ABSTRACT

Title of dissertation: The cohomology rings of the special affine group of $\mathbb{F}_p^2$ and of $PSL(3,p)$

Jane Holsapple Long, Doctor of Philosophy, 2008

Dissertation directed by: Professor James A. Schafer
Department of Mathematics

The special affine group of $\mathbb{F}_p^2$, p an odd prime, denoted $Qd(p)$, plays an important role in the search for free actions by finite groups on products of spheres. The mod-p and integral cohomology rings of $Qd(p)$ are computed and, as an extension of these results, the mod-p and p-primary part of the integral cohomology of the simple group $PSL(3,p)$ are computed. Various properties of these rings are discussed.

The cohomology rings of the special affine group of $\mathbb{F}_p^2$ and of $PSL(3, \mathbb{F}_p)$

by

Jane Holsapple Long

Dissertation submitted to the Faculty of the Graduate School of the
University of Maryland, College Park in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
2008

Advisory Committee:
Professor James A. Schafer, Chair/Advisor
Professor Joel Cohen
Professor Jonathan Rosenberg
Professor Lawrence Washington
Professor Theodore Jacobson

Dedication

For Nick & Nixon, and Mom & Dad.

Acknowledgments

This work would not have been possible without help from very many people. Mathematically speaking, I'd like to thank all of the mathematicians whose work is cited here. On a personal level, there are lots more:

First, all of my wonderful teachers from kindergarten to the present, who made such an impression on me that I still remember their names and faces 20 years later. In particular, Mr. Nickerson at Lincoln Southeast High School and all of my math professors at Bryn Mawr: Helen Grundman, Rhonda Hughes, Paul Melvin, and, most of all, Lisa Traynor.

My time at Bryn Mawr was fulfilling and inspiring, both because of the stimulating atmosphere and high expectations and because of the wonderful friends I made there (Goose, Kitty, Lena, and Jess, you know who you are). Likewise, Madge helped me survive high school.

Here at Maryland, there have been many friends and administrators who made life in general more pleasant. Justin Wyss-Gallifent and Dr. David Lay helped me learn how to teach. Linette Berry, Haydée Hidalgo, Celeste Regalado, and Patty Woodwell (who is missed!) minimized my stress by helping with all sorts of things. Besides being good friends, Chris Truman helped me learn topology, Fernando Galaz Garcia was always willing to talk about math, and he and the incomparable Dave Shoup helped me study for quals. Chris Zorn, Eric Errthum, and Dave Bourne provided inspiration. Juliana Belding (whom I'd like to thank most especially for all of her company this last month), Julie Staub, Heather McDonough, and Avi Dalal cultivated a lively and comfortable "math

house." Kate Truman and Brian and Margo Alford were also great friends.

Thanks very much to Dr. Karsten Grove, who supported my research efforts this semester, the Maryland Math Department for various types of financial support for the past 6 years, and the U.S. Department of Education and the GAANN program for considerable support. In addition, the math department and the university libraries have created a fine collection of resources which helped me in my studies.

I don't feel that I can properly explain how much help my family has given me. Sue Long, my Mom, and Jeremy Greenfield helped enormously with babysitting; Duffy Chapman has been a fantastic aunt and hostess for many years; Ariel Woods and Jeremy Greenfield (and now Ada!), the Gentry, Chapman, and Montgomery families have been wonderful neighbors; the Foardes adopted me when I moved here; the Longs and Lloyds welcomed me; and Mom, Dad, and Em were indescribably supportive. Nixon made home a joyful place to be, and, last and best of all, there's Nick, whom I hope already knows how important he is.

Lastly, thanks very much to Dr. Schafer for all of his help and guidance, for being so understanding about my informal maternity leave, and for not giving up on me when I made stupid mistakes.

Table of Contents

List of Abbreviations

$\mathbb{F}_p$	the finite field of p elements
$\mathbb{F}_p^\times$	the units in $\mathbb{F}_p$
$\mathbb{Z}$	the ring of integers
$\mathbb{Z}_n$	the cyclic group of order n
$(\mathbb{Z}_n)^k$	the product of k copies of $\mathbb{Z}_n$
$SL(n,p)$	the $n \times n$ determinant-1 matrices with entries in $\mathbb{F}_p$
$H^*(X;M)$	the cohomology of the topological space X with coefficients in M
$H^*(G;M)$	the cohomology of the finite group G with coefficients in M
$X \simeq Y$	the topological space X is homotopic to the topological space Y
$X \times Y$	the product of the topological spaces X and Y
S^n	the n-dimensional sphere
$(S^n)^k$	the product of k copies of S^n

Chapter 0

Introduction

One of the most interesting examples of the interplay between algebra and topology is the study of group actions on topological spaces. Perhaps the best result to date is the solution by Madsen, Thomas, and Wall (see [41] for the proof and [17] for a survey) in 1976 of the topological spherical space form problem (see [21]): classify all manifolds whose universal cover is a sphere. This is equivalent to finding all free actions by finite groups on spheres. The solution by Madsen, Thomas, and Wall was to classify the finite groups which can act freely on a sphere, and to construct an action in each case. Their investigation was motivated by the following results, for which we need a definition:

Definition 0.0.1. A finite group G is **periodic**, that is, its cohomology is periodic of period n, if and only if

$$H^i(G;\mathbb{Z}) \cong H^{i+n}(G;\mathbb{Z})$$

for all $i \geq 1$ (assuming G acts trivially on $\mathbb{Z}$).

Theorem 0.0.1 (P. A. Smith, [38])**.** *If a finite group G acts freely on a sphere, the cohomology of G must be periodic.*

Theorem 0.0.2 (Milnor, [31])**.** *If a finite group G acts freely on a sphere, every involution in G must be central.*

Theorem 0.0.3 (Swan, [39])**.** *A finite group G acts freely on a finite complex X homotopy equivalent to S^{n-1} if and only if the cohomology of G is periodic.*

Finally, Madsen, Thomas, and Wall showed:

Theorem 0.0.4 ([41])**.** *A finite group G acts freely on a sphere if and only if the cohomology of G is periodic and every involution in G is central.*

Swan's theorem is a converse of Smith's theorem for the case of finite complexes homotopy equivalent to spheres. Eventually, for a group G satisfying Smith and Milnor's conditions, Madsen, Thomas, and Wall were able to obtain a free action on a manifold using surgery-theoretic techniques on Swan's free G-complex X. That is, they showed that the Smith and Milnor conditions were necesssary and sufficient.

It can be shown (see [16] Theorem XII.11.6) that the cohomology of a group G is periodic if and only if every abelian subgroup of G is cyclic, a condition which can be rephrased in terms of the rank of the group:

Definition 0.0.2. The **rank** of a finite group G, denoted $r(G)$, is the maximum value of the p-ranks,

$$r_p(G) = max\{n|(\mathbb{Z}_p)^n \hookrightarrow G\},$$

for all p dividing the order of G.

A group of the form $(\mathbb{Z}_p)^n$ is called an **elementary abelian group** of rank n.

Hence, a group G is periodic if and only if $r(G) = 1$. Since a rank-1 group acts freely on a finite complex having the homotopy type of one sphere, this led to the following conjecture (appearing in [7]):

Conjecture 0.0.5. *A finite group of rank n acts freely on a finite complex homotopy equivalent to* $S^{m_1} \times \cdots \times S^{m_n}$.

It should be noted that free actions arising from representations (called "linear spheres") can always be used to find such free actions for p-groups, but cannot be used in general (see [36]).

We will outline the present state of research on this problem and, after some necessary background is given, explain the contributions that the present work makes. In particular, we will restrict attention to the problem of finding free actions by rank-2 finite groups on finite complexes homotopy equivalent to a product of two spheres.

0.1 Euler classes

Before proceeding with a discussion of results pertaining to Conjecture 0.0.5, we will need the following definitions:

Definition 0.1.1. Given an action of a group G on a topological space X, the following fibration is called the **Borel Construction**:

$$\begin{array}{ccccc} X & \hookrightarrow & X \times_G EG & = & X \times EG/((x,e) \backsim (gx,ge)) \\ & & \downarrow p & & \\ & & BG & & \end{array}$$

where BG is the classifying space for the group G (so that $H^*(G;M) \cong H^*(BG;M)$ for any coefficients M) and EG is a principal G-bundle over BG (in the case of a discrete group, this is just the universal cover of BG). If M is any G-module, the Leray-Serre (also called "Serre") spectral sequence associated to this fibration is a first quadrant spectral sequence with

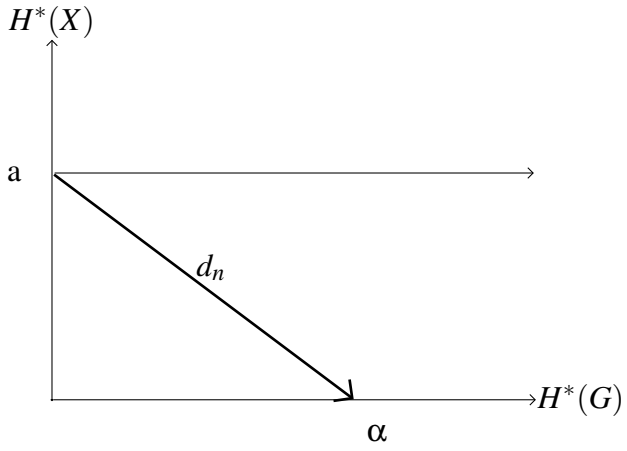
$$E_2^{p,q} \cong H^p(G;H^q(X;M))$$

and converging to the **equivariant cohomology** $H^*_G(X;M)$. Note that, if the G-action is free, the equivariant cohomology is isomorphic to the cohomology of the quotient space: $H^*_G(X;M) \cong H^*(X/G;M)$.

In the case of an orientable G-action on $X \simeq S^{n-1}$, we have

$$H^*(X;\mathbb{Z}) \cong H^*(S^{n-1};\mathbb{Z}) \cong \mathbb{Z}$$

with generator a, so that the E_2 page of the spectral sequence is:



Recall that the **transgression** is the map

$$d_n : E_n^{0,n-1} \to E_n^{n,0}$$

from the vertical axis to the horizontal axis. The class $d_n(a) = \alpha \in H^n(G)$ is the **Euler class** of the action.

Remark 0.1.1.

