta, 2a+\delta] \times [0, 2a-\delta] \\ 1 & \text{when } (x,y)^T \in [0, 2a-\delta]^2 \backslash [0, a+\frac{\delta}{2}]^2 \\ 1 & \text{when } (x,y)^T \in [a-\frac{\delta}{2}, a+\frac{\delta}{2}]^2 \text{ where } 2a \leq x+y 2a+\delta \\ \theta^2\left(\frac{a-y}{\delta/2}\right) + \theta^2\left(\frac{y-a}{\delta/2}\right) & \text{when } (x,y)^T \in [0, a-\frac{\delta}{2}] \times [a-\frac{\delta}{2}, a+\frac{\delta}{2}] \\ \theta^2\left(\frac{x+y-2a+\delta/2}{\delta/2}\right) + & \\ \quad \theta^2\left(\frac{2a-x-y-\delta/2}{\delta/2}\right) & \text{when } (x,y)^T \in [a-\frac{\delta}{2}, a+\frac{\delta}{2}]^2 \text{ where } 2a-\delta \leq x+y \leq 2a \\ \theta^2\left(\frac{a-x}{\delta/2}\right) + \theta^2\left(\frac{x-a}{\delta/2}\right) & \text{when } (x,y)^T \in [a-\frac{\delta}{2}, a+\frac{\delta}{2}] \times [0, a-\frac{\delta}{2}] \\ 1 & \text{when } (x,y)^T \in [0, a-\frac{\delta}{2}]^2 \backslash [0, \frac{a}{2}+\frac{\delta}{4}]^2 \\ 1 & \text{when } (x,y)^T \in [\frac{a}{2}-\frac{\delta}{4}, \frac{a}{2}+\frac{\delta}{4}]^2 \text{ where } a \leq x+y a+\frac{\delta}{2} \\ \theta^2\left(\frac{y-a/2}{\delta/4}\right) & \text{when } (x,y)^T \in [0, \frac{a}{2}-\frac{\delta}{4}] \times [\frac{a}{2}-\frac{\delta}{4}, \frac{a}{2}+\frac{\delta}{4}] \\ \theta^2\left(\frac{x+y-a+\delta/4}{\delta/4}\right) & \text{when } (x,y)^T \in [\frac{a}{2}-\frac{\delta}{4}, \frac{a}{2}+\frac{\delta}{4}]^2 \text{ where } a-\frac{\delta}{2} \leq x+y \leq a \\ \theta^2\left(\frac{x-a/2}{\delta/4}\right) & \text{when } (x,y)^T \in [\frac{a}{2}-\frac{\delta}{4}, \frac{a}{2}+\frac{\delta}{4}] \times [0, \frac{a}{2}-\frac{\delta}{4}] \\ 0 & \text{otherwise,} \end{cases}$$

$$= \begin{cases} \theta^2\left(\frac{2a-y}{\delta}\right) & \text{when } (x,y)^T \in [0,2a-\delta]\times[2a-\delta,2a+\delta] \\ \theta^2\left(\frac{4a-x-y-\delta}{\delta}\right) & \text{when } (x,y)^T \in [2a-\delta,2a+\delta]^2 \\ \theta^2\left(\frac{2a-x}{\delta}\right) & \text{when } (x,y)^T \in [2a-\delta,2a+\delta]\times[0,2a-\delta] \\ 1 & \text{when } (x,y)^T \in [0,2a-\delta]^2\backslash[0,\frac{a}{2}+\frac{\delta}{4}]^2 \\ 1 & \text{when } (x,y)^T \in [\frac{a}{2}-\frac{\delta}{4},\frac{a}{2}+\frac{\delta}{4}]^2 \text{ when } a \le x+y \le a+\frac{\delta}{2} \\ \theta^2\left(\frac{y-a/2}{\delta/4}\right) & \text{when } (x,y)^T \in [0,\frac{a}{2}-\frac{\delta}{4}]\times[\frac{a}{2}-\frac{\delta}{4},\frac{a}{2}+\frac{\delta}{4}] \\ \theta^2\left(\frac{x+y-a+\delta/4}{\delta/4}\right) & \text{when } (x,y)^T \in [\frac{a}{2}-\frac{\delta}{4},\frac{a}{2}+\frac{\delta}{4}]^2 \text{ where } a-\frac{\delta}{2} \le x+y \le a \\ \theta^2\left(\frac{x-a/2}{\delta/4}\right) & \text{when } (x,y)^T \in [\frac{a}{2}-\frac{\delta}{4},\frac{a}{2}+\frac{\delta}{4}]\times[0,\frac{a}{2}-\frac{\delta}{4}] \\ 0 & \text{otherwise,} \end{cases}$$

$$= f^2_{([-2a,2a]^2\backslash[-\frac{a}{2},\frac{a}{2}]^2;\delta,\frac{\delta}{4})}(\vec{x}),$$

as desired. □

**Proposition 61.** *Let* $0 < a < \frac{1}{4}$*, set* $L = [-2a,2a]^2\backslash[-a,a]^2$*, and pick a* $0 < \delta < \frac{1}{2}\min\{1-4a, a\}$*. Let* $\hat{\psi}_\delta = f_{(L_a;\delta,\frac{\delta}{2})}$*. Then* $\hat{\psi}_\delta \in C_c(\widehat{\mathbb{R}}^2)\backslash C_c^1(\widehat{\mathbb{R}}^2)$ *and* $\mathcal{W}(\psi_\delta)$ *is a Parseval frame for* $L^2(\mathbb{R}^2)$*.*

*Proof.* Since

$$\operatorname{supp}\hat{\psi}_\delta \subseteq [-2a-\delta,2a+\delta]^2 \subseteq \left(-2a-\frac{1}{2}(1-4a), 2a+\frac{1}{2}(1-4a)\right)^2 = \left(-\frac{1}{2},\frac{1}{2}\right)^2,$$

for all $n \in \mathbb{N}\cup\{0\}$ and $k \in \mathbb{Z}^2\backslash\{0\}$,

$$\hat{\psi}_\delta\left(2^n(\vec{x}+k)\right)\overline{\hat{\psi}_\delta(2^n\vec{x})} = 0 \text{ a.e.},$$

where $\vec{x} = (x,y)$. In order to utilize Theorem 51, we would like to show that

$$\sum_{n\in\mathbb{Z}}\left|\hat{\psi}_\delta(2^n\vec{x})\right|^2 = 1 \text{ a.e.}$$

We will accomplish this by showing that

$$\hat{\psi}_\delta^2(\vec{x}) + \hat{\psi}_\delta^2(2\vec{x}) = f^2_{([-2a,2a]^2\setminus[-\frac{a}{2},\frac{a}{2}]^2;\delta,\frac{\delta}{4})}(\vec{x}).$$

Then it will follow from iteration that

$$\sum_{n=M}^{N}\left|\hat{\psi}_\delta(2^n\vec{x})\right|^2 = f^2_{([-2^{1-M}a,2^{1-M}a]^2\setminus[-2^{-N}a,2^{-N}a]^2;2^{-M}\delta,2^{-1-N}\delta)}(\vec{x}),$$

which is 1 on $[2^{1-M}a+2^{-M}\delta, 2^{1-M}a-2^{-M}\delta]^2\setminus[2^{-N}a+2^{-1-N}\delta, 2^{-N}a-2^{-1-N}\delta]^2$, where

$$2^{1-M}a - 2^{-M}\delta = 2^{-M}(2a-\delta) \to \infty \quad \text{as } M \to -\infty$$

and

$$2^{-N}a - 2^{-1-N}\delta = 2^{-N}(a-\frac{\delta}{2}) \to 0 \quad \text{as } N \to \infty$$

So $\sum_{n\in\mathbb{Z}}\left|\hat{\psi}_\delta(2^n\vec{x})\right|^2 = 1$ a.e. By symmetry, it will suffice to show that

$$\hat{\psi}_\delta^2(\vec{x}) + \hat{\psi}_\delta^2(2\vec{x}) = f^2_{([-2a,2a]^2\setminus[-\frac{a}{2},\frac{a}{2}]^2;\delta,\frac{\delta}{4})}(\vec{x})$$

for positive $x$ and $y$. The preceding proposition is a proof of this fact.

Thus, $\mathcal{W}(\psi_\delta)$ is a Parseval frame, but $\hat{\psi}_\delta$ has cusps along $\{(2a-\delta+t, 2a-\delta)^T : 0 \le t \le 2\delta\}$, as well as other edges. So $\hat{\psi}_\delta \notin C_c^1(\widehat{\mathbb{R}}^2)$. $\square$

Thus we have found a method to smooth the Parseval frame wavelets $\check{\mathbb{1}}_{L_a}$ for $0 < a < \frac{1}{4}$, which is analogous to Han's method, but it does not yield Parseval frame wavelets with good temporal decay like Schwartz functions. It seems that this method should generalize to other Parseval frame wavelet sets in $\widehat{\mathbb{R}}^2$ which have piecewise horizontal and vertical boundaries. However, there does not seem to be an easy way to write an explicit formula that works in general. Furthermore, only a relatively small number of Parseval frame wavelet sets have such a boundary. Perhaps not all is lost. Instead of trying to smooth $\mathbb{1}_K$ for some Parseval frame wavelet set $K$ resulting from the neighborhood mapping construction, we now try to build Schwartz class Parseval frames for $L^2(\mathbb{R}^2)$ directly from the $C_c^\infty$ bump functions over $\widehat{\mathbb{R}}$.

## 3.3 A construction in higher dimensions

In the preceding work, problems arose around the corners of the boundary of $L_a = [-2a, 2a]^2 \backslash [-a, a]^2$ when we tried to smooth $\mathbb{1}_{L_a}$. What if there were no corners to deal with? For $0 < a < \frac{1}{4}$, we define

$$f_{([a,2a]\times S^1;\frac{\delta}{2},\delta)}(x, y) = f_{([a,2a];\frac{\delta}{2},\delta)}(\sqrt{x^2 + y^2}),$$

where yet again $f_{([a,2a];\frac{\delta}{2},\delta)}(\cdot)$ is as in Definition 53.

**Proposition 62.** *Let $0 < a < \frac{1}{4}$. For any $0 < \delta < \frac{1}{2}\min\{1 - 4a, a\}$, define $\hat{\psi}_\delta : \mathbb{R}^2 \to \mathbb{R}$ by $\hat{\psi}_\delta(x, y) = f_{([a,2a];\frac{\delta}{2},\delta)}(\sqrt{x^2 + y^2})$. Then, $\hat{\psi} \in \mathscr{S}(\mathbb{R}^2)$ and $\mathcal{W}(\psi_\delta)$ is a Parseval frame for $L^2(\mathbb{R}^2)$.*

*Proof.* By construction, $\hat{\psi}_\delta \in C_c^\infty(\widehat{\mathbb{R}}^2) \Rightarrow \psi_\delta \in \mathscr{S}(\mathbb{R}^2)$. Since $\delta < \frac{1}{2}(1 - 4a)$, $\Delta(\operatorname{supp} \hat{\psi}_\delta) < 0$. So for all $n \in \mathbb{N} \cup \{0\}$ and $k \in \mathbb{Z}\backslash\{0\}$,

$$\hat{\psi}_\delta(2^n(\vec{x} + k))\overline{\hat{\psi}(2^n\vec{x})} = 0 \text{ a.e.}$$

where $\vec{x} = (x, y)^T$. Hence, in order to prove that $\mathcal{W}(\psi_\delta)$ is a Parseval frame, it suffices to show that $\sum_{n\in\mathbb{Z}} \left|\hat{\psi}_\delta(2^n\vec{x})\right|^2 = 1$ a.e. We compute

$$\begin{aligned}\sum_{n\in\mathbb{Z}} \left|\hat{\psi}(2^n\vec{x})\right|^2 &= \sum_{n\in\mathbb{Z}} \left|f_{([a,2a];\frac{\delta}{2},\delta)}\left(\sqrt{(2^n x)^2+(2^n y)^2}\right)\right|^2 \\ &= \sum_{n\in\mathbb{Z}} \left|f_{([a,2a];\frac{\delta}{2},\delta)}(2^n z)\right|^2 \text{ for } z=\sqrt{x^2+y^2} \qquad (3.4)\end{aligned}$$

We know that (3.4) $= 1$ for almost all non-negative $z$, specifically for $z > 0$. So $\sum_{n\in\mathbb{Z}} \left|\hat{\psi}_\delta(2^n\hat{x})\right|^2 = 1$ for $\widehat{\mathbb{R}}^2 \ni \vec{x} \neq 0$. Thus $\mathcal{W}(\psi_\delta)$ is a Parseval frame for $L^2(\mathbb{R}^2)$. $\square$

This result and proof generalize to $\mathbb{R}^d$, $d > 2$.

**Corollary 63.** *Let $0 < a < \frac{1}{4}$. For any $0 < \delta < \frac{1}{2}\min\{1-4a, a\}$, define $\hat{\psi}_\delta : \mathbb{R}^d \to \mathbb{R}$ by $\hat{\psi}_\delta(x) = f_{([a,2a];\frac{\delta}{2},\delta)}(\|x\|)$. Then, $\hat{\psi} \in \mathscr{S}(\mathbb{R}^d)$ and $\mathcal{W}(\psi_\delta)$ is a Parseval frame for $L^2(\mathbb{R}^d)$.*

*Proof.* The proof is as above. $\square$

We now have Schwartz class Parseval frames for $L^2(\mathbb{R}^d)$, $d > 1$ which are elementary to describe.

## 3.4 Partitions of unity

$C^\infty$ partitions of unity are important tools in analysis and differential topology. While the topic of $C^\infty$ partitions of unity is outside the scope of this thesis, we shall utilize a class of functions which is commonly used in conjunction with that subject, e.g.: [81].

**Definition 64.** Let $f : \widehat{\mathbb{R}} \to \mathbb{R}$ be the function $f(\gamma) = e^{-\frac{1}{\gamma}}\mathbb{1}_{(0,\infty)}$. Also, let $b, m > 0$

be such that $b - \frac{1}{m} > 0$. Define $\hat{\varphi} \in C_c^\infty(\widehat{\mathbb{R}})$ as

$$\hat{\varphi}_{b,m}(\gamma) = \frac{f(b + \frac{1}{m} - |\gamma|)}{f(b + \frac{1}{m} - |\gamma|) + f(|\gamma| - b + \frac{1}{m})}.$$

If $a > \frac{4}{m}$ is clear from the context, we shall write $\hat{\varphi}_m = \hat{\varphi}_{\frac{a}{4},m}$.

$\hat{\varphi}_{b,m}$ is a smooth function which takes the value 1 on the disk $(-b + \frac{1}{m}, b - \frac{1}{m})$ and the value 0 outside the disk $D(-b - \frac{1}{m}, b + \frac{1}{m})$. We shall now prove that the function is actually monotonic over the positive reals.

**Lemma 65.** Fix $b - \frac{1}{m} > 0$. Then $\hat{\varphi}_{b,m}$ is increasing over $(-\infty, 0)$ and decreasing over $(0, \infty)$.

*Proof.* Let $\gamma > 0$. We calculate $\hat{\varphi}'_{b,m}(\gamma)$

$$= \frac{\left(f(b + \frac{1}{m} - \gamma) + f(\gamma - b + \frac{1}{m})\right)\left(-f'(b + \frac{1}{m} - \gamma)\right) - f(b + \frac{1}{m} - \gamma)\left(-f'(b + \frac{1}{m} - \gamma) + f'(\gamma - b + \frac{1}{m})\right)}{\left(f(b + \frac{1}{m} - \gamma) + f(\gamma - b + \frac{1}{m})\right)^2}$$

$$= \frac{-\left(f(\gamma - b + \frac{1}{m})\right)f'(b + \frac{1}{m} - \gamma) + f(b + \frac{1}{m} - \gamma)f'(\gamma - b + \frac{1}{m})\right)}{\left(f(b + \frac{1}{m} - \gamma) + f(\gamma - b + \frac{1}{m})\right)^2}.$$

For all $\gamma \in \widehat{\mathbb{R}}$, $f(\gamma) \geq 0$ and $f'(\gamma) = \frac{1}{\gamma^2} e^{-\frac{1}{\gamma}} \mathbb{1}_{(0,\infty)} \geq 0$. Hence for $\gamma \geq 0$, $\hat{\varphi}'_{b,m}(\gamma) \leq 0$. Since $\hat{\varphi}_{b,m}$ is even, this implies that $\hat{\varphi}'_{b,m}(\gamma) \geq 0$ for $\gamma \leq 0$. □

**Theorem 66.** *Let $0 < \alpha < \frac{1}{2}$ and $m > \max\{\frac{6}{a}, \frac{2}{1-2a}\}$. Define*

$$\hat{\psi}_m = \hat{\varphi}_m(\gamma + \frac{3a}{4}) + \hat{\varphi}_m(\gamma - \frac{3a}{4}) \in C_c^\infty(\widehat{\mathbb{R}}).$$

*Then $\psi_m \in \mathcal{S}(\mathbb{R})$ and $\mathcal{W}(\psi_m)$ forms a frame with bounds $A_m$ and $B_m$. For all $m$, $A_m \leq \frac{1}{2}$, but as $m \to \infty$, $B_m \to 1$.*

*Proof.* As $m > \frac{4}{a}$, $\hat{\varphi}_m$ is well-defined, and it follows from the definition of $\hat{\varphi}_m$ that

$$\operatorname{supp} \hat{\psi}_m = [-a - \frac{1}{m}, -\frac{a}{2} + \frac{1}{m}] \cup [\frac{a}{2} - \frac{1}{m}, a + \frac{1}{m}].$$

Since $m > \frac{2}{1-2a}$, $\operatorname{supp}\hat{\psi}_m \subset (-\frac{1}{2}, \frac{1}{2})$. By continuity, there exists $\epsilon > 0$ such that $[-a, -\frac{a}{2}) \cup [\frac{a}{2}, a) \subseteq \operatorname{supp}_\epsilon \hat{\psi}_m$. Thus, it follows from Theorem 33 that $\mathcal{W}(\psi_m)$ forms a frame with lower frame bound $A_m = \underline{K}_{\psi_m}$ and upper frame bound $B_m = \overline{K}_{\psi_m}$. As $\hat{\psi}_m$ is even, it suffices to optimize $\kappa_{\psi_m}$ over any positive dyadic interval. We shall use $[\frac{a}{2} + \frac{1}{2}, a + \frac{1}{m})$. Since $m > \frac{5}{a}$, $\kappa_{\psi_m}(\gamma) = (\hat{\psi}_m(\gamma))^2 + (\hat{\psi}_m(\frac{\gamma}{2}))^2$ for $\gamma \in [\frac{a}{2} + \frac{1}{2m}, a + \frac{1}{m})$. Also $m > \frac{6}{a}$ implies that $a - \frac{2}{m} > \frac{a}{2} + \frac{1}{m}$. Hence, over $[\frac{a}{2} + \frac{1}{2m}, a + \frac{1}{m})$,

$$
\begin{aligned}
\kappa_{\psi_m} &= (\hat{\psi}_m(\gamma))^2 + (\hat{\psi}_m(\frac{\gamma}{2}))^2 \\
&= \begin{cases}
(\hat{\psi}_m(\gamma))^2 + 0 & \text{for } \frac{a}{2} + \frac{1}{2m} \leq \gamma < \frac{a}{2} + \frac{1}{m} \\
1 + 0 & \text{for } \frac{a}{2} + \frac{1}{m} \leq \gamma < a - \frac{2}{m} \\
1 + (\hat{\psi}_m(\frac{\gamma}{2}))^2 & \text{for } a - \frac{2}{m} \leq \gamma < a - \frac{1}{m} \\
(\hat{\psi}_m(\gamma))^2 + (\hat{\psi}_m(\frac{\gamma}{2}))^2 & \text{for } a - \frac{1}{m} \leq \gamma < a + \frac{1}{m}
\end{cases} \\
&= \begin{cases}
(\hat{\varphi}_m(\gamma - \frac{3a}{4}))^2 & \text{for } \frac{a}{2} + \frac{1}{2m} \leq \gamma < \frac{a}{2} + \frac{1}{m} \\
1 & \text{for } \frac{a}{2} + \frac{1}{m} \leq \gamma < a - \frac{2}{m} \\
1 + (\hat{\varphi}_m(\frac{\gamma}{2} - \frac{3a}{4}))^2 & \text{for } a - \frac{2}{m} \leq \gamma < a - \frac{1}{m} \\
(\hat{\varphi}_m(\gamma - \frac{3a}{4}))^2 + (\hat{\psi}(\frac{\gamma}{2} - \frac{3a}{4}))^2 & \text{for } a - \frac{1}{m} \leq \gamma < a + \frac{1}{m}
\end{cases}.
\end{aligned}
$$

Note that

- $\gamma - \frac{3a}{4} < 0$ for $\frac{a}{2} + \frac{1}{2m} \leq \gamma < \frac{a}{2} + \frac{1}{m}$ since $m > \frac{4}{a}$,
- $\frac{\gamma}{2} - \frac{3a}{4} < 0$ for $a - \frac{2}{m} \leq \gamma < a - \frac{1}{m}$,
- $\gamma - \frac{3a}{4} > 0$ for $a - \frac{1}{m} \leq \gamma < a + \frac{1}{m}$ since $m > \frac{4}{a}$, and
- $\frac{\gamma}{2} - \frac{3a}{4} < 0$ for $a - \frac{1}{m} \leq \gamma < a + \frac{1}{m}$ since $m > \frac{2}{a}$.

Thus, $\kappa_{\psi_m}$ is increasing over $\frac{a}{2} + \frac{1}{2m} \leq \gamma \leq a - \frac{1}{m}$, but is not monotonic over $a - \frac{1}{m} < \gamma < a + \frac{1}{m}$. Hence

$$\begin{aligned}
\min_{\frac{a}{2}+\frac{1}{2} \leq \gamma \leq a - \frac{1}{m}} \kappa_{\psi_m}(\gamma) &= \kappa_{\psi_m}(\frac{a}{2} + \frac{1}{2}) \\
&= \left(\hat{\varphi}_m((\frac{a}{2} + \frac{1}{2m}) - \frac{3a}{4})\right)^2 \\
&= \left(\hat{\varphi}_m(-\frac{a}{4} + \frac{1}{2m})\right)^2 \\
&= \left(\frac{e^{-2m/3}}{e^{-2m/3} + e^{-2m}}\right)^2 \\
&= \left(\frac{1}{1 + e^{-4m/3}}\right)^2,
\end{aligned}$$

and

$$\begin{aligned}
\max_{\frac{a}{2}+\frac{1}{2} \leq \gamma \leq a - \frac{1}{m}} \kappa_{\psi_m}(\gamma) &= \kappa_{\psi_m}(a - \frac{1}{m}) \\
&= 1 + \left(\hat{\varphi}_m(\frac{1}{2}(a - \frac{1}{m}) - \frac{3a}{4})\right)^2 \\
&= 1 + \left(\frac{e^{-2m}}{e^{-2m} + e^{-2m/3}}\right)^2 \\
&= 1 + \left(\frac{1}{1 + e^{4m/3}}\right)^2.
\end{aligned}$$

Note that as $m \to \infty$,

$$\begin{aligned}
\min_{\frac{a}{2}+\frac{1}{2} \leq \gamma \leq a - \frac{1}{m}} \kappa_{\psi_m}(\gamma) &\to 1 \\
\max_{\frac{a}{2}+\frac{1}{2} \leq \gamma \leq a - \frac{1}{m}} \kappa_{\psi_m}(\gamma) &\to 1
\end{aligned}$$

We shall now consider $\kappa_{\psi_m}$ over $(a - \frac{1}{m}, a + \frac{1}{m})$. We start by substituting $\gamma = a + \frac{t}{m}$,

$t \in (-1,1)$ and expanding $\kappa_{\psi_m}$ over these values:

$$\begin{aligned}
\kappa_{\psi_m}(a+\frac{t}{m}) &= \left(\hat{\varphi}_m(a+\frac{t}{m}-\frac{3a}{4})\right)^2 + \left(\hat{\varphi}_m(\frac{a}{2}+\frac{t}{2m}-\frac{3a}{4})\right)^2 \\
&= \left(\hat{\varphi}_m(\frac{a}{4}+\frac{t}{m})\right)^2 + \left(\hat{\varphi}_m(-\frac{a}{4}+\frac{t}{2m})\right)^2 \\
&= \left(\frac{e^{-m/(1-t)}}{e^{-m/(1-t)}+e^{-m/(1+t)}}\right)^2 + \left(\frac{e^{-2m/(2+t)}}{e^{-2m/(2+t)}+e^{-2m/(2-t)}}\right)^2 \\
&= \left(\frac{1}{1+e^{2mt/(1-t^2)}}\right)^2 + \left(\frac{1}{1+e^{-4mt/(4-t^2)}}\right)^2
\end{aligned}$$

At $t=0$, $\kappa_{\psi_m}$ takes the value $\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2=\frac{1}{2}$. We claim that for any $0<\delta<\frac{1}{2}$, $\kappa_{\psi_m}(a+\frac{t}{m})$ converges uniformly to 1 over $[\delta,1-\delta]$. Choose an arbitrary $0<\epsilon<2$. We claim that for any

$$m > \max\left\{\frac{(1-\delta^2)\ln(\sqrt{\frac{2}{\epsilon}}-1)}{\delta}, \frac{(\delta^2-4)\ln(1-\sqrt{1-\frac{\epsilon}{2}})}{4\delta}\right\}$$

$|\kappa_{\psi_m}(a+\frac{t}{m})-1|<\epsilon$ for all $t\in[\delta,1-\delta]$. A routine application of the triangle inequality yields

$$\begin{aligned}
|\kappa_{\psi_m}(a+\frac{t}{m})-1| &= \left|\left(\frac{1}{1+e^{2mt/(1-t^2)}}\right)^2 + \left(\frac{1}{1+e^{-4mt/(4-t^2)}}\right)^2 - 1\right| \\
&\leq \left|\left(\frac{1}{1+e^{2mt/(1-t^2)}}\right)^2\right| + \left|\left(\frac{1}{1+e^{-4mt/(4-t^2)}}\right)^2 - 1\right|.
\end{aligned}$$

Since $m>\frac{(1-\delta^2)\ln(\sqrt{\frac{2}{\epsilon}}-1)}{\delta}$, for all $t\in[\delta,1-\delta]$,

$$\begin{aligned}
& \sqrt{\frac{2}{\epsilon}}-1 < e^{2m\delta/(1-\delta^2)} \leq e^{2mt/(1-t^2)} \\
\Rightarrow\ & \frac{2}{\epsilon} < (1+e^{2mt/(1-t^2)})^2 \\
\Rightarrow\ & \left|\left(\frac{1}{1+e^{2mt/(1-t^2)}}\right)^2\right| < \frac{\epsilon}{2}.
\end{aligned}$$

Since $m > \frac{(\delta^2-4)\ln(1-\sqrt{1-\frac{\epsilon}{2}})}{4\delta}$,

$$\begin{aligned}
& 1-\sqrt{1-\frac{2}{\epsilon}} > e^{-4m\delta/(4-\delta^2)} \geq e^{-4mt/(4-t^2)} \\
\Rightarrow \quad & (e^{-4mt/(4-t^2)})^2 - 2(e^{-4mt/(4-t^2)}) + \frac{\epsilon}{2} > 0 \\
\Rightarrow \quad & |(e^{-4mt/(4-t^2)})^2 - 2(e^{-4mt/(4-t^2)})| < \frac{\epsilon}{2} \\
\Rightarrow \quad & |1-(1+e^{-4mt/(4-t^2)})^2| < \frac{\epsilon}{2} \\
\Rightarrow \quad & \left|\frac{1-(1+e^{-4mt/(4-t^2)})^2}{(1+e^{-4mt/(4-t^2)})^2}\right| < \frac{\epsilon}{2} \\
\Rightarrow \quad & \left|\frac{1}{(1+e^{-4mt/(4-t^2)})^2} - 1\right| < \frac{\epsilon}{2}.
\end{aligned}$$

Thus, $|\kappa_{\psi_m}(a+\frac{t}{m})-1| < \epsilon$ for all $t \in [\delta, 1-\delta]$. It is also true that for any $0 < \delta < \frac{1}{2}$, $\kappa_{\psi_m}(a+\frac{t}{m})$ converges uniformly to 1 over $[-1+\delta, -\delta]$. The proof works in the same manner, except the triangle inequality is used in the following way

$$\begin{aligned}
|\kappa_{\psi_m}(a+\frac{t}{m})-1| &= \left|\left(\frac{1}{1+e^{2mt/(1-t^2)}}\right)^2 + \left(\frac{1}{1+e^{-4mt/(4-t^2)}}\right)^2 - 1\right| \\
&\leq \left|\left(\frac{1}{1+e^{2mt/(1-t^2)}}\right)^2 - 1\right| + \left|\left(\frac{1}{1+e^{-4mt/(4-t^2)}}\right)^2\right|.
\end{aligned}$$

Combining this convergence with our knowledge of the values $\kappa_{\psi_m}(0) = \frac{1}{2}$, $\kappa_{\psi_m}(a+\frac{1}{m}) = \left(\frac{1}{1+e^{-4m/3}}\right)^2$, and $\kappa_{\psi_m}(a-\frac{1}{m}) = 1+\left(\frac{1}{1+e^{4m/3}}\right)^2$, we conclude that

$$\lim_{m\to\infty} B_m = \lim_{m\to\infty} \overline{K}_{\psi_m} = 1.$$

$\square$

Since the $\frac{t}{1-t^2}$ exponent makes computer calculations nearly impossible, these smooth frames are interesting, but not usable in applications. As there is no upper frame bound gap, it follows from Theorem 48 that these functions which are commonly used in mathematics are not the result of convolutional smothing.

