ABSTRACT

Title of dissertation:	Abstract Elementary Classes With Löwenheim-Skolem Number Cofinal with ω
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An abstract elementary class is a class $\mathbb{K}$ of structures for some vocabulary L together with a "strong substructure" relation $\prec_{\mathbb{K}}$ on $\mathbb{K}$ satisfying certain axioms. Abstract elementary classes include elementary classes with elementary substructure and classes axiomatizable in $L_{\infty,\omega}$ with elementary substructure relative to some fragment of $L_{\infty,\omega}$. For every abstract elementary class there is some number κ, called the Löwenheim-Skolem number, so that every structure in the class has a strong substructure of cardinality $\leq \kappa$.

We study abstract elementary classes with Löwenheim-Skolem number κ, where κ is cofinal with ω, which have finite character. We generalize results obtained by Kueker for $\kappa = \omega$. In particular we show that $\mathbb{K}$ is closed under $L_{\infty,\kappa}$-elementary equivalence and obtain sufficient conditions for $\mathbb{K}$ to be $L_{\infty,\kappa}$-axiomatizable. The results depend on developing an appropriate concept of κ-a.e.

Abstract Elementary Classes with Löwenheim-Skolem Number Cofinal with ω

by

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Dedication

For my daughter, Gabrielle Noella Goss.

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Table of Contents

Chapter 0

Preface

In model theory, the term *elementary class* refers to the class of all models of a first order theory. The class of all infinite sets, the class of all torsion-free, divisible abelian groups and the class of all algebraically closed fields in a fixed characteristic are all examples of elementary classes. Much work in model theory over the past 50 years has focused on the study of elementary classes and their structure. This field of study has yielded strong results such as Morley's categoricity theorem ([12]) and brand new areas of research such as geometric stability theory. One weakness of elementary classes is that there are mathematically interesting classes of structures that first order logic is not sufficiently strong to define. Examples of such classes include Zilber's class of algebraically closed fields with pseudo-exponentiation ([15]) and the class of all solvable groups, among others. Infinitary logics were later introduced to deal with such classes of structures because of their greater expressive power. While there have been some results using infinitary logics, they appear to have very little structure and no single non-first order logic is suitable to define all mathematically interesting classes of structures.

Abstract elementary classes were introduced in the 1980's by Saharon Shelah ([14]) as generalizations of elementary classes. Abstract elementary classes, which consist of a class of models along with a notion of a strong substructure, were proposed as the broadest possible class of structures to potentially have a feasible model theory. In particular, Shelah conjectured that Morley's categoricity theorem could be generalized to abstract elementary classes. That is, if an abstract elementary class, $\mathbb{K}$, is categorical in some sufficiently large cardinal, then it is categorical in all sufficiently large cardinals. In the past decade there has been an explosion of work exploring the potential of these classes, thanks in part to Baldwin ([1]), Grossberg and VanDieren ([4]).

In 2005, Tapani Hyttinen and Meeri Kesälä introduced *finitary abstract elementary classes* ([5]) which require, among other things, that the corresponding notion of strong substructure is a local property. This criteria of finitary abstract elementary classes is referred to as *finite character*. The assumption of finite character has proven very fruitful towards the analysis of abstract elementary classes. Many of the non-elementary classes that are of mathematical interest exhibit finite character and thus the assumption is not overly restrictive.

Recently, Kueker has employed the techniques of infinitary logics and countable approximations to analyze abstract elementary classes ([10]). This technique has been very successful and has proven a large variety of results. The most notable results that Kueker has recently proven relate to abstract elementary classes that have Löwenheim-Skolem number ω and exhibit finite character. Letting $(\mathbb{K}, \prec_{\mathbb{K}})$ be an abstract elementary class with a Löwenheim-Skolem number of ω and finite character, we list some of the major results.

1. $\mathbb{K}$ is closed under $L_{\infty,\omega}$-equivalence.

2. If $\mathcal{M} \in \mathbb{K}$ and $\mathcal{M} \prec_{\infty,\omega} \mathcal{N}$ then $\mathcal{M} \prec_{\mathbb{K}} \mathcal{M}$.

3. $\mathbb{K}$ is axiomatizable by a sentence of the game theoretic logic $L(\omega)$.

4. (AP, etc.) If $\mathbb{K}$ is λ-categorical for some $\lambda \geq \omega$ then there is a complete sentence σ of $L_{\omega_1,\omega}$ such that for every $\mathcal{M}$ with $|\mathcal{M}| \geq \lambda$, $\mathcal{M} \in \mathbb{K}$ IFF $\mathcal{M} \models \sigma$.

In this paper we endeavor to show that most of the recent results of Kueker's were not dependent on the Löwnheim-Skolem number being countable, but simply that the Löwenheim-Skolem number had a countable cofinality. The main crux of this program is determining an appropriate analogue to a higher cardinality of what it means for a property to occur in almost all approximations.

In chapter 1 we introduce the background necessary for this material. We provide a brief overview of abstract elementary classes and some of the more important corresponding properties. We also define the terminology used in first order infinitary logics and the game theoretic characterizations of some of these concepts. Finally, we discuss the definitions, techniques and results of Kueker's countable approximations.

In chapter 2 we define κ-approximations and a filter on subsets of size κ using the game theoretic characterization of Kueker's filter. We go on to prove that this filter exhibits many of the desirable properties of Kueker's filter and use it to prove many of our intended results. The main results of this chapter are:

Theorem 0.0.1. *Assume* $(\mathbb{K}, \prec_{\mathbb{K}})$ *is an abstract elementary class with finite character and a Löwenheim-Skolem number of* κ*, where* κ *has a countable cofinality.. If* $\mathcal{M} \in \mathbb{K}$ *and* $\mathcal{M} \equiv_{\infty,\kappa} \mathcal{N}$ *then* $\mathcal{N} \in \mathbb{K}$.

Theorem 0.0.2. *Assume* $(\mathbb{K}, \prec_{\mathbb{K}})$ *is an abstract elementary class with finite character and a Löwenheim-Skolem number of* κ*, where* κ *has a countable cofinality. If* $\mathcal{M} \in \mathbb{K}$ *and* $\mathcal{M} \prec_{\infty,\kappa} \mathcal{N}$ *then* $\mathcal{M} \prec_{\mathbb{K}} \mathcal{N}$*.*

Theorem 0.0.3. *Assume* $(\mathbb{K}, \prec_{\mathbb{K}})$ *is an abstract elementary class with finite character and a Löwenheim-Skolem number of* κ*, where* κ *has a countable cofinality. If* $\mathbb{K}$ *has at most* λ*-many models of cardinality* λ *for some* λ *such that* $\lambda^{<\kappa} = \lambda$ *then* $\mathbb{K} = Mod(\sigma)$ *for some* $\sigma \in L_{\infty,\kappa}$*.*

In chapter 3 we demonstrate that having a Löwnheim-Skolem number with countable cofinality and having finite character is enough to prove axiomatizability by a sentence in a game theoretic logic. To achieve this result, we first define an appropriate game theoretic logic, $L(\kappa)$, which extends $L_{\infty,\kappa}$. We then proceed to use $L(\kappa)$ to write formulas stating that almost all approximations of a model are in the abstract elementary class. Using our game theoretic logic and our game theoretic definition of a filter, we're able to prove the following axiomatizability result:

Theorem 0.0.4. *Let* $(\mathbb{K}, \prec_{\mathbb{K}})$ *be an abstract elementary class with finite character and a Löwenheim-Skolem number of* κ*, where* κ *has a countable cofinality. There is a sentence* $\theta \in L(\kappa)$ *such that for all structures* $\mathcal{N}$*,* $\mathcal{N} \models \theta$ *IFF* $\mathcal{N} \in \mathbb{K}$*.*

In chapter 4 we provide a new definition of a galois saturated model over sets (consistent with the older definitions). In this context we use infinitary logics to analyze galois saturated models and provide biconditional strengthenings of many our previous theorems (in particular, Theorems 0.0.1 and 0.0.2). Using these results and assuming categoricity at a cardinal with a cofinality bigger than the

Löwenheim-Skolem number we are able to prove there is a complete $L_{\infty,\kappa}$-sentence closely approximating the abstract elementary class.

Theorem 0.0.5. (AP, etc.) *Let* $(\mathbb{K}, \prec_{\mathbb{K}})$ *be an abstract elementary class with a Löwenheim-Skolem number of* κ*, where* κ *has a countable cofinality. Assume* $\mathbb{K}$ *is* λ*-categorical for* $\lambda > \kappa$ *and* $cof(\lambda) > \kappa$*. Then there is a complete sentence* $\sigma \in L_{\infty,\kappa}$ *such that:*

1. $Mod(\sigma) \subseteq \mathbb{K}$ *and* σ *has a model of cardinality* κ^+*.*

2. $\mathbb{K}$ *and* $Mod(\sigma)$ *contain precisely the same models of cardinality* $\geq \lambda$*.*

3. *If* $\mathcal{M}, \mathcal{N} \models \sigma$ *then* $\mathcal{M} \prec_{\mathbb{K}} \mathcal{N}$ *IFF* $\mathcal{M} \prec_{\infty,\kappa} \mathcal{N}$*.*

Chapter 1

Introduction

Throughout this paper we assume a basic knowledge of model theory which can be found in [11] and [2] among other places. We also assume a minimal knowledge of set theory. Background on the necessary set theory can be found in [3]. The notation used is consistent with the notation found in these books.

1.1 Abstract Elementary Classes

The major results of this paper concern abstract elementary classes. We begin by defining an abstract elementary class and the corresponding definition of an embedding. These definitions are due to Saharon Shelah and can be found in [14].

Definition 1.1.1. For a given vocabulary L, an *Abstract Elementary Class* (or AEC), $(\mathbb{K}, \prec_{\mathbb{K}})$, is a family of L-structures together with a binary relation on $\mathbb{K}$, $\prec_{\mathbb{K}}$, satisfying the following axioms:

(1) If $\mathcal{M} \in \mathbb{K}$ and $\mathcal{N} \cong \mathcal{M}$ then $\mathcal{N} \in \mathbb{K}$; if $\mathcal{M} \prec_{\mathbb{K}} \mathcal{N}$ and $(\mathcal{N}, \mathcal{M}) \cong (\mathcal{N}', \mathcal{M}')$ then $\mathcal{M}' \prec_{\mathbb{K}} \mathcal{N}'$.

(2) If $\mathcal{M} \prec_{\mathbb{K}} \mathcal{N}$ then $\mathcal{M} \subseteq \mathcal{N}$.

(3) If $\mathcal{M} \in \mathbb{K}$ then $\mathcal{M} \prec_{\mathbb{K}} \mathcal{M}$; if $\mathcal{M}_0 \prec_{\mathbb{K}} \mathcal{M}_1$ and $\mathcal{M}_1 \prec_{\mathbb{K}} \mathcal{M}_2$ then $\mathcal{M}_0 \prec_{\mathbb{K}} \mathcal{M}_2$.

(4) **(Löwenheim-Skolem Axiom)** There is an infinite cardinal number called the

Löwenheim-Skolem number, denoted $LS(\mathbb{K})$, such that for every $\mathcal{M} \in \mathbb{K}$ and for every subset $A \subseteq \mathcal{M}$ there is some $\mathcal{M}' \prec_{\mathbb{K}} \mathcal{M}$ such that $A \subseteq \mathcal{M}'$ and $|\mathcal{M}'| \leq max\{|A|, LS(\mathbb{K})\}$.

(5) **(Union Axiom)** Let $\{\mathcal{M}_i\}_{i<\delta}$ be a continuous $\prec_{\mathbb{K}}$-chain. Then:

(i) $\bigcup_{i<\delta} \mathcal{M}_i \in \mathbb{K}$.

(ii) For each $j < \delta$, $\mathcal{M}_j \prec_{\mathbb{K}} \bigcup_{i<\delta} \mathcal{M}_i$.

(iii) If $\mathcal{M}_i \prec_{\mathbb{K}} \mathcal{N}$ for all $i < \delta$ then $\bigcup_{i<\delta} \mathcal{M}_i \prec_{\mathbb{K}} \mathcal{N}$

(6) **(Coherence Axiom)** If $\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2 \in \mathbb{K}$, $\mathcal{M}_0 \prec_{\mathbb{K}} \mathcal{M}_2$, $\mathcal{M}_1 \prec_{\mathbb{K}} \mathcal{M}_2$ and $\mathcal{M}_0 \subseteq \mathcal{M}_1$ then $\mathcal{M}_0 \prec_{\mathbb{K}} \mathcal{M}_1$.

Definition 1.1.2. Let $(\mathbb{K}, \prec_{\mathbb{K}})$ be an AEC. If $\mathcal{M}, \mathcal{N} \in \mathbb{K}$ and $f : \mathcal{M} \to \mathcal{N}$ is an embedding such that $f(\mathcal{M}) \prec_{\mathbb{K}} \mathcal{N}$ then we say that f is a $\mathbb{K}$-*embedding* of $\mathcal{M}$ into $\mathcal{N}$.

Shelah proved in [14] that if a class of L-structures satisfies the axioms of 1.1.1 then the union axiom can be generalized to unions of $\prec_{\mathbb{K}}$-directed families. We refer to a set of models S as a $\prec_{\mathbb{K}}$-directed family if for any $\mathcal{M}_0$, $\mathcal{M}_1 \in S$ there exists $\mathcal{M}_2 \in S$ such that $\mathcal{M}_0$, $\mathcal{M}_1 \prec_{\mathbb{K}} \mathcal{M}_2$.

Lemma 1.1.3. *Let $(\mathbb{K}, \prec_{\mathbb{K}})$ be an AEC and let S be a $\prec_{\mathbb{K}}$-directed family of models from $\mathbb{K}$. Let $\mathcal{N} = \bigcup S$. Then the following hold:*

(a) $\mathcal{N} \in \mathbb{K}$.

(b) $\mathcal{M} \prec_{\mathbb{K}} \mathcal{N}$ *for all* $\mathcal{M} \in S$.

(c) Given a model $\mathcal{A} \in \mathbb{K}$, if $\mathcal{M} \prec_{\mathbb{K}} \mathcal{A}$ for all $\mathcal{M} \in S$ then $\mathcal{N} \prec_{\mathbb{K}} \mathcal{A}$.

In the study of AECs, we frequently restrict ourselves to AECs with two additional "nice" properties.

Definition 1.1.4. Let $\mathbb{K}$ be an Abstract Elementary Class.

1. $\mathbb{K}$ has the *amalgamation property* if for all models $\mathcal{M}, \mathcal{N}_1, \mathcal{N}_2 \in \mathbb{K}$ such that $\mathcal{M} \prec_{\mathbb{K}} \mathcal{N}_1$ and $\mathcal{M} \prec_{\mathbb{K}} \mathcal{N}_2$ there is a model $\mathcal{N} \in \mathbb{K}$ and $\mathbb{K}$-embeddings f_1, f_2 such that f_i maps $\mathcal{N}_i$ into $\mathcal{N}$ and $f_1(\mathcal{M}) = f_2(\mathcal{M})$.

2. $\mathbb{K}$ has the *joint embedding property* if for all $\mathcal{M}_1, \mathcal{M}_2 \in \mathbb{K}$, there is a model $\mathcal{N} \in \mathbb{K}$ and $\mathbb{K}$-embeddings f_i of $\mathcal{M}_i$ into $\mathcal{N}$.

Abstract Elementary Classes are not guaranteed to have arbitrarily large models in the way Elementary Classes are. However, as a consequence of Shelah's Presentation Theorem for AECs (see [1], [14]), it can be shown that for every abstract elementary class $\mathbb{K}$, there is a number $\mathcal{H}$ (the Hanf number of $\mathbb{K}$), which depends only on $LS(\mathbb{K})$ and $|L|$, such that if $\mathbb{K}$ has a model of at least size $\mathcal{H}$ then $\mathbb{K}$ has arbitrarily large models.

Under the assumptions of amalgamation, joint embedding and arbitrarily large models we can prove the existence of strongly homogeneous models. The following definitions of homogeneity are from [1], though the concepts originate from other sources.

Definition 1.1.5. Let $(\mathbb{K}, \prec_{\mathbb{K}})$ be an abstract elementary class and μ be a cardinal number bigger than $LS(\mathbb{K})$.

1. A model $\mathcal{M} \in \mathbb{K}$ is *μ-model homogeneous* iff for every $\mathcal{M}_0 \prec_{\mathbb{K}} \mathcal{M}$ and every $\mathcal{N} \in \mathbb{K}$ such that $\mathcal{M}_0 \prec_{\mathbb{K}} \mathcal{N}$ and $|\mathcal{N}| < \mu$ there is a $\mathbb{K}$-embedding of $\mathcal{N}$ into $\mathcal{M}$ fixing $\mathcal{M}_0$ pointwise.

2. A model $\mathcal{M} \in \mathbb{K}$ is *strongly μ-model homogeneous* iff for every $\mathcal{M}_0, \mathcal{M}_1 \prec_{\mathbb{K}} \mathcal{M}$ such that $|\mathcal{M}_0|, |\mathcal{M}_1| < \mu$ any isomorphism from $\mathcal{M}_0$ onto $\mathcal{M}_1$ extends to an automorphism of $\mathcal{M}$.

3. A model $\mathcal{M} \in \mathbb{K}$ is *strongly model homogeneous* iff it is strongly homogeneous in its own cardinality.

We will use (AP, etc.) throughout this paper to denote that the AEC in question is assumed to satisfy amalgamation, joint embedding and arbitrarily large models. Under these assumptions, we can assume that all $\mathbb{K}$-structures in question are $\mathbb{K}$-substructures of a *monster model*, $\mathbb{C}$. The following Theorem from [1] along with (AP, etc.) allows this assumption.

Theorem 1.1.6. (AP, etc) *For each cardinal number λ, and each model $\mathcal{M} \in \mathbb{K}$ of cardinality λ, there is a strongly λ-model homogeneous model $\mathcal{N} \in \mathbb{K}$ containing $\mathcal{M}$.*

In addition to the above properties, most of the AECs that we will consider in this paper will have *finite character*. Finite character was introduced by Hyttinen and Kesala ([5]) in order to indicate that the definition of strong substructure in the AEC is a local property. The following definition, formulated by Kueker ([10]), is not the same as the notion introduced by Hyttinen and Kesala, but it's equivalent under the assumption of amalgamation.

Definition 1.1.7. An AEC $(\mathbb{K}, \prec_{\mathbb{K}})$ has *finite character* iff for all models $\mathcal{M}, \mathcal{N} \in \mathbb{K}$, $\mathcal{M} \prec_{\mathbb{K}} \mathcal{N}$ whenever $\mathcal{M} \subseteq \mathcal{N}$ and for every finite tuple $a_0, \ldots, a_n \in \mathcal{M}$ there is a $\mathbb{K}$-embedding of $\mathcal{M}$ into $\mathcal{N}$ fixing $a_0, \ldots, a_n$ pointwise.

Many useful and mathematically interesting classes of structures have finite character. For instance, any class where the strong substructure notion $\prec_{\mathbb{K}}$ is either first order elementary substructure, $\prec$, or basic substructure, $\subseteq$, since both of these notions of substructure are only dependent on checking a finite amount of information at a time. In addition, excellent classes, homogeneous classes and classes of structures modeling a sentence or sentences from $L_{\lambda,\omega}$ (where $\prec_{\mathbb{K}}=\prec_{\lambda,\omega}$) all exhibit finite character. However, it's important to note that many AECs do not satisfy finite character. The following example, due to Kueker, illustrates a very simple case of an AEC where finite character fails.

Example 1.1.8. Define the vocabulary $L = \{P\}$ where P is a unary predicate symbol and let μ be an infinite cardinal number. Let $\mathbb{K} = \{\mathcal{M} : \mathcal{M}$ is an L-structure, $|P^{\mathcal{M}}| = \mu$, $|\neg P^{\mathcal{M}}| \geq \mu\}$. In addition, define $\prec_{\mathbb{K}}$ as $\mathcal{M} \prec_{\mathbb{K}} \mathcal{N}$ IFF $\mathcal{M} \subseteq \mathcal{N}$ and $P^{\mathcal{M}} = P^{\mathcal{N}}$.

First we will show that $\mathbb{K}$ satisfies the axioms of an abstract elementary class. From the definition of $(\mathbb{K}, \prec_{\mathbb{K}})$ it is pretty clear that axioms 1, 2 and 3 hold. In order to show the Löwenheim-Skolem axiom holds let $\mathcal{N}$ be an arbitrary $\mathbb{K}$-structure and $A \subseteq \mathcal{N}$ be an arbitrary subset. Further, let $\mathcal{M} \subseteq \mathcal{N}$ such that $A \subseteq \mathcal{M}$, $P^{\mathcal{N}} \subseteq \mathcal{M}$ and $|\neg P^{\mathcal{M}}| = max\{\mu, |A|\}$. Then $\mathcal{M} \in \mathbb{K}$, $\mathcal{M} \prec_{\mathbb{K}} \mathcal{N}$ and $|\mathcal{M}| = max\{\mu, |A|\}$ and hence $LS(\mathbb{K}) = \mu$.