This construction also gives the familiar Gysin sequence for any trivial G-module

M:

$$\cdots \to H^i(G;M) \xrightarrow{\smile \alpha} H^{i+n}(G;M) \xrightarrow{p^*} H_G^{i+n}(X;M) \to H^{i+1}(G;M) \to \cdots .$$

In the case of a free G-action on some $X \simeq S^{n-1}$, recall that the cohomology of G must be periodic, and in fact the Euler class α is a periodicity generator for the cohomology of G. That is, isomorphism on cohomology is realized by cup product with α. The spectral sequence associated to a free G-action on a product of equidimensional spheres is not much more complicated than the case of a single sphere, and led to some results in this case. For example (also proved by D. Benson and J. Carlson ([7]) and Browder ([11])):

Theorem 0.1.1 (G. Carlsson, [14])**.** *Suppose that a finite group G acts freely on a finite complex $X \simeq (S^{n-1})^r$ with trivial action on homology. Then for any prime p the p-rank of G is at most r. Moreover, the complex of cellular chains on X is G-chain homotopic to a tensor product of n complexes, where each complex has the homology of a sphere.*

The first part of this theorem is a converse of Conjecture 0.0.5 in the special case of a product of equidimensional spheres. It should be noted, however, that we cannot expect every rank-n group to act freely on a finite complex homotopy equivalent to a product of n equidimensional spheres: Oliver (see [34]) showed that not every rank-2 group can act on a finite complex having the homotopy type of a product of two spheres of the same dimension. In particular, he showed that A_4, which contains $\mathbb{Z}_2 \times \mathbb{Z}_2$, cannot act freely on such a space. Since every rank-2 nonabelian simple group contains A_4, this implies that no rank-2 simple group can act freely on such a space (see [4] section 5). His main tool was the machinery of the Steenrod algebra, which is discussed in the following subsection.

Due to the increasing complexity of the above spectral sequence and related issues, the arguments in [7], [14], and [11] do not easily generalize to products of spheres of different dimensions. More tools are needed to obtain information in these cases.

0.2 The Steenrod Algebra

The mod-p Steenrod Algebra, $\mathcal{A}_p$, is generated by a collection of cohomology operations which are natural with respect to maps of topological spaces and commute with the suspension of topological spaces. These operations are called $\mathfrak{P}$-power operations, except in the case of the familiar Bockstein map, which is also a generator. In later chapters, we will use some properties of the $\mathfrak{P}$-power operations and the Bockstein, and will explain their use in context. We will need a few results concerning the Steenrod Algebra.

Theorem 0.2.1. *$H^*(X;\mathbb{F}_p)$ is a module over $\mathcal{A}_p$ for every topological space X.*

Thus, the Steenrod Algebra contains obstructions to a graded commutative $\mathbb{F}_p$-algebra being the cohomology ring structure of a topological space. There are some famous non-existence results proved using the above fact, including the non-existence of $\mathbb{Z}[\alpha]/(\alpha^m)$, $m > 3$ and $|\alpha|$ other than 2 or 4, as the cohomology ring of a topological space (see [20] for a discussion of this and some other applications of the properties of $\mathcal{A}_p$). Recalling the Borel construction for a G-action on a space X, and noting that transgressions in the Leray-Serre spectral sequence commute with the action of $\mathcal{A}_p$ (see [29] Corollary 6.9), we see that:

FACT 0.2.1. Let G be a finite group. The Euler class of a G-action on a space $X \simeq S^n$ must generate an $\mathcal{A}_p$-invariant ideal in $H^*(G;\mathbb{F}_p)$.

In particular, this implies that not every cohomology class in $H^*(G;\mathbb{F}_p)$ can be the Euler class of an action on a space homotopy equivalent to a sphere. This fact will be very important.

0.3 Effective Euler Classes

More recently, in 2001, Adem and Smith (see [4]) constructed actions satisfying the conditions of Conjecture 0.0.5 for some large classes of rank-2 groups. Their construction takes a certain type of (necessarily non-free) action by a rank-2 finite group G on a single sphere and builds a free G-action on a finite complex homotopy equivalent to a product of two spheres:

Theorem 0.3.1 ([4])**.** *Let G be a finite group and X a finite dimensional G-CW complex such that all of the isotropy subgroups of G have periodic cohomology. Then there exists a finite dimensional CW-complex Y with a free G-action such that $Y \simeq S^N \times X$. If X is simply connected and finitely dominated, then we can assume Y is a finite complex.*

They also showed that any rank-2 p-group and every rank-2 simple group except possibly $PSL(3,p)$ (p an odd prime) yields to this construction. Finding the required type of G-action is equivalent to finding an action with a particular type of Euler class:

Lemma 0.3.2 (see [4], Lemma 4.5)**.** *The following conditions on an Euler class $\alpha \in H^N(G;\mathbb{Z})$ of a G-action on a finite-dimensional $X \simeq S^{N-1}$ are equivalent:*

1. *every maximal rank elementary abelian subgroup of G acts without stationary points;*

2. $\alpha|_E \neq 0$ *for all maximal rank elementary abelian subgroups* E *of* G.

Such an Euler class is said to be ***effective***.

An effective Euler class need not always exist; Adem and Smith noted the possibility that $PSL(3,p)$ did not have one in [4] and, in [22], Jackson completely characterized the groups which have effective Euler classes:

Theorem 0.3.3 (Jackson, [22])**.** *A rank-2 group has an effective Euler class if and only if it does not* p'*-involve the group* $\mathbf{Qd}(\mathbf{p})$*, for* p *an odd prime.* $Qd(p)$ *is given by the split extension*

$$0 \to \mathbb{F}_p^2 \to \mathbf{Qd}(\mathbf{p}) \to SL(2,p) \to 1,$$

with $SL(2,p)$ *acting on the vectors of* $\mathbb{F}_p^2$ *by multiplication on the left.*

A group G *is said to* $\mathbf{p'}$***-involve*** *another group* L *if there are subgroups* $K \triangleleft H \leq G$ *such that* $H/K \cong L$ *and the order of* K *is relatively prime to* p.

In particular, $Qd(p)$ does not have an effective Euler class and, since $Qd(p) \hookrightarrow PSL(3,p)$ (see Chapter 2) with the same Sylow-p subgroup, $PSL(3,p)$ does not have one, either. Therefore, Adem and Smith's construction fails for the groups $Qd(p)$ and $PSL(3,p)$, and free actions by these groups (if they exist) must be obtained by different means.

0.4 Other Constructions of Free Actions

Essentially, 0.3.1 is the only effective method for constructing the desired free actions. One other possibility for constructing a free G-action ($r(G) = n$) as in Conjec-

ture 0.0.5 is mentioned by D. Benson and J. Carlson in [7]. In their proof of Theorem 0.1.1, they give a means for constructing (from n carefully chosen classes in $H^*(G;\mathbb{Z})$) a chain complex F which is the tensor product of chain complexes having the homology of a sphere and is naturally endowed with a free G-action. A finite *topological realization* of this complex, that is, a finite complex X whose cellular chain complex is G-chain homotopic to F, would have the homotopy type of a product of n spheres. D. Benson and J. Carlson give no suggestions as to how to find such a realization, noting only that there are obstructions, including one in $\mathcal{A}_p$. In particular, if we study the Serre spectral sequence associated to a free action on a product of spheres, we see that the transgressions of the fundamental classes must generate an $\mathcal{A}_p$-invariant ideal. An algorithm for finding a topological realization of a chain complex is given by J. Smith in [37]. A realization need not always exist and, in fact, Smith's theorem contains necessary and sufficient conditions for the existence of a realization. Although the algorithm is not easy to implement, it may illuminate other obstructions to the existence of a free action.

The mod-p reductions of the n integral classes in $H^*(G;\mathbb{Z})$ must form a *homogeneous system of parameters* for $H^*(G;\mathbb{F}_p)$ (see [33] Appendix A and [10]). The computations in Chapter 1 make it possible to find homogeneous systems of parameters for $Qd(p)$ and $PSL(3,p)$.

0.5 Results

The main results of this paper are:

Theorem 0.5.1. *The following are computed:*

- *$H^*(Qd(p);\mathbb{F}_p)$ and $H^*(Qd(p);\mathbb{Z})$;*
- *$H^*(PSL(3,p);\mathbb{F}_p)$ and $H^*(PSL(3,p);\mathbb{Z})_{(p)}$;*
- *The $\mathcal{A}_p$-invariant prime ideals in $H^*(Syl_p(Qd(p));\mathbb{F}_p)$, $H^*(Qd(p);\mathbb{F}_p)$, and $H^*(PSL(3,p);\mathbb{F}_p)$.*

The computation of $\mathcal{A}_p$-invariant prime ideals yields information about pairs of cohomology classes which can possibly arise as transgressions in the spectral sequence associated to the Borel construction which, by the construction of [7] discussed in Subsection 0.4, could generate the desired free $Qd(p)$- or $PSL(3,p)$-complex.

Aside from questions in the realm of group actions, the $Qd(p)$ groups are an interesting class of groups in their own right. Indeed, they are the special affine group of $\mathbb{F}_p^2$. They exhibit a necessary condition for having Cohen-Macaulay group cohomology (every maximal elementary abelian subgroup has equal rank) although $H^*(Qd(p);\mathbb{F}_p)$ is not a Cohen-Macaulay ring (see Chapter 3). $PSL(3,p)$ is a finite, simple, rank-2 group of Lie type, whose cohomology in characteristic p had not been computed before this work.

The fact that $Qd(p)$ has all maximal elementary abelian p-subgroups of equal rank and has no effective Euler class suggested that perhaps this group might be a counterexample to the following "depth conjecture":

Conjecture 0.5.2 (see [13])**.** *The depth of $A = H^*(G;k)$ is equal to $dim(A/\mathfrak{q})$ for some associated prime ideal $\mathfrak{q}$.*

However, $Qd(p)$ is not a counterexample (see Proposition 3.0.8). In order to show this, we use our calculation of $H^*(Qd(p);\mathbb{F}_p)$ to exhibit the required associated prime

ideal. In general, there is not an easy way to find which associated prime realizes the depth, and there does not seem to be another proof possible without the calculations in Chapter 1.

Chapter 1

The $Qd(p)$ Groups and Their Cohomology

The $Qd(p)$ groups are given by the split extension:

$$0 \longrightarrow (\mathbb{Z}_p)^2 \longrightarrow Qd(p) \rightleftarrows SL(2,p) \longrightarrow 1$$

with $SL(2,p)$ acting on $(\mathbb{Z}_p)^2$ by matrix multiplication on the left.

