ABSTRACT

Title of dissertation: Density properties of
Euler characteristic -2 surface group,
$\mathbb{PSL}(2, \mathbb{R})$ character varieties.

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In 1981, Dr. William Goldman proved that surface group representations into $\mathbb{PSL}(2, \mathbb{R})$ admit hyperbolic structures if and only if their Euler class is maximal in the Milnor-Wood interval. Furthermore the mapping class group of the prescribed surface acts properly discontinuously on its set of extremal representations into $\mathbb{PSL}(2, \mathbb{R})$. However, little is known about either the geometry of, or the mapping class group action on, the other connected components of the space of surface group representations into $\mathbb{PSL}(2, \mathbb{R})$. This article is devoted to establishing a few results regarding this.

Density properties of Euler characteristic -2 surface group, $\mathbb{PSL}(2,\mathbb{R})$ character varieties.

by

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CONTENTS

1. INTRODUCTION

1.1 Motivations for work and results obtained

$\mathbb{PSL}(2,\mathbb{R})$ and $\mathbb{PSL}(2,\mathbb{C})$ act on $\mathbb{H}^2$ and $\mathbb{CP}^1$ respectively by Möbius transformations. If Σ is a closed oriented surface and

$$\rho : \pi_1(\Sigma) \longrightarrow \mathbb{PSL}(2,\mathbb{R})$$

is a representation, let $e(\rho)$ be the Euler class of the flat bundle over Σ with fibre $\mathbb{H}^2$, structure group $\mathbb{PSL}(2,\mathbb{R})$ and holonomy ρ. $e(\rho)$ is a member of $H^2(\Sigma,\mathbb{Z})$ and therefore can be thought of as an integer.

Similarly if

$$\rho : \pi_1(\Sigma) \longrightarrow \mathbb{PSL}(2,\mathbb{C})$$

is a representation, let $w(\rho)$ be the top Stiefel-Whitney class of the flat bundle over Σ with fibre $\mathbb{CP}^1$, structure group $\mathbb{PSL}(2,\mathbb{C})$ and holonomy ρ. $w(\rho)$ is a member of $H^2(\Sigma,\mathbb{Z}/2\mathbb{Z})$ but can be thought of as an integer modulo 2.

By results of Milnor and Wood, $|e(\rho)| \leq -\chi(\Sigma)$, [11], [14]. Furthermore if

$$\chi(\Sigma) \leq n \leq -\chi(\Sigma),$$

then n occurs as the Euler class of some representation,

$$\rho : \pi_1(\Sigma) \longrightarrow \mathbb{PSL}(2,\mathbb{R}),$$

[3]. $e(\rho)$ parameterizes the path components of $\mathsf{Hom}(\pi_1(\Sigma), \mathbb{PSL}(2,\mathbb{R}))$ [5], each of which can be realized as a complex, rank $g-1+e(\rho)$, vector bundle over $Sym^d(\Sigma)$ and is therefore a homotopy equivalent to Σ [9]. ρ occurs as the holonomy of a hyperbolic structure on Σ if and only if $|e(\rho)| = -\chi(\Sigma)$,[3]. The mapping class group of Σ (the group of isotopy classes of homeomorphisms of Σ) acts properly discontinuously on this pair of components of $\mathsf{Hom}(\pi_1(\Sigma), \mathbb{PSL}(2,\mathbb{R}))$ only.

Similarly $w(\rho)$ parameterizes the path components of $\mathsf{Hom}(\pi_1(\Sigma), \mathbb{PSL}(2,\mathbb{C}))$.

$$\rho : \pi_1(\Sigma) \longrightarrow \mathbb{PSL}(2,\mathbb{C})$$

occurs as the holonomy of a complex projective structure if and only if the image of ρ is non-elementary and $w(\rho) = 0$, [2]. It is worth noting that when a representation,

$$\rho : \pi_1(\Sigma) \longrightarrow \mathbb{PSL}(2,\mathbb{R}),$$

is viewed as a representation,

$$\rho : \pi_1(\Sigma) \longrightarrow \mathbb{PSL}(2,\mathbb{C}),$$

$w(\rho) = e(\rho) \mathsf{\ mod\ } 2$. Therefore there are $\mathbb{PSL}(2,\mathbb{R})$ representations that do not occur as the holonomy of hyperbolic structures yet do occur as the holonomy of complex projective structures on Σ.

Let $\mathsf{k} = \mathbb{C}$ or $\mathbb{R}$ and let $X = \mathbb{H}^2$ or $\mathbb{CP}^1$ respectively.

$$\rho : \pi_1(\Sigma) \longrightarrow \mathbb{PSL}(2,\mathsf{k})$$

is said to admit a branched hyperbolic or complex projective structure if there is a branched ρ-equivariant map, D_ρ, from the universal cover of Σ to X. In addition to characterizing representations,

$$\rho : \pi_1(\Sigma) \longrightarrow \mathbb{PSL}(2, \mathbb{C}),$$

that occur as the holonomy of complex projective structures on Σ, Gallo, Kapovich and Marden also proved that

$$\rho : \pi_1(\Sigma) \longrightarrow \mathbb{PSL}(2, \mathbb{C})$$

admits a branched complex projective structure on Σ if and only if its image is non-elementary and $w(\rho) = 0 \text{ mod } 2$ [2].

Despite the great success in understanding when $\mathbb{PSL}(2, \mathbb{C})$ representations admit branched complex projective structures, it is not known when representations,

$$\rho : \pi_1(\Sigma) \longrightarrow \mathbb{PSL}(2, \mathbb{R}),$$

admit branched hyperbolic structures. Ser Tan Peow found an example of an Euler class 2 representation of the genus-3 surface group into $\mathbb{PSL}(2, \mathbb{R})$ that does not admit a branched hyperbolic structure but is arbitrarily close to representations that do, [12]. Furthermore Goldman conjectured that if $e(\rho) = \pm(-\chi(\Sigma) + 1)$, it admits a branched hyperbolic structure, [unpublished]. Until recently, there has been little progress on Goldman's conjecture.

In 2001, while trying to prove Goldman's conjecture, Daniel Virgil Mathews obtained the following partial results.

Let Σ_g be the genus-g surface and (for later) let $\Sigma_{g,h}$ be the genus-g surface with h holes.

Moreover, let S_g be the set of Euler class $\pm(\chi(\Sigma_g)+1)$ representations of the Σ_g group into $\mathbb{PSL}(2,\mathbb{R})$ that takes a separating simple closed curve to a non-hyperbolic isometry.

Let N_g be the set of Euler class $\pm(\chi(\Sigma_g)+1)$ representations of the Σ_g group into $\mathbb{PSL}(2,\mathbb{R})$ that takes a non-separating simple closed curve to a elliptic isometry.

Let B_g be the set of Euler class $\pm(\chi(\Sigma_g)+1)$ representations of the Σ_g surface group into $\mathbb{PSL}(2,\mathbb{R})$ admitting a branched hyperbolic structure.

Mathews established Goldman's conjecture for members of S_2. Although S_2 is not necessarily the entire Euler class 1 component of the space of Σ_2 group representations, it has non-empty interior.

Theorem 1. *Every Euler class $\pm(\chi(\Sigma_2)+1)$ representation of the genus-2 surface group into $\mathbb{PSL}(2,\mathbb{R})$ that takes a separating simple closed curve to a non-hyperbolic isometry admits a branched hyperbolic structure, [10].*

Mathews established Goldman's conjecture for a dense subset of N_g, namely $B_g \cap N_g$ is dense in B_g.

Theorem 2. *The set of Euler class $\pm(\chi(\Sigma_g)+1)$ representations of the genus-g surface group into $\mathbb{PSL}(2,\mathbb{R})$ that admits a branched hyperbolic structure is dense in the set of Euler class $\pm(\chi(\Sigma_g)+1)$ representations of the genus-g surface group that takes a non-separating simple closed curve to an elliptic isometry, [10].*

Theorems 1 and 2 imply that B_2 is dense in the open subset of Euler class 1 representations of the Σ_2 surface group into $\mathbb{PSL}(2,\mathbb{R})$ taking a simple closed curve to an elliptic isometry.

This article is devoted to better understanding the relationships between Theorems 1 and 2. In particular the following assertions will be proved:

Theorem 3. *Let P be the set of Euler class $\pm(\chi(\Sigma_2)+1)$, genus-2 surface group representations into $\mathbb{PSL}(2,\mathbb{R})$ that take a separating simple closed curve to a parabolic isometry. Let E be the set of Euler class $\pm(\chi(\Sigma_2)+1)$, genus-2 surface group representations into $\mathbb{PSL}(2,\mathbb{R})$ that take a non-separating simple closed curve to an elliptic isometry. Then $P \cap E$ is dense in P.*

The proof of the above theorem involves pulling ρ back by certain homeomorphisms of Σ and applying the resulting representation to a canonical non-separating simple closed curve.

Theorem 4. *Let either $\Sigma \simeq \Sigma_{1,2}$ or $\Sigma \simeq \Sigma_2$. If a representation,*

$$\rho : \pi_1(\Sigma) \longrightarrow \mathbb{PSL}(2,\mathbb{R}),$$

takes all boundary components to non-identity isometries and takes a non-separating simple closed curve to an elliptic isometry, then ρ is arbitrarily close to a representation, $\overline{\rho}$ (with the same boundary data as ρ), that takes a separating simple closed curve to a unipotent isometry.

In other words, the set of Σ group representations that takes all boundary components to non-identity isometries and takes a separating simple closed curve to a unipotent isometry is dense in the set of Σ group representations that take a non-separating simple closed curve to an elliptic isometry.

Corollary. If $\Sigma \simeq \Sigma_2$ and if the Euler class 1 representation,

$$\rho : \pi_1(\Sigma) \longrightarrow \mathbb{PSL}(2, \mathbb{R}),$$

takes some non-separating simple closed curve to an elliptic isometry, then ρ is arbitrarily close to a representation,

$$\overline{\rho} : \pi_1(\Sigma) \longrightarrow \mathbb{PSL}(2, \mathbb{R}),$$

that takes a separating simple closed curve to a parabolic isometry.

The proof of Theorem 4 involves first understanding when certain 4-holed sphere group representations take non-peripheral simple closed curves to parabolic isometries and then extending them to 2-holed torus and genus-2 surface group representations.

A noteworthy corollary to Theorems 3 and 4:

Corollary. Let $\mathsf{Simp} \subset \pi_1(\Sigma_2)$ be the set of classes represented by non-separating simple closed curves. If the Euler class ± 1 homomorphism,

$$\rho : \pi_1(\Sigma_2) \longrightarrow \mathbb{PSL}(2, \mathbb{R}),$$

takes a non-separating simple closed curve to an elliptic isometry, then ρ is arbitrarily close to a homomorphism,

$$\overline{\rho} : \pi_1(\Sigma_2) \longrightarrow \mathbb{PSL}(2, \mathbb{R}),$$

where the set $\{|\mathsf{Tr}(\overline{\rho}(\gamma))|\}_{\gamma \in \mathsf{Simp}}$ is dense in $[0, \infty)$.

The above corollary can be proved using results of Goldman but the proof in this article is independent.

Theorems 3 and 4 will be proved in Chapter 2.

In chapter 3, the following two theorems about boundary-parabolic, relative Euler class 1, 4-holed sphere,$\Sigma_{0,4}$, group representations are proved using methods similar to those used to prove Theorems 3 and 4 .

Theorem 5. *If a boundary parabolic, relative Euler class 1 representation,*

$$\rho : \pi_1(\Sigma_{0,4}) \longrightarrow \mathbb{PSL}(2, \mathbb{R}),$$

takes a simple closed curve to an elliptic isometry, then ρ is arbitrarily close to a representation,

$$\overline{\rho} : \pi_1(\Sigma_{0,4}) \longrightarrow \mathbb{PSL}(2, \mathbb{R}),$$

so that there is a decomposition of

$$\Sigma_{0,4} = \Sigma^1 \bigoplus_{\gamma} \Sigma^2$$

into three holed spheres, Σ^1 and Σ^2, so that

- $\overline{\rho}_{|\pi_1(\Sigma^1)}$ *is abelian*

  and

- $\overline{\rho}_{|\pi_1(\Sigma^2)}$ *is the holonomy of a cusped hyperbolic structure.*

The relative Euler class will be defined in section 1.6.2.

Theorem 6. *There are infinitely many irreducible, non-discrete, relative Euler class 1 homomorphisms of the 4-holed sphere group into $\mathbb{PSL}(2, \mathbb{R})$ that take all simple closed curves to hyperbolic isometries.*

Theorem 6 is quite unexpected seeing that irreducible, non-discrete representations take some curve to an elliptic isometry.

1.2 Notation and conventions

The term, "surface", denotes a compact oriented surface with possibly non-empty boundary while the term, "closed surface", refers to a surface with empty boundary.

If Σ is a surface, $\widetilde{\Sigma}$ is its universal cover.

Definition 7. A curve, γ, is said to be peripheral if it is either null-homotopic or freely homotopic to a boundary component, otherwise, γ is called non-peripheral.

Definition 8. Let Σ be a surface. If the non-peripheral simple closed curve, γ, separates Σ into surfaces, Σ^1 and Σ^2, with non-empty boundary, then $\Sigma = \Sigma^1 \bigoplus_\gamma \Sigma^2$.

If the surfaces, S_1 and S_2, are homeomorphic, then $S_1 \simeq S_2$.

Depending on the context, $\Sigma_{g,h}$ is either the compact oriented genus-g surface with h disks removed, or the oriented genus-g surface with h punctures.

- If $\Sigma \simeq \Sigma_{0,3}$, unless otherwise stated, assume that $\pi_1(\Sigma)$ has the following presentation:

  $$\pi_1(\Sigma_{0,3}) = \langle A, B, C | A \cdot B \cdot C \rangle.$$

  Here A, B and C represent boundary components of $\Sigma_{0,3}$.

- If $\Sigma \simeq \Sigma_{1,1}$, unless otherwise stated, assume that $\pi_1(\Sigma)$ has the following presentation:

  $$\pi_1(\Sigma_{1,1}) = \langle A, B, C | [A, B] \cdot C \rangle.$$

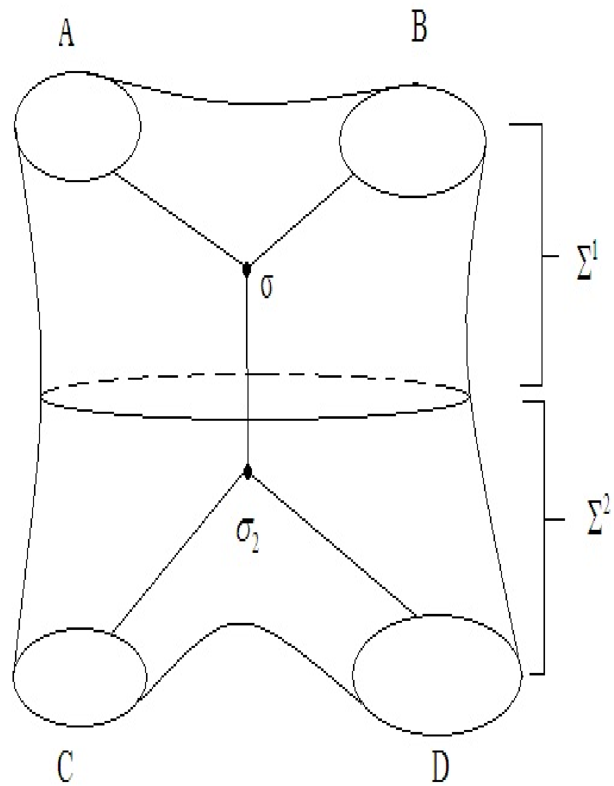
  Here A and B represent non-separating simple closed curves that intersect one another exactly once. $[A, B]$ represents the boundary component of $\Sigma_{1,1}$.

- If $\Sigma \simeq \Sigma_{0,4} = \Sigma^1 \bigoplus_\gamma \Sigma^2$, then both $\Sigma^1 \simeq \Sigma^2 \simeq \Sigma_{0,3}$.

  Unless otherwise stated, assume that $\pi_1(\Sigma_{0,4})$ has following presentation:

  $$\pi_1(\Sigma_{0,4}) = \langle A, B, C, D | A \cdot B \cdot C \cdot D \rangle.$$

  Here A, B, C and D represent boundary components of $\Sigma_{0,4}$.

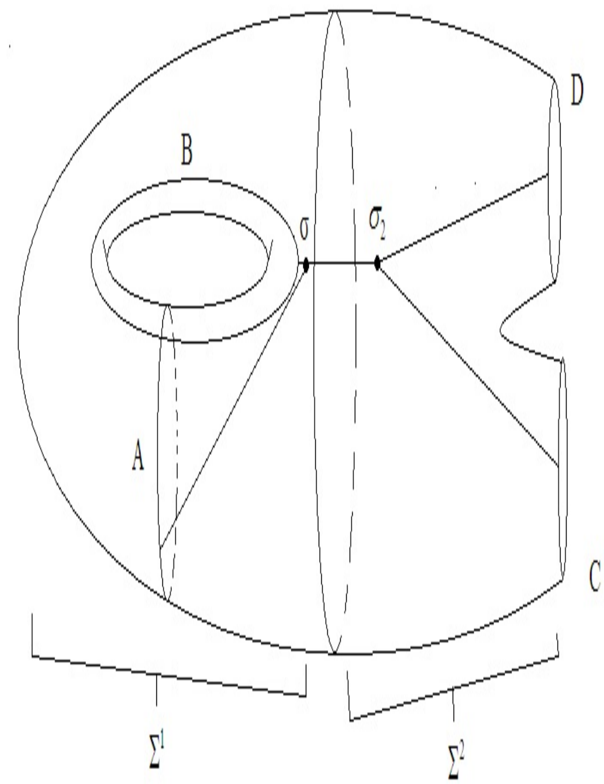


- If $\Sigma \simeq \Sigma_{1,2} = \Sigma^1 \bigoplus_\gamma \Sigma^2$, then $\Sigma^1 \simeq \Sigma_{1,1}$ and $\Sigma^2 \simeq \Sigma_{0,3}$. (Unless stated, assume this convention)

  Unless otherwise stated, assume that $\pi_1(\Sigma_{1,2})$ has following presentation:

$$\pi_1(\Sigma_{1,2}) = \langle A, B, C, D | [A, B] \cdot C \cdot D \rangle.$$

  C and D represent boundary components of $\Sigma_{1,2}$ while A and B represent non-separating simple closed curves that intersect each other exactly once while not intersecting either C or D.



