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1 Pareto-Evolutionary Robustness

The seminal work of *Maynard Prince* and *Price* in 1973 [3] laid the foundations in establishing the concepts of evolutionarily stable game strategies. It attempts to select strategies which are robust to evolutionary selection pressures. However we observe that most of the literature has evolved is concentrated on single objective games. In this work we extend the notion of evolutionary stabilty to games with vector payoff functions. We trust that this notion of Multicriteria evolutionary stability, models a much larger class of interactions in social, economic and biological problems.

As with single objective games, we focus on symmetric pairwise interactions within a single large population. It does not involve interactions between more than two individuals at a time. We observe that there are many ways of defining the score function which yield different notions of robustness.

1.1 Notations

The analysis concerns symmetric two player games. However these games are different in the sense that the payoff function for each player is vector valued. Since the games are symmetric we can denote both the players pure strategy set as $K = \{1, 2, \cdots, k\}$ and the associated mixed strategy set as $\Delta = \{x \in \mathcal{R}^k_+ : \sum_{i \in K} x_i = 1\}$. In this work, we introduce the notion of *Multiple Matrix Games* (MMG) which is a natural extension to the *Bi-Matrix Games* in the single objective setting. The same matrix notation has also been considered by [2]. Associated with each player is a sequence of matrix payoff functions $(A_1, A_2, \cdots, A_l)$ and $(B_1 = A_1^T, B_2 = A_2^T, \cdots, B_l = A_l^T)$ (for *player I* and *II* respectively). We assume that *player I* is the row-player and *player II* is column player. The polyhedron of mixed strategy profiles is then $\Theta = \Delta^2$, and the vector payoff to strategy $x \in \Delta$, when played against $y \in \Delta$, is given by $u(x, y) = (u^1, u^2, \cdots, u^l)(x, y) = (x.A_1y, x.A_2y, \cdots, x.A_ly)$. The Pareto best response replies to any strategy $y \in \Delta$ is denoted by $\beta^{P*}(y) \subset \Delta$. The set of all symmetric Pash Nash equilibria is denoted by Δ^{PNE}

1.2 Evolutionary Stability

Assume there exists a large population which plays the same incumbent strategy $x \in \Delta$. Let us suppose small group of mutants which are programmed to play some mutant strategy $y \in \Delta$ are introduced into the original population. Let the share of mutants in the *post-entry* population be $\epsilon \in (0, 1)$. Pairs of individuals

Notation	Definition	Name
$x \gg y$	$x_i > y_i \quad i = 1, 2, .., l$	Strict component-wise order
$x > y$	$x_i \geq y_i \quad i = 1, 2, .., l$ and $x \neq y$	Component-wise order
$x >_{lex} y$	$x_j > y_j \quad j = \min\{i : x_i \neq y_i\} \quad j \leq l$	Lexicographic component-wise order
$x_\lambda > y_\lambda$	$\lambda^T x > \lambda^T y \quad \lambda \geq 0$	λ-scalarized order

Table 1: Table of Orders in $\mathcal{R}^l$

are drawn from this hybrid population to play a game. For any individual the probability that the opponent will play the mutant population $y$is ϵ and the probability that the opponent will play the incumbent strategy x is $1 - \epsilon$. Thus the *post-entry payoff* to the incumbent strategy is $u(x, \epsilon y + (1 - \epsilon)x)$ and that of the mutant strategy is $u(y, \epsilon y + (1 - \epsilon)x)$.

Definition A strategy $x \in \Delta$ is said to be *evolutionary stable* in the $\succ$ ordering if $\forall y \in \Delta, y \neq x, \exists$ some ϵ_y such that $\forall \epsilon \in (0, \epsilon_y)$

$$u(x, \epsilon y + (1 - \epsilon)x) \succ u(y, \epsilon y + (1 - \epsilon)x)$$

The various orderings $\succ$ on $\mathcal{R}^l$ that we consider are shown in table 1. We find that various orderings gives i rise to several definitions for evolutionary stability in the multi-objective setting.