To see $(\mathbb{K}, \prec_{\mathbb{K}})$ satisfies the union axioms, assume $\mathcal{M}_i \prec_{\mathbb{K}} \mathcal{M}_{i+1}$ for all $i \in \delta$. Then, $\mathcal{M}_i \subseteq \mathcal{M}_{i+1}$ and $P^{\mathcal{M}_i} = P^{\mathcal{M}_{i+1}}$ for all $i \in \delta$. Hence, $\mathcal{M}_i \subseteq \bigcup_{j\in\delta} \mathcal{M}_j$ for all $i \in \delta$ and $P^{\mathcal{M}_i} = P^{\bigcup \mathcal{M}_j}$ for all $i \in \delta$. Furthermore, since $|\neg P^{\mathcal{M}_i}| \geq \mu$ for all $i \in \delta$, we get $|\neg P^{\bigcup \mathcal{M}_j}| \geq \mu$. Hence, $\bigcup_{j\in\delta} \mathcal{M}_j \in \mathbb{K}$ and $\mathcal{M}_i \prec_{\mathbb{K}} \bigcup_{j\in\delta} \mathcal{M}_j$. Moreover, from the definition of $\prec_{\mathbb{K}}$ it is clear that if $\mathcal{M}_i \prec_{\mathbb{K}} \mathcal{N}$ for all $i \in \delta$ then $\bigcup_{j\in\delta} \mathcal{M}_j \prec_{\mathbb{K}} \mathcal{N}$.

Finally, for the coherence axiom, assume $\mathcal{M}_0, \mathcal{M}_1 \prec_{\mathbb{K}} \mathcal{M}_2$ and $\mathcal{M}_0 \subseteq \mathcal{M}_1$. Then, $P^{\mathcal{M}_0} = P^{\mathcal{M}_2} = P^{\mathcal{M}_1}$ and hence $\mathcal{M}_0 \prec_{\mathbb{K}} \mathcal{M}_1$.

We should note that $(\mathbb{K}, \prec_{\mathbb{K}})$ satisfies the amalgamation and joint embedding properties and has arbitrarily large models. It remains to show that $\mathbb{K}$ fails to have finite character.

Proof.

Let $\mathcal{M}, \mathcal{N} \in \mathbb{K}$ be such that $\mathcal{M} \subseteq \mathcal{N}$ and there is just a single element $b \in P^{\mathcal{N}} \setminus P^{\mathcal{M}}$. For any $n \in \omega$ and $a_0, \ldots, a_n \in \mathcal{M}$ there is a $\mathbb{K}$-embedding $f : \mathcal{M} \to \mathcal{N}$ fixing $a_0, \ldots, a_n$ (since $|P^{\mathcal{M}} \setminus \{a_0 \ldots a_n\}| = |P^{\mathcal{N}} \setminus \{a_0 \ldots a_n\}| = \mu$). However, $P^{\mathcal{M}} \neq P^{\mathcal{N}}$ and thus $\mathcal{M} \not\prec_{\mathbb{K}} \mathcal{N}$. Therefore, $\mathbb{K}$ fails to have finite character. □

1.2 Infinitary Logics

Infinitary logics are the primary tools used in this paper to analyze abstract elementary classes. We will heavily apply concepts of first order infinitary logic allowing either infinitely many conjunctions and disjunctions or infinitely many variables (or both). The essential definitions and results are given here. Further reading on this subject can be found in [9] and [6].

Definition 1.2.1. Given an arbitrary vocabulary L we define the infinitary logic $L_{\infty,\mu}$ inductively as follows:

- If φ is atomic then $\varphi \in L_{\infty,\mu}$.
- If $\varphi \in L_{\infty,\mu}$ then $\neg\varphi \in L_{\infty,\mu}$.
- If $\Phi \subseteq L_{\infty,\mu}$ and Φ has $< \mu$-many free variables then $\bigvee \Phi \in L_{\infty,\mu}$ and $\bigwedge \Phi \in L_{\infty,\mu}$.
- If $\varphi \in L_{\infty,\mu}$ and V is a set of variables with cardinality less than μ then $\forall V \varphi$ and $\exists V \varphi \in L_{\infty,\mu}$.

We further define the infinitary logic $L_{\chi,\mu}$ by restricting formulas of $L_{\infty,\mu}$ to having conjunctions and disjunctions of size less than χ.

Remark 1.2.2.

1. $L_{\omega,\omega}$ is equivalent to the set of formulas in standard first order logic.
2. $L_{\infty,\mu}$ and $L_{\chi,\mu}$ have $< \mu$-many free variables.

The notion of $L_{\infty,\mu}$ or $L_{\chi,\mu}$-elementary substructure is defined as the natural extension of first order elementary substructure.

Definition 1.2.3. Given two L-structures $\mathcal{M}$ and $\mathcal{N}$, we say $\mathcal{M} \prec_{\infty,\mu} \mathcal{N}$ (or $\mathcal{M} \prec_{\chi,\mu} \mathcal{N}$) IFF $\mathcal{M} \subseteq \mathcal{N}$ and for every formula $\varphi(\bar{x}) \in L_{\infty,\mu}$ (or $L_{\chi,\mu}$) and for any sequence $\bar{a} \subseteq \mathcal{N}$ such that $lh(\bar{a}) = lh(\bar{x})$, $\mathcal{M} \models \varphi(\bar{a})$ iff $\mathcal{N} \models \varphi(\bar{a})$.

In $L_{\infty,\mu}$ and $L_{\chi,\mu}$ there are two schools of thought on how to define elementary equivalence. We will use the more restrictive definition of elementary equivalence in which $(\mathcal{M}, \bar{a})$ does not add new constants to the language for $\bar{a}$ but merely refers to formulas of $L_{\infty,\mu}$ (or $L_{\chi,\mu}$) applied to elements of the sequence $\bar{a}$. We state below the definitions of $L_{\infty,\mu}$-elementary equivalence and $L_{\chi,\mu}$-elementary equivalence that will be used throughout this paper.

Definition 1.2.4. Given L-structures $\mathcal{M}$ and $\mathcal{N}$ let $\bar{a} \subseteq \mathcal{M}$ and $\bar{b} \subseteq \mathcal{N}$ be sequences of the same length. Then $(\mathcal{M}, \bar{a}) \equiv_{\infty,\mu} (\mathcal{N}, \bar{b})$ IFF for every $\varphi(\bar{x}) \in L_{\infty,\mu}$ with $lh(\bar{x}) = \delta$, $\mathcal{M} \models \varphi(\langle a_{i(j)} \rangle_{j\in\delta}) \Leftrightarrow \mathcal{N} \models \varphi(\langle b_{i(j)} \rangle_{j\in\delta})$ for every $i \in {}^{\delta}lh(\bar{a})$. Note that $\delta < \mu$ necessarily, since $\varphi(\bar{x}) \in L_{\infty,\mu}$. We define $L_{\chi,\mu}$-elementary equivalence the same way except we restrict φ to be in $L_{\chi,\mu}$.

For each cardinal κ there is a useful game characterization of $L_{\infty,\kappa}$-equivalence that we will implement throughout this paper.

Definition 1.2.5. Let $\mathcal{M}$ and $\mathcal{N}$ be two L-structures, for some vocabulary L.

1. A map, h, is a *partial isomorphism* of $\mathcal{M}$ into $\mathcal{N}$ if:

    - h is one-to-one.
    - $dom(h) \subseteq \mathcal{M}$ and $ran(h) \subseteq \mathcal{N}$.
    - For any n-ary relation symbol $R \in L$ and any $a_1, \dots a_n \in dom(h)$, $R^{\mathcal{M}}(a_1, \dots a_n)$ IFF $R^{\mathcal{N}}(h(a_1), \dots h(a_n))$.
    - For any n-ary function symbol $f \in L$ and any $a_1, \dots a_n, a \in dom(h)$, $f^{\mathcal{M}}(a_1, \dots a_n) = a$ IFF $f^{\mathcal{N}}(h(a_1), \dots h(a_n)) = h(a)$.
    - For any constant symbol $c \in L$ and $c^{\mathcal{M}} \in dom(h)$, $c^{\mathcal{N}} = h(c^{\mathcal{M}})$.

2. A κ-partial isomorphism system, P_κ, is a collection of partial isomorphisms from $\mathcal{M}$ to $\mathcal{N}$ such that

   - For all $f \in P_\kappa$ and $\bar{a} \subseteq \mathcal{M}$ of cardinality $< \kappa$, there is a partial isomorphism $g \in P_\kappa$ such that $g \supseteq f$ and $\bar{a} \subseteq dom(g)$.

   - For all $f \in P_\kappa$ and $\bar{b} \subseteq \mathcal{N}$ of cardinality $< \kappa$, there is a partial isomorphism $g \in P_\kappa$ such that $g \supseteq f$ and $\bar{b} \subseteq ran(g)$.

3. The game $G_\kappa(\mathcal{M}, \mathcal{N})$ is a 2-person ω-length game such that players I and II alternately choose sequences $\bar{a}^n \subseteq \mathcal{M}$ and $\bar{b}^n \subseteq \mathcal{N}$ of length less than κ. Player II wins if the map h defined as $h(\bar{a}^i) = \bar{b}^i$ for all $i \in \omega$ is a partial isomorphism.

The following theorem can be found in [9]. Since this will be essential machinery throughout this paper, we work out the details of the proof below.

Theorem 1.2.6. *Let κ be an infinite cardinal. For L-structures $\mathcal{M}$ and $\mathcal{N}$, $\mathcal{M} \equiv_{\infty,\kappa} \mathcal{N}$ IFF player II has a winning strategy in the game $G_\kappa(\mathcal{M}, \mathcal{N})$.*

Proof. The following claim is essential to proving one direction of this theorem.

Claim: If $\mathcal{M} \equiv_{\infty,\kappa} \mathcal{N}$ then for every $\bar{a} \subseteq \mathcal{M}$ of length $< \kappa$ there is a $\bar{b} \subseteq \mathcal{N}$ such that $(\mathcal{M}, \bar{a}) \equiv_{\infty,\kappa} (\mathcal{N}, \bar{b})$.

Proof.

Let S be the set of all $\bar{c} \subseteq \mathcal{N}$ of length $|\bar{a}|$ such that $(\mathcal{M}, \bar{a}) \not\equiv_{\infty,\kappa} (\mathcal{N}, \bar{c})$.

For each $\bar{c} \in S$, let $\varphi_{\bar{c}}(\bar{x}) \in L_{\infty,\kappa}$ such that $\mathcal{M} \models \varphi_{\bar{c}}(\bar{a})$ and $\mathcal{N} \models \neg\varphi_{\bar{c}}(\bar{c})$.

Let $\psi(\bar{x}) = \bigwedge_{\bar{c} \in S} \varphi_{\bar{c}}(\bar{x})$. Since $\mathcal{M} \models \psi(\bar{a})$ and $\mathcal{M} \equiv_{\infty,\kappa} \mathcal{N}$ there is a $\bar{b} \subseteq \mathcal{N}$ such that $\mathcal{N} \models \psi(\bar{b})$. Thus, $\bar{b} \notin S$ and $(\mathcal{M}, \bar{a}) \equiv_{\infty,\kappa} (\mathcal{N}, \bar{b})$.

$\square$

($\Rightarrow$): Suppose $\mathcal{M} \equiv_{\infty,\kappa} \mathcal{N}$. We will construct a winning strategy for player II by induction.

Base Case: $n = 0$

Let player I choose $\bar{a}^0 \subseteq \mathcal{M}$ (or $\bar{b}^0 \subseteq \mathcal{N}$) of length $< \kappa$ in the game $G_\kappa(\mathcal{M}, \mathcal{N})$. By the previous claim, there is a $\bar{b}^0 \subseteq \mathcal{N}$ (or $\bar{a}^0 \subseteq \mathcal{M}$) such that $(\mathcal{M}, \bar{a}^0) \equiv_{\infty,\kappa} (\mathcal{N}, \bar{b}^0)$. Let player II choose $\bar{b}^0$ (or $\bar{a}^0$).

Successor Stage: Suppose $\bar{a}^i, \bar{b}^i$ have been chosen for $i < n+1$ such that

$(\mathcal{M}, \bar{a}^0, \ldots \bar{a}^n) \equiv_{\infty,\kappa} (\mathcal{N}, \bar{b}^0, \ldots \bar{b}^n)$.

Let player I choose $\bar{a}^{n+1} \subseteq \mathcal{M}$ (or $\bar{b}^{n+1} \subseteq \mathcal{N}$) of length $< \kappa$. By the inductive hypothesis $(\mathcal{M}, \bar{a}^0, \ldots \bar{a}^n) \equiv_{\infty,\kappa} (\mathcal{N}, \bar{b}^0, \ldots, \bar{b}^n)$, the previous claim implies there is a $\bar{b}^{n+1} \subseteq \mathcal{N}$ (or $\bar{a}^{n+1} \subseteq \mathcal{M}$) such that $(\mathcal{M}, \bar{a}^0, \ldots, \bar{a}^{n+1}) \equiv_{\infty,\kappa} (\mathcal{N}, \bar{b}^0, \ldots, \bar{b}^{n+1})$. Let player II choose $\bar{b}^{n+1}$ (or $\bar{a}^{n+1}$).

By construction, the map defined as $h(\bar{a}^i) = \bar{b}^i$ for all $i \in \omega$ is a partial isomorphism and thus player II wins the game as desired.

($\Leftarrow$): Suppose player II has a winning strategy in the game $G_\kappa(\mathcal{M}, \mathcal{N})$. We first claim there is a κ-partial isomorphism system, P_κ, from $\mathcal{M}$ to $\mathcal{N}$ where every $f \in P$ has a domain of cardinality less than κ.

Let P_κ be the set of all maps $f(\bar{a}^i) = \bar{b}^i$ where $\bar{a}^0, \ldots \bar{a}^n, \bar{b}^0, \ldots \bar{b}^n$ are the results of some play of $G_\kappa(\mathcal{M}, \mathcal{N})$ at a finite stage of the game and player II used his winning strategy. Since player II used his winning strategy, each $f \in P_\kappa$ is a partial L-embedding. Additionally, since player I can play any $< \kappa$-sequence from either $\mathcal{M}$ or $\mathcal{N}$ at stage $n+1$, there is a $g \in P_\kappa$ extending f such that for any $< \kappa$-sequence $\bar{a} \subseteq \mathcal{M}$ (or $\bar{b} \subseteq \mathcal{N}$) $\bar{a} \subseteq dom(g)$ (or $\bar{b} \subseteq ran(g)$).

We show that for every $\varphi(\bar{x}) \in L_{\infty,\kappa}$, for every $\bar{a} \subseteq \mathcal{M}$ with $lh(\bar{a}) = lh(\bar{x})$ and for every partial isomorphism $h \in P_\kappa$, if $\bar{a} \subseteq dom(h)$ then $\mathcal{M} \models \varphi(\bar{a})$ IFF $\mathcal{N} \models \varphi(h(\bar{a}))$.

Let $\varphi \in L_{\infty,\kappa}$ and, without loss of generality, let $\bar{a} \subseteq \mathcal{M}$ be an arbitrary sequence of length $< \kappa$. By induction on the complexity of formulas we will show that for any partial isomorphism $h \in P$, if $\bar{a} \subseteq dom(h)$ then $\mathcal{M} \models \varphi(\bar{a})$ IFF $\mathcal{N} \models \varphi(h(\bar{a}))$.

Atomic Formulas:

Assume $\varphi \in L_{\infty,\kappa}$ is atomic. Since atomic formulas are preserved under isomorphism, this is clear.

Conjunction/Disjunction:

Assume $\varphi = \bigwedge_{i\in\delta} \varphi_i$ or $\varphi = \bigvee_{i\in\delta} \varphi_i$ where δ is an ordinal. Further assume that for any $\bar{a}$ and any partial isomorphism $h \in P_\kappa$ such that $\bar{a} \subseteq dom(h)$, $\mathcal{M} \models \varphi_i(\bar{a})$ IFF $\mathcal{N} \models \varphi_i(h(\bar{a}))$. Clearly, for any such h, $\mathcal{M} \models \varphi(\bar{a})$ IFF $\mathcal{N} \models \varphi(h(\bar{a}))$.

Negation:

Assume $\varphi = \neg\psi$ and that for any partial isomorphism $h \in P_\kappa$, $\mathcal{M} \models \psi(\bar{a})$ IFF $\mathcal{N} \models \psi(h(\bar{a}))$. Clearly, for any such h, $\mathcal{M} \models \varphi(\bar{a})$ IFF $\mathcal{N} \models \varphi(h(\bar{a}))$.

Existential:

Assume $\varphi = \exists\bar{y}\psi(\bar{x},\bar{y})$ where $\bar{y}$ is a sequence of length $< \kappa$ variables. Further assume that for any $\bar{a}$, $\bar{b}$ and any partial isomorphism $h \in P_\kappa$ such that $\bar{a}, \bar{b} \subseteq dom(h)$, $\mathcal{M} \models \psi(\bar{a}, \bar{b})$ IFF $\mathcal{N} \models \psi(h(\bar{a}), h(\bar{b}))$.

If $\mathcal{M} \models \varphi(\bar{a})$ then $\mathcal{M} \models \psi(\bar{a}, \bar{b})$ for some $< \kappa$-sequence $\bar{b} \subseteq \mathcal{M}$. Let $h \in P_\kappa$ be such that $\bar{a} \subseteq dom(h)$. There is a partial isomorphism $h' \in P_\kappa$ extending h such that $\bar{b} \subseteq dom(h')$. Thus $\mathcal{N} \models \psi(h'(\bar{a}), h'(\bar{b}))$ by the inductive hypothesis. Since $h' \supseteq h$, $\mathcal{N} \models \psi(h(\bar{a}), h'(\bar{b}))$. Thus, $\mathcal{N} \models \varphi(h(\bar{a}))$.

For the converse, let $h \in P_\kappa$ such that $\bar{a} \subseteq dom(h)$. If $\mathcal{N} \models \varphi(h(\bar{a}))$ then $\mathcal{N} \models \psi(h(\bar{a}), \bar{c})$ for some $< \kappa$-sequence $\bar{c} \subseteq \mathcal{N}$. There is a partial isomorphism $h' \in P_\kappa$ with $h' \supseteq h$ and $\bar{c} \subseteq ran(h')$. Let $\bar{b} = h'^{-1}(\bar{c})$. Thus $\mathcal{M} \models \psi(\bar{a}, \bar{b})$ by the inductive hypothesis. Hence, $\mathcal{M} \models \varphi(\bar{a})$.

Since universal quantification can be defined from negation and existential quantification, this case is taken care of already. Hence, $\mathcal{M} \equiv_{\infty,\kappa} \mathcal{N}$ as desired. □

1.3 Countable Approximations

In [8], David Kueker first introduced the idea of a countable approximation to a model. Recent work of Kueker's ([10]) has yielded many new results applying countable approximations to abstract elementary classes. We will define countable approximations and briefly review a few of his results.

Throughout this paper, for any set s, any countable vocabulary L, and any L-structure $\mathcal{M}$, we use the notation $\mathcal{M}^s$ to denote the substructure of $\mathcal{M}$ generated by $(\mathcal{M} \cap s)$. If s is countable then we call $\mathcal{M}^s$ a *countable approximation* to $\mathcal{M}$. Additionally, for any set C, we construct a filter on $\mathcal{P}_{\omega_1}(C)$ (the set of countable subsets of C) in order to define a notion of *almost all* $s \subseteq C$.

Definition 1.3.1. Fix a set C and let $X \subseteq \mathcal{P}_{\omega_1}(C)$.

(a) X is *ω-closed* IFF for all $\{s_i\}_{i\in\omega} \subseteq X$ such that $s_i \subseteq s_{i+1}$ for all $i \in \omega$, $\bigcup_{i\in\omega} s_i \in X$. (i.e. X is closed under unions of countable chains)

(b) X is *ω-unbounded* iff for every $s_0 \in \mathcal{P}_{\omega_1}(C)$ there is an $s \in X$ such that $s_0 \subseteq s$.

Definition 1.3.2. The *filter* $D_{\omega_1}(C)$ is the set of all $X \subseteq \mathcal{P}_{\omega_1}(C)$ such that X contains an ω-closed and ω-unbounded subset.

We note that $D_{\omega_1}(C)$ is defined in such a way to guarantee ω_1-completeness and closure under diagonalization for sets indexed by finite sequences. These properties are crucial to most of the results obtained using the filter and analogues of them will need to hold when defining filters in higher cardinalities.

Definition 1.3.3. A property of one or more models and/or formulas is said to hold *almost everywhere* (or a.e.) IFF it holds for all $s \in X$ for some $X \in D_{\omega_1}(C)$.

The filter $D_{\omega_1}(C)$ has a game theoretic characterization that is useful in proving many results regarding countable approximations and is integral to the generalization of the filter to higher cardinalities.

Given a set C and a subset $X \subseteq \mathcal{P}_{\omega_1}(C)$, we define the ω-length game $G_\omega(X)$ by having player I_X and player II_X alternately choose single elements $a_i \in C$. Player II_X *wins* the game if $\{a_i\}_{i\in\omega} \in X$.

Theorem 1.3.4. [10] *Fix a set C and let $X \subseteq \mathcal{P}_{\omega_1}(C)$. $X \in D_{\omega_1}(C)$ IFF player II_X has a winning strategy in the game $G_\omega(X)$.*

Using countable approximations Kueker proved many results regarding abstract elementary classes with $LS(\mathbb{K}) = \omega$. We briefly mention some of his major results that we will endeavor to generalize to higher cardinalities. For the following results, assume $(\mathbb{K}, \prec_{\mathbb{K}})$ is an AEC with $LS(\mathbb{K}) = \omega$.