- If $\Sigma \simeq \Sigma_2 = \Sigma^1 \bigoplus \Sigma^2$, then $\Sigma^1 \simeq \Sigma^2 \simeq \Sigma_{1,1}$. Unless otherwise stated, assume that $\pi_1(\Sigma_2)$ has following presentation:

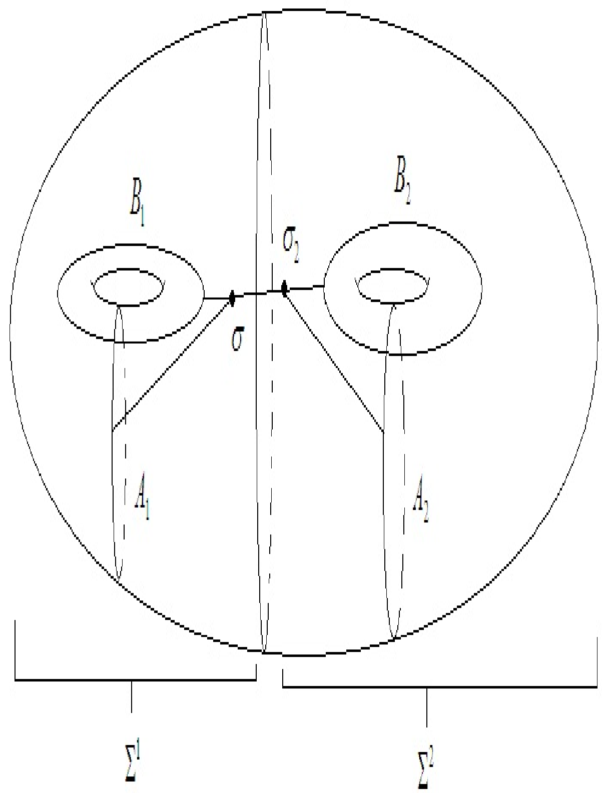
$$\pi_1(\Sigma_2) = \langle A_1, B_1, A_2, B_2 | [A_1, B_1] \cdot [A_2, B_2] \rangle.$$

A_1, B_1, A_2 and B_2 represent non-separating simple closed curves with

$$i(A_1, B_1) = i(A_2, B_2) = 1$$

while

$$i(A_1, A_2) = i(A_1, B_2) = i(B_1, A_2) = i(B_1, B_2) = 0.$$



- $\pi_1(\Sigma) := \pi_1(\Sigma, \sigma)$. ($\sigma$ is the prescribed base-point for $\pi_1(\Sigma)$.)

$\sigma = \sigma_1 \in \Sigma^1$ and $\sigma_2 \in \Sigma^2$. σ_1 is joined to σ_2 by a simple arc. If i is the inclusion of $\pi_1(\Sigma^2, \sigma_2)$ into $\pi_1(\Sigma, \sigma)$ given by the above mentioned simple arc then,

- if either $\Sigma \simeq \Sigma_{0,4}$ or $\Sigma \simeq \Sigma_{1,2}$,

$$\pi_1(\Sigma^1, \sigma) = \pi_1(\Sigma^1) = \langle A, B\rangle$$

and

$$i \circ \pi_1(\Sigma^2, \sigma_2) := \pi_1(\Sigma^2) = \langle C, D\rangle,$$

- if $\Sigma \simeq \Sigma_2$,

$$\pi_1(\Sigma^1, \sigma) = \pi_1(\Sigma^1) = \langle A_1, B_1 \rangle$$

and

$$i \circ \pi_1(\Sigma^2, \sigma_2) := \pi_1(\Sigma^2) = \langle A_2, B_2 \rangle.$$

1.3 Definition of a geometry

Definition 9. Let G be a path-connected, finite dimensional Lie group. Let $H \leq G$ be a closed Lie subgroup of G and let $X = G \diagup H$. When this is the case,

- G acts transitively on the homogeneous space, X, by left translation,
- X is an analytic manifold

  and
- G acts on X by analytic homeomorphisms.

Any such pair (X, G) is called a geometry.

Definition 10. Two geometries, (X_1, G_1) and (X_2, G_2), are said to be isomorphic if there is a Lie group isomorphism,

$$\phi : G_1 \longrightarrow G_2,$$

and a ϕ-equivariant homeomorphism,

$$h : X_1 \longrightarrow X_2.$$

There is a G-invariant Riemannian metric on X if and only if H is compact, [13]. Let G_1 and G_2 be path-connected, finite dimensional Lie groups. Let H_1 and H_2 be compact (and therefore closed) Lie subgroups of G_1 and G_2 respectively. Let

$$X_1 = G_1 \diagup H_1$$

and let

$$X_2 = G_2 \diagup H_2.$$

If (G_1, X_1) is isomorphic to (G_2, X_2), then their corresponding Riemannian geometries can be chosen to be isometric.

1.4 The hyperbolic plane

1.4.1 Standard models of the hyperbolic plane

$\mathbb{H}^2$ is the hyperbolic plane and $\mathsf{Isom}^+(\mathbb{H}^2)$ is its set of orientation preserving isometries. All of the following geometries are isomorphic (and isometric) and yield different models of the $(\mathbb{H}^2, \mathsf{Isom}^+(\mathbb{H}^2))$ geometry.

- **The Poincaré upper Half Plane Model** The underlying set, $\mathbb{H}^2$, is the upper half plane,

$$\{x + iy \in \mathbb{C} : y > 0\} \subset \mathbb{C} \subset \mathbb{CP}^1.$$

$$\mathsf{Isom}^+(\mathbb{H}^2) = \mathbb{PSL}(2, \mathbb{R}) = \mathbb{SL}(2, \mathbb{R}) \diagup \{\pm \mathbb{I}\}.$$

$\mathbb{PSL}(2,\mathbb{R})$ acts on the upper half plane as follows:

If $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{SL}(2,\mathbb{R})$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}.$$

The above $\mathbb{SL}(2,\mathbb{R})$ action on $\mathbb{H}^2$ descends to a $\mathbb{PSL}(2,\mathbb{R})$ action. The isotropy group of point is Lie group isomorphic to the compact Lie group, $\mathbb{SO}(2,\mathbb{R})/\{\pm\mathbb{I}\}$. Therefore $\mathbb{H}^2$ possesses an $\mathsf{Isom}^+(\mathbb{H}^2)$ invariant metric,

$$ds^2 = \frac{dx^2+dy^2}{y^2}.$$

It is possible to uniquely write any $\alpha \in \mathbb{SL}(2,\mathbb{R})$ as follows:

$$\alpha = A \cdot B,$$

where $A \in \mathbb{SL}(2,\mathbb{R})$ is a positive definite symmetric matrix and $B \in \mathbb{SO}(2)$. It follows that $\mathbb{SL}(2,\mathbb{R})$ and $\mathbb{PSL}(2,\mathbb{R}) \simeq \mathsf{Isom}^+(\mathbb{H}^2)$ are topological solid tori.

Geodesics are either circular arcs that intersect $\mathbb{R}$ orthogonally or vertical lines in $\mathbb{H}^2$.

- **The Poincaré Unit Disk Model** The underlying set, $\mathbb{H}^2$, is the interior of the unit disk in $\mathbb{C}$.

$$\mathsf{Isom}^+(\mathbb{H}^2) = \mathbb{PSU}(1,1) = \{\begin{pmatrix} a & \overline{c} \\ c & \overline{a} \end{pmatrix} : |a|^2 - |c|^2 = 1\}/\{\pm\mathbb{I}\}.$$

As in the Poincaré Upper Half Plane Model, $\mathbb{PSU}(1,1)$ acts on $\mathbb{H}^2$ as follows: If $\begin{pmatrix} a & \overline{c} \\ c & \overline{a} \end{pmatrix} \in \mathbb{SU}(1,1)$, then

$$\begin{pmatrix} a & \overline{c} \\ c & \overline{a} \end{pmatrix} \cdot z = \frac{az+\overline{c}}{cz+\overline{a}}.$$

The above action descends to a $\mathbb{PSU}(1,1)$ action on $\mathbb{H}^2$.

Geodesics in this model are circular arcs that intersect the unit circle orthogonally.

Remark 11. It is well known that $\mathbb{PSL}(2,\mathbb{C})$ acts on $\mathbb{CP}^1$. The underlying sets for the above two models of $\mathbb{H}^2$ are subsets of $\mathbb{CP}^1$ and each realization of $\mathsf{Isom}^+(\mathbb{H}^2)$ includes into $\mathbb{PSL}(2,\mathbb{C})$. Each inclusion map is equivariant with respect to the $\mathsf{Isom}^+(\mathbb{H}^2)$ actions on $\mathbb{H}^2$ and $\mathbb{CP}^1$.

- **The Lorentz Hyperboloid Model** Let $\mathbb{R}^{2,1}$ denote $\mathbb{R}^3$ with the indefinite signature $(2,1)$ metric,

$$< (x,y,z),(w,u,v) >= -xw+yu+zv.$$

The underlying set, $\mathbb{H}^2$, is

$$\{\overline{x} = (x_1,x_2,x_3) \in \mathbb{R}^{2,1} :< \overline{x},\overline{x} >= -1, x_1 > 0\}.$$

$\mathsf{Isom}^+(\mathbb{H}^2) = \mathbb{PSO}(2,1)$ (the set of linear transformations of $\mathbb{R}^3$ that leave $<,>$ invariant and preserve the sign of x_1) acts on $\mathbb{H}^2$ in the obvious way.

Geodesics are the intersections of 2 dimensional linear vector spaces with $\mathbb{H}^2$.

- **The Klein Projective Model** Radially project the Lorentz Hyperboloid Model onto the unit disk $\mathbb{H}^2 = \{(x,y,1) : y^2 + z^2 < 1\}$. $\mathsf{Isom}^+(\mathbb{H}^2) = \mathbb{PSO}(2,1)$. Geodesics are chords through $\mathbb{H}^2$.

Unless otherwise stated, the Poincaré Upper Half Plane Model will be used when doing calculations while pictures will be drawn in the Poincaré Unit Disk Model.

If $\alpha \in \mathbb{SL}(2,\mathbb{R})$, then $\mathsf{Tr}(\alpha)$ denotes the trace of α while $|\mathsf{Tr}(\alpha)|$ refers to the absolute value of the trace of α. If $\alpha \in \mathbb{PSL}(2,\mathbb{R})$ then, $|\mathsf{Tr}(\alpha)|$ is well defined.

1.4.2 Isometries of the hyperbolic plane

The orientation preserving isometries of $\mathbb{H}^2$ fall into exactly 1 of the following 4 categories:

- **The Identity Transformation** Not much to be said here except that throughout this article $\mathbb{I}$ will denote the Identity transformation.

- **Hyperbolic Transformations** leave exactly 1 geodesic, g_T, invariant and have exactly two fixed points in $\overline{\mathbb{H}^2}$. Depending on the model, either $\overline{\mathbb{H}^2} \subseteq \mathbb{CP}^1$ (as in Poincaré Unit Disk and Upper Half-Plane Models) or $\overline{\mathbb{H}^2} \subseteq \mathbb{RP}^2$ (as in the Klein Projective Model). The hyperbolic transformation, T, translates every point on g_T by the same hyperbolic length l_T. In the Poincaré Models, the absolute value of the trace of a corresponding matrix equals $2\cosh(\frac{l_T}{2}) > 2$.

Two hyperbolic isometries with the same trace are conjugate in $\mathsf{Isom}^+(\mathbb{H}^2)$.

- **Elliptic Transformations** fix exactly 1 point, $p_T \in \mathbb{H}^2$, and leave each hyperbolic circle centered at p_T invariant. Unlike hyperbolic transformations, these transformations have exactly one fixed point in $\overline{\mathbb{H}^2}$. Each non-fixed point in $\mathbb{H}^2$ is rotated by an angle, θ_T (that depends only on T), about the fixed point, p_T. In the Poincaré Models, the absolute value of the trace of a corresponding matrix equals $2\cos(\frac{\theta_T}{2}) < 2$. Two elliptic isometries with the same trace fall in one of two $\mathsf{Isom}^+(\mathbb{H}^2)$ conjugacy classes.

- **Parabolic Transformations** are non-identity transformations that neither fix a point in $\mathbb{H}^2$ nor leave a geodesic invariant. These transformations have exactly one fixed point in $\overline{\mathbb{H}^2}$. Parabolic transformations fall into one of two $\mathsf{Isom}^+(\mathbb{H}^2)$ conjugacy classes. The absolute value of the trace of a parabolic transformation is 2.

If $\alpha \in \mathbb{PSL}(2,\mathbb{R})$ is a hyperbolic element, α_* is the repeller of α while α^* is the attractor of α.

If $\alpha \in \mathbb{PSL}(2,\mathbb{R})$ is either an elliptic or a parabolic element, α_* is its fixed point in $\overline{\mathbb{H}^2}$ (the closure of $\mathbb{H}^2$).

Definition 12. $\alpha \in \mathbb{PSL}(2,\mathbb{R})$ is said to be unipotent if it is either parabolic or the identity.

Definition 13. For $p \in \overline{\mathbb{H}^2}$,

$$\mathsf{Stab}(p) := \{\alpha \in \mathsf{Isom}^+(\mathbb{H}^2) : \alpha \cdot p = p\}$$

is the stabilizer of p.

1.5 Development and holonomy

Let Σ be a compact oriented surface with possibly non-empty boundary. A hyperbolic structure on Σ is a metric, $<,>$, on Σ that is locally isometric to the metric on $\mathbb{H}^2$. Each hyperbolic structure comes with a homomorphism,

$$\rho : \pi_1(\Sigma) \longrightarrow \mathsf{Isom}^+(\mathbb{H}^2)$$

(its holonomy representation) and a map,

$$D_\rho : \widetilde{\Sigma} \longrightarrow \mathbb{H}^2$$

(its developing map), that is

- equivariant with respect to the LEFT $\pi_1(\Sigma)$ actions on $\widetilde{\Sigma}$ and $\mathbb{H}^2$

  and

- a homeomorphism onto its image.

[See [13] for explicit definition.]

Prescribing a hyperbolic structure on Σ is equivalent to assuming a holonomy representation and compatible developing map.

Definition 14. If ρ is realized as the holonomy of a hyperbolic structure on Σ, ρ is said to admit a hyperbolic structure on Σ.

Not all homomorphisms,

$$\rho : \pi_1(\Sigma) \longrightarrow \mathsf{Isom}^+(\mathbb{H}^2),$$

admit hyperbolic structures. For example, the trivial representation,

$$\mathbf{1} : \pi_1(\Sigma) \longrightarrow \mathsf{Isom}^+(\mathbb{H}^2),$$

cannot because unless Σ is simply connected, $\mathbf{1}$-equivariant maps,

$$D_{\mathbf{1}}\widetilde{\Sigma} \longrightarrow \mathbb{H}^2,$$

are never injective.

Question: Which closed oriented surface group representations into $\mathsf{Isom}^+(\mathbb{H}^2)$ admit hyperbolic structures?

In 1981 Dr. William Goldman answered this question. To precisely express Dr. Goldman's solution, one must understand the Euler class of a closed surface group representation into $\mathsf{Isom}^+(\mathbb{H}^2)$.

1.6 Euler class and relative Euler class of a surface group representation

1.6.1 Euler class of a closed surface group representation

Assume that Σ_g is a closed oriented genus-g surface.

$$\pi_1(\Sigma_g) = \langle A_1, B_1, \ldots, A_g, B_g | \prod_{1 \leq i \leq g} [A_i, B_i] \rangle.$$

$$R(A_1, B_1, \ldots, A_g, B_g) := \prod_{1 \leq i \leq g} [A_i, B_i].$$

In order to give the set of representations of $\pi_1(\Sigma_g)$ into $\mathsf{Isom}^+(\mathbb{H}^2) \simeq \mathbb{PSL}(2, \mathbb{R})$ a topology, view it as a closed subset of $\mathbb{PSL}(2, \mathbb{R})^{2g}$. $\mathsf{Isom}^+(\mathbb{H}^2) \simeq \mathbb{PSL}(2, \mathbb{R})$ acts on this subset as follows:

if $\alpha \in \mathsf{Isom}^+(\mathbb{H}^2)$ and

$$\rho : \pi_1(\Sigma) \longrightarrow \mathsf{Isom}^+(\mathbb{H}^2)$$

is a homomorphism, then define the homomorphism,

$$\alpha \cdot \rho : \pi_1(\Sigma_g) \longrightarrow \mathsf{Isom}^+(\mathbb{H}^2),$$

as follows:

$$(\alpha \cdot \rho)(\gamma) := \alpha \cdot \rho(\gamma) \cdot \alpha^{-1}$$

for $\gamma \in \pi_1(\Sigma_g)$.

To form the $\mathsf{Isom}^+(\mathbb{H}^2)$, genus-$g$ surface group character variety

$$\mathsf{Hom}(\pi_1(\Sigma_g), \mathsf{Isom}^+(\mathbb{H}^2)) / \mathsf{Isom}^+(\mathbb{H}^2)$$

identify two representations if and only if the closure of their orbits under the above action intersect.

Let

$$\rho : \pi_1(\Sigma_g) \longrightarrow \mathsf{Isom}^+(\mathbb{H}^2)$$

be a homomorphism. Define the Euler class of ρ, $e(\rho) \in \mathbb{Z}$, as follows:

Definition 15. $e(\rho)$ is computed as follows [11]:

Consider the following short exact sequence of groups:

$$1 \longrightarrow \pi_1(\mathsf{Isom}^+(\mathbb{H}^2)) \longrightarrow \widetilde{\mathsf{Isom}^+(\mathbb{H}^2)} \longrightarrow \mathsf{Isom}^+(\mathbb{H}^2) \longrightarrow 1.$$

(The first non-trivial homomorphism is the standard inclusion, i, of $\pi_1(\mathsf{Isom}^+(\mathbb{H}^2))$ into $\widetilde{\mathsf{Isom}^+(\mathbb{H}^2)}$ while the second is the universal covering homomorphism,

$$p : \widetilde{\mathsf{Isom}^+(\mathbb{H}^2)} \longrightarrow \mathsf{Isom}^+(\mathbb{H}^2).)$$

For each $i \leq g$, choose lifts of $\rho(A_i)$ and $\rho(B_i)$, (respectively) $\widetilde{\rho(A_i)}, \widetilde{\rho(B_i)} \in \widetilde{\mathsf{Isom}^+(\mathbb{H}^2)}$.

Because the universal covering map,

$$\widetilde{\mathsf{Isom}^+(\mathbb{H}^2)} \longrightarrow \mathsf{Isom}^+(\mathbb{H}^2),$$

is a homomorphism and the above sequence is exact,

$$R(\widetilde{\rho(A_1)}, \widetilde{\rho(A_2)}, \ldots, \widetilde{\rho(A_g)}, \widetilde{\rho(B_g)}) \in i \circ \pi_1(\Sigma) \simeq \mathbb{Z}.$$

Define

$$e(\rho) := i^{-1} \circ R(\widetilde{\rho(A_1)}, \widetilde{\rho(A_2)}, \ldots, \widetilde{\rho(A_g)}, \widetilde{\rho(B_g)}).$$

Lemma 16. *$e(\rho)$ does not depend on the choice of lifts of $\rho(A_i)$ and $\rho(B_i)$.*

Proof. This follows from the facts that $i(\pi_1(\mathsf{Isom}^+(\mathbb{H}^2)))$ is central in $\widetilde{\mathsf{Isom}^+(\mathbb{H}^2)}$ and R is a product of commutators.