1.3 Strong Ideal Evolutionary Stability

When order $\succ$ is chosen to be the strict component-wise order we obtain the strong ideal evolutionary stability. Let us denote all the strongly ideal evolutionary stable strategies by Δ^{SIESS}. To characterize this evolutionary robustness let us consider the sequence of auxillary games which arise from the components of the vector payoff function.

The sequence of symmetric *component games* is given by the payoff matrices $A_m, B_m = A_m^T \quad 1 \leq m \leq l$ which correspond to the matrices in the original multiple objective game. We denote each game as $G_m^c, 1 \leq m \leq l$ and the corresponding best replies for each of these games by $\beta^{m*}(y) \quad 1 \leq m \leq l$. Let us denote the symmetric nash equilibria for each of these games by $\Delta^{NE(G_m^c)}$. The evolutionarily stable equilibria for each of these games is denoted by $\Delta^{ESS(G_m^c)}$.

Proposition 1.1 $\Delta^{SIESS} = \cap_m \Delta^{ESS(G_m^c)}$

Proof If $x \in \Delta_{SIESS}$
$\Rightarrow u(x, \epsilon y + (1 - \epsilon)x) \gg u(y, \epsilon y + (1 - \epsilon)x)$
$\Rightarrow u(x, \epsilon y + (1 - \epsilon)x) \gg u(y, \epsilon y + (1 - \epsilon)x)$
$\Rightarrow u^m(x, \epsilon y + (1 - \epsilon)x) > u^m(y, \epsilon y + (1 - \epsilon)x) \quad 1 \leq m \leq l$
$\Rightarrow x \in \Delta^{ESS(G_m^c)} \quad 1 \leq m \leq l$

1.4 Ideal Evolutionary Stability

With component-wise order we obtain ideal evolutionary stability. Let us denote all the ideal evolutionary stable strategies by Δ^{IESS}. Let us consider the *component games* introduced in subsection 1.3.

Proposition 1.2
$\Delta^{IESS} = \{x \in \cap_m \Delta^{NE(G_m^c)} \quad | \quad u(y,y) < u(x,y) \quad \forall y \in \cap_m \beta^{*m}(x), y \neq x\}$

Proof Let us define the score function for the strategy $x \in \Delta$ $f(\epsilon, y) = u(x, \epsilon y + (1-\epsilon)x) - u(y, \epsilon y + (1-\epsilon)x)$. If $x \in \Delta$ is ideally evolutionary stable then

$$
\begin{aligned}
& f(\epsilon, y) > 0 \\
\Rightarrow \quad & u(x, \epsilon y + (1-\epsilon)x) - u(y, \epsilon y + (1-\epsilon)x) > 0 \\
\Rightarrow \quad & u(x-y, x) + \epsilon u(x-y, y-x) > 0 \\
\Rightarrow \quad & u(x-y, x) \geq 0 \textit{ and if } u(x-y,x) = 0, \textit{ then } u(y,y) < u(x,y) \\
\Rightarrow \quad & x \in \cap_m \Delta^{NE(G_m^c)} \textit{ and if } y \textit{ is alternate best reply} \\
& \qquad \textit{for all the component games then } u(y,y) < u(x,y)
\end{aligned}
$$

1.5 Lexicographic Evolutionary Stability

With lexicographic order we obtain ideal evolutionary stability. Let us denote all the lexicographic evolutionary stable strategies by Δ^{LESS}. We again consider the *component games* introduced in subsection 1.3.