Theorem 1.3.5. [10] *Assume that* $(\mathbb{K}, \prec_{\mathbb{K}})$ *has finite character.*

(a) If $\mathcal{M} \in \mathbb{K}$ *and* $\mathcal{M} \equiv_{\infty,\omega} \mathcal{N}$ *then* $\mathcal{N} \in \mathbb{K}$.

(b) If $\mathcal{M} \in \mathbb{K}$ *and* $\mathcal{M} \prec_{\infty,\omega} \mathcal{N}$ *then* $\mathcal{M} \prec_{\mathbb{K}} \mathcal{N}$.

Let $L(\omega)$ denote Keisler's game theoretic logic extending $L_{\infty,\omega}$ ([7]). Under the assumption of finite character, $\mathbb{K}$ can be axiomatized by a sentence of $L(\omega)$.

Theorem 1.3.6. [10] *Assume* $(\mathbb{K}, \prec_{\mathbb{K}})$ *has finite character. Then* $\mathbb{K} = Mod(\theta)$ *for some* $\theta \in L(\omega)$.

A class of structures, $\mathbb{K}$, is called λ-categorical iff for all structures $\mathcal{M}$ and $\mathcal{N}$ from $\mathbb{K}$ of cardinality λ there is an isomorphism from $\mathcal{M}$ to $\mathcal{N}$. Restricting to $L_{\infty,\omega}$, Kueker proved that if $(\mathbb{K}, \prec_{\mathbb{K}})$ is λ-categorical for some infinite cardinal λ then $\mathbb{K}$ is almost axiomatizable by a sentence of $L_{\omega_1,\omega}$.

Theorem 1.3.7. [10] *Assume* $(\mathbb{K}, \prec_{\mathbb{K}})$ *is an AEC satisfying (AP, etc.) and finite character. In addition, assume* $(\mathbb{K}, \prec_{\mathbb{K}})$ *is* λ*-categorical for some* $\lambda \geq \omega$*. Then there is a complete sentence* $\sigma \in L_{\omega_1,\omega}$ *such that for all* L*-structures* $\mathcal{M}$ *with* $|\mathcal{M}| \geq \lambda$, $\mathcal{M} \in \mathbb{K}$ *IFF* $\mathcal{M} \models \sigma$.

Chapter 2

κ-Approximations

From now on assume κ **is an infinite cardinal with cofinality** ω**. We will choose (and fix) a countable, increasing sequence of infinite cardinals** $\langle \kappa_i \rangle_{i \in \omega}$ **such that** $\kappa = \bigcup_{i \in \omega} \kappa_i$. Any exceptions to this assumption will be explicitly noted.

2.1 The Filter

For an arbitrary set C, we will define a filter on $\mathcal{P}_{\kappa^+}(C)$ in an analogous way to the filter used for countable approximations. In order to do this, we must first generalize the game $G_\omega(X)$ that characterized $D_{\omega_1}(C)$ (Theorem 1.3.4). There are several natural choices for how to define $G_\kappa(X)$. We will define the four most likely choices.

Definition 2.1.1. Let C be a set and $X \subseteq \mathcal{P}_{\kappa^+}(C)$. We define:

1. $G_\kappa(X)$ as the ω-length game in which players I_X and II_X alternately choose $s_i \in \mathcal{P}_\kappa(C)$. We say player II_X *wins* the game $G_\kappa(X)$ iff $\bigcup_{i \in \omega} s_i \in X$.

2. $G^*_\kappa(X)$ as the ω-length game in which player I^*_X and II^*_X alternately choose $s_i \in \mathcal{P}_\kappa(C)$ such that $|s_{2n}|, |s_{2n+1}| \leq \kappa_n$. We say player II^*_X *wins* iff $\bigcup_{i \in \omega} s_i \in X$.

3. $G_\kappa^\dagger(X)$ as the κ-length game in which players $\mathrm{I}_X^\dagger$ and $\mathrm{II}_X^\dagger$ alternately choose single elements $a_i \in C$. We say player $\mathrm{II}_X^\dagger$ *wins* iff $\{a_i\}_{i\in\kappa} \in X$.

4. $G_{\kappa^+}(X)$ as the ω-length game in which players I_X^+ and II_X^+ alternately choose $s_i \in \mathcal{P}_{\kappa^+}(X)$. We say player II_X^+ *wins* iff $\bigcup_{i\in\omega} s_i \in X$.

Remark 2.1.2. In the previous definition, number 4 is exactly the same as number 1, with κ replaced by κ^+.

The following theorem proves that if player II has a winning strategy in any one these games then player II has a winning strategy in all of the other games.

Theorem 2.1.3. *The following are equivalent:*

1. *Player II_X has a winning strategy in $G_\kappa(X)$.*

2. *Player II_X^* has a winning strategy in $G_\kappa^*(X)$.*

3. *Player $II_X^\dagger$ has a winning strategy in $G_\kappa^\dagger(X)$.*

4. *Player II_X^+ has a winning strategy in $G_{\kappa^+}(X)$.*

Proof.

(1 ⇒ 2): Suppose player II_X has a winning strategy in $G_\kappa(X)$. We use this winning strategy to define a winning strategy for player II_X^*.

<u>Base Case</u>: $n = 0$

Assume player I_X^* chooses $s_0^* \in \mathcal{P}_\kappa(C)$ with $|s_0^*| \le \kappa_0$. Let player I_X choose $s_0 = s_0^*$ and player II_X will use his winning strategy to choose $s_1 \in \mathcal{P}_\kappa(C)$.

- If $|s_1| \le \kappa_0$, let player II_X^* choose $s_1^* = s_1$.

- If $|s_1| > \kappa_0$ then player II_X^* chooses $s_1^* = \emptyset$.

Successor Case: Assume we have defined s_i and s_i^* for all $i < 2(n+1)$. Further assume player I_X^* has chosen $s_{2(n+1)}^* \in \mathcal{P}_\kappa(C)$ with $|s_{2(n+1)}^*| \leq \kappa_{n+1}$. Let player I_X choose $s_{2(n+1)} = s_{2(n+1)}^*$ and player II_X will use his winning strategy to choose $s_{2(n+1)+1} \in \mathcal{P}_\kappa(C)$. Let player II_X^* choose $s_{2(n+1)+1}^* = \bigcup\{s_{2i+1} : i \leq n, |s_{2i+1}| \leq \kappa_{n+1}\}$.

Since player II_X used his winning strategy, $\bigcup_{i\in\omega} s_i \in X$. By construction, $\bigcup_{i\in\omega} s_i = \bigcup_{i\in\omega} s_i^*$. Hence, player II_X^* has a winning strategy in $G_\kappa^*(X)$.

(2 $\Rightarrow$ 3): Suppose player II_X^* has a winning strategy in $G_\kappa^*(X)$. Assume player $\mathrm{I}_X^\dagger$ has chosen $a_0 \in C$. Let player I_X^* choose $s_0 = \{a_0\}$. Player II_X^* then uses his winning strategy to choose $s_1 \in \mathcal{P}_\kappa(C)$ of size $\leq \kappa_0$. Player $\mathrm{II}_X^\dagger$ then proceeds to play the elements of s_1 as his next κ_0-many moves. In the meantime, player $\mathrm{I}_X^\dagger$ chooses $a_2, \ldots, a_{2n}, \ldots \in C$ for $n \in \kappa_0$. At stage κ_0, player $\mathrm{I}_X^\dagger$ chooses $a_{\kappa_0} \in C$. Let Player I_X^* choose $s_2 = \{a_{2n}\}_{n\in\kappa_0} \cup \{a_{\kappa_0}\}$ (which has cardinality $\leq \kappa_1$). Player II_X^* uses his winning strategy to choose $s_3 \in \mathcal{P}_\kappa(C)$ of size $\leq \kappa_1$. Player $\mathrm{II}_X^\dagger$ then proceeds to play the elements of s_3 as his next κ_1-many moves. Continue in this manner for the remaining moves.

Since player II_X^* used his winning strategy, $s = \bigcup_{i\in\omega} s_i \in X$. By construction, $s = \bigcup_{i\in\kappa}\{a_i\}$. Hence, $\bigcup_{i\in\kappa}\{a_i\} \in X$ and thus player $\mathrm{II}_X^\dagger$ has a winning strategy.

(3 $\Rightarrow$ 1): Suppose player $\mathrm{II}_X^\dagger$ has a winning strategy in $G_\kappa^\dagger(X)$. Assume player

I_X has chosen $s_0 \in \mathcal{P}_\kappa(C)$ such that $|s_0| = \lambda_0 < \kappa$. Let $\delta_0 = \max\{\lambda_0, \kappa_0\}$ and denote s_0 as a sequence indexed by the even numbers less than δ_0, $\{a_{2i}\}_{i\in\delta_0}$ (if $\lambda_0 < \kappa_0$ then repeat an a_{2i} term enough to get a sequence of length κ_0).

At this point, players $\mathrm{I}^\dagger_X$ and $\mathrm{II}^\dagger_X$ will play their first δ_0-many moves of $G^\dagger_\kappa(X)$. Player $\mathrm{I}^\dagger_X$ chooses a_0 as his first move. Player $\mathrm{II}^\dagger_X$ uses his winning strategy to choose $a_1 \in C$. Then, player $\mathrm{I}^\dagger_X$ chooses a_2 as his next move. Player $\mathrm{II}^\dagger_X$ continues to use his winning strategy to choose $a_3 \in C$. This gameplay continues for δ_0-many moves. Player II_X now chooses $s_1 = \langle a_{2i+1}\rangle_{i\in\delta_0}$ in response to player I_X's choice of s_0.

Next, assume player I_X has chosen $s_2 \in \mathcal{P}_\kappa(C)$ such that $|s_2| = \lambda_1 < \kappa$. Let $\delta_1 = \max\{\lambda_1, \kappa_1\}$ and continue as before.

Since player $\mathrm{II}^\dagger_X$ used his winning strategy, $s = \bigcup_{i\in\kappa} a_i \in X$. By construction, $s = \bigcup_{i\in\omega} s_i$. Hence, player II_X has a winning strategy in the game $G_\kappa(X)$.

(4 $\Rightarrow$ 1): Suppose player II^+_X has a winning strategy in $G_{\kappa^+}(X)$. Assume player I_X has chosen $s_0 \in \mathcal{P}_\kappa(C)$. Let player I^+_X choose $s^+_0 = s_0$. Player II^+_X will then use his winning strategy to respond with $s^+_1 \in \mathcal{P}_{\kappa^+}(C)$. Since κ is cofinal with ω, let $s^+_1 = \bigcup_{i\in\omega} t^i_1$ such that $t^i_1 \subseteq t^{i+1}_1$ and $|t^i_1| < \kappa$ for all $i \in \omega$. Player II_X then responds to player I_X with $s_1 = t^0_1$.

For the inductive step, suppose s_i and s^+_i for all $i < 2n$ and t^j_{2i+1} for all $i < n$ and all $j \in \omega$ have been determined already. Assume player I_X has chosen $s_{2n} \in \mathcal{P}_\kappa(C)$. Let player I^+_X choose $s^+_{2n} = s_{2n}$. Player II^+_X will then use his winning strategy to choose $s^+_{2n+1} \in \mathcal{P}_{\kappa^+}(C)$. Again, let $s^+_{2n+1} = \bigcup_{i\in\omega} t^i_{2n+1}$ such that $t^i_{2n+1} \subseteq t^{i+1}_{2n+1}$ and $|t^i_{2n+1}| < \kappa$ for all $i \in \omega$. Player II_X then responds to player I_X with

$s_{2n+1} = \bigcup_{j \leq n} t^n_{2j+1}$.

By construction, $s = \bigcup_{i \in \omega} s^+_i = \bigcup_{i \in \omega} s_i$ and $s \in X$ since player II^+_X used his winning strategy. Thus player II_X also wins the game, as desired.

(1 $\Rightarrow$ 4): This proof proceeds exactly as the previous proof, with the roles of player I and II from the previous proof reversed.

Suppose player II_X has a winning strategy in $G_\kappa(X)$. Assume player I^+_X has chosen $s^+_0 \in \mathcal{P}_{\kappa^+}(C)$. Since κ is cofinal with ω, let $s^+_0 = \bigcup_{i \in \omega} t^i_0$ such that $t^i_0 \subseteq t^{i+1}_0$ and $|t^i_0| < \kappa$ for all $i \in \omega$. Let player I_X choose $s_0 = t^0_0$. Player II_X will then use his winning strategy to respond with $s_1 \in \mathcal{P}_\kappa(C)$. Player II^+_X then responds to player I^+_X with $s^+_1 = s_1$.

For the inductive step, suppose s_i and s^+_i for all $i < 2n$ and t^j_{2i} for all $i < n$ and all $j \in \omega$ have been determined already. Assume player I^+_X has chosen $s^+_{2n} \in \mathcal{P}_{\kappa^+}(C)$. Again, let $s^+_{2n} = \bigcup_{i \in \omega} t^i_{2n}$ such that $t^i_{2n} \subseteq t^{i+1}_{2n}$ and $|t^i_{2n}| < \kappa$ for all $i \in \omega$. Let player I_X choose $s_{2n} = \bigcup_{j \leq n} t^n_{2j}$. Player II_X will then use his winning strategy to choose $s_{2n} \in \mathcal{P}_\kappa(C)$. Player II^+_X then responds to player I^+_X with $s^+_{2n} = s_{2n}$.

By construction, $s = \bigcup_{i \in \omega} s_i = \bigcup_{i \in \omega} s^+_i$ and $s \in X$ since player II_X used his winning strategy. Thus player II^+_X also wins the game, as desired. □

We can now define the set $D_{\kappa^+}(C)$, which will be our filter, based on the game theoretic characterization of the filter $D_{\omega_1}(C)$ from Theorem 1.3.4. Lemma 2.1.7 will provide the necessary verification that $D_{\kappa^+}(C)$ is indeed a filter.

Definition 2.1.4. Given a set C, define the set $D_{\kappa^+}(C)$ such that:

$$D_{\kappa^+}(C) = \{X \subseteq \mathcal{P}_{\kappa^+}(C) : \mathrm{II}_X \text{ has a winning strategy in } G_\kappa(X)\}$$

Remark 2.1.5. By Theorem 2.1.3, for each proof we are free to choose whichever of the games in Definition 2.1.1 that is most convenient.

Note that it is not true that each $X \in D_{\kappa^+}(C)$ contains a κ-closed and κ-unbounded subset. However, it is true that if X is a κ-closed and κ-unbounded subset of $\mathcal{P}_{\kappa^+}(C)$ then X is in the filter. (In fact, this is true if X is merely ω-closed and κ-unbounded)

Lemma 2.1.6. *Let C be a set and $X \subseteq \mathcal{P}_{\kappa^+}(C)$. If X is ω-closed and κ-unbounded, then $X \in D_{\kappa^+}(C)$.*

Proof. Let $X \subseteq \mathcal{P}_{\kappa^+}(C)$ be an ω-closed and κ-unbounded set. It suffices to show that player II_X has a winning strategy in the game $G_\kappa(X)$. We describe player II_X's strategy by induction.

<u>Base Case</u>: $n = 0$.

Assume player I_X has chosen $s_0 \in \mathcal{P}_\kappa(C)$. By the κ-unboundedness of X there exists $t_1 \in X$ such that $s_0 \subseteq t_1$. Since the cofinality of κ is ω, we may write $t_1 = \bigcup_{i\in\omega} t_1^i$ such that $|t_1^i| < \kappa_i$ and $t_1^i \subseteq t_1^{i+1}$. Player II_X then responds to player I_X's choice of s_0 with $s_1 = t_1^0$.

<u>Successor Stage</u>: Suppose we have defined s_i for $i < 2n$.

Assume player I_X has chosen $s_{2n} \in \mathcal{P}_\kappa(C)$. By the κ-unboundedness of X there exists $t_{2n+1} \in X$ such that $s_{2n} \cup t_{2n-1} \subseteq t_{2n+1}$. Again, we may write $t_{2n+1} =$

$\bigcup_{i\in\omega} t^i_{2n+1}$ such that $|t^i_{2n+1}| < \kappa_i$ and $t^i_{2n+1} \subseteq t^{i+1}_{2n+1}$. Player II_X then responds to player I_X's choice of s_{2n} with $s_{2n+1} = \bigcup_{j\leq n} t^n_{2j+1}$.

Since $t_{2i+1} \subseteq t_{2(i+1)+1}$ and X is ω-closed, $\bigcup_{i\in\omega} t_{2i+1} \in X$. By construction, $\bigcup_{i\in\omega} s_i = \bigcup_{i\in\omega} t_{2i+1}$. Hence, $\bigcup_{i\in\omega} s_i \in X$ and player II_X has a winning strategy in the game $G_\kappa(X)$ as desired. $\square$

We proceed to show some other desirable properties that our filter exhibits. First we show that it is closed under κ-many intersections.

Lemma 2.1.7. *$D_{\kappa^+}(C)$ is κ^+-complete.*

Proof. Let $X_\alpha \in D_{\kappa^+}(C)$ for $\alpha \in \kappa$ and let $Y = \bigcap_{\alpha\in\kappa} X_\alpha$. It suffices to show that player II^*_Y has a winning strategy in the game $G^*_\kappa(Y)$. We will do this by playing κ-many concurrent games and use these to describe player II^*_Y's strategy. It is important to note how this gameplay proceeds. At the time player I^*_Y plays his first move, we start the first κ_0-many games, $G^*_\kappa(X_\alpha)$ for $\alpha < \kappa_0$. When player I^*_Y plays his second move, the first κ_0-many games continue and the games $G^*_\kappa(X_\alpha)$ start for $\kappa_0 \leq \alpha < \kappa_1$. We continue to stagger the beginning of each game $G^*_\kappa(X_\alpha)$ in this manner.

<u>Base Case</u>: $n = 0$

Assume player I^*_Y has chosen $s_0 \in \mathcal{P}_\kappa(C)$ with $|s_0| \leq \kappa_0$. We break our gameplay into 2 cases.

- For $\alpha < \kappa_0$, let player $\mathrm{I}^*_{X_\alpha}$ choose $s^\alpha_0 = s_0$. Player $\mathrm{II}^*_{X_\alpha}$ then uses his winning strategy to choose $s^\alpha_1 \in \mathcal{P}_\kappa(C)$ of cardinality $\leq \kappa_0$.

- For $\alpha \geq \kappa_0$, player $\mathrm{I}^*_{X_\alpha}$ doesn't start playing the game yet. For the sake of simplicity, we will denote this as $s^\alpha_0 = \emptyset$ and $s^\alpha_1 = \emptyset$.

Player II^*_Y now chooses $s_1 = \bigcup_{\alpha \in \kappa_0} s^\alpha_1$ in response to player I_Y's choice of s_0. Note that $|s_1| \leq \kappa_0$.

Successor Stage: Suppose we have defined s_i and s^α_i for all $i < 2n$.

Assume player I^*_Y has chosen $s_{2n} \in \mathcal{P}_\kappa(C)$ with $|s_{2n}| \leq \kappa_n$. Again we consider 2 cases.

- For $\alpha < \kappa_n$, let player $I^*_{X_\alpha}$ choose $s^\alpha_{2n} = \bigcup_{i \leq 2n} s_i$ (note that $|s^\alpha_{2n}| \leq \kappa_n$). Player $\mathrm{II}^*_{X_\alpha}$ then uses his winning strategy to choose $s^\alpha_{2n+1} \in \mathcal{P}_\kappa(C)$ with $|s^\alpha_{2n+1}| \leq \kappa_n$.

- For $\alpha \geq \kappa_n$, player I_{X_α} still hasn't started playing the game. Again, for simplicity sake, we denote this as $s^\alpha_{2n} = \emptyset$ and $s^\alpha_{2n+1} = \emptyset$.

By construction, $s = \bigcup_{i \in \omega} s_i = \bigcup_{i \in \omega} s^\alpha_i$ for all $\alpha \in \kappa$. Since each player $\mathrm{II}^*_{X_\alpha}$ used his winning strategy once the game started, $s \in X_\alpha$ for all $\alpha \in \kappa$. Hence, $s \in Y$ as desired. □

Next we show that our filter is closed under the diagonalization of sets indexed by finite sequences. When considering AECs with finite character, the following lemma will prove very useful.

Lemma 2.1.8. *$D_{\kappa^+}(C)$ is closed under diagonalization for sets indexed by finite sequences. That is, if $X_{\langle i_0, \ldots i_n \rangle} \in D_{\kappa^+}(C)$ for all $n \in \omega$ and for every $i_0, \ldots i_n \in I$, where $I \subseteq C$, then $\bar{X} \in D_{\kappa^+}(C)$ where $\bar{X} = \{s \in \mathcal{P}_{\kappa^+}(C) : s \in X_{\langle i_0, \ldots i_n \rangle}$ for all $n \in \omega$ and for all $i_0, \ldots i_n \in (I \cap s)\}$.*

Proof. We will construct a winning strategy for player $\mathrm{II}_{\bar{X}}$ in $G_\kappa(\bar{X})$ by playing ω-many games.