$\square$

$e(\rho)$ is an integer valued function of

$$\mathsf{Hom}(\pi_1(\Sigma_g), \mathsf{Isom}^+(\mathbb{H}^2)) \big/ \mathsf{Isom}^+(\mathbb{H}^2).$$

When thought of this way, $e(\rho)$ is continuous and parameterizes the set of path components of the genus-g surface group character variety [5]. By the results of Milnor and Wood,

$$|e(\rho)| \leq -\chi(\Sigma_g).$$

This bound is known as the **Milnor-Wood Bound**.

Goldman proved in his Ph.D thesis that ρ admits a hyperbolic structure if and only if $e(\rho) = \pm\chi(\Sigma_g)$. When this is the case, ρ is said to be **extremal**. Otherwise ρ is **non-extremal**. The path components of

$$\mathsf{Hom}(\pi_1(\Sigma_g), \mathsf{Isom}^+(\mathbb{H}^2))/\mathsf{Isom}^+(\mathbb{H}^2)$$

that contain extremal representations are called **extremal components** while all other components are called **non-extremal components**.

Later Goldman conjectured that every Euler class $\pm(\chi(\Sigma_g)+1)$ representation,

$$\rho : \pi_1(\Sigma_g) \longrightarrow \mathsf{Isom}^+(\mathbb{H}^2),$$

admits a branched hyperbolic structure.

Definition 17.

$$\rho : \pi_1(\Sigma_g) \longrightarrow \mathsf{Isom}^+(\mathbb{H}^2)$$

is said to admit a branched hyperbolic structure if there is a branched map,

$$D_\rho : \widetilde{\Sigma_g} \longrightarrow \mathbb{H}^2,$$

that is equivariant with respect to the LEFT $\pi_1(\Sigma_g)$ actions on $\widetilde{\Sigma}$ and $\mathbb{H}^2$.

1.6.2 The relative Euler class of a surface group representation with non-elliptic boundary

Slightly modify the above construction for $\Sigma_{g,h\neq 0}$:

Definition 18. [10]

$$\pi_1(\Sigma_{g,h}) =$$

$$\langle A_1, B_1, \dots, A_g, B_g, C_1, \dots, C_h | \prod_{1\leq i\leq g} [A_i, B_i] \cdot \prod_{1\leq j\leq h} C_j \rangle$$

$$R(A_1, B_1, \dots, A_g, B_g, C_1, \dots, C_h) := \prod_{1\leq i\leq g} [A_i, B_i] \cdot \prod_{1\leq j\leq h} C_j.$$

Definition 19. A homomorphism,

$$\rho : \pi_1(\Sigma_{g,h}) \longrightarrow \mathsf{Isom}^+(\mathbb{H}^2),$$

is said to be boundary-non-elliptic if ρ takes all boundary components to non-elliptic isometries.

For any boundary-non-elliptic homomorphism,

$$\rho : \pi_1(\Sigma_{g,h}) \longrightarrow \mathsf{Isom}^+(\mathbb{H}^2),$$

there is a canonical simplest lift of $\rho(C_i)$ to $\widetilde{\mathsf{Isom}^+(\mathbb{H}^2)}$, $\widetilde{\rho(C_i)}$, (See [10]). Choose any lifts, $\widetilde{\rho(A_i)}$ and $\widetilde{\rho(B_i)}$, of $\rho(A_i)$ and $\rho(B_i)$. The relative Euler class of ρ, $e(\rho) \in \mathbb{Z}$, is defined as follows:

$$e(\rho) := i^{-1} \circ R(\widetilde{\rho(A_1)}, \widetilde{\rho(B_1)}, \dots, \widetilde{\rho(A_g)}, \widetilde{\rho(B_g)}, \widetilde{\rho(C_1)}, \dots, \widetilde{\rho(C_h)}).$$

As is $e(\rho)$ for

$$\rho : \pi_1(\Sigma_g) \longrightarrow \mathsf{Isom}^+(\mathbb{H}^2),$$

$e(\rho)$ is well defined and can be thought of as a continuous, integer valued function on the space of boundary non-elliptic homomorphisms

Furthermore the relative Euler class of a boundary non-elliptic representation,

$$\rho : \pi_1(\Sigma_{g,h}) \longrightarrow \mathsf{Isom}^+(\mathbb{H}^2),$$

is additive. More precisely, if

- γ is a simple closed curve on $\Sigma_{g,h}$,
- $$\Sigma_{g,h} = \Sigma^1 \bigoplus_{\gamma} \Sigma^2$$

  and
- $\rho(\gamma)$ is non-elliptic,

then

$$e(\rho) = e(\rho_{\pi_1(\Sigma^1)}) + e(\rho_{\pi_1(\Sigma^2)}).$$

(If $h = 0$, $e(\rho)$ is the Euler class of ρ.)

As with closed surfaces,

$$|e(\rho)| \leq -\chi(\Sigma_{g,h}).$$

This bound is also called the **Milnor-Wood Bound**.

The following important definitions end this section:

Definition 20. Let $C_1, \ldots, C_h \in \pi_1(\Sigma_{g,h})$ be represented by the boundary components of $\Sigma_{g,h}$. Then

$$\rho_1 : \pi_1(\Sigma_{g,h}) \longrightarrow \mathsf{Isom}^+(\mathbb{H}^2)$$

and

$$\rho_2 : \pi_1(\Sigma_{g,h}) \longrightarrow \mathsf{Isom}^+(\mathbb{H}^2)$$

are said to have the same boundary data if for each $i \leq h$, $\rho_1(C_i)$ is conjugate to $\rho_2(C_i)$

Definition 21. Let $C_1, \ldots, C_h \in \pi_1(\Sigma_{g,h})$ be represented by the boundary components of $\Sigma_{g,h}$. Then

$$\rho : \pi_1(\Sigma_{g,h}) \longrightarrow \mathsf{Isom}^+(\mathbb{H}^2)$$

is said to be boundary parabolic if $\rho(C_i)$ is parabolic for each $i \leq h$.

1.7 *Simple closed curves on a surface with possibly non-empty boundary*

If γ is a simple closed curve on $\Sigma_{g,h}$, then one of the following is true:

- $\Sigma_{g,h} - \gamma$ is connected, in which case γ is called non-separating. Given another non-separating simple closed curve, γ_1, there is a homeomorphism of $\Sigma_{g,h}$ taking γ to γ_1.

  or

- $\Sigma_{g,h} - \gamma$ consists of exactly two connected components, Σ^1 and Σ^2, with

$$\chi(\Sigma^1) + \chi(\Sigma^2) = \chi(\Sigma_{g,h}).$$

(Here $\chi(\Sigma_{g,h}) = 2 - 2g + h$ is Euler characteristic of $\Sigma_{g,h}$.)

If

$$\Sigma = \Sigma^1 \bigoplus_{\gamma} \Sigma^2 = \overline{\Sigma^1} \bigoplus_{\gamma_1} \overline{\Sigma^2}$$

so that

- Σ^1 is homeomorphic to $\overline{\Sigma^1}$

  and

- Σ^2 is homeomorphic to $\overline{\Sigma^2}$,

then there is a homeomorphism of $\Sigma_{g,h}$ taking γ to γ_1.

Twist flows along simple closed curves

- Let

$$\rho : \pi_1(\Sigma_{g,h}) \longrightarrow \mathsf{Isom}^+(\mathbb{H}^2)$$

be a homomorphism and let γ be a separating simple closed curve so that

$$\Sigma_{g,h} = \Sigma^1 \bigoplus_{\gamma} \Sigma^2$$

(as usual, let the prescribed base-point be in Σ^1). If α centralizes $\rho(\gamma)$, define the representation,

$$\rho[\gamma, \alpha] : \pi_1(\Sigma_{g,h}) \longrightarrow \mathsf{Isom}^+(\mathbb{H}^2),$$

as follows:

$$\rho[\gamma, \alpha]_{|\pi_1(\Sigma^1)}(\omega) := \rho(\omega)$$

$$\rho[\gamma, \alpha]_{|\pi_1(\Sigma^2)}(\omega) := \alpha \cdot \rho(\omega) \cdot \alpha^{-1}.$$

Lemma 22. *$\rho[\gamma, \alpha]$ defines an representation of $\pi_1(\Sigma)$.*

Proof. Recall that

$$\pi_1(\Sigma) = \langle A_1, B_1, \ldots, A_g, B_g | \prod_{1 \leq i \leq g} [A_i, B_i] \rangle.$$

It suffices to show that

$$\rho[\gamma, \alpha](\prod_{1 \leq i \leq g} [A_i, B_i]) = \mathbb{I}.$$

Without loss of generality,

$$\gamma = \prod_{1 \leq i \leq k} [A_i, B_i],$$

for some $k < g$. From the definition of $\rho[\gamma, \alpha]$,

$$\rho[\gamma, \alpha](\gamma) = \rho(\gamma) = \alpha \cdot \rho(\gamma) \cdot \alpha^{-1}$$

and

$$\rho[\gamma, \alpha]_{\pi_1(\Sigma^2)} = \alpha \cdot \rho_{\pi_1(\Sigma^2)} \cdot \alpha^{-1}.$$

Therefore since

$$\rho(\prod_{1 \leq i \leq g} [A_i, B_i]) = \mathbb{I},$$

it follows that

$$\rho[\gamma, \alpha](\prod_{1 \leq i \leq g} [A_i, B_i]) = \mathbb{I}$$

as well. □

Because $\pi_1(\Sigma)$ is generated by $\pi_1(\Sigma^1)$ and $\pi_1(\Sigma^2)$, $\rho[\gamma, \alpha]$ is uniquely determined.

- If γ is a non-separating simple closed curve, ρ is as above and α centralizes $\rho(\gamma)$, define the representation,

$$\rho[\gamma, \alpha] : \pi_1(\Sigma_{g,h}) \longrightarrow \mathsf{Isom}^+(\mathbb{H}^2),$$

as follows:

If ω is represented by a simple closed curve that intersects γ exactly once, then,

$$\rho[\gamma, \alpha](\omega) := \rho(\omega) \cdot \alpha$$

while if ω is represented by a simple closed curve that does not intersect γ, then

$$\rho[\gamma, \alpha](\omega) := \rho(\omega).$$

Lemma 23. *$\rho[\gamma, \alpha]$ defines a representation from $\pi_1(\Sigma_g)$ to $\mathsf{Isom}^+(\mathbb{H}^2)$.*

Proof. If γ is a non-separating simple closed curve, without loss of generality, $\gamma = A_1$. Then

$$\rho[\gamma, \alpha](B_1) = \rho(B_1) \cdot \alpha,$$

$$\rho[\gamma, \alpha](A_i) = \rho(A_i)$$

for $1 \le i \le g$ and

$$\rho[\gamma, \alpha](B_i) = \rho(B_i)$$

for $2 \le i \le g$.

It follows from the definition of $\rho[\gamma, \alpha]$ that

$$\rho[\gamma, \alpha]([A_1, B_1]) = \rho(A_1) \cdot \rho(B_1) \cdot \alpha \cdot \rho(A_1)^{-1} \cdot \alpha^{-1} \cdot \rho(B_1)^{-1}.$$

Because α centralizes $\rho(A_1)$, α also centralizes $\rho(A_1)^{-1}$, therefore

$$\rho[\gamma, \alpha]([A_1, B_1]) = \rho(A_1) \cdot \rho(B_1) \cdot \alpha \cdot \alpha^{-1} \cdot \rho(A_1)^{-1} \cdot \rho(B_1)^{-1} = \rho([A_1, B_1]) = \mathbb{I}.$$

Therefore since

$$\rho(\prod_{1 \leq i \leq g} [A_i, B_i]) = \mathbb{I},$$

it follows that

$$\rho[\gamma, \alpha](\prod_{1 \leq i \leq g} [A_i, B_i]) = \mathbb{I}$$

as well. □

Because $\pi_1(\Sigma_{g,h})$ is generated by simple closed curves that either

- intersect γ exactly once

  or

- do not intersect γ,

$\rho[\gamma, \alpha]$ is uniquely determined.

$\rho[\gamma, \alpha]$ is called the **twist flow** along the curve γ by α. $\rho[\gamma, \alpha]$ is said to be a small twist flow if α is close to $\mathbb{I}$.

1.7.1 Certain homeomorphisms of $\Sigma_{g,h}$

By applying homeomorphisms to certain "canonical simple closed curves", it is possible to generate many simple closed curves of a desired type.

Dehn twists

Let $\gamma \subset \Sigma_{g,h}$ be a non-peripheral simple closed curve and let N be a closed annular neighborhood of γ. N is homeomorphic to the set, (written in polar coordinates),

$$\{(r, \theta) : 1 \leq r \leq 2, 0 \leq \theta \leq 2\pi\} \subseteq \mathbb{R}^2.$$

The homeomorphism,

$$D_\gamma(r, \theta) = (r, 2\pi(r-1) + \theta)$$

of the above annular region yields a homeomorphism of N that fixes its boundary. Thus, D_γ yields a homeomorphism of $\Sigma_{g,h}$ (also called D_γ). D_γ is not isotopic to the identity as it does not induce an inner automorphism of $\pi_1(\Sigma_{g,h})$.

From now on, if S is an oriented surface with possibly non-empty boundary and ω is a simple closed curve on S, D_ω is the homeomorphism of S obtained by Dehn twisting along ω. (Often times notation will not distinguish between D_ω and its induced map on the fundamental group of S.)

If S is a surface with boundary,

$$\psi : \pi_1(S, s) \longrightarrow \mathbb{PSL}(2, \mathbb{R})$$

is a homomorphism and

$$\varphi : S \longrightarrow S$$

is a homeomorphism that fixes s, then

$$(\varphi^* \psi)(\alpha) := (\psi \circ (\varphi_*)^{-1})(\alpha)$$

(φ_* is the automorphism of $\pi_1(S, s)$ induced by φ).

A few simple examples:

Simple closed curves on the 4-holed sphere $g = 0, h = 4$

Recall that

$$\pi_1(\Sigma_{0,4}) = \langle A, B, C, D | A \cdot B \cdot C \cdot D \rangle.$$

A non-peripheral simple closed curve, γ, separates the boundary components of $\Sigma_{0,4}$, A, B, C and D into pairs and thus separates $\Sigma_{0,4}$ into two 3-holed spheres, Σ^1, Σ^2. If the simple closed curves on $\Sigma_{0,4}$, γ_1 and γ_2, separate the boundary components of $\Sigma_{0,4}$ into the same pairs, then γ_1 and γ_2 are said to be in the **same class**.

Without loss of generality, let $\gamma = A \cdot B$. Let

- Σ^1 have boundary components A, B and $A \cdot B$
- Σ^2 have boundary components, $A \cdot B = (C \cdot D)^{-1}, C$ and D

  and

- let the base-point for $\pi_1(\Sigma_{0,4})$ be in the interior of Σ^1.

Then,

$$D_{\gamma_*}(A) = A$$

$$D_{\gamma_*}(B) = B$$

$$D_{\gamma_*}(C) = (A \cdot B) \cdot C \cdot (A \cdot B)^{-1}.$$

Simple closed curves on the two holed torus $g = 1, h = 2$

Recall that

$$\pi_1(\Sigma_{1,2}) = \langle A, B, C, D | [A, B] \cdot C \cdot D \rangle.$$

A non-peripheral simple closed curve, γ, on $\Sigma_{1,2}$ is either non-separating or separates $\Sigma_{1,2}$ into

- a 1-holed torus Σ^1 with boundary component, γ,

  and

- a three holed sphere, Σ^2, with boundary components C, D and γ.

When γ is non-separating, without loss of generality, let $\gamma = B$. A intersects γ exactly once while C and D do not intersect γ.

$$D_{\gamma_*}(A) = A \cdot B$$

$$D_{\gamma_*}(B) = B$$

$$D_{\gamma_*}(C) = C.$$

When γ is separating, without loss of generality, $\gamma = [A, B]$ and the base-point of $\pi_1(\Sigma_{1,2})$ is in Σ^1.

$$D_{\gamma_*}(A) = A$$

$$D_{\gamma_*}(B) = B$$

$$D_{\gamma_*}(C) = [A, B] \cdot C \cdot [A, B]^{-1}.$$

Simple closed curves on the genus two surface $g = 2, h = 0$

Recall that

$$\pi_1(\Sigma_2 = \langle A_1, B_1, A_2, B_2 | [A_1, B_1] \cdot [A_2, B_2] \rangle.$$

A simple closed curve, γ, on Σ_2 is either non-separating or separates Σ_2 into two 1-holed tori, Σ^1 and Σ^2. When γ is non-separating, let $\gamma = B_1$. Then

$$D_{\gamma_*}(A) = A_1 \cdot B_1$$

$$D_{\gamma_*}(B) = B_1$$

$$D_{\gamma_*}(C) = A_2$$

$$D_{\gamma_*}(B_2) = B_2.$$

When γ is separating, let $\gamma = [A_2, B_2]$ and let the base-point of $\pi_1(\Sigma_2)$ be in the 1-holed torus containing curves A_1 and B_1,

$$D_{\gamma_*}(A_1) = A_1$$

$$D_{\gamma_*}(B_1) = B_1$$

$$D_{\gamma_*}(A_2) = [A_1, B_1] \cdot A_2 \cdot [A_1, B_1]^{-1}$$

$$D_{\gamma_*}(D) = [A_1, B_1] \cdot B_2 \cdot [A_1, B_1]^{-1}.$$

The rest of this article will assume the Poincaré Unit Disk Model of $\mathbb{H}^2$.

2. GENUS-2 SURFACE GROUP REPRESENTATIONS WITH ELLIPTIC NON-SEPARATING SIMPLE CLOSED CURVES

The following two theorems will be proved in this chapter.

Theorem 24. *Let P be the set of Euler class 1, genus-2 surface group representations into $\mathbb{PSL}(2,\mathbb{R})$ that take a separating simple closed curve to a parabolic isometry. Let E be the set of Euler class 1, genus-2 surface group representations into $\mathbb{PSL}(2,\mathbb{R})$ that take a non-separating simple closed curve to an elliptic isometry. Then $P \cap E$ is dense in P.*

In other words, every representation in P is arbitrarily close to a member of $P \cap E$. (This is Theorem 3 in the introduction.)