Proposition 1.3
$\Delta^{LESS} = \{x \in \cap_{m=1}^{j} \Delta^{NE(G_m^c)} \quad | \quad u^1(y,y) \leq u^1(x,y), u^2(y,y) \leq u^2(x,y), \cdots, u^{j-1}(y,y) \leq u^{j-1}(x,y), u^j(y,y) < u^j(x,y) \quad \forall y \in \cap_{m=1}^{j} \beta^{*m}(x), y \neq x\}$

Proof The proof is again based on the vector score function,

$$
\begin{aligned}
& f(\epsilon, y) = u(x-y, x) + \epsilon u(x-y, y-x) >_{lex} 0 \\
\Rightarrow \quad & \begin{array}{c} f^1(\epsilon,y) = 0 \\ f^2(\epsilon,y) = 0 \\ \vdots \\ f^{j-1}(\epsilon,y) = 0 \\ f^j(\epsilon,y) > 0 \end{array} \\
\Rightarrow \quad & \begin{array}{c} x \in \cap_{m=1}^{j} \Delta^{NE(G_m^c)} \\ \forall y \in \cap_{m=1}^{j} \beta^{*m}(x), y \neq x \\ u^1(y,y) \leq u^1(x,y), \\ u^2(y,y) \leq u^2(x,y), \\ \vdots, \\ u^{j-1}(y,y) \leq u^{j-1}(x,y), \\ u^j(y,y) < u^j(x,y) \end{array}
\end{aligned}
$$

1.6 Biased Evolutionary Stability

The λ-scalarized order gives rise what we call as the *Biased Evolutionary Stability*. Let us denote these strategies by Δ^{BESS}.

This ordering is typically used to compare vectors wherein one component has relatively higher importance in the ordering when compared to the others. This gives rise to the perceived bias in the payoff functions of the game. To characterize the biased evolutionary stable strategies, we consider an auxiliary symmetric single objective game $G(\lambda)$. The payoff function for strategy $x \in$

Δ against $y \in \Delta$ in game $G(\lambda)$ is given by $\lambda^T u(x,y)$ (where $u(x,y)$ is the corresponding vector payoff in the multi-objective game).

Proposition 1.4
$\Delta^{BESS} = \{x \in \Delta^{PNE} \quad | \quad y \neq x, y \in \beta^{*P}(x) \Rightarrow x \text{ } dominates \text{ } y$ *as a reply to opponent strategy* $y\}$

Proof The proof is based on the score function of the auxiliary game.

$$
\begin{aligned}
& f(\epsilon, y) = \lambda^T u(x-y,x) + \epsilon\lambda^T u(x-y,y-x) > 0 \\
\Rightarrow \quad & \lambda^T u(x-y,x) \geq 0 \\
\Rightarrow \quad & \lambda^T u(x,x) \geq \lambda^T u(y,x) \\
\Rightarrow \quad & x \in \Delta^{PNE} \\
& \mathit{if}\ y \in \beta^{*P}(x) \Rightarrow \lambda^T u(x-y,y-x) > 0 \\
& \Rightarrow \lambda^T u(x,y) - \lambda^T u(y,y) < 0 \\
& \Rightarrow x \text{ } dominates \text{ } y \text{ } as \text{ } a \text{ } reply \text{ } to \text{ } strategy \text{ } y
\end{aligned}
$$

Remark The λ-scalarized ordering yields a form of evolutionary stability that closely resembles single objective evolutionary stable strategy set.

2 Achieving Pareto ESS - Scalarized Replicator Dynamics

Replicator dynamics have been studied extensively in the single objective settings and their relations to evolutionary stability has been well established (Chapter 3 of [1]). In this section we extend from first principles the population dynamics in a multi-objective setting and establish a new form of replicator dynamics. We also attempt to provided some insights into the dynamic stability of the multi-criteria replicator dynamics.