The Sylow-p subgroup of $Qd(p)$ is the extra-special p-group of order p^3 and exponent p, which has the following presentation:

$$P = \langle A,B,C | A^p = B^p = C^p = 1, C = [A,B], AC = CA, BC = CB \rangle,$$

where $C = [A,B] = A^{-1}B^{-1}AB$. P may be expressed as a group extension in two different ways (see [40]):

$$1 \longrightarrow C_p^C \longrightarrow P \longrightarrow C_p^A \times C_p^B \longrightarrow 1 \text{ (non-split)}$$

$$1 \longrightarrow C_p^B \times C_p^C \longrightarrow P \rightleftarrows C_p^A \longrightarrow 1 \text{ (split)}$$

and embeds in $Qd(p)$ as:

$$A = \left(0, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right), B = \left(e_2, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right), C = \left(e_1, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right),$$

with 0 viewed as the zero vector and e_1, e_2 as the standard basis vectors. There are $p+1$ copies of P in $Qd(p)$ since there are $p+1$ copies of $\mathbb{Z}_p$ in $SL(2,p)$. In P, the subgroup $\langle B,C \rangle \cong (\mathbb{Z}_p)^2$ is normal. We will refer to the subgroup $\langle B,C \rangle$ as H. Note also that $Z(P) = \langle C \rangle$.

1.1 Computation of $H^*(Qd(p);\mathbb{F}_p)$

The strategy is given by the following

Lemma 1.1.1. *Let G be a finite group, P a Sylow-p subgroup of G, and $C = \bigcap_{g\in G} gPg^{-1}$. If the map $Res_C^{P\cap gPg^{-1}}$ is a monomorphism for all $g \notin N_G(P)$, then for any G-module M, $H^*(G;M)_{(p)} \cong [Res_C^P]^{-1}(H^*(C;M)^{G/C}) \cap H^*(P;M)^{W_G(P)}$.*

Proof: Recall that, for a subgroup K of H, the quotient $W_HK = N_HK/K$ is the *Weyl group* of K. This group is the quotient of N_HK having potentially nontrivial action on cohomology, since conjugation by $k \in K$ induces the identity map. Since $C = \bigcap_{g\in G} gPg^{-1}$ is clearly a normal subgroup, $W_G(C) \cong G/C$.

Recall the following well-known "stability theorem":

Theorem 1.1.2. *(see [2] Theorem II.6.6) Let $G \supseteq H \supseteq Syl_p(G)$, where $Syl_p(G)$ is a p-Sylow subgroup of G. Then, for any $\mathbb{Z}G$-module M,*

$$Res_H^G : H^*(G;M)_{(p)} \to H^*(H;M)_{(p)}$$

is injective, and its image consists of the stable elements in $H^*(H;M)_{(p)}$, that is, the subring of $H^*(H;M)_{(p)}$ for which the following diagram commutes for all $g \in G$:

$$\begin{array}{ccc} H^*(H;M)_{(p)} & \xrightarrow{\quad c_g^* \quad} & H^*(gHg^{-1};M)_{(p)} \\ & {\scriptstyle res}\searrow \quad \swarrow{\scriptstyle res} & \\ & H^*(H\cap gHg^{-1};M)_{(p)} & \end{array}$$

So the restriction map from a group to any subgroup containing its Sylow-p subgroup gives an isomorphism of the stable elements and the p-primary part of the cohomology ring. So we want to find the stable elements of $H^*(P;M)$, that is, the elements making the following diagram commute for all $g \in G$:

$$\begin{array}{ccc} H^*(P;M) & \xrightarrow{\quad c_g^* \quad} & H^*(gPg^{-1};M) \\ & {\scriptstyle res} \searrow \quad \swarrow {\scriptstyle res} & \\ & H^*(P\cap gPg^{-1};M) & \end{array}$$

If $g \in N_G(P)$, $P \cap gPg^{-1} = P$. The diagram commutes for all $g \in N_G(P)$ on the fixed points under conjugation by g, which are $H^*(P;M)^{W_G(P)}$, so all stable elements must be contained in this set of fixed points.

If $g \notin N_G(P)$, $gPg^{-1} \cap P \neq P$. The following commutative diagram will be helpful in illustrating our argument:

$$\begin{array}{ccc} H^*(P) & \xrightarrow{\quad c_g^* \quad} & H^*(gPg^{-1}) \\ \downarrow {\scriptstyle res} & & \downarrow {\scriptstyle res} \\ H^*(P\cap gPg^{-1}) & \xrightarrow{c_g^*} & H^*(g(P\cap gPg^{-1})g^{-1}) \\ \downarrow {\scriptstyle res} & & \downarrow {\scriptstyle res} \\ H^*(C) & \xrightarrow{\quad c_g^* \quad} & H^*(C) \end{array}$$

Suppose that an element $x \in H^*(P;M)$ is stable. Then

$$Res^{gPg^{-1}}_{gPg^{-1}\cap P}(c_g^*(x)) = Res^{P}_{gPg^{-1}\cap P}(x)$$

for all $g \in G$. Restricting to C, we have

$$Res^{gPg^{-1}\cap P}_{C} Res^{gPg^{-1}}_{gPg^{-1}\cap P}(c_g^*(x)) = Res^{gPg^{-1}\cap P}_{C} Res^{P}_{gPg^{-1}\cap P}(x).$$

But since c_g^* commutes with restriction maps, this shows that

$$c_g^*(Res^{P\cap gPg^{-1}}_{C}(Res^{P}_{P\cap gPg^{-1}}(x)) = Res^{P\cap gPg^{-1}}_{C}(Res^{P}_{gPg^{-1}\cap P}(x)).$$

Hence $Res^P_C(x)$ is fixed under conjugation by g for all $g \in G$ and

$$x \in [Res^P_C]^{-1}(H^*(C;M)^G) \cap H^*(P;M)^{W_G(P)}.$$

For the converse: by assumption, the map c_g^* is the identity on $Res_C^P(x)$ for all $g \in G$. Now,

$$Res_C^P(x) = Res_C^{P\cap gPg^{-1}}(Res^P_{P\cap gPg^{-1}}(x))$$

and, for $g \notin N_G(P)$, $Res_C^{P\cap gPg^{-1}}$ is a monomorphism, implying that

$$\begin{aligned} Res_C^{P\cap gPg^{-1}}(Res^P_{P\cap gPg^{-1}}(x)) &= c_g^*(Res_C^{P\cap gPg^{-1}}(Res^P_{P\cap gPg^{-1}}(x)) \\ &= Res_C^{P\cap gPg^{-1}}(c_g^*(Res^P_{P\cap gPg^{-1}}(x))) \\ \Rightarrow Res^P_{P\cap gPg^{-1}}(x) &= c_g^*(Res^P_{P\cap gPg^{-1}}(x)) \\ &= Res^{gPg^{-1}}_{P\cap gPg^{-1}}(c_g^*(x)) \end{aligned}$$

by commutativity of c_g^* and restriction maps. Hence x is stable, as desired. □

Corollary 1.1.3. $H^*(Qd(p);\mathbb{F}_p) \cong [Res^P_H]^{-1}(H^*(H;\mathbb{F}_p)^{SL(2,p)}) \cap H^*(P;\mathbb{F}_p)^{W_G(P)}$.

Remark 1.1.1.

In particular, the lemma gives a means for finding the cohomology of any normal extension G given by

$$(\mathbb{Z}_p)^n = H \xrightarrow{\triangleleft} G \longrightarrow N$$

with the *p-rank* of G,

$$r_p(G) = max\{m|(\mathbb{Z}_p)^m \hookrightarrow G\},$$

being equal to n, and $|Syl_p(G)| = p^{n+1}$ since in this case $\bigcap_{g\in G} gSyl_p(G)g^{-1} = H$. Thus $Res_C^{P\cap gPg^{-1}}$ is clearly a monomorphism for $g \notin N_G P$.

1.2 The Ring $H^*(P;\mathbb{F}_p)$

Recall that $H^*(Qd(p);\mathbb{F}_p) \cong H^*(P;\mathbb{F}_p)^{stab(Qd(p))}$, so we will need to understand the cohomology of P in order to proceed. The following theorem of I.J. Leary, who refers to P as $P(3)$, is taken from [25]; note that the case $p = 3$ will be considered separately.

Theorem 1.2.1 (Leary, [25]). *Let p be a prime greater than 3. Then $H^*(P;\mathbb{F}_p)$ is generated by elements*

$$y, y', x, x', Y, Y', X, X', d_4, \ldots, d_p, c_4, \ldots, c_{p-1}, z,$$

with

$$deg(y) = deg(y') = 1,\ deg(x) = deg(x') = deg(Y) = deg(Y') = 2,$$

$$deg(X) = deg(X') = 3,\ deg(d_i) = 2i-1,\ deg(c_i) = 2i,\ deg(z) = 2p.$$

The generators are defined as follows: identify the class y with the cocycle $y(A^iB^jC^k) = i$ and the class y' with the cocycle $y'(A^iB^jC^k) = j$, Y with the Massey triple product $\langle y, y, y'\rangle$, Y' with $\langle y', y', y\rangle$, and

$$d_i = \begin{cases} Cor^P_{\langle B,C\rangle}(c'^{i-1}d') & i < p-1 \\ Cor^P_{\langle B,C\rangle}(c'^{p-2}d') - x^{p-2}y & i = p-1 \\ Cor^P_{\langle B,C\rangle}(c'^{p-1}d') + x^{p-2}X & i = p \end{cases}$$

where d' denotes the cocycle $d'(B^rC^s) = s$ and $c' = \sigma(d')$. The generator z is the mod-p reduction of the top-dimensional integral Chern class of a p-dimensional irreducible representation of P restricting to the center $\langle C\rangle \cong \mathbb{Z}_p$ as p copies of the identity. Finally,

$$\sigma(y) = x, \quad \sigma(y') = x',$$

$$\sigma(Y) = X, \quad \sigma(Y') = X'.$$

$$\sigma(d_i) = \begin{cases} c_i & i < p \\ 0 & i = p. \end{cases}$$

The generators satisfy the relations:

$$yy' = 0,\ xy' = x'y,\ yY = y'Y' = 0,\ yY' = y'Y,$$

$$Y^2 = Y'^2 = YY' = 0,\ yX = xY,\ y'X' = x'Y',$$

$$Xy' = 2xY' + x'Y,\ X'y = 2x'Y + xY',$$

$$XY = X'Y' = 0,\ XY' = -X'Y,\ xX' = -x'X,$$

$$x(xY' + x'Y) = x'(xY' + x'Y) = 0,$$

$$x^p y' - x'^p y = 0,\ x^p x' - x'^p x = 0,$$

$$x^p Y' + x'^p Y = 0,\ x^p X' + x'^p X = 0,$$

$$c_i y = \begin{cases} 0 & i < p-1 \\ -x^{p-1}y & i = p-1 \end{cases} \qquad c_i y' = \begin{cases} 0 & i < p-1 \\ -x'^{p-1}y' & i = p-1 \end{cases}$$

$$c_i x = \begin{cases} 0 & i < p-1 \\ -x^p & i = p-1 \end{cases} \qquad c_i x' = \begin{cases} 0 & i < p-1 \\ -x'^p & i = p-1 \end{cases}$$