Theorem 25. *Let either $\Sigma \simeq \Sigma_{1,2}$ or $\Sigma \simeq \Sigma_2$. If a representation,*

$$\rho : \pi_1(\Sigma) \longrightarrow \mathbb{PSL}(2,\mathbb{R}),$$

takes all boundary components to non-identity isometries and takes a non-separating simple closed curve to an elliptic isometry, then ρ is arbitrarily close to a representation, $\overline{\rho}$, that takes a separating simple closed curve to a unipotent isometry.

In other words, the set of Σ group representations that take all boundary components to non identity isometries and that take a separating simple closed curve to a unipotent isometry is dense in the set of Σ group representations that take a non-separating simple closed curve to an elliptic isometry. (This is Theorem 4 in the introduction.)

An important corollary:

Corollary. If the Euler class 1 homomorphism,

$$\rho : \pi_1(\Sigma_2) \longrightarrow \mathbb{PSL}(2, \mathbb{R}),$$

takes some non-separating simple closed curve to an elliptic isometry, then it is arbitrarily close to a representation that takes a separating simple closed curve to a parabolic isometry.

The structure of this article is as follows:

Section 1 is devoted to establishing a certain canonical form for non-abelian reducible representations,

$$\rho : \mathbb{F}^2 \simeq \pi_1(\Sigma_{1,1}) \longrightarrow \mathbb{PSL}(2, \mathbb{R}).$$

Theorem 1 is proved in section 2.

If E is the set of $\Sigma_{0,4}$ group representations that take some non-peripheral simple closed curve to an elliptic isometry and if U is the set of $\Sigma_{0,4}$ group representations that take some non-peripheral simple closed curve to a unipotent isometry, then the Elliptic-Parabolic Lemma, proved in section 2, relates E to U.

The Elliptic-Parabolic Lemma will be used later to prove Theorem 25. Section 4 is devoted to constructing machinery for

- extending certain $\Sigma_{0,4}$ group representations to $\Sigma_{1,2}$ group representations

and

- extending certain $\Sigma_{1,2}$ group representations to Σ_2 group representations.

Theorem 25 is proved in section 5.

2.1 Basic facts about non-abelian reducible $\mathbb{PSL}(2,\mathbb{R})$ *representations of the rank two free group*

Definition 26. $\mathbb{F}^2 =< A, B >$ is the free group on two generators, A and B.

Definition 27. If $\alpha_1, \alpha_2 \in \mathbb{F}^2$ freely generate $\mathbb{F}^2$, then both α_1 and α_2 are called primitives.

To prove Theorem 24, it is necessary to find a certain canonical form for reducible non-abelian representations of $\mathbb{F}^2 \simeq \pi_1(\Sigma_{1,1})$ into $\mathbb{PSL}(2,\mathbb{R})$.

If the homomorphism,

$$\rho : \mathbb{F}^2 \to \mathbb{PSL}(2,\mathbb{R}),$$

is non-abelian and reducible, then ρ is $\mathbb{PSL}(2,\mathbb{R})$ conjugate to an upper triangular representation of the following form:

$$\rho(A) = \begin{pmatrix} e^s & \star \\ 0 & e^{-s} \end{pmatrix},$$
$$\rho(B) = \begin{pmatrix} e^{\alpha s} & \star \\ 0 & e^{-\alpha s} \end{pmatrix}.$$

If $\alpha \in \mathbb{Q}$, ρ is said to satisfy the **Rational Case**, otherwise, ρ satisfies the **Irrational Case**.

The goal of this section is to prove the following lemma which will be important to the proof of Theorem 24:

Lemma 28 (Canonical Form)**.** *If $\rho : \mathbb{F}^2 \longrightarrow \mathbb{PSL}(2, \mathbb{R})$ is non-abelian and reducible, then there is an automorphism, ϕ, of $\mathbb{F}^2$, that fixes $[A, B]$ so that $\phi^*\rho$ is of one of the following forms:*

1. ρ ***satisfies the Rational Case***

$$\phi^*\rho(A) = \begin{pmatrix} 1 & \star \\ 0 & 1 \end{pmatrix}$$

while

$$\phi^*\rho(B) = \begin{pmatrix} e^u & \star \\ 0 & e^{-u} \end{pmatrix}$$

for some $u \neq 0 \in \mathbb{R}$

2. ρ ***satisfies the Irrational Case***

$$\phi^*\rho(A) = \begin{pmatrix} e^\epsilon & \star \\ 0 & e^{-\epsilon} \end{pmatrix}$$

for some ϵ arbitrarily close to 0

while

$$\phi^*\rho(B) = \begin{pmatrix} e^u & \star \\ 0 & e^{-u} \end{pmatrix}$$

for some $u \neq 0 \in \mathbb{R}$.

Remark 29. Although the proof of the Rational Case of Lemma 28 is not needed, it is included for completeness.

Proof. **Rational Case** Let s and t be real numbers so that t is a rational multiple of s. In other words $t = \frac{p}{q}s$, where $p, q \in \mathbb{Z}$ and $(p,q) = 1$. Since $(p,q) = 1$, $(-p,q) = 1$ as well. Because $(-p,q) = 1$, there is a primitive, $w(A,B) \in \mathbb{F}^2$, where the sum of the powers of A in $w(A,B)$ is $-p$ and the sum of the powers of B in $w(A,B)$ is q. Since ρ is an upper triangular representation of $\mathbb{F}^2$ into $\mathbb{PSL}(2,\mathbb{R})$, the diagonal entries of $\rho(w(A,B))$ are the same as those of $\rho(A^{-p} \cdot B^q)$.

Without loss of generality,

$$\rho(A) = \begin{pmatrix} e^s & \star \\ 0 & e^{-s} \end{pmatrix},$$

$$\rho(B) = \begin{pmatrix} e^t & \star \\ 0 & e^{-t} \end{pmatrix}.$$

$$\rho(A^{-p} \cdot B^q) = \begin{pmatrix} e^{-ps} & \star \\ 0 & e^{ps} \end{pmatrix} \cdot \begin{pmatrix} e^{\frac{p}{q}qs} & \star \\ 0 & e^{-\frac{p}{q}qs} \end{pmatrix} =$$

$$\begin{pmatrix} e^{-ps} & 0 \\ 0 & e^{ps} \end{pmatrix} \cdot \begin{pmatrix} e^{ps} & 1 \\ 0 & e^{-ps} \end{pmatrix} = \begin{pmatrix} 1 & \star \\ 0 & 1 \end{pmatrix}.$$

Since the diagonal entries of $\rho(w(A,B))$ are the same as those of $\rho(A^{-p} \cdot B^q)$,

$$\rho(w(A,B)) = \begin{pmatrix} 1 & \star \\ 0 & 1 \end{pmatrix}$$

is parabolic.

Because $w(A, B)$ is primitive, there is a $\overline{w}(A, B) \in \mathbb{F}^2$ so that the set,

$$\{w(A, B), \overline{w}(A, B)\},$$

freely generates $\mathbb{F}^2$. It follows that there is an automorphism of $\mathbb{F}^2$, φ, where

$$\varphi(A) = w(A, B)$$

and

$$\varphi(B) = \overline{w}(A, B).$$

By Nielsen's Theorem, [7], $\varphi([A, B])$ is conjugate to $[A, B]^{\pm 1}$, so there is an $\alpha \in \mathbb{F}^2$ where

$$\alpha \cdot \varphi([A, B]) \cdot \alpha^{-1} = [A, B]^{\pm 1}.$$

If $\alpha \cdot \varphi([A, B]) \cdot \alpha^{-1} = [A, B]$, define

$$\phi(\beta) := \alpha \cdot \varphi^{-1}(\beta) \cdot \alpha^{-1}$$

for $\beta \in \mathbb{F}^2$.

Define the automorphism, $\mathsf{inv} : \mathbb{F}^2 \longrightarrow \mathbb{F}^2$, as follows:

$$\mathsf{inv}(A) := A^{-1}$$

$$\mathsf{inv}(B) := B$$

If $\alpha \cdot \varphi([A, B]) \cdot \alpha^{-1} = [A, B]^{-1}$, define

$$\phi^{-1}(\beta) := A \cdot \mathsf{inv}(\alpha \cdot \varphi(\beta) \cdot \alpha^{-1}) \cdot A^{-1}$$

for $\beta \in \mathbb{F}^2$.

$$\phi^*\rho(A) = \begin{pmatrix} 1 & \star \\ & \\ 0 & 1 \end{pmatrix}$$

and

$$\phi^*\rho(B) = \begin{pmatrix} e^u & \star \\ & \\ 0 & e^{-u} \end{pmatrix}.$$

Irrational Case

Suppose $\alpha \notin \mathbb{Q}$, then there is a sequence of rational numbers, $\{\frac{p_i}{q_i}\} \to \alpha$, where for each i, $(p_i, q_i) = 1 = (-p_i, q_i)$. $p_i \to q_i\alpha$, therefore $e^{q_i\alpha - p_i} \to 1$. Consequently the diagonal entries of

$$\rho(A^{-p_i} \cdot B^{q_i}) = \begin{pmatrix} e^{(q_i\alpha - p_i)s} & \star \\ & \\ 0 & e^{(p_i - q_i\alpha)s} \end{pmatrix}$$

approach 1. Since for each i, $(-p_i, q_i) = 1$, there is a primitive, $w_i(A, B) \in \mathbb{F}^2$, with homology $(-p_i, q_i)$. As in the Rational case,

$$\rho(w_i(A, B)) = \begin{pmatrix} e^{q_i\alpha - p_i} & \star \\ & \\ 0 & e^{-(q_i\alpha - p_i)} \end{pmatrix}.$$

Proceeding as in the Rational Case, there is an automorphism,

$$\phi : \mathbb{F}^2 \longrightarrow \mathbb{F}^2,$$

fixing $[A, B]$, where

$$\phi^*\rho(A) = \begin{pmatrix} e^{q_i\alpha - p_i} & \star \\ & \\ 0 & e^{-(q_i\alpha - p_i)} \end{pmatrix}$$

for the real number, $q_i\alpha - p_i$, with arbitrarily small absolute value and

$$\phi^*\rho(B) = \begin{pmatrix} e^u & \star \\ 0 & e^{-u} \end{pmatrix}$$

for some non-zero real number, u.

$\square$

The following lemma will be important later.

Lemma 30. *Suppose the upper triangular, non-abelian representation,*

$$\rho : \mathbb{F}^2 \longrightarrow \mathbb{PSL}(2, \mathbb{R}),$$

satisfies the Rational Case, then ρ is arbitrarily close to an upper triangular, non-abelian representation,

$$\overline{\rho} : \mathbb{F}^2 \longrightarrow \mathbb{PSL}(2, \mathbb{R}),$$

that satisfies the Irrational Case so that $\overline{\rho}([A, B]) = \rho([A, B])$

Proof. Let

$$\rho(A) = \begin{pmatrix} e^s & \star \\ 0 & e^{-s} \end{pmatrix}$$

and

$$\rho(B) = \begin{pmatrix} e^{\alpha s} & \star \\ 0 & e^{-\alpha s} \end{pmatrix}.$$

Without loss of generality, $\rho([A, B]) = \begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix}$.

Let

$$\overline{\rho}(A) = \begin{pmatrix} e^{s} & \star \\ 0 & e^{-s} \end{pmatrix}$$

and

$$\overline{\rho}(B) = \begin{pmatrix} e^{(\alpha+\epsilon)s} & \star \\ 0 & e^{-(\alpha+\epsilon)s} \end{pmatrix}$$

The non-zero off-diagonal entry of $\rho([A,B])$ is a continuous function of the entries of $\rho(A)$ and $\rho(B)$, so for $\epsilon \in \mathbb{R}$ with arbitrarily small absolute value,

$$\overline{\rho}([A,B]) = \begin{pmatrix} 1 & \pm(1+\delta) \\ 0 & 1 \end{pmatrix}$$

for some $\delta \in \mathbb{R}$ arbitrarily close to 0. If $\varrho = \begin{pmatrix} \pm|1+\delta|^{\frac{1}{2}} & 0 \\ 0 & \pm|1+\delta|^{-\frac{1}{2}} \end{pmatrix}$, then $\varrho \cdot \overline{\rho}([A,B]) \cdot \varrho^{-1} = \rho([A,B])$. Furthermore if δ is close to 0, then $|1+\delta|^{\frac{1}{2}}$ is close to 1. □

2.2 Euler class 1 representations of the genus-2 surface group, with parabolic separating simple closed curve

Throughout this section let Σ be a closed oriented genus-2 surface. Recall

$$\pi = \pi_1(\Sigma, \sigma) = \pi_1(\Sigma) \simeq < A_1, B_1, A_2, B_2 | [A_1, B_1] \cdot [A_2, B_2] > .$$

With the above presentation, $\Sigma = \Sigma^1 \bigoplus_{[A_1,B_1]} \Sigma^2$ where Σ^1 and Σ^2 are two 1-holed tori separated by the simple closed curve, $\kappa = [A_1, B_1] \in \pi$, $(\sigma \in \Sigma^1)$. Let $\pi_1(\Sigma^1) =< A_1, B_1 >$ and $\pi_1(\Sigma^2) =< A_2, B_2 >$.

2.2.1 Important lemmas

The following lemmas will be important to the proof of Theorem 24.

Lemma 31. *Let $\rho : \pi \to \mathbb{PSL}(2,\mathbb{R})$ be an Euler class 1 representation with $\rho(\kappa)$ parabolic. Without loss of generality, $\rho_{|\pi_1(\Sigma^1)}$ is the holonomy of a cusped hyperbolic structure and $\rho_{|\pi_1(\Sigma^2)}$ is a non-abelian reducible representation.*

Proof. Without loss of generality, $\rho([A_1, B_1])$ is parabolic. For $i \in \{1,2\}$, $\rho_{|\pi_1(\Sigma^i)}$ is therefore either the holonomy of a cusped hyperbolic structure on Σ^i or is reducible and non-abelian, [7]. $e(\rho_{|\pi_1(\Sigma^i)}) = \pm 1$ if and only if $\rho_{|\pi_1(\Sigma^i)}$ is the holonomy of a hyperbolic structure on Σ^i and $e(\rho_{|\pi_1(\Sigma^i)}) = 0$ if and only if $\pi_1(\Sigma^i)$ is reducible and non-abelian, [7]. By the additivity of $e(\rho)$, the result holds. $\square$

Lemma 32. *Suppose*

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{SL}(2,\mathbb{R})$$

and

$$Y = \begin{pmatrix} \lambda & t \\ 0 & \lambda^{-1} \end{pmatrix} \in \mathbb{SL}(2,\mathbb{R}).$$

If $c \neq 0$, then $X \cdot Y$ projects to an elliptic isometry in $\mathbb{PSL}(2,\mathbb{R})$ if and only if either

$$t \in \Big(\frac{-2-(a\lambda + d\lambda^{-1})}{c}, \frac{2-(a\lambda + d\lambda^{-1})}{c}\Big)$$

or

$$\Big(\frac{2-(a\lambda + d\lambda^{-1})}{c}, \frac{-2-(a\lambda + d\lambda^{-1})}{c}\Big).$$

Proof.

$$\mathsf{Tr}(X \cdot Y) = \mathsf{Tr}(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} \lambda & t \\ 0 & \lambda^{-1} \end{pmatrix}) = a\lambda + d\lambda^{-1} + ct.$$

$$t \in \{\frac{2 - (a\lambda + d\lambda^{-1})}{c}, \frac{-2 - (a\lambda + d\lambda^{-1})}{c}\},$$

if and only if $X \cdot Y$ is unipotent. Furthermore $\mathsf{Tr}(X \cdot Y)$ is a linear and bijective real valued function of t and for t with large absolute value $X \cdot Y$ is hyperbolic. □

Observation 33. The length of interval in Lemma 32,

$$|\frac{2 - (a\lambda + d\lambda^{-1})}{c} - \frac{-2 - (a\lambda + d\lambda^{-1})}{c}| = \frac{4}{|c|}$$

and therefore only depends on X.

Definition 34. If I_1 and I_2 are distinct real numbers while

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{SL}(2, \mathbb{R})$$

and

$$Y = \begin{pmatrix} \lambda & t \\ 0 & \lambda^{-1} \end{pmatrix} \in \mathbb{SL}(2, \mathbb{R}),$$

then

$$I_{I_1, I_2, X, Y} := (\frac{I_1 - (a\lambda + d\lambda^{-1})}{c}, \frac{I_2 - (a\lambda + d\lambda^{-1})}{c}).$$

Observation 35. Notice that in order for $\mathsf{Trace}(X \cdot Y)$ to be in the interval, (I_1, I_2), t must be in the interval,

$$I_{I_1, I_2, X, Y} = (\frac{I_1 - (a\lambda + d\lambda^{-1})}{c}, \frac{I_2 - (a\lambda + d\lambda^{-1})}{c}).$$

$I_{I_1,I_2,X,Y}$, has length $\frac{|I_1-I_2|}{|c|}$.

Lemma 36. *Suppose $c \neq 0 \in \mathbb{R}$. Let $r, t \in \mathbb{R}$ and $|t| < \frac{2}{|c|}$, then there is an integer, n, so that $r + nt \in I_{\mp 2, \pm 2, X, Y}$*

Proof. Without loss of generality, $c > 0$. Because the subset,

$$\{r + nt\} \subset \mathbb{R},$$

is discrete, there is a member, $r + n_0 t$, of minimum distance from the interval

$$I_{-2,2,X,Y} = (\frac{-2 - (a\lambda + d\lambda^{-1})}{c}, \frac{2 - (a\lambda + d\lambda^{-1})}{c}).$$

That minimum distance cannot be greater than t or else the distance from either $r + (n_0 + 1)t$ or $r + (n_0 - 1)t$ to $I_{-2,2,X,Y}$ is less than the distance from $r + n_0 t$ to $I_{-2,2,X,Y}$. It is now clear that either $r + (n_0 + 1)t$ or $r + (n_0 - 1)t$ is in the prescribed interval.

$\square$

2.2.2 The proof of Theorem 24

Let $\rho : \pi_1(\Sigma) \to \mathbb{PSL}(2, \mathbb{R})$ be an Euler class 1 homomorphism where for some real number, α and real number, $s \neq 0$,

1.

$$\rho(A_1) = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

$$\rho(A_2) = \begin{pmatrix} \pm e^s & t_0 \\ 0 & \pm e^{-s} \end{pmatrix},$$

$$\rho(B_2) = \begin{pmatrix} \pm e^{\alpha s} & r \\ 0 & \pm e^{-\alpha s} \end{pmatrix}$$

and

2.

$$\rho([A_2, B_2]) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

$\rho_{|\pi_1(\Sigma^1)}$ is a discrete embedding, so without loss of generality $c \neq 0$.

By virtue of Lemma 30, it suffices show that if α is irrational, then ρ takes a non-separating simple closed curve to an elliptic isometry. Assume α is irrational.

The proof of Theorem 24.