<u>Base Case</u>: $m = 0$

Assume player $\mathrm{I}_{\bar{X}}$ has chosen $s_0 \in \mathcal{P}_\kappa(C)$. Let $Y^0 = \bigcap\{X_{\langle i_0,\ldots i_n\rangle} : n \in \omega \text{ and } i_0, \ldots i_n \in (I \cap s_0)\}$. $Y^0 \in D_{\kappa^+}(C)$ by κ^+-completeness. Hence, player II_{Y^0} has a winning strategy in the game $G_\kappa(Y^0)$. Let player I_{Y^0} choose $t_0^0 = s_0$. Player II_{Y^0} then uses his winning strategy to choose $t_1^0 \in \mathcal{P}_\kappa(C)$. Player $\mathrm{II}_{\bar{X}}$ finally responds to player $I_{\bar{X}}$ with $s_1 = t_1^0 \cup t_0^0$.

<u>Successor Stage</u>: Suppose we have defined s_i for each $i < 2m$ and both t_{2i}^j and t_{2i+1}^j for $(i + j) < m$.

Assume player $\mathrm{I}_{\bar{X}}$ has chosen $s_{2m} \in \mathcal{P}_\kappa(C)$. Let $Y^m = \bigcap\{X_{\langle i_0,\ldots i_n\rangle} : n \in \omega, i_0, \ldots i_n \in (I \cap (\bigcup_{j\leq 2m} s_j))\}$. $Y^m \in D_{\kappa^+}(C)$ by κ^+-completeness. Let player I_{Y^m} choose $t_0^m = \bigcup_{j\leq 2m} s_j$, player $\mathrm{I}_{Y^{m-1}}$ choose $t_2^{m-1} = \bigcup_{j\leq 2m} s_j$, $\ldots$, and player I_{Y^0} choose $t_{2m}^0 = \bigcup_{j\leq 2m} s_j$. Players II_{Y^k} for $k \leq m$ then use their winning strategies to choose $t_1^m, \ldots t_{2m+1}^0 \in \mathcal{P}_\kappa(C)$. Player $\mathrm{II}_{\bar{X}}$ finally responds to player $\mathrm{I}_{\bar{X}}$ with $s_{2m+1} = t_1^m \cup \ldots \cup t_{2m+1}^0$.

By construction, $s = \bigcup_{i\in\omega} s_i = \bigcup_{i\in\omega} t_i^j$ for all $j \in \omega$. If $i_0, \ldots i_n \in I \cap s$ then there is a $k_0 \in \omega$ such that $i_0, \ldots i_n \in I \cap (\bigcup_{j\leq 2k} s_j)$ for all $k \geq k_0$. Thus, $s \in Y^k$ for all $k \geq k_0$. Hence $s \in X_{\langle i_0,\ldots i_n\rangle}$ for all $i_0, \ldots i_n \in I \cap s$ and $s \in \bar{X}$ as desired. $\square$

2.2 Closure under $\equiv_{\infty,\kappa}$

Now that we have defined a filter on $\mathcal{P}_{\kappa^+}(C)$, we are able to define what it means for a property to hold almost always on κ-size approximations.

Definition 2.2.1. For any set s, any countable vocabulary L, and any L-structure $\mathcal{M}$, we define the L-structure $\mathcal{M}^s$ to be the substructure of $\mathcal{M}$ generated by $\mathcal{M} \cap s$. If s is of size κ then we call the structure $\mathcal{M}^s$ a *κ-approximation of* $\mathcal{M}$.

Definition 2.2.2. A property of approximations to one or more models and/or formulas is said to hold *κ-almost everywhere* (or κ-a.e.) iff it holds for all $s \in X$ for some $X \in D_{\kappa^+}(C)$, where C is large enough to approximate all the structures and/or formulas involved.

An interesting initial consequence of these definitions is that if two structures are $L_{\infty,\kappa}$-equivalent then their κ-approximations are almost always isomorphic.

Lemma 2.2.3. *If $\mathcal{M}$ and $\mathcal{N}$ are L-structures and $\mathcal{M} \equiv_{\infty,\kappa} \mathcal{N}$ then $\mathcal{M}^s \cong \mathcal{N}^s$ κ-a.e.*

Proof. Without loss of generality, we may assume $\mathcal{M} \cap \mathcal{N} = \emptyset$. Let $X = \{s \in \mathcal{P}_{\kappa^+}(\mathcal{M} \cup \mathcal{N}) : \mathcal{M}^s \cong \mathcal{N}^s\}$. It suffices to prove that player II_X has a winning strategy in the game $G_\kappa(X)$.

Assume player I_X begins by choosing $s_0 \in \mathcal{P}_\kappa(\mathcal{M} \cup \mathcal{N})$. We can write $s_0 = s_0^{\mathcal{M}} \cup s_0^{\mathcal{N}}$ where $s_0^{\mathcal{M}} \subseteq \mathcal{M}$ and $s_0^{\mathcal{N}} \subseteq \mathcal{N}$. Since $\mathcal{M} \equiv_{\infty,\kappa} \mathcal{N}$, by Theorem 1.2.6 we know there exists $s_1^{\mathcal{N}} \in \mathcal{P}_\kappa(\mathcal{N})$ such that $(\mathcal{M}, s_0^{\mathcal{M}}) \equiv_{\infty,\kappa} (\mathcal{N}, s_1^{\mathcal{N}})$. In addition, there exists $s_1^{\mathcal{M}} \in \mathcal{P}_\kappa(\mathcal{M})$ such that $(\mathcal{M}, s_0^{\mathcal{M}}, s_1^{\mathcal{M}}) \equiv_{\infty,\kappa} (\mathcal{N}, s_1^{\mathcal{N}}, s_0^{\mathcal{N}})$. Player II_X now responds to player I_X with $s_1 = s_1^{\mathcal{M}} \cup s_1^{\mathcal{N}}$. Players I_X and II_X continue in this manner for the rest of the game.

Let $s = \bigcup_{i\in\omega} s_i$. We can write $s = s^{\mathcal{M}} \cup s^{\mathcal{N}}$ where $s^{\mathcal{M}} = \bigcup_{i\in\omega} s_i^{\mathcal{M}}$ and $s^{\mathcal{N}} = \bigcup_{i\in\omega} s_i^{\mathcal{N}}$. Denote $s^{\mathcal{M}}$ as $\langle a_i \rangle_{i\in\kappa}$ and $s^{\mathcal{N}}$ as $\langle b_i \rangle_{i\in\kappa}$ in such a way so that $(\mathcal{M}, \langle a_i \rangle_{i\in\kappa}) \equiv_{\infty,\omega} (\mathcal{N}, \langle b_i \rangle_{i\in\kappa})$, which player II_X made possible with his strategy. Note that we can not conclude $L_{\infty,\kappa}$-elementary equivalence because of the cofinality of κ, but we can conclude $L_{\infty,\omega}$-elementary equivalence because $L_{\infty,\omega}$ formulas only have finitely many free variables. However, $L_{\infty,\omega}$-elementary equivalence is sufficient for our proof. From it, we get that the map f defined as $f(a_i) = b_i$ is a partial isomorphism which will extend uniquely to an isomorphism of $\mathcal{M}^s$ and $\mathcal{N}^s$, since $\mathcal{M}^s$ and $\mathcal{N}^s$ are generated by $\langle a_i \rangle_{i\in\kappa}$ and $\langle b_i \rangle_{i\in\kappa}$, respectively. Thus, $s \in X$ as desired. $\square$

The following application of κ-approximations is implied by the Löwenheim-Skolem axiom, but will be more helpful to us stated in this form. For the following proof it will be useful to note that given a model $\mathcal{M}$ of cardinality greater than or equal to κ, the set $\{s : (\mathcal{M} \cap s) = \mathcal{M}^s\}$ is ω-closed and κ-unbounded. Hence, by Lemma 2.1.6, $\{s : (\mathcal{M} \cap s) = \mathcal{M}^s\} \in D_{\kappa^+}(\mathcal{M})$.

Lemma 2.2.4. *Let* $(\mathbb{K}, \prec_{\mathbb{K}})$ *be an AEC.*

1. *If* $\mathcal{M} \in \mathbb{K}$ *then* $\mathcal{M}^s \prec_{\mathbb{K}} \mathcal{M}$*,* κ*-a.e.*

2. *If* $\mathcal{M} \in \mathbb{K}$ *and* $\mathcal{M}_0 \prec_{\mathbb{K}} \mathcal{M}$ *such that* $|\mathcal{M}_0| = \kappa$ *then* $\mathcal{M}_0 \prec_{\mathbb{K}} \mathcal{M}^s$*,* κ*-a.e.*

Proof.

1. Let $X = \{s \in \mathcal{P}_{\kappa^+}(\mathcal{M}) : \mathcal{M}^s \prec_{\mathbb{K}} \mathcal{M}, \mathcal{M} \cap s = \mathcal{M}^s\}$. By Lemma 2.1.6 it

suffices to show X is ω-closed and κ-unbounded.

Let $\langle s_i \rangle_{i\in\omega}$ be an increasing sequence in X. That is, $\mathcal{M}^{s_i} \prec_{\mathbb{K}} \mathcal{M}$ for all $i \in \omega$ and $\mathcal{M}^{s_i} \subseteq \mathcal{M}^{s_{i+1}}$. Then $\mathcal{M}^{s_i} \prec_{\mathbb{K}} \mathcal{M}^{s_{i+1}}$ for all $i \in \omega$ by the coherence axiom (Axiom 6 of Definition 1.1.1). By the chain axiom, $\bigcup_{i\in\omega} \mathcal{M}^{s_i} \prec_{\mathbb{K}} \mathcal{M}$. Since $\mathcal{M}^{s_i} = \mathcal{M} \cap s_i$ for all $i \in \omega$, $\bigcup_{i\in\omega} \mathcal{M}^{s_i} = \mathcal{M}^{\cup s_i}$. Hence, $\bigcup_{i\in\omega} s_i \in X$ and X is ω-closed.

Let $s \in \mathcal{P}_{\kappa^+}(\mathcal{M})$. By the Löwenheim-Skolem axiom, there exists $\mathcal{M}' \in \mathbb{K}$ of size $\leq \kappa$ such that $\mathcal{M} \cap s \subseteq \mathcal{M}' \prec_{\mathbb{K}} \mathcal{M}$. Let $s' = ran(\mathcal{M}') \in \mathcal{P}_{\kappa^+}(\mathcal{M})$. $\mathcal{M}^{s'} = \mathcal{M} \cap s' = \mathcal{M}'$ and $\mathcal{M}^{s'} \prec_{\mathbb{K}} \mathcal{M}$. Thus, $s' \in X$ and X is κ-unbounded.

2. Note that $\mathcal{M}_0 \subseteq \mathcal{M}^s$ κ-a.e. From part (a) we have that $\mathcal{M}^s \prec_{\mathbb{K}} \mathcal{M}$ κ-a.e. By the coherence axiom, $\mathcal{M}_0 \prec_{\mathbb{K}} \mathcal{M}^s$ κ-a.e as desired.

$\square$

Lemma 2.2.5. *Assume $\mathcal{M} \in \mathbb{K}$, $\mathcal{M}_0 \prec_{\mathbb{K}} \mathcal{M}$ of cardinality κ, $n \in \omega$ and $\bar{a}_0, \ldots \bar{a}_{n-1} \subseteq \mathcal{M}_0$ are sequences of length $< \kappa$. Let $\mathcal{N}$ be an arbitrary L-structure and $\bar{b}_0, \ldots \bar{b}_{n-1} \subseteq \mathcal{N}$ be such that $(\mathcal{M}, \langle \bar{a}_i \rangle_{i<n}) \equiv_{\infty,\kappa} (\mathcal{N}, \langle \bar{b}_i \rangle_{i<n})$. Then [there is a $\mathbb{K}$-embedding h of $\mathcal{M}_0$ into $\mathcal{N}^s$ such that $h(\bar{a}_i) = \bar{b}_i$ for all $i < n$] κ-a.e.*

Proof. Let $Y = \{s \in \mathcal{P}_{\kappa^+}(\mathcal{N})$: there is a $\mathbb{K}$-embedding h of $\mathcal{M}_0$ into $\mathcal{N}^s$ s.t. $h(\bar{a}_i) = \bar{b}_i \forall i < n\}$. We will show player II_Y has a winning strategy in the game $G_\kappa(Y)$.

By Lemma 2.2.4, we know that $\mathcal{M}_0 \prec_{\mathbb{K}} \mathcal{M}^s$ κ-a.e. Therefore $X = \{s \in \mathcal{P}_{\kappa^+}(\mathcal{M}) : \mathcal{M}_0 \prec_{\mathbb{K}} \mathcal{M}^s, \mathcal{M}^s = s\} \in D_{\kappa^+}(\mathcal{M})$. Thus, player II_X has a winning strategy in the game $G_\kappa(X)$. We will use this winning strategy to construct a winning strategy for player II_Y.

Assume player I_Y has chosen $\bar{d}_0 \in \mathcal{P}_\kappa(N)$. Note that $\bar{d}_0$ can be viewed as a sequence from $\mathcal{N}$ of length $< \kappa$ (arrange it in any order). By Theorem 1.2.6, there exists $\bar{c}_0 \subseteq \mathcal{M}$ such that $(\mathcal{M}, \langle \bar{a}_i \rangle_{i<n}, \bar{c}_0) \equiv_{\infty,\kappa} (\mathcal{N}, \langle \bar{b}_i \rangle_{i<n}, \bar{d}_0)$. Let player I_X choose $ran(\bar{c}_0)$ in the game $G_\kappa(X)$. Player II_X then uses his winning strategy to choose $\bar{c}_1 \subseteq \mathcal{M}$. By Theorem 1.2.6 again, there exists $\bar{d}_1 \subseteq \mathcal{N}$ such that $(\mathcal{M}, \langle \bar{a}_i \rangle_{i<n}, \bar{c}_0, \bar{c}_1) \equiv_{\infty,\kappa} (\mathcal{N}, \langle \bar{b}_i \rangle_{i<n}, \bar{d}_0, \bar{d}_1)$. Now, let player II_Y choose $ran(\bar{d}_1)$ in response to player I_Y's choice of $\bar{d}_0$. Continue this process for all $i \in \omega$.

Let $\bar{c} = \bigcup_{i\in\omega} \bar{c}_i \subseteq \mathcal{M}$. Similarly, let $\bar{d} = \bigcup_{i\in\omega} \bar{d}_i \subseteq \mathcal{N}$. Since $cof(\kappa) = \omega$, it is not necessarily true that $(\mathcal{M}, \langle \bar{a}_i \rangle_{i<n}, \bar{c}) \equiv_{\infty,\kappa} (\mathcal{N}, \langle \bar{b}_i \rangle_{i<n}, \bar{d})$. However, we can say that $(\mathcal{M}, \langle \bar{a}_i \rangle_{i<n}, \bar{c}) \equiv_{\infty,\omega} (\mathcal{N}, \langle \bar{b}_i \rangle_{i<n}, \bar{d})$ since $L_{\infty,\omega}$-formulas only have finitely many free variables and $(\mathcal{M}, \langle \bar{a}_i \rangle_{i<n}, \bar{c}_0, \ldots \bar{c}_k) \equiv_{\infty,\kappa} (\mathcal{N}, \langle \bar{b}_i \rangle_{i<n}, \bar{d}_0, \ldots \bar{d}_k)$ for every $k \in \omega$. Since player II_X used his winning strategy, $ran(\bar{c}) = s_0 \in X$. Thus $\mathcal{M}_0 \prec_{\mathbb{K}} \mathcal{M}^{s_0}$ and $\mathcal{M}^{s_0} = s_0 = ran(\bar{c})$.

Let $s_1 = ran(\bar{d})$. Define $g : \mathcal{M}^{s_0} \to \mathcal{N}$ by $g(\bar{c}_i) = \bar{d}_i$ for all $i \in \omega$. Then g is an isomorphism of $\mathcal{M}^{s_0}$ onto a substructure $\mathcal{N}^{s_1}$ of $\mathcal{N}$ such that $\mathcal{N}^{s_1} = s_1$. If we let $\mathcal{N}_0 = g(\mathcal{M}_0)$ then $\mathcal{N}_0 \prec_{\mathbb{K}} \mathcal{N}^{s_1}$ because $\prec_{\mathbb{K}}$ is preserved under isomorphism and $\mathcal{M}_0 \prec_{\mathbb{K}} \mathcal{M}^{s_0}$. In addition, $g(\bar{a}_i) = \bar{b}_i$ for all $i < n$. If we let $h = g \restriction \mathcal{M}_0$ then h is a $\mathbb{K}$-embedding of $\mathcal{M}_0$ into $\mathcal{N}^{s_1}$ such that $h(\bar{a}_i) = \bar{b}_i$ for all $i < n$. Therefore $s_1 \in Y$ and player II_Y has a winning strategy as desired. $\square$

Lemma 2.2.6. *Assume* $(\mathbb{K}, \prec_{\mathbb{K}})$ *has finite character. Let* $\mathcal{M} \in \mathbb{K}$*,* $\mathcal{M}_0 \prec_{\mathbb{K}} \mathcal{M}$ *where*

$|\mathcal{M}_0| \leq \kappa$ *and* $\bar{a} \subseteq \mathcal{M}$ *such that* $ran(\bar{a}) = \mathcal{M}_0$. *Let* $\mathcal{N}$ *be an arbitrary* L*-structure and let* $\bar{b} \subseteq \mathcal{N}$ *such that* $\bar{a}$ *and* $\bar{b}$ *have the same length. If* $(\mathcal{M}, a_{i_0}, \ldots a_{i_n}) \equiv_{\infty,\kappa} (\mathcal{N}, b_{i_0}, \ldots b_{i_n})$ *for all* $i_0, \ldots i_n \in |\bar{a}|$ *and for all* $n \in \omega$ *then* $ran(\bar{b}) = \mathcal{N}_0$ *where* $\mathcal{N}_0 \prec_{\mathbb{K}} \mathcal{N}^s$ κ*-a.e. and* $\mathcal{M}_0 \cong \mathcal{N}_0$.

Proof. Let $Y^{b_{i_0},\ldots b_{i_n}} = \{s \in \mathcal{P}_{\kappa^+}(\mathcal{N}) :$ there exists a $\mathbb{K}$-embedding $h : \mathcal{M}_0 \to \mathcal{N}^s$ s.t. $h(a_{i_k}) = b_{i_k}\ \forall k \leq n\}$. Lemma 2.2.5 implies that $Y^{b_{i_0},\ldots b_{i_n}} \in D_{\kappa^+}(\mathcal{N})$ for all finite sequences $\langle b_{i_0}, \ldots b_{i_n} \rangle \subseteq \bar{b}$. Thus $Z = \bigcap Y^{b_{i_0},\ldots b_{i_n}} \in D_{\kappa^+}(\mathcal{N})$ by κ^+-completeness.

Define the map $g : \mathcal{M}_0 \to \mathcal{N}$ as $g(a_i) = b_i$ for all $i \in \kappa$. As in the previous proofs, we know $(\mathcal{M}, \bar{a}) \equiv_{\infty,\omega} (\mathcal{N}, \bar{b})$ and thus g is an isomorphism of $\mathcal{M}_0$ onto some substructure $\mathcal{N}_0 \subseteq \mathcal{N}$ where $ran(\bar{b}) = \mathcal{N}_0$. Fix $s \in Z$ then for any finite sequence $\langle b_{i_0}, \ldots b_{i_n} \rangle \subseteq \mathcal{N}_0$ the map $h \circ g^{-1}$ is a $\mathbb{K}$-embedding of $\mathcal{N}_0$ into $\mathcal{N}^s$ fixing $b_{i_0}, \ldots b_{i_n}$. Hence, by finite character, $\mathcal{N}_0 \prec_{\mathbb{K}} \mathcal{N}^s$. Therefore $\mathcal{N}_0 \prec_{\mathbb{K}} \mathcal{N}^s$ κ-a.e., as desired. □

Lemma 2.2.7. *Assume* $(\mathbb{K}, \prec_{\mathbb{K}})$ *has finite character. Let* $\mathcal{M} \in \mathbb{K}$ *and assume* $\mathcal{M} \equiv_{\infty,\kappa} \mathcal{N}$ *for some* L*-structure* $\mathcal{N}$*. Then for every subset* $B_0 \subseteq \mathcal{N}$ *of cardinality* $\leq \kappa$*, there is a substructure* $\mathcal{N}_0 \subseteq \mathcal{N}$ *of cardinality* κ *such that* $B_0 \subseteq \mathcal{N}_0$ *and* $\mathcal{N}_0 \prec_{\mathbb{K}} \mathcal{N}^s$ κ*-a.e.*

Proof. Let $X = \{s \in \mathcal{P}_{\kappa^+}(\mathcal{M}) : \mathcal{M}^s \prec_{\mathbb{K}} \mathcal{M}$ and $\mathcal{M}^s = s\}$. Lemma 2.2.4 implies that $X \in D_{\kappa^+}(\mathcal{M})$ and thus player II_X has a winning strategy in the game $G_\kappa(X)$.

Enumerate B_0 as $\langle \bar{b}_{2i} \rangle_{i\in\omega}$ such that $\bar{b}_{2i} \subseteq \bar{b}_{2(i+1)}$ and $|\bar{b}_{2i}| < \kappa$ for all $i \in \omega$. This is possible since $cof(\kappa) = \omega$.