$$c_i Y = \begin{cases} 0 & i < p-1 \\ -x^{p-1}Y & i = p-1 \end{cases} \qquad c_i Y' = \begin{cases} 0 & i < p-1 \\ -x'^{p-1}Y' & i = p-1 \end{cases}$$

$$c_i X = \begin{cases} 0 & i < p-1 \\ -x^{p-1}X & i = p-1 \end{cases} \qquad c_i X' = \begin{cases} 0 & i < p-1 \\ -x'^{p-1}X' & i = p-1 \end{cases}$$

$$c_i c_j = \begin{cases} 0 & i + j < 2p-2 \\ x^{2p-2} + x'^{2p-2} - x^{p-1}x'^{p-1} & i = j = p-1 \end{cases}$$

$$d_i y = \begin{cases} 0 & i < p \\ -x^{p-1}Y & i = p \end{cases} \qquad d_i y' = \begin{cases} 0 & i < p \\ -x'^{p-1}Y' & i = p \end{cases}$$

$$d_i x = \begin{cases} 0 i < p-1 & \\ -x^{p-1}y & i = p-1 \\ x^{p-1}X & i = p \end{cases} \qquad d_i x' = \begin{cases} 0 & i < p-1 \\ -x'^{p-1}y' & i = p-1 \\ -x'^{p-1}X' & i = p \end{cases}$$

$$d_i Y = 0, \qquad d_i Y' = 0,$$

$$d_i X = \begin{cases} 0 & i \neq p-1 \\ -x^{p-1}Y & i = p-1 \end{cases} \qquad d_i X' = \begin{cases} 0 & i \neq p-1 \\ -x'^{p-1}Y' & i = p-1 \end{cases}$$

$$d_i d_j = \begin{cases} 0 & i < p-1 \text{ or } j < p-1 \\ x^{2p-3}Y - x'^{2p-3}Y' + x^{p-1}x'^{p-2}Y' & i = p, j = p-1 \end{cases}$$

$$d_i c_j = \begin{cases} 0 & i < p-1 \text{ or } j < p-1 \\ x^{2p-3}y + x'^{2p-3}y' - x^{p-1}x'^{p-2}y' & i = j = p-1 \\ -x^{2p-3}X + x'^{2p-3}X' - x^{p-1}x'^{p-2}X' & i = p, j = p-1 \end{cases}$$

The action of the Steenrod algebra $\mathcal{A}_p$ is completed by the following:

$$P^1(z) = zc_{p-1},\ P^1 c_i = \begin{cases} izc_{p-1} & i < p-1 \\ -zc_{p-2} + x^{2p-2} + x'^{2p-2} - x^{p-1}x'^{p-1} & i = p-1 \end{cases}$$

$$P^1(X) = x^{p-1}X + zy, \qquad P^1(X') = x'^{p-1}X' - zy'.$$

Remark 1.2.1.

In [8], Benson and Carlson attempt to find a more compact description, if not a nicer presentation, of $H^*(P;\mathbb{F}_p)$ using symplectic forms. However, as acknowledged in [9], their description was not complete. Others, including Milgram and Tezuka (see [30] for

the case $p = 3$ to follow) and Lewis (see [26]), have also studied the cohomology of this group.

Regarding the action of $\mathcal{A}_p$: since we know the Bocksteins of the generators and Leary computes the action of P^1 above, we can use the Adem relations to get P^n from P^1 for $n \leq p-1$. This completes the action for all generators except d_i and z. But since $deg(z) = 2p$, we know that $P^p(z) = z^p$ and $P^m(z) = 0$ for $m > p$. Although Leary does not compute the action of P^1 on the d_i, it is obtained by a straightforward calculation using Frobenius reciprocity:

$$\begin{aligned} P^1(Cor^P_{\langle B,C\rangle}(c'^{i-1}d')) &= Cor^P_{\langle B,C\rangle}(P^1(c^{i-1}d')) \\ &= (i-1)Cor^P_{\langle B,C\rangle}(c'^{p+i-2}d') \\ &= (i-1)Cor^P_{\langle B,C\rangle}(c'^{i-2}d'Res(z) - x'^{p-1}c'^{i-1}d') \\ &= (i-1)zCor^P_{\langle B,C\rangle}(c'^{i-2}d') + (i-1)x'^{p-1}Cor^P_{\langle B,C\rangle}(c'^{i-1}d'), \end{aligned}$$

so that

$$P^1(d_i) = \begin{cases} (i-1)zd_{i-1} + (i-1)x'^{p-1}d_i & i < p-1 \\ -2zd_{p-2} + 2x'^{2p-3}y' - 2x'^{p-1}x^{p-2}y + 2x^{2p-3}y & i = p-1, \\ -zd_{p-1} - x'^{2p-3}X' + x'^{p-1}x^{p-2}X & i = p. \end{cases}$$

A similar argument also shows that $\sigma(d_p) = 0$: since the Bockstein commutes with corestrictions, we have

$$\begin{aligned} \sigma(d_p) &= Cor(c'^p) \\ &= Cor(Res(z) + c'Res(x'^{p-1})) \\ &= x'^{p-1}Cor(c') \\ &= x'^{p-1}\sigma(Cor(d')). \end{aligned}$$

by Frobenius Reciprocity and the fact that $Cor \circ Res$ is multiplication by the index $[P:H] = p$ since $H \triangleleft P$. It remains to show that $Cor(d') = 0$. This can be done directly, viewing d' as the cocycle in $d'(A^iB^jC^k) = k$ in $H^*(P;\mathbb{F}_p)$. A set of coset representatives for H in P is given by A^s, $0 \leq s \leq p-1$, so the corestriction is

$$\begin{aligned}
\sum_{s=0}^{p-1} {}^{A^s}d'(A^iB^jC^k) &= \sum_{s=0}^{p-1} d'(A^s(A^iB^jC^k)A^{s^{-1}}) \text{ since the action of } A^s \text{ on } \mathbb{F}_p \text{ is trivial} \\
&= \sum_{s=0}^{p-1} d'(A^i(A^sB^jA^{s^{-1}})C^k) \\
&= \sum_{s=0}^{p-1} d'(A^i(BC^s)^jC^k) \\
&= \sum_{s=0}^{p-1} d'(A^iB^jC^{sj+k}) \\
&= \sum_{s=0}^{p-1} k + sj \\
&= j\sum_{s=0}^{p-1} s \\
&= j\left(\frac{p(p-1)}{2}\right) \\
&= 0.
\end{aligned}$$

Regarding the class z: Leary first computes the integral cohomology of P (see [24]), then the mod-p cohomology. The class z is first defined as the top integral Chern class of an irreducible p-dimensional representation of P restricting to the center $\langle C \rangle$ as p copies of the identity. Leary shows that the additive order of z in integral cohomology is p^2; the z in mod-p cohomology is the reduction mod-p of the integral class z. We will need this fact when we compute the integral cohomology of $Qd(p)$ later. (Note that Lewis's computation of $H^*(P;\mathbb{Z})$ in [26] agrees with that of Leary, although Lewis defines the generator in degree $2p$ differently.) Lastly, note that, since $z \in H^*(P;\mathbb{F}_p)$ is the restriction of an integral class, it maps to zero under σ. $\square$

In our computations, we will use the following

FACT 1.2.1. : $H^*(P;\mathbb{F}_p)$ has generators over $\mathbb{F}_p$:

Degree	**Elements**
1	y, y'
2	Y, Y', x, x'
3	$X, X', yY', xy, xy', x'y'$
4	$xY, xY', x'Y, x'Y', x'^2, x^2, xx'$
5	$x^2y, x^2y', x'^2y, x'^2y', xX, x'X, x'X', XY'$
6	$XX', x^2Y, x^2Y', x'^2Y, x'^2Y', x^3, x'^3, x^2x', xx'^2$
>6 even	$f_1, f_2Y, f_3Y', c_ig_i, XX'g_m$
>6 odd	$f_1y, f_2y', f_3X, f_4X', d_rg_r$

where $4 \leq i \leq p-1$, $4 \leq r \leq p$, the f_j are homogeneous polynomials in x, x', z and the g_k are homogeneous polynomials in z.

Note that, since $deg(c_i) = 2i$ and $deg(d_i) = 2i-1$, the elements c_ig_i and d_rg_r only appear in certain degrees. Furthermore, note that products of the c_i and d_r, including powers of the non-nilpotent element c_{p-1}, can be written in terms of the above generators. This shows that elements of even degree have one of the two following forms:

$$f_1 + f_2Y + f_3Y' + XX'g_k \qquad \text{or} \qquad f_1 + f_2Y + f_3Y' + c_ig_i$$

for some $4 \leq i \leq p-1$ and elements of odd degree have one of the two following forms:

$$f_1y + f_2y' + f_3X + f_4X' + XY'g_s \qquad \text{or} \qquad f_1y + f_2y' + f_3X + f_4X' + d_rg_r$$

for some $4 \leq r \leq p$. This **FACT** can be checked directly using the relations in $H^*(P;\mathbb{F}_p)$. Leary makes this verification for elements not involving d_i, c_i, or z, but it is easy to check

that any element in $H^*(P;\mathbb{F}_p)$ can be written in one of the above forms.

1.3 The Map Res^P_H.

Recall that H is the subgroup $\langle B,C\rangle$ of P and $[Res^P_H]^{-1}(H^*(H;\mathbb{F}_p)^{SL(2,p)})$ is an ingredient in the computation of $H^*(Qd(p);\mathbb{F}_p)$ by Lemma 1.1.1. Let Res denote the restriction map Res^P_H, Cor the corestriction Cor^P_H, and σ the Bockstein map.

Proposition 1.3.1. *The images of the generators of $H^*(P;\mathbb{F}_p)$ under the map Res are:*

Element	**Image under** *Res*	**Element**	**Image under** *Res*
y	0	y'	u
x	0	x'	β
Y	0	Y'	uv
X	0	X'	$v\beta - u\gamma$
d_i	0	c_i	$0, \quad i < p-1$
d_{p-1}	$-\beta^{p-2}u$	c_{p-1}	$-\beta^{p-1}$
d_p	$-\beta^{p-2}(v\beta - u\gamma)$	z	$\gamma^p - \gamma\beta^{p-1}$

Table 1.1: Images under the map *Res*

Proof: Recall that $H \lhd P$ denotes the subgroup $\langle B,C\rangle$ with cohomology ring $H^*(H;\mathbb{F}_p) \cong \mathbb{F}_p[\beta,\gamma] \otimes \wedge(u,v)$, where $\sigma(u) = \beta$ and $\sigma(v) = \gamma$. The class u represents the cocycle $\bar{y}'(B^rC^s) = r$ and v represents the cocycle $d'(B^rC^s) = s$.