Proof. By Lemma 28 assume that s is arbitrarily close to 0, so that $|e^s - e^{-s}|$ is arbitrarily close to 0. Without loss of generality, let

$$|e^s - e^{-s}| < |\frac{4}{2c}| = |\frac{2}{c}|.$$

For each integer, n, $A_1 \cdot \kappa^n \cdot A_2 \cdot \kappa^{-n}$ is represented by a non-separating simple closed curve on Σ. It suffices to show that there is an integer, n, where $\rho(A_1 \cdot \kappa^n \cdot A_2 \cdot \kappa^{-n})$ is elliptic.

Since

$$\rho([A_2, B_2]) = -\rho([A_1, B_1]^{-1}) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix},$$

a simple calculation shows that

$$\rho(\kappa^{-n} \cdot A_2 \cdot \kappa^n) = \begin{pmatrix} \pm e^s & n(e^{-s} - e^s) + t_0 \\ 0 & \pm e^{-s} \end{pmatrix}.$$

Because

$$|e^{-s} - e^{s}| < \frac{2}{|c|}$$

there is, by Lemma 36, an integer, n, so that the non-zero off-diagonal entry of $\rho(\kappa^n \cdot A_2 \cdot \kappa^{-n})$ is in the interval,

$$\left(\frac{2 - (ae^s + de^{-s})}{c}, \frac{-2 - (ae^s + de^{-s})}{c}\right).$$

By Lemma 32, $\rho(A_1 \cdot \kappa^{-n} \cdot A_2 \cdot \kappa^n)$ is therefore elliptic. Since every member of P is arbitrarily close to a representation, ρ, where $\rho_{|\pi_1(\Sigma^2)}$ satisfies the Irrational Case, Theorem 24 is proved.

□

Summing up the proof of Theorem 24

To obtain a non-separating simple closed curve, γ, where $\rho(\gamma)$ is elliptic, it is necessary to:

1. first perturb ρ so that $\rho_{|\pi_1(\Sigma^2)}$ satisfies the Irrational Case,

   then

2. apply a homeomorphism, ϕ, of Σ that fixes $\pi_1(\Sigma^1)$ so that the diagonal elements of $\phi^*\rho(A_2)$ are as close to 1 as is needed,

   and finally

3. apply an appropriate power of $D_{[A_1,B_1]}$ to the non-separating simple closed curve, $A_1 \cdot A_2$, so that ρ takes the resulting non-separating simple closed curve to an elliptic isometry. By the calculations above, if the diagonal elements of $\phi^*\rho(A_2)$ are close enough to 1, this is possible.

The above proof of Theorem 24 generalizes to a proof of the following theorem.

Theorem 37. *Let I_1 and I_2 be distinct real numbers. If E_{I_1,I_2} is the set of Euler class 1 representations of the genus-2 surface group into $\mathbb{PSL}(2,\mathbb{R})$ that take a non-separating simple closed curve to an isometry with trace in (I_1, I_2) and if P is the set of Euler class 1 representations of the genus-2 surface group into $\mathbb{PSL}(2,\mathbb{R})$ that take a separating simple closed curve to a parabolic isometry, then $P \cap \bigcap_{I_1 \neq I_2}(E_{I_1,I_2})$ is dense in P.*

2.3 The Elliptic-Parabolic Lemma

2.3.1 The statement and proof of the Elliptic-Parabolic Lemma

The following lemma is key to the proof of Theorem 25.

Proposition 38 (The Weak Elliptic-Parabolic Lemma)**.** *Consider the following hypothesis' on the homomorphism,*

$$\rho : \pi_1(\Sigma_{0,4}) \longrightarrow \mathbb{PSL}(2,\mathbb{R}) :$$

1. $|\mathit{Tr}(\rho(A))| = |\mathit{Tr}(\rho(C))| \geq 2$

2. $\rho(A), \rho(C) \neq \mathbb{I}$

3. $\rho(A \cdot B)$ *is an elliptic isometry of infinite order.*

If ρ satisfies hypothesis' 1 through 3, then there is

- *a non-peripheral simple closed curve, γ, of the same class as $A \cdot C$*

  and

- *a representation, $\overline{\rho}$, with the same boundary data as and is arbitrarily close to ρ*

so that

$\overline{\rho}(\gamma)$ *is unipotent.*

Proof. Hypotheses 1 and 2 guarantee the existence of the fixed points,

$$\rho(A)_*, \rho(A)^*, \rho(C)_*, \rho(C)^* \in \partial\mathbb{H}^2,$$

(if $|\mathsf{Tr}(A)| = 2$, then $\rho(A)^* = \rho(A)_*$ and $\rho(C)^* = \rho(C)_*$).

Since $\rho(A \cdot B)$ is an elliptic isometry of infinite order,

- $\rho(A \cdot B)$ has a fixed point, $\rho(A \cdot B)_* \in \mathbb{H}^2$

  and

- the cyclic group, $\langle \rho(A \cdot B) \rangle$, is dense in $\mathsf{Stab}(\rho(A \cdot B)_*)$.

Furthermore there is an elliptic isometry, $\beta \in \mathsf{Stab}(\rho(A \cdot B)_*)$, that takes $\rho(C)_*$ to $\rho(A)^*$.

Since

- $\langle \rho(A \cdot B) \rangle$ is dense in the stabilizer of $\rho(A \cdot B)_*$

  and

- β stabilizes $\rho(A \cdot B)_*$,

there is a sequence of integers, $\{n_i\}$, where

$$\rho(A \cdot B)^{n_i} \to \beta \in \mathbb{PSL}(2, \mathbb{R}).$$

It follows that

$$\lim_{i\to\infty}(\rho(A \cdot B)^{n_i} \cdot (\rho(C)_*)) = (\rho(A \cdot B)^{n_i} \cdot \rho(C) \cdot \rho(A \cdot B)^{-n_i})_* = \rho(A)^*.$$

Without loss of generality, $\rho(A)^* = \infty$.

Therefore

- $$\rho(A) = \begin{pmatrix} e^{\cosh^{-1}(\frac{\mathrm{Tr}(\rho(A))}{2})} & \star \\ 0 & e^{-\cosh^{-1}(\frac{\mathrm{Tr}(\rho(A))}{2})} \end{pmatrix}$$

  and

- $$\lim_{i\to\infty} \rho(A \cdot B)^{n_i} \cdot \rho(C) \cdot \rho(A \cdot B)^{-n_i} = \begin{pmatrix} e^{-\cosh^{-1}(\frac{\mathrm{Tr}(\rho(A))}{2})} & \star \\ 0 & e^{\cosh^{-1}(\frac{\mathrm{Tr}(\rho(A))}{2})} \end{pmatrix}.$$

This follows from

- hypothesis' 1 and 2

  and

- $$\infty = \rho(A)^* = \lim_{i\to\infty}(\rho(A \cdot B)^{n_i} \cdot \rho(C) \cdot \rho(A \cdot B)^{-n_i})_*.$$

Therefore

$$\lim_{i\to\infty}(\rho(A)\cdot\rho(A\cdot B)^{n_i}\cdot\rho(C)\cdot\rho(A\cdot B)^{-n_i}) = \rho(A)\cdot\beta\cdot\rho(C)\cdot\beta^{-1} =$$

$$\begin{pmatrix} 1 & \star \\ 0 & 1 \end{pmatrix}$$

is unipotent.

If necessary, first perform an arbitrarily small twist flow along $A\cdot B$ so that there is some, (possibly very large) integer, n_i, where

$$(\rho(A\cdot B)^{n_i}\cdot\rho(C)\cdot\rho(A\cdot B)^{-n_i})^* = \rho(A)_*.$$

Since twist flowing ρ along any simple closed curve preserves boundary data, the result follows.

$\square$

If $\rho(A\cdot B)$ has finite order, (since $\rho(C)$ is either parabolic or hyperbolic, therefore $\rho_{|\pi_1(\Sigma^1)}$ and $\rho_{|\pi_1(\Sigma^2)}$ are irreducible) it is possible to perturb each representation, $\rho_{|\pi_1(\Sigma^1)}$ and $\rho_{|\pi_1(\Sigma^2)}$, by an arbitrarily small perturbation, to representations, $\overline{\rho_{|\pi_1(\Sigma^1)}}$ and $\overline{\rho_{|\pi_1(\Sigma^2)}}$ so that

1. $\overline{\rho_{|\pi_1(\Sigma^1)}}(A\cdot B)$ and $\overline{\rho_{|\pi_1(\Sigma^1)}}(C\cdot D)^{-1}$ are of infinite order and are $\mathbb{PSL}(2,\mathbb{R})$ conjugate by an isometry arbitrarily close to $\mathbb{I}$,

2. $\mathsf{Tr}(\rho(A)) = \mathsf{Tr}(\overline{\rho_{|\pi_1(\Sigma^1)}}(A)) = \mathsf{Tr}(\overline{\rho_{|\pi_1(\Sigma^2)}}(C)) = \mathsf{Tr}(\rho(C))$

   and

3. $\mathsf{Tr}(\overline{\rho_{|\pi_1(\Sigma^2)}}(D)) = \mathsf{Tr}(\rho(D))$ and $\mathsf{Tr}(\overline{\rho_{|\pi_1(\Sigma^1)}}(B)) = \mathsf{Tr}(\rho(B))$

The elliptic isometries, $\overline{\rho_{|\pi_1(\Sigma^2)}}(A \cdot B)$ and $\overline{\rho_{|\pi_1(\Sigma^2)}}(C \cdot D^{-1})$, may or may not coincide. However by condition 1 it is possible to conjugate $\overline{\rho_{|\pi_1(\Sigma^2)}}$ by a small $\mathbb{PSL}(2,\mathbb{R})$ element so that $\rho(A \cdot B)$ and $\rho(C \cdot D)^{-1}$ coincide. Therefore

Proposition 39 (The Elliptic-Parabolic Lemma)**.** *Consider the following hypothesis' on*

$$\rho : \pi_1(\Sigma_{0,4}) \longrightarrow \mathbb{PSL}(2,\mathbb{R}) :$$

1. $|\mathit{Tr}(\rho(A))| = |\mathit{Tr}(\rho(C))| \geq 2$

2. $\rho(A), \rho(C) \neq \mathbb{I}$

3. $\rho(A \cdot B)$ *is an elliptic isometry.*

If ρ satisfies hypothesis' 1 through 3, then there is

- *a non-peripheral simple closed curve, γ, of the same class as $A \cdot C$*

  and

- *a representation, $\overline{\rho}$, with the same boundary data as and is arbitrarily close to ρ*

so that

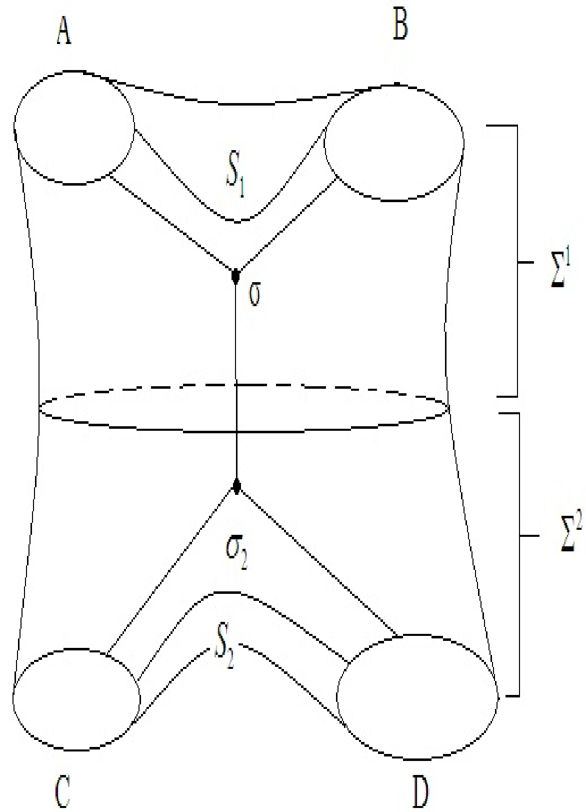
$\overline{\rho}(\gamma)$ *is unipotent.*

2.4 *Relating representations of Euler characteristic -2 surface groups*

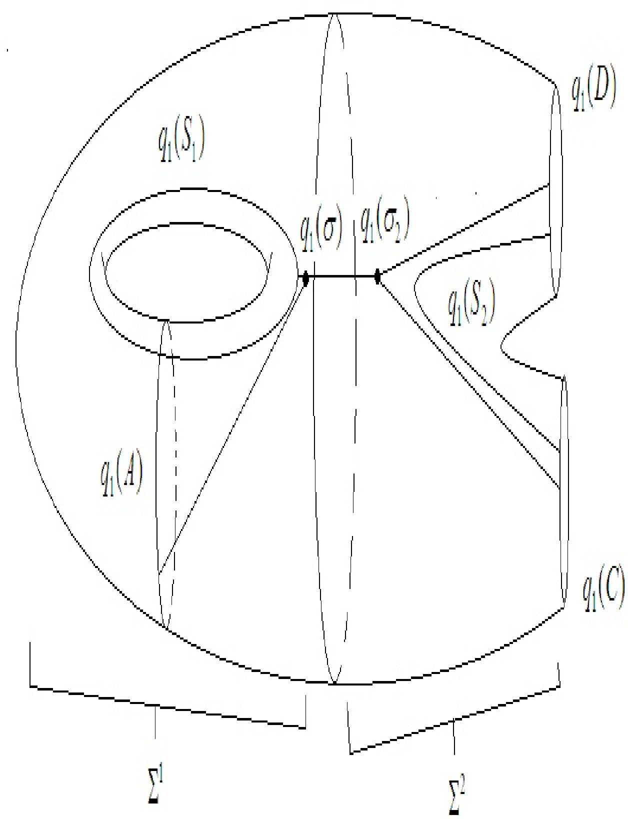
2.4.1 *Conventions*

The following conventions will be used in the next two sections:

Let $\Sigma \simeq \Sigma_{0,4}$ have boundary components A, B, C and D.



Form $\overline{\Sigma} \simeq \Sigma_{1,2}$ by identifying the boundary components of Σ, A and B, by an orientation reversing homeomorphism. $q_1 : \Sigma \longrightarrow \overline{\Sigma}$ is the corresponding quotient map.

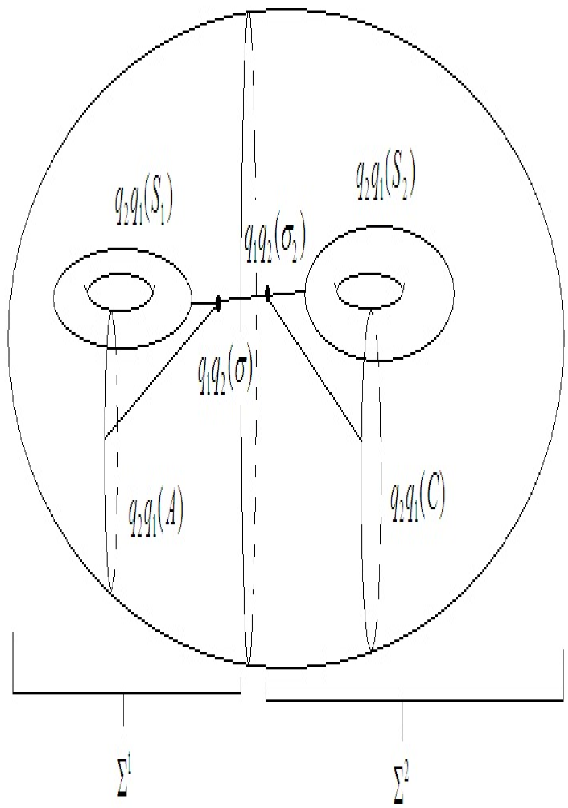


Form $\overline{\overline{\Sigma}} \simeq \Sigma_2$ by identifying the boundary the components of $\overline{\Sigma}$, $q_1(C)$ and $q_1(D)$, by an orientation reversing homeomorphism. $q_2 : \overline{\Sigma} \longrightarrow \overline{\overline{\Sigma}}$ is the corresponding quotient map.

Let S_1 be a segment (disjoint from $A \cdot B$) on Σ that joins the boundary components, A and B, so that $q_1(S_1)$ is a non-separating simple closed curve on $\overline{\Sigma}$ that intersects $q_1(A)$ exactly once.

Let S_2 be a segment (disjoint from $A \cdot B$) on Σ that joins the boundary components, C and D, so that $q_2(q_1(S_2))$ is a non-separating simple closed curve on $\overline{\overline{\Sigma}}$

that intersects $q_2 q_1(C)$ exactly once.



Recall that

$$\pi_1(\Sigma) = \langle A, B, C, D | A \cdot B \cdot C \cdot D \rangle.$$

$$\pi_1(\overline{\Sigma}) = \langle (q_1)_*(A), q_1(S_1), q_{1*}(C), q_{1*}(D) | [q_{1*}(A), q_1(S_1)] \cdot q_{1*}(C) \cdot q_{1*}(D) \rangle$$

and

$$\pi_1(\overline{\overline{\Sigma}}) = \langle q_{2*}q_{1*}(A), q_{2*}q_1(S_1), q_{2*}q_{1*}(C), q_2 q_1(S_2) |$$

$$[q_{2*}q_{1*}(A), q_{2*}q_1(S_1)] \cdot [q_{2*}q_{1*}(C), q_2 q_1(S_2)] \rangle.$$

Definition 40. If

$$\rho : \pi_1(\Sigma) \longrightarrow \mathbb{PSL}(2, \mathbb{R})$$

is a homomorphism where $\rho(A)$ and $\rho(B^{-1})$ are $\mathbb{PSL}(2, \mathbb{R})$ conjugate, then ρ is said to be **extendible**.

(For example, this is true if $\rho(A)$ and $\rho(B)$ are both hyperbolic with equal trace.)

Definition 41. For an extendible homomorphism,

$$\rho : \pi_1(\Sigma) \longrightarrow \mathbb{PSL}(2, \mathbb{R}),$$

if $\tau \in \mathbb{PSL}(2, \mathbb{R})$ and

$$\tau \cdot \rho(A) \cdot \tau^{-1} = \rho(B^{-1}),$$

τ is said to satisfy the ρ **Extension Condition**.

Observation 42. If

- ρ is an extendible 4-holed sphere group representation
- τ satisfies the ρ Extension Condition,
- a centralizes $\rho(A)$ and
- b centralizes $\rho(B)$,

then $b \cdot \tau \cdot a$ also satisfies the ρ Extension Condition.

In fact, if $\tau_1, \tau_2 \in \mathbb{PSL}(2,\mathbb{R})$ satisfy the ρ Extension Condition, then either $\tau_1 = \tau_2 \cdot a$, (for some a that centralizes $\rho(A)$), or $\tau_1 = b \cdot \tau_2$ (for some b centralizing $\rho(B)$).