By Theorem 1.2.6 we can pick $\bar{a}_0 \subseteq \mathcal{M}$ such that $(\mathcal{M}, \bar{a}_0) \equiv_{\infty,\kappa} (\mathcal{N}, \bar{b}_0)$. Have player I_X choose $\bar{a}_0$ as his first move in the game $G_\kappa(X)$. Player II_X will then use his winning strategy to choose $\bar{a}_1 \in \mathcal{P}_\kappa(\mathcal{M})$. By Theorem 1.2.6 again we can pick $\bar{b}_1 \subseteq \mathcal{N}$ such that $(\mathcal{M}, \bar{a}_0, \bar{a}_1) \equiv_{\infty,\kappa} (\mathcal{N}, \bar{b}_0, \bar{b}_1)$. Continue in this manner for all $n \in \omega$.

Since player II_X used his winning strategy we know $\{\bar{a}_i\}_{i\in\omega} \in X$. Thus, $ran(\bar{a}) = \mathcal{M}_0$ where $\mathcal{M}_0 = \mathcal{M}^s \prec_{\mathbb{K}} \mathcal{M}$.

By construction, $B_0 \subseteq ran(\bar{b})$ and for any $i_0, \ldots i_n \in \kappa$ and any $n \in \omega$ we know that $(\mathcal{M}, a_{i_0}, \ldots a_{i_n}) \equiv_{\infty,\kappa} (\mathcal{N}, b_{i_0}, \ldots b_{i_n})$. By Lemma 2.2.6 we can conclude that $ran(\bar{b}) = \mathcal{N}_0$ where $\mathcal{N}_0 \prec_{\mathbb{K}} \mathcal{N}^s$ κ-a.e. and $B_0 \subseteq \mathcal{N}_0$ as desired. □

We are now able to use κ-approximations to prove that AECs with finite character and a Löwenheim-Skolem number of κ are closed under $L_{\infty,\kappa}$-equivalence.

Theorem 2.2.8. *Assume $(\mathbb{K}, \prec_{\mathbb{K}})$ has finite character. Let $\mathcal{M} \in \mathbb{K}$ and $\mathcal{N}$ be an arbitrary L-structure. If $\mathcal{M} \equiv_{\infty,\kappa} \mathcal{N}$ then $\mathcal{N} \in \mathbb{K}$.*

Proof. Let $S = \{\mathcal{N}_0 \subseteq \mathcal{N} : |\mathcal{N}_0| = \kappa,\ \mathcal{N}_0 \prec_{\mathbb{K}} \mathcal{N}^s\ \kappa\text{-a.e.}\}$. By Lemma 1.1.3(a), it suffices to show that S is a family of $\mathbb{K}$-structures directed under $\prec_{\mathbb{K}}$ and that $\bigcup S = \mathcal{N}$.

Assume $\mathcal{N}_0, \mathcal{N}_1 \in S$. By Lemma 2.2.7 there is a $\mathbb{K}$-structure $\mathcal{N}_2$ such that $\mathcal{N}_2 \subseteq \mathcal{N}$, $\mathcal{N}_0, \mathcal{N}_1 \subseteq \mathcal{N}_2$, $|\mathcal{N}_2| = \kappa$ and $\mathcal{N}_2 \prec_{\mathbb{K}} \mathcal{N}^s$ κ-a.e. Thus $\mathcal{N}_2 \in S$ and it follows that S is a family of κ-size $\mathbb{K}$-structures directed under $\subseteq$. Furthermore, if $\mathcal{N}_0$, $\mathcal{N}_1 \in S$ and $\mathcal{N}_0 \subseteq \mathcal{N}_1$ then there will be some $\mathcal{N}^s \subseteq \mathcal{N}$ such that $\mathcal{N}_0 \prec_{\mathbb{K}} \mathcal{N}^s$, and $\mathcal{N}_1 \prec_{\mathbb{K}} \mathcal{N}^s$. Hence, $\mathcal{N}_0 \prec_{\mathbb{K}} \mathcal{N}_1$ by the coherence axiom. Therefore, S is a family of κ-size $\mathbb{K}$-structures directed under $\prec_{\mathbb{K}}$. In addition, by Lemma 2.2.7, for every $b \in \mathcal{N}$ there exists $\mathcal{N}_0 \in S$ such that $b \in \mathcal{N}_0$. Hence $\bigcup S = \mathcal{N}$ and S is as desired. $\square$

Under the assumption of finite character we also get that $\prec_{\mathbb{K}}$ is preserved by $L_{\infty,\kappa}$-equivalence.

Corollary 2.2.9. *Assume* $(\mathbb{K}, \prec_{\mathbb{K}})$ *has finite character. Further assume* $\mathcal{M}_0 \prec_{\mathbb{K}} \mathcal{M}$ *and* $\bar{a} \subseteq \mathcal{M}$ *such that* $ran(\bar{a}) = \mathcal{M}_0$*. If* $\bar{b}$ *is a sequence of length* $|\bar{a}|$ *from a model* $\mathcal{N}$ *and* $(\mathcal{M}, a_{i_0}, \dots a_{i_n}) \equiv_{\infty,\kappa} (\mathcal{N}, b_{i_0}, \dots b_{i_n})$ *for all* $i_0, \dots i_n \in |\bar{a}|$ *and for all* $n \in \omega$ *then* $\mathcal{N}_0 \prec_{\mathbb{K}} \mathcal{N}$ *where* $ran(\bar{b}) = \mathcal{N}_0$ *and* $\mathcal{M}_0 \cong \mathcal{N}_0$*.*

Proof. Since we assume $\mathcal{M} \equiv_{\infty,\kappa} \mathcal{N}$, Theorem 2.2.8 implies $\mathcal{N} \in \mathbb{K}$. Thus, by Lemma 2.2.4, $X = \{s \in \mathcal{P}_{\kappa^+}(\mathcal{N}) : \mathcal{N}^s \prec_{\mathbb{K}} \mathcal{N}\} \in D_{\kappa^+}(\mathcal{N})$. We break this up into 2 cases.

<u>Case 1</u>: $|\mathcal{M}_0| = \kappa$

Lemma 2.2.6 implies that $ran(\bar{b}) = \mathcal{N}_0$, $\mathcal{M}_0 \cong \mathcal{N}_0$ and $\mathcal{N}_0 \prec_{\mathbb{K}} \mathcal{N}^s$ κ-a.e. Let $Z = \{s \in \mathcal{P}_{\kappa^+}(\mathcal{N}) : \mathcal{N}_0 \prec_{\mathbb{K}} \mathcal{N}^s\}$.

Since X and Z are in $D_{\kappa^+}(\mathcal{N})$, $X \cap Z \in D_{\kappa^+}(\mathcal{N})$. Therefore, for any $s \in X \cap Z$, $\mathcal{N}_0 \prec_{\mathbb{K}} \mathcal{N}^s$ and $\mathcal{N}^s \prec_{\mathbb{K}} \mathcal{N}$. Hence $\mathcal{N}_0 \prec_{\mathbb{K}} \mathcal{N}$ as desired.

<u>Case 2</u>: $|\mathcal{M}_0| > \kappa$

$\mathcal{M}_0 = \bigcup S$ where $S = \{\mathcal{A}_i \prec_{\mathbb{K}} \mathcal{M}_0 : |\mathcal{A}_i| = \kappa\}$. Note that S is a directed family of $\mathbb{K}$-structures under $\prec_{\mathbb{K}}$. For each $\mathcal{A}_i \in S$, let $\mathcal{A}_i = ran(\langle a_{i_j} \rangle_{j \in \kappa})$. By case 1, there exists $\mathcal{B}_i \prec_{\mathbb{K}} \mathcal{N}$ such that $\mathcal{B}_i = ran(\langle b_{i_j} \rangle_{j \in \kappa})$. Define S' as the family consisting of the $\mathcal{B}_i$'s corresponding to each $\mathcal{A}_i \in S$. S' is then also a directed family of $\mathbb{K}$-structures under $\prec_{\mathbb{K}}$. Let $\mathcal{N}_0 = \bigcup S'$. Then $ran(\bar{b}) = \mathcal{N}_0$ and $\mathcal{N}_0 \in \mathbb{K}$. Lemma

1.1.3(b) implies $\mathcal{N}_0 \prec_{\mathbb{K}} \mathcal{N}$ and thus $\mathcal{N}_0$ is as desired. $\square$

A final corollary of Theorem 2.2.8 is that $L_{\infty,\kappa}$-substructures are also $\mathbb{K}$-substructures assuming only that $\mathbb{K}$-substructure exhibits finite character.

Corollary 2.2.10. *Assume* $(\mathbb{K}, \prec_{\mathbb{K}})$ *has finite character. If* $\mathcal{M} \in \mathbb{K}$ *and* $\mathcal{M} \prec_{\infty,\kappa} \mathcal{N}$ *then* $\mathcal{M} \prec_{\mathbb{K}} \mathcal{N}$.

Proof. In the context of Corollary 2.2.9, let $\bar{a} = \bar{b}$ list all the elements of $\mathcal{M}_0 = \mathcal{M}$. Since $\mathcal{M} \prec_{\infty,\kappa} \mathcal{N}$, $(\mathcal{M}, a_{i_0}, \dots a_{i_n}) \equiv_{\infty,\kappa} (\mathcal{N}, b_{i_0}, \dots b_{i_n})$ for all $i_0, \dots i_n \in |\bar{a}|$ and all $n \in \omega$. Corollary 2.2.9 implies that $\mathcal{M} \prec_{\mathbb{K}} \mathcal{N}$ as desired (since $\mathcal{M}_0 = \mathcal{M} \prec_{\mathbb{K}} \mathcal{M}$). $\square$

2.3 Examples

In this section we will provide several examples to show that the assumptions made in the previous section are necessary and that closure under $L_{\infty,\kappa}$-equivalence is the best possible result.

First, we will show that if we remove the assumption of finite character, we can not assure closure under $L_{\infty,\kappa}$-equivalence.

Example 2.3.1. Define $(\mathbb{K}, \prec_{\mathbb{K}})$ as in Example 1.1.8 with $\mu = \kappa$. We have already shown that this is an AEC satisfying (AP, etc.) and does not have finite character. We claim $\mathbb{K}$ is not closed under $L_{\infty,\kappa}$-equivalence.

Let $\mathcal{M}, \mathcal{N}$ be L-structures such that $|P^{\mathcal{M}}| = \kappa$ and $|\neg P^{\mathcal{M}}| = \kappa^+$ but $|P^{\mathcal{N}}| = \kappa^+$ and $|\neg P^{\mathcal{N}}| = \kappa^+$. Thus, $\mathcal{M} \in \mathbb{K}$ and $\mathcal{N} \notin \mathbb{K}$. However, $\mathcal{M} \equiv_{\infty,\kappa} \mathcal{N}$. Therefore, $\mathbb{K}$ is not closed under $L_{\infty,\kappa}$-equivalence.

Next we will show that the singularity of κ is essential to assuring closure under $L_{\infty,\kappa}$-equivalence. Michael Morley ([13]) provided the following example of 2 models of size $\aleph_1$ that are L_{∞,ω_1}-equivalent but are not isomorphic. We will use this example to construct an AEC with $LS(\mathbb{K}) = \aleph_1$, that has finite character but is not closed under L_{∞,ω_1}-equivalence. Similar examples will work for any regular κ.

Example 2.3.2. There exists a well-founded tree of cardinality $\aleph_1$, $\mathcal{M}$, such that:

1. Every element has exactly ω_1 immediate successors.
2. For every $a_0 \in \mathcal{M}$, $\mathcal{M} \cong \mathcal{M} \restriction \{a : a_0 \leq a\}$.
3. Every branch is countable but there are arbitrarily long countable branches.

Define $\mathcal{N}$ by starting with $(\omega_1, <)$ and putting a copy of $\mathcal{M}$ above every element of ω_1. Thus, $|\mathcal{N}| = \aleph_1$, $\mathcal{M} \equiv_{\infty,\omega_1} \mathcal{N}$ by a back-and-forth argument which we omit, but $\mathcal{M} \ncong \mathcal{N}$ (since $\mathcal{M}$ has only countable branches, but $\mathcal{N}$ has one branch of length ω_1).

Let $\mathbb{K} = \{\mathcal{A} : \mathcal{N} \cong\subseteq \mathcal{A}\}$ and define $\prec_{\mathbb{K}}$ as $\mathcal{A} \prec_{\mathbb{K}} \mathcal{B}$ if $\mathcal{A} \subseteq \mathcal{B}$ (i.e. ordinary substructure).

We first claim $(\mathbb{K}, \prec_{\mathbb{K}})$ is an AEC satisfying amalgamation, joint embedding and arbitrarily large models. Clearly $\mathbb{K}$ is closed under isomorphism, $LS(\mathbb{K}) = \aleph_1$, $\prec_{\mathbb{K}}$ is transitive and reflexive and $\mathcal{A} \prec_{\mathbb{K}} \mathcal{B}$ implies $\mathcal{A} \subseteq \mathcal{B}$. In addition, since $\prec_{\mathbb{K}}$ is the same as substructure, the coherence axiom clearly holds. It remains to show that $(\mathbb{K}, \prec_{\mathbb{K}})$ satisfies the union axioms. To do this assume $\mathcal{A}_i \in \mathbb{K}$ and $\mathcal{A}_i \prec_{\mathbb{K}} \mathcal{A}_{i+1}$ for all $i \in \delta$. Therefore $\mathcal{N} \cong\subseteq \mathcal{A}_i$ for all $i \in \delta$. In particular, there exists a map

$f : \mathcal{N} \cong\subseteq \mathcal{A}_0$ and this same map is an isomorphic embedding of $\mathcal{N}$ into $\mathcal{A}_i$ for all $i \in \delta$. By definition of substructure, this same map f isomorphically embeds $\mathcal{N}$ into $\bigcup_{i\in\delta} \mathcal{A}_i$. Hence, $\bigcup_{i\in\delta} \mathcal{A}_i \in \mathbb{K}$. Furthermore, $\mathcal{A}_i \prec_{\mathbb{K}} \bigcup_{i\in\delta} \mathcal{A}_i$ for all $i \in \delta$, and if $\mathcal{A}_i \prec_{\mathbb{K}} \mathcal{B}$ for each $i \in \delta$ then $\bigcup_{i\in\delta} \mathcal{A}_i \prec_{\mathbb{K}} \mathcal{B}$ because $\prec_{\mathbb{K}}=\subseteq$. Thus $(\mathbb{K}, \prec_{\mathbb{K}})$ is an AEC.

$(\mathbb{K}, \prec_{\mathbb{K}})$ also has finite character since $\prec_{\mathbb{K}}=\subseteq$ and substructure has finite character. In addition, $(\mathbb{K}, \prec_{\mathbb{K}})$ clearly has arbitrarily large models and exhibits both the amalgamation and joint embedding.

Note that $\mathcal{M} \notin \mathbb{K}$ since there is no copy of $\mathcal{N}$ in $\mathcal{M}$. Hence $\mathbb{K}$ is not closed under L_{∞,ω_1}-equivalence.

Finally, we provide an example illustrating that under the assumptions in the previous section, closure under $L_{\infty,\kappa}$-equivalence is the best possible result. In particular, we can find an example of an AEC with finite character, arbitrarily large models and Löwenheim-Skolem number κ (where $cof(\kappa) = \omega$) admitting amalgamation and joint embedding that is not closed under $L_{\infty,\tau}$-equivalence for any ordinal $\tau < \kappa$.

Example 2.3.3. Define $(\mathbb{K}, \prec_{\mathbb{K}})$ in a similar manner to examples 2.3.1 and 1.1.8 with $\mathbb{K} = \{\mathcal{M} : |P^{\mathcal{M}}|, |\neg P^{\mathcal{M}}| \geq \kappa\}$, but define $\prec_{\mathbb{K}}$ as ordinary substructure. $(\mathbb{K}, \prec_{\mathbb{K}})$ can be show to be an AEC with (AP, etc.) in an almost identical proof to that of Example 1.1.8. Also, $(\mathbb{K}, \prec_{\mathbb{K}})$ has finite character since $\prec_{\mathbb{K}}=\subseteq$. Thus, by Theorem 2.2.8, $\mathbb{K}$ is closed under $L_{\infty,\kappa}$-equivalence.

We claim that $\mathbb{K}$ is not closed under $L_{\infty,\tau}$-equivalence for $\tau < \kappa$. To show this, let $\mathcal{M}$ and $\mathcal{N}$ be L-structures such that $|P^{\mathcal{M}}| = \kappa$, $|\neg P^{\mathcal{M}}| = \kappa$, $|P^{\mathcal{N}}| = \tau$ and $|\neg P^{\mathcal{N}}| = \kappa$. Thus $\mathcal{M} \in \mathbb{K}$ and $\mathcal{N} \notin \mathbb{K}$. However, $\mathcal{M} \equiv_{\infty,\tau} \mathcal{N}$ and therefore $(\mathbb{K}, \prec_{\mathbb{K}})$ is not closed under $L_{\infty,\tau}$-elementary equivalence.

2.4 Axiomatizability

In this section we raise the question of whether or not Theorem 2.2.8 can be improved to yield axiomatizability. First we we will prove that there are 2^{2^κ}-many AECs with Löwenheim-Skolem number κ and finite character, and thus too many for every one to be axiomatizable in $L_{\kappa^+,\kappa}$. First observe that, regardless of the cofinality of κ, Shelah's Lemma 1.1.3 implies that there are at most 2^{2^κ}-many such AECs.

Theorem 2.4.1. *There are 2^{2^κ}-many AECs with $LS(\mathbb{K}) = \kappa$ and finite character and satisfying (AP, etc.).*

Proof. Let L_0 be a language consisting solely of κ-many unary predicates R_i for each $i \in \kappa$. For every infinite $A \subseteq \kappa$, let the first order theory T_A have the following axiom scheme:

- $\forall x(\neg(R_i(x) \wedge R_j(x)))$ for each $i \neq j$ with $i, j \in \kappa$.
- $R_i(x)$ is infinite for each $i \in A$.
- $R_j(X)$ is empty for each $j \notin A$.

For $A \neq B \subseteq \kappa$, T_A and T_B are distinct, complete first order L_0-theories. Consequently we have created 2^κ-many complete first order L_0-theories. Let E be a new binary predicate and let $L = L_0 \cup \{E\}$. For every non-empty $S \subseteq 2^\kappa$, we will construct an AEC $(\mathbb{K}_S, \prec_{\mathbb{K}_S})$ with finite character so that different S's define different AECs.

Let $\mathbb{K}_S = \{\mathcal{M} | \mathcal{M}$ is an L-structure, $E^\mathcal{M}$ is an equivalence relation and every $E^\mathcal{M}$ class is a model of T_A for some $A \in S\}$. If $\mathcal{M} \in \mathbb{K}_S$ and $a \in \mathcal{M}$, we define $\mathcal{M}_a$ as the L_0-reduct of the substructure of $\mathcal{M}$ whose universe is the equivalence class $E^\mathcal{M}(x, a)$. Each $\mathcal{M}_a \models T_A$ for some $A \in S$. For $\mathcal{M}, \mathcal{N} \in \mathbb{K}_S$ define $\prec_{\mathbb{K}_S}$ as $\mathcal{M} \prec_{\mathbb{K}_s} \mathcal{N}$ IFF $\mathcal{M} \subseteq \mathcal{N}$ and $\mathcal{M}_a \prec \mathcal{N}_a$ for all $a \in \mathcal{M}$.

$(\mathbb{K}_S, \prec_{\mathbb{K}_S})$ clearly has finite character since $\prec_{\mathbb{K}_S}$ only depends on first order substructure and first order elementary substructure, both which have finite character. Additionally, it is easy to see that $(\mathbb{K}_S, \prec_{\mathbb{K}_S})$ is an AEC and that if $S, S' \subseteq 2^\kappa$ such that $S \neq S'$ then $\mathbb{K}_S \neq \mathbb{K}_{S'}$. Since there are 2^{2^κ}-many such S, we get 2^{2^κ}-many $\mathbb{K}_S$'s, as desired. □

By the previous theorem, we know there are AECs, $(\mathbb{K}, \prec_\mathbb{K})$, with finite character and $LS(\mathbb{K}) = \kappa$ that are not axiomatizable in $L_{\kappa^+,\kappa}$. The next theorem will provide sufficient conditions to axiomatize such an AEC with a sentence from $L_{\infty,\kappa}$. This will be the first of several axiomatizability results in this paper.

Theorem 2.4.2. *Assume κ is cofinal with ω. Let $(\mathbb{K}, \prec_\mathbb{K})$ be an AEC with $LS(\mathbb{K}) = \kappa$ and satisfying finite character. Assume that $\mathbb{K}$ has at most λ-many models of cardinality λ for some λ such that $\lambda^{<\kappa} = \lambda$. Then $\mathbb{K} = Mod(\sigma)$ for some $\sigma \in L_{\infty,\kappa}$.*

The proof of Theorem 2.4.2 depends on two previously proven results. For the first theorem, we reference a result of Kueker's and use the notation λ^*-a.e. This is not the obvious generalization of κ-a.e. as previously defined. Instead, we define the filter $D^*_{\lambda^+}(C)$ to be the set of all $X \subseteq \mathcal{P}_{\lambda^+}(C)$ such that player $\mathrm{II}^\dagger_X$ has a winning strategy in the λ-length game $G^\dagger_\lambda(X)$. Thus a property occurs λ^*-a.e. iff it occurs for all $s \in X$ for some $X \in D^*_{\lambda^+}(C)$. The only important property of this definition of the filter is that every λ-closed and λ-unbounded subset of $\mathcal{P}_{\lambda^+}(C)$ is in $D^*_{\lambda^+}(C)$. See [8] for details.