Recall that the class y' represents the cocycle $y'(A^iB^jC^k) = j$ and Y' by the Massey product $Y' = \langle y', y', y\rangle$. Clearly y' restricts to $u \in H^*(P;\mathbb{F}_p)$. Leary computes the restriction $Res(Y') = uv$ directly from the bar construction, an argument we will not reproduce

here. Thus $Res(x') = \beta$ and $Res(X') = \sigma(Res(Y')) = \sigma(uv) = v\beta - u\gamma$. It is also a direct consequence of the definitions that the elements y, Y, x, X restrict to 0 in $H^*(H; \mathbb{F}_p)$.

Recalling the correspondence

$$d' \leftrightarrow v \quad y' \leftrightarrow u$$

$$c' \leftrightarrow \gamma \quad x' \leftrightarrow \beta,$$

the definition of the d_i can be written as

$$d_i = \begin{cases} Cor^P_{\langle B,C\rangle}(\gamma^{i-1}v) & i < p-1 \\ Cor^P_{\langle B,C\rangle}(\gamma^{p-2}v) - x^{p-2}y & i = p-1 \\ Cor^P_{\langle B,C\rangle}(\gamma^{p-1}v) - x^{p-2}X & i = p. \end{cases}$$

We now compute the restriction of the d_i. Since $H \triangleleft P$, $Res \circ Cor$ is multiplication by the norm element N in $\mathbb{Z}[< A >]$, which is $N = 1 + A + \cdots + A^{p-1}$. It can be computed directly that $A^k(v) = v + ku$, so that

$$Res \circ Cor(\gamma^j v) = \sum_{k=0}^{p-1} (\gamma + k\beta)^j (v + ku).$$

Expanding,

$$\sum_{k=0}^{p-1} (\gamma + k\beta)^j = \sum_{i=0}^{j} \begin{pmatrix} j \\ i \end{pmatrix} \gamma^{j-i}\beta^i \left(\sum_{k=0}^{p-1} k^i \right),$$

which is zero for $i < p-1$ and -1 for $i = p-1$ by Newton's formula. Therefore

$$Res \circ Cor(\gamma^j v) = v \sum_{i=0}^{j} \begin{pmatrix} j \\ i \end{pmatrix} \gamma^{j-i}\beta^i \left(\sum_{k=0}^{p-1} k^i \right) + u \sum_{i=0}^{j} \begin{pmatrix} j \\ i \end{pmatrix} \gamma^{j-i}\beta^i \left(\sum_{k=0}^{p-1} k^{i+1} \right)$$

$$= \begin{cases} 0 & j < p-2 \\ -\beta^{p-2}u & j = p-2 \\ -\beta^{p-2}(v\beta - u\gamma) & j = p-1. \end{cases}$$

Furthermore, since x and X are in the kernel of Res, we see that $Res(d_{p-1}) = -\beta^{p-2}u$, implying that $Res(c_{p-1}) = -\beta^{p-1}$ and $Res(d_p) = -\beta^{p-2}(v\beta - u\gamma)$.

For the generator z: recall the definition of z as the p^{th} Chern class of an irreducible p-dimensional representation of P, which restricts to H as a sum of one copy of each of the representations of H restricting to $\langle C \rangle$ as the identity. These representations have first Chern classes $\gamma + i\beta$ for each choice of i, so

$$Res(z) = \prod_{i=0}^{p-1}(\gamma + i\beta) = \gamma^p - \gamma\beta^{p-1}.$$

Recalling the definitions of the generators and using the above table, we can see:

Proposition 1.3.2. *The image of Res is the subalgebra of $H^*(H;\mathbb{F}_p)$ generated by $\beta, \gamma^p - \gamma\beta^{p-1}, u, uv, v\beta - u\gamma$ and is generated by the restrictions of y', x', Y', X', d_{p-1}, c_{p-1}, d_p, and z. As a module over $\mathcal{A}_p$, this is the subalgebra generated by the restrictions of y', Y', d_{p-1}, d_p, and z.*

1.4 Computation of $Res^{-1}(H^*(H;\mathbb{F}_p)^{SL(2,p)})$

This will proceed in steps: first, we determine the fixed points $H^*(H;\mathbb{F}_p)^{SL(2,p)}$, then the elements in $H^*(P;\mathbb{F}_p)$ which restrict to the fixed points (modulo the kernel), and, finally, the kernel of Res.

1.4.1 Computation of $H^*(H;\mathbb{F}_p)^{SL(2,p)}$

The ring

$$H^*(H;\mathbb{F}_p) \cong \mathbb{F}_p[\gamma,\beta] \otimes \wedge(u,v),$$

where γ, β have degree 2, v, u have degree 1, and the image of the Bockstein map σ is $\sigma(v) = \gamma$, $\sigma(u) = \beta$. $SL(2,p)$ acts in the standard way on $H^1(H;\mathbb{F}_p)$, with generators

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, D = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

(see [5]). The action of A and D on $\mathbb{F}_p[\gamma,\beta] \otimes \wedge(u,v)$ is given by:

$$\begin{aligned} A(v) &= u+v & D(u) &= v \\ A(u) &= u & D(v) &= -u \\ A(\gamma) &= \gamma+\beta & D(\gamma) &= -\beta \\ A(\beta) &= \beta & D(\beta) &= \gamma. \end{aligned}$$

Proposition 1.4.1 (Wilkerson, [42])**.**

$$\mathbb{F}_p[\gamma,\beta]^{SL(2,p)} \cong \mathbb{F}_p[\sum_{i=0}^{p} (\gamma^{p-1})^{p-i}(\beta^{p-1})^i, \gamma\beta^p - \beta\gamma^p].$$

Proof: The fixed-point set $\mathbb{F}_p[\gamma,\beta]^{GL(2,p)}$ is well-known; the generators over $\mathbb{F}_p$ are the *Dickson invariants*, call them $c_{2,1} = \sum_{i=0}^{p} (\gamma^{p-1})^{p-i}(\beta^{p-1})^i$ of degree $2p(p-1)$ and $c_{2,0} = (\gamma\beta^p - \beta\gamma^p)^{p-1}$ of degree $2(p+1)(p-1)$. Wilkerson gives algorithms for computing $c_{2,1}$ and $c_{2,0}$. He also shows that $\mathbb{F}_p[\beta,\gamma]^{SL(2,p)}$ has generators $c_{2,1}$ and $w = \gamma\beta^p - \beta\gamma^p$; note that $w^{p-1} = c_{2,0}$.

Claim 1.4.2. *The fixed points of even degree are* $\mathbb{F}_p[\beta,\gamma]^{SL(2,p)} \otimes \wedge(vu)$.

Proof: It remains to show that vu is fixed under the action of $SL(2,p)$:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} v \\ u \end{bmatrix} = \begin{bmatrix} av+bu \\ cv+du \end{bmatrix}, \text{ so } vu \mapsto (ad-bc)vu = vu.$$

Theorem 1.4.3 (Mui, [32]). *The fixed points of odd degree are generated by* $v\beta - u\gamma$ *and* $v\beta^p - u\gamma^p$ *over* $\mathbb{F}_p[\gamma,\beta]^{SL(2,p)}$.

Proof: Note that $v\beta - u\gamma$ and $v\beta^p - u\gamma^p$ are invariant and linearly independent over $\mathbb{F}_p[\gamma,\beta]$: let

$$f = (v\beta - u\gamma)g + (v\beta^p - u\gamma^p)h$$

where g,h are homogeneous polynomials in β,γ. Then if $f = 0$,

$$0 = (v\beta - u\gamma)f = vu(\gamma\beta^p - \beta\gamma^p)h \Rightarrow h = 0$$

and similarly

$$0 = (v\beta^p - u\gamma^p)f = -vu(\gamma\beta^p - \beta\gamma^p)g \Rightarrow g = 0.$$

Hence, an element of the form $(v\beta - u\gamma)g + (v\beta^p - u\gamma^p)h$ is invariant if and only if g,h are invariant.

An arbitrary element of odd degree has the form $f = vf_1(\gamma,\beta) + uf_2(\gamma,\beta)$; suppose that f is invariant under the action of $SL(2,p)$. Invariance of f under $SL(2,p)$ implies invariance under the subgroup generated by A. Applying A to f and equating coefficients of u and v, we see that

$$f_2(\gamma,\beta) - f_2(\gamma+\beta,\beta) = f_1(\gamma+\beta,\beta),$$

so f_1 is divisible by β, and

$$f_1(\gamma,\beta) = f_1(\gamma+\beta,\beta),$$

so f_1 is invariant under A. We now rewrite

$$f = (v\beta - u\gamma)f_1' + u\gamma f_1' + uf_2,$$

where $f_1 = \beta f_1'$ with f_1' invariant under A. Furthermore,

$$f(v,0;\gamma,0) = 0.$$

Since f must also be invariant under $D = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$,

$$D(f) = -uf_1(-\beta,\gamma) + vf_2(-\beta,\gamma)$$

therefore $f(v,0;\gamma,0) = 0$ implies that f_2 is divisible by γ. Rewrite f as

$$f = (v\beta - u\gamma)f_1' + u\gamma f_3.$$

Invariance of f and $(v\beta - u\gamma)f_1'$ under A imply invariance of γf_3 under A.

Notice that an element which is divisible by γ and invariant under A must be divisible by $\gamma^p - \gamma\beta^{p-1}$, for the image of γ under A^k is $\gamma + k\beta$ and

$$\prod_{k\in\mathbb{F}_p} \gamma + k\beta = \gamma^p - \gamma\beta^{p-1}.$$

Rewrite f again as

$$f = (v\beta - u\gamma)f_1' + u(\gamma^p - \gamma\beta^{p-1})f_3'.$$

Finally, note that

$$u(\gamma^p - \gamma\beta^{p-1}) = (v\beta - u\gamma)\beta^{p-1} - (v\beta^p - u\gamma^p),$$

showing that

$$f = (v\beta - u\gamma)(f_1' + \beta^{p-1}f_3') - (v\beta^p - u\gamma^p)f_3'.$$

Linear independence of $v\beta - u\gamma$ and $v\beta^p - u\gamma^p$ shows that $f_1' + \beta^{p-1}f_3'$ and f_3' are invariant under $SL(2,p)$, as desired. □

There is a simpler way to write the results shown above: in terms of the Steenrod algebra $\mathcal{A}_p$. The Steenrod algebra is a collection of cohomology operations satisfying certain axioms, including

$$P^k(x) = \begin{cases} x^p & |x| = 2k \\ 0 & |x| < 2k. \end{cases}$$

The cohomology ring of a group is a module over the Steenrod algebra $\mathcal{A}_p$ and so must be the fixed points $(H^*(H;\mathbb{F}_p))^{SL(2,p)}$. Hence, we may state our results in terms of $\mathcal{A}_p$; we will do so throughout. In the present case, we have

Proposition 1.4.4. *The fixed point set* $(\mathbb{F}_p[\gamma,\beta]\otimes\wedge(v,u))^{SL(2,p)}$ *has* $\mathcal{A}_p$*-invariant generator* vu *over* $\mathbb{F}_p[\gamma,\beta]^{SL(2,p)}$.