Definition 43. If τ satisfies the ρ Extension Condition, it is possible construct a homomorphism,

$$\rho_\tau : \pi_1(\overline{\Sigma}) \longrightarrow \mathbb{PSL}(2,\mathbb{R}),$$

as follows:

$$\rho_\tau((q_1)_*(A)) := \rho(A)$$

$$\rho_\tau(q_1(S)) := \tau$$

$$\rho_\tau((q_1)_*(C)) := \rho(C)$$

$$\rho_\tau((q_1)_*(D)) := \rho(D).$$

Definition 44. To obtain a canonical 4-holed sphere group representation,

$$\dot{\rho} : \pi_1(\Sigma) \longrightarrow \mathbb{PSL}(2,\mathbb{R}),$$

from a 2-holed torus group representation,

$$\rho : \pi_1(\overline{\Sigma}) \longrightarrow \mathbb{PSL}(2,\mathbb{R}),$$

define

$$\dot{\rho} : \pi_1(\Sigma) \longrightarrow \mathbb{PSL}(2,\mathbb{R})$$

as follows:

$$\dot{\rho}(A) := \rho((q_1)_*(A))$$

$$\dot{\rho}(B) := \rho(q_1(S_1)) \cdot \rho((q_1)_*(A^{-1})) \cdot \rho(q_1(S_1))^{-1}$$

$$\dot{\rho}(C) := \rho((q_1)_*(C))$$

$$\dot{\rho}(D) := \rho((q_1)_*(D)).$$

$$\dot{\rho}(A \cdot B \cdot C \cdot D) = \rho([(q_1)_*(A), q_1(S_1)] \cdot q_1(C) \cdot q_1(D)) = \mathbb{I},$$

thus $\dot{\rho}$ is an extendible 4-holed sphere group representation where $\rho \circ q_* = \dot{\rho}$.

Lemma 45. *If ρ is extendible and τ satisfies the ρ Extension Condition, then ρ_τ is an extension of ρ by $(q_1)_*$.*

Proof. Because

- A, B, C and D generate $\pi_1(\Sigma)$,
- $A \cdot B \cdot C \cdot D = 1$

  and
- $B = A^{-1}D^{-1}C^{-1}$,

each curve in $\pi_1(\Sigma)$ can be expressed as a word in A, C and D.

If $\omega \in \pi_1(\Sigma)$ is a word in A, C and D, then $(q_1)_*(\omega)$ is a word in $(q_1)_*(A), (q_1)_*(C)$ and $(q_1)_*(D)$. Recall that

$$\rho_\tau((q_1)_*(A)) = \rho(A),$$

$$\rho_\tau((q_1)_*(C)) = \rho(C)$$

and

$$\rho_\tau((q_1)_*(D)) = \rho(D).$$

Since ρ, ρ_τ and q_{1*} are homomorphisms, then

$$\rho_\tau((q_1)_*(\omega)) = \rho(\omega).$$

$\square$

In particular if ω is a simple closed curve on Σ, then

- $(q_1)_*(\omega)$ is a simple closed curve on $\overline{\Sigma}$

  and

- $\rho_\tau((q_1)_*(\omega)) = \rho(\omega)$.

2.4.3 Perturbing extensions of 4-holed sphere group representations

It will be necessary to extend perturbed 4-holed sphere group, (and later two holed torus group), representations to perturbed 2-holed torus group, (genus-2 surface group), representations.

Let

$$\rho : \pi_1(\Sigma) \longrightarrow \mathbb{PSL}(2, \mathbb{R})$$

be extendible and let τ satisfy the ρ Extension Condition. If

- $\rho(A)$ and $\rho(B)$, are not involutions

  and

- one chooses to perturb ρ to $\overline{\rho}$ by a small perturbation,

then it is possible to choose

$$\overline{\rho_\tau} : \pi_1(\overline{\Sigma}) \longrightarrow \mathbb{PSL}(2,\mathbb{R})$$

that extends $\overline{\rho}$ and is close to ρ_τ. More precisely,

Lemma 46. *Let $\{\rho_i\}$ and ρ be a sequence of 4-holed sphere group representations and a 4-holed sphere group representation respectively,*

where

- $\lim_{i\to\infty} \rho_i = \rho$,

- *for each i, $\rho_i(A), \rho_i(C^{-1}), \rho(A), \rho(C^{-1})$ are all in the same $\mathbb{PSL}(2,\mathbb{R})$ conjugacy class*

  and

- *$\rho(A)$ and $\rho(C)$ are not involutions.*

let τ satisfy the ρ Extension Condition, then there is a sequence, $\{\tau_i\} \to \tau$, of members of $\mathbb{PSL}(2,\mathbb{R})$ that satisfy the ρ_i Extension Condition.

Proof. The proof of this lemma will be separated into the following 4 cases:

1. $\rho(A)$ and $\rho(B)$ are both hyperbolic

2. $\rho(A)$ and $\rho(B)$ are both parabolic

3. $\rho(A)$ and $\rho(B)$ are both elliptic of non-zero trace

4. $\rho(A)$ and $\rho(B)$ are both the identity matrix.

Case 1. $\rho(A)$ and $\rho(B)$ are hyperbolic

For each i, τ_i satisfies the identity

$$\rho_i(B)^{-1} = \tau_i \cdot \rho(A) \cdot {\tau_i}^{-1}$$

if and only if both

1. $\tau_i \cdot \rho_i(A)_* = \rho_i(B)^*$

   and

2. $\tau_i \cdot \rho_i(A)^* = \rho_i(B)_*$.

It suffices to find a sequence, $\tau_i \to \tau \in \mathbb{PSL}(2, \mathbb{R})$, so that identities 1 and 2 hold for all large i.

There is a point, $p \in \partial(\mathbb{H}^2)$, where

$$p, \tau \cdot p \notin \{\rho(A)^*, \rho(A)_*, \rho(B)^*, \rho(B)_*\}.$$

Because $\rho(A)$ and $\rho(C)$ are both hyperbolic, $\rho(A)_* \neq \rho(A)^*$ and $\rho(C)_* \neq \rho(C)^*$. Choose open intervals I^A, I_A, I^B, I_B about $\rho(A)^*, \rho(A)_*, \rho(B)^*, \rho(B)_*$ respectively so that

- $p, \tau \cdot p \notin \overline{I^A} \cup \overline{I_A} \cup \overline{I^B} \cup \overline{I_B}$

  and

- $\overline{I^A} \cap \overline{I_A} = \overline{I^B} \cap \overline{I_B} = \emptyset$. (For interval I, $\overline{I}$ is its closure.)

If 3-tuples of points in $\partial\mathbb{H}^2$, (x_1, y_1, z_1) and (x_2, y_2, z_2), consist of 3 distinct points define

$T[(x_1, x_2), (y_1, y_2), (z_1, z_2)]$ to be the unique member of $\mathbb{PGL}(2, \mathbb{C})$ that takes

$$x_1 \mapsto x_2,$$

$$y_1 \mapsto y_2$$

and

$$z_1 \mapsto z_2.$$

Since $x_1, x_2, y_1, y_2, z_1, z_2 \in \partial\mathbb{H}^2$, $T[(x_1, x_2), (y_1, y_2), (z_1, z_2)] \in \mathbb{PGL}(2, \mathbb{R})$.

Let

$$S(\rho) = \{(X, Y) \in \mathbb{PSL}(2, \mathbb{R}) \times \mathbb{PSL}(2, \mathbb{R}) :$$
$$|\mathsf{Tr}(X)|, |\mathsf{Tr}(Y)| > 2, X^* \in I^A, X_* \in I_A, Y^* \in I^B, Y_* \in I_B\}.$$

$S(\rho)$ is open in $\mathbb{PSL}(2, \mathbb{R}) \times \mathbb{PSL}(2, \mathbb{R})$. Since $\rho_i \to \rho \in \mathbb{PSL}(2, \mathbb{R})$ and $(\rho(A), \rho(B)) \in S(\rho)$, it follows that for large i,

$$(\rho_i(A), \rho_i(B)) \in S(\rho).$$

Define $\Phi : S(\rho) \longrightarrow \mathbb{PGL}(2, \mathbb{R})$ as follows:

$$\phi(X, Y) := T[(X^*, Y_*), (X_*, Y^*), (p, \tau \cdot p)].$$

Φ is continuous on $S(\rho)$ and $\Phi(\rho(A), \rho(B)) = \tau$. For large i, define

$$\tau_i := \Phi(\rho_i(A), \rho_i(B)).$$

Then, $\tau_i \cdot \rho_i(A) \cdot {\tau_i}^{-1} = \rho_i(B^{-1})$. Furthermore since $\rho_i \to \rho$, it follows that

$$\rho_i(A) \to \rho(A)$$

and

$$\rho_i(B) \to \rho(B).$$

Therefore

$$\tau_i = \Phi(\rho_i(A), \rho_i(B)) \to \Phi(\rho(A), \rho(B)) = \tau.$$

Because $\tau \in \mathbb{PSL}(2, \mathbb{R})$, $\tau_i \in \mathbb{PSL}(2, \mathbb{R})$ for large i.

Case 2. $\rho(A)$ and $\rho(B)$ are parabolic

If X and Y are parabolic isometries and $\alpha \in \mathbb{PGL}(2, \mathbb{R})$, then $\alpha \cdot X \cdot \alpha^{-1} = Y^{\pm 1}$ if and only if $\alpha \cdot X_* = Y_*$. Let p and q be points in $\partial \mathbb{H}^2$ so that no two members of the sets, $\{\rho(A)_*, p, q\}$ and $\{\rho(B)_*, \gamma \cdot p, \gamma \cdot q\}$, coincide. Choose disjoint intervals, I_A and I_B, about $\rho(A)_*$ and $\rho(B)_*$ respectively with closures disjoint from the sets, $\{p, q\}$ and $\{\tau \cdot p, \tau \cdot q\}$, respectively. Since $\rho_i \to \rho$ and $\rho_i(A)$ is parabolic,

$$\rho_i(A)_* \to \rho(A)_*$$

and

$$\rho_i(B)_* \to \rho(B)_*.$$

For large i, define

$$\tau_i := T[(\rho_i(A)_*, \rho_i(B)_*), (p, \tau \cdot p), (q, \tau \cdot q)].$$

As in the previous case, $\tau_i \to \tau$ in $\mathbb{PGL}(2, \mathbb{R})$ and

$$\tau_i \cdot \rho_i(A)_* = \rho_i(B)_*.$$

Because $\tau_i \to \tau$ and $\tau \in \mathbb{PSL}(2, \mathbb{R})$, then both $\tau_i \in \mathbb{PSL}(2, \mathbb{R})$ for large i and

$$\tau_i \cdot \rho_i(A) \cdot {\tau_i}^{-1} \to \tau \cdot \rho(A) \cdot \tau^{-1} = \rho(B)^{-1}.$$

$\rho(B)$ and $\rho_i(B)$ are not involutions, so for large i, $\tau_i \cdot \rho_i(A) \cdot {\tau_i}^{-1} = \rho_i(B^{-1})$.

Case 3. $\rho(A)$ **and** $\rho(B)$ **are elliptic of non-zero trace**

If X and Y are elliptic members of $\mathbb{PSL}(2,\mathbb{R})$, let $\overline{X_*, Y_*}$ be the geodesic segment joining X_* and Y_*. Let

$$F(X,Y) : \{(X,Y) \in \mathbb{PSL}(2,\mathbb{R}) \times \mathbb{PSL}(2,\mathbb{R}) : |\mathsf{Tr}(X)|, |\mathsf{Tr}(X)| < 2\} \longrightarrow \mathbb{PSL}(2,\mathbb{R})$$

be the translation along $\overline{X_*, Y_*}$ taking X_* to Y_*. F is continuous. Observe

- $\gamma, F(\rho(A)_*, \rho(B)_*) \in \mathbb{PSL}(2,\mathbb{R})$

  and

- $\tau \cdot \rho(A) \cdot \tau^{-1} = \rho(B)^{-1}$

A transformation, $\alpha \in \mathbb{PSL}(2,\mathbb{R})$, takes $\rho(A)_*$ to $\rho(B)_*$ if and only if α is in the path connected set, $\mathsf{Stab}(\rho(B)_*) \cdot F(\rho(A)_*, \rho(B)_*)$.

Because $\rho_i \to \rho$, it follows that

$$\rho_i(A)_* \to \rho(A)_*$$

and

$$\rho_i(B)_* \to \rho(B)_*.$$

Furthermore

$$F(\rho_i(A)_*, \rho_i(B)_*) \to F(\rho(A)_*, \rho(B)_*).$$

Let $s_i \in \mathsf{Stab}(\rho_i(B)_*)$ be so that $s_i \to s$. If

$$\tau_i := s_i \cdot F(\rho_i(A)_*, \rho_i(B)_*),$$

then $\tau_i \to \tau$ and $\tau_i \cdot \rho_i(A)_* = \rho_i(B)_*$. So for each i, either

$$\tau_i \cdot \rho_i(A) \cdot {\tau_i}^{-1} = \rho_i(B)^{-1}$$

or

$$\tau_i \cdot \rho_i(A) \cdot {\tau_i}^{-1} = \rho_i(B).$$

Since $\mathsf{Tr}(\rho(A)) = \mathsf{Tr}(\rho(B)) \neq 0$, it follows that

$$\rho(B) \neq \rho(B)^{-1}.$$

As in the previous two cases, it follows that

$$\tau_i \cdot \rho_i(A) \cdot {\tau_i}^{-1} = \rho_i(B)^{-1}$$

for large i.

Case 4. $\rho(A)$ and $\rho(B)$ are the identity isometry

In this case, any member of $\mathbb{PSL}(2,\mathbb{R})$ centralizes both $\rho(A)$ and $\rho(B)$ so choose any sequence $\tau_i \to \tau$.

□

Lemma 47 (The $\Sigma, \overline{\Sigma}$, Lifting Lemma)**.** *Let P be a property of extendible $\pi_1(\Sigma)$ representations and let Q be a property of $\pi_1(\overline{\Sigma})$ representations, where for the extendible Σ group representation,*

$$\rho : \pi_1(\Sigma) \longrightarrow \mathbb{PSL}(2,\mathbb{R}) :$$

If τ satisfies the

$$\rho : \pi_1(\Sigma) \longrightarrow \mathbb{PSL}(2,\mathbb{R})$$

Extension Condition, then $P(\rho) \Rightarrow Q(\rho_\gamma)$,

then if

- *any open neighborhood,* U*, of*

$$\rho : \pi_1(\Sigma) \longrightarrow \mathbb{PSL}(2, \mathbb{R})$$

contains a representation, $\overline{\rho}$ *(with the same boundary data as* ρ*), satisfying* $P(\overline{\rho})$

and

- $\rho(A)$ *is not an involution,*

it follows that any open neighborhood, V*, of*

$$\rho_\tau : \pi_1(\overline{\Sigma}) \longrightarrow \mathbb{PSL}(2, \mathbb{R})$$

contains a representation, $\widetilde{\rho_\tau}$ *(with the same boundary data as* ρ*), satisfying* $Q(\widetilde{\rho_\tau})$*.*

Proof. By hypothesis 1, construct a sequence of $\pi_1(\Sigma)$ representations, $\rho_i \longrightarrow \rho$ that satisfy property $P(\rho_i)$. By Lemma 46, it is possible to construct a set of extensions, $\rho_{i_{\tau_i}} \rightarrow \rho_\tau$. By hypothesis, $Q(\rho_{i_{\tau_i}})$ is true. $\square$

2.4.4 Relating 2-holed torus group Representations to genus-2 surface group representations

A homomorphism,

$$\rho : \pi_1(\overline{\Sigma} \simeq \Sigma_{1,2}) \longrightarrow \mathbb{PSL}(2, \mathbb{R}),$$

is **extendible** if $\rho(q_{1*}(C))$ is $\mathbb{PSL}(2,\mathbb{R})$ conjugate to $\rho(q_{1*}(D))^{-1}$.

For an extendible homomorphism, ρ, $\tau \in \mathbb{PSL}(2,\mathbb{R})$ satisfies the ρ **Extension Condition** if

$$\tau \cdot \rho(C) \cdot \tau^{-1} = \rho(D^{-1}).$$

Given ρ and τ, it is possible to define a representation,

$$\rho^{\tau} : \pi_1(\overline{\overline{\Sigma}}) \longrightarrow \mathbb{PSL}(2, \mathbb{R}),$$

as follows:

$$\rho^{\tau}(q_{2*}q_{1*}(A)) := \rho((q_1)_*(A))$$

$$\rho^{\tau}(q_{2*}q_1(S_1)) := \rho(q_1(S_1))$$

$$\rho^{\tau}(q_{2*}q_{1*}(C)) := \rho(C)$$

$$\rho^{\tau}(q_2q_1(S_2)) := \tau.$$

As in the previous section, ρ^{τ}

- is an extension of ρ

  and

- $\rho^{\tau}([q_{2*}q_{1*}(A), q_{2*}q_1(S_1)][q_{2*}q_{1*}(C), q_2q_1(S_2)]) = \mathbb{I}.$

It is also possible to lift a genus-2 surface group representation to a 2-holed torus group representation.

If

$$\rho : \pi_1(\overline{\overline{\Sigma}}) \longrightarrow \mathbb{PSL}(2, \mathbb{R})$$

is a homomorphism, define $\ddot{\rho} : \pi_1(\overline{\Sigma}) \longrightarrow \mathbb{PSL}(2, \mathbb{R})$ as follows:

$$\ddot{\rho}((q_1)_*(A)) := \rho(q_{2*}q_{1*}(A))$$

$$\ddot{\rho}(q_1(S_1)) := \rho(q_{2*}q_1(S_1))$$

$$\ddot{\rho}((q_1)_*(C)) := \rho(q_{2*}q_{1*}(C)).$$

This will force

$$\ddot{\rho}(q_{1*}(D)) = \rho(q_2 q_1(S_2)) \cdot \rho(q_{2*}q_{1*}(C^{-1})) \cdot \rho(q_2 q_1(S_2))^{-1}.$$

Therefore ρ can be canonically lifted to a 2-holed torus group representation.