Theorem 2.4.3. [8] *Let $\sigma \in L_{\mu^+,\kappa}$ and let λ be such that $\lambda \geq \mu$ and $\lambda^{<\kappa} = \lambda$. Then $\mathcal{M} \models \sigma$ IFF $\mathcal{M}^s \models \sigma$ λ-a.e.*

An easy generalization of Scott's Theorem shows that for every $\kappa \geq \omega$ and every $\mathcal{M}$ there is some $\sigma \in L_{\infty,\kappa}$ such that for every $\mathcal{N}$, $\mathcal{N} \models \sigma$ iff $\mathcal{M} \equiv_{\infty,\kappa} \mathcal{N}$. In fact we have the following theorem.

Theorem 2.4.4. [9] *Let $\mathcal{M}$ be an infinite model for a language with at most $|\mathcal{M}|$-many symbols, and let $\lambda = |\mathcal{M}|^{<\kappa}$, where $\kappa \geq \omega$. Then there is a sentence θ of $L_{\lambda^+,\kappa}$ such that for every $\mathcal{N}$, $\mathcal{N} \models \theta$ IFF $\mathcal{N} \equiv_{\infty,\kappa} \mathcal{M}$.*

Using these theorems we can now proceed to prove our axiomatizability result.

Proof. (Theorem 2.4.2)

Let $\{\mathcal{M}_i : i \in I\}$ list all models of $\mathbb{K}$ of cardinality λ, up to isomorphism. By Theorem 2.4.4, since $\lambda^{<\kappa} = \lambda$, for each $i \in I$, let $\theta_i \in L_{\lambda^+,\kappa}$ determine $\mathcal{M}_i$ up to $L_{\infty,\kappa}$-elementary equivalence. Let $\sigma_1 = \bigvee_{i \in I} \theta_i$, which is in $L_{\lambda^+,\kappa}$ since $|I| \leq \lambda$. Clearly every model in $\mathbb{K}$ of cardinality λ models σ_1. We claim that every model in $\mathbb{K}$ of cardinality greater than λ also models σ_1.

Let $\mathcal{N} \in \mathbb{K}$ such that $|\mathcal{N}| > \kappa$. Suppose $\mathcal{N} \not\models \sigma_1$. Then $\mathcal{N} \models \neg\sigma_1$. Theorem 2.4.3 implies that $\mathcal{N}^s \models \neg\sigma_1$ λ^*-a.e. However, $\mathcal{N}^s \in \mathbb{K}$ λ^*-a.e. and $|\mathcal{N}^s| = \lambda$, λ^*-a.e because $\{s : \mathcal{N}^s \in \mathbb{K}, |\mathcal{N}^s| = \lambda\}$ is λ-closed and λ-unbounded. Thus there is a substructure $\mathcal{N}^s$ such that $\mathcal{N}^s \models \neg\sigma_1$, $\mathcal{N}^s \in \mathbb{K}$ and $|\mathcal{N}^s| = \lambda$, which is a contradiction.

Let $\{\mathcal{A}_j : j \in J\}$ list all models $\mathbb{K}$ of cardinality less than λ, up to isomorphism. By Theorem 2.4.4, let θ'_j determine $\mathcal{A}_j$ up to $L_{\infty,\kappa}$-elementary equivalence. Let $\sigma_2 = \bigvee_{j \in J} \theta'_j$. Finally, let $\sigma = \sigma_1 \vee \sigma_2$. It is clear that any structure in $\mathbb{K}$ models σ. Conversely, suppose $\mathcal{M} \models \sigma$, for some L-structure $\mathcal{M}$. Then $\mathcal{M} \equiv_{\infty,\kappa} \mathcal{N}$ for some $\mathcal{N} \in \mathbb{K}$, by the definition of σ. Since $(\mathbb{K}, \prec_{\mathbb{K}})$ has finite character, Theorem 2.2.8 implies that $\mathcal{M} \in \mathbb{K}$, as desired. $\square$

Chapter 3

Infinitary Game Logic

For the entirety of this chapter, assume $(\mathbb{K}, \prec_{\mathbb{K}})$ is an abstract elementary class with a Löwenheim-Skolem number of κ where κ is cofinal with ω. The main goal of this chapter is to find an axiomatization of $\mathbb{K}$ by a sentence incorporating game quantification.

3.1 $L(\kappa)$

The following definition is a generalization of Keisler's game logic $L(\omega)$ ([7]).

Definition 3.1.1. Define the infinitary game logic $L(\kappa)$ as follows:

1. All atomic formulas are in $L(\kappa)$.

2. If $\varphi \in L(\kappa)$ then $\neg\varphi \in L(\kappa)$.

3. If $\Phi \subseteq L(\kappa)$ then $\bigwedge \Phi, \bigvee \Phi \in L(\kappa)$.

4. If $\varphi \in L(\kappa)$ then $\exists\bar{x}\varphi, \forall\bar{x}\varphi \in L(\kappa)$ where $|\bar{x}| < \kappa$ provided they have just finitely many free variables.

5. If $\varphi \in L(\kappa)$ then $Q_0\bar{x}_0 \ldots Q_n\bar{x}_n \ldots \varphi \in L(\kappa)$ for $n < \omega$ where each Q_n is either $\forall$ or $\exists$ and $|\bar{x}_i| < \kappa$ provided the formula has just finitely many free variables.

Let $\mathcal{M}_0$ be an arbitrary $\mathbb{K}$-structure of cardinality κ and let $\bar{a}$ be a κ-sequence from $\mathcal{M}_0$ such that $ran(\bar{a}) = \mathcal{M}_0$. The following two lemmas prove the existence of specific quantifier-free formulas in $L(\kappa)$ necessary to create the sentence axiomatizing $\mathbb{K}$.

Lemma 3.1.2. *There exists a quantifier-free formula $\alpha^{\mathcal{M}_0,\bar{a}}(\bar{x},\bar{y}) \in L(\kappa)$ such that for every L-structure $\mathcal{N}$ and all κ-sequences $\bar{c}$, $\bar{d}$ from $\mathcal{N}$, $\mathcal{N} \models \alpha^{\mathcal{M}_0,\bar{a}}(\bar{c},\bar{d})$ IFF the map defined by $h(a_i) = c_i$ for all $i \in \kappa$ defines an isomorphism of $\mathcal{M}_0$ onto $\mathcal{N}_0 = ran(\bar{c})$ and $\mathcal{N}_0 \prec_{\mathbb{K}} \mathcal{N}_1$ where $\mathcal{N}_1 = ran(\bar{d})$ and is of cardinality κ.*

Proof. Let $\{(\mathcal{A}_i, \mathcal{M}_0) : i \in I\}$ list, up to isomorphism, all pairs of $\mathbb{K}$-structures such that $\mathcal{M}_0 \prec_{\mathbb{K}} \mathcal{A}_i$ and $|\mathcal{A}_i| = \kappa$. For any $i \in I$ and $\bar{b}_i \subseteq \mathcal{A}_i$ a κ-sequence such that $ran(\bar{b}_i) = \mathcal{A}_i$, define:

$$\alpha_i^{\bar{b}_i}(\bar{x},\bar{y}) = \bigwedge\{\beta(\bar{x},\bar{y}) : \beta \text{ is a basic open formula and } \mathcal{A}_i \models \beta(\bar{a},\bar{b}_i)\}$$

Observe that an L-structure $\mathcal{N} \models \alpha_i^{\bar{b}_i}(\bar{c},\bar{d})$ if and only if $ran(\bar{c}) = \mathcal{N}_0$, $ran(\bar{d}) = \mathcal{B}_i$ and $(\mathcal{A}_i, \mathcal{M}_0) \cong (\mathcal{B}_i, \mathcal{N}_0)$ under the map defined by $h(b_{i_j}) = d_j$ for all $j \in \kappa$ (which, of course, implies $h(a_i) = c_i$ for all $i \in \kappa$).

Let $\{\bar{b}_i^j : j \in J\}$ list all enumerations of $\mathcal{A}_i$. Define $\alpha_i^{\mathcal{M}_0,\bar{a}}(\bar{x},\bar{y}) = \bigvee\{\alpha_i^{\bar{b}_i^j}(\bar{x},\bar{y}) : j \in J\}$. Furthermore, define $\alpha^{\mathcal{M}_0,\bar{a}} = \bigvee\{\alpha_i^{\mathcal{M}_0,\bar{a}}(\bar{x},\bar{y}) : i \in I\}$. This formula then clearly has the desired properties. $\square$

Lemma 3.1.3. *For each $n \in \omega$ and $i_0 < \ldots < i_n < \kappa$, there is a quantifier-free formula $\gamma_{i_0,\ldots i_n}^{\mathcal{M}_0,\bar{a}}(x_{i_0},\ldots x_{i_n},\bar{y}) \in L(\kappa)$ such that for any L-structure $\mathcal{N}$, $\mathcal{N} \models \gamma_{i_0,\ldots i_n}^{\mathcal{M}_0,\bar{a}}(c_0,\ldots c_n,\bar{b})$ IFF $c_k = b_{i_k}$ for all $k \leq n$, $ran(\bar{b}) = \mathcal{N}_0$ and $(\mathcal{M}_0, a_{i_0},\ldots a_{i_n}) \cong (\mathcal{N}_0, c_0,\ldots c_n)$.*

Proof. As in the previous lemma, define:

$$\gamma^{\mathcal{M}_0,\bar{a}}_{i_0,\dots i_n} = \bigwedge\{\beta(x_{i_0},\dots x_{i_n},\bar{y}) : \beta \text{ basic open formula s.t. } \mathcal{M}_0 \models \beta(a_{i_0},\dots a_{i_n},\bar{a})\}$$

By construction, if $\mathcal{N} \models \gamma^{\mathcal{M}_0,\bar{a}}_{i_0,\dots i_n}(c_0,\dots c_n,\bar{b})$ then $ran(\bar{b}) = \mathcal{N}_0$ such that $(\mathcal{M}_0, a_{i_0},\dots a_{i_n}) \cong (\mathcal{N}_0, c_0,\dots c_n)$. Furthermore, the c_k's must correspond to the same positions in the sequence $\bar{b}$ that the a_{i_k}'s do in the sequence $\bar{a}$. Thus, $c_k = b_{i_k}$ for all $k \leq n$.

Conversely, if $ran(\bar{b}) = \mathcal{N}_0$, $(\mathcal{M}_0, a_{i_0},\dots a_{i_n}) \cong (\mathcal{N}_0, c_0,\dots c_n)$ and $c_k = b_{i_k}$ for all $k \leq n$ then $\mathcal{M}_0 \models \beta(a_{i_0},\dots a_{i_n},\bar{a})$ IFF $\mathcal{N} \models \beta(c_0,\dots c_n,\bar{b})$ for every β. Thus, $\mathcal{N} \models \gamma^{\mathcal{M}_0,\bar{a}}_{i_0,\dots i_n}(c_0,\dots c_n,\bar{b})$. □

The following lemma combines the formulas from the previous two lemmas to show the existence of a formula determining that almost all κ-approximations contain a copy of $\mathcal{M}_0$ and and the elements satisfying the formula are the images of $a_{i_0},\dots a_{i_n}$.

Lemma 3.1.4. *For each $n \in \omega$ and $i_0 < \dots < i_n < \kappa$, there exists a formula $\varphi^{\mathcal{M}_0,a_{i_0},\dots a_{i_n}} \in L(\kappa)$ such that for every L-structure $\mathcal{N}$ and $c_0,\dots c_n \in \mathcal{N}$, $\mathcal{N} \models \varphi^{\mathcal{M}_0,a_{i_0},\dots a_{i_n}}(c_0,\dots c_n)$ IFF [there exists a $\mathbb{K}$-embedding h of $\mathcal{M}_0$ into $\mathcal{N}^s$ such that $h(a_{i_k}) = c_k$ for all $k \leq n$] κ-a.e.*

Proof. Define the following quantifier-free formula from $L(\kappa)$:

$$\chi^{\mathcal{M}_0,\bar{a}}_{i_0,\dots i_n}(x_{i_0},\dots x_{i_n},\bar{y}) = \bigvee_{f:\kappa\to\kappa}\left[\gamma^{\mathcal{M}_0,\bar{a}}_{i_0,\dots i_n}(x_{i_0},\dots x_{i_n},y_{f(0)},\dots) \wedge \alpha^{\mathcal{M}_0,\bar{a}}(y_{f(0)},\dots,\bar{y})\right]$$

Observe that $\mathcal{N} \models \chi^{\mathcal{M}_0,\bar{a}}_{i_0,\dots i_n}(c_0,\dots c_n,\bar{b})$ if and only if for some function $f : \kappa \to \kappa$, $c_k = b_{f(i_k)}$ for all $k \leq n$, $ran(b_{f(0)},\dots) = \mathcal{N}_0$, the map $h(a_i) = b_{f(i)}$ is an isomorphism of $\mathcal{M}_0$ onto $\mathcal{N}_0$ and $\mathcal{N}_0 \prec_{\mathbb{K}} \mathcal{N}_1$ where $\mathcal{N}_1 = ran(\bar{b})$ by the previous 2 lemmas.

Define:

$$\varphi^{\mathcal{M}_0, a_{i_0}, \ldots a_{i_n}}(x_{i_0}, \ldots x_{i_n}) = \forall \bar{y}_0 \exists \bar{y}_1 \ldots \forall \bar{y}_{2n} \exists \bar{y}_{2n+1} \ldots \chi^{\mathcal{M}_0, \bar{a}}_{i_0, \ldots i_n}(x_{i_0}, \ldots x_{i_n}, \bar{y})$$

where $|\bar{y}_{2j}| = |\bar{y}_{2j+1}| = \kappa_j$ for all $j \in \omega$.

($\Rightarrow$): Assume $\mathcal{N} \models \varphi^{\mathcal{M}_0, a_{i_0}, \ldots a_{i_n}}(c_0, \ldots c_n)$. We desire to show [there is a $\mathbb{K}$-embedding h of $\mathcal{M}_0$ into $\mathcal{N}^s$ such that $h(a_{i_k}) = c_k$ for all $k \leq n$] κ-a.e.

Let $X = \{s \in \mathcal{P}_{\kappa^+}(\mathcal{N})$: there exists a $\mathbb{K}$-embedding h of $\mathcal{M}_0$ into $\mathcal{N}^s$ s.t. $h(a_{i_k}) = c_k\}$. We will show $X \in D_{\kappa^+}(\mathcal{N})$ by demonstrating a winning strategy for player II^*_X in the game $G^*_\kappa(X)$. We proceed by playing 2 parallel games and describing the strategy by induction.

<u>Base Case</u>: $n = 0$

Assume player I^*_X has chosen $s_0 \in \mathcal{P}_\kappa(\mathcal{N})$ of cardinality $\leq \kappa_0$. Without loss of generality we may assume $|s_0| = \kappa_0$ (if $|s_0| < \kappa_0$ then allow an element of s_0 to repeat κ_0-many times in $\bar{y}_0$ below). Since $\mathcal{N} \models \varphi^{\mathcal{M}_0, a_{i_0}, \ldots a_{i_n}}(c_0, \ldots c_n)$, player II_φ has a winning strategy in the ω-length game defined by $\varphi^{\mathcal{M}_0, a_{i_0}, \ldots a_{i_n}}(c_0, \ldots c_n)$. Allow player I_φ to choose $\bar{y}_0 = s_0$ (in any order). Player II_φ can then use his winning strategy to choose $\bar{y}_1 \subseteq \mathcal{N}$ of cardinality κ_0. Player II^*_X will then respond to player I^*_X with $s_1 = ran(\bar{y}_1)$.

<u>Successor Stage</u>: Suppose s_i and $\bar{y}_i$ have been chosen for each $i < 2n$.

Assume player I^*_X has chosen $s_{2n} \in \mathcal{P}_\kappa(\mathcal{N})$ of cardinality κ_n. Allow player I_φ to choose $\bar{y}_{2n} = s_{2n}$ (in any order) and player II_φ will use his winning strategy to respond with $\bar{y}_{2n+1} \subseteq \mathcal{N}$ of cardinality κ_n. Player II^*_X will then respond to player I^*_X with $s_{2n+1} = ran(\bar{y}_{2n+1})$.

Let $s = \bigcup_{n\in\omega} s_n$. By our observation above, there is a $\mathbb{K}$-embedding h of $\mathcal{M}_0$ into $\mathcal{N}_1$ such that $h(a_{i_k}) = c_k$ for all $k \leq n$ and $\mathcal{N}_1 = \mathcal{N}^s$. Thus $s \in X$ as desired.

($\Leftarrow$): Fix $c_0, \ldots c_k \in \mathcal{N}$. Let

$$X = \{s \in \mathcal{P}_{\kappa^+}(\mathcal{N}) : \exists \text{ } \mathbb{K}\text{-embedding } h \text{ of } \mathcal{M}_0 \text{ into } \mathcal{N}^s \text{ s.t. } h(a_{i_k}) = c_k\}$$

We assume $X \in D_{\kappa^+}(\mathcal{N})$ and thus player II^*_X has a winning strategy in the game $G^*_\kappa(X)$. We desire to show that player II_φ has a winning strategy in the ω-length game defined by $\varphi^{\mathcal{M}_0,a_{i_0},\ldots a_{i_n}}(c_0,\ldots c_n)$ which will imply $\mathcal{N} \models \varphi^{\mathcal{M}_0,a_{i_0},\ldots a_{i_n}}(c_0,\ldots c_k)$. We again proceed by playing 2 parallel games and defining the strategy by induction.

Base Case: $n = 0$

Assume player I_φ has chosen $\bar{y}_0 \subseteq \mathcal{N}$ of length κ_0. Allow player I^*_X to choose $s_0 = ran(\bar{y}_0)$ and player II^*_X will use his winning strategy to respond with $s_1 \in \mathcal{P}_\kappa(\mathcal{N})$ of cardinality $\leq \kappa_0$ (again, without loss of generality, we may assume $|s_1| = \kappa_0$). Player II_φ will then respond to player I_φ with $\bar{y}_1 = s_1$ (in any order).

Successor Stage: Suppose s_i and $\bar{y}_i$ have been chosen for each $i < 2n$.

Assume player I_φ has chosen $\bar{y}_{2n} \subseteq \mathcal{N}$ of length κ_n. Allow player I^*_X to choose $s_{2n} = ran(\bar{y}_{2n})$ and player II^*_X will use his winning strategy to respond with $s_{2n+1} \in \mathcal{P}_\kappa(n)$ of length κ_n. Player II_φ will then respond to player I_φ with $\bar{y}_{2n+1} = s_{2n+1}$ (in any order).

Let $s = \bigcup_{i\in\omega} s_i$. Note that $\mathcal{N}^s = ran(\bar{y})$ and there is a $\mathbb{K}$-embedding h of $\mathcal{M}_0$ into $\mathcal{N}^s$ such that $h(a_{i_k}) = c_k$ for all $k \leq n$. Thus we get a function $f : \kappa \to \kappa$ such that $c_k = y_{f(i_k)}$ for all $k \leq n$ and $ran(y_{f(0)}, \ldots) = h(\mathcal{M}_0)$. From our observation above, it is clear that player II_φ has a winning strategy in the game defined by $\varphi^{\mathcal{M}_0, a_{i_0}, \ldots a_{i_n}}(c_0, \ldots c_n)$. $\square$

3.2 Axiomatizability

For the remainder of the chapter we continue to assume that $(\mathbb{K}, \prec_{\mathbb{K}})$ is an AEC with $LS(\mathbb{K}) = \kappa$ and $cof(\kappa) = \omega$ and we further assume that $\mathbb{K}$ has finite character. With this assumption, we will be able to use the formula from Lemma 3.1.4 to axiomatize $\mathbb{K}$ by a sentence of $L(\kappa)$. The following lemma is the key step to proving this result.

Lemma 3.2.1. *There is a formula $\varphi^{\mathcal{M}_0, \bar{a}}(\bar{x}) \in L(\kappa)$ such that for every L-structure $\mathcal{N}$ and every κ-sequence $\bar{b}$ from $\mathcal{N}$, $\mathcal{N} \models \varphi^{\mathcal{M}_0, \bar{a}}(\bar{b})$ IFF the mapping g defined by $g(a_k) = b_k$ for all $k \in \kappa$ is an isomorphism of $\mathcal{M}_0$ onto some $\mathcal{N}_0$ such that $\mathcal{N}_0 \prec_{\mathbb{K}} \mathcal{N}^s$ κ-a.e.*

Proof. Let $\varphi^{\mathcal{M}_0, \bar{a}}(\bar{x}) = \bigwedge_{n \in \omega} \bigwedge_{i_0 < \ldots i_n < \kappa} \varphi^{\mathcal{M}_0, a_{i_0}, \ldots a_{i_n}}(x_{i_0}, \ldots x_{i_n})$.