2.1 Population Dynamics under biased fitness function

Consider a large but finite population of individuals who are programmed to pure strategies $i \in K$ in a symmetric multi-objective two player game with the vector payoff function u. Let $p_i(t) \geq 0 \quad t \geq 0$ denote the number of of individuals who are currently programmed to the pure strategy $i \in K$, and $p(t) = \sum_{i \in K} p_i(t) \geq 0$. The population state is given by the population share vector $x(t) = (x_1(t), x_2(t), \cdots, x_k(t))$, where each component $x_i(t) = p_i(t)/p(t)$. Thus $x(t) \in \Delta$. The expected vector payoff to any pure strategy i at a random match, when the population is in state $x \in \Delta$, is accordingly $u(e^i, x)$. The associated *population average payoff* is given by

$$u(x,x) = \sum_{i=1}^{k} x_i u(e^i, x)$$

It is common in evolutionary biology to describe the population dynamics using a fitness function. The fitness function usually describes the number of offspring per unit time. The fitness usually reflects the payoffs. A particular population changes its population share based on its fitness in the current population state. For this work, we assume that the fitness is governed by a biased payoff function. We assume that each of the component payoffs has a scale of relative importance in governing the population growth or death. Let suppose the payoff enters the fitness in the form $\lambda^T u$. Let us also suppose there is a background birth and death rate $\beta \geq 0$ and $\delta \geq 0$. This results in the following population dynamics.

$$\dot{p_i} = [\beta + \lambda^T u(e^i, x) - \delta] p_i$$

The corresponding dynamics for the population shares x_i becomes

$$\dot{x_i} = [\lambda^T u(e^i, x) - \lambda^T u(x, x)] x_i \qquad \cdots (BRE(\lambda)) \tag{1}$$

Since these dynamics are based on the fitness function obtained from the biased payoff, we refer to $BRE(\lambda)$ as the *Biased Replicator Dynamics.*

Remark It should be noted here that this form of dynamics corresponds to interactions between two randomly chosen entities from a large population with partitions which play a particular pure strategy $i \in K$. Further this dynamics corresponds to a game where all the players choose the *same* trade-off weights among the various objectives of their vector payoff functions.

Pareto Nash Equilibria

The set of Pareto Nash Equilibria is for the vector payoff symmetric two player game is given by

$$\Delta^{PNE} = \{x \in \Delta \quad | \quad x = arg \overset{P}{\max_{z \in \Delta}} u(z, x)\} \tag{2}$$

There is an alternative representation for the Pareto Nash Equilibria given by the following lemma. Let $\mathcal{U}(z, x)$ denote the decision space for player I, when player II chooses to play strategy $x \in \Delta$. From the multicriteria linear programming perspective we know that $\mathcal{U}$ is a convex polyhedron with vertices in $u(e^i, x) \quad i \in K$.

Lemma 2.1

$$\begin{aligned} \Delta^{PNE} = \{ \qquad & x \in \Delta | \quad \lambda^T u(e^i, x) = \lambda^T u(x, x) \\ & i \in C(x) \ \textit{and} \ u(e^i, x) \in \ \textit{Pareto Dominating Face of} \ \mathcal{U} \\ & \textit{where} \ \lambda \in \mathcal{R}^l_{>0} \ \textit{is the normal to the corresponding face.}\} \end{aligned}$$

Proof Every face of the convex polyhedron is supported by a hyperplane and thus there $\exists \lambda \in \mathcal{R}^l_{>0}$ such that $\lambda^T(u(e^i, x) - u(y, x)) = 0$, for all y in a particular face of convex polyhedron. If face of polyhedron contains x, then $\lambda^T(u(e^i, x) - u(x, x)) = 0$. Further $x \in$ Pareto dominating face of the convex polyhedron then $x \in \Delta^{PNE}$.

Pareto Nash Equilibria with a given trade-off

In any symmetric game it is very much possible for both the player to choose strategies which are asymmetric. However for the random interactions game which we have considered, the set of symmetric strategies is sufficient to characterize the dynamics. However for these symmetric games we observe that symmetry also in the trade-off function.