## 3.5 Other Methods

### 3.5.1 MSF smoothing

As mentioned above, E. Hernàndez, X. Wang, and G. Weiss, [69], created the theory of MSF wavelets. They characterize wavelets $\psi$ for which $\hat{\psi}$ has support in $[-\frac{8}{3}\alpha, 2-\frac{4}{3}\alpha]$, for $0 < \alpha \leq \frac{1}{2}$, and prove that these are all associated with a multiresolution analysis (MRA). The authors then smoothed these MSF wavelets [70]. Their smoothing procedure was accomplished by deforming given low-pass filters to obtain new filters. This process sometimes results in non-bandlimited orthonormal wavelets.

This process will not work to improve the frame wavelet set wavelets which were created in [14] because this process relies heavily on the associated MRA structure, which requires an orthonormal basis. Further, some of the sets generated in the neighborhood mapping construction do not have support lying in $[-\frac{8}{3}\alpha, 2-\frac{4}{3}\alpha]$, for some $0 < \alpha \leq \frac{1}{2}$. For example, $K_1 \backslash A_1$ in the construction of the 1-$d$ Journé set is $[-\frac{9}{4}, -2) \cup [-\frac{1}{2}, -\frac{9}{32}) \cup [\frac{9}{32}, \frac{1}{2}) \cup [2, \frac{9}{4})$. The support of this set is spread due to the fact that it is approximating a wavelet set which is not associated with an MRA.

### 3.5.2 Baggett, Jorgensen, Merrill, and Packer smoothing

A different smoothing idea is employed in [5]. The authors smooth the 1-$d$ Journé wavelet using a Generalized Multiresolution Analysis. Frame wavelets for

$L^2(\mathbb{R})$ are constructed which have the same dimension function,

$$\sum_{k\in\mathbb{Z}}\sum_{n=1}^{\infty}|\hat{\psi}\left(2^n\left(x+k\right)\right)|^2,$$

as the Journé wavelet set but are arbitrarily differentiable and have $C^\infty$ Fourier transforms. As in [70], they do not regularize the members of the frame directly but rather define auxillary functions which build wavelets sharing certain traits with the original wavelet. Since they construct Parseval frame wavelets, we know from Theorem 48 that their functions cannot result from convolutional smoothing on the frequency domain. They note that if $\psi$ is a frame wavelet contructed using the methods of [5], then $\{T_k\psi : k \in \mathbb{Z}\}$ does not form a frame for $\overline{\text{span}}\{T_k\psi : k \in \mathbb{Z}\}$. They reference [13] in their comment. The following theorem from [13] may be used to prove the same result for frames found using our methods.

**Proposition 67.** *Let $\psi \in L^2(\mathbb{R})$ and let $\Psi(\gamma) = \sum_{k\in\mathbb{Z}}|\hat{\psi}(\gamma+k)|^2 \in L^\infty(\mathbb{T})$. The sequence $\{T_k\psi : k \in \mathbb{Z}\}$ is a frame for $\overline{\text{span}}\{T_k\psi : k \in \mathbb{Z}\}$ if and only if there are positive constants $A$ and $B$ such that*

$$A \le \Psi(\gamma) \le B \qquad a.e.\ on\ \mathbb{T}\backslash N,$$

*where $N = \{\gamma \in \mathbb{T} : \Psi(\gamma) = 0\}$.*

**Proposition 68.** *Let $\psi$ be as in Corollary 36. Then $\{T_k\psi : k \in \mathbb{Z}^d\}$ is not a frame for $\overline{\text{span}}_{k\in\mathbb{Z}^d}T_k\psi$.*

*Proof.* Since $\psi$ is the result of a convolution, it is continuous. Let $\mathbf{\frac{1}{2}}$ denote the vector in $\widehat{\mathbb{R}}^d$ for which every component is $\frac{1}{2}$. Then $\Psi(\mathbf{\frac{1}{2}}) = \sum_{k\in\mathbb{Z}}|\hat{\psi}(\mathbf{\frac{1}{2}}+k)|^2 = 0$

by the construction of $\psi$. Since $\hat{\psi}$ is continuous, for any $\epsilon > 0$ which is sufficiently small, there exists a $\gamma \in \widehat{\mathbb{R}}^d$ such that $\epsilon = \Psi(\gamma)$. The conclusion now follows from Proposition 67. $\square$

### 3.5.3 Smoothing by time domain convolution

In [1], the authors make use of the following two results in order to smooth the Haar wavelet by means of convolution on the time domain.

**Proposition 69** ([50])**.** *Let $f$ be a bounded variation function with total variation $V(f)$. Assume that* $\operatorname{supp} f \subseteq I$*, where $I$ is an interval of length less than* 1*, and* $\int f(t)dt = 0$*. Then $\mathcal{W}(f)$ is a Bessel sequence, with bound*

$$M = 11\|f\|_\infty \left(V(f) + \|f\|_\infty\right)|I|.$$

**Theorem 70.** *If $\{e_n : n \in \mathbb{N}\}$ is an orthonormal basis for $L^2(\mathbb{R})$ and $\{e_n - f_n : n \in \mathbb{N}\}$ is a Bessel sequence with bound $M < 1$, then $\{f_n : n \in \mathbb{N}\}$ is a Riesz basis with bounds $A$ and $B$ satisfying $A \geq \left(1 - \sqrt{M}\right)^2$ and $B \leq \left(1 + \sqrt{M}\right)^2$.*

The main idea of [1] is to convolve the Haar function $\psi = \mathbb{1}_{[0,1/2)} - \mathbb{1}_{[1/2,1)}$ with a function $\phi \in W^{1,j}(\mathbb{R}) = \{\phi \in L^1(\mathbb{R}) : \phi^{(j)} \in L^1(\mathbb{R})\}$. This results in a $C^{(m)}(\mathbb{R})$ function $\psi * \phi$. If $\phi$ has additional properties, then Proposition 69 can be used to show that $\mathcal{W}(\psi - \psi * \phi)$ is a Bessel sequence with bound $M < 1$. It follows from Theorem 70 that $\psi * \phi$ is a $C^{(m)}(\mathbb{R})$ function which generates a Riesz basis with bounds $\left(1 - \sqrt{M}\right)^2$ and $\left(1 + \sqrt{M}\right)^2$. Thus, convolutional smoothing on the temporal domain does not yield frame bound gaps, in contrast to smoothing on the frequency domain.

### 3.5.4 Operator interpolation

Let $K$ and $L$ be (orthonormal) wavelet sets. By Proposition 12, $K$ and $L$ are $\mathbb{Z}^d$-translation congruent and tile $\widehat{\mathbb{R}}^d$ by dyadic dilation. Dai and Larson use these facts to construct a unitary operator $U$ on $L^2(\widehat{\mathbb{R}}^d)$ in [33]. If the group generated by $U$ commutes with the Fourier transformed dilation and translation operators when applied to $\mathbb{1}_K$, then the wavelet sets admit *operator interpolation*. This means that polynomials in $U$, with coefficients satisfying certain conditions, applied to $\mathbb{1}_K$ yield the Fourier transform of a mother wavelet. If $K$ and $L$ satisfy futher conditions, then this interpolated wavelet may also be continuous in the frequency domain. In [64], this process is extended to Parseval (sub-) frame wavelet sets. The "sub-" prefix means that the generated sequence of functions form a Parseval frame for their span, rather than necessarily for all of $L^2(\mathbb{R}^d)$. The loosened restrictions allow for $K$ and $L$ to be Parseval sub-frame wavelet sets which are $\mathbb{Z}^d$-translation congruent. While the sets $K_m \backslash A_m$ from the neighborhood mapping construction are Parseval frame wavelets, operator interpolation does not seem to be a viable method to obtain frame wavelets with good decay from these sets. Initially, if we fix a neighborhood $K_0$ and map $S$, then $K_{m_1} \backslash A_{m_1}$ and $K_{m_2} \backslash A_{m_2}$ will have different measure if $m_1 \neq m_2$. Hence, these sets can not be $\mathbb{Z}^d$-translation congruent. If we look to sets created using different neighborhoods $K_0$ and $\tilde{K}_0$ and maps $S$ and $\tilde{S}$, we may obtain Parseval frame wavelet sets $K_{m_1} \backslash A_{m_1}$ and $\tilde{K}_{m_2} \backslash \tilde{A}_{m_2}$ which have equal measure and thus may potentially be $\mathbb{Z}^d$-translation congruent. However, calculations have not yet yielded any pairs of these equal measure sets which are actually $\mathbb{Z}^d$-translation congruent. Hence, we are unable to interpolate between the corresponding Parseval frame wavelets. Although operator interpoliation is a clever application of von Neumann algebra theory to regularization of (sub-frame) wavelet set wavelets, it is not helpful in our endeavor.

### 3.5.5 Stability results and an error in [46]

While attempting to modify the methods of [1] in order to use them, we considered a number of various stability results. Stability results give conditions under which perturbations of a frame or Riesz basis is again a frame or Riesz basis. Theorem 70 is a very special case of the following stability proven by Chistensen and Heil for Banach frames [25]. The result as it applies to Hilbert spaces has a remarkably simple proof which uses the triangle inequality.

**Theorem 71.** *Let $\{e_n\}$ be a frame for a Hilbert space $\mathcal{H}$ with frame bounds $A$ and $B$. Let $\{f_n\} \subseteq \mathcal{H}$. If $\{e_n - f_n\}$ is a Bessel sequence for $\mathcal{H}$ with bound $M < A$, then $\{f_n\}$ is a frame with frame bounds $\tilde{A}$ and $\tilde{B}$ satisfying $A\left(1 - \sqrt{M/A}\right)^2 \leq \tilde{A}$ and $\tilde{B} \leq B\left(1 + \sqrt{M/B}\right)^2$.*

While the hypothesis for this result is much weaker than many pre-existing basis-type assumptions and, thus, is more applicable, the use of the triangle inequality in its proof means that, in general, the constants in Theorem 71 are not optimal. In fact, convolving $K_0 \backslash A_0$ from the 2-$d$ Journé construction with elements of an approximate identity failed to yield bounds $M < 1$, meaning that Theorem 71 cannot be applied at all in this case. Fortunately, Corollary 19 provides frame bound estimation that is simple to compute. There are other stability results in [46], but, unfortunately, one of the major results of the paper is incorrect.

**Definition 72.** $\{D_j T_k \phi\}$ is *semiorthogonal* if for any $j \neq m$ and any $k, n \in \mathbb{Z}^d$, $\langle D_j T_k \phi, D_m T_n \phi \rangle = 0$.

They write in [46]

> Theorem 10: Let $\Phi$ be a semiorthogonal sequence in $L^2(\mathbb{R}^d)$, and let $\psi$ be any function in the closure of the linear span of $\{T_k \phi : k \in \mathbb{Z}^d\}$. If $\Phi$ is a wavelet frame (wavelet Riesz basis) in $L^2(\mathbb{R}^d)$ with bounds $A$ and $B$ and $\|\psi - \phi\| < A^{3/2}B$, then $\Psi = \{\phi_{j,k} = D_j T_k \phi : j, k \in \mathbb{Z}^d\}$ is a semiorthogonal wavelet frame wavelet (wavelet Riesz basis) in $L^2(\mathbb{R}^d)$ with bounds $\left(\left[1 - \left(\frac{B}{A^{3/2}}\right)\right] \|\psi - \phi\|\right)^2 A$ and $\left(\left[1 + \left(\frac{B^{1/2}}{A}\right)\right] \|\psi - \phi\|\right)^2 B$.

Obviously, they mean $j \in \mathbb{Z}$. Also, there is a typo in the statement of the new bounds. Since $B \geq A$, the bounds as stated would always yield lower frame bounds $\leq 0$. Based on the results that they cited in the proof, they meant to write $\left(1 - \left(\frac{B}{A^{3/2}}\right) \|\psi - \phi\|\right)^2 A$ and $\left(1 + \left(\frac{B^{1/2}}{A}\right) \|\psi - \phi\|\right)^2 B$.

> Proof: The semiorthogonality is trivial. Let $g = \phi - \psi$, and let $S$ be the frame operator, i.e.:
>
> $$Sf = \sum_{j \in \mathbb{Z}^d} \sum_{k \in \mathbb{Z}^d} \langle f, \phi_{j,k} \rangle \phi_{j,k}.$$

Again the $j$ should lie in $\mathbb{Z}$.

> The hypotheses imply that if $j \neq 0$, then $\langle g, \phi_{j,k} \rangle = 0$. Thus, $g = \sum_{m \in \mathbb{Z}^d} a_m \phi_{0,m}(x)$, for $a_m = \langle S^{-1} g, \phi_{0,m} \rangle$.

While it is true that $g = \sum_n \sum_m \langle S^{-1}g, \phi_{n,m}\rangle \phi_{n,m}$, there is no reason for $\langle S^{-1}g, \phi_{m,n}\rangle = 0$ for all $n \neq 0$. If the rest of the proof were correct, then this problem would most likely be fixed by writing $g = \sum_{m\in\mathbb{Z}^d} b_m S^{-1}\phi_{0,m}$, for $b_m = \langle g, \phi_{0,m}\rangle$.

Let $\{c_{n,k}\}$ be any finite sequence of scalars, then

$$\begin{aligned} \left\|\sum_j \sum_k c_{j,k} g_{j,k}\right\|^2 &= \left\|\sum_j \sum_k \sum_{m\in\mathbb{Z}^d} a_m c_{j,k} \phi_{m,j+k}\right\|^2 \\ &\leq B \sum_j \sum_k \sum_m |a_m c_{j,k}|^2 \\ &= B \sum_m |a_m|^2 \sum_j \sum_k |c_{j,k}|^2 . \end{aligned}$$

Initially, there is an error in the indices, $\left\|\sum_j \sum_k \sum_{m\in\mathbb{Z}^d} a_m c_{j,k}\phi_{m,j+k}\right\|^2$ should be $\left\|\sum_j \sum_k \sum_{m\in\mathbb{Z}^d} a_m c_{j,k}\phi_{j,m+k}\right\|^2$. More crucially, the authors miss the subtle fact that $m$ and $k$ are taking values all over $\mathbb{Z}^d$, so $m_1+k_1 = m_2+k_2$ for numerous values of $m_i, k_i$. Using the substitution $n = m+k$, the correct inequality would be

$$\begin{aligned} \left\|\sum_j \sum_k \sum_{m\in\mathbb{Z}^d} a_m c_{j,k} \phi_{j,m+k}\right\|^2 &= \left\|\sum_j \sum_n \sum_{m\in\mathbb{Z}^d} a_m c_{j,n-m} \phi_{j,n}\right\|^2 \\ &\leq B \sum_j \sum_n \left|\sum_m c_{j,n-m} a_m\right|^2 . \end{aligned}$$

Since in general $|\sum_m x_m|^2 \not\leq \sum_m |x_m|^2$, their desired result does not follow. Their proof isn't repairable, which of course doesn't mean that the theorem is also incorrect; however, calculations using other methods indicate that a class of functions which satisfy the hypotheses do not form frames. Let $\Phi = \mathcal{W}(\mathbb{1}_{[-1,-1/2)\cup[1/2,1)})$. $\Phi$ is an orthonormal basis and is thus trivially a semiorthogonal Riesz basis with frame bound 1. Set

$$\hat{\psi} = (\frac{1}{2}I - \frac{1}{2}M_2)\mathbb{1}_{[-1,-1,2)\cup[1/2,1)},$$

then $\psi \in \operatorname{span}\{T_k \mathbb{1}_{[-1,-1,2)\cup[1/2,1)}\}$. Also,

$$\begin{aligned}
\|\psi - \mathbb{1}^{\vee}_{[-1,-1/2)\cup[1/2,1)}\|^2 &= \|\hat{\psi} - \mathbb{1}_{[-1,-1/2)\cup[1/2,1)}\|^2 \\
&= \left\|\left((\frac{1}{2} - \frac{1}{2}e^{4\pi i x}) - 1\right) \mathbb{1}_{[-1,-1/2)\cup[1/2,1)}(x)\right\|^2 \\
&= \int_{-1}^{-1/2} \frac{1}{4}(1 + 2\cos 4\pi i x + 1)dx + \int_{1/2}^{1} \frac{1}{4}(1 + 2\cos 4\pi i x + 1)dx \\
&= \frac{1}{2}(-\frac{1}{2} - (-1) + 1 - \frac{1}{2}) \\
&= \frac{1}{2} \\
&< 1.
\end{aligned}$$

Thus, the hypotheses of Theorem 10 in [46] are satisfied. However, $\psi$ is a continuous function which vanishes at every point $\gamma = \pm\frac{1}{2^k}$. Thus $\kappa_\psi$ is also a continuous function which vanishes at every point $\gamma = \pm\frac{1}{2^k}$, and

$$\inf_{\gamma \in \widehat{\mathbb{R}}} \kappa_\psi(\gamma) = 0.$$

It now follows from Proposition 18 that $\mathcal{W}(\psi)$ is not a frame.

### 3.5.6 Conclusion

Theorem 44 in Chapter 2 showed that convolutional smoothing of Parseval frame wavelet set wavelets on the frequency domain yields systems with upper frame bounds which increase away from 1 as the dimension increases. This theme of Chapter 2 and [11] is continued in this chapter. We see in Section 3.2.2 that natural generalizations of Bin Han's construction of smooth Parseval wavelets in $L^2(\mathbb{R})$ to $\mathbb{R}^2$ also fail. Thus, the question remains whether there exist continuous functions $\hat{\psi}_n$ for which $\mathcal{W}(\psi_n)$ has frame bounds converging to 1 and for which $\|\mathbb{1}_{[-a,a]^2\setminus[-a/2,a/2]^2} -$

$\hat{\psi}_n\|_{L^2(\widehat{\mathbb{R}}^2)}$ converges to 0 as $n \to \infty$ for some $a < 1/2$. Furthermore, we also explicitly construct smooth frame wavelets which have upper frame bounds converging to 1 in Section 3.3. This construction, combined with results from Chapter 2 shows that basic functions used in differential topology are not the result of convolutional smoothing.

# Chapter 4

# Shearlet Analogues for $L^2(\mathbb{R}^d)$

## 4.1 Introduction

### 4.1.1 Problem

Typically, multidimensional data has been analyzed using tensor products of 1-dimensional wavelets; however, these methods do not yield any information about directional components or trends. For example, if one image was a rotation of another image, we would like the coefficients of the wavelet representation to indicate that. A number of new representations have sprung up in an attempt to solve this problem. Contourlets ([42]), curvelets and ridgelets ([21]), bandlets ([89]), wedgelets ([43]), and shearlets ([57] and [80]) are just a few examples. There is a Fast Shearlet Transform, which makes the shearlets especially desirable. However, these transforms are for 2-dimensional data sets. It is becoming more common for higher dimensional data sets to appear, which need to be analyzed. Inspired by the work of Cordero, DeMari, Nowak, and Tabacco ([28], [28], [39], and [40]), we would like to exploit the representation theory of the extended metaplectic group in order to construct analogs of the shearlet transform for $L^2(\mathbb{R}^d)$, $d > 2$.

### 4.1.2 Shearlets

Shearlets were introduced by Labate, Guo, Kutyniok and Weiss in [57] and [80] and are a specific type of *composite dilation wavelet* ([58]).

**Definition 73.** Given $\psi \in L^2(\mathbb{R}^2)$, the *continuous shearlet system* is

$$\{T_y D_{(S_\ell A_a)^{-1}}\psi = a^{-3/4}\psi(A_a^{-1}S_\ell^{-1}(\cdot - y)) : a > 0, \ell \in \mathbb{R}, y \in \mathbb{R}^2\},$$

where $A_a$ is the parabolic scaling matrix $\begin{pmatrix} a & 0 \\ 0 & \sqrt{a} \end{pmatrix}$ and $S_\ell$ is the shearing matrix $\begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix}$.

*Shearlet bases and frames* are formed by selecting a discrete subcollection of a continuous shearlet system. The most common subcollection is generated by the indices

$$\{(a, \ell, y) = (4^j, k2^j, S_{k2^j}A_{4^j}m) : j, k \in \mathbb{Z}, m \in \mathbb{Z}^2\}.$$

By the commutation properties of the dilation and translation operators and shearing and parabolic scaling matrices,

$$\begin{aligned} T_{S_{k2^j}A_{4^j}m} D_{(S_{k2^j}A_{4^j})^{-1}} &= D_{(S_{k2^j}A_{4^j})^{-1}} T_m \\ &= D_{(A_{4^j}S_k)^{-1}} T_m, \end{aligned}$$

which looks exactly like the operators used to generate wavelets in Sections 2 and 3, with dilation by powers of 2 replaced with dilation by the products of matrix powers $A_{4^j}S_k$.

**Definition 74.** Let $\mathcal{B} = \{B_j\}_{j\in\mathcal{J}}$ and $\mathcal{C} = \{C_k\}_{k\in\mathcal{K}}$ be subsets of $\mathrm{GL}(d, \mathbb{R})$. Let $\psi \in L^2(\mathbb{R}^d)$. The *composite dilation wavelet system* corresponding to $\mathcal{B}$ and $\mathcal{C}$ is

$$\{D_{(B_jC_k)^{-1}}T_m\psi : j \in \mathcal{J}, k \in \mathcal{K}, m \in \mathbb{Z}^d\}.$$

A shearlet system is a composite dilation wavelet system with $B_j = A_{4^j}$ and $C_k = S_k$. Typically, more restraints are put on the sets $\mathcal{B}$ and $\mathcal{C}$. For example, it is common for $\mathcal{B}$ to be a collection of invertible matrices with integer entries and eigenvalues with moduli greater than 1 and for $\mathcal{C}$ to be a group of matrices, each with determinant 1. However, these constraints are not always necessary, see [58]. Many authors have published results about shearlets and composite dilation wavelets, which in some sense emulate wavelet theory and applications. To name a few examples, there is a shearlet multiresolution analysis theory and decomposition algorithm ([78]), there is an FFT-based method to compute the Continuous Shearlet Transform ([79]), and there are composite dilation wavelet sets ([16]). Many of these results should hopefully extend to the higher dimensional shearlets.

### 4.1.3 Outline and Results

Our goal is to integrate shearlet theory into the theory created by Cordero, DeMari, Nowak and Tabacco. To this end we also present some of the main results of Cordero *et al.* concerning reproducing subgroups of the metaplectic group and their connection to known function transforms in Section 4.2. In Section 4.3, we generalize the Translation-Dilation-Shearing group to $L^2(\mathbb{R}^d)$ for arbitrarily large $d$. The Translation-Dilation-Shearing group maps under a certain representation to a group of operators that are related to the operators used in the shearlet transform. We also fit the actual group of shearlet operators into the framework set up by Cordero *et al.* in Section 4.4. We conclude with Section 4.5, which contains references to other generalizations of shearlets and future directions of research.

## 4.2 Reproducing subgroups

### 4.2.1 Preliminaries

In a general sense, a reproducing formula is simply an integral representation of the identity on some function space. Reproducing formulas are used throughout mathematics and the sciences. The Cauchy integral formula of complex analysis ([51]), reconstruction in computed axial tomography ([87]), and resolutions of the identity are all examples of such formulas. We shall be particularly interested in the following types of reproducing formulas, which hold for all $f \in L^2(\mathbb{R}^d)$,

$$f = \int_H \langle f, \mu_e(h)\phi\rangle \mu_e(h)\phi dh, \tag{4.1}$$

where $H$ is a Lie subgroup of a particular Lie group, $\mu_e$ is a particular representation of that group, $\phi$ is a suitable window in $L^2(\mathbb{R}^d)$, and the integral is interpreted weakly. Equations which have the form of (4.1) arise in such areas as time-frequency analysis, wavelet theory, and quantum mechanics ([53], [47]). If such a $\phi$ exists, we shall call $H$ a *reproducing subgroup* and any function $\phi$ which satisfies Eqn (4.1) a *reproducing function* for $H$. We begin by introducing the Heisenberg group(s) and Schrödinger representation.

**Definition 75.** For $d \geq 1$, the *Heisenberg (Lie) group* $\mathbb{H}^d$ is a central extension of $\mathbb{R}^{2d}$ by $\mathbb{R}$, endowed with the manifold structure of $\mathbb{R} \times \mathbb{R}^{2d}$ and the product

$$(\lambda, x, \xi) \cdot (\tilde{\lambda}, \tilde{x}, \tilde{\xi}) = (\lambda + \tilde{\lambda} + \frac{1}{2}(\langle \tilde{x}, \xi \rangle - \langle x, \tilde{\xi} \rangle), x + \tilde{x}, \xi + \tilde{\xi}).$$

The Schrödinger representation $\rho$ is a unitary representation of the Heisenberg group. That is, $\rho$ is a strongly continuous group homomorphism from $\mathbb{H}^d$ to the unitary operators on $L^2(\mathbb{R}^d)$ defined by $\rho(\lambda, x, \xi) = e^{2\pi i \lambda} e^{\pi i \langle x, \xi \rangle} T_x M_\xi$.