The $\overline{\Sigma}, \overline{\overline{\Sigma}}$ Lifting Lemma

Lemma 48. *Let P be a property of extendible $\pi_1(\overline{\Sigma})$ representations and let Q be a property of $\pi_1(\overline{\overline{\Sigma}})$ representations, where for the extendible Σ group representation,*

$$\rho : \pi_1(\overline{\Sigma}) \longrightarrow \mathbb{PSL}(2, \mathbb{R})$$

:

If γ satisfies the ρ Extension Condition, then $P(\rho) \Rightarrow Q(\rho_\gamma)$,

then if

- *any open neighborhood, U, of*

$$\rho : \pi_1(\overline{\Sigma}) \longrightarrow \mathbb{PSL}(2, \mathbb{R})$$

contains a representation,

$$\overline{\rho} : \pi_1(\overline{\Sigma}) \longrightarrow \mathbb{PSL}(2, \mathbb{R})$$

(with the same boundary data as ρ), satisfying $P(\overline{\rho})$

and

- *$\rho(A)$ is not an involution,*

it follows that any open neighborhood, V, of

$$\rho_\gamma : \pi_1(\overline{\overline{\Sigma}}) \longrightarrow \mathbb{PSL}(2, \mathbb{R})$$

contains a representation,

$$\widetilde{\rho_\gamma} : \pi_1(\overline{\overline{\Sigma}}) \longrightarrow \mathbb{PSL}(2, \mathbb{R})$$

(with the same boundary data as ρ), satisfying $Q(\widetilde{\rho_\gamma})$.

2.5 The proof of Theorem 25

The Curve Lengthening Lemma

Lemma 49 (The Curve Lengthening Lemma)**.** *Let*

$$\rho : \pi_1(\overline{\Sigma}) \longrightarrow \mathbb{PSL}(2, \mathbb{R})$$

be a homomorphism and let γ *and* β *be non-peripheral and non-separating simple closed curves on* $\overline{\Sigma}$ *so that:*

- $i(\gamma, \beta) = 0$,

- $\rho(\beta) \neq \mathbb{I}$ *is non-elliptic*

  and

- $\rho(\gamma)$ *is elliptic,*

  then there is a

- *separating simple closed curve,* ξ*, on* $\overline{\Sigma}$

  and

- *a* $\overline{\Sigma}$ *group representation,* $\overline{\rho}$*, that is arbitrarily close to and has the same boundary data as* ρ

so that $\overline{\rho}(\xi)$ *is unipotent.*

Proof. Since $i(\gamma, \beta) = 0$, there is a homeomorphism, ϕ (fixing the prescribed basepoint of $\overline{\Sigma}$), taking γ to $q_{1*}(A \cdot C)^{\pm 1}$ and taking β to $q_{1*}(A)^{\pm 1}$. Furthermore

$$\phi^{-1^*} : \mathsf{Hom}(\pi_1(\Sigma), \mathbb{PSL}(2, \mathbb{R})) \longrightarrow \mathsf{Hom}(\pi_1(\Sigma), \mathbb{PSL}(2, \mathbb{R}))$$

is continuous. So if ρ is arbitrarily close to a representation that takes a separating simple closed curve to a unipotent isometry, then so is $\phi^{-1*}(\rho)$. Without loss of generality, assume that $\rho((q_1)_*(A \cdot C))$ is elliptic and $\rho(q_{1*}(A)) \neq \mathbb{I}$ is non-elliptic.

It suffices to show that when this is the case, there is a representation, $\overline{\rho}$, that is both arbitrarily close to ρ and takes a separating simple closed curve to a unipotent isometry.

Observe that

- $\rho(q_{1*}(A)) = \dot{\rho}(A) \neq \mathbb{I}$ and $\dot{\rho}(B) \neq \mathbb{I}$ are $\mathbb{PSL}(2, \mathbb{R})$ conjugate and non-elliptic

  while

- $\rho(q_{1*}(A \cdot C)) = \dot{\rho}(A \cdot C)$ is elliptic.

By the Elliptic-Parabolic Lemma, the 4-holed sphere group representation, $\dot{\rho}$, is arbitrarily close to a 4-holed sphere group representation, $\dot{\overline{\rho}}$ (with the same boundary data as ρ), that takes a non-peripheral simple closed curve ζ in the class of $A \cdot B$ to a unipotent isometry.

Let $P(\eta)$ be the following property of extendible $\pi_1(\Sigma)$ representations:

- "η takes a simple closed curve in the class of $A \cdot B$ to a unipotent isometry

  and

- $\eta(A)$ is either hyperbolic or parabolic"

  and let $Q(\zeta)$ be the following property of $\pi_1(\overline{\Sigma})$ representations:

- "ζ takes a separating simple closed curve to a unipotent isometry".

For an extendible 4-holed sphere group representation, η and for $\gamma \in \mathbb{PSL}(2,\mathbb{R})$ that satisfies the η Extension Condition,

$$P(\eta) \Rightarrow Q(\eta_\gamma).$$

It was just shown that in any open neighborhood of $\dot{\rho}$ there is a representation, $\dot{\overline{\rho}}$, that satisfies $P(\dot{\overline{\rho}})$ and has the same boundary data as $\dot{\rho}$. By the $\Sigma, \overline{\Sigma}$ Lifting Lemma, in any open neighborhood of ρ there is a representation, $\overline{\rho}$ (with the same boundary data as ρ), that satisfies $Q(\overline{\rho})$. That is, $\overline{\rho}$ takes a separating simple closed curve to a unipotent isometry.

$\square$

Theorem 50 (The 2-holed Torus Group Theorem)**.** *If the representation,*

$$\rho : \pi_1(\overline{\Sigma}) \longrightarrow \mathbb{PSL}(2,\mathbb{R}),$$

takes all boundary components to non-identity isometries and takes a non-peripheral non-separating simple closed curve, γ, to an elliptic isometry, then ρ is arbitrarily close to a representation,

$$\overline{\rho} : \pi_1(\overline{\Sigma}) \longrightarrow \mathbb{PSL}(2,\mathbb{R}),$$

that takes a separating simple closed curve to a unipotent isometry.

Without loss of generality, $\gamma = q_{1*}(A)$.

In light of the Curve Lengthening Lemma, the following fact is necessary.

Lemma 51. *If ρ satisfies the hypothesis' of Theorem 50 and if*

$$\rho_{|\pi_1(\Sigma^1)} = \langle q_{1*}(A), q_1(S_1) \rangle$$

is non-abelian, then there is

- *a representation,* $\overline{\rho}$*, that is arbitrarily close to and has the same boundary data as* ρ

  and

- *a non-separating simple closed curve,* ζ*, on* $\overline{\Sigma}$

  so that

- $i(\zeta, \gamma) = 0$

  and

- $\overline{\rho}(\zeta)$ *is hyperbolic.*

Proof. Assume that both $\rho(q_{1*}(A))$ is elliptic and $\rho(q_{1*}(A \cdot C))$ is not hyperbolic. Since $\rho_{|\pi_1(\Sigma^1)}$ is both non-abelian and takes $\gamma = q_{1*}(A)$ to an elliptic isometry, it follows that $\rho([q_{1*}(A), q_1(S_1)])$ is hyperbolic, [7]. Therefore without loss of generality,

$$\rho([q_{1*}(A), q_1(S_1)]) = \rho(q_{1*}(C \cdot D)^{-1}) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix},$$

where $\lambda \neq 0, \pm 1 \in \mathbb{R}$,

$$\rho(q_{1*}(A)) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and

$$\rho(q_{1*}(C)) = \begin{pmatrix} u & v \\ w & z \end{pmatrix}.$$

Because $\rho(q_{1*}(A))$ is elliptic, both $b \neq 0$ and $c \neq 0$.

$$\omega_n := D_{[q_{1*}(A), q_1(S_1)]_*}{}^n(q_{1*}(A \cdot C))$$

is represented by a non-separating simple closed curve on $\overline{\Sigma}$ that does not intersect $q_1(A)$ on $\overline{\Sigma}$.

$$|\mathsf{Tr}(\rho(\omega_n))| = |au + zd + cv\lambda^{-2n} + bw\lambda^{2n}|.$$

Because both $b \neq 0$ and $c \neq 0$, if either $v \neq 0$ or $w \neq 0$ (i.e. $\rho(q_{1*}(C))$ is not diagonal), then there is an integer, $n \geq 0$, so that $\rho(\omega_n)$ is hyperbolic. Therefore (ω_n) is a non-separating simple closed curve on $\overline{\Sigma}$ where:

- $\rho((\omega_n))$ is hyperbolic

  and

- $i(\omega_n, (D^n_{[q_{1*}(A), q_1(S_1)]}(q_{1*}(A)))) = i(\omega_n, q_{1*}(A)) = 0$ on $\overline{\Sigma}$.

It suffices to show that ρ is arbitrarily close to a $\overline{\Sigma}$ group representation (with the same boundary data as ρ), $\overline{\rho}$, where $\overline{\rho}(q_{1*}(C))$ is not diagonal.

By hypothesis, $\rho(q_{1*}(C)), \rho(q_{1*}(D)) \neq \mathbb{I}$. Assume $\rho(q_{1*}(C))$ is diagonal:

$$\rho(q_{1*}(C)) = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}.$$

Recall that

$$\rho(q_{1*}(C \cdot D))^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}.$$

Define

$$\overline{\rho} : \pi_1(\overline{\Sigma}) \longrightarrow \mathbb{PSL}(2, \mathbb{R})$$

as follows:

$$\overline{\rho}(q_{1*}(A)) := \rho(q_{1*}(A))$$

$$\overline{\rho}(q_1(S_1)) := \rho(q_1(S_1))$$

choose a non-zero real number, δ, with arbitrarily small absolute value so that:

$$\overline{\rho}(q_{1*}(C)) := \begin{pmatrix} u & -\delta \\ 0 & u^{-1} \end{pmatrix}$$

$$\overline{\rho}(q_{1*}(D)) := \begin{pmatrix} \lambda^{-1}u^{-1} & \delta\lambda \\ 0 & \lambda u \end{pmatrix}.$$

(Note that since $\rho(q_{1*}(C)), \rho(q_{1*}(D)) \neq \mathbb{I}$, it follows that $\overline{\rho}(q_{1*}(C))$ is conjugate to $\rho(q_{1*}(C))$ and $\overline{\rho}(q_{1*}(D))$ is conjugate to $\rho(q_{1*}(D))$.)

Then,

$$\overline{\rho}(q_{1*}(C \cdot D))^{-1} = \begin{pmatrix} \lambda & -(u\delta\lambda - \lambda u\delta) = 0 \\ 0 & \lambda^{-1} \end{pmatrix} = \rho(q_{1*}(C \cdot D))^{-1}.$$

$\overline{\rho}(q_{1*}(C))$ is not diagonal, so $\overline{\rho}$ can be chosen to be arbitrarily close to and to have the same boundary data as ρ.

□

Proof of the 2-holed Torus Group Theorem

Proof. Without loss of generality, $\gamma = \rho(q_{1*}(A))$ is elliptic. Either $\rho_{|\pi_1(\Sigma^1)}$ is abelian, in which case $\rho([q_{1*}(A), q_1(S_1)]) = \mathbb{I}$, or not. If so, the result is established. If not, apply Lemma 51 to find a 2-holed torus group representation, ρ_1, that is arbitrarily close to and has the same boundary data as ρ, so that there is a non-separating simple closed curve, ζ, where

- $i(\zeta, q_{1*}(A)) = 0$

  and

- $\rho_1(\zeta)$ is hyperbolic.

Apply the Curve Lengthening Lemma to obtain a 2-holed torus group representation, $\overline{\rho}$, that is arbitrarily close to and has the same boundary data as ρ_1, so that $\overline{\rho}$ takes a separating simple closed curve to a unipotent isometry.

$\square$

Unfortunately it is not clear when a relative Euler class 1, 2-holed torus group representation, $\overline{\rho}$, is obtained by gluing a reducible representation of the 1-holed torus group to a Fuchsian representation of the 3-holed sphere group or not.

Open Question: Can any relative Euler class 1, 2-holed torus group representation taking a non-separating simple closed curve to an elliptic element be perturbed by an arbitrarily small perturbation to a representation obtained by gluing a reducible 1-holed torus group representation to a 3-holed sphere group representation?

The Genus-2 Surface Group Theorem

Theorem 52 (The Genus-2 Surface Group Theorem)**.** *If*

$$\rho : \pi_1(\Sigma_2) \longrightarrow \mathbb{PSL}(2, \mathbb{R})$$

takes a non-separating simple closed curve to an elliptic isometry, then ρ is arbitrarily close to a representation,

$$\overline{\rho} : \pi_1(\Sigma_2) \longrightarrow \mathbb{PSL}(2, \mathbb{R}),$$

that takes a separating simple closed curve to a unipotent isometry.

Proof of the Genus-2 Theorem

Proof. Without loss of generality, $\gamma = q_{2*}q_{1*}(A)$ and $\rho(\gamma)$ is elliptic. Either

$$\rho([q_{2*}q_{1*}(A), q_{2*}q_1(S_1)]) = \mathbb{I}$$

or not. If so, the result holds. If not, then both $\rho_{|\pi_1(\Sigma^1)}$ and $\rho_{|\pi_1(\Sigma^2)}$ are non-abelian and as in the proof of Theorem 50, $\rho([q_{2*}q_{1*}(A), q_{2*}q_1(S_1)])$ is hyperbolic (and without loss of generality, diagonal). Since $\rho_{|\pi_1(\Sigma^2)}$ is non-abelian, without loss of generality, $\rho(q_{2*}q_{1*}(C))$ is not diagonal. In this case, proceed as in the proof of Lemma 51 to find a non-separating simple closed curve ζ that does not intersect γ where $\rho(\zeta)$ is hyperbolic. Without loss of generality (after applying an appropriate homeomorphism to $\overline{\overline{\Sigma}}$), assume $\zeta = q_{2*}q_{1*}(C)$ and γ is still equal to $q_{2*}q_{1*}(A)$. Apply Theorem 50 to the 2-holed torus group representation, $\ddot{\rho}$, then apply the $\overline{\Sigma}, \overline{\overline{\Sigma}}$ Lifting

Theorem to the representation obtained by perturbing $\ddot{\rho}$ (by Theorem 50) to prove the result as follows:

$\rho(q_{2*}q_{1*}(C)) = \ddot{\rho}(q_{1*}(C))$ and $\ddot{\rho}(q_{1*}(D))$ are hyperbolic and $\mathbb{PSL}(2,\mathbb{R})$ conjugate. Therefore it is possible to apply Theorem 50 to $\ddot{\rho}$ to obtain a representation, $\overline{\ddot{\rho}}$, that

- is arbitrarily close to and has the same boundary data as $\ddot{\rho}$

  and

- takes a separating simple closed curve to a unipotent isometry.

Since $\ddot{\rho}(q_{1*}(C)) = \rho(q_{2*}q_{1*}(C))$ is not an involution, neither is $\overline{\ddot{\rho}}(q_{1*}(C))$. Apply the $\overline{\Sigma}, \overline{\overline{\Sigma}}$ Lifting Theorem to $\overline{\ddot{\rho}}$ to obtain a representation,

$$\overline{\rho} : \pi_1(\overline{\overline{\Sigma}}) \longrightarrow \mathbb{PSL}(2,\mathbb{R}),$$

that is arbitrarily close to ρ and takes a separating simple closed curve to a unipotent isometry.

$\square$

Corollary. If $\Sigma \simeq \Sigma_2$ and if the Euler class 1 representation,

$$\rho : \pi_1(\Sigma) \longrightarrow \mathbb{PSL}(2,\mathbb{R}),$$

takes some non-separating simple closed curve to an elliptic isometry, then ρ is arbitrarily close to a representation,

$$\overline{\rho} : \pi_1(\Sigma) \longrightarrow \mathbb{PSL}(2,\mathbb{R}),$$

that takes a separating simple closed curve to a parabolic isometry.

Proof. By the Genus-2 Surface Group Theorem, ρ is arbitrarily close to $\overline{\rho}$ that takes a separating simple closed curve to a unipotent isometry. $e(\overline{\rho})$ is also 1. If $\overline{\rho}$: $\pi_1(\Sigma) \longrightarrow \mathbb{PSL}(2,\mathbb{R})$ takes a separating simple closed curve (say $[q_{2*}q_{1*}(A), q_{2*}q_1(S_1)]$) to $\mathbb{I}$, then $\overline{\rho}_{\pi_1(\Sigma^1)}$ and $\overline{\rho}_{\pi_1(\Sigma^2)}$ are both abelian, [10]. Therefore $e(\overline{\rho}_{\pi_1(\Sigma^1)}) = e(\overline{\rho}_{\pi_1(\Sigma^2)}) = 0$. By the additivity of $e(\overline{\rho})$, $e(\rho) = e(\overline{\rho}) = 0$. This contradicts the hypothesis that $e(\rho) = 1$. $\square$

Corollary. Let $\mathsf{Simp} \subset \pi_1(\Sigma_2)$ be the set of classes represented by non-separating simple closed curves. If the Euler class ± 1 homomorphism,

$$\rho : \pi_1(\Sigma_2) \longrightarrow \mathbb{PSL}(2,\mathbb{R}),$$

takes a non-separating simple closed curve to an elliptic isometry, then ρ is arbitrarily close to a homomorphism,

$$\overline{\rho} : \pi_1(\Sigma_2) \longrightarrow \mathbb{PSL}(2,\mathbb{R}),$$

where the set, $\{|\mathsf{Tr}(\overline{\rho}(\gamma)|)\}_{\gamma\in\mathsf{Simp}}$, is dense in $[0,\infty)$.

Proof. This follows from Corollary 2.5 and Theorem 37. $\square$

3. BOUNDARY PARABOLIC 4-HOLED SPHERE GROUP REPRESENTATIONS

3.1 Boundary parabolic 4-holed sphere group representations with an elliptic simple closed curve

Theorem 53. *If the relative Euler class 1, boundary-parabolic representation,*

$$\rho : \pi_1(\Sigma_{0,4}) \longrightarrow \mathbb{PSL}(2, \mathbb{R}),$$

takes a non-peripheral simple closed curve to an elliptic isometry, then there is

- *a non-peripheral simple closed curve, γ, that separates $\Sigma_{0,4}$ into two 3-holed spheres, Σ^1 and Σ^2,*

  and

- *a homomorphism,*

  $$\overline{\rho} : \pi_1(\Sigma_{0,4}) \longrightarrow \mathbb{PSL}(2, \mathbb{R}),$$

  that is both arbitrarily close to and has the same boundary conditions as ρ so that the following is true:

  - *$\overline{\rho}_{\pi_1(\Sigma^1)}$ is the holonomy of a cusped hyperbolic structure on Σ^1*

    while

– $\overline{\rho}_{\pi_1(\Sigma^2)}$ *is an abelian unipotent representation.*

Proof. Recall,

$$\pi_1(\Sigma_{0,3}) = \langle A, B, C, | A \cdot B \cdot C \rangle,$$

where A, B and C represent the boundary components of $\Sigma_{0,3}$.

Lemma 54. *If*

$$\zeta : \pi_1(\Sigma_{0,3}) \longrightarrow \mathbb{PSL}(2, \mathbb{R})$$

is boundary-parabolic, then either

- ζ *is abelian, in which case its relative Euler class,* $e(\zeta)$*, is 0*

  or

- ζ *is the holonomy of a cusped hyperbolic structure on* $\Sigma_{0,3}$*, in which case its relative Euler class,* $e(\zeta)$*, is* ± 1.