Proof: Using the axioms for $\mathcal{A}_p$, we see that $\sigma(vu) = -(v\beta - u\gamma)$ and $P^1(v\beta - u\gamma) = (v\beta^p - u\gamma^p)$. In particular, $P^1(\beta) = \beta^p$ and $P^1(\gamma) = \gamma^p$ since the degree of these elements is 2.

1.4.2 The Inverse Image of the Fixed Points Under *Res*.

We begin by computing the inverse image in even degrees.

Claim 1.4.5. $[Res^{-1}](\mathbb{F}_p[\beta\gamma^p - \gamma\beta^p, \sum_{i=0}^{p}\gamma^{(p-1)(p-i)}\beta^{(p-1)i}])$ *is the subalgebra*

$$\mathbb{F}_p[x'z, z^{p-1} + x'^{p(p-1)}] + ker(Res)$$

of $H^{ev}(P;\mathbb{F}_p)$.

Proof: It is clear that $Res(x'z) = \beta\gamma^p - \gamma\beta^p$ and

$$Res(z^{p-1} + x'^{p(p-1)}) = \sum_{i=0}^{p}(\gamma^{p-1})^{p-i}(\beta^{p-1})^i :$$

$$
\begin{aligned}
Res(z^{p-1}+x'^{p(p-1)}) &= (\gamma^p-\gamma\beta^{p-1})^{p-1}+\beta^{p(p-1)} \\
&= \frac{(\gamma^p-\gamma\beta^{p-1})^p}{\gamma^p-\gamma\beta^{p-1}}+\beta^{p(p-1)} \\
&= \frac{\gamma^{p^2}-\gamma^p\beta^{p(p-1)}+\beta^{p(p-1)}(\gamma^p-\gamma\beta^{p-1})}{\gamma^p-\gamma\beta^{p-1}} \\
&= \frac{\gamma^{p^2}-\gamma\beta^{(p+1)(p-1)}}{\gamma^p-\gamma\beta^{p-1}} \\
&= \frac{\gamma^{(p+1)(p-1)}-\beta^{(p+1)(p-1)}}{\gamma^{p-1}-\beta^{p-1}} \\
&= \sum_{i=0}^{p}(\gamma^{p-1})^{p-i}(\beta^{p-1})^i.
\end{aligned}
$$

Using the form of even-degree elements in $H^*(P;\mathbb{F}_p)$, the claim follows from:

Lemma 1.4.6. *The map* $\mathbb{F}_p[x',z] \xrightarrow{g} \mathbb{F}_p[\beta,\gamma]$ *given by*

$$
g: \begin{cases} x' \mapsto \beta \\ z \mapsto \gamma^p-\gamma\beta^{p-1} \end{cases}
$$

is a monomorphism on homogeneous elements.

Proof: Recall that the degree of x' is 2 and the degree of z is $2p$; β and γ both have degree 2. Now, g is clearly a monomorphism on homogeneous polynomials in degree $n<2p$. For $n\geq 2p$, write $n=2pm+2r$, where $r<p$. Then a homogeneous polynomial h of degree n can be written as

$$
h(x',z)=\sum_{i=0}^{m} a_i x'^{pi+r} z^{m-i}.
$$

If $r>0$, we can factor out x'^r, so it suffices to show the case $n=2pm$. Proceed by induction; the cases $m=0$ and $m=1$ are clear. For the inductive case, assume that g is a monomorphism on homogeneous polynomials of degree less than or equal to m. In degree $m+1$, we have

$$0 = g(h) = \sum_{i=0}^{m+1} a_i \beta^{p(m+1-i)} (\gamma^p - \gamma\beta^{p-1})^i$$
$$= a_0 \beta^{p(m+1)} + \sum_{i=1}^{m} a_i \beta^{p(m+1-i)} (\gamma^p - \gamma\beta^{p-1})^i + a_{m+1} (\gamma^p - \gamma\beta^{p-1})^{m+1}$$

$\Rightarrow a_0, a_{m+1} = 0$, since the coefficients of $\beta^{p(m+1)}$ and $\gamma^{p(m+1)}$ must be zero. Therefore

$$0 = \sum_{i=1}^{m} a_i \beta^{p(m+1-i)} (\gamma^p - \gamma\beta^{p-1})^i$$
$$= \beta^p (\gamma^p - \gamma\beta^{p-1}) \sum_{i=0}^{m-1} a_{i+1} \beta^{p(m-1-i)} (\gamma^p - \gamma\beta^{p-1})^i$$

$$\Rightarrow \sum_{i=0}^{m-1} a_{i+1} \beta^{p(m-1-i)} (\gamma^p - \gamma\beta^{p-1})^i = 0.$$

But this is the image of $\sum_{i=0}^{m-1} a_{i+1} x'^{p(m-1-i)} z^i$, a homogeneous polynomial of degree $m-1$, so each a_{i+1} must be zero. Hence $h = 0$. □

Since there are no relations in $H*(P;\mathbb{F}_p)$ involving z, Lemma 1.4.6 essentially gives the following picture:

$$\begin{array}{ccccc}
\mathbb{F}_p[x'z, z^{p-1} + x'^{p(p-1)}] & \hookrightarrow & \mathbb{F}_p[x', z] & \hookrightarrow & H^*(P;\mathbb{F}_p) \\
\downarrow \cong & & \downarrow \cong & & \downarrow Res \\
\mathbb{F}_p[\beta\gamma^p - \gamma\beta^p, \sum_{i=0}^{p} \gamma^{(p-1)(p-i)} \beta^{(p-1)i}] & \hookrightarrow & \mathbb{F}_p[\beta, \gamma^p - \gamma\beta^{p-1}] & \hookrightarrow & H^*(H;\mathbb{F}_p)
\end{array}$$

Having computed the image of *Res* in Proposition 1.3.1 above, and using the forms of arbitrary elements in $H^*(P;\mathbb{F}_p)$, it is now easy to verify:

Proposition 1.4.7. *$Res^{-1}[H^*(H;\mathbb{F}_p)^{SL(2,p)}]$ is the $\mathbb{F}_p$-subalgebra of $H^*(P;\mathbb{F}_p)$ generated by*

$$Y', X', x'z, x'^{p-1}X' - zy', z^{p-1} + x'^{p(p-1)}$$

and the kernel of Res.

1.4.3 The Kernel of *Res*.

Proposition 1.4.8. *The kernel of Res is the ideal in $H^*(P;\mathbb{F}_p)$ generated by the $\mathbb{F}_p$ generators*

$$y, x, Y, X, d_i, c_i, x'^{p-2}y' + d_{p-1}, x'^{p-1} + c_{p-1}, x'^{p-2}X' + d_p,$$

where $4 \leq i < p-1$.

Proof: Clearly, the elements y, x, Y, X, and, for $4 \leq i < p-1$, d_i and c_i are in the kernel (see Proposition 1.3.1). It remains to show that the proposed set generates $ker(Res)$. For elements of degree less than or equal to 6, we can find the elements of the kernel by inspection. To compute restrictions of elements of larger degree, we will need:

Generators for $H^*(P)$ over $\mathbb{F}_p$ in degrees greater than 6 are given above and we shall use them now to find the inverse image of 0 under *Res*; this will suffice to determine a generating set for the kernel. An element of even degree may be written (non-uniquely) as

$$f_1 + f_2Y + f_3Y' + g_kXX' \qquad \text{or} \qquad f_1 + f_2Y + f_3Y' + g_ic_i,$$

where $4 < i \leq p-1$ and $f_j = f_j(x,x',z)$ and $g_r = g_r(z) = a_rz^r$ are homogeneous polynomials. Write $Res(f_j(x,x',z)) = \bar{f}_j(0,\beta,\gamma^p - \gamma\beta^{p-1})$ and $Res(g_r(z)) = \bar{g}_r(\gamma^p - \gamma\beta^{p-1})$. We would like to determine when

$$Res(f_1 + f_2Y + f_3Y' + g_kXX') = \bar{f}_1 + \bar{f}_3uv$$

is zero. Since $\bar{f}_1$ and $\bar{f}_3$ are polynomials in β and γ, $\bar{f}_1 + \bar{f}_3uv = 0$ exactly when $\bar{f}_1 = 0$ and $\bar{f}_3 = 0$. By Lemma 1.4.6, *Res* is a monomorphism on the $f_i(0,x',z)$, so f_1 and f_3 must either be zero or multiples of x.

Next, observe that

$$Res(f_1 + f_2Y + f_3Y' + g_ic_i) = \begin{cases} \bar{f}_1 + \bar{f}_3uv & 4 \le i < p-1 \\ \bar{f}_1 + \bar{f}_3uv - \beta^{p-1}\bar{g}_{p-1} & i = p-1. \end{cases}$$

The case $4 \le i < p-1$ is the same as the argument shown above. Similarly, for the case $i = p-1$, we must have $\bar{f}_1 - \beta^{p-1}\bar{g}_{p-1} = 0$ and $\bar{f}_3 = 0$. Since x is in the kernel, it suffices to consider $\bar{f}_1 = f_1(0, \beta, \gamma^p - \gamma\beta^{p-1})$, giving

$$\bar{f}_1 - \beta^{p-1}\bar{g}_{p-1} = \beta^r \sum_{i=0}^{m} a_i \beta^{p(m-i)} (\gamma^p - \gamma\beta^{p-1})^i - b\beta^{p-1}(\gamma^p - \gamma\beta^{p-1})^n$$

for some $r < p$, which has degree $2r + 2pm = 2(p-1) + 2pn$. Thus $r \equiv -1 \bmod p$, so $r = p-1$ and $m = n$. Rewrite

$$\bar{f}_1 - \beta^{p-1}\bar{g}_{p-1} = \beta^{p-1}\left(\sum_{i=0}^{n} a_i \beta^{p(n-i)} (\gamma^p - \gamma\beta^{p-1})^i - b(\gamma^p - \gamma\beta^{p-1})^n\right) = 0$$

$\Rightarrow a_i = 0$ for $0 \le i < n$ and $a_n = b$. This shows that $\bar{f}_1 = \beta^{p-1}(\gamma^p - \gamma\beta^{p-1})^n$. But this is the restriction of a homogeneous polynomial in x' and z, so the lemma shows that $f_1 = x'^{p-1}z^n$. The element $f_1 + c_{p-1}g_{p-1}$ must therefore be a multiple of $(x'^{p-1} + c_{p-1})z^n$. Hence $x'^{p-1} + c_{p-1}$ is a generator of the kernel.