($\Rightarrow$): Assume $\mathcal{N} \models \varphi^{\mathcal{M}_0, \bar{a}}(\bar{b})$. By Lemma 3.1.4 and κ^+-completeness the map g is a $\mathbb{K}$-embedding of $\mathcal{M}_0$ into $\mathcal{N}^s$ κ-a.e. Then $\mathcal{N}_0 = g(\mathcal{M}_0)$ is as desired.

($\Leftarrow$): By assumption $X \in D_{\kappa^+}(\mathcal{N})$ where:

$$X = \{s \in \mathcal{P}_{\kappa^+}(\mathcal{N}) : \mathcal{N}_0 \prec_{\mathbb{K}} \mathcal{N}^s \text{ where } \mathcal{N}_0 = g(\mathcal{M}_0) \text{ defined by } g(a_k) = b_k \ \forall k \in \kappa\}$$

Hence for any $n \in \omega$ and $i_0 < \ldots < i_n < \kappa$, g defines a $\mathbb{K}$-embedding of $\mathcal{M}_0$ into $\mathcal{N}^s$ such that $g(a_{i_k}) = b_{i_k}$ for all $k \leq n$, κ-a.e. Lemma 3.1.4 implies

$\mathcal{N} \models \varphi^{\mathcal{M}_0, a_{i_0}, \ldots a_{i_n}}(b_{i_0}, \ldots b_{i_n})$ for all $n \in \omega$ and $i_0 < \ldots < i_n < \kappa$. Hence, $\mathcal{N} \models \varphi^{\mathcal{M}_0, \bar{a}}(\bar{b})$. $\square$

Lemma 3.2.2. *There exists a formula $\varphi(\bar{x}) \in L(\kappa)$ such that for every L-structure $\mathcal{N}$ and every κ-sequence $\bar{b}$ from $\mathcal{N}$, $\mathcal{N} \models \varphi(\bar{b})$ IFF $ran(\bar{b}) = \mathcal{N}_0$ for some $\mathcal{N}_0$ such that $\mathcal{N}_0 \prec_{\mathbb{K}} \mathcal{N}^s$ κ-a.e.*

Proof. Let $\varphi(\bar{x}) = \bigvee\{\varphi^{\mathcal{M}_0, \bar{a}}(\bar{x}) : \mathcal{M}_0 \in \mathbb{K}, |\mathcal{M}_0| = \kappa \text{ and } ran(\bar{a}) = \mathcal{M}_0\}$. By Lemma 3.2.1 we obtain the desired result. $\square$

Theorem 3.2.3. *There exists a sentence $\theta \in L(\kappa)$ such that for any L-structure, $\mathcal{N}$, $\mathcal{N} \models \theta$ IFF $\mathcal{N} \in \mathbb{K}$.*

Proof. Let $\theta = \forall \bar{x}_0 \exists \bar{x}_1 \ldots \forall \bar{x}_{2n} \exists \bar{x}_{2n+1} \ldots \varphi(\bar{x})$ where $|\bar{x}_{2j}| = |\bar{x}_{2j+1}| = \kappa_j$ for all $j \in \omega$.

($\Rightarrow$): Assume $\mathcal{N} \models \theta$. This means that player II_θ has a winning strategy in the ω-length game defined by θ. Let $S = \{\mathcal{N}_0 \subseteq \mathcal{N} : \mathcal{N}_0 \prec_{\mathbb{K}} \mathcal{N}^s \ \kappa\text{-a.e.}\}$. We desire to show that S is a directed family under $\prec_{\mathbb{K}}$ and that $\bigcup S = \mathcal{N}$.

First we claim that S is a directed family under $\subseteq$. Assume $\mathcal{M}_0, \mathcal{M}_1 \in S$. It suffices to show that there is a model $\mathcal{M}_2 \in S$ such that $\mathcal{M}_0$, $\mathcal{M}_1 \subseteq \mathcal{M}_2$. To do this, let player I_θ play the game defined by θ by listing all the elements of $\mathcal{M}_0 \cup \mathcal{M}_1$ while player II_θ always uses his winning strategy. As a result, we obtain $\bar{b} \subseteq \mathcal{N}$ such that $\mathcal{M}_0 \cup \mathcal{M}_1 \subseteq ran(\bar{b})$ and $\mathcal{N} \models \varphi(\bar{b})$. By Lemma 3.2.2, $ran(\bar{b}) = \mathcal{M}_2$ such that $\mathcal{M}_2 \prec_{\mathbb{K}} \mathcal{N}^s$ κ-a.e. Hence, $\mathcal{M}_2 \in S$ and is as desired.

Next we claim S is a directed family under $\prec_{\mathbb{K}}$. By the previous claim, it suffices to show that if $\mathcal{M}_0$, $\mathcal{M}_1 \in S$ and $\mathcal{M}_0 \subseteq \mathcal{M}_1$ then $\mathcal{M}_0 \prec_{\mathbb{K}} \mathcal{M}_1$. Since

$\mathcal{M}_0 \prec_{\mathbb{K}} \mathcal{N}^s$ κ-a.e. and $\mathcal{M}_1 \prec_{\mathbb{K}} \mathcal{N}^s$ κ-a.e., there exists a κ-approximation $\mathcal{N}^{s_0}$ such that $\mathcal{M}_0, \mathcal{M}_1 \prec_{\mathbb{K}} \mathcal{N}^{s_0}$. By the coherence axiom, we can conclude $\mathcal{M}_0 \prec_{\mathbb{K}} \mathcal{M}_1$.

Since player II_θ has a winning strategy in the game defined by θ, it is clear that for any $b \in \mathcal{N}$ there exists $\mathcal{N}_0 \in S$ such that $b \in \mathcal{N}_0$. Therefore, $\mathcal{N} = \bigcup S$ and hence $\mathcal{N} \in \mathbb{K}$ by Lemma 1.1.3.

($\Leftarrow$): Assume $\mathcal{N} \in \mathbb{K}$. We will demonstrate a winning strategy for player II_θ in the game defined by θ (and thus $\mathcal{N} \models \theta$). We proceed by induction.

<u>Base Case</u>: $n = 0$

Assume player I_θ has chosen $\bar{x}_0$ a sequence from $\mathcal{N}$ such that $|\bar{x}_0| = \kappa_0$. By the Löwenheim-Skolem axiom, there exists a model $\mathcal{N}_0 \prec_{\mathbb{K}} \mathcal{N}$ such that $ran(\bar{x}_0) \subseteq \mathcal{N}_0$ and $|\mathcal{N}_0| = \kappa$. Since $cof(\kappa) = \omega$, we can let $\mathcal{N}_0 = \bigcup_{i \in \omega} \bar{b}^0_i$ such that $\bar{b}^0_i \subseteq \bar{b}^0_{i+1}$ and $|\bar{b}^0_i| = \kappa_i$ for all $i \in \omega$. Player II_θ then responds with $\bar{x}_1 = \bar{b}^0_0$ (in any order).

<u>Successor Stage</u> Suppose $\bar{x}_i$ has been determined for each $i < 2n$.

Assume player I_θ has chosen $\bar{x}_{2n}$ a sequence from $\mathcal{N}$ such that $|\bar{x}_{2n}| = \kappa_n$. By the Löwenheim-Skolem axiom again, there exists a model $\mathcal{N}_n \prec_{\mathbb{K}} \mathcal{N}$ such that $\mathcal{N}_{n-1} \cup ran(\bar{x}_{2n}) \subseteq \mathcal{N}_n$ and $|\mathcal{N}_n| = \kappa$. Let $\mathcal{N}_n = \bigcup_{i \in \omega} \bar{b}^n_i$ such that $\bar{b}^n_i \subseteq \bar{b}^n_{i+1}$ and $|\bar{b}^n_i| = \kappa_i$ for all $i \in \omega$. Player II_θ then responds with $\bar{x}_{2n+1} = \bar{b}^0_n \cup \ldots \cup \bar{b}^n_n$ (in any order).

By construction, $\bigcup_{i \in \omega} ran(\bar{x}_i) = \bigcup_{i \in \omega} \mathcal{N}_i = \mathcal{M} \in \mathbb{K}$ and $\mathcal{M}$ is of cardinality κ. Furthermore, by the union axiom we get $\mathcal{M} \prec_{\mathbb{K}} \mathcal{N}$. Lemma 2.2.4 tells us that $\mathcal{M} \prec_{\mathbb{K}} \mathcal{N}^s$ κ-a.e. Lemma 3.2.2 further implies $\mathcal{N} \models \varphi(\bar{b})$ where $\bar{b} = \bigcup_{i \in \omega} ran(\bar{x}_i)$ (in any order). Hence, $\mathcal{N} \models \theta$, as desired. $\square$

Chapter 4

Type Saturation and Categoricity

For the duration of this chapter, we assume that $(\mathbb{K}, \prec_{\mathbb{K}})$ is an AEC satisfying the amalgamation and joint embedding properties and has arbitrarily large models. We will still assume that $LS(\mathbb{K}) = \kappa$ but we will no longer make the assumption that κ has a cofinality of ω unless explicitly stated. Furthermore, we will always assume that all models from $\mathbb{K}$ are $\mathbb{K}$-substructures of the monster model, $\mathbb{C}$.

4.1 Types and Saturation

In [14], Shelah defined a *galois type* of a sequence over a $\mathbb{K}$-structure as a generalization of first order types. We provide a more general definition of a galois type of a sequence over a set.

Definition 4.1.1. For a small sequence $\bar{a}$ from the monster model $\mathbb{C}$ (sufficiently smaller than $|\mathbb{C}|$) and a small subset B of $\mathbb{C}$, the *galois type of $\bar{a}$ over B*, denoted $tp^g(\bar{a}/B)$, is the set of images of $\bar{a}$ under automorphisms of $\mathbb{C}$ fixing B pointwise.

In addition, Shelah provided a definition of λ-galois saturated structures over models when $\lambda > \kappa$. We will also extend this definition of λ-galois saturated models to include the case where $\lambda = \kappa$. The case where $\lambda = \kappa = \omega$ can be found in both [5] and [10]. We provide both definitions below.

Definition 4.1.2.

1. *(Model Saturation)* A model $\mathcal{M}$ is *λ-galois saturated* for $\lambda > \kappa$ if for every $\mathcal{N} \prec_{\mathbb{K}} \mathcal{M}$ with $|\mathcal{N}| < \lambda$ and for every element $a \in \mathbb{C}$, $\mathcal{M} \models tp^g(a/\mathcal{N})$.

2. *(Set Saturation)* A model $\mathcal{M}$ is *λ-galois saturated* for $\lambda \geq \kappa$ if for every $< \lambda$-sequence $\bar{a} \subseteq \mathcal{M}$ and every $< \lambda$-sequence $\bar{b} \subseteq \mathbb{C}$, there exists $\bar{c} \subseteq \mathcal{M}$ such that $tp^g(\bar{a}\bar{b}/\emptyset) = tp^g(\bar{a}\bar{c}/\emptyset)$.

It is necessary now to show that the two definitions of galois saturation stated above are equivalent when $\lambda > \kappa$. The following lemma is essential to the proof. Details on this lemma can be found in [1].

Lemma 4.1.3. *A model $\mathcal{M} \in \mathbb{K}$ is λ-galois saturated as in definition 4.1.2(1) IFF $\mathcal{M}$ is λ-model homogeneous.*

Lemma 4.1.4. *For $\lambda > \kappa$, the two definitions from 4.1.2 agree.*

Proof.

$(1 \Rightarrow 2)$: Assume $\mathcal{M}$ is λ-galois saturated as in definition 4.1.2(1). Then $\mathcal{M}$ is λ-model homogeneous by Lemma 4.1.3. Let $\bar{a} \subseteq \mathcal{M}$ and $\bar{b} \subseteq \mathbb{C}$ be $< \lambda$-sequences. Since the Löwenheim-Skolem number is less than λ, there exists $\mathcal{N} \prec_{\mathbb{K}} \mathcal{M}$ such that $\bar{a} \subseteq \mathcal{N}$ and $|\mathcal{N}| < \lambda$. By the same reasoning there exists $\mathcal{N}' \in \mathbb{K}$ such that $\mathcal{N} \prec_{\mathbb{K}} \mathcal{N}' \prec_{\mathbb{K}} \mathbb{C}$ and $\bar{b} \subseteq \mathcal{N}'$ and $|\mathcal{N}'| < \lambda$. By the λ-model homogeneity of $\mathcal{M}$, there is a $\mathbb{K}$-embedding h of $\mathcal{N}'$ into $\mathcal{M}$ fixing $\mathcal{N}$ (and thus fixing $\bar{a}$) pointwise. This isomorphism naturally extends to an automorphism of $\mathbb{C}$. Let $\bar{c} = h(\bar{b}) \subseteq \mathcal{M}$. Thus, by definition, $tp^g(\bar{a}\bar{b}/\emptyset) = tp^g(\bar{a}\bar{c}/\emptyset)$.

$(2 \Rightarrow 1)$: Assume $\mathcal{M}$ is λ-galois saturated as in definition 4.1.2(2). Assume further that $\mathcal{N} \prec_{\mathbb{K}} \mathcal{M}$ with $|\mathcal{N}| < \lambda$ and $b \in \mathbb{C}$. Write $\mathcal{N}$ as a sequence $\bar{a}$ such that

$|\bar{a}| < \lambda$. Since $\mathcal{M}$ is λ-galois saturated in the sense of (2), there exists $c \in \mathcal{M}$ such that $tp^g(\bar{a}b/\emptyset) = tp^g(\bar{a}c/\emptyset)$. Hence there exists $h \in aut(\mathbb{C})$ such that h fixes $\bar{a}$ pointwise (and thus fixes $\mathcal{N}$ pointwise) and sends b to c. Therefore, $\mathcal{M} \models tp^g(b/\mathcal{N})$ as desired. $\square$

Remark 4.1.5. While model saturation and set saturation agree when $\lambda > \kappa$, set saturation is usually a stronger assumption when $\lambda = \kappa$, since $\mathbb{K}$ may contain no models of cardinality less than λ.

We proceed to develop properties of κ-galois saturated models.

Theorem 4.1.6. ($cof(\kappa) = \omega$) *Let $\mathcal{M}$, $\mathcal{N} \in \mathbb{K}$ and $\bar{a}$, $\bar{b}$ be $< \kappa$-sequences of the same length from $\mathcal{M}$ and $\mathcal{N}$, respectively. If $(\mathcal{M}, \bar{a}) \equiv_{\infty,\kappa} (\mathcal{N}, \bar{b})$ then $tp^g(\bar{a}/\emptyset) = tp^g(\bar{b}/\emptyset)$.*

Proof. By assumption, $\mathcal{M}$, $\mathcal{N} \prec_{\mathbb{K}} \mathbb{C}$ and $(\mathcal{M}, \bar{a}) \equiv_{\infty,\kappa} (\mathcal{N}, \bar{b})$. From Lemma 2.2.3 applied to $(\mathcal{M}, \bar{a})$ and $(\mathcal{N}, \bar{b})$, we obtain $(\mathcal{M}^s, \bar{a}) \cong (\mathcal{N}^s, \bar{b})$ κ-a.e. In addition, by Lemma 2.2.4, $\mathcal{M}^s \prec_{\mathbb{K}} \mathcal{M}$ κ-a.e. and $\mathcal{N}^s \prec_{\mathbb{K}} \mathcal{N}$ κ-a.e. Thus, for some s, $\mathcal{M}^s \prec_{\mathbb{K}} \mathcal{M}$, $\mathcal{N}^s \prec_{\mathbb{K}} \mathcal{N}$ and $(\mathcal{M}^s, \bar{a}) \cong (\mathcal{N}^s, \bar{b})$. This isomorphism of $\mathbb{K}$-structures naturally extends to an automorphism of $\mathbb{C}$ since $\mathcal{M}^s$, $\mathcal{N}^s \prec_{\mathbb{K}} \mathbb{C}$ and $\mathbb{C}$ is strongly homogeneous. Hence, $tp^g(\bar{a}/\emptyset) = tp^g(\bar{b}/\emptyset)$. $\square$

The following lemma will enable us to prove many properties using back-and-forth arguments.

Lemma 4.1.7. *Assume $\mathcal{M} \in \mathbb{K}$ is κ-galois saturated. Let $\mathcal{N} \in \mathbb{K}$, and let $\bar{a} \subseteq \mathcal{M}$ and $\bar{b} \subseteq \mathcal{N}$ be $< \kappa$-sequences of the same length. Further assume that $tp^g(\bar{a}/\emptyset) =$*

$tp^g(\bar{b}/\emptyset)$. *Then, for any* $< \kappa$*-sequence* $\bar{d} \subseteq \mathcal{N}$ *there exists* $\bar{c} \subseteq \mathcal{M}$ *such that* $tp^g(\bar{a}\bar{c}/\emptyset) = tp^g(\bar{b}\bar{d}/\emptyset)$.

Proof. By assumption, $\mathcal{M}$, $\mathcal{N} \prec_{\mathbb{K}} \mathbb{C}$ and $tp^g(\bar{a}/\emptyset) = tp^g(\bar{b}/\emptyset)$. Thus there exists $h \in aut(\mathbb{C})$ such that $h(\bar{a}) = \bar{b}$. If we let $\bar{c}' = h^{-1}(\bar{d})$ then $tp^g(\bar{a}\bar{c}'/\emptyset) = tp^g(\bar{b}\bar{d}/\emptyset)$. Since $\mathcal{M}$ is κ-galois saturated, there exists $\bar{c} \subseteq \mathcal{M}$ such that $tp^g(\bar{a}\bar{c}/\emptyset) = tp^g(\bar{a}\bar{c}'/\emptyset)$, as desired. □

Assuming that $\mathcal{M}$ is κ-galois saturated, we are now able to prove the following biconditional strengthening of Theorem 2.2.8.

Theorem 4.1.8. (Finite Character) *Assume that* $\mathcal{M} \in \mathbb{K}$ *is* κ*-galois saturated.*

1. *If* $\mathcal{N} \in \mathbb{K}$ *is* κ*-galois saturated then* $\mathcal{M} \equiv_{\infty,\kappa} \mathcal{N}$

2. $(cof(\kappa) = \omega)$ *If* $\mathcal{N} \equiv_{\infty,\kappa} \mathcal{M}$ *then* $\mathcal{N}$ *is a* κ*-galois saturated* $\mathbb{K}$*-structure.*

Proof.

(1): Assume $\mathcal{N} \in \mathbb{K}$ is κ-galois saturated. By Theorem 1.2.6 it suffices to show that player II always has a winning strategy in the ω-length game $G_\kappa(\mathcal{M},\mathcal{N})$.

<u>Base Case</u>: $n = 0$

Assume player I has chosen $\bar{a}^0 \subseteq \mathcal{M}$ (or $\bar{b}^0 \subseteq \mathcal{N}$) a sequence of length $< \kappa$. Since $\mathcal{M}$ and $\mathcal{N}$ are both κ-galois saturated, there exists $\bar{b}^0 \subseteq \mathcal{N}$ (or $\bar{a}^0 \subseteq \mathcal{M}$) such that $tp^g(\bar{a}^0/\emptyset) = tp^g(\bar{b}^0/\emptyset)$. Player II will then choose $\bar{b}^0$ (or $\bar{a}^0$) in response to player I's move.

<u>Successor Stage</u>: Suppose $\bar{a}^i$ and $\bar{b}^i$ have been chosen already for $i \leq n$ so that $tp^g(\bar{a}^0 \ldots \bar{a}^n/\emptyset) = tp^g(\bar{b}^0 \ldots \bar{b}^n/\emptyset)$.

Without loss of generality, assume player I has chosen $\bar{a}^{n+1} \subseteq \mathcal{M}$ a sequence of length $< \kappa$. By the inductive hypothesis, $tp^g(\bar{a}^0 \ldots \bar{a}^n/\emptyset) = tp^g(\bar{b}^0 \ldots \bar{b}^n/\emptyset)$. Lemma 4.1.7 implies that there exists $\bar{b}^{n+1} \subseteq \mathcal{N}$ such that $tp^g(\bar{a}^0 \ldots \bar{a}^{n+1}/\emptyset) = tp^g(\bar{b}^0 \ldots \bar{b}^{n+1}/\emptyset)$. Player II will then choose $\bar{b}^{n+1}$ in response to player I's move.

At any finite stage there is an automorphism of $\mathbb{C}$ sending $\bar{a}^0, \ldots \bar{a}^n$ to $\bar{b}^0, \ldots \bar{b}^n$, hence the map h such that $h(\bar{a}^i) = \bar{b}^i$ for all $i \in \omega$ is a partial isomorphism and thus player II wins. Hence, $\mathcal{M} \equiv_{\infty,\kappa} \mathcal{N}$.

(2): Assume $\mathcal{M} \equiv_{\infty,\kappa} \mathcal{N}$ and thus $\mathcal{N} \in \mathbb{K}$ by Theorem 2.2.8.