We are now able to be more precise in our description of equation (4.1). We would like to first clarify of which Lie group $H$ is a subgroup. In order to do this, we must define the symplectic group.

**Definition 76.** For $d \geq 1$, let $\mathcal{J} = \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix}$. The *symplectic group* $\mathrm{Sp}(d, \mathbb{R})$ is the subgroup of $2d \times 2d$ matrices $g \in \mathrm{M}(2d, \mathbb{R})$ which satisfy ${}^t g \mathcal{J} g = \mathcal{J}$, where ${}^t g$ denotes the transpose of $g$.

We shall make a few quick comments concerning the pertinent representation theory. A detailed exposition may be found in [53]. Each symplectic matrix induces a unitary representation of the Heisenberg group which, by the Stone-von Neumann Theorem is equivalent to the Schrödinger representation. By considering the relationship between each symplectic matrix and the corresponding intertwining operators, one obtains a projective unitary representation of $\mathrm{Sp}(d, \mathbb{R})$. By passing to the double cover of the $\mathrm{Sp}(d, \mathbb{R})$, the *metaplectic group* $\mathrm{Mp}(d, \mathbb{R})$, one obtains an actual unitary representation $\mu$, the *metaplectic representation.* Matrix multiplication defines an action of $\mathrm{Sp}(d, \mathbb{R})$ in $\mathbb{R}^{2d}$ which in turn induces an action of

$\mathrm{Mp}(d, \mathbb{R})$ on $\mathbb{H}^d$. The metaplectic representation extends to the *extended metaplectic representation* $\mu_e$ on $\mathbb{H}^d \rtimes \mathrm{Mp}(d, \mathbb{R})$. We shall define $\mu_e$ to be

$$\mu_e(x, \xi, A) = \rho(\lambda, x, \xi)\mu(A), \quad (\lambda, x, \xi, A) \in \mathbb{H}^d \rtimes \mathrm{Mp}(d, \mathbb{R}). \tag{4.2}$$

We use the standard model of the metaplectic representation which is given by the following formulas:

$$\mu\begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix} f(x) = D_{A^{-1}} f(x), \quad A \in \mathrm{GL}(d, \mathbb{R}), \tag{4.3}$$

$$\mu\begin{pmatrix} I & 0 \\ C & I \end{pmatrix} f(x) = e^{i\pi\langle Cx, x\rangle} f(x), \quad C \in \mathrm{M}(d, \mathbb{R}), C = {}^tC \tag{4.4}$$

$$\mu(\mathcal{J}) = (-i)^{d/2}\mathcal{F}.$$

$\mathrm{Sp}(d, \mathbb{R})$ is generated by finite products of

$$\left\{\begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix} : A \in \mathrm{GL}(d, \mathbb{R})\right\}, \left\{\begin{pmatrix} I & 0 \\ C & I \end{pmatrix} : C \in \mathrm{M}(d, \mathbb{R}), C = {}^tC\right\}, \text{ and } \mathcal{J}$$

(see, [47]). It follows from Schur's Lemma that this definition of $\mu$ on generators of $\mathrm{Sp}(d, \mathbb{R})$ is ambiguous up to sign; however, for the $H$ we shall consider, (4.3) and (4.4) explicitly define a representation.

The $H$ in (4.1) shall be Lie subgroups of $\mathrm{Sp}(d, \mathbb{R})$ and $\mu_e$ defined as in (4.2). In Theorems 89 and 107, we shall prove which $\phi \in L^2(\mathbb{R}^d)$ satisfy Eqn (4.1) for given Lie subgroups $H$. The $H$ that we shall use are related to the shearlet transform.

### 4.2.2 Lie subgroups of $\mathbb{R}^2 \rtimes \mathrm{Sp}(1, \mathbb{R})$

DeMari and Nowak classified all connected Lie subgroups, up to inner conjugation, of $\mathbb{H} \rtimes \mathrm{Sp}(1, \mathbb{R})$ and $\mathbb{R}^2 \rtimes \mathrm{Sp}(1, \mathbb{R})$ in [40]. They then analyzed those groups in [39] and determined which ones were reproducing. It turns out that the reproducing subgroups correspond to the known transforms of $L^2(\mathbb{R})$, namely, the Gabor-Weyl-Heisenberg, wavelet and chirp transforms. One may hope that by characterizing the reproducing subgroups of $\mathbb{R}^{2d} \rtimes \mathrm{Sp}(d, \mathbb{R})$, new transforms for $L^2(\mathbb{R}^d)$ may be discovered. However, the structure of $\mathbb{R}^{2d} \rtimes \mathrm{Sp}(2d, \mathbb{R})$ is incredibly complicated for $d > 1$. For example, the affine (or $ax + b$) group embeds into $\mathbb{R}^{2d} \rtimes \mathrm{Sp}(d, \mathbb{R})$. The representation of the affine group over $\mathbb{R}$ as the product of translation and dilation operators splits into only two irreducible representations. However, the analogous representation for the affine group over $\mathbb{R}^d$, $d > 1$, is highly reducible ([53]). Also, in order to characterize the Lie subgroups of $\mathbb{R}^2 \rtimes \mathrm{Sp}(1, \mathbb{R})$, DeMari and Nowak used a characterization of the subgroups of $\mathrm{Sp}(1, \mathbb{R})$ that does not exist over higher dimensions. Thus, it may not be possible to characterize all of the reproducing subgroups of $\mathbb{R}^{2d} \rtimes \mathrm{Sp}(d, \mathbb{R})$ for $d > 1$. However, the theory may still be used in order to find new transformations. In [29] and [30], the authors prove that 2 different subgroups of $\mathbb{R}^4 \rtimes \mathrm{Sp}(2, \mathbb{R})$ are indeed reproducing. We will generalize one of those, the Translation-Dilation-Shearing group, to higher dimensions in the following section. There are some known dimension bounds for reproducing subgroups in $\mathbb{R}^{2d} \rtimes \mathrm{Sp}(d, \mathbb{R})$ for arbitrary $d$ ([28]). The bounds are sharp, but it is still not known whether reproducing subgroups exist of the intermediate dimensions between the upper and lower bounds, even when $d = 2$.

## 4.3 Translation-dilation-shearing group

We begin with the following definition from [29] and [28].

**Definition 77.** The *Translation-Dilation-Shearing Group* is

$$\left\{ A_{t,\ell,y} = \begin{pmatrix} t^{-1/2} S_{\ell/2} & 0 \\ t^{-1/2} B_y S_{\ell/2} & t^{1/2} ({}^t S_{-\ell/2}) \end{pmatrix} : t > 0, \ell \in \mathbb{R}, y \in \mathbb{R}^2 \right\},$$

where

$$B_y = \begin{pmatrix} 0 & y_1 \\ y_1 & y_2 \end{pmatrix}, \quad y = {}^t(y_1, y_2) \in \mathbb{R}^2; \quad S_\ell = \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix}, \quad \ell \in \mathbb{R}.$$

The matrix $S_\ell$ is called a *shearing matrix*.

This group is a reproducing subgroup of $Sp(2, \mathbb{R})$. We would like to generalize the Translation-Dilation-Shearing group to higher dimensions. Under this extension, a subgroup of $\mathrm{Sp}(1, \mathbb{R})$ which corresponds to wavelets in a certain way is a lower dimensional analogue of the Translation-Dilation-Shearing group.

**Definition 78.** We define the following collection of sets, which we shall show are Lie subgroups of $\mathrm{Sp}(k, \mathbb{R})$ for $k \geq 1$:

$$(\mathrm{TDS})_k = \left\{ A_{t,\ell,y} = \begin{pmatrix} t^{-1/2} S_{\ell/2} & 0 \\ t^{-1/2} B_y S_{\ell/2} & t^{1/2} ({}^t S_{-\ell/2}) \end{pmatrix} : t > 0, \ell \in \mathbb{R}^{k-1}, y \in \mathbb{R}^k \right\}$$

where

$$B_y = \left( \begin{cases} y_k & ; \quad i = j = k \\ y_j & ; \quad i = k, j < k \\ y_i & ; \quad i < k, j = k \\ 0 & ; \quad \text{else} \end{cases} \right)_{i,j} = \begin{pmatrix} 0 & 0 & \dots & y_1 \\ 0 & 0 & \dots & y_2 \\ \vdots & \vdots & \ddots & \vdots \\ y_1 & y_2 & \dots & y_k \end{pmatrix}, \quad y = {}^t(y_1, y_2, \cdots, y_k) \in \mathbb{R}^k.$$

For $k \geq 2$ and $\ell = {}^t(\ell_1, \ell_2, \dots \ell_{k-1}) \in \mathbb{R}^{k-1}$, $S_\ell$ is the shearing matrix

$$S_\ell = \left( \begin{cases} 1 & ; \quad i = j \\ \ell_j & ; \quad 1 \leq i \leq k-1, j = k \\ 0 & ; \quad \text{else} \end{cases} \right)_{i,j} = \begin{pmatrix} 1 & 0 & \dots & 0 & \ell_1 \\ 0 & 1 & \dots & 0 & \ell_2 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \dots & 1 & \ell_{k-1} \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

For $k = 1$, we formally define $S_\ell$ for $\ell \in \mathbb{R}^0$ to be simply 1.

Note that when $k = 1$,

$$(\mathrm{TDS})_1 = \left\{ A_{t,y} = \begin{pmatrix} t^{-1/2} & 0 \\ t^{-1/2} B_y & t^{1/2} \end{pmatrix} : t > 0, y \in \mathbb{R} \right\}.$$

We will use both $A_{t,y}$ and the formally defined $A_{t,\ell,y}$ as notation in the case $k = 1$ while hopefully keeping the exposition clear. Furthermore, when $k > 1$, as a geometric operator, $S_\ell$ fixes the $k$th dimension and stretches the first $k-1$ coordinates parallel to the $k$th coordinate. Mundane computations show that for $\ell, \tilde{\ell} \in \mathbb{R}^{k-1}$, $S_\ell S_{\tilde{\ell}} = S_{\ell+\tilde{\ell}}$ and for $y, \tilde{y} \in \mathbb{R}^k$, $B_y + B_{\tilde{y}} = B_{y+\tilde{y}}$.

**Proposition 79.** *For any $k \geq 1$, $(\mathrm{TDS})_k$ is a Lie subgroup of $\mathrm{Sp}(k, \mathbb{R})$ of dimension $2k$.*

*Proof.* We first show that $(\mathrm{TDS})_k \subseteq \mathrm{Sp}(k,\mathbb{R})$. Let 0 denote the matrix of all 0s of the appropriate dimensions.

$$\begin{aligned}
{}^tA_{t,\ell,y}\mathcal{J}A_{t,\ell,y} &= \begin{pmatrix} t^{-1/2}({}^tS_{\ell/2}) & t^{-1/2}({}^tS_{\ell/2})B_y \\ 0 & t^{1/2}S_{-\ell/2} \end{pmatrix}\begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix} A_{t,\ell,y} \\
&= \begin{pmatrix} -t^{-1/2}({}^tS_{\ell/2})B_y & t^{-1/2}({}^tS_{\ell/2}) \\ -t^{1/2}S_{-\ell/2} & 0 \end{pmatrix}\begin{pmatrix} t^{-1/2}S_{\ell/2} & 0 \\ t^{-1/2}B_yS_{\ell/2} & t^{-1/2}({}^tS_{-\ell/2}) \end{pmatrix} \\
&= \begin{pmatrix} -t^{-1}({}^tS_{\ell/2})B_yS_{\ell/2} + t^{-1}({}^tS_{\ell/2})B_yS_{\ell/2} & {}^tS_{\ell/2}{}^tS_{\ell/2} \\ -S_{-\ell/2}S_{\ell/2} & 0 \end{pmatrix} \\
&= \mathcal{J}.
\end{aligned}$$

We now show that $(\mathrm{TDS})_k$ is closed under multiplication. A quick computation shows that

$$A_{t,\ell,y}A_{\tilde{t},\tilde{\ell},\tilde{y}} = \begin{pmatrix} (t\tilde{t})^{-1/2}S_{\ell/2+\tilde{\ell}/2} & 0 \\ (t\tilde{t})^{-1/2}\left(B_yS_{\ell/2+\tilde{\ell}/2} + t({}^tS_{-\ell/2})B_{\tilde{y}}S_{\tilde{\ell}/2}\right) & (t\tilde{t})^{1/2}({}^tS_{-\ell/2-\tilde{\ell}/2}) \end{pmatrix}.$$

We would like to be able to write this in the form $A_{\bar{t},\bar{\ell},\bar{y}}$, where $\bar{t} > 0$, $\bar{\ell} \in \mathbb{R}^{k-1}$, $\bar{y} \in \mathbb{R}^k$. If we are able to do this, then $\bar{t} = t\tilde{t}$, $\bar{\ell} = \ell + \tilde{\ell}$ and $\bar{y}$ needs to be such that $B_yS_{\ell/2+\tilde{\ell}/2} + t({}^tS_{-\ell/2})B_{\tilde{y}}S_{\tilde{\ell}/2} = B_{\bar{y}}S_{\ell/2+\tilde{\ell}/2}$. We start with the base calculation

$$B_yS_\ell = \left(\begin{cases} y_i & ; \quad 1 \le i \le k-1, j = k \\ y_j & ; \quad i = k, 1 \le j \le k-1 \\ y_k + \sum_{m=1}^{k-1}\ell_m y_m & ; \quad i = j = k \\ 0 & ; \quad \text{else} \end{cases}\right)_{i,j}.$$

So

$$B_y S_{\ell/2+\tilde{\ell}/2} = \left( \begin{cases} y_i & ; \quad 1 \le i \le k-1, j = k \\ y_j & ; \quad i = k, 1 \le j \le k-1 \\ y_k + \sum_{m=1}^{k-1} \frac{\ell_m + \tilde{\ell}_m}{2} y_m & ; \quad i = j = k \\ 0 & ; \quad \text{else} \end{cases} \right)_{i,j}, \text{ and}$$

$$\begin{aligned} {}^t S_{-\ell/2} B_{\tilde{y}} S_{\tilde{\ell}/2} &= {}^t S_{-\ell/2} \left( \begin{cases} \tilde{y}_i & ; \quad 1 \le i \le k-1, j = k \\ \tilde{y}_j & ; \quad i = k, 1 \le j \le k-1 \\ \tilde{y}_k + \sum_{m=1}^{k-1} \frac{\tilde{\ell}_m}{2} \tilde{y}_m & ; \quad i = j = k \\ 0 & ; \quad \text{else} \end{cases} \right)_{i,j} \\ &= \left( \begin{cases} \tilde{y}_i & ; \quad 1 \le i \le k-1, j = k \\ \tilde{y}_j & ; \quad i = k, 1 \le j \le k-1 \\ \tilde{y}_k + \left( \sum_{m=1}^{k-1} \frac{\tilde{\ell}_m}{2} \tilde{y}_m \right) + \sum_{m=1}^{k-1} -\frac{\ell_m}{2} \tilde{y}_m & ; \quad i = j = k \\ 0 & ; \quad \text{else} \end{cases} \right)_{i,j}. \end{aligned}$$

Thus $B_y S_{\ell/2+\tilde{\ell}/2} + t({}^t S_{-\ell/2}) B_{\tilde{y}} S_{\tilde{\ell}/2}$

$$= \left( \begin{cases} y_i + t\tilde{y}_i & ; \quad 1 \le i \le k-1, j = k \\ y_j + t\tilde{y}_j & ; \quad i = k, 1 \le j \le k-1 \\ y_k + \sum_{m=1}^{k-1} \frac{\ell_m + \tilde{\ell}_m}{2} y_m + t \left( \tilde{y}_k + \sum_{m=1}^{k-1} \frac{-\ell_m + \tilde{\ell}_m}{2} \tilde{y}_m \right) & ; \quad i = j = k \\ 0 & ; \quad \text{else} \end{cases} \right)_{i,j}$$

$$= \left( \begin{cases} y_i + t\tilde{y}_i & ; \quad 1 \le i \le k-1, j = k \\ y_j + t\tilde{y}_j & ; \quad i = k, 1 \le j \le k-1 \\ (y_k + t\tilde{y}_k) + \left( \sum_{m=1}^{k-1} \frac{\ell_m + \tilde{\ell}_m}{2} (y_m + t\tilde{y}_m) \right) - 2\sum_{m=1}^{k-1} \frac{\ell_m}{2} t\tilde{y}_m & ; \quad i = j = k \\ 0 & ; \quad \text{else} \end{cases} \right)_{i,j}.$$

If $B_y S_{\ell/2+\tilde{\ell}/2} + t({}^t S_{-\ell/2}) B_{\tilde{y}} S_{\tilde{\ell}/2} = B_{\overline{y}} S_{\ell/2+\tilde{\ell}/2}$, then

$$\overline{y}_i = y_i + t\tilde{y}_i, \quad \text{for } 1 \leq i \leq k-1 \text{ and}$$

$$\overline{y}_k + \sum_{m=1}^{k-1} \frac{\ell_m + \tilde{\ell}_m}{2} \overline{y}_m = (y_k + t\tilde{y}_k) + \left( \sum_{m=1}^{k-1} \frac{\ell_m + \tilde{\ell}_m}{2} (y_m + t\tilde{y}_m) \right) - 2 \sum_{m=1}^{k-1} \frac{\ell_m}{2} t\tilde{y}_m$$
$$\Rightarrow \overline{y}_k = y_k + t\tilde{y}_m - \sum_{m=1}^{k-1} \ell_m t\tilde{y}_m.$$

So each $(\mathrm{TDS})_k$ corresponds to a subgroup of $\mathrm{Sp}(k, \mathbb{R})$ with a group operation parameterized by

$$(t, \ell, y)(\tilde{t}, \tilde{\ell}, \tilde{y}) = (t\tilde{t}, \ell + \tilde{\ell}, {}^t(y_1 + t\tilde{y}_1, \ldots, y_{k-1} + t\tilde{y}_{k-1}, y_k + t\tilde{y}_k - \sum_{m=1}^{k-1} \ell_m t\tilde{y}_m)),$$

when $k > 1$ and

$$(t, y)(\tilde{t}, \tilde{y}) = (t\tilde{t}, y + t\tilde{y}),$$

when $k = 1$, with identity the identity matrix corresponding to $(1, 0, 0)$ for any $k \geq 1$. Hence for each $k \geq 1$, $(\mathrm{TDS})_k$ is a matrix group. In particular, each $(\mathrm{TDS})_k$ is a Lie subgroup of $\mathrm{Sp}(k, \mathbb{R})$. Finally, $(\mathrm{TDS})_k$ is of dimension $1+(k-1)+k = 2k$. $\square$

We now define the first of three unitary representations of $(\mathrm{TDS})_k$, the *wavelet representation*. This representation is called the wavelet representation because each element of the Lie group is mapped to a product of a translation and a dilation operator.

**Proposition 80.** *The mapping $\nu$ defined on each $(\mathrm{TDS})_k$ by*

$$\nu(A_{t,\ell,y}) = T_y D_{t^{-1}({}^t S_\ell)}$$

*is a unitary representation.*

*Proof.* Let $y, \tilde{y} \in \mathbb{R}^k$, $t, \tilde{t} > 0$, $\ell, \tilde{\ell} \in \mathbb{R}^{k-1}$. We shall compute $(T_y D_{t^{-1}({}^tS_\ell)})(T_{\tilde{y}} D_{\tilde{t}^{-1}({}^tS_{\tilde{\ell}})})$. For $f \in L^2(\mathbb{R}^k)$,

$$
\begin{aligned}
(T_y D_{t^{-1}({}^tS_\ell)})(T_{\tilde{y}} D_{\tilde{t}^{-1}({}^tS_{\tilde{\ell}})}) f(x) &= \tilde{t}^{-k/2} T_y D_{t^{-1}({}^tS_\ell)} f(\tilde{t}^{-1}({}^tS_{\tilde{\ell}}(x - \tilde{y})) \\
&= (t\tilde{t})^{-k/2} f((t\tilde{t})^{-1}({}^tS_{\ell+\tilde{\ell}}(x - (y + t\,{}^tS_{-\ell}\tilde{y}))).
\end{aligned}
$$

We compute

$$
\begin{aligned}
{}^tS_{-\ell}\tilde{y} &= \begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \\ \vdots \\ \tilde{y}_k - \sum_{m=1}^{k-1} \ell_m \tilde{y}_m \end{pmatrix} \\
y + t\,{}^tS_{-\ell}\tilde{y} &= \begin{pmatrix} y_1 + t\tilde{y}_1 \\ y_2 + t\tilde{y}_2 \\ \vdots \\ y_k + t\tilde{y}_k - \sum_{m=1}^{k-1} \ell_m t\tilde{y}_m \end{pmatrix}
\end{aligned}
$$

Hence

$$
(T_y D_{t^{-1}({}^tS_\ell)})(T_{\tilde{y}} D_{\tilde{t}^{-1}({}^tS_{\tilde{\ell}})}) = T_{\overline{y}} D_{(t\tilde{t})^{-1}({}^tS_{\ell+\tilde{\ell}})},
$$

where $\overline{y}_i = y_i + t\tilde{y}_i$ for $1 \le i \le k-1$ and $\overline{y}_k = y_k + \tilde{y}_k - \sum_{m=1}^{k-1} \ell_m t\tilde{y}_m$. Thus $\nu(A_{t,\ell,y})\nu(A_{\tilde{t},\tilde{\ell},\tilde{y}}) = \nu(A_{t,\ell,y}A_{\tilde{t},\tilde{\ell},\tilde{y}})$. Strong continuity of the representation follows from the strong continuity of the translation and dilation operators. It follows that each $\nu$ is a unitary representation. $\square$

Thus, the $(\mathrm{TDS})_k$ groups generate composite dilation wavelets. The second representation is derived from the first.

**Definition 81.** Conjugate $\nu$ with $\mathcal{F}$ to obtain the $\pi$ representation of each $(\mathrm{TDS})_k$

$$\begin{aligned}\pi(A_{t,y})f(u) &= \mathcal{F}\circ\nu(A_{t,y})\circ\mathcal{F}^{-1}f(u) = e^{-2\pi iyu}D_tf(u), \quad k=1\\ \pi(A_{t,\ell,y})f(u) &= \mathcal{F}\circ\nu(A_{t,\ell,y})\circ\mathcal{F}^{-1}f(u) = e^{-2\pi i\langle y,u\rangle}D_{tS_{-\ell}}f(u), \quad k>1\end{aligned}$$

The final representation that we consider in this section is the metaplectic representation. We make use of the factorization

$$A_{t,\ell,y} = D_{t,\ell}L_{t,y,\ell} = \begin{pmatrix} t^{-1/2}S_{\ell/2} & 0 \\ 0 & t^{1/2}({}^tS_{-\ell/2}) \end{pmatrix}\begin{pmatrix} I & 0 \\ t^{-1}({}^tS_{\ell/2})B_yS_{\ell/2} & I \end{pmatrix}.$$

Then for $f\in L^2(\mathbb{R}^k)$ by Eqns (4.3) and (4.4),

$$\begin{aligned}\mu(A_{t,\ell,y})f(x) &= \left[\mu(D_{t,\ell})\mu(L_{t,y,\ell})f\right](x)\\ &= t^{k/4}\left[\mu(L_{t,y,\ell})f\right](t^{1/2}S_{-\ell/2}x)\\ &= t^{k/4}e^{-i\pi\langle {}^tS_{\ell/2}B_yx,S_{-\ell/2}x\rangle}f(t^{1/2}S_{-\ell/2}x)\\ &= t^{k/4}e^{-i\pi\langle B_yx,x\rangle}f(t^{1/2}S_{-\ell/2}x).\end{aligned}$$

We record the Haar measures of the $(\mathrm{TDS})_k$ for future use.

**Proposition 82.** *The left Haar measures, up to normalization, of the* $(\mathrm{TDS})_k$ *are* $d\tau = \frac{dt}{t^2}dy$ *for* $k=1$ *and* $d\tau = \frac{dt}{t^{k+1}}dyd\ell$ *for* $k>1$, *where* $dt$, $dy$ *and* $d\ell$ *are the Lebesgue measures over* $\mathbb{R}^+$, $\mathbb{R}^k$ *and* $\mathbb{R}^{k-1}$, *respectively.*

*Proof.* Assume that $k>1$. Let $f$ be integrable with respect to $d\tau$. We shall use the notation $f(t,\ell,y) = f(A_{t,\ell,y})$. Then

$$\begin{aligned}
\int_{(\mathrm{TDS})_k} f(A_{\tilde{t},\tilde{\ell},\tilde{y}}A_{t,\ell,y})\frac{dt}{t^{k+1}}dyd\ell &= \int_{\mathbb{R}^{k-1}}\int_{\mathbb{R}^k}\int_0^\infty f(\tilde{t}t,\tilde{\ell}+\ell,\tilde{y}+\tilde{t}({}^tS_{-\tilde{\ell}})y)\frac{dt}{t^{k+1}}dyd\ell \\
&= \int_{\mathbb{R}^{k-1}}\int_{\mathbb{R}^k}\int_0^\infty f(\overline{t},\tilde{\ell}+\ell,\tilde{y}+\tilde{t}({}^tS_{-\tilde{\ell}})y)\frac{1}{\tilde{t}}\frac{d\overline{t}}{(\overline{t}/\tilde{t})^{k+1}}dyd\ell \\
&= \int_{\mathbb{R}^{k-1}}\int_{\mathbb{R}^k}\int_0^\infty f(\overline{t},\tilde{\ell}+\ell,\overline{y})\frac{\tilde{t}^k d\overline{t}}{\overline{t}^{k+1}}\frac{d\overline{y}}{\tilde{t}^k}d\ell \\
&= \int_{\mathbb{R}^{k-1}}\int_{\mathbb{R}^k}\int_0^\infty f(\overline{t},\overline{\ell},\overline{y})\frac{d\overline{t}}{\overline{t}^{k+1}}d\overline{y}d\overline{\ell} \\
&= \int_{(\mathrm{TDS})_k} f(A_{\overline{t},\overline{\ell},\overline{y}})\frac{d\overline{t}}{\overline{t}^{k+1}}d\overline{y}d\overline{\ell} \\
&= \int_{(\mathrm{TDS})_k} f(A_{t,\ell,y})\frac{dt}{t^{k+1}}dyd\ell.
\end{aligned}$$

The $\ell$ component did not affect the $t$ or $y$ components. Thus, the calculation above also proves the claim for $k = 1$ as a degenerate case. $\square$

We shall now build a collection of auxiliary results in order to prove admissibility conditions for the $(\mathrm{TDS})_k$.

**Definition 83.** For $k = 1$ define the sets $\mathbb{R}_\pm = \dot{\mathbb{R}}_\pm = \{\pm x > 0\} \subset \mathbb{R}$, and for $k > 1$

$$\begin{aligned}
\dot{\mathbb{R}}^k_\pm &= \{(x_1, x_2, \ldots, x_k) : x_1, x_2, \ldots, x_{k-1} \neq 0, \pm x_k > 0\} \subset \mathbb{R}^k \quad \text{and} \\
\mathbb{R}^k_\pm &= \{(x_1, x_2, \ldots, x_k) : \pm x_k > 0\} \subset \mathbb{R}^k.
\end{aligned}$$

Further define the mapping $\Psi : \mathbb{R}^k \to \mathbb{R}^k$ by

$$x \mapsto (x_1x_k, x_2x_k, \ldots, x_{k-1}x_k, \frac{x_k^2}{2}) \text{ for } k \geq 2, \text{ and} \tag{4.5}$$

$$x \mapsto \frac{x^2}{2} \text{ for } k = 1. \tag{4.6}$$

The following proposition is an extension of Proposition 4.1 in [29].