Lemma 2.2 *At $x \in \Delta^{PNE}$, a symmetric Pareto Nash equilibrium both the players strategies are supported by the same exiting hyperplane.*

Proof Suppose $\lambda \in \mathcal{R}^l_{>0}$ is the normal of the existing hyperplane $\Rightarrow x = arg\max_{zin\Delta}{}^P \lambda^T u(z,x)$. Suppose u_2 is the payoff of player II, by the symmetric of the payoffs we have $u(z,x) = u_2(x,z)$. Thus strategies of both the players are supported by the same exiting hyperplane λ.

We observe that the game strategies are symmetric even in the sense of trade-off. For a given trade-off vector λ, let us denote the set of *trade-off equivalent* symmetric strategies are $\Delta^{PNE}(\lambda)$.

$$\Delta^{PNE}(\lambda) = x \in \Delta| \quad x = arg\max_{z\in\Delta} \lambda^T u(z,x)$$

It should noted that there is only a finite set of trade-off vectors which are non-degenerate. Further

$$\Delta^{PNE} = \cup_{\lambda \in \mathcal{R}^l_{>0}} \Delta^{PNE}(\lambda) \tag{3}$$

Thus all the Pareto Nash symmetric equilibria can be characterized using auxiliary single objective games of the form above. This observation is helpful in characterizing the stationary states of BRE.

Stationary States

The stationary states of the autonomous dynamics given by $BRE(\lambda)$ is given by

$$\Delta^o(\lambda) = \{x \in \Delta | \lambda^T u(e^i, x) = \lambda^T u(x,x) \quad \forall i \in C(x)\} \tag{4}$$

Let $\Delta^{oo}(\lambda)$ denote the set of interior stationary states of $BRE(\lambda)$. i.e. $\Delta^{oo}(\lambda) = \Delta^o(\lambda) \cap int(\Delta)$

We present a series of propositions to characterize the stationary and stable set of the BRE. The proofs are based on the observation in Equation 3. They are obtained trivially by considering the single objective symmetric game with a scalar payoff $\lambda^T u(x,y)$. Refer to chapter 3 of [1] for the detailed proof of the single objective case.

Proposition 2.3 $\{e^1, e^2, \cdots, e^k\} \cup \Delta^{PNE}(\lambda) \subset \Delta^o(\lambda)$. $\Delta^{oo}(\lambda) = \Delta^{PNE}(\lambda) \cap int(\Delta)$.

Proposition 2.4 *If $x \in \Delta$ is Lyapunov stable in $BRE(\lambda)$, then $x \in \Delta^{PNE}(\lambda)$.*

Let suppose that $\zeta_\lambda(t, x_0)$ is a solution to BRE with $\zeta_\lambda(0, x_0) = x_0$.

Proposition 2.5 *If $x_0 \in int(\Delta)$ and $\zeta(t, x_0) \to x$, then $x \in \Delta^{PNE}(\lambda)$.*

Both the aforementioned propositions are related to the fact that Lyapunov stable states must be stationary states of the dynamics. We proceed to characterise the asymptotically stable states of the dynamics. We begin by introducing a candidate Lyapunov function. To define its domain, consider the neighbourhood set

$$Q_x = \{y \in \Delta | \quad C(x) \subset C(y)\}$$

The function $H_x : Q_x \to R$

$$H_x(y) = \sum_{i \in C(x)} x_i log(\frac{x_i}{y_i})$$

behaves as a Lyapunov function the biased replicator dynamics. The time derivative along the dynamics is given by

$$\dot{H_x}(y) = -\lambda^T [u(x, y) - u(y, y)]$$

Proposition 2.6 *If $x \in \Delta^{BESS}$ then $BRE(\lambda)$ is asymptotically stable.*

Proof The proof is based on *Lyapunov's direct method*. Since $x \in \Delta^{BESS}$ we have $\dot{H_x}(y) < 0 \quad \forall y \in N_x \cup Q_x$, where N_x is some neighbourhood of x.

References

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