Proof. Let $x = \mathsf{Tr}(\zeta(A)), y = \mathsf{Tr}(\zeta(B))$ and $z = \mathsf{Tr}(\zeta(A \cdot B = C^{-1}))$. Then,

$$\mathsf{Tr}(\zeta([A, B])) = x^2 + y^2 + z^2 - xyz - 2.$$

Since ζ is boundary parabolic, $x = \pm 2, y = \pm 2$ and $z = \pm 2$. Therefore

$$x^2 + y^2 + z^2 = 4 + 4 + 4 = 12$$

and depending on the signs of x, y and z,

$$xyz = \pm 8.$$

If $xyz = 8$, then $\mathsf{Tr}(\zeta([A, B])) = 2$ and if $xyz = -8$, then $\mathsf{Tr}(\zeta([A, B])) = 18$.

Let $\kappa = \rho([A, B])$.

- If $\mathsf{Tr}(\kappa) = 2$, then the unipotent representation,

$$\zeta : \pi_1(\Sigma_{0,3}) \longrightarrow \mathbb{PSL}(2, \mathbb{R}),$$

  is reducible and abelian and therefore has relative Euler class 0.

- If $\mathsf{Tr}(\kappa = 18)$, ζ is the holonomy of a cusped hyperbolic structure on $\Sigma_{0,3}$ and therefore has relative Euler class ± 1, [10], Lemma 8.2.5.

$\square$

Since $|\mathsf{Tr}(\rho(A))| = |\mathsf{Tr}(\rho(B))| = 2$, by the Elliptic-Parabolic Lemma, there is

- a non-peripheral simple closed curve, γ, that separates $\Sigma_{0,4}$ into two 3-holed spheres, Σ^1 and Σ^2,

  and

- a homomorphism,

$$\overline{\rho} : \pi_1(\Sigma_{0,4}) \longrightarrow \mathbb{PSL}(2, \mathbb{R})$$

  (with the same boundary data as ρ), so that

$\overline{\rho}(\gamma)$ is unipotent. Without loss of generality, $\gamma = A \cdot C$. Let Σ^1 be the 3-holed sphere with boundary components A, C and $A \cdot C$ and let Σ^2 be the 3-holed sphere with boundary components, B, D and $(A \cdot C)^{-1}$. Since $\overline{\rho}_{|\pi_1(\Sigma^1)}$ and $\overline{\rho}_{|\pi_1(\Sigma^2)}$ are both boundary parabolic,

$$e(\overline{\rho}) = e(\overline{\rho}_{|\pi_1(\Sigma^1)}) + e(\overline{\rho}_{|\pi_1(\Sigma^2)}).$$

Since the relative Euler class is a continuous, integer valued function on the set of boundary-non-elliptic 4-holed sphere group representations into $\mathbb{PSL}(2,\mathbb{R})$,

$$e(\rho) = e(\overline{\rho}).$$

Therefore by Lemma 54, one of $\overline{\rho}_{|\pi_1(\Sigma^1)}$ and $\overline{\rho}_{|\pi_1(\Sigma^1)}$ is the holonomy of a cusped hyperbolic structure on a 3-holed sphere while the other is an abelian unipotent representation. This proves Theorem 5 (as listed in the introduction) or Theorem 53 (as listed in this chapter). $\square$

3.2 Irreducible, non-discrete 4-holed sphere group representations with no simple closed elliptic

Let $\Sigma^1 \subset \Sigma_{0,4}$ be the 3-holed sphere with boundary components A, B and $A \cdot B$ while $\Sigma^2 \subset \Sigma_{0,4}$ is the 3-holed sphere with boundary components $A \cdot B$, C and D. Define the 1-parameter family of homomorphisms,

$$\rho_t : \pi_1(\Sigma_{0,4}) \longrightarrow \mathbb{PSL}(2,\mathbb{R}),$$

for $t \in \mathbb{R}$, as follows:

$$\rho_t(A) := \begin{pmatrix} -2 & \frac{1}{4} \\ -4 & 0 \end{pmatrix}$$

$$\rho_t(B) := \begin{pmatrix} 0 & -\frac{1}{4} \\ 4 & 2 \end{pmatrix}.$$

$$\rho_t(C) := \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

$$\rho_t(D = (A \cdot B \cdot C)^{-1}) := \begin{pmatrix} 1 & -(1+t) \\ 0 & 1 \end{pmatrix}.$$

$\rho_{t_{\pi_1(\Sigma^1)}}$ is the holonomy of a cusped hyperbolic structure on Σ^1 with

$$\rho_t(A \cdot B) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

while $\rho_{t_{\pi_1(\Sigma^2)}}$ is abelian and all unipotent.

A pair of simple calculations yield

$$\mathsf{Tr}(\rho_t(B \cdot C)) = 2 + 4t$$

and

$$\mathsf{Tr}(\rho_t(A \cdot C)) = -(2 + 4t).$$

Theorem 55. *If $t > 0$, then ρ_t takes all non-peripheral simple closed curves to hyperbolic isometries.*

To prove this result, let $\mathsf{Mod}(\Sigma_{0,4})$ be the group of isotopy classes of homeomorphisms of $\Sigma_{0,4}$. Define the subgroup of $\mathsf{Mod}(\Sigma_{0,4})$, G, as follows:

$$G := \langle D_{A \cdot B}, D_{B \cdot C} \rangle.$$

Lemma 56. *Every non-peripheral simple closed curve on $\Sigma_{0,4}$ is freely homotopic to a member of $G \cdot \{(A \cdot B)^{\pm 1}, (A \cdot C)^{\pm 1}, (B \cdot C)^{\pm 1}\}$.*

Proof. The 4-holed sphere, $\Sigma_{0,4}$, is embedded into a quadruply punctured sphere $\overline{\Sigma_{0,4}}$ via a homotopy equivalence so that

- all simple closed curves in $\Sigma_{0,4}$ embed as simple closed curves in $\overline{\Sigma_{0,4}}$

  and

- there is a strong deformation retraction of $\overline{\Sigma_{0,4}}$ onto $\Sigma_{0,4}$ that happens to be an isotopy. Therefore any simple closed curve on $\overline{\Sigma_{0,4}}$ can be isotoped to a simple closed curve on $\Sigma_{0,4}$.

Following [1], $\mathsf{PMod}(\overline{\Sigma_{0,4}})$ is the subgroup of $\mathsf{Mod}(\overline{\Sigma_{0,4}})$ that fixes each puncture.

The Birman Exact sequence of $\overline{\Sigma_{0,4}}$,

$$1 \longrightarrow \pi_1(\Sigma_{0,3}) \longrightarrow \mathsf{PMod}(\overline{\Sigma_{0,4}}) \longrightarrow \mathsf{PMod}(\Sigma_{0,3}) \longrightarrow 1,$$

is exact.

The first non-trivial map is the "point-pushing map" P_B obtained by pushing the puncture (that corresponds to) B around the prescribed member of $\pi_1(\Sigma_{0,3})$. The second non-trivial map is obtained by forgetting the puncture, B. Since $\mathsf{PMod}(\Sigma_{0,3})$ is trivial, P_B is an isomorphism. Therefore $\mathsf{PMod}(\overline{\Sigma_{0,4}})$ is freely generated by

$$\mathsf{P}_B(A) = D_A {D_{A\cdot B}}^{-1}$$

and

$$\mathsf{P}_B(C) = D_A {D_{B\cdot C}}^{-1},$$

[1]. Since A and C are homotopic to boundary components (actually punctures) of $\overline{\Sigma_{0,4}}$,

$$\mathsf{id} = D_{A*}, D_{C*} : \pi_1(\overline{\Sigma_{0,4}}) \longrightarrow \pi_1(\overline{\Sigma_{0,4}}).$$

Therefore

$$\mathsf{PMod}(\overline{\Sigma_{0,4}}) = \langle D_{A\cdot B}, D_{B\cdot C}\rangle = G.$$

To establish Lemma 56, every non-peripheral simple closed curve in the 4-holed sphere is freely homotopic to a member of the $G = \mathsf{PMod}(\overline{\Sigma_{0,4}})$ orbit of the set,

$$\{A \cdot B^{\pm 1}, A \cdot C^{\pm 1}, B \cdot C^{\pm 1}\}.$$

□

Lemma 57. *If $\omega \in \pi_1(\Sigma_{0,4})$, then $\rho_t(D_{A\cdot B*}(\omega)) = \rho(\omega)$.*

Proof. Recall that

$$\pi_1(\Sigma_{0,4}) \simeq \langle A, B, C, D | A \cdot B \cdot C \cdot D\rangle$$

is freely generated by the set,

$$\{A, B, C\}.$$

Each word in $\pi_1(\Sigma_{0,4})$ is of the following form:

$$C^{n_1} \cdot W_1(A, B) \cdot C^{n_2} \cdot \ldots \cdot W_{k-1}(A, B) \cdot C^{n_k},$$

where $n_i \neq 0$ for $1 < i < k$ and for each i, $W_i(A, B)$ is a word in $\langle A, B\rangle$.

$$D_{A\cdot B*}(C^{n_1} \cdot W_1(A, B) \cdot C^{n_2} \cdot \ldots \cdot W_{k-1}(A, B) \cdot C^{n_k}) =$$

$$(A \cdot B) \cdot C^{n_1} \cdot (A \cdot B)^{-1} \cdot W_1(A, B) \cdot (A \cdot B) \cdot C^{n_2} \cdot (A \cdot B)^{-1} \cdot$$

$$\ldots \cdot W_{k-1}(A, B) \cdot (A \cdot B) C^{n_k} \cdot (A \cdot B)^{-1}.$$

$\rho_t(A \cdot B)$ centralizes $\rho_t(C)$. Since ρ_t is a homomorphism, the lemma is proved.

□

In particular if $\omega \in \pi_1(\Sigma_{0,4})$, then $\rho_t(D_{A \cdot B *}(\omega))$ is elliptic if and only if $\rho_t(\omega)$ is elliptic. Therefore it suffices to consider the simple closed curves,

$$D_{B \cdot C *}{}^b(A \cdot B),$$

$$D_{B \cdot C *}{}^b(A \cdot C)$$

and

$$D_{B \cdot C *}(B \cdot C)$$

for $b \in \mathbb{Z}$. Because $D_{B \cdot C *}{}^b(B \cdot C)$ is conjugate to $B \cdot C$, if ρ_t takes the simple closed curves,

$$D_{B \cdot C *}{}^b(A \cdot B)$$

and

$$D_{B \cdot C *}{}^b(A \cdot C),$$

to hyperbolic isometries, then ρ_t takes all simple closed curves on $\Sigma_{0,4}$ to either parabolic or hyperbolic isometries.

Lemma 58. *If $t > 0$ and $\omega \in \pi_1(\Sigma_{0,4})$ is represented by a non-peripheral simple closed curve, then $\rho_t(\omega)$ is hyperbolic.*

Proof. Let

$$\beta_2 = ((1+2t) - 2(t^2+t)^{\frac{1}{2}}),$$

$$\beta_1 = ((1+2t) + 2(t^2+t)^{\frac{1}{2}})$$

and

$$\alpha = \frac{\beta_2}{\beta_1}.$$

By a "Mathematica" calculation,

$$\mathsf{Tr}(\rho_t(D_{B\cdot C*}{}^b(A\cdot B))) = \alpha^b + \alpha^{-b}.$$

Therefore

$$|\mathsf{Tr}(\rho_t(D_{B\cdot C_*}^b(A\cdot B)))| \geq 2$$

and equals 2 if and only if $t \in \{0, -\frac{1}{2}, -1\}$.

Notice

$$\beta_1 + \beta_2 = 2 + 4t$$

and

$$\beta_1 - \beta_2 = 4(t+t^2)^{\frac{1}{2}}.$$

By another "Mathematica" calculation, if $b \in \mathbb{Z}$,

$$\mathsf{Tr}(D_{(B\cdot C)_*}{}^b(A\cdot C)) = \frac{-1}{(\beta_1\beta_2)^b}(\beta_1^{2b} + \beta_2^{2b} + (2(t+t^2)^{\frac{1}{2}}(\beta_2^{2b} - \beta_1^{2b})) + 2t(\beta_1^{2b} + \beta_2^{2b})).$$

Regrouping terms and simplifying,

$$\mathsf{Tr}(D_{(B\cdot C)_*}{}^b(A\cdot C)) = -(\frac{\beta_2^b}{\beta_1^b} + 2t\frac{\beta_2^b}{\beta_1^b} + 2(t+t^2)^{\frac{1}{2}}\frac{\beta_2^b}{\beta_1^b} + \frac{\beta_1^b}{\beta_2^b} + 2t\frac{\beta_1^b}{\beta_2^b} - 2(t+t^2)^{\frac{1}{2}}\frac{\beta_1^b}{\beta_2^b}) =$$

$$-((1+2t+2(t+t^2)^{\frac{1}{2}})\frac{\beta_2^b}{\beta_1^b}+(1+2t-2(t+t^2)^{\frac{1}{2}})\frac{\beta_1^b}{\beta_2^b})=$$

$$-(\beta_1\frac{\beta_2^b}{\beta_1^b}+\beta_2\frac{\beta_1^b}{\beta_2^b}).$$

Expand β_1 and β_2 out to obtain

$$\mathsf{Tr}(D_{(B\cdot C)_*}{}^b(A\cdot C))=-(\frac{(1+2t-2(t+t^2)^{\frac{1}{2}})^b}{(1+2t+2(t+t^2)^{\frac{1}{2}})^{b-1}}+\frac{(1+2t+2(t+t^2)^{\frac{1}{2}})^b}{(1+2t-2(t+t^2)^{\frac{1}{2}})^{b-1}}).$$

Replace b with $-b$ to obtain

$$\mathsf{Tr}(D_{(B\cdot C)_*}{}^{-b}(A\cdot C))=-(\frac{(1+2t-2(t+t^2)^{\frac{1}{2}})^{b+1}}{(1+2t+2(t+t^2)^{\frac{1}{2}})^b}+\frac{(1+2t+2(t+t^2)^{\frac{1}{2}})^{b+1}}{(1+2t-2(t+t^2)^{\frac{1}{2}})^b}=$$

$$\mathsf{Tr}(D_{(B\cdot C)_*}{}^{b+1}(A\cdot C))).$$

Therefore without loss of generality, let $b>0$.

Add the two summands in most recent expression for $\mathsf{Tr}(D_{(B\cdot C)_*}{}^b(A\cdot C))$ to obtain

$$\mathsf{Tr}(D_{(B\cdot C)_*}{}^b(A\cdot C))=-(\frac{(1+2t-2(t+t^2)^{\frac{1}{2}})^{2b-1}+(1+2t+2(t+t^2)^{\frac{1}{2}})^{2b-1}}{(1+2t+2(t+t^2)^{\frac{1}{2}})^{b-1}(1+2t-2(t+t^2)^{\frac{1}{2}})^{b-1}}).$$

Lemma 59. *The denominator of the above expression is 1.*

Proof. The denominator of the above expression is

$$(1+2t+2(t+t^2)^{\frac{1}{2}})^{b-1}(1+2t-2(t+t^2)^{\frac{1}{2}})^{b-1}=$$

$$(((1+2t)+2(t+t^2)^{\frac{1}{2}})((1+2t)-2(t+t^2)^{\frac{1}{2}}))^{b-1}=$$

$$((1+2t)^2 - (2(t+t^2)^{\frac{1}{2}})^2)^{b-1} =$$

$$(1+4t+4t^2-4t-4t^2)^{b-1} = 1^{b-1} = 1.$$

$\square$

Therefore the equation,

$$\mathsf{Tr}(D_{(B\cdot C)_*}{}^b(A\cdot C)) = -(\frac{(1+2t-2(t+t^2)^{\frac{1}{2}})^{2b-1} + (1+2t+2(t+t^2)^{\frac{1}{2}})^{2b-1}}{(1+2t+2(t+t^2)^{\frac{1}{2}})^{b-1}(1+2t-2(t+t^2)^{\frac{1}{2}})^{b-1}}),$$

reduces to

$$\mathsf{Tr}(D_{(B\cdot C)_*}{}^b(A\cdot C)) = -((1+2t-2(t+t^2)^{\frac{1}{2}})^{2b-1} + (1+2t+2(t+t^2)^{\frac{1}{2}})^{2b-1}) =$$

$$-(\sum_{0\leq i<2b}\binom{2b-1}{i}(1+2t)^{2b-1-i}(-2(t+t^2)^{\frac{1}{2}})^i + \sum_{0\leq i<2b}\binom{2b-1}{i}(1+2t)^{2b-1-i}(2(t+t^2)^{\frac{1}{2}})^i).$$

Notice that the terms in the above binomial expansions that correspond to the odd powers of $2(t+t^2)^{\frac{1}{2}}$ cancel, so that

$$\mathsf{Tr}(D_{(B\cdot C)_*}{}^b(A\cdot C)) = -(\sum_{0\leq 2i<2b} 2\binom{2b-1}{2i}(1+2t)^{2b-1-2i}(2(t+t^2)^{\frac{1}{2}})^{2i}).$$

Therefore $\mathsf{Tr}(D_{(B\cdot C)_*}{}^b(A\cdot C))$ can be expressed as a polynomial in t with all positive coefficients.

The first term, $c_0^b(t)$, of the expression,

$$\mathsf{Tr}(D_{(B\cdot C)_*}{}^b(A\cdot C)) = -(\sum_{0\leq 2i<2b} 2\binom{2b-1}{2i}(1+2t)^{2b-1-2i}(2(t+t^2)^{\frac{1}{2}})^{2i}),$$

(as a polynomial in $1+2t$ and $2(t+t^2)^{\frac{1}{2}}$) is

$$c_0^b(t) = -2\binom{2b-1}{0}(1+2t)^{2b-1} = 2(1+2t)^{2b-1}.$$

Because $t, b > 0$, it follows that $|c_0^b(t)| > 2$. Therefore

$$|\mathsf{Tr}(\rho(D_{B\cdot C*}{}^b(A\cdot C)))| > 2.$$

$\square$

Theorem 55 follows from Lemma 56, Lemma 57 and Lemma 58. Furthermore if t is irrational, the group $\langle \rho_t(A\cdot B), \rho_t(C)\rangle$ is not discrete, therefore

Theorem 60. *If $t > 0$ is irrational, then ρ_t takes infinitely many curves in $\pi_1(\Sigma_{0,4})$ to elliptic isometries but takes all simple closed curves to hyperbolic isometries.*

The following question remains open:

Open Question: If $\rho : \pi_1(\Sigma_{0,4}) \longrightarrow \mathbb{PSL}(2,\mathbb{R})$ takes all boundary components to hyperbolic isometries, are there non-discrete representations that take all simple closed curves to non-elliptic isometries?

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