**Proposition 84.** *The mappings defined on lines (4.5) and (4.6) restrict to diffeomorphisms* $\Psi : \dot{\mathbb{R}}^k_\pm \to \dot{\mathbb{R}}^k_+$ *and is such that* $\Psi(-x) = \Psi(x)$*. Further, they satisfy*

*a. The Jacobian of* $\Psi$ *at* $x = (x_1, x_2, \ldots, x_k) \in \dot{\mathbb{R}}^k_\pm$ *is* $J_\Psi(x) = x_k^k$*;*

*b. The Jacobian of* $\Psi^{-1} : \dot{\mathbb{R}}^k_+ \to \dot{\mathbb{R}}^k_+$ *at* $u = (u_1, u_2, \ldots, u_k) = \Psi(x_1, x_2, \ldots, x_k)$ *is* $J_{\Psi^{-1}}(u) = (2u_k)^{-k/2} = x_k^{-k}$*;*

*c.* $\Psi^{-1}(t^2 S_{2\ell} u) = tS_\ell \Psi^{-1}(u)$ *for every* $u \in \mathbb{R}^k_\pm$*; and*

*d.* $\langle B_y x, x\rangle = \langle 2y, \Psi(x)\rangle$ *for every* $x \in \dot{\mathbb{R}}^k_\pm$ *and every* $y \in \mathbb{R}^k$*.*

*Proof.* Clearly, $\Psi(-x) = \Psi(x)$. We calculate the matrices of the partials.

$$\frac{\partial \Psi(x)_i}{\partial x_j} = \begin{cases} x_k & ; \quad i = j \\ x_i & ; \quad j = k \\ 0 & ; \quad \text{else} \end{cases}$$

and

$$\frac{\partial \Psi^{-1}(u)_i}{\partial u_j} = \begin{cases} (2u_k)^{-1/2} & ; \quad i = j \\ -u_i(2u_k)^{-3/2} & ; \quad j = k, i < k \\ 0 & ; \quad \text{else} \end{cases}$$

Hence the Jacobian of $\Psi$ at $x$ is $\det\left(\frac{\partial \Psi(x)_i}{\partial x_j}\right)_{i,j} = x_k^k$, and the Jacobian of $\Psi^{-1}$ at $u = \Psi(x)$ is $\det\left(\frac{\partial \Psi^{-1}(u)_i}{\partial u_j}\right)_{i,j} = (2u_k)^{-k/2} = x_k^{-k}$. We now prove parts (c) and (d).

$$\begin{aligned} \Psi^{-1}(t^2 S_{2\ell} u) &= \Psi^{-1}(t^2({}^t(u_1 + 2\ell_1 u_k, \ldots, u_{k-1} + 2\ell_{k-1} u_k, u_k)) \\ &= {}^t(t^2(u_1 + 2\ell_1 u_k/\sqrt{2t^2 u_k}), \ldots, t^2(u_{k-1} + 2\ell_{k-1} u_k/\sqrt{2t^2 u_k}), \sqrt{t^2 u_k}) \\ &= t^t(\frac{u_1}{\sqrt{2u_k}} + \ell_1\sqrt{2u_k}, \ldots, \frac{u_{k-1}}{\sqrt{2u_k}} + \ell_{k-1}\sqrt{2u_k}, \sqrt{u_k}) \\ &= tS_\ell \Psi^{-1}(u). \end{aligned}$$

When $k = 1$, the preceding computation simply consists of the $k$th coordinate. Also,

$$\begin{aligned}
\langle B_y x, x\rangle &= {}^t(y_1 x_k, \ldots, y_{k-1}x_k, \sum_{i=1}^{k} y_i x_i) \\
&= \sum_{i=1}^{k-1} y_i x_i x_k + \sum_{i=1}^{k} y_i x_i x_k \\
&= 2\left[ (\sum_{i=1}^{k-1} y_i x_i x_k) + y_k(\frac{1}{2}x_k^2) \right] \\
&= \langle 2y, \Psi(x)\rangle.
\end{aligned}$$

Since $\Psi$ and $\Psi^{-1}$ are smooth, bijective self maps of $\dot{\mathbb{R}}^k_+$ with non-vanishing Jacobians, they are diffeomorphisms. By considering $-\Psi^{-1}$, we can see that $\Psi : \dot{\mathbb{R}}^k_- \to \dot{\mathbb{R}}^k_+$ is also a diffeomorphism. $\square$

The following function class will be useful in a number of proofs.

**Definition 85.** Define $\mathcal{L}_2(\mathbb{R}^k)$ to be the collection

$$\{f \in L^2(\mathbb{R}^k) : \operatorname{supp} f \subseteq \{\|u_1, \ldots, u_{k-1}\| < C\} \times \{c < \|u_k\| < C\}, 0 < c < C < \infty\}.$$

**Lemma 86.** Let $\mathbb{1}_+ = \mathbb{1}_{\dot{\mathbb{R}}^k_+}$. If $h \in \mathcal{L}_2(\mathbb{R}^k)$, then

$$\frac{\mathbb{1}_+(u)}{(2u_k)^{k/2}}(h(\Psi^{-1}(u)) + h(-\Psi^{-1}(u)) \in L^1 \cap L^2(\mathbb{R}^k).$$

*Proof.* We first show that $h(\Psi^{-1}(\cdot)) \in L^2(\dot{\mathbb{R}}^k_+)$:

$$\begin{aligned}
\int_{\dot{\mathbb{R}}^k_+} |h(\Psi^{-1}(u))|^2 du &= \int_{\dot{\mathbb{R}}^k_+} |h(x)|^2 x_k^k dx \\
&\leq \|h\|^2_{L^2(\mathbb{R}^k)} \operatorname{esssup}_{x \in \operatorname{supp} h} |x_k|^k \\
&< \infty.
\end{aligned}$$

We examine the support of $h(\pm\Psi^{-1}(\cdot))$. For almost all $u$ with $u_k \neq 0$:

$$\begin{aligned} h(\pm\Psi^{-1}(u)) \neq 0 \quad &\Rightarrow \quad \|\frac{u_1}{\sqrt{2u_k}}, \ldots, \frac{u_{k-1}}{\sqrt{2u_k}}\| < C \text{ and } c < \sqrt{2u_k} < C \\ &\Leftrightarrow \quad \sqrt{\frac{\sum_{i=1}^{k-1} u_i^2}{2u_k}} < C \text{ and } \frac{c^2}{2} < u_k < \frac{C^2}{2} \\ &\Leftrightarrow \quad \sqrt{\sum_{i=1}^{k-1} u_i^2} < C^2 \text{ and } \frac{c^2}{2} < u_k < \frac{C^2}{2}. \end{aligned}$$

Hence, $f(u) = \mathbb{1}_+(u)[h(\Psi^{-1}(u))+h(-\Psi^{-1}(u))] \in L^2(\mathbb{R}^k)$ has support in the compact set

$$\{^t(u_1, \ldots, u_{k-1}) : \|u_1, \ldots, u_{k-1}\| < C^2\} \times \{u_k : \frac{c^2}{2} < \|u_k\| < \frac{C^2}{2}\}.$$

Thus, if we show that $\frac{f(u)}{(2u_k)^{k/2}} \in L^2(\mathbb{R}^k)$, we will have proven the claim. We compute

$$\int_{\mathbb{R}^k} \frac{|f(u)|^2}{(2u_k)^k} du < \frac{2^k}{c^{2k}} \|f\|_{L^2(\mathbb{R}^k)} < \infty.$$

□

The following lemma is a generalization of Lemma 4.4 in [29], which is about functions in $L^2(\mathbb{R}^2)$. The preceding lemma is neither stated nor proven in [29], but it is necessary to justify the proof of Lemma 4.4. Also, the hypotheses of the following lemma in the case that $k = 2$ are slightly different that than the hypotheses on Lemma 4.4 in [29]. The hypotheses in [29] allow for possible divergence of the integrals and make certain lines in the proof purely formal calculations. Other than those points, the proof of the following lemma closely mimics the proof of Lemma 4.4 in [29].

**Lemma 87.** Let $\mathbb{1}_+ = \mathbb{1}_{\dot{\mathbb{R}}^k_+}$. Let $h \in \mathcal{L}_2(\mathbb{R}^k)$. Then

$$\int_{\mathbb{R}^k} \left| \int_{\mathbb{R}^k} h(x) e^{2\pi i \langle y, \Psi(x) \rangle} dx \right|^2 dy = \int_{\mathbb{R}^k_+} |h(x) + h(-x)|^2 \frac{dx}{x_k^k}.$$

*Proof.* By the properties of $\Psi$ listed in Proposition 84,

$$\begin{aligned}
\int_{\mathbb{R}^k} h(x)e^{2\pi i\langle y,\Psi(x)\rangle}dx &= \left(\int_{\dot{\mathbb{R}}^k_+} + \int_{\dot{\mathbb{R}}^k_-}\right) h(x)e^{2\pi i\langle y,\Psi(x)\rangle}dx \\
&= \int_{\dot{\mathbb{R}}^k_+} h(x)e^{2\pi i\langle y,\Psi(x)\rangle}dx + \int_{\dot{\mathbb{R}}^k_+} h(-x)e^{2\pi i\langle y,\Psi(-x)\rangle}dx \\
&= \int_{\dot{\mathbb{R}}^k_+} [h(x)+h(-x)]e^{2\pi i\langle y,\Psi(x)\rangle}dx \\
&= \int_{\dot{\mathbb{R}}^k_+} [h(\Psi^{-1}(u))+h(-\Psi^{-1}(u))]e^{2\pi i\langle y,u\rangle}\frac{du}{(2u_k)^{k/2}} \\
&= \int_{\mathbb{R}^k} \frac{\mathbb{1}_+(u)}{(2u_k)^{k/2}}[h(\Psi^{-1}(u))+h(-\Psi^{-1}(u))]e^{2\pi i\langle y,u\rangle}du \qquad (4.7)
\end{aligned}$$

Since $\frac{\mathbb{1}_+(u)}{(2u_k)^{k/2}}[h(\Psi^{-1}(u))+h(-\Psi^{-1}(u))]e^{2\pi i\langle y,u\rangle} \in L^1\cap L^2(\mathbb{R}^k)$ by Lemma 86, it follows from Parseval's equality that

$$\begin{aligned}
\|(\text{Eqn } 4.7)\|^2_{L^2(\mathbb{R}^k)} &= \int_{\mathbb{R}^k}\left|\int_{\mathbb{R}^k}\frac{\mathbb{1}_+(u)}{(2u_k)^{k/2}}[h(\Psi^{-1}(u))+h(-\Psi^{-1}(u))]e^{2\pi i\langle y,u\rangle}du\right|^2 dy \\
&= \int_{\mathbb{R}^k}\left|\frac{\mathbb{1}_+(u)}{(2u_k)^{k/2}}[h(\Psi^{-1}(u))+h(-\Psi^{-1}(u))]\right|^2 du \\
&= \int_{\dot{\mathbb{R}}^k}\left|h(\Psi^{-1}(u))+h(-\Psi^{-1}(u))\right|^2\frac{du}{(2u_k)^{k/2}} \\
&= \int_{\dot{\mathbb{R}}^k}|h(x)+h(-x)|^2\frac{x_k^k dx}{x_k^{2k}} \\
&= \int_{\dot{\mathbb{R}}^k}|h(x)+h(-x)|^2\frac{dx}{x_k^k}.
\end{aligned}$$

□

Proposition 84 can be used to prove the equivalence the wavelet and metaplectic representations of the $(\text{TDS})_k$ groups.

**Theorem 88.** *Let $u \in \dot{\mathbb{R}}^k_+$ and extend the map $Qf(u) = |2u_k|^{-k/4}f(\Psi^{-1}(u))$ as an even function to $\mathbb{R}^k\backslash\{x_1x_2\dots x_k = 0\}$. This is a unitary map of $L^2_{even}(\mathbb{R}^k)$ onto itself that intertwines the representations $\pi$ and $\mu$.*

*Proof.* Let $k > 1$. Initially note that $Q^{-1}f(u) = |u_k|^{k/2}f(\Psi(u))$. Let $f \in L^2_{\text{even}}(\mathbb{R}^k)$. We use the Jacobian calculation from Proposition 84 to determine that

$$\begin{aligned}
\|Qf\|^2_{L^2(\mathbb{R}^k)} &= \int_{\mathbb{R}^k} |Qf(u)|^2 du \\
&= 2\int_{\mathbb{R}^k_+} \frac{1}{(2u_k)^{k/2}} |f(\Psi^{-1}(u))|^2 du \\
&= 2\int_{\mathbb{R}^k_+} |f(x)|^2 dx \\
&= \|f\|^2_{L^2(\mathbb{R}^k)}.
\end{aligned}$$

As a surjective isometry $Q$ is a unitary on $L^2_{\text{even}}(\mathbb{R}^k)$. By Proposition 84.c and the fact that the $k$th coordindate of a vector is fixed by multiplication by a shearing matrix, we obtain

$$\begin{aligned}
\pi(A_{t,\ell,y})(Qf)(u) &= t^{k/2}e^{-2\pi i\langle y,u\rangle}(Qf)(tS_{-\ell}u) \\
&= \frac{t^{k/2}}{|2tu_k|^{k/4}}e^{-2\pi i\langle y,u\rangle}f(\Psi^{-1}(tS_{-\ell}u)) \\
&= \frac{t^{k/4}}{|2u_k|^{k/4}}e^{-2\pi i\langle y,u\rangle}f(t^{1/2}S_{-\ell/2}\Psi^{-1}(u)).
\end{aligned}$$

We apply lines (4.3) and (4.4) and Proposition 84.d to obtain

$$\begin{aligned}
Q(\mu(A_{t,\ell,y})f) &= \frac{1}{|2u_k|^{k/4}}(\mu(A_{t,\ell,y})f)(\Psi_{-1}(u)) \\
&= \frac{t^{k/4}}{|2u_k|^{k/4}}e^{-i\pi\langle B_y\Psi^{-1}(u),\Psi^{-1}(u)\rangle}f(t^{1/2}S_{-\ell/2}\Psi^{-1}(u)) \\
&= \frac{t^{k/4}}{|2u_k|^{k/4}}e^{-2\pi i\langle y,u\rangle}f(t^{1/2}S_{-\ell/2}\Psi^{-1}(u)) \\
&= \pi(A_{t,\ell,y})(Qf)(u),
\end{aligned}$$

as desired. ∎

Finally, for $H = (\text{TDS})_k$, we characterize the $\phi$ which satisfy Eqn (4.1).

**Theorem 89.**

$$\|f\|^2_{L^2(\mathbb{R}^k)} = \int_{(TDS)_k} |\langle f, \mu(A_{t,\ell,y})\phi\rangle|^2 \frac{dt}{t^{k+1}} dy d\ell \textit{ for all } f \in L^2(\mathbb{R}^k)$$

*if and only if*

$$\begin{aligned} 2^{-k} &= \int_{\mathbb{R}^k_+} |\phi(y)|^2 \frac{dy}{y_k^{2k}} = \int_{\mathbb{R}^k_+} |\phi(-y)|^2 \frac{dy}{y_k^{2k}} && (4.8) \\ 0 &= \int_{\mathbb{R}^2_+} \overline{\phi}(y)\phi(-y)\frac{dy}{y_k^{2k}}. && (4.9) \end{aligned}$$

*Proof.* The case $k = 1$ was proven in [39]. Let $k > 1$. Assume that for all $f \in L^2(\mathbb{R}^k)$,

$$\|f\|^2_{L^2(\mathbb{R}^k)} = \int_{(\mathrm{TDS})_k} |\langle f, \mu(A_{t,\ell,y})\phi\rangle|^2 \frac{dt}{t^{k+1}} dy d\ell.$$

Then, in particular, this holds for all $f \in L^\infty \cap \mathcal{L}_2(\mathbb{R}^2)$. Let $f \in L^\infty \cap \mathcal{L}_2(\mathbb{R}^k)$ and set $h(x) = f(x)\overline{\phi}(t^{1/2}S_{-\ell/2}x)$. Since $\phi \in L^2(\mathbb{R}^k)$, $h \in L^\infty \cap \mathcal{L}^2(\mathbb{R}^k)$, we use Proposition 84 to obtain

$$\begin{aligned} \|f\|^2_{L^2(\mathbb{R}^k)} &= \int_{\mathbb{R}^{k-1}} \int_{\mathbb{R}^k} \int_0^\infty \left| \int_{\mathbb{R}^k} h(x) e^{2\pi i \langle y, \Psi(x)\rangle} dx \right|^2 \frac{dt}{t^{(k/2)+1}} dy d\ell \\ &= \int_{\mathbb{R}^{k-1}} \int_0^\infty \int_{\dot{\mathbb{R}}^k_+} |h(x) + h(-x)|^2 \frac{dx}{x_k^k} \frac{dt}{t^{(k/2)+1}} d\ell \\ &= \int_{\mathbb{R}^{k-1}} \int_0^\infty \int_{\dot{\mathbb{R}}^k_+} (|h(x)|^2 + 2\Re h(x)\overline{h}(-x) + |h(-x)|^2) \frac{dx}{x_k^k} \frac{dt}{t^{(k/2)+1}} d\ell \\ &= A + B + C, \end{aligned}$$

where

$$\begin{aligned} A &= \int_{\mathbb{R}^{k-1}} \int_0^\infty \int_{\dot{\mathbb{R}}^k_+} |f(x)|^2 |(\phi(t^{1/2}S_{-\ell/2}x)|^2 \frac{dx}{x_k^k} \frac{dt}{t^{(k/2)+1}} d\ell \\ B &= \int_{\mathbb{R}^{k-1}} \int_0^\infty \int_{\dot{\mathbb{R}}^k_+} |f(-x)|^2 |(\phi(-t^{1/2}S_{-\ell/2}x)|^2 \frac{dx}{x_k^k} \frac{dt}{t^{(k/2)+1}} d\ell \\ C &= 2\Re \int_{\mathbb{R}^{k-1}} \int_0^\infty \int_{\dot{\mathbb{R}}^k_+} f(x)\overline{f}(-x)\overline{\phi}(t^{1/2}S_{-\ell/2}x)\phi(-t^{1/2}S_{-\ell/2}x) \frac{dx}{x_k^k} \frac{dt}{t^{(k/2)+1}} d\ell. \end{aligned}$$

Assume further that $f$ vanishes on $\dot{\mathbb{R}}^k_-$. Then $B = C = 0$ and $\|f\|_{L^2(\mathbb{R}^k)} = A$. We perform a change of variables

$$(t,\ell) \mapsto y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{k-1} \\ y_k \end{pmatrix} = t^{1/2} S_{-\ell/2} x = \begin{pmatrix} t^{1/2}(x_1 - \frac{\ell_1}{2} x_k) \\ t^{1/2}(x_2 - \frac{\ell_2}{2} x_k) \\ \vdots \\ t^{1/2}(x_{k-1} - \frac{\ell_{k-1}}{2} x_k) \\ t^{1/2} x_k \end{pmatrix}.$$

The matrix of mixed partials is

$$\begin{pmatrix} \frac{1}{2}t^{-1/2}(x_1 - \frac{\ell_1}{2}x_k) & \frac{1}{2}t^{-1/2}(x_2 - \frac{\ell_2}{2}x_k) & \dots & \frac{1}{2}t^{-1/2}(x_{k-1} - \frac{\ell_{k-1}}{2}x_k) & \frac{1}{2}t^{-1/2}x_k \\ -\frac{t^{1/2}x_k}{2} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & -\frac{t^{1/2}x_k}{2} & \dots & 0 & 0 \\ 0 & 0 & \dots & -\frac{t^{1/2}x_k}{2} & 0 \end{pmatrix}.$$

So $dtd\ell = \frac{2^k}{t^{(k-2)/2} x_k^k} dy$ and $t^{-k} = \frac{x_k^{2k}}{y_k^{2k}}$. Thus,

$$\begin{aligned} \|f\|^2_{L^2(\mathbb{R}^k)} &= \int_{\dot{\mathbb{R}}^k_+} \int_{\dot{\mathbb{R}}^k_+} |f(x)|^2 |\phi(y)|^2 \frac{dx}{x_k^k} \left(\frac{2^k}{x_k^k}\right) \left(\frac{x_k^{2k}}{y_k^{2k}}\right) dy \\ &= \int_{\dot{\mathbb{R}}^k_+} \int_{\dot{\mathbb{R}}^k_+} |f(x)|^2 |\phi(y)|^2 \frac{2^k}{y_k^{2k}} dxdy \\ &= \|f\|^2_{L^2(\mathbb{R}^k)} \int_{\dot{\mathbb{R}}^k_+} |\phi(y)|^2 \frac{2^k}{y_k^{2k}} dy. \end{aligned}$$

Similarly, now assume that $f \in L^\infty \cap \mathcal{L}_2(\mathbb{R}^k)$ is supported in $\ddot{\mathbb{R}}^k_-$. Then,

$$\begin{aligned} \|f\|^2_{L^2(\mathbb{R}^k)} &= B \\ &= \int_{\dot{\mathbb{R}}^k_+} \int_{\dot{\mathbb{R}}^k_-} |f(x)|^2 |\phi(-y)|^2 \frac{2^k}{y_k^{2k}} dxdy \\ &= \|f\|^2_{L^2(\mathbb{R}^k)} \int_{\dot{\mathbb{R}}^k_+} |\phi(-y)|^2 \frac{2^k}{y_k^{2k}} dy. \end{aligned}$$

Thus any reproducing function must satisfy (4.8). Now simply assume that $f \in L^\infty \cap \mathcal{L}_2(\mathbb{R}^k)$ and assume that $\phi$ satisfies (4.8). Then

$$\|f\|^2_{L^2(\mathbb{R}^k)} = \|f\|^2_{L^2(\ddot{\mathbb{R}}^k_+)} + \|f\|^2_{L^2(\ddot{\mathbb{R}}^k_-)} + C,$$

which implies that $C = 0$. Also

$$0 = C = 2\Re \int_{\dot{\mathbb{R}}^k_+} \int_{\dot{\mathbb{R}}^k_+} f(x)\overline{f}(-x)\overline{\phi}(y)\phi(-y)\frac{2^k}{y_k^{2k}}dydx. \tag{4.10}$$

Assume that $\int_{\ddot{\mathbb{R}}^k_+} f(x)\overline{f}(-x)dx \neq 0$. If $f$ is also real valued, then (4.10) implies that $\Re \int_{\dot{\mathbb{R}}^k_+} \overline{\phi}(y)\phi(-y)\frac{dy}{y_k^{2k}} = 0$. If $f$ is purely imaginary, then (4.10) implies that $\Im \int_{\dot{\mathbb{R}}^k_+} \overline{\phi}(y)\phi(-y)\frac{dy}{y_k^{2k}} = 0$. Hence $\phi$ must satisfy (4.9).

Conversely, assume that $f \in L^\infty \cap \mathcal{L}_2(\mathbb{R}^k)$. By the preceding arguments,

$$\begin{aligned}\int_{(\mathrm{TDS})_k} |\langle f, \mu(A_{t,\ell,y})\phi\rangle|^2 \frac{dt}{t^{k+1}}dyd\ell &= \int_{\dot{\mathbb{R}}^k_+} |f(x)|^2dx \int_{\dot{\mathbb{R}}^k_+} |\phi(y)|^2\frac{2^k}{y_k^{2k}}dy \\ &+ \int_{\dot{\mathbb{R}}^k_-} |f(x)|^2dx \int_{\dot{\mathbb{R}}^k_+} |\phi(-y)|^2\frac{2^k}{y_k^{2k}}dy \\ &+ 2\Re \int_{\dot{\mathbb{R}}^k_+} f(x)\overline{f}(-x)dx \int_{\dot{\mathbb{R}}^k_+} \overline{\phi}(y)\phi(-y)\frac{2^k}{y_k^{2k}}dy.\end{aligned}$$

If $\phi$ satisfies (4.8) and (4.9), then

$$\int_{(\mathrm{TDS})_k} |\langle f, \mu(A_{t,\ell,y})\phi\rangle|^2 \frac{dt}{t^{k+1}}dyd\ell = \left(\int_{\dot{\mathbb{R}}^k_-} + \int_{\dot{\mathbb{R}}^k_+}\right)|f(x)|^2dx + 0 = \|f\|^2_{L^2(\mathbb{R}^k)}.$$

Now let $f$ be any arbitrary function in $L^2(\mathbb{R}^k)$ where $\phi$ still satisfies (4.8) and (4.9). Since $L^\infty \cap \mathcal{L}_2(\mathbb{R}^k)$ is dense in $L^2(\mathbb{R}^k)$,we can chose a $\{f_n\} \subset L^\infty \cap \mathcal{L}_2(\mathbb{R}^k)$ which converges to $f$ in $L^2(\mathbb{R}^k)$. For any $f_n$, $f_m$, the difference $f_n - f_m$ lies in $L^\infty \cap \mathcal{L}_2(\mathbb{R}^k)$. Hence

$$\|\langle f_n, \mu((A_{t,\ell,y})\phi\rangle - \langle f_m, \mu((A_{t,\ell,y})\phi\rangle\|^2_{L^2((\mathrm{TDS})_k)} = \|f_n - f_m\|^2_{L^2(\mathbb{R}^k)} \to 0$$

as $n, m \to \infty$. Hence, $\{\langle f_n, \mu(A_{t,\ell,y})\phi\rangle\}_n$ is a Cauchy sequence in $L^2((\mathrm{TDS})_k)$. Also, it follows from the Cauchy-Schwarz inequality over $L^2(\mathbb{R}^k)$ that $\{\langle f_n, \mu(A_{t,\ell,y})\phi\rangle\}_n$ converges pointwise to $\langle f, \mu(A_{t,\ell,y})\phi\rangle$. A sequence which is Cauchy in norm and additionally converges pointwise also converges in norm to the pointwise limit. Thus,

$$\begin{aligned}
\|f\|^2_{L^2(\mathbb{R}^k)} &= \lim_{n\to\infty} \|f_n\|^2_{L^2(\mathbb{R}^k)} \\
&= \lim_{n\to\infty} \int_{(\mathrm{TDS})_k} |\langle f_n, \mu(A_{t,\ell,y})\phi\rangle|^2 \frac{dt}{t^{k+1}} dy d\ell \\
&= \int_{(\mathrm{TDS})_k} |\langle f, \mu(A_{t,\ell,y})\phi\rangle|^2 \frac{dt}{t^{k+1}} dy d\ell,
\end{aligned}$$

as desired. □

### 4.3.1 Building Reproducing Functions for $(\mathrm{TDS})_k$

We would like to present some simple methods to build reproducing functions for $(\mathrm{TDS})_k$. We are inspired by [30], which contains the following two theorems.