Let $\bar{b} \subseteq \mathcal{N}$ and $\bar{d}' \subseteq \mathbb{C}$ be sequences of length $< \kappa$. By Theorem 1.2.6 there exists $\bar{a} \subseteq \mathcal{M}$ such that $(\mathcal{M}, \bar{a}) \equiv_{\infty,\kappa} (\mathcal{N}, \bar{b})$. Hence, by Theorem 4.1.6, $tp^g(\bar{a}/\emptyset) = tp^g(\bar{b}/\emptyset)$. Since $\mathcal{M}$ is κ-galois saturated, by Lemma 4.1.7 we know there exists $\bar{c} \subseteq \mathcal{M}$ such that $tp^g(\bar{a}\bar{c}/\emptyset) = tp^g(\bar{a}\bar{d}'/\emptyset)$. Using Theorem 1.2.6 again, we can find $\bar{d} \subseteq \mathcal{N}$ such that $(\mathcal{M}, \bar{a}\bar{c}) \equiv_{\infty,\kappa} (\mathcal{N}, \bar{b}\bar{d})$. Theorem 4.1.6 implies $tp^g(\bar{a}\bar{c}/\emptyset) = tp^g(\bar{b}\bar{d}/\emptyset)$. Thus $\bar{d}$ is as desired to show $\mathcal{N}$ is κ-galois saturated. □

Under the assumption that $\mathcal{M}$ and $\mathcal{N}$ are κ-galois saturated, we can also find a biconditional strengthening of several of our previous theorems. The following is a strengthening of Theorem 4.1.6

Corollary 4.1.9. *Let $\mathcal{M}$, $\mathcal{N} \in \mathbb{K}$ be κ-galois saturated. Also, let $\bar{a} \subseteq \mathcal{M}$ and $\bar{b} \subseteq \mathcal{N}$ be sequences of length $< \kappa$.*

1. *If $tp^g(\bar{a}/\emptyset) = tp^g(\bar{b}/\emptyset)$ then $(\mathcal{M}, \bar{a}) \equiv_{\infty,\kappa} (\mathcal{N}, \bar{b})$.*

2. *$(cof(\kappa) = \omega)$ If $(\mathcal{M}, \bar{a}) \equiv_{\infty,\kappa} (\mathcal{N}, \bar{b})$ then $tp^g(\bar{a}/\emptyset) = tp^g(\bar{b}/\emptyset)$.*

Proof.

(1): The proof is identical to the proof of part (1) of Theorem 4.1.8. Following the same steps it can easily be seen that we can construct a winning strategy for player II in the game $G_\kappa((\mathcal{M},\bar{a}),(\mathcal{N},\bar{b}))$. Hence $(\mathcal{M},\bar{a}) \equiv_{\infty,\kappa} (\mathcal{N},\bar{b})$.

(2): Done by Theorem 4.1.6 $\square$

The next corollary strengthens Theorem 2.2.9

Corollary 4.1.10. (Finite Character) *Let $\mathcal{M} \in \mathbb{K}$ be κ-galois saturated and $\mathcal{M}_0 \prec_{\mathbb{K}} \mathcal{M}$. Further let $\bar{a}$ be a sequence from $\mathcal{M}$ such that $ran(\bar{a}) = \mathcal{M}_0$. Finally, let $\mathcal{N}$ be an arbitrary L-structure and $\bar{b} \subseteq \mathcal{N}$ be a sequence of length $|\bar{a}|$.*

1. *($cof(\kappa) = \omega$) If $(\mathcal{M}, a_{i_0}, \dots a_{i_n}) \equiv_{\infty,\kappa} (\mathcal{N}, b_{i_0}, \dots b_{i_n})$ for all $i_0, \dots i_n \in |\bar{a}|$ and for all $n \in \omega$ then the map defined by $h(a_i) = b_i$ for all $i \in |\bar{a}|$ is a $\mathbb{K}$-embedding of $\mathcal{M}_0$ into $\mathcal{N}$.*

2. *If the map defined by $h(a_i) = b_i$ for all $i \in |\bar{a}|$ is a $\mathbb{K}$-embedding of $\mathcal{M}_0$ into $\mathcal{N}$ and $\mathcal{N}$ is κ-galois saturated then $(\mathcal{M}, a_{i_0}, \dots a_{i_n}) \equiv_{\infty,\kappa} (\mathcal{N}, b_{i_0}, \dots b_{i_n})$ for all $i_0, \dots i_n \in |\bar{a}|$ and for all $n \in \omega$.*

Proof.

(1): Done by Theorem 2.2.9

(2): Since h is a $\mathbb{K}$-embedding by assumption, for all $n \in \omega$ $tp^g(a_{i_0}, \dots a_{i_n}/\emptyset) = tp^g(b_{i_0}, \dots b_{i_n}/\emptyset)$. By Corollary 4.1.9 $(\mathcal{M}, a_{i_0}, \dots a_{i_n}) \equiv_{\infty,\kappa} (\mathcal{N}, b_{i_0}, \dots b_{i_n})$. $\square$

The final corollary is a strengthening of Corollary 2.2.10.

Corollary 4.1.11. (Finite Character) *Assume that $\mathcal{M}$, $\mathcal{N} \in \mathbb{K}$ are κ-galois saturated.*

1. *If $\mathcal{M} \prec_{\mathbb{K}} \mathcal{N}$ then $\mathcal{M} \prec_{\infty,\kappa} \mathcal{N}$.*

2. *($cof(\kappa) = \omega$) If $\mathcal{M} \prec_{\infty,\kappa} \mathcal{N}$ then $\mathcal{M} \prec_{\mathbb{K}} \mathcal{N}$.*

Proof.

(1): Assume $\mathcal{M} \prec_{\mathbb{K}} \mathcal{N}$. Let $\bar{a} \subseteq \mathcal{M}$ be an arbitrary sequence of length $< \kappa$. Since $\mathcal{M}$ is a $\mathbb{K}$-substructure of $\mathcal{N}$, $tp^g_{\mathcal{M}}(\bar{a}/\emptyset) = tp^g_{\mathcal{N}}(\bar{a}/\emptyset)$. Corollary 4.1.10 implies $(\mathcal{M}, \bar{a}) \equiv_{\infty,\kappa} (\mathcal{N}, \bar{a})$. Since $\bar{a}$ was an arbitrary $< \kappa$-sequence, we get $\mathcal{M} \prec_{\infty,\kappa} \mathcal{N}$.

(2): Done by Corollary 2.2.10. □

4.2 Categoricity And Axiomatizability

In this section we will prove that under the assumptions of finite character, $cof(\kappa) = \omega$, (AP, etc.) and categoricity, we can find a complete sentence of $L_{\infty,\kappa}$ closely approximating $(\mathbb{K}, \prec_{\mathbb{K}})$. First, we define the notion of stability in the context of an AEC.

Definition 4.2.1. Let $\lambda \geq \kappa$. We say that $(\mathbb{K}, \prec_{\mathbb{K}})$ is λ-*galois stable* IFF for every structure $\mathcal{N} \in \mathbb{K}$ such that $|\mathcal{N}| \leq \lambda$, there are at most λ-many galois types of *finite tuples* over $\mathcal{N}$.

The following technical lemma about stability can be found in [1]. We state it without proof.

Lemma 4.2.2. (AP, etc.) *If $\mathbb{K}$ is λ-categorical for $\lambda > \kappa$ then $\mathbb{K}$ is σ-galois stable for all $\sigma < \lambda$.*

If $(\mathbb{K}, \prec_{\mathbb{K}})$ is σ-galois stable for $LS(\mathbb{K}) \leq \sigma$, then for every $\mathbb{K}$-structure $\mathcal{M}_0$ of cardinality σ, there is a σ-universal model over $\mathcal{M}_0$ of cardinality σ. We provide the definition and proof of existence below, which can also be found in [1].

Definition 4.2.3. Let $\mathcal{M}_0, \mathcal{M}_1 \in \mathbb{K}$. We say $\mathcal{M}_1$ is *σ-universal over $\mathcal{M}_0$* if $\mathcal{M}_0 \prec_{\mathbb{K}} \mathcal{M}_1$ and whenever $\mathcal{M}_0 \prec_{\mathbb{K}} \mathcal{N}$ with $|\mathcal{N}| \leq \sigma$, there exists a $\mathbb{K}$-embedding of $\mathcal{N}$ into $\mathcal{M}_1$ fixing $\mathcal{M}_0$.

Theorem 4.2.4. *Let $(\mathbb{K}, \prec_{\mathbb{K}})$ be an AEC that is κ-galois stable for $LS(\mathbb{K}) \leq \kappa$. Assume $\mathcal{M}_0 \in \mathbb{K}$ of cardinality κ. Then there exists a model $\mathcal{M} \in \mathbb{K}$ that is κ-universal over $\mathcal{M}_0$ and of cardinality κ.*

Proof. Fix $\mathcal{M}_0 \prec_{\mathbb{K}} \mathbb{C}$ of cardinality κ. By κ-galois stability and the Löwenheim-Skolem axiom we can find a model $\mathcal{M}_1 \in \mathbb{K}$ such that $\mathcal{M}_0 \prec_{\mathbb{K}} \mathcal{M}_1 \prec_{\mathbb{K}} \mathbb{C}$, $|\mathcal{M}_1| = \kappa$ and $\mathcal{M}_1$ realizes all 1-galois types over $\mathcal{M}_0$. Continue this construction for κ-many steps, taking unions at limits, and let $\mathcal{M} = \bigcup_{i \in \kappa} \mathcal{M}_i$. We claim $\mathcal{M}$ is as desired.

Let $\mathcal{N} \in \mathbb{K}$ be such that $\mathcal{M}_0 \prec_{\mathbb{K}} \mathcal{N} \prec_{\mathbb{K}} \mathbb{C}$ and $|\mathcal{N}| = \kappa$. Enumerate $\mathcal{N} \setminus \mathcal{M}_0$ as $\{a_j\}_{j \in \kappa}$. We will create a κ-chain of isomorphisms extending each other and find the desired $\mathbb{K}$-embedding by taking the union. We proceed by induction.

Base Case: Let $\mathcal{N}_0 = \mathcal{M}_0$ and $f_0 = id_{\mathcal{M}_0}$. Let $\hat{f}_0 = id_{\mathbb{C}}$.

Successor Stage: Suppose we have constructed and defined $\mathcal{N}_i \prec_{\mathbb{K}} \mathbb{C}$ and an isomorphism f_i of $\mathcal{M}_i$ onto $\mathcal{N}_i$. Let $\hat{f}_i$ be an automorphism of $\mathbb{C}$ extending f_i.

Let j be the least such that $a_j \notin \mathcal{N}_i$. Note that $tp^g(\hat{f}_i^{-1}(a_j)/f_i^{-1}(\mathcal{N}_i)) = tp^g(\hat{f}_i^{-1}(a_j)/\mathcal{M}_i)$. By construction there exists an element $b_j \in \mathcal{M}_{i+1}$ such that $b_j \models tp^g(\hat{f}_i^{-1}(a_j)/\mathcal{M}_i)$. Hence there exists an automorphism g of $\mathbb{C}$ extending f_i and sending b_j to a_j. Let $f_{i+1} = g \restriction \mathcal{M}_{i+1}$ and $\mathcal{N}_{i+1} = f_{i+1}(\mathcal{M}_{i+1})$.

Limit Stage: For $\delta < \kappa$ a limit ordinal, let $\mathcal{M}_\delta = \bigcup_{i<\delta} \mathcal{M}_i$, $f_\delta = \bigcup_{i<\delta} f_i$ and $\mathcal{N}_\delta = \bigcup_{i<\delta} \mathcal{N}_i = f_\delta(\mathcal{M}_\delta)$.

Let $f = \bigcup_{i<\kappa} f_i$ and $\mathcal{N}' = \bigcup_{i<\kappa} \mathcal{N}_i$. Then $\mathcal{M}_0 \prec_{\mathbb{K}} \mathcal{N} \prec_{\mathbb{K}} \mathbb{C}$ and $\mathcal{M}_0 \prec_{\mathbb{K}} \mathcal{N}' = f(\mathcal{M}) \prec_{\mathbb{K}} \mathbb{C}$ and $\mathcal{N} \subseteq \mathcal{N}'$. The coherence axiom implies that $\mathcal{N} \prec_{\mathbb{K}} \mathcal{N}'$. Then f^{-1} is the desired map because $\mathcal{M}_0 \prec_{\mathbb{K}} f^{-1}(\mathcal{N} \prec_{\mathbb{K}} \mathcal{M} = f^{-1}(\mathcal{N}')$. □

The next lemma can also be found in [1]. We provide the details of the proof below to both illustrate the standard techniques used in AECs to prove saturation and to incorporate our new (equivalent) definition of galois saturation.

Lemma 4.2.5. *Assume $\mathbb{K}$ is λ-categorical for $\lambda > \kappa$ and $cof(\lambda) \geq \kappa$. Then the model of size λ is κ-galois saturated.*

Proof. Choose $\mathcal{M}_0 \prec_{\mathbb{K}} \mathbb{C}$ of cardinality less than λ, by the Löwenheim-Skolem axiom.

By Lemma 4.2.2 $\mathbb{K}$ is σ-galois stable for each $\sigma < \lambda$, thus by Lemma 4.2.4 we can find $\mathcal{M}_1 \in \mathbb{K}$ such that $\mathcal{M}_1$ is universal over $\mathcal{M}_0$ and also has the same cardinality as $\mathcal{M}_0$. Continue this process, taking unions at limits, for each $i < \lambda$ and let $\mathcal{M} = \bigcup_{i\in\lambda} \mathcal{M}_i$ (note that $|\mathcal{M}| = \lambda$).

Let $\bar{a} \subseteq \mathcal{M}$ be a sequence of length less than κ and $\bar{b} \subseteq \mathbb{C}$ be another sequence of length less than κ. Since $cof(\lambda) \geq \kappa$, for some $i \in \lambda$, $\bar{a} \subseteq \mathcal{M}_i$. Let

$\mathcal{M}_i \prec_{\mathbb{K}} \mathcal{N} \prec_{\mathbb{K}} \mathbb{C}$ such that $\bar{b} \subseteq \mathcal{N}$ and $|\mathcal{N}| = |\mathcal{M}_i|$ by the Löwenheim-Skolem axiom. By construction, $|\mathcal{M}_{i+1}| = |\mathcal{M}_i|$ and $\mathcal{M}_{i+1}$ is universal over $\mathcal{M}_i$. Let f be a $\mathbb{K}$-embedding of $\mathcal{N}$ into $\mathcal{M}_{i+1}$ fixing $\mathcal{M}_i$ pointwise (and, in particular, fixing $\bar{a}$ pointwise). Thus, $tp^g(\bar{a}\bar{b}/\emptyset) = tp^g(\bar{a}f(\bar{b})/\emptyset)$ and $f(\bar{b}) \subseteq \mathcal{M}$. Hence $\mathcal{M}$ is κ-galois saturated, as desired. □

Remark 4.2.6. In the previous lemma, the same proof will show that the model of size λ is $cof(\lambda)$-galois saturated. In particular, if λ is regular, then the model of size λ is λ-galois saturated.

From these lemmas we can show that all models at or above the categoricity cardinal, λ, are κ-galois saturated when $cof(\lambda) \geq \kappa$. As in the remark, if the cofinality of λ is regular then all models at or above the categoricity cardinal are λ-galois saturated.

Theorem 4.2.7. *Assume $\mathbb{K}$ is λ-categorical for $\lambda > \kappa$ and $cof(\lambda) \geq \kappa$. If $\mathcal{M} \in \mathbb{K}$ and $|\mathcal{M}| \geq \lambda$ then $\mathcal{M}$ is κ-galois saturated.*

Proof. Let $\bar{a} \subseteq \mathcal{M}$ and $\bar{b} \subseteq \mathbb{C}$ be sequences of length less than κ. By the Löwenheim-Skolem axiom, there exists $\mathcal{M}_0 \prec_{\mathbb{K}} \mathcal{M}$ of size λ such that $\bar{a} \subseteq \mathcal{M}_0$. By Lemma 4.2.5, $\mathcal{M}_0$ is κ-galois saturated. Hence, there exists $\bar{c} \subseteq \mathcal{M}_0$ such that $tp^g(\bar{a}\bar{c}/\emptyset) = tp^g(\bar{a}\bar{b}/\emptyset)$. Clearly $\bar{c} \subseteq \mathcal{M}$ and thus $\mathcal{M} \models tp^g(\bar{a}\bar{b}/\emptyset)$. □

Corollary 4.2.8. *Assume $\mathbb{K}$ is λ-categorical for $\lambda > \kappa$ and $cof(\lambda) \geq \kappa$. If $\mathcal{M}$, $\mathcal{N} \in \mathbb{K}$ and $|\mathcal{M}|$, $|\mathcal{N}| \geq \lambda$ then $\mathcal{M} \equiv_{\infty,\kappa} \mathcal{N}$.*

Proof. By Theorem 4.2.7, $\mathcal{M}$ and $\mathcal{N}$ are both κ-galois saturated. Since Theorem

4.1.8(1) didn't require any assumption on the cofinality of κ, it follows that $\mathcal{M} \equiv_{\infty,\kappa} \mathcal{N}$. $\square$

We continue to work towards our next axiomatizability result with the following lemma.

Lemma 4.2.9. *Assume $(\mathbb{K}, \prec_{\mathbb{K}})$ has finite character and the cofinality of κ is ω. If $\mathbb{K}$ is λ-categorical for $\lambda > \kappa$ and $cof(\lambda) > \kappa$ then there exists a sentence $\sigma \in L_{\infty,\kappa}$ such that for every L-structure $\mathcal{M}$, $\mathcal{M} \models \sigma$ IFF $\mathcal{M} \in \mathbb{K}$ and $\mathcal{M}$ is κ-galois saturated.*

Proof. Let $\mathcal{N}$ be the unique model in $\mathbb{K}$ of cardinality λ. By Lemma 4.2.5, $\mathcal{N}$ is κ-galois saturated. Let $\sigma \in L_{\infty,\kappa}$ describe $\mathcal{N}$ up to $L_{\infty,\kappa}$-equivalence by Theorem 2.4.4.

($\Rightarrow$): Suppose $\mathcal{M} \models \sigma$. Then $\mathcal{M} \equiv_{\infty,\kappa} \mathcal{N}$ by Theorem 2.4.4. Hence, $\mathcal{M} \in \mathbb{K}$ and is κ-galois saturated by Theorem 4.1.8.

($\Leftarrow$): Suppose $\mathcal{M} \in \mathbb{K}$ and κ-galois saturated. Then $\mathcal{M} \equiv_{\infty,\kappa} \mathcal{N}$ by Theorem 4.1.8. Hence $\mathcal{M} \models \sigma$ by Theorem 2.4.4. $\square$

The following theorem is implied by Lemma 4.2.9 and the results of section 1.

Theorem 4.2.10. *Assume $(\mathbb{K}, \prec_{\mathbb{K}})$ has finite character and $cof(\kappa) = \omega$. Let $\mathbb{K}$ be λ-categorical for $\lambda > \kappa$ and $cof(\lambda) > \kappa$. Then there is a complete sentence $\sigma \in L_{\infty,\kappa}$ such that:*

1. *$Mod(\sigma) \subseteq \mathbb{K}$ and σ has a model of cardinality κ^+.*

2. *$\mathbb{K}$ and $Mod(\sigma)$ contain precisely the same models of cardinality $\geq \lambda$.*

3. *If $\mathcal{M}, \mathcal{N} \models \sigma$ then $\mathcal{M} \prec_{\mathbb{K}} \mathcal{N}$ IFF $\mathcal{M} \prec_{\infty,\kappa} \mathcal{N}$.*

Proof. Define σ as in Lemma 4.2.9

1. If $\mathcal{M} \models \sigma$ then $\mathcal{M} \in \mathbb{K}$ and $\mathcal{M}$ is κ-galois saturated by Lemma 4.2.9. In addition, since $\lambda > \kappa$, Lemma 4.2.2 implies that $\mathbb{K}$ is κ-galois stable. In the same manner of the proof of Lemma 4.2.5, we can construct a model, $\mathcal{M}$, of cardinality κ^+ that is κ-galois saturated (this is also proved in [1]). Lemma 4.2.9 then implies that $\mathcal{M} \models \sigma$, as desired.

2. If $\mathcal{M} \in \mathbb{K}$ and $|\mathcal{M}| \geq \lambda$ then $\mathcal{M}$ is κ-galois saturated by Theorem 4.2.7. By Lemma 4.2.9 again implies that $\mathcal{M} \models \sigma$. Part (1) of this theorem implies that if $\mathcal{M} \models \sigma$ and $|\mathcal{M}| \geq \lambda$ then $\mathcal{M} \in \mathbb{K}$.

3. If $\mathcal{M}, \mathcal{N} \models \sigma$ then $\mathcal{M}$ and $\mathcal{N}$ are κ-galois saturated by Lemma 4.2.9. Corollary 4.1.11 implies that $\mathcal{M} \prec_{\mathbb{K}} \mathcal{N}$ IFF $\mathcal{M} \prec_{\infty,\kappa} \mathcal{N}$.

□

Remark 4.2.11. It is still an open question as to whether or not σ must have a model of cardinality κ.

We can now axiomatize $\mathbb{K}$ by taking our sentence σ from Lemma 4.2.9 and disjuncting it with each sentence describing the models below the categoricity cardinal. This will be the same argument as the proof of Theorem 2.4.2.

Theorem 4.2.12. *Assume $(\mathbb{K}, \prec_{\mathbb{K}})$ has finite character and $cof(\kappa) = \omega$. Let $\mathbb{K}$ be λ-categorical for $\lambda > \kappa$ and $cof(\lambda) > \kappa$. Then there is $\theta \in L_{\infty,\kappa}$ such that $\mathbb{K} = Mod(\theta)$.*

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