An element of odd degree can be written as

$$f_1y + f_2y' + f_3X + f_4X' + g_id_i$$

or

$$f_1y + f_2y' + f_3X + f_4X' + XY'g_s$$

where $4 \le i \le p$, $f_j = f_j(x, x', z)$ and $g_i = g_i(z) = a_rz^r$ are homogeneous polynomials.

Observe that

$$Res(f_1y+f_2y'+f_3X+f_4X'+g_id_i)=\begin{cases}\bar{f}_2u+\bar{f}_4(v\beta-u\gamma) & i<p-1\\ \bar{f}_2u+\bar{f}_4(v\beta-u\gamma)-\beta^{p-2}u\bar{g}_{p-1} & i=p-1\\ \bar{f}_2u+\bar{f}_4(v\beta-u\gamma)-\beta^{p-2}(v\beta-u\gamma)\bar{g}_p & i=p.\end{cases}$$

and

$$Res(f_1y+f_2y'+f_3X+f_4X'+XY'g_s)=\bar{f}_2u+\bar{f}_4(v\beta-u\gamma).$$

To determine when these restrictions are zero, observe that u and $v\beta-u\gamma$ are linearly independent over $\mathbb{F}_p[\gamma,\beta]$, so $\bar{f}_2$ and $\bar{f}_4$ must be zero when $i<p-1$; as above, f_2 and f_4 must be 0 or multiples of x. For the case $i=p-1$, proceeding as above, we see that $\bar{f}_4$ must be zero and $f_2y'+d_{p-1}g_{p-1}$ is a multiple of $(x'^{p-2}y'+d_{p-1})z^n$. Similarly, in the case $i=p$, $\bar{f}_2=0$ and $f_4X'+d_pg_p$ is a multiple of $(x'^{p-2}X'+d_p)z^n$. □

1.5 The Action of the Weyl Group

Now we find the fixed points of $H^*(P;\mathbb{F}_p)$ under the action of $W_{Qd(p)}(P)$. Recall the presentation for P:

$$P=\langle A,B,C|A^p=B^p=C^p=1,C=[A,B],AC=CA,BC=CB\rangle$$

embedding in $Qd(p)$ as

$$A=\left(0,\begin{pmatrix}1&1\\0&1\end{pmatrix}\right),B=\left(e_2,\begin{pmatrix}1&0\\0&1\end{pmatrix}\right),C=\left(e_1,\begin{pmatrix}1&0\\0&1\end{pmatrix}\right).$$

By inspection of the elements in $SL(2,p)$, we see that $W_{Qd(p)}(P)\cong\mathbb{Z}_{p-1}\cong\mathbb{F}_p^\times$, generated by $\alpha=\left(0,\begin{pmatrix}\alpha&0\\0&\alpha^{-1}\end{pmatrix}\right)$ where α is a primitive root of $\mathbb{F}_p^\times$. Under the

action of α, that is, $\alpha(w) = \alpha w \alpha^{-1}$,

$$A \mapsto A^{\alpha^2}, B \mapsto B^{\alpha^{-1}}, C \mapsto C^{\alpha}.$$

Clearly the action of α induces an automorphism of P.

We now determine the action of $W_{Qd(p)}(P)$ on $H^*(P;\mathbb{F}_p)$. By the definitions of y and y' as cocycles $y(A^iB^jC^k) = i$ and $y'(A^iB^jC^k) = j$, we see that $y \mapsto \alpha^2 y$ and $y' \mapsto \alpha^{-1}y'$. Since $\beta(y) = x$ and $\beta(y') = x'$, this implies that $x \mapsto \alpha^2 x$ and $x' \mapsto \alpha^{-1}x'$.

To find the action on Y and Y', recall that Massey triple products are natural with respect to group automorphisms, so that the product is linear in each factor. These properties give the action on Y and Y':

$$\begin{aligned} Y &= \langle y, y, y' \rangle \mapsto \langle \alpha^2 y, \alpha^2 y, \alpha^{-1} y' \rangle \\ &= \alpha^{2+2-1} \langle y, y, y' \rangle \\ &= \alpha^3 Y \end{aligned}$$

Similarly, $Y' \mapsto Y'$. Since X and X' are the images under the Bockstein of Y and Y', $X \mapsto \alpha^3 X$ and X' is fixed.

Since the action of $W_{Qd(p)}(P)$ induces an automorphism of P sending $C \mapsto C^{\alpha}$, we may use:

Lemma 1.5.1 (Leary, [25]). *In mod-p cohomology, the effect of an automorphism of P restricting to $Z(P)$ as $C \mapsto C^j$ sends d_i to $j^i d_i$, c_i to $j^i c_i$, and z to jz.*

Our results are summarized in the following table:

Element	$\mathbf{W_{Qd(p)}(P)}$**-Image**	**Element**	$\mathbf{W_{Qd(p)}(P)}$**-Image**
y	$\alpha^2 y$	y'	$\alpha^{-1}y'$
x	$\alpha^2 x$	x'	$\alpha^{-1}x'$
Y	$\alpha^3 Y$	Y'	Y'
X	$\alpha^3 X$	X'	X'
d_i	$\alpha^i d_i$	c_i	$\alpha^i c_i$
z	αz		

Table 1.2: Images under $W_{Qd(p)}(P)$-action

Proposition 1.5.2. *An even-degree element of* $H^*(P;\mathbb{F}_p)^{W_{Qd(p)}(P)}$ *has the form*

$$f_1 + f_2 z^{p-4} Y + f_3 Y' + \left\{ \begin{array}{c} g_1 z^{p-4} XX' \\ c_i z^{p-1-i} g_i \end{array} \right\}$$

and an odd-degree element has the form

$$f_1 \left\{ \begin{array}{c} x'^2 y \\ x^n z^{p-3-2n} y \end{array} \right\} + f_2 \left\{ \begin{array}{c} zy' \\ x'^{p-2} y' \end{array} \right\} + f_3 z^{p-4} X + f_4 X' + \left\{ \begin{array}{c} z^{p-4} XY' g_s \\ d_i z^{p-1-i} g_i \\ d_p z^{p-2} g_p \end{array} \right\},$$

with braces indicating possible choices (recalling that the elements must be homogeneous), and

- *the* f_j *are homogeneous polynomials in* $x'^2 x$, $x'z$, x'^{p-1}, $x^k z^{p-1-2k}$, *and* z^{p-1};
- *the* g_r *are homogeneous polynomials in* z^{p-1};
- $0 \leq n \leq \frac{p-3}{2}$;

- $0 < k \leq \frac{p-1}{2}$;
- $4 \leq i \leq p-1$.

Proof: Since the $W_{Qd(p)}(P)$-action maps each generator to itself times some power of α, linear combinations of generators are fixed if and only if each constituent monomial is. To find the fixed points in $H^*(P;\mathbb{F}_p)$, we inspect the form of arbitrary elements in each degree. The inspection proceeds quickly in degrees less than or equal to 6. In degrees greater than 6, recall that $H^*(P;\mathbb{F}_p)$ has the following generators over $\mathbb{F}_p$:

$$f_1,\ f_2Y,\ f_3Y',\ c_ig_i,\ XX'g_k,\ f_1y,\ f_2y',\ f_3X,\ f_4X',\ d_ih_i,\ XY'g_s$$

where $4 \leq i \leq p-1$, the f_j are homogeneous polynomials in x, x', z and the g_k are homogeneous polynomials in z, so it suffices to check these.

For example, we determine when the monomial $x'^r x^s z^t$ is fixed under the action of $W_{Qd(p)}(P)$ as follows: notice that $x^{\frac{p-1}{2}}$, x'^{p-1}, and z^{p-1} are all fixed, so that we may restrict attention to $0 \leq s \leq \frac{p-1}{2}$ and $0 \leq r,t \leq p-1$. Under the $W_{Qd(p)}(P)$-action, $x'^r x^s z^t \mapsto \alpha^{2s+t-r} x'^r x^s z^t$, and finding the fixed points is equivalent to determining when $2s+t-r$ is a multiple of $p-1$. In the case $r=0$, we have $2s+t = k(p-1)$, where $k \in \{0,1,2\}$. Thus, the possibilities are $x^n z^{p-1-2n}$ and $x^{p-1}x^n z^{p-1-2n}$, $0 \leq n \leq \frac{p-1}{2}$. Subtracting $r=2$ means we must add one to s or two to t, likewise subtracting $r=1$ means we must add one to t. Hence, the fixed monomials in x', x, and z must be products of $x^{\frac{p-1}{2}}$, x'^{p-1}, z^{p-1}, x'^2x, $x'z$, and $x^n z^{p-1-2n}$. The arguments for other generators over $\mathbb{F}_p$ are similar, so we will not include all of them here.

We can easily check that the elements listed below, in conjunction with their images under σ and P^1, yield all of the fixed monomials found in the tables above. We have shown

Proposition 1.5.3. *$H^*(P;\mathbb{F}_p)^{W_{Qd(p)}(P)}$ has $\mathcal{A}_p$-invariant algebra generators Y', zy', $x'^{p-2}y'$, z^{p-1}, x'^2y, $x^nz^{p-3-2n}y$, $z^{p-4}Y$, d_iz^{p-1-i}, and d_pz^{p-2}, where $0 \leq n \leq \frac{p-3}{2}$ and $4 \leq i \leq p-1$.*

1.6 The Ring $H^*(Qd(p);\mathbb{F}_p)$

We are finally in a position to prove

Theorem 1.6.1. *Let p be a prime greater than 3. Then $H^*(Qd(p);\mathbb{F}_p)$ is isomorphic as an $\mathbb{F}_p$-algebra to the subalgebra of $H^*(P;\mathbb{F}_p)$ generated by $x'z$, $z^{p-1}+x'^{p(p-1)}$, $x'^{p-1}+c_{p-1}$, x'^2x, x^kz^{p-1-2k}, Y', X', $x'^{p-1}X'-zy'$, $d_{p-1}+x'^{p-2}y'$, x'^2y, c_iz^{p-1-i}, d_iz^{p-1-i}, $x^nz^{p-3-2n}y$, $z^{p-4}Y$, $z^{p-4}X$ where $0 < k \leq \frac{p-1}{2}$, $0 \leq n \leq \frac{p-3}{2}$, and $4 \leq i < p-1$. The $\mathcal{A}_p$-invariant generators are Y', $z^{p-1}+x'^{p(p-1)}$, $d_{p-1}+x'^{p-2}y'$, x'^2y, d_iz^{p-1-i}, $x^nz^{p-3-2n}y$, and $z^{p-4}Y$.*