**Theorem 90.** *$\phi_0 \in L^2(\mathbb{R})$ is a reproducing function for $\mathbb{R}^2 \times \{I\} \leq \mathbb{R}^2 \rtimes \mathrm{Sp}(1, \mathbb{R})$ if and only if $\phi_0 \in L^2(\mathbb{R})$ and $\|\phi_0\| = 1$. $\phi_1 \in L^2(\mathbb{R})$ is a reproducing function for $(\mathrm{TDS})_1 \leq \mathbb{R}^2 \rtimes \mathrm{Sp}(1, \mathbb{R})$ if and only if $\phi_1 \in L^2(\mathbb{R})$ and*

$$\int_0^\infty |\phi_1(x)|^2 \frac{dx}{x^2} = \int_0^\infty |\phi_1(-x)|^2 \frac{dx}{x^2} = \frac{1}{2}, \quad \int_0^\infty \phi_1(x)\overline{\phi_1(-x)} \frac{dx}{x^2} = 0.$$

**Theorem 91.** *Let $\phi_0 \in L^2(\mathbb{R})$ be a reproducing function for $\mathbb{R}^2 \times \{I\} \leq \mathbb{R}^2 \rtimes \mathrm{Sp}(1, \mathbb{R})$ and $\phi_1 \in L^2(\mathbb{R})$ be a reproducing function for $(\mathrm{TDS})_1 \leq \mathbb{R}^2 \rtimes \mathrm{Sp}(1, \mathbb{R})$. Then if $\tilde{\phi}_1(y) = y\phi(y)$,*

$$\phi(x) = \frac{1}{\sqrt{2}} (\phi_0 \otimes \tilde{\phi}_1)(x), \quad x \in \mathbb{R}^2,$$

*is a reproducing function for $(\mathrm{TDS})_2$.*

We generalize this result to obtain the following theorem.

**Theorem 92.** *Let $\phi_0 \in L^2(\mathbb{R})$ be a reproducing function for $\mathbb{R}^2 \times \{I\} \leq \mathbb{R}^2 \rtimes \mathrm{Sp}(1, \mathbb{R})$ and let $\phi_{k-1} \in L^2(\mathbb{R}^{k-1})$ be a reproducing function for $(\mathrm{TDS})_{k-1}$. Then*

$$\phi(x) = \frac{x_k}{\sqrt{2}}(\phi_0 \otimes \phi_{k-1})(x), \quad x \in \mathbb{R}^k$$

*is a reproducing function for $(\mathrm{TDS})_k$.*

*Proof.* By Theorem 90 $\phi_0$ satisfies $\|\phi_0\| = 1$ and $\phi_{k-1}$ satisfies

$$\begin{aligned}
1 &= 2^{k-1} \int_{\mathbb{R}_+^{k-1}} |\phi_{k-1}(x)|^2 \frac{dx}{x_{k-1}^{2(k-1)}}, \\
1 &= 2^{k-1} \int_{\mathbb{R}_+^{k-1}} |\phi_{k-1}(-x)|^2 \frac{dx}{x_{k-1}^{2(k-1)}}, \text{ and} \\
0 &= 2^{k-1} \int_{\mathbb{R}_+^{k-1}} \phi_{k-1}(x)\overline{\phi_{k-1}(-x)} \frac{dx}{x_{k-1}^{2(k-1)}}.
\end{aligned}$$

So

$$\begin{aligned}
2^k \int_{\mathbb{R}_+^k} |\phi(x)|^2 \frac{dx}{x_k^{2k}} &= 2^k \int_{\mathbb{R}_+^k} |\frac{x_k}{\sqrt{2}}(\phi_0 \otimes \phi_{k-1})(x)|^2 \frac{dx}{x_k^{2k}} \\
&= 2^{k-1} \int_{\mathbb{R}} |\phi_0(x_1)|^2 dx_1 \int_{\mathbb{R}_+^{k-1}} |\phi_{k-1}(x)|^2 \frac{dx}{x_{k-1}^{2(k-1)}} \\
&= 2^{k-1} \cdot 1 \cdot \frac{1}{2^{k-1}} \\
&= 1.
\end{aligned}$$

Similarly, $2^k \int_{\mathbb{R}_+^k} |\phi(-x)|^2 \frac{dx}{x_k^{2k}} = 1$. Finally,

$$\begin{aligned}
\int_{\mathbb{R}_+^k} \phi(x)\overline{\phi(-x)} \frac{dx}{x_k^{2k}} &= \frac{1}{2} \int_{\mathbb{R}} \phi_0(x_1)\overline{\phi_0(-x_1)} dx_1 \int_{\mathbb{R}_+^{k-1}} \phi_{k-1}(x)\overline{\phi_{k-1}}(-x) \frac{dx}{x_{k-1}^{2(k-1)}} \\
&= \frac{1}{2} \left( \int_{\mathbb{R}} \phi_0(x_1)\overline{\phi_0(-x_1)} dx_1 \right) \cdot 0 \\
&= 0.
\end{aligned}$$

□

This leads to the following corollary.

**Corollary 93.** *Let* $\phi_0 \in L^2(\mathbb{R})$ *be a reproducing function for* $\mathbb{R}^2 \times \{I\} \leq \mathbb{R}^2 \rtimes \mathrm{Sp}(1,\mathbb{R})$ *and let* $\phi_1 \in L^2(\mathbb{R})$ *be a reproducing function for* $(\mathrm{TDS})_1$*. Then*

$$\phi(x) = \frac{1}{2^{(k-1)/2}}\left((\otimes_{i=1}^{k-1}\phi_0) \otimes \tilde{\phi}_1\right)(x), \quad x \in \mathbb{R}^k,$$

*where* $\tilde{\phi}_1(y) = y^{k-1}\phi_1(y)$*, is a reproducing function for* $(\mathrm{TDS})_k$*.*

We shall now now explicitly construct a class of functions which are reproducing functions for $(\mathrm{TDS})_1$.

**Theorem 94.** *Let* $f : \mathbb{R} \to \mathbb{R}$ *be supported in some interval* $[0, b]$*,* $b > 0$ *and satisfy* $\int f^2(x)dx = \frac{1}{4}$*. For* $a > 0$*, define*

$$\phi(x) = x\left(f(x-a) - f(-x+a+2b) + f(x+a+b) + f(-x-a-b)\right).$$

*Then* $\phi$ *is a reproducing function for* $(\mathrm{TDS})_1$*.*

*Proof.* By Theorem 90, it suffices to show that

$$\begin{aligned} 0 &= \int_0^\infty \phi(x)\overline{\phi(-x)}\frac{dx}{x^2} \text{ and} \\ \frac{1}{2} &= \int_0^\infty |\phi(x)|^2\frac{dx}{x^2} = \int_0^\infty |\phi(-x)|^2\frac{dx}{x^2}. \end{aligned}$$

We note that the following shifts have supports contained in

| shift | supp |
|---|---|
| $f(\cdot - a)$ | $[a, a+b]$ |
| $f(-\cdot + a + 2b)$ | $[a+b, a+2b]$ |
| $f(\cdot + a + b)$ | $[-a-b, -a]$ |
| $f(-\cdot - a - b)$ | $[-a-2b, -a-b]$. |

So

$$\int_0^\infty \phi(x)\overline{\phi(-x)}\frac{dx}{x^2} = \int_a^{a+b} f(x-a)\overline{\phi(-x)}\frac{dx}{x} - \int_{a+b}^{a+2b} f(-x+a+2b)\overline{\phi(-x)}\frac{dx}{x}$$

$$\begin{aligned}
&= \int_a^{a+b} f(x-a)f(-x+a+b)dx - \int_{a+b}^{a+2b} f(-x+a+2b)f(x-a-b)dx \\
&= \int_a^{a+b} f(x-a)f(-x+a+b)dx - \int_a^{a+b} f(-(u+b)+a+2b)f((u+b)-a-b)du \\
&= 0.
\end{aligned}$$

Also,

$$\begin{aligned}
\int_0^\infty |\phi(x)|^2\frac{dx}{x^2} &= \int_0^\infty f^2(x-a)+f^2(-x+a+2b)dx \\
&= \frac{1}{4}+\frac{1}{4} \\
&= \frac{1}{2} \text{ and}
\end{aligned}$$

$$\begin{aligned}
\int_0^\infty |\phi(-x)|^2\frac{dx}{x^2} &= \int_0^\infty f^2(-x+a+b)+f^2(x-a-b)dx \\
&= \frac{1}{4}+\frac{1}{4} \\
&= \frac{1}{2}.
\end{aligned}$$

□

The restrictions on $f$ are so light that we are able to grant $\phi$ certain properties.

**Example 95.**

- Let $f(x) = \mathbb{1}_{[0,\frac{1}{4}]}$, then $f$ trivially satisfies the hypotheses of Theorem 94.
- If $f = \frac{1}{\sqrt{\pi}}\cos\cdot\mathbb{1}_[0,\frac{\pi}{2}]$, then a simple calculation shows that $f$ may be used as in Theorem 94.

- If $f \in C_c^\infty(\mathbb{R})$ has support in $[0, b]$ and is scaled so that $\int f^2 = \frac{1}{4}$, then the resulting $\phi$ will also lie in $C_c^\infty$.

## 4.4 Shearlets and the extended metaplectic group

Inspired by our work on the translation-dilation-shearing groups, whose very names suggest something shearlet-like, we would like to connect the continuous shearlet transform to a subgroup of $\mathrm{Sp}(2, \mathbb{R})$ in such a way that the methods of section 4.3 may be applied. We hope that this novel approach will yield a natural generalization of shearlets to higher dimension. To this end, we present the following definition and proposition.

**Definition 96.** The *continuous shearlet group*, $(\mathrm{CSG})_2$, is

$$(\mathrm{CSG})_2 = \{\mathscr{S}_{a,\ell,y} = \begin{pmatrix} S_{-\ell/2}\mathcal{A}_a & 0 \\ \mathcal{B}_y S_{-\ell/2}\mathcal{A}_a & {}^t S_{\ell/2}\mathcal{A}_a^{-1} \end{pmatrix} : a > 0, \ell \in \mathbb{R}, y \in \mathbb{R}^2\},$$

where $S_\ell = \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix}$, $\mathcal{A}_a = \begin{pmatrix} 1 & 0 \\ 0 & a^{-1/2} \end{pmatrix} = \frac{1}{a} A_a$, and $\mathcal{B}_y = \begin{pmatrix} 0 & y_2 \\ y_2 & y_1 \end{pmatrix} = B_{\binom{y_2}{y_1}}$.

**Proposition 97.** $(\mathrm{CSG})_2$ *is a Lie subgroup of* $\mathrm{Sp}(2, \mathbb{R})$ *of dimension* 4.

*Proof.* Let $\mathscr{S}_{a,\ell,y}$ and $\mathscr{S}_{\tilde{a},\tilde{\ell},\tilde{y}}$ be arbitrary elements of $(\mathrm{CSG})_2$. We first show that

$(\mathrm{CSG})_2 \subset \mathrm{Sp}(2,\mathbb{R})$.

$$
\begin{aligned}
{}^t\mathscr{S}_{a,\ell,y}\mathcal{J}\mathscr{S}_{a,\ell,y} &= \begin{pmatrix} \mathcal{A}_a{}^t S_{-\ell/2} & \mathcal{A}_a{}^t S_{-\ell/2}\mathcal{B}_y \\ 0 & {}^t\mathcal{A}_a^{-1} S_{\ell/2} \end{pmatrix} \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} \mathscr{S}_{a,\ell,y} \\
&= \begin{pmatrix} -\mathcal{A}_a{}^t S_{-\ell/2}\mathcal{B}_y & \mathcal{A}_a{}^t S_{-\ell/2} \\ -{}^t\mathcal{A}_a^{-1} S_{\ell/2} & 0 \end{pmatrix} \begin{pmatrix} S_{-\ell/2}\mathcal{A}_a & 0 \\ \mathcal{B}_y S_{-\ell/2}\mathcal{A}_a & {}^t S_{\ell/2}\mathcal{A}_a^{-1} \end{pmatrix} \\
&= \begin{pmatrix} -\mathcal{A}_a{}^t S_{-\ell/2}\mathcal{B}_y S_{-\ell/2}\mathcal{A}_a + \mathcal{A}_a{}^t S_{-\ell/2}\mathcal{B}_y S_{-\ell/2}\mathcal{A}_a & \mathcal{A}_a{}^t S_{-\ell/2}{}^t S_{\ell/2}\mathcal{A}_a^{-1} \\ -{}^t\mathcal{A}_a^{-1} S_{\ell/2} S_{-\ell/2}\mathcal{A}_a & 0 \end{pmatrix} \\
&= \mathcal{J}.
\end{aligned}
$$

We now compute

$$
\begin{aligned}
\mathscr{S}_{a,\ell,y}\mathscr{S}_{\tilde{a},\tilde{\ell},\tilde{y}} &= \begin{pmatrix} S_{-\ell/2}\mathcal{A}_a & 0 \\ \mathcal{B}_y S_{-\ell/2}\mathcal{A}_a & {}^t S_{\ell/2}\mathcal{A}_a^{-1} \end{pmatrix} \begin{pmatrix} S_{-\tilde{\ell}/2}\mathcal{A}_{\tilde{a}} & 0 \\ \mathcal{B}_{\tilde{y}} S_{-\tilde{\ell}/2}\mathcal{A}_{\tilde{a}} & {}^t S_{\tilde{\ell}/2}\mathcal{A}_{\tilde{a}}^{-1} \end{pmatrix} \\
&= \begin{pmatrix} S_{-\ell/2}\mathcal{A}_a S_{-\tilde{\ell}/2}\mathcal{A}_{\tilde{a}} & 0 \\ \mathcal{B}_y S_{-\ell/2}\mathcal{A}_a S_{-\tilde{\ell}/2}\mathcal{A}_{\tilde{a}} + {}^t S_{\ell/2}\mathcal{A}_a^{-1}\mathcal{B}_{\tilde{y}} S_{-\tilde{\ell}/2}\mathcal{A}_{\tilde{a}} & {}^t S_{\ell/2}\mathcal{A}_a^{-1\,t} S_{\tilde{\ell}/2}\mathcal{A}_{\tilde{a}}^{-1} \end{pmatrix} \\
&= \begin{pmatrix} S_{-\ell/2} S_{-a^{1/2}\tilde{\ell}/2}\mathcal{A}_a\mathcal{A}_{\tilde{a}} & 0 \\ (\mathcal{B}_y S_{-\ell/2}\mathcal{A}_a + {}^t S_{\ell/2}\mathcal{A}_a^{-1}\mathcal{B}_{\tilde{y}}) S_{-\tilde{\ell}/2}\mathcal{A}_{\tilde{a}} & {}^t S_{\ell/2}{}^t S_{a^{1/2}\tilde{\ell}/2}\mathcal{A}_a^{-1}\mathcal{A}_{\tilde{a}}^{-1} \end{pmatrix},
\end{aligned}
$$

where $(\mathcal{B}_y S_{-\ell/2}\mathcal{A}_a + {}^t S_{\ell/2}\mathcal{A}_a^{-1}\mathcal{B}_{\tilde{y}})S_{-\tilde{\ell}/2}\mathcal{A}_{\tilde{a}}$

$$= \left(\begin{pmatrix} 0 & a^{-1/2}y_2 \\ y_2 & \frac{-a^{-1/2}\ell}{2}y_2 + a^{-1/2}y_1 \end{pmatrix} + \begin{pmatrix} 0 & \tilde{y}_2 \\ a^{1/2}\tilde{y}_2 & \frac{\ell}{2}\tilde{y}_2 + a^{1/2}\tilde{y}_1 \end{pmatrix}\right)\begin{pmatrix} 1 & \frac{-\tilde{a}^{-1/2}\tilde{\ell}}{2} \\ 0 & \tilde{a}^{-1/2} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & a^{-1/2}y_2 + \tilde{y}_2 \\ y_2 + a^{1/2}\tilde{y}_2 & \frac{-a^{-1/2}\ell}{2}y_2 + a^{-1/2}y_1 + \frac{\ell}{2}\tilde{y}_2 + a^{1/2}\tilde{y}_1 \end{pmatrix}\begin{pmatrix} 1 & \frac{-\tilde{a}^{-1/2}\tilde{\ell}}{2} \\ 0 & \tilde{a}^{-1/2} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & (a\tilde{a})^{-1/2}(y_2 + a^{1/2}\tilde{y}_2) \\ y_2 + a^{1/2}\tilde{y}_2 & \frac{-(a\tilde{a})^{-1/2}}{2}(\ell + a^{1/2}\tilde{\ell})(y_2 + a^{1/2}\tilde{y}_2) + (a\tilde{a})^{-1/2}(y_1 + a\tilde{y}_1 + a^{1/2}\ell\tilde{y}_2) \end{pmatrix}$$

$$= \mathcal{B}_{\binom{y_1 + a\tilde{y}_1 + a^{1/2}\ell\tilde{y}_2}{y_2 + a^{1/2}\tilde{y}_2}} S_{\frac{-(\ell + a^{1/2}\tilde{\ell})}{2}}\mathcal{A}_{a\tilde{a}}.$$

So $\mathscr{S}_{a,\ell,y}\mathscr{S}_{\tilde{a},\tilde{\ell},\tilde{y}} = \mathscr{S}_{\overline{a},\overline{\ell},\overline{y}}$, where $\overline{a} = a\tilde{a}$, $\overline{\ell} = \ell + a^{1/2}\tilde{\ell}$ and $\overline{y} = \binom{y_1 + a\tilde{y}_1 + a^{1/2}\ell\tilde{y}_2}{y_2 + a^{1/2}\tilde{y}_2}$. Thus $(\mathrm{CSG})_2$ is a Lie subgroup of $\mathrm{Sp}(2,\mathbb{R})$. The parameterization of $(\mathrm{CSG})_2$ yields the dimension. □

As we did in Section 4.3, we shall again present three representations of $(\mathrm{CSG})_2$. We shall begin with a "wavelet representation," which we shall also denote with a $\nu$.

**Definition 98.** Define the *wavelet representation* of $(\mathrm{CSG})_2$ by

$$\nu(\mathscr{S}_{a,\ell,y}) = T_y D_{(S_\ell A_a)^{-1}}.$$

**Proposition 99.** *$\nu$ is a unitary representation of* $(\mathrm{CSG})_2$.

*Proof.* Each $\nu(\mathscr{S}_{a,\ell,y})$ is a unitary operator on $L^2(\mathbb{R}^2)$. In the proof of Proposition 97, we showed

$$\mathscr{S}_{a,\ell,y}\mathscr{S}_{\tilde{a},\tilde{\ell},\tilde{y}} = \mathscr{S}_{\overline{a},\overline{\ell},\overline{y}},$$

where $\overline{a} = a\tilde{a}$, $\overline{\ell} = \ell + a^{1/2}\tilde{\ell}$ and $\overline{y} = \binom{y_1 + a\tilde{y}_1 + a^{1/2}\ell\tilde{y}_2}{y_2 + a^{1/2}\tilde{y}_2}$. Thus, it suffices to show that $T_y D_{(S_\ell A_a)^{-1}} T_{\tilde{y}} D_{(S_{\tilde{\ell}} A_{\tilde{a}})^{-1}} = T_{\overline{y}} D_{(S_{\overline{\ell}} A_{\overline{a}})^{-1}}$. Commutation relations yield that

$$\begin{aligned}
T_y D_{(S_\ell A_a)^{-1}} T_{\tilde{y}} D_{(S_{\tilde{\ell}} A_{\tilde{a}})^{-1}} &= T_y T_{S_\ell A_a \tilde{y}} D_{(S_\ell A_a)^{-1}} D_{(S_{\tilde{\ell}} A_{\tilde{a}})^{-1}} \\
&= T_{y + S_\ell A_a \tilde{y}} D_{(S_\ell A_a S_{\tilde{\ell}} A_{\tilde{a}})^{-1}} \\
&= T_{\binom{y_1 + a\tilde{y}_1 + a^{1/2}\ell\tilde{y}_2}{y_2 + a^{1/2}\tilde{y}_2}} D_{(S_{\ell + a^{1/2}\tilde{\ell}} A_{a\tilde{a}})^{-1}},
\end{aligned}$$

as desired. Strong continuity follows from traits of the dilation and translation operators. $\square$

We obtain the second representation by conjugating $\nu$ with $\mathcal{F}$.

**Definition 100.** The unitary representation $\pi$ of $(\mathrm{CSG})_2$ is

$$\pi(\mathscr{S}_{a,\ell,y}) f(u) = \mathcal{F} \circ \nu(\mathscr{S}_{a,\ell,y}) \circ \mathcal{F}^{-1} f(u).$$

Finally, we consider the metaplectic representation of $(\mathrm{CSG})_2$. For $f \in L^2(\mathbb{R}^2)$, it follows from lines (4.3) and (4.4) that

$$\begin{aligned}
\mu(\mathscr{S}_{a,\ell,y}) f(x) &= \left[ \mu \begin{pmatrix} S_{-\ell/2}\mathcal{A}_a & 0 \\ 0 & {}^t S_{\ell/2}\mathcal{A}_a^{-1} \end{pmatrix} \mu \begin{pmatrix} I & 0 \\ \mathcal{A}_a {}^t S_{-\ell/2} \mathcal{B}_y S_{-\ell/2} \mathcal{A}_a & I \end{pmatrix} f \right](x) \\
&= a^{1/4} \left[ \mu \begin{pmatrix} I & 0 \\ \mathcal{A}_a {}^t S - \ell/2 \mathcal{B}_y S_{-\ell/2} \mathcal{A}_a & I \end{pmatrix} f \right] (\mathcal{A}_a^{-1} S_{\ell/2} x) \\
&= a^{1/4} e^{-\pi i \langle (\mathcal{A}_a {}^t S_{-\ell/2} \mathcal{B}_y S_{-\ell/2} \mathcal{A}_a)(\mathcal{A}_a^{-1} S_{\ell/2} x), \mathcal{A}_a^{-1} S_{\ell/2} x \rangle} f(\mathcal{A}_a^{-1} S_{\ell/2} x) \\
&= a^{1/4} e^{-\pi i \langle {}^t S_{-\ell/2} \mathcal{B}_y x, S_{\ell/2} x \rangle} f(\mathcal{A}_a^{-1} S_{\ell/2} x) \\
&= a^{1/4} e^{-\pi i \langle \mathcal{B}_y x, x \rangle} f(\mathcal{A}_a^{-1} S_{\ell/2} x).
\end{aligned}$$

We move on to presenting a Haar measure of $(\mathrm{CSG})_2$.

**Proposition 101.** *Let $da$, $dy$ and $d\ell$ be the Lebesgue measures over $\mathbb{R}_+$, $\mathbb{R}^2$ and $\mathbb{R}$, respectively. Then $\frac{da}{a^3}dyd\ell$ is a left Haar measure of* $(\mathrm{CSG})_2$.

*Proof.* Let $f$ be integrable with respect to $\frac{da}{a^3}dyd\ell$. We shall use the notation

$$f(a,\ell,y) = f(\mathscr{S}_{a,\ell,y}).$$

Then

$$\begin{aligned}
\int_{(\mathrm{CSG})_2} f(\mathscr{S}_{\tilde{a},\tilde{\ell},\tilde{y}}\mathscr{S}_{a,\ell,y})\frac{da}{a^3}dyd\ell &= \int_{\mathbb{R}}\int_{\mathbb{R}^2}\int_0^\infty f(\tilde{a}a, \tilde{\ell}+\tilde{a}^{-1/2}\ell, \tilde{y}+S_{\tilde{\ell}}A_{\tilde{a}}y)\frac{da}{a^3}dyd\ell \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}^2}\int_0^\infty f(\overline{a}, \tilde{\ell}+\tilde{a}^{-1/2}\ell, \tilde{y}+S_{\tilde{\ell}}A_{\tilde{a}}y)\frac{d\overline{a}}{\tilde{a}}\frac{1}{(\overline{a}/\tilde{a})^3}dyd\ell \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}^2}\int_0^\infty f(\overline{a}, \tilde{\ell}+\tilde{a}^{-1/2}\ell, \overline{y})\frac{d\overline{a}}{\overline{a}^3}\tilde{a}^2\frac{d\overline{y}}{\tilde{a}^{1/2}}d\ell \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}^2}\int_0^\infty f(\overline{a}, \overline{\ell}, \overline{y})\frac{d\overline{a}}{\overline{a}^3}\tilde{a}^{3/2}\frac{d\overline{y}}{\tilde{a}^{1/2}}\frac{d\overline{\ell}}{\tilde{a}^{3/2}} \\
&= \int_{(\mathrm{CSG})_2} f(a,\ell,y)\frac{da}{a^3}dyd\ell.
\end{aligned}$$

□

Also, in analog to line (4.5), define the mapping

$$\Phi : \mathbb{R}^2 \to \mathbb{R}^2 \text{ by } x \mapsto \begin{pmatrix} \frac{1}{2}x_2^2 \\ x_1x_2 \end{pmatrix}.$$

Also define

$$\ddot{\mathbb{R}}^2_\pm = \{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : x_2 \neq 0, \pm x_1 > 0\}.$$

**Proposition 102.** *The mapping $\Phi$ induces diffeomorphisms $\Phi : \ddot{\mathbb{R}}^2_\pm \to \ddot{\mathbb{R}}^2_+$ and is such that $\Phi(-x) = \Phi(x)$. Further, it satisfies*

*a. The Jacobian of $\Phi$ at $x \in \ddot{\mathbb{R}}^2_\pm$ is $J_\Phi(x) = -x_2^2$;*

*b. The Jacobian of $\Phi^{-1} : \ddot{\mathbb{R}}^2_+ \to \ddot{\mathbb{R}}^2_+$ at $u = \Phi(x)$ is $J_{\Phi^{-1}}(u) = \frac{-1}{2u_1}$;*

*c.* $\Phi^{-1}(A_a{}^t S_\ell u) = \mathcal{A}_a^{-1} S_{\ell/2} \Phi^{-1}(u)$ *for every* $u \in \ddot{\mathbb{R}}^2_+$*; and*

*d.* $\langle \mathcal{B}_y x, x\rangle = 2\langle y, \Phi(x)\rangle$ *for every* $x \in \ddot{\mathbb{R}}^2_\pm$ *and every* $y \in \mathbb{R}^2$*.*

*Proof.* By definition, $\Phi$ is even. Note that $\Phi^{-1}(u) = \binom{u_2/\sqrt{2u_1}}{\sqrt{2u_1}}$. We now calculate the matrices of the partials:

$$\frac{\partial \Phi(x)_i}{\partial x_j} = \begin{cases} x_2 & ; \quad i = 3-j \\ x_1 & ; \quad i = j = 2 \\ 0 & ; \quad \text{else} \end{cases} \quad \text{and}$$

$$\frac{\partial \Phi^{-1}(u)_i}{\partial u_j} = \begin{cases} \frac{1}{(2u_1)^{1/2}} & ; \quad i = 3-j \\ \frac{-2u_2}{(2u_1)^{3/2}} & ; \quad i = j = 2 \\ 0 & ; \quad \text{else} \end{cases} .$$

Hence

$$\begin{aligned} J_\Phi(x) &= \det\left(\frac{\partial \Phi(x)_i}{\partial x_j}\right)_{i,j} = -x_2^2 \text{ and} \\ J_{\Phi^{-1}}(u) &= \det\left(\frac{\partial \Phi^{-1}(u)_i}{\partial u_j}\right)_{i,j} = \frac{-1}{2u_1}. \end{aligned}$$

We now prove parts (c) and (d):

$$\begin{aligned} \Phi^{-1}(A_a{}^t S_\ell u) &= \Phi^{-1}\begin{pmatrix} a u_1 \\ a^{1/2}\ell u_1 + a^{1/2} u_2 \end{pmatrix} \\ &= \begin{pmatrix} \frac{u_1 + a^{1/2} u_2}{\sqrt{2au_1}} \\ \sqrt{2au_1} \end{pmatrix} \\ &= \begin{pmatrix} \frac{\ell}{2}\sqrt{2u_1} + \frac{u_2}{\sqrt{2u_1}} \\ a^{1/2}\sqrt{2u_1} \end{pmatrix} \\ &= \mathcal{A}_a^{-1} S_{\ell/2} \begin{pmatrix} \frac{u_2}{\sqrt{2u_1}} \\ \sqrt{2u_1} \end{pmatrix} \\ &= \mathcal{A}_a^{-1} S_{\ell/2} \Phi^{-1}(u), \end{aligned}$$

and

$$\begin{aligned}
\langle \mathcal{B}_y x, x\rangle &= \langle \begin{pmatrix} 0 & y_2 \\ & \\ y_2 & y_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \rangle \\
&= \langle \begin{pmatrix} x_2 y_2 \\ x_1 y_2 + x_2 y_1 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \rangle \\
&= x_1 x_2 y_2 + x_1 x_2 y_2 + x_2^2 y_1 \\
&= 2\langle \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} x_2^2 \\ x_1 x_2 \end{pmatrix} \rangle \\
&= 2\langle y, \Phi(x)\rangle.
\end{aligned}$$

□

We now use these results to show that the wavelet and metaplectic repsentations of $(\mathrm{CSG})_2$ are equivalent.

**Theorem 103.** *Let $u \in \ddot{\mathbb{R}}^2_+$ and extend the map*

$$\mathcal{P} f(u) = \frac{1}{|2u_1|^{1/2}} f(\Phi^{-1}(u))$$

*as an even function to $\mathbb{R}^2 \backslash \{x_1 x_2 = 0\}$. This is a unitary map of $L^2_{even}(\mathbb{R}^2)$ onto itself that intertwines the representations $\pi$ and $\mu$.*

*Proof.* Note that $\mathcal{P}^{-1} f(x) = x_2 f(\Phi(x))$. Let $f \in L^2_{\text{even}}(\mathbb{R}^2)$. It follows from the Jacobian calculation in Proposition 102 that

$$\begin{aligned}
\|\mathcal{P} f\|^2_{L^2(\mathbb{R}^2)} &= \int_{\mathbb{R}^2} |\mathcal{P} f(u)|^2 du \\
&= 2\int_{\ddot{\mathbb{R}}^2_+} \left(\frac{1}{2u_1}\right) |f(\Phi^{-1}(u))|^2 du \\
&= 2\int_{\ddot{\mathbb{R}}^2_+} |f(x)|^2 dx \\
&= \|f\|^2_{L^2(\mathbb{R}^2)}.
\end{aligned}$$

$\mathcal{P}$ is a unitary operator on $L^2_{\text{even}}(\mathbb{R}^2)$ since it is a surjective isometry. By Proposition 102, (4.3), (4.4) and Definition 100

$$\begin{aligned}
\pi(\mathscr{S}_{a,\ell,y})(\mathcal{P}f)(u) &= e^{-2\pi i\langle y,u\rangle} D_{A_a{}^tS_\ell}(\mathcal{P}f)(u) \\
&= a^{3/4}e^{-2\pi i\langle y,u\rangle}(\mathcal{P}f)(A_a{}^tS_\ell u) \\
&= a^{3/4}e^{-2\pi i\langle y,u\rangle}\frac{1}{|2au_1|^{1/2}}f(\Phi^{-1}(A_a{}^tS_\ell u)) \\
&= \frac{a^{1/4}}{|2u_1|^{1/2}}e^{-2\pi i\langle y,u\rangle}f(\mathcal{A}_a^{-1}({}^tS_{\ell/2})\Phi^{-1}(u)) \\
&= \frac{a^{1/4}}{|2u_1|^{1/2}}e^{-\pi i(2\langle y,\Phi(\Phi^{-1}(u))\rangle)}f(\mathcal{A}_a^{-1}({}^tS_{\ell/2})\Phi^{-1}(u)) \\
&= \frac{a^{1/4}}{|2u_1|^{1/2}}e^{-\pi i\langle \mathcal{B}_y\Phi^{-1}(u),\Phi^{-1}(u)\rangle}f(\mathcal{A}_a^{-1}({}^tS_{\ell/2})\Phi^{-1}(u)) \\
&= \frac{1}{|2u_1|^{1/2}}(\mu(\mathscr{S}_{a,\ell,y})f)(\Phi^{-1}(u)) \\
&= \mathcal{P}(\mu(\mathscr{S}_{a,\ell,y})f)(u).
\end{aligned}$$

□

**Definition 104.** Define

$$\mathcal{L}_2'(\mathbb{R}^k) = \{f(x_1,x_2,\ldots,x_k) \text{ mapping } \mathbb{R}^k \to \mathbb{C} : f(x_2,\ldots,x_k,x_1) \in \mathcal{L}_2\}$$

The next two lemmas are the workhorses needed to prove that $(\text{CSG})_2$ is reproducing.

**Lemma 105.** Let $\mathbb{1}_+ = \mathbb{1}_{\ddot{\mathbb{R}}^2_+}$. If $h \in \mathcal{L}_2'(\mathbb{R}^2)$ then

$$\frac{\mathbb{1}_+(u)}{2u_1}[h(\Phi^{-1}(u)) + h(-\Phi^{-1}(u))] \in L^1 \cap L^2(\mathbb{R}^2).$$

*Proof.* The proof is exactly like the proof of Lemma 86, but with $\Phi$ instead of $\Psi$, $x_1$ and $x_2$ swapped, $u_1$ and $u_2$ swapped, and $\ddot{\mathbb{R}}^2_+$ instead of $\dot{\mathbb{R}}^2_+$. □

**Lemma 106.** Let $h \in \mathcal{L}_2'(\mathbb{R}^2)$. Then

$$\int_{\mathbb{R}^2} \left| \int_{\mathbb{R}^2} h(x) e^{2\pi i \langle y, \Phi(x) \rangle} dx \right|^2 dy = \int_{\ddot{\mathbb{R}}^2_+} |h(x) + h(-x)|^2 \frac{dx}{x_2^2}.$$

*Proof.* By Proposition 102

$$\begin{aligned}
\int_{\mathbb{R}^2} h(x) e^{2\pi i \langle y, \Phi(x) \rangle} dx &= (\int_{\ddot{\mathbb{R}}^2_+} + \int_{\ddot{\mathbb{R}}^2_-}) h(x) e^{2\pi i \langle y, \Phi(x) \rangle} dx \\
&= \int_{\ddot{\mathbb{R}}^2_+} h(x) e^{2\pi i \langle y, \Phi(x) \rangle} dx + \int_{\ddot{\mathbb{R}}^2_+} h(-x) e^{2\pi i \langle y, \Phi(-x) \rangle} dx \\
&= \int_{\ddot{\mathbb{R}}^2_+} [h(x) + h(-x)] e^{2\pi i \langle y, \Phi(x) \rangle} dx \\
&= \int_{\ddot{\mathbb{R}}^2_+} [h(\Phi^{-1}(u)) + h(-\Phi^{-1}(u))] e^{2\pi i \langle y, u \rangle} \frac{du}{2u_1} \\
&= \int_{\mathbb{R}^2} \frac{\mathbb{1}_+(u)}{2u_1} [h(\Phi^{-1}(u)) + h(-\Phi^{-1}(u))] e^{2\pi i \langle y, u \rangle} du \quad (4.11)
\end{aligned}$$

Since the integrand is in $L^1 \cap L^2(\mathbb{R}^2)$, we may apply Parseval's equality to obtain

$$\begin{aligned}
\|\text{Eqn}(4.11)\|^2_{L^2(\mathbb{R}^2)} &= \int_{\mathbb{R}^2} \left| \int_{\mathbb{R}^2} \frac{\mathbb{1}_+(u)}{2u_1} [h(\Phi^{-1}(u)) + h(-\Phi^{-1}(u))] e^{2\pi i \langle y, u \rangle} du \right|^2 dy \\
&= \int_{\mathbb{R}^2} | \frac{\mathbb{1}_+(u)}{2u_1} [h(\Phi^{-1}(u)) + h(-\Phi^{-1}(u))] |^2 du \\
&= \int_{\ddot{\mathbb{R}}^2_+} |h(\Phi^{-1}(u)) + h(-\Phi^{-1}(u))|^2 \frac{du}{(2u_1)^2} \\
&= \int_{\ddot{\mathbb{R}}^2_+} |h(x) + h(-x)|^2 \frac{x_2^2 dx}{(x_2^2)^2} \\
&= \int_{\ddot{\mathbb{R}}^2_+} |h(x) + h(-x)|^2 \frac{dx}{x_2^2}.
\end{aligned}$$

$\square$

We show that $(\mathrm{CSG})_2$ is reproducing and characterize the reproducing functions.

**Theorem 107.** *If $\phi \in L^2(\mathbb{R}^2)$, then*

$$\|f\|^2_{L^2(\mathbb{R}^2)} = \int_{(CSG)_2} |\langle f, \mu(\mathscr{S}_{a,\ell,y}) \phi \rangle|^2 \frac{da}{a^3} dy d\ell \text{ for all } f \in L^2(\mathbb{R}^2)$$

*if and only if*

$$\frac{1}{4} = \int_{\ddot{\mathbb{R}}^2_+} |\phi(y)|^2 \frac{dy}{y_2^4} = \int_{\ddot{\mathbb{R}}^2_+} |\phi(-y)|^2 \frac{dy}{y_2^4} \tag{4.12}$$

$$0 = \int_{\ddot{\mathbb{R}}^2_+} \overline{\phi}(y)\phi(-y)\frac{dy}{y_2^4}. \tag{4.13}$$

*Proof.* Assume that for all $f \in L^2(\mathbb{R}^2)$,

$$\|f\|^2_{L^2(\mathbb{R}^2)} = \int_{(\mathrm{CSG})_2} |\langle f, \mu(\mathscr{S}_{a,\ell,y})\phi\rangle|^2 \frac{da}{a^3} dy d\ell.$$

Then, in particular, this holds for all $f \in L^\infty \cap \mathcal{L}_2'(\mathbb{R}^2)$. Let $f \in L^\infty \cap \mathcal{L}_2'(\mathbb{R}^2)$ and set $h(x) = f(x)\overline{\phi}(\mathcal{A}_a^{-1}S_{\ell/2}x)$. Since $\phi \in L^2(\mathbb{R}^2)$, $h \in L^1 \cap \mathcal{L}'^2(\mathbb{R}^2)$. Thus, we obtain

$$\begin{aligned}
\|f\|^2_{L^2(\mathbb{R}^2)} &= \int_{\mathbb{R}}\int_{\mathbb{R}^2}\int_0^\infty \left|\int_{\mathbb{R}^2} h(x)e^{2\pi i\langle y,\Phi(x)\rangle}dx\right|^2 \frac{da}{a^{5/2}}dyd\ell \\
&= \int_{\mathbb{R}}\int_0^\infty\int_{\ddot{\mathbb{R}}^2_+} |h(x)+h(-x)|^2 \frac{dx}{x_2^2}\frac{da}{a^{5/2}}d\ell \\
&= \int_{\mathbb{R}}\int_0^\infty\int_{\ddot{\mathbb{R}}^2_+} (|h(x)|^2 + 2\Re h(x)\overline{h}(-x) + |h(-x)|^2)\frac{dx}{x_2^2}\frac{da}{a^{5/2}}d\ell \\
&= A + B + C,
\end{aligned}$$

where

$$\begin{aligned}
A &= \int_{\mathbb{R}}\int_0^\infty\int_{\ddot{\mathbb{R}}^2_+} |f(x)|^2|(\phi(\mathcal{A}_a^{-1}S_{\ell/2}x)|^2\frac{dx}{x_2^2}\frac{da}{a^{5/2}}d\ell \\
B &= \int_{\mathbb{R}}\int_0^\infty\int_{\ddot{\mathbb{R}}^2_+} |f(-x)|^2|(\phi(-\mathcal{A}_a^{-1}S_{\ell/2}x)|^2\frac{dx}{x_2^2}\frac{da}{a^{5/2}}d\ell \\
C &= 2\Re\int_{\mathbb{R}}\int_0^\infty\int_{\ddot{\mathbb{R}}^2_+} f(x)\overline{f}(-x)\overline{\phi}(\mathcal{A}_a^{-1}S_{\ell/2}x)\phi(-\mathcal{A}_a^{-1}S_{\ell/2}x)\frac{dx}{x_2^2}\frac{da}{a^{5/2}}d\ell.
\end{aligned}$$

Assume further that $f$ vanishes on $\ddot{\mathbb{R}}^2_-$. Then $B = C = 0$ and $\|f\|_{L^2(\mathbb{R}^2)} = A$. We perform a change of variables

$$(a,\ell) \mapsto y = (y_1,y_2) = \mathcal{A}_a^{-1}S_{\ell/2}x = (x_1 + \frac{\ell}{2}x_2, a^{1/2}).$$

The matrix of mixed partials is

$$\begin{pmatrix} 0 & \frac{1}{2}a^{-1/2}x_2 \\ \frac{x_2}{2} & 0 \end{pmatrix}.$$

So $dad\ell = \frac{1}{\frac{1}{4}a^{-1/2}x_2^2}dy_1dy_2 = \frac{4a^{1/2}}{x_2^2}dy_1dy_2$ and $a^{-2} = \frac{x_2^4}{y_2^4}$. Thus,

$$\begin{aligned} \|f\|^2_{L^2(\mathbb{R}^2)} &= \int_{\ddot{\mathbb{R}}^2_+}\int_{\ddot{\mathbb{R}}^2_+} |f(x)|^2|\phi(y)|^2\frac{dx}{x_2^2}\left(\frac{4}{x_2^2}\right)\left(\frac{x_2^4}{y_2^4}\right)dy \\ &= \int_{\ddot{\mathbb{R}}^2_+}\int_{\ddot{\mathbb{R}}^2_+} |f(x)|^2|\phi(y)|^2\frac{4}{y_2^4}dxdy \\ &= \|f\|^2_{L^2(\mathbb{R}^2)}\int_{\ddot{\mathbb{R}}^2_+}|\phi(y)|^2\frac{4}{y_2^4}dy. \end{aligned}$$

Similarly, now assume that $f \in L^\infty \cap \mathcal{L}_2'(\mathbb{R}^2)$ is supported in $\ddot{\mathbb{R}}^2_-$. Then,

$$\begin{aligned} \|f\|^2_{L^2(\mathbb{R}^2)} &= B \\ &= \int_{\ddot{\mathbb{R}}^2_+}\int_{\ddot{\mathbb{R}}^2_-} |f(x)|^2|\phi(-y)|^2\frac{4}{y_2^4}dxdy \\ &= \|f\|^2_{L^2(\mathbb{R}^2)}\int_{\ddot{\mathbb{R}}^2_+}|\phi(-y)|^2\frac{4}{y_2^4}dy. \end{aligned}$$

Thus any reproducing function must satisfy (4.12). Now simply assume that $f \in L^\infty \cap \mathcal{L}_2'(\mathbb{R}^2)$ and assume that $\phi$ satisfies (4.12). Then

$$\|f\|^2_{L^2(\mathbb{R}^2)} = \|f\|^2_{L^2(\ddot{\mathbb{R}}^2_+)} + \|f\|^2_{L^2(\ddot{\mathbb{R}}^2_-)} + C,$$

which implies that $C = 0$. Also

$$0 = C = 2\Re\int_{\ddot{\mathbb{R}}^2_+}\int_{\ddot{\mathbb{R}}^2_+} f(x)\overline{f}(-x)\overline{\phi}(y)\phi(-y)\frac{4}{y_2^4}dydx. \tag{4.14}$$

Assume that $\int_{\ddot{\mathbb{R}}^2_+} f(x)\overline{f}(-x)dx \neq 0$. If $f$ is also real valued, then (4.14) implies that $\Re\int_{\ddot{\mathbb{R}}^2_+}\overline{\phi}(y)\phi(-y)\frac{dy}{y_2^4} = 0$. If $f$ is purely imaginary, then (4.14) implies that $\Im\int_{\ddot{\mathbb{R}}^2_+}\overline{\phi}(y)\phi(-y)\frac{dy}{y_2^4} = 0$. Hence $\phi$ must satisfy (4.13).

Conversely, assume that $f \in \mathcal{L}_2'(\mathbb{R}^2)$. By the preceding arguments,

$$\begin{aligned}\int_{(\mathrm{CSG})_2} |\langle f, \mu(\mathscr{S}_{a,\ell,y})\phi\rangle|^2 \frac{da}{a^3} dy d\ell \quad = \quad & \int_{\ddot{\mathbb{R}}^2_+} |f(x)|^2 dx \int_{\ddot{\mathbb{R}}^2_+} |\phi(y)|^2 \frac{4}{y_2^4} dy \\ & + \int_{\ddot{\mathbb{R}}^2_-} |f(x)|^2 dx \int_{\ddot{\mathbb{R}}^2_+} |\phi(-y)|^2 \frac{4}{y_2^4} dy \\ & + 2\Re \int_{\ddot{\mathbb{R}}^2_+} f(x)\overline{f}(-x) dx \int_{\ddot{\mathbb{R}}^2_+} \overline{\phi}(y)\phi(-y) \frac{4}{y_2^4} dy.\end{aligned}$$

If $\phi$ satisfies (4.12) and (4.13), then

$$\int_{(\mathrm{CSG})_2} |\langle f, \mu(\mathscr{S}_{a,\ell,y})\phi\rangle|^2 \frac{da}{a^3} dy d\ell = \left( \int_{\ddot{\mathbb{R}}^2_-} + \int_{\ddot{\mathbb{R}}^2_+} \right) |f(x)|^2 dx + 0 = \|f\|^2_{L^2(\mathbb{R}^2)}.$$

Now let $f$ be any arbitrary function in $L^2(\mathbb{R}^2)$ where $\phi$ still satisfies (4.12) and (4.13). Since $L^\infty \cap \mathcal{L}_2'(\mathbb{R}^2)$ is dense in $L^2(\mathbb{R}^2)$, we can chose a $\{f_n\} \subset L^\infty \cap \mathcal{L}_2'(\mathbb{R}^2)$ which converges to $f$ in $L^2(\mathbb{R}^2)$. For any $f_n$, $f_m$, the difference $f_n - f_m$ lies in $L^\infty \cap \mathcal{L}_2'(\mathbb{R}^2)$. Hence

$$\|\langle f_n, \mu(\mathscr{S}_{a,\ell,y})\phi\rangle - \langle f_m, \mu(\mathscr{S}_{a,\ell,y})\phi\rangle\|^2_{L^2((\mathrm{CSG})_2)} = \|f_n - f_m\|^2_{L^2(\mathbb{R}^2)} \to 0$$

as $n, m \to \infty$. Hence, $\{\langle f_n, \mu(\mathscr{S}_{a,\ell,y})\phi\rangle\}_n$ is a Cauchy sequence in $L^2((\mathrm{CSG})_2)$. Also, it follows from the Cauchy-Schwarz inequality over $L^2(\mathbb{R}^2)$ that $\{\langle f_n, \mu(\mathscr{S}_{a,\ell,y})\phi\rangle\}_n$ converges pointwise to $\langle f, \mu(\mathscr{S}_{a,\ell,y})\phi\rangle$. A sequence which is Cauchy in norm and additionally converges pointwise also converges in norm to the pointwise limit. Thus,

$$\begin{aligned}\|f\|^2_{L^2(\mathbb{R}^2)} &= \lim_{n\to\infty} \|f_n\|^2_{L^2(\mathbb{R}^2)} \\ &= \lim_{n\to\infty} \int_{(\mathrm{CSG})_2} |\langle f_n, \mu(\mathscr{S}_{a,\ell,y})\phi\rangle|^2 \frac{da}{a^3} dy d\ell \\ &= \int_{(\mathrm{CSG})_2} |\langle f, \mu(\mathscr{S}_{a,\ell,y})\phi\rangle|^2 \frac{da}{a^3} dy d\ell,\end{aligned}$$

as desired. $\square$

## 4.5 Conclusion

Dahlke *et al.* created shearlet analogs for $L^2(\mathbb{R}^d)$, $d > 2$ in [31] and [32]. One paper deals with anisotropic dilations, which generalize the parabolic scaling matrix, and one paper uses isotropic dilations. They work from the perspective of co-orbit space theory and obtain a different generalization of the shearing matrix. It is curious because no intuitive generalization of the shearing matrix, other than the one employed in Section 4.3, yields a reproducing subgroup. We plan on using co-orbit space theory to analyze each $(\mathrm{TDS})_k$ in order to find discrete, implementable transformations. The generalization of the Translation-Dilation-Shearing group in Section 4.2 may be used to analyze multidimensional data and images. We are currently working to find the appropriate definition of $(\mathrm{CDS})_k$ for $k > 2$, generalizing the results of Section 4.4. Finally, we would like to completely characterize the dimensions of the reproducing subgroups of $\mathbb{R}^4 \rtimes \mathrm{Sp}(2, \mathbb{R})$. The results of this chapter successfully integrate shearlet theory into the theory created by Cordero, DeMari, Nowak and Tabacco and hopefully will lay the foundation of new transforms in various applied fields.

# Chapter 5

# Grassmannian fusion frames

## 5.1 Introduction

When data is transmitted over a communication line, the received message may be corrupted by noise and data loss. As an oversimplified example, if I send you the message `1729`, you could receive the noisy message `1728` or nothing at all. Representing data in a way that is resilient to such problems is clearly desirable. Expressing data using a redundant frame provides some protection, but some frames work better than others. Goyal *et al.* proved that a normalized frame is *optimally robust against* (*o.r.a.*) noise and one erasure if the frame is tight. Furthermore, a normalized frame is o.r.a multiple erasures if it is Grassmannian ([96], [7]).

**Definition 108.** For $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, define

$\mathscr{F}(N, \mathbb{F}^M) = \{\{e_i\}_{i=1}^N \subset \mathbb{F}^M : \|e_i\| = 1$ for all $1 \leq i \leq N$ and $\{e_i\}$ is a frame for $\mathbb{F}^M\}$.

The *maximal frame correlation* is

$$\mathcal{M}_\infty(\{e_i\}_{i=1}^N) = \max_{1 \leq i < j \leq N} \{|\langle e_i, e_j \rangle|\}.$$

A sequence of unit norm vectors $\{u_i\}_{i=1}^N \subseteq \mathbb{F}^M$ is called a *Grassmannian frame* if it is a solution to

$$\min_{\{e_i\} \in \mathscr{F}(N, \mathbb{F}^M)} \{\mathcal{M}_\infty(\{e_i\}_{i=1}^N)\}.$$

If $N = M$, the Grassmannian frames are precisely the orthonormal bases for $\mathbb{F}^M$. If $N = 3$ and $M = 2$, the 2-dimensional vectors representing the cubic roots of unity are a Grassmannian frame. However, the vectors representing the fourth roots of unity do not form a Grassmannian frame for $N = 4$ and $M = 2$. Since $|\langle(1,0),(-1,0)\rangle| = 1$, the fourth roots of unity actually have the maximum possible maximal frame correlation. The following theorem is proven in a number of classical texts, see [96] for one proof and citations of other methods.

**Theorem 109.** *Let $\{e_i\}_{i=1}^N$ be a normalized frame for $\mathbb{F}^M$. Then*

$$\mathcal{M}_\infty(\{e_i\}_{i=1}^N) \geq \sqrt{\frac{N-M}{M(N-1)}} \tag{5.1}$$

Equality holds in (5.1) if and only if $\{e_i\}$ is an equiangular tight frame. However, equality can only hold for certain $N$ and $M$. Bodmann and Paulsen proved a functorial equivalence between real equiangular frames and $\alpha$-regular 2-graphs, where $\alpha$ depends on $N$ and $M$ [20]. An $\alpha$-regular 2-graph is a particular type of hypergraph. This correspondence can be used to characterize when equiangular frames exist. Other than the case $N = M + 1$, there are very few known pairs $(N, M)$ which yield equiangular frames and, further, there are many pairs $(N, M)$ for which it has been proven that no equiangular frames exist. When equiangular frames do not exist, it can be complicated to construct Grassmannian frames.

**Definition 110.** For $1 \leq m \leq M$, set $G(M,m)$ to be the collection of $m$ dimensional subspaces of $\mathbb{F}^M$. As a manifold, $G(M,m) = O(M)/(O(m) \times O(M-m))$, where $O(M)$ denotes elements of the $\mathbb{R}$ orthogonal group or the $\mathbb{C}$ unitary group. $G(M,m)$ is also an algebraic projective variety.

$G(M, m)$ is called a *Grassmannian*. The *Grassmannian packing problem* is the problem of finding $N$ points in $G(M, m)$ so that the minimal distance between any two of them is as large as possible. Finding Grassmannian frames is equivalent to solving the Grassmannian packing problem for $G(M, 1)$. It is natural to ask what analytic structures one obtains by considering the Grassmannian packing problem for $m > 1$.

*Fusion frames*, originally called *frames of subspaces*, were introduced by Casazza and Kutyniok in [22]. There are many potentially exciting applications of fusion frames, in areas such as coding theory [19], distributed sensing [23], and neurology [91].

**Definition 111.** A *fusion frame* for $\mathbb{F}^M$ is a finite collection of subspaces $\{\mathcal{W}_i\}_{i=1}^N$ in $\mathbb{F}^M$ such that there exist $0 < A \leq B < \infty$ satisfying

$$A\|x\|^2 \leq \sum_{i=1}^{N} \|P_i x\|^2 \leq B\|x\|^2,$$

where $P_i$ is an orthogonal projection onto $\mathcal{W}_i$. If $A = B$, we say that the fusion frame is *tight*.

For two subspaces $\mathcal{W}_i, \mathcal{W}_j \subset \mathbb{F}^M$ of equal dimension $m$, define the *chordal distance*

$$\operatorname{dist}(\mathcal{W}_i, \mathcal{W}_j) = [m - \operatorname{tr}(P_i P_j)]^{1/2},$$

where $P_i$ is an orthogonal projection onto $\mathcal{W}_i$. A tight fusion frame consisting of equal dimensional subspaces with equal pairwise chordal distances is an *equi-distance tight fusion frame*.

Similar to (1.3), a fusion frame is tight with bound $A$ if and only if

$$\sum_{i=1}^{N} P_i = AI_M. \tag{5.2}$$

Fusion frames may either be viewed as generalizations of frames or special types of frames. In the former sense, we are merely replacing the projections (modulo a constant multiples) of vectors $x \in \mathcal{H}$ (on line (1.2)) onto the subspace spanned by each frame vector with projections onto spaces of dimensions possibly higher than 1. In the latter sense, we may see a fusion frame as a frame with sub-collections of frame vectors which group in *nice* ways. As is common in the literature, *nice* is an oversimplification of some very deep properties. In fact, splitting frames into such sub-collections is related to the (in)famous Feichtinger Conjecture. A fusion frame is o.r.a. against noise if it is tight ([77]), a tight fusion frame is o.r.a. one subspace erasure if the dimensions of the subspaces are equal ([19]), and a tight fusion frame is o.r.a. multiple subspace erasures if the subspaces have equal chordal distances. An equi-distance tight fusion frame is a solution to the Grassmannian packing problem ([77]). The following definition does not exist in the literature but seems quite natural.