Proof: By Lemma 1.1.1,

$$H^*(Qd(p);\mathbb{F}_p) \cong [Res^P_H]^{-1}(H^*(H;\mathbb{F}_p)^{SL(2,p)}) \cap H^*(P;\mathbb{F}_p)^{W_G(P)}.$$

By Proposition 1.4.7, this is the following subalgebra of $H^*(P;\mathbb{F}_p)$:

$$(\mathbb{F}_p[Y', X', x'z, x'^{p-1}X'-zy', z^{p-1}+x'^{p(p-1)}] + ker(Res)) \cap H^*(P;\mathbb{F}_p)^{W_G(P)}.$$

Examining table 1.2, we see that

$$Y', X', x'z, x'^{p-1}X'-zy', z^{p-1}+x'^{p(p-1)} \subseteq H^*(P;\mathbb{F}_p)^{W_G(P)},$$

therefore the $\mathbb{F}_p$-subalgebra these elements generate in $H^*(P;\mathbb{F}_p)$ is also contained in $H^*(P;\mathbb{F}_p)^{W_G(P)}$. By the modular law, we now have the subalgebra

$$\mathbb{F}_p[Y', X', x'z, x'^{p-1}X'-zy', z^{p-1}+x'^{p(p-1)}] + ker(Res)^{W_G(P)}.$$

All that remains is to find a generating set for this subalgebra over $H^*(P;\mathbb{F}_p)$. Comparing the form of elements of $H^*(P;\mathbb{F}_p)^{W_{Qd(p)}(P)}$ given in Proposition 1.5.2 and the kernel of *Res* computed in Proposition 1.4.8, we obtain the set of generators listed in the hypothesis. Lastly, verification of the $\mathcal{A}_p$-invariant generators is straightforward.

Remark 1.6.1.

We now know that an even-degree element of $H^*(Qd(p);\mathbb{F}_p)$ has the following form:

$$f_1 + f_2 z^{p-4} Y + f_3 Y' + \left\{ \begin{array}{c} z^{r_1(p-1)+p-4} XX' \\ c_i z^{r_i(p-1)+p-1-i} \\ (x'^{p-1} + c_{p-1}) z^{r(p-1)} \end{array} \right\}$$

and an odd-degree element has the form:

$$f_1 \left\{ \begin{array}{c} x'^2 y \\ x^n z^{p-3-2n} y \end{array} \right\} + f_2(x'^{p-1}X' - zy') + f_3 z^{p-4} X + f_4 X' + \left\{ \begin{array}{c} z^{p-4} XY' g_s \\ d_i z^{p-1-i} g_i \\ x'^{p-2} y' + d_{p-1} \end{array} \right\}$$

with braces indicating possible choices (recalling that the elements must be homogeneous), and

- the f_j are homogeneous polynomials in $x'^2 x$, $x'z$, $z^{p-1} + x'^{p(p-1)}$, and $x^k z^{p-1-2k}$;
- the g_r are homogeneous polynomials in z^{p-1};
- $0 \leq n \leq \frac{p-3}{2}$;
- $0 < k \leq \frac{p-1}{2}$;
- $4 \leq i \leq p-1$.

These conditions represent a refinement of the conditions of Proposition 1.5.2.

1.7 The Rings $H^*(Qd(p);\mathbb{Z})$ and $H^*(Qd(p);\mathbb{F}_q)$, $q \neq p$.

Recall that, for a finite group G,

$$H^*(G;\mathbb{Z}) \cong \bigoplus_{q||G|} H^*(G;\mathbb{Z})_{(q)}$$

in dimensions greater than 0; $H^0(G;\mathbb{Z}) \cong \mathbb{Z}$. As before, for each prime p, the p-primary part $H^*(G;\mathbb{Z})_{(p)}$ consists of the stable elements in the integral cohomology of the Sylow-p subgroup.

First, we compute the most interesting case of $q = p$. This can be done in the same manner as the computation for mod-p cohomology using Leary's computation of $H^*(P;\mathbb{Z})$ (see [24]) and our lemma showing

$$H^*(Qd(p);\mathbb{Z})_{(p)} \cong Res^{-1}[H^*(H;\mathbb{Z})^{SL(2,p)}] \cap H^*(P;\mathbb{Z})^{W_{Qd(p)}(P)}.$$

The ring

$$H^*(H;\mathbb{Z}) \cong \mathbb{F}_p[\gamma,\beta] \otimes \wedge(v\beta - u\gamma)$$

so clearly

$$H^*(H;\mathbb{Z})^{SL(2,p)} \cong \mathbb{F}_p[\sum_{i=0}^{p} \gamma^{(p-1)(p-i)}\beta^{(p-1)i}, \beta\gamma^p - \gamma\beta^p] \otimes \wedge(v\beta - u\gamma).$$

We now need to compute the inverse image of this ring and the action of $W_{Qd(p)}(P)$. Most of the computations will carry over from the mod-p case, a fact which we can show by exhibiting the relationship between $H^*(P;\mathbb{F}_p)$ and $H^*(P;\mathbb{Z})$. The ring $H^*(Qd(p);\mathbb{Z})_{(p)}$ is isomorphic to the subring of elements in mod-p cohomology lifting to integral coho-

mology. That is, the coefficient sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{p} & \mathbb{Z} & \xrightarrow{\pi} & \mathbb{F}_p & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{F}_p & \xrightarrow{p} & \mathbb{Z}/p^2\mathbb{Z} & \xrightarrow{\bar{\pi}} & \mathbb{F}_p & \longrightarrow & 0 \end{array}$$

yields a pair of long exact sequences on cohomology

$$\begin{array}{ccccccccc} H^n(P;\mathbb{Z}) & \xrightarrow{\pi_*} & H^n(P;\mathbb{F}_p) & \xrightarrow{\delta} & H^{n+1}(P;\mathbb{Z}) & \xrightarrow{p} & H^{n+1}(P;\mathbb{Z}) \\ \downarrow & & \downarrow = & \searrow^{\sigma} & \downarrow \pi_* & & \downarrow \\ H^n(P;\mathbb{Z}/p^2\mathbb{Z}) & \xrightarrow{\bar{\pi}_*} & H^n(P;\mathbb{F}_p) & \xrightarrow{\sigma} & H^{n+1}(P;\mathbb{F}_p) & \xrightarrow{p} & H^{n+1}(P;\mathbb{Z}/p^2\mathbb{Z}) \end{array}$$

where δ and σ are the respective Bockstein maps and the diagram is commutative. A mod-p cohomology class *lifts to integral cohomology* if it is in the image of π_*.

Commutativity of the triangle implies that $im(\sigma)$ lifts and $im(\pi_*) \subseteq ker(\sigma)$. Hence the classes x, x', X, X', and c_i for all $4 \leq i \leq p-1$ lift and, furthermore, these elements have order p in integral cohomology. The only other generators which could possibly lift are d_p and z. Inspection of the ring $H^*(P;\mathbb{Z})$ (see [24]) shows that d_p does not lift; z lifts by definition.

Lastly, we examine arbitrary elements of $H^*(P;\mathbb{F}_p)$, using the set of generators for the ring over $\mathbb{F}_p$ in each degree. In view of the relation $x'X = -xX'$, we see that $x'Y + xY'$ is in the kernel of σ. In fact, this class lifts (see [25] p. 69).

It is easy to see that the action of $W_{Qd(p)}(P)$ is the same for the elements x, x', X, X', and c_i, $4 \leq i \leq p-1$ in mod-p cohomology and in integral cohomology. The action on mod-p cohomology also determines the action on $x'Y + xY'$, so it remains only to find the action on z. Recalling that $C \mapsto C^{\alpha}$, where α is the generator of $W_{Qd(p)}(P)$, and z restricts to $\langle C \rangle$ as p copies of the identity, it is clear that $z \mapsto \alpha^p z \equiv \alpha z$, so that the action on the

integral class z is the same as the action on the mod-p class z. Now it is clear that

Proposition 1.7.1. *$H^*(Qd(p);\mathbb{Z})_{(p)}$ is generated as an algebra over $\mathbb{Z}$ by the elements $z^{p-1}+x'^{p(p-1)}$, $x'z$, $x'^{p-1}+c_{p-1}$, X', x'^2x, c_iz^{p-1-i}, x^nz^{p-1-2n}, $z^{p-4}X$, $z^{p-3}(x'Y+xY')$ where $4\leq i<p-1$ and $1\leq n\leq\frac{p-1}{2}$, subject to the relations in $H^*(P;\mathbb{F}_p)$ for the generators of the same names and the additional relation*

$$px=px'=pX'=pX=pc_i=p(x'Y+xY')=p^2z=0.$$

Remark 1.7.1.

The class $x'Y+xY'$ lifts to $H^*(P;\mathbb{Z})_{(p)}$ but is not fixed under the action of $W_{Qd(p)}(P)$. However, $z^{p-3}(x'Y+xY')$ is fixed, and is contained in $H^*(Qd(p);\mathbb{F}_p)$ since it can be expressed in the form $(xz^{p-3})Y'+(x'z)(z^{p-4}Y)$.

Now we compute $H^*(Qd(p);\mathbb{Z})_{(q)}$, $q\neq p$. We will use

Claim 1.7.2. *For $p\neq q$ and any coefficients M, $H^*(Qd(p);M)_{(q)}\cong H^*(SL(2,p);M)_{(q)}$.*

Proof: This follows immediately from the spectral sequence for the group extension

$$0\longrightarrow(\mathbb{Z}_p)^2\longrightarrow Qd(p)\rightleftarrows SL(2,p)\longrightarrow 1\ ,$$

which collapses in $\mathbb{F}_q$-coefficients.

The following proof for the case $q\neq 2$ uses properties of stable elements: again, we look for the stable elements using Lemma 1.1.1. Recalling that $Qd(p)$ is defined as the semidirect product $(\mathbb{Z}_p)^2\rtimes SL(2,p)$, we see that the Sylow-$q$ subgroups of $Qd(p)$ are exactly the Sylow-q subgroups of $SL(2,p)$. These are isomorphic to $\mathbb{Z}_q$ and we will

call them Q. Recall that the stable elements are the cohomology classes in $H^*(Q;M)_{(q)}$ making the following diagram commute for all $g \in Qd(p)$:

$$\begin{array}{ccc} H^*(Q;M) & \xrightarrow{\quad c_g^* \quad} & H^*(gQg^{-1};M) \\ & {\scriptstyle res} \searrow \quad \swarrow {\scriptstyle res} & \\ & H^*(Q \cap gQg^{-1};M) & \end{array}$$

Now, if $g \in N_{Qd(p)}Q$, then $Q \cap gQg^{-1} = Q$. The stable elements under $N_{Qd(p)}Q$ are the fixed points under its action. If $g \notin N_{Qd(p)}Q$, then $Q