**Definition 112.** A fusion frame $\{\mathcal{W}_i\}_{i=1}^{N}$ for $\mathbb{R}^M$ consisting of $d$-dimensional subspaces shall be called a *Grassmannian fusion frame* if it is a solution to the Grassmannian packing problem for $N$ points in $G(M, m)$.

We would like to construct such objects. The idea is very simple and makes use of Hadamard matrices.

## 5.2 Hadamard matrices

The first Hadamard matrices were discovered by Sylvester in 1867 ([97]). In 1893, Hadamard first defined and started to characterize Hadamard matrices, which have the maximal possible determinant amongst matrices with entries from $\{\pm 1\}$ ([61]).

**Definition 113.** A *Hadamard Matrix* of order $n$ is an $n \times n$ matrix $H$ with entries from $\{\pm 1\}$ such that $H({}^tH) = nI_n$.

In the original paper by Hadamard, it was proven that Hadamard matrices must have order equal to 2 or a multiple of 4. It is still an open conjecture as to whether Hadamard matrices exist for every dimension divisible by 4. However, there are many constructions of Hadamard matrices, which use methods from number theory, group cohomology and other areas of math. Horadam's book [71] is an excellent resource. One class of Hadamard matrices are formed from Walsh functions.

**Definition 114.** The *Walsh functions* $\omega_j : [0, 1] \to \{\pm 1\} \subset \mathbb{R}$, $j \geq 0$, are piecewise constant functions which have the following properties:

1. $\omega_j(0) = 1$ for all $j \geq 0$,
2. $\omega_j$ has precisely $j$ sign changes (zero crossings), and
3. $\langle \omega_j, \omega_k \rangle = \delta_{jk}$.

Walsh functions have been used for over 100 years by communications engineers to minimize cross talk. The first four Walsh functions are

$$\begin{aligned}
\omega_0 &= \mathbb{1}_{[0,1]} \\
\omega_1 &= \mathbb{1}_{[0,1/2)} - \mathbb{1}_{[1/2,1]} \\
\omega_2 &= \mathbb{1}_{[0,1/4)} - \mathbb{1}_{[1/4,3/4)} + \mathbb{1}_{[3/4,1]} \\
\omega_3 &= \mathbb{1}_{[0,1/4)} - \mathbb{1}_{[1/4,1/2)} + \mathbb{1}_{[1/2,3/4)} - \mathbb{1}_{[3/4,1]}.
\end{aligned}$$

By sampling the first $2^n$ Walsh functions at the points $\frac{k}{2^n}$, one obtains a Hadamard matrix; that is

$$W_n = \left(\omega_j\left(\frac{k}{2^n}\right)\right)_{0\leq j,k<2^n}$$

is a $2^n \times 2^n$ Hadamard matrix. There is a speedy algorithm, the Fast Hadamard (or Walsh) Transform, for multiplying a vector by such a matrix.

## 5.3 Construction

The idea behind the new construction of the Grassmannian fusion frames is very simple. We remove the first $i$ rows of a Hadamard matrix $H$ to obtain a submatrix $H'$. The columns of $H'$ will then be partitioned into spanning sets for subspaces. Since the elements of the matrix are $\pm 1$, the computations should be simplified and the resulting (fusion) frame should be easy to implement. It is well-known (see, for example, [71]) that any Hadamard matrix can be normalized so that first row consists solely of 1s.

**Theorem 115.** *Let $H$ be a $2^n \times 2^n$ Hadamard matrix indexed by $0, \ldots, 2^n - 1$ which* *has been normalized so that the first row consists solely of* 1*s. Then*

$$\{e_k = \frac{1}{\sqrt{2^n - 1}}(H(j,k))_{1\le j\le 2^n-1} : 0 \le k \le 2^n - 1\}$$

*is a Grassmannian frame for* $\mathbb{F}^{2^n-1}$.

*Proof.* For any $0 \le k_1, k_2 \le 2^n - 1$,

$$\begin{aligned}
\langle H(j,k_1))_{0\le j\le 2^n-1}, H(j,k_2))_{0\le j\le 2^n-1}\rangle &= 2^n\delta_{k_1,k_2} \text{ and} \\
\langle H(k_1,j))_{0\le j\le 2^n-1}, H(k_2,j))_{0\le j\le 2^n-1}\rangle &= 2^n\delta_{k_1,k_2}.
\end{aligned}$$

Thus $\langle e_{k_1}, e_{k_2}\rangle = \frac{1}{2^n-1}(2^n\delta_{k_1,k_2} - 1)$, and the collection is equiangular. We now show that the $e_k$ do indeed form a frame. Let $x \in \mathbb{F}^{2^n-1}$ be arbitrary. We verify that (1.3) holds. Let

$$L = \left(\frac{1}{\sqrt{2^n - 1}}(H(j,k))\right)_{0\le k\le 2^n-1, 1\le j\le 2^n-1}.$$

Then, by the orthogonality of the columns,

$$\sum_{k=0}^{2^n-1}\langle x, e_k\rangle e_k = {}^*LLx = \frac{2^n}{2^n-1}x.$$

□

We follow it up with the construction of a class of Grassmannian fusion frames.

**Theorem 116.** *Let* $W_n$ *be the* $2^n \times 2^n$ *Walsh-Hadamard matrix indexed by* $0, \ldots, 2^n - 1$*. Then*

$$\left\{\mathcal{W}_k = \operatorname{span}\left\{(W_n(j,k))_{2\le j\le 2^n-1}, (W_n(j,k+2^{n-1}))_{2\le j\le 2^n-1}\right\} : 0 \le k \le 2^{n-1}\right\}$$

*is a tight Grassmannian fusion frame for* $\mathbb{F}^{2^n-2}$.

*Proof.* Since

$$W_n(0,k) = 1 \text{ for } 0 \le k \le 2^n - 1$$
$$W_n(1,k) = \begin{cases} 1 & \text{for } 0 \le k \le 2^{n-1} - 1 \\ -1 & \text{for } 2^{n-1} \le k \le 2^n - 1 \end{cases},$$

For any $0 \le k_1, k_2 \le 2^n - 1$, $\langle W_n(j,k_1))_{2 \le j \le 2^n - 1}, W_n(j,k_2))_{2 \le j \le 2^n - 1}\rangle$

$$= \begin{cases} 2^n - 2 & k_1 = k_2 \\ -2 & \text{for } 0 \le k_1, k_2 \le 2^{n-1} - 1 \text{ or } 2^{n-1} \le k_1, k_2 \le 2^n - 1 \\ 0 & \text{else.} \end{cases}$$

Thus, for each $0 \le k \le 2^{n-1}$,

$$\left\{ \frac{1}{\sqrt{2^n - 2}} (W_n(j,k))_{2 \le j \le 2^n - 1}, \frac{1}{\sqrt{2^n - 2}} (W_n(j, k + 2^{n-1}))_{2 \le j \le 2^n - 1} \right\}$$

is an orthonormal basis for $\mathcal{W}_k$, and $P_k = {}^*L_k L_k$, where

$$L_k = \left( \frac{1}{\sqrt{2^n - 2}} (W_n(j, k + i)) \right)_{i = 0,4, 1 \le j \le 2^n - 1}.$$

For $0 \le k_1, k_2 \le 2^{n-1} - 1$,

$$\begin{aligned} \operatorname{tr}(P_{k_1} P_{k_2}) &= \operatorname{tr}({}^*L_{k_1} L_{k_1} ({}^*L_{k_2} L_{k_2})) \\ &= \operatorname{tr}({}^*L_{k_1} (\frac{-2}{2^n - 2} I_2) L_{k_2}) \\ &= \frac{-2}{2^n - 2} \operatorname{tr}({}^*L_{k_1} L_{k_2}) \\ &= \frac{-2}{2^n - 2} (\frac{-2}{2^n - 2} + \frac{-2}{2^n - 2}) \\ &= \frac{2}{(2^{n-1} - 1)^2}. \end{aligned}$$

Thus the $\mathcal{W}_k$ have pairwise equal chordal distance. We now show that the $\mathcal{W}_k$ do indeed form a frame by verifying that (5.2) holds. Let $x \in \mathbb{F}^{2^n}$ be arbirary. Then,

$$\begin{aligned}\left(\sum_{k=0}^{2^{n-1}-1} P_k\right) x &= \sum_{k=0}^{2^n-1} \langle x, \frac{1}{\sqrt{2^n-2}}(W_n(j,k))_{2\leq j\leq 2^n-1}\rangle \frac{1}{\sqrt{2^n-2}}(W_n(j,k))_{2\leq j\leq 2^n-1} \\ &= \frac{2^n}{2^n-2}x.\end{aligned}$$

□

## 5.4 Future Work

Removing the first 3 rows of a Walsh-Hadamard matrix does not yield an equi-distance fusion frame. However, there has been recent work done to numerically find optimal Grassmannian packings ([41]) which works well in some dimensions. However, in other dimensions the convergence is incredibly slow. The algorithm uses a random initial configuration. Perhaps by seeding the algorithm with a Hadamard submatrix, the convergence would accelerate. On the other hand, removing the first 4 rows of a Walsh-Hadamard matrix does yield an equi-distance fusion frame. The proof is very similar to the proof of Theorem 116 in the preceding section. We conjecture that removing the first $2^k$ rows from $W_n$, $n > k$ will always yield an equi-distance fusion frame. Finally, we would like develop an algorithm for a fast implementation of the Walsh-Hadamard-generated fusion frames utilizing the Fast Hadamard Transform.

# Chapter 6

# $p$-adic wavelets

## 6.1 Introduction

Kurt Hensel introduced the $p$-adic numbers $\mathbb{Q}_p$, $p$ prime, in 1897 motivated by the connections between algebraic field theory for numbers and for functions ([68]). The $p$-adic numbers still play a big role in algebraic number and function theory ([74]), but there is a growing movement to incorporate $p$-adic numbers into pure and applied analysis. The strange topology of $\mathbb{Q}_p$ seems ideal to model quantum physical phenomena (see, for example, [73]). There even seems to be a relation between distance in a $p$-adic model and genetic code degeneracy ([44]). A good physical model will contain some sort of differentiation. One cannot define a derivative on $p$-adic function spaces, but the $p$-adics are locally compact abelian groups, so there exists a Haar measure. Thus, one can define pseudo-differential operators over the $p$-adics. It ends up that $p$-adic wavelets diagonalize certain pseudo-differential operators ([2], [92]).

Up until now, all $p$-adic wavelet systems have been formed using dilations by $p$. However, there is no reason to believe that other matrix dilations could not be used. Furthermore, it is well known (see, for example [36]) that a wavelet basis formed from a real multiresolution analysis (MRA) generated by dilations by a matrix $A$ contains $|\det A| - 1$ generating wavelets. Thus, we would like to generate an MRA with a $p$-adic matrix $A$ for which $|\det A|_2 = 2$.

## 6.2 Preliminaries

### 6.2.1 $p$-adic numbers

We begin by defining a valuation.

**Definition 117.** Let $\mathbb{F}$ be a field. A *valuation* $|\cdot|$ is a map $|\cdot| : \mathbb{F} \to \mathbb{R}$ satisfying :

1. $|\alpha| \geq 0$ for a $\alpha \in \mathbb{F}$, and $|\alpha| = 0$ if an only if $\alpha = 0$,
2. For all $\alpha$, $\beta \in \mathbb{F}$, $|\alpha\beta| = |\alpha||\beta|$, and
3. For all $\alpha$, $\beta \in \mathbb{F}$, $|\alpha + \beta| \leq |\alpha| + |\beta|$.

A valuation is called *non-Archimedean* if it satisfies the *strong triangle inequality*; that is, for all $\alpha$, $\beta \in \mathbb{F}$

$$|\alpha + \beta| \leq \max\{|\alpha|, |\beta|\}.$$

Two valuations $|\cdot|_1$, $|\cdot|_2$ are said to be equivalent if there exists a positive real number $s$ such that for all $\alpha \in \mathbb{F}$, $|\alpha|_2 = |\alpha|_1^s$.

We now define the $p$-adic valuation.

**Definition 118.** Given a prime $p$, every non-zero rational number $r$ may be written uniquely as $p^i m/n$, where $\gcd(m, n) = 1$ and $p \nmid m, n$. The *$p$-adic absolute value* $|\cdot|_p$ is a valuation on $\mathbb{Q}$ defined as

$$|r|_p = \begin{cases} 0 & \text{if } r = 0 \\ p^{-i} & \text{if } r \neq 0 \end{cases}$$

The *p-adic numbers* $\mathbb{Q}_p$ are $\mathbb{Q}$ analytically completed with respect to the $p$-adic absolute value.

The $p$-adic valuation is non-Archimedian, and, thus, $\mathbb{Q}_p$ has an interesting topology. For example, every ball is both topologically closed and open, and any two balls are either nested or disjoint. Over $\mathbb{Q}$, all of the valuations are known; this is one of a handful of results in number theory known as Ostrowski's theorem.

**Theorem 119.** *Every non-trivial valuation of $\mathbb{Q}$ is equivalent to either a $p$-adic valuation or the Euclidean absolute value.*

The proof of Ostrowski's theorem is quite simple, given certain elementary facts of valuation theory, which we will not state here.

For every prime $p$, the additive group of $\mathbb{Q}_p$ is a locally compact abelian group which contains the compact open subgroup $\mathbb{Z}_p$. The group $\mathbb{Z}_p$ is the set $\{\alpha \in \mathbb{Q}_p \Big| |\alpha|_p \leq 1\}$, the unit ball in $\mathbb{Q}_p$. Equivalently, $\mathbb{Z}_p$ is the subgroup generated by 1. It is well-known ([90]) that since $\mathbb{Q}_p$ is a locally compact abelian group with a compact open subgroup, it has a Haar measure normalized so that the measure of $\mathbb{Z}_p$ is 1. For simplicity, we shall denote this measure by $dx$. The metric on $\mathbb{Q}_p$ is induced by the valuation.

Every $p$-adic rational number $x$ may be expanded

$$x = \sum_{k=K}^{\infty} \alpha_k p^k,$$

where $|x|_p = p^{-K}$ and for all $K \leq k < \infty$, $\alpha_k \in \{0, 1, \ldots, p-1\}$. Notice that the series

$$\frac{1}{1-p} = \sum_{k=0}^{\infty} p^k$$

converges. Thus, for each prime $p$,

$$-1 = \frac{p-1}{1-p} = \sum_{k=0}^{\infty} (p-1)p^k.$$

**Definition 120.** For any $x = \sum_{k=K}^{\infty} \alpha_k p^k \in \mathbb{Q}_p$, we define the *fractional part* of $x$, $\{x\} = \sum_{k=K}^{-1} \alpha_k p^k$, where the sum is formally 0 if $K \geq 0$ and the *integer part* of $x$, $[x] = \sum_{k=0}^{\infty} \alpha_k p^k$. We also define the set

$$I_p = \{x \in \mathbb{Q}_p : \{x\} = x\}.$$

$I_p$ is a set of coset representatives for $\mathbb{Q}_p/\mathbb{Z}_p$. Since $\mathbb{Z}_p$ is open, $I_p$ is discrete. This set will be very useful in the work that follows because $\mathbb{Q}_p$ has no discrete subgroup. We mention the $p$-adic valuation of elements of finite algebraic extensions, which we shall use to analyze eigenvalues.

**Definition 121.** If $\alpha$ is the root of a monic, irreducible polynomial $x^n + a_{n-1}x^{n-1} + \ldots + a_0 \in \mathbb{Q}_p[x]$, we define $|\alpha|_p = \sqrt[n]{|a_o|_p}$.

Also, for any $k \geq 1$, $\mathbb{Q}_p^k$ is a vector space over $\mathbb{Q}_p$. We shall denote the norm with single bars, rather than double bars, and define it as

**Definition 122.** For any ${}^t(x_1, x_2, \ldots, x_k) \in \mathbb{Q}_p^k$, the norm is

$$|{}^t(x_1, x_2, \ldots, x_k)|_p = \max_{1 \leq i \leq k} \{|x_i|_p\}.$$

### 6.2.2 $p$-adic wavelets

There are two main function theories over the $p$-adics. One deals with functions $\mathbb{Q}_p \to \mathbb{C}$ and the other with $\mathbb{Q}_p \to \mathbb{Q}_p$. We shall only deal with the former theory. The latter differs dramatically from the former. For example, $p$-adic valued functions have no Haar measure, but there is differentiation. For an analytic treatment of both function theories, see [73].

In 2002, Kozyrev published the first example of a $p$-adic wavelet ([76])

$$\{D_{p^j} T_a e^{\{p^{-1}kx\}} \mathbb{1}_{\mathbb{Z}_p}(x) = p^{-j/2} e^{\{p^{-1}k(p^j x - a)\}} \mathbb{1}_{\mathbb{Z}_p}(p^j x - a) : k = 1, 2, \ldots, p-1, j \in \mathbb{Z}, a \in I_p\}.$$

Since $\mathbb{Q}_p$ does not have a discrete subgroup, he translated by a discrete set of coset representatives. This approach is typical, not only in $p$-adic wavelet analysis, but also in Gabor theory over locally compact abelian groups ([54]). Also notice the sign of the exponent of the constant that comes out during dilation

$$D_p f(x) = p^{-1/2} f(px),$$

in contrast to the real case. This is due to the $p$-adic valuation.

After Kozyrev's publication, Shelkovich and Skopina created a $p$-adic MRA theory ([92]), and Benedetto and Benedetto introduced $p$-adic wavelet set theory ([15], [10]). We shall be concerned with $p$-adic MRA. We now define a $p$-adic MRA, generalized for this thesis.

**Definition 123.** Let $A \in GL(k, \mathbb{Q}_p)$. If the $p$-adic valuations of the eigenvalues are all strictly greater than 1 and $|\det A|_p \in \mathbb{N}$ then $A$ is an *expansive* $p$-adic matrix.

**Definition 124.** Let $A \in GL(k, \mathbb{Q}_p)$ be an expansive $p$-adic matrix. A collection of closed spaces $V_j \subset L^2(\mathbb{Q}_p^k)$ is called a *multiresolution analysis* (*MRA*) for $L^2(\mathbb{Q}_p^k)$ if the following hold:

a. $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$,

b. $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{Q}_p)$,

c. $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$,

d. $f(\cdot) \in V_J \Leftrightarrow f(A\cdot) \in V_{j+1}$ for all $j \in \mathbb{Z}$, and

e. there exists a function $\phi \in V_0$, called the *scaling function*, such that the system $\{\phi(\cdot - a) : a \in I_p^k\}$ is an orthonormal basis for $V_0$.

The structure is very similar to an MRA over $\mathbb{R}^k$. Notice that by definition, the functions $\{|\det A|_p^{1/2}\phi(A^j x - a) : a \in I_p^k\}$ form an orthonormal basis for $V_j$. Mimicking real MRA, for each $j \in \mathbb{Z}$, we define *wavelet spaces* $W_j$ by

$$V_{j+1} = V_j \oplus W_j.$$

It then follows from Definition 124.b-d that

$$L^2(\mathbb{Q}_p^k) = \bigoplus_{j\in\mathbb{Z}} W_j.$$

Thus, if $\psi \in W_0$ is such that $\{\psi(x - a) : a \in I_p^k\}$ is an orthonormal basis for $W_0$, then

$$\{|\det A|^{1/2}\psi(A^j x - a) : j \in \mathbb{Z}, a \in I_p^k\}$$

is an *orthonormal (wavelet) basis* for $L^2(\mathbb{Q}_p^k)$ We call such a $\psi$ a *(p-adic) wavelet* function. It follows from Definition 124.a,d that a scaling function $\phi$ must satisfy

$$\phi = \sum_{a \in I_p^k} \alpha_a \phi(A \cdot - a)$$

for some $\alpha_a \in \mathbb{C}$. We call such a function *refinable*. Assume that the $I_p^k$ translations of $\phi \in L^2(\mathbb{Q}_p^k)$ form an orthonormal set and that $\phi$ is refinable. Then define

$$V_j = \overline{\mathrm{span}}\{\phi(A \cdot - a) : a \in I_p^k\}.$$

Clearly, parts (d) and (e) of Definition 124 are satisfied. In the real case, the refinability of $\phi$ would also give us (a). However, that is not true in the $p$-adic case because $I_p^k$ does not form a group. In order for (a) to be true, $\phi(\cdot - b)$ must be refinable for each $b \in I_p^k$.

The number of wavelet functions needed to create an orthonormal basis corresponding to an MRA is equal to $|\det A| - 1$, where the bars represent whichever valuation one is working with. The only previously known wavelet bases for $L^2(\mathbb{Q}_2^2)$ required 3 wavelet generators. We would like to construct an MRA for $L^2(\mathbb{Q}_2^2)$ using a matrix $A$ for which $|\det A|_2 = 2$. We now have the tools we need to construct this new MRA. We shall construct the MRA in Section 6.3. We shall also present a wavelet for this MRA in 6.4. Finally, in Section 6.5, we review future directions for the research.

## 6.3 MRA construction

Let

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix},$$

the inverse of the well-known quincunx matrix. In [52], a multiresolution analysis was presented for $L^2(\mathbb{R}^2)$ which used dilations by the quincunx. Since $|\det A^{-1}| = 2$, there was only one wavelet generator. However, its support was a fractal, the twin dragon fractal. In contrast, we shall see that the MRA for $L^2(\mathbb{Q}_2^2)$ associated to this matrix corresponds to a wavelet with a simple support, namely, $\mathbb{Z}_2^2$.

The inverse of the quincunx matrix is expansive since the eigenvalues $\frac{1\pm i}{2}$ have 2-adic valuation $\sqrt{2} > 1$. Furthermore,

$$|\det A|_2 = \left|\frac{1}{2}\right|_2 = 2,$$

so we hope to obtain an MRA with a single generating wavelet. We first prove a lemma that will be useful in computations which will follow later in the thesis.

**Lemma 125.** Let $x_1, x_2 \in \mathbb{Q}_2$.

- If $|x_1|_2 > |x_2|_2$ then $|x_1 + x_2|_2 = |x_1 - x_2|_2 = |x_1|_2$;
- If $|x_1|_2 < |x_2|_2$ then $|x_1 + x_2|_2 = |x_1 - x_2|_2 = |x_2|_2$; and
- If $|x_1|_2 = |x_2|_2$ then $|x_1+x_2|_2 \leq \frac{1}{2}|x_1-x_2|_2 = \frac{1}{4}|x_1|_2$ or $|x_1-x_2|_2 \leq \frac{1}{2}|x_1+x_2|_2 = \frac{1}{4}|x_1|_2$.

*Proof.* Set $2^{-K_i} = |x_i|_2$ for $i = 1, 2$. There exist $\alpha_k \in \{0, 1\}$ and $\beta_k \in \{0, 1\}$ such that

$$\begin{aligned} x_1 &= 2^{K_1} + \sum_{k=K_1+1}^{\infty} \alpha_k 2^k \text{ and} \\ x_2 &= 2^{K_2} + \sum_{k=K_2+1}^{\infty} \beta_k 2^k. \end{aligned}$$

Since $-1 = \sum_{k=0}^{\infty} 2^k$,

$$-x_2 = 2^{K_2} + \sum_{k=K_2+1}^{\infty} (1 + \sum_{j=K_2+1}^{k} \beta_j) 2^k.$$

Assume that $|x_1|_2 > |x_2|_2$, then $K_2 \geq K_1 + 1$. So

$$\begin{aligned} |x_1 + x_2|_2 &= \left| 2^{K_1} + \sum_{k=K_1+1}^{K_2-1} \alpha_k 2^k + (1+\alpha_k) 2^{K_2} + \sum_{k=K_2+1}^{\infty} (\alpha_k + \beta_k) 2^k \right|_2 \\ &= |x_1|_2, \end{aligned}$$

and

$$\begin{aligned} |x_1 - x_2|_2 &= \left| 2^{K_1} + \sum_{k=K_1+1}^{K_2-1} \alpha_k 2^k + (1+\alpha_k) 2^{K_2} + \sum_{k=K_2+1}^{\infty} (\alpha_k + (1 + \sum_{j=K_1+1}^{k} \beta_j)) 2^k \right|_2 \\ &= |x_1|_2. \end{aligned}$$

By symmetry, if $|x_1|_2 < |x_2|_2$ then

$$|x_1 + x_2|_2 = |x_1 - x_2|_2 = |x_2|_2.$$

Now assume that $|x_1|_2 = |x_2|_2$. Then

$$\begin{aligned} |x_1 + x_2|_2 &= \left| (1 + \alpha_{K_1+1} + \beta_{K_2+1}) 2^{K_1+1} + \sum_{k=K_1+2}^{\infty} (\alpha_k + \beta_k) 2^k \right|_2 \\ &\begin{cases} = \frac{|x_1|_2}{2} & \text{for } \alpha_{K_1+1} = \beta_{K_2+1} \\ \leq \frac{|x_1|_2}{4} & \text{for } \alpha_{K_1+1} \neq \beta_{K_2+1}. \end{cases} \end{aligned}$$

Similarly,

$$
\begin{aligned}
|x_1 - x_2|_2 &= \left| (2 + \alpha_{K_1+1} + \beta_{K_2+1})2^{K_1+1} + \sum_{k=K_1+2}^{\infty} (\alpha_k + 1 + \sum_{j=K_2+1}^{k} \beta_j) 2^k \right|_2 \\
&\begin{cases} = \frac{|x_1|_2}{2} & \text{for } \alpha_{K_1+1} \neq \beta_{K_2+1} \\ \leq \frac{|x_1|_2}{4} & \text{for } \alpha_{K_1+1} = \beta_{K_2+1}. \end{cases}
\end{aligned}
$$

□

We wish to find a candidate for a scaling function from which we can build an MRA.

**Proposition 126.**

$$\mathbb{Z}_2^2 = A^{-1}\mathbb{Z}_2^2 \sqcup \left( A^{-1}\mathbb{Z}_2^2 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right),$$

*where* $\sqcup$ *represents disjoint union.*

*Proof.* We decompose $\mathbb{Z}_2^2$ into

$$\mathbb{Z}_2^2 = S_1 \sqcup S_2 \sqcup S_3 \sqcup S_4,$$

where

$$
\begin{aligned}
S_1 &= \{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{Z}_2^2 : |x_1|_2 = |x_2|_2 = 1 \} \\
S_2 &= \{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{Z}_2^2 : |x_1|_2 = 1, |x_2|_2 < 1 \} \\
S_3 &= \{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{Z}_2^2 : |x_2|_2 = 1, |x_1|_2 < 1 \} \text{ and} \\
S_4 &= \{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{Z}_2^2 : |x_1|_2, |x_2|_2 < 1 \}.
\end{aligned}
$$

It follows from simple parity arguments that

$$A^{-1}S_1 \sqcup A^{-1}S_4 \subseteq S_4, \quad A^{-1}S_2 \sqcup A^{-1}S_3 \subseteq S_1.$$

Also, $S_1 + \binom{0}{1} = S_2$ and $S_4 + \binom{0}{1} = S_3$. We shall now prove that

$$AS_1 \subseteq S_2 \sqcup S_3 \text{ and } AS_4 \subseteq S_1 \sqcup S_4.$$

Assume that $x \in S_1$ then $Ax = \frac{1}{2}\binom{x_1+x_2}{-x_1+x_2}$. It follows from Lemma 125 that one of $|\frac{1}{2}x_1 + \frac{1}{2}x_2|_2$ and $|-\frac{1}{2}x_1 + \frac{1}{2}x_2|_2$ is $\frac{1}{2}|\frac{1}{2}x_1|_2 = 1$ and one is $\leq \frac{1}{4}|\frac{1}{2}x_1|_2 = \frac{1}{2}$. Either way, $Ax \in S_2 \sqcup S_3$. Thus , $S_1 = A^{-1}S_2 \sqcup A^{-1}S_3$.

Now assume that $x \in S_4$. If either $|x_1|_2, |x_2|_2 < \frac{1}{2}$ or if $|x_1|_2 = |x_2|_2 = \frac{1}{2}$, then it follows from Lemma 125 that

$$\left|\frac{1}{2}x_1 + \frac{1}{2}x_2\right|_2, \left|-\frac{1}{2}x_1 + \frac{1}{2}x_2\right|_2 < 1.$$

If instead $|x_1|_2 < |x_2|_2 = \frac{1}{2}$ or $|x_2|_2 < |x_1|_2 = \frac{1}{2}$ then

$$\left|\frac{1}{2}x_1 + \frac{1}{2}x_2\right|_2 = \left|-\frac{1}{2}x_1 + \frac{1}{2}x_2\right|_2 = 1.$$

Thus, $AS_4 \subseteq S_1 \sqcup S_4$, which implies that $S_4 = A^{-1}S_1 \sqcup A^{-1}S_4$. Hence, $A^{-1}\mathbb{Z}_2^2 = S_1 \sqcup S_4$. Furthermore,

$$\begin{aligned} A^{-1}\mathbb{Z}_2^2 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} &= (S_1 \sqcup S_4) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= (S_1 + \begin{pmatrix} 0 \\ 1 \end{pmatrix}) \sqcup (S_4 + \begin{pmatrix} 0 \\ 1 \end{pmatrix}) \\ &= S_2 \sqcup S_3. \end{aligned}$$

□

Define

$$\phi(x) = \mathbb{1}_{\mathbb{Z}_2^2}(x)$$

and

$$V_j = \overline{\operatorname{span}}\{\phi(A^j x - a) : a \in I_2^2\} \text{ for } j \in \mathbb{Z}. \tag{6.1}$$

We shall show that $\phi$ and the $V_j$ form a multiresolution analysis.

**Proposition 127.** *Let $V_j$ be defined as in (6.1). For all $j \in \mathbb{Z}$, $V_j \subseteq V_{j+1}$.*

*Proof.* The proposition is equivalent to the statement

$$\phi(A^j x - a) \in \overline{\text{span}}\{\phi(A^{j+1}x - b) : b \in I_2^2\}$$

for all $j \in \mathbb{Z}$ and $a \in I_2^2$. Further, it suffices to prove this statement for the case $j = 0$. The remaining cases will follow from the substitution $x \mapsto A^j x$. It follows from Proposition 126 that $\phi$ satisfies the refinement equation

$$\phi(x) = \phi(Ax) + \phi(Ax + \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}).$$

Furthermore, $\phi$ is $\mathbb{Z}_2^2$-periodic. Let $a \in I_2^2$. Then

$$\begin{aligned} \phi(x - a) &= \phi(Ax - Aa) + \phi(Ax - Aa + \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}) \\ &= \phi(Ax - \{Aa\}) + \phi(Ax - \{Aa - \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}\}). \end{aligned}$$

$\square$

We now make note of an important lemma.

**Lemma 128.** For each $x \in \mathbb{Q}_2^2$, there exists $N \in \mathbb{Z}$ such that

$$|A^N x|_2 = 1.$$

*Proof.* We first employ a (real) linear algebra trick to compute $A^N x$. Namely, we view $A$ as a dilation and rotation. We write

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos(\frac{\pi}{4}) & \sin(\frac{\pi}{4}) \\ \sin(-\frac{\pi}{4}) & \cos(\frac{\pi}{4}) \end{pmatrix}.$$

So

$$A^k = 2^{-k/2} \begin{pmatrix} \cos(\frac{k\pi}{4}) & \sin(\frac{k\pi}{4}) \\ \sin(-\frac{k\pi}{4}) & \cos(\frac{k\pi}{4}) \end{pmatrix}.$$

In particular, if $k$ is even,

$$A^k = \begin{cases} 2^{-4j} I & ; \quad \text{if } k = 8j \\ 2^{-(4j+1)} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & ; \quad \text{if } k = 8j+2 \\ 2^{-(4j+2)} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} & ; \quad \text{if } k = 8j+4 \\ 2^{-(4j+3)} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & ; \quad \text{if } k = 8j+6 \end{cases}.$$

Hence, if $k \in \mathbb{Z}$ and $x \in \mathbb{Q}_2^2$,

$$|A^{2k}x|_2 = 2^k |x|_2. \tag{6.2}$$

Set $N$ to satisfy $2^{-N/2} = |x|_2$. □

**Corollary 129.**

$$\bigcup_{N \in \mathbb{Z}} A^N \mathbb{Z}_2^2 = \mathbb{Q}_2^2.$$

*Proof.* The corollary immediately follows from the lemma. □

The following proposition is a generalization of a similar theorem in [3], which itself is an extension of a theorem well known in real wavelet theory [38].

**Proposition 130.** *Let $\varphi \in L^2(\mathbb{Q}_2^2)$. Define the spaces $V_j$, $j \in \mathbb{Z}$ be defined as $\overline{\text{span}}\{\varphi(A^j x - a) : a \in I_2^2\}$, $j \in \mathbb{Z}$. Also assume that $\varphi(\cdot - b) \in \cup_{j \in \mathbb{Z}} V_j$ for any*

$b \in \mathbb{Q}_2^2$. *Then*

$$\bigcup_{j\in\mathbb{Z}} V_j \text{ is dense in } L^2(\mathbb{Q}_2^2)$$

*if and only if*

$$\bigcup_{i\in\mathbb{Z}} \operatorname{supp} \hat{\varphi}(A^{-j}\cdot) = \widehat{\mathbb{Q}}_2^2.$$

*Proof.* The proof is exactly like the proof of theorem 2.4 in [3] with $\mathbb{Q}_p$ replaced with $\widehat{\mathbb{Q}}_2^2$, $I_p$ replaced with $I_2^2$, and $p^{-j}$ replaced with $A^j$. $\square$

**Corollary 131.** *For $\phi = \mathbb{1}_{\mathbb{Z}_2^2}$ and $V_j$ defined as in (6.1),*

$$\overline{\cup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{Q}_2^2).$$

*Proof.* It follows from the $\mathbb{Z}_2^2$-periodicity of $\phi$ and Corollary 129 that the hypotheses of Proposition 130 are satisfied. $\square$

We are almost done showing that $\phi$ and the $V_j$ form an MRA.

**Proposition 132.** *Let $V_j$ be defined as in (6.1). Then*

$$\bigcap_{j\in\mathbb{Z}} V_j = \{0\}.$$

*Proof.* Let $f \in \cap_{j\in\mathbb{Z}} V_j$. Thus, for each $j \in \mathbb{Z}$, there exists $\{c_{j,a}\}_{a\in I_2^2}$ satisfying

$$f(x) = \sum_{a\in I_2^2} c_{j,a}\phi(A^j x - a).$$

Fix $y \in \mathbb{Q}_2^2$. By line (6.2) in the proof of Lemma 128, if $|y|_2 = 2^{-\frac{N}{2}}$, then $|A^N x|_2 = 1$. Also note that $|Ay|_2 \leq 2|y|_2$. Making use of line (6.2) again, we note that if $M$ is even and $M < N$ then

$$|A^M y|_2 = 2^{\frac{M}{2}}|y|_2 = 2^{\frac{M-N}{2}} < 1.$$

If $M = 2K+1$ and $M < N-1$ then

$$|A^M y|_2 \leq 2|A^{2K} y|_2 = 2^{K+1-\frac{N}{2}} < 1.$$

Thus, if $M < N$, then $A^M x \in \mathbb{Z}_2^2$, which implies that $|A^M y - a|_2 > 1$ for all $a \in I_2^2$, $a \neq 0$. So $\phi(A^M y - a) = 0$ for all $M < N$ and $a \in I_2^2 \backslash \{0\}$, while $\phi(A^M y) = 1$ for all $M < N$. Thus, $f(x) = c_{M,0}$ for each $M < N$. Similarly for another point $y'$, there exists $N' \in \mathbb{Z}$ satisfying $|A^{N'} y'|_2 = 1$, implying that $f(y') = c_{M,0}$ for all $M < N'$. By considering small enough $M$, we obtain $f(y) = f(y')$. Since $y$ and $y'$ were arbitrary, $f$ is constant. The only constant function in $L^2(\mathbb{Q}_2^2)$ is the 0 function, as desired. □

The following lemma is implicit in many other works on $p$-adic wavelet theory.

**Lemma 133.** If $V_0$ is defined as in (6.1), then $\{\phi(\cdot - a) : a \in I_2^2\}$ is an orthonormal basis for $V_0$.

*Proof.* Since $I_2^2$ is a set of coset representatives for $\mathbb{Q}_2^2/\mathbb{Z}_2^2$, for distinct $a, b \in I_2^2$

$$\operatorname{supp} \phi(\cdot - a) \cap \operatorname{supp} \phi(\cdot - b) = \emptyset.$$

We use a Haar measure normalized so that the measure of $\mathbb{Z}_2^2$ is 1. Thus, $\{\phi(\cdot - a) : a \in I_2^2\}$ is an orthonormal set. Since $V_0$ is the closed span of $\{\phi(\cdot - a) : a \in I_2^2\}$, the claim is proven. □

We are finally prepared to prove that $\phi$ and the $V_j$ form a MRA.

**Theorem 134.** *Let $\phi = \mathbb{1}_{\mathbb{Z}_2^2}$. Further define*

$$\begin{aligned} A &= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \quad \textit{and} \\ V_j &= \overline{\operatorname{span}}\{\phi(A^j x - a) : a \in I_2^2\}, \quad j \in \mathbb{Z}. \end{aligned}$$

*Then $\{V_j\}$ is a multiresolution analysis for $L^2(\mathbb{Q}_2^2)$ and $\phi$ is a scaling function for this MRA.*

*Proof.* Using the lettering from Definition 124, (a) follows from Proposition 127, (b) follows from Corollary 131, (c) follows from Proposition 132, (d) follows from the definition of the $V_j$ and (e) follows from Lemma 133. $\square$

## 6.4 Wavelet construction

We define the function $\psi(x) = \phi(Ax) - \phi(Ax - \binom{1/2}{1/2})$. We would like to show that $\psi$ is a wavelet corresponding to the MRA in Theorem 134. We make note that

$$\phi(Ax) = \frac{1}{2}(\phi(x) + \psi(x)) \tag{6.3}$$

$$\phi(Ax - \begin{pmatrix}1/2\\1/2\end{pmatrix}) = \frac{1}{2}(\phi(x) - \psi(x)). \tag{6.4}$$

**Lemma 135.** Let $a, b \in I_2^2$. If $A(b-a)$ or $A(b-a) + \binom{1/2}{1/2}$ is in $\mathbb{Z}_2^2$, then $a = b$.

*Proof.* If $A(b-a) \in \mathbb{Z}_2^2$ then $b - a \in \mathbb{Z}_2^2$ since $A^{-1}\mathbb{Z}_2^2 \subset \mathbb{Z}_2^2$. As $I_2^2$ is a set of coset respresentativies of $\mathbb{Q}_2^2/\mathbb{Z}_2^2$, $a = b$. If

$$A(b-a) + \begin{pmatrix}1/2\\1/2\end{pmatrix} = A(b - a + \begin{pmatrix}1/2\\1/2\end{pmatrix}) \in \mathbb{Z}_2^2,$$

then $b - a + \binom{0}{1} \in \mathbb{Z}_2^2$ and thus $a = b$. $\square$

We need to prove another lemma before proving the final theorem.

**Lemma 136.** Define

$$\begin{aligned} U_1 &= \{Ab \in I_2^2 : b \in I_2^2\}, \\ U_2 &= \{Ab + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in I_2^2 : b \in I_2^2\}, \\ U_3 &= \{Ab + \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} \in I_2^2 : b \in I_2^2\}, \text{ and} \\ U_4 &= \{Ab + \begin{pmatrix} -1/2 \\ 1/2 \end{pmatrix} \in I_2^2 : b \in I_2^2\}. \end{aligned}$$

Then $I_2^2 = U_1 \sqcup U_2 \sqcup U_3 \sqcup U_4$.

*Proof.* We begin by making a comment about notation. For any prime $p$ it is impossible (see, for example [73]) to define a partial order structure on $\mathbb{Q}_p$ such that

- $1 \geq 0 \geq -1$,
- if $a, b \geq 0$, then $a + b \geq 0$, and
- if $a_n \geq 0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = a$, then $a \geq 0$

all hold. However, in order to prove this lemma, we shall use the symbols "$<$" and "$\leq$." When those symbols appear, they will be used between 2-adic numbers which are also non-negative dyadic rational numbers. The symbols should then be interpreted as coming from the canonical strict total and total orderings placed on the non-negative dyadic rational numbers as a subset of the real numbers.

Assume that $a = \binom{a_1}{a_2} \in I_2^2$. We analyze when $A^{-1}a$, $A^{-1}(a - \binom{0}{1})$, $A^{-1}(a - \binom{1/2}{1/2})$ and $A^{-1}(a - \binom{-1/2}{1/2})$ lie in $I_2^2$.

$$A^{-1}a = \begin{pmatrix} a_1 - a_2 \\ a_1 + a_2 \end{pmatrix}$$

lies in $I_2^2$ if and only if $a_2 \leq a_1$ and $0 \leq a_1 + a_2 < 1$. Similarly,

$$A^{-1}(a - \begin{pmatrix} 0 \\ 1 \end{pmatrix}) = \begin{pmatrix} a_1 - a_2 + 1 \\ a_1 + a_2 - 1 \end{pmatrix}$$

lies in $I_2^2$ if and only if $a_1 < a_2$ and $1 \leq a_1 + a_2 < 2$,

$$A^{-1}(a - \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}) = \begin{pmatrix} a_1 - a_2 \\ a_1 + a_2 - 1 \end{pmatrix}$$

lies in $I_2^2$ if and only if $a_2 \leq a_1$ and $1 \leq a_1 + a_2 < 2$, and

$$A^{-1}(a - \begin{pmatrix} -1/2 \\ 1/2 \end{pmatrix}) = \begin{pmatrix} a_1 - a_2 + 1 \\ a_1 + a_2 \end{pmatrix}$$

lies in $I_2^2$ if and only if $a_1 < a_2$ and $0 \leq a_1 + a_2 < 1$. Thus,

$$\begin{aligned} U_1 &= \{a \in I_2^2 : a_2 \leq a_1 \text{ and } 0 \leq a_1 + a_2 < 1\}, \\ U_2 &= \{a \in I_2^2 : a_1 < a_2 \text{ and } 1 \leq a_1 + a_2 < 2\}, \\ U_3 &= \{a \in I_2^2 : a_2 \leq a_1 \text{ and } 1 \leq a_1 + a_2 < 2\}, \text{ and} \\ U_4 &= \{a \in I_2^2 : a_1 < a_2 \text{ and } 0 \leq a_1 + a_2 < 1\}, \end{aligned}$$

which implies that $I_2^2 = U_1 \sqcup U_2 \sqcup U_3 \sqcup U_4$. □

We introduce the following notation. Let

$$\begin{aligned} \tilde{U}_1 &= \{b \in I_2^2 : Ab \in U_1\}, \\ \tilde{U}_2 &= \{b \in I_2^2 : Ab + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in U_2\}, \\ \tilde{U}_3 &= \{b \in I_2^2 : Ab + \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} \in U_3\}, \text{ and} \\ \tilde{U}_4 &= \{b \in I_2^2 : Ab + \begin{pmatrix} -1/2 \\ 1/2 \end{pmatrix} \in U_4\}. \end{aligned}$$

Note that $\tilde{U}_1 \cap \tilde{U}_2, \tilde{U}_3 \cap \tilde{U}_4 = \emptyset$, but the other pairwise intersections are not empty. Thus, unlike the $U_i$, the $\tilde{U}_i$ are not disjoint. However,

$$I_2^2 \subset \tilde{U}_1 \cup \tilde{U}_2 \subset \tilde{U}_1 \cup \tilde{U}_2 \cup \tilde{U}_3 \cup \tilde{U}_4. \tag{6.5}$$

**Theorem 137.** *$\psi = \phi(A\cdot) - \phi(A\cdot - \binom{1/2}{1/2})$ is a wavelet corresponding to the MRA in Theorem 134. That is, $\{\psi(\cdot - a) : a \in I_2^2\}$ is an orthonormal basis for $W_0 = V_1 \ominus V_0$.*

*Proof.* We first show that $\{\psi(\cdot - a) : a \in I_2^2\}$ is an orthonormal set. We compute for $a, b \in I_2^2$, using $\mathbb{Z}_2^2$-periodicity of $\phi$,

$$
\begin{aligned}
&2\langle \psi(x-a), \psi(x-b)\rangle = |\det A|_2 \langle \psi(x-a), \psi(x-b)\rangle \\
= \quad & |\det A|_2 \langle \phi(Ax - Aa) - \phi(Ax - (Aa + \begin{pmatrix}1/2\\1/2\end{pmatrix})), \phi(Ax - Ab) - \phi(Ax - (Ab + \begin{pmatrix}1/2\\1/2\end{pmatrix}))\rangle \\
= \quad & \langle \phi(x - Aa), \phi(x - Ab) - \phi(x - (Ab + \begin{pmatrix}1/2\\1/2\end{pmatrix}))\rangle \\
& \quad + \langle \phi(x - (Aa + \begin{pmatrix}1/2\\1/2\end{pmatrix})), -\phi(x - Ab) + \phi(x - (Ab + \begin{pmatrix}1/2\\1/2\end{pmatrix}))\rangle \\
= \quad & \langle \phi(x - Aa), \phi(x - Ab) - \phi(x - (Ab + \begin{pmatrix}1/2\\1/2\end{pmatrix})) + \phi(x - Ab) - \phi(x - (Ab + \begin{pmatrix}1/2\\1/2\end{pmatrix}))\rangle \\
= \quad & 2\langle \phi(x - Aa), \phi(x - Ab) - \phi(x - (Ab + \begin{pmatrix}1/2\\1/2\end{pmatrix}))\rangle \\
= \quad & 2\langle \phi(x), \phi(x - A(b-a)) - \phi(x - (A(b-a) + \begin{pmatrix}1/2\\1/2\end{pmatrix}))\rangle \\
= \quad & \begin{cases} 2 & \text{if } b = a \\ 0 & \text{if } b \neq a \end{cases}
\end{aligned}
$$

by Lemma 135. Hence $\{\psi(\cdot - a) : a \in I_2^2\}$ is an orthonormal set. We now show that $\psi(x - a) \in V_0^{\perp}$ for each $a \in I_2^2$. We compute for $a, b \in I_2^2$, using the $\mathbb{Z}_2^2$-periodicity of $\phi$,

$$
\begin{aligned}
& |\det A|_2 \langle \psi(x-a), \phi(x-b)\rangle \\
= \quad & |\det A|_2 \langle \phi(Ax - Aa) - \phi(Ax - (Aa + \begin{pmatrix}1/2\\1/2\end{pmatrix})), \phi(Ax - Ab) + \phi(Ax - (Ab + \begin{pmatrix}1/2\\1/2\end{pmatrix}))\rangle \\
= \quad & \langle \phi(x - Aa), \phi(x - Ab) + \phi(x - (Ab + \begin{pmatrix}1/2\\1/2\end{pmatrix}))\rangle \\
& \quad + \langle \phi(x - (Aa + \begin{pmatrix}1/2\\1/2\end{pmatrix})), -\phi(x - Ab) - \phi(x - (Ab + \begin{pmatrix}1/2\\1/2\end{pmatrix}))\rangle \\
= \quad & \langle \phi(x - Aa), \phi(x - Ab) + \phi(x - (Ab + \begin{pmatrix}1/2\\1/2\end{pmatrix})) - \phi(x - Ab) - \phi(x - (Ab + \begin{pmatrix}1/2\\1/2\end{pmatrix}))\rangle \\
= \quad & 0,
\end{aligned}
$$

as desired. We finally claim that if $f \in V_1$, $f \in V_0^{\perp}$ and $f \perp \psi(\cdot - a)$ for each $a \in I_2^2$, then $f = 0$. Assume that $f \in V_1$. Then there exists $\{c_a\} \in \ell^2(I_2^2)$ such that we may use Lemma 136 as well as lines (6.3) and (6.4) to expand $f$ as

$$\begin{aligned}
f(x) &= \sum_{a \in I_2^2} c_a \phi(Ax - a) \\
&= \sum_{b \in \tilde{U}_1} c_{Ab} \phi(Ax - Ab) + \sum_{b \in \tilde{U}_2} c_{Ab + \binom{0}{1}} \phi(Ax - (Ab + \begin{pmatrix} 0 \\ 1 \end{pmatrix})) + \\
&\quad \sum_{b \in \tilde{U}_3} c_{Ab + \binom{1/2}{1/2}} \phi(Ax - (Ab + \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix})) + \sum_{b \in \tilde{U}_4} c_{Ab + \binom{-1/2}{1/2}} \phi(Ax - (Ab + \begin{pmatrix} -1/2 \\ 1/2 \end{pmatrix})) \\
&= \frac{1}{2} \sum_{b \in \tilde{U}_1} c_{Ab} (\phi(x - b) + \psi(x - b)) + \frac{1}{2} \sum_{b \in \tilde{U}_2} c_{Ab + \binom{0}{1}} (\phi(x - b) + \psi(x - b)) \\
&\quad + \frac{1}{2} \sum_{b \in \tilde{U}_3} c_{Ab + \binom{1/2}{1/2}} (\phi(x - b) - \psi(x - b)) + \frac{1}{2} \sum_{b \in \tilde{U}_4} c_{Ab + \binom{-1/2}{1/2}} (\phi(x - b) - \psi(x - b))
\end{aligned}$$

If $f \in V_0^{\perp}$, then for all $a \in I_2^2$, $0 = \langle f, \phi(\cdot - a) \rangle$, which implies that

$$\begin{aligned}
0 &= \sum_{b \in \tilde{U}_1} c_{Ab} \langle \phi(x - b) + \psi(x - b), \phi(x - a) \rangle + \sum_{b \in \tilde{U}_2} c_{Ab + \binom{0}{1}} \langle \phi(x - b) + \psi(x - b), \phi(x - a) \rangle \\
&\quad + \sum_{b \in \tilde{U}_3} c_{Ab + \binom{1/2}{1/2}} \langle \phi(x - b) - \psi(x - b), \phi(x - a) \rangle \\
&\quad + \sum_{b \in \tilde{U}_4} c_{Ab + \binom{-1/2}{1/2}} \langle \phi(x - b) - \psi(x - b), \phi(x - a) \rangle \\
&= \begin{cases}
c_{Aa} & \text{for } a \in \tilde{U}_1 \cap \tilde{U}_3^c \cap \tilde{U}_4^c \\
c_{Aa} + c_{Aa + \binom{1/2}{1/2}} & \text{for } a \in \tilde{U}_1 \cap \tilde{U}_3 \\
c_{Aa} + c_{Aa + \binom{-1/2}{1/2}} & \text{for } a \in \tilde{U}_1 \cap \tilde{U}_4 \\
c_{Aa + \binom{0}{1}} & \text{for } a \in \tilde{U}_2 \cap \tilde{U}_3^c \cap \tilde{U}_4^c \\
c_{Aa + \binom{0}{1}} + c_{Aa + \binom{1/2}{1/2}} & \text{for } a \in \tilde{U}_2 \cap \tilde{U}_3 \\
c_{Aa + \binom{0}{1}} + c_{Aa + \binom{-1/2}{1/2}} & \text{for } a \in \tilde{U}_2 \cap \tilde{U}_4 \\
c_{Aa + \binom{1/2}{1/2}} & \text{for } a \in \tilde{U}_3 \cap \tilde{U}_1^c \cap \tilde{U}_2^c \\
c_{Aa + \binom{-1/2}{1/2}} & \text{for } a \in \tilde{U}_4 \cap \tilde{U}_1^c \cap \tilde{U}_2^c
\end{cases}
\end{aligned}$$

By line (6.5), these cases exhaust all of the $a \in I_2^2$. Hence,

$$f(x) = \sum_{b \in \tilde{U}_1 \cap (\tilde{U}_3 \cup \tilde{U}_4)} c_{Ab}\psi(x-b) \quad + \sum_{b \in \tilde{U}_2 \cap (\tilde{U}_3 \cup \tilde{U}_4)} c_{Ab+\binom{0}{1}}\psi(x-b).$$

If further, $f \perp \psi(\cdot - a)$ for all $a \in I_2^2$, then

$$\begin{aligned} 0 &= \sum_{b \in \tilde{U}_1 \cap (\tilde{U}_3 \cup \tilde{U}_4)} c_{Ab}\langle \psi(x-b), \psi(x-a)\rangle + \sum_{b \in \tilde{U}_2 \cap (\tilde{U}_3 \cup \tilde{U}_4)} c_{Ab+\binom{0}{1}}\langle \psi(x-b), \psi(x-a)\rangle \\ &= \begin{cases} c_{Aa} & \text{for } a \in \tilde{U}_1 \cap (\tilde{U}_3 \cup \tilde{U}_4) \\ c_{Aa+\binom{0}{1}} & \text{for } a \in \tilde{U}_2 \cap (\tilde{U}_3 \cup \tilde{U}_4) \end{cases}. \end{aligned}$$

Hence $f$ is identically 0. □

Thus, $\{D_{A^n}T_a\psi : n \in \mathbb{Z}, a \in I_2^2\}$ is an orthonormal basis for $L^2(\mathbb{Q}_2^2)$ generated by a single wavelet.

## 6.5 Future work

It has been proven that the Haar MRA is the only MRA that exists for $L^2(\mathbb{Q}_p)$ under dilation by $p$ ([3]). One possible choice for the scaling function of such an MRA is $\mathbb{1}_{\mathbb{Z}_p}$. The known MRAs for $L^2(\mathbb{Q}_p^k)$ are tensor products of the 1-dimensional systems. It follows from the definition of the $\mathbb{Q}_p^k$ metric that $\bigotimes_{i=1}^k \mathbb{1}_{\mathbb{Z}_p} = \mathbb{1}_{\mathbb{Z}_p^k}$, which for $p = 2$ and $k = 2$ was the scaling function used in this thesis. It remains to be seen if there exists a $p$-adic MRA for which $\mathbb{1}_{\mathbb{Z}_p^k}$ cannot be a scaling function. We would like to construct MRAs using different dilations in order to try to answer this problem.

Benedetto and Bendetto initiated the study of local field wavelet sets, which included the construction of $p$-adic wavelet sets ([10], [15]). We are also interested in constructing wavelet sets associated to this new dilation. Unlike in the real setting, $p$-adic wavelet set wavelets are already "smooth," so there is no need to smooth them as we did in Chapters 2 and 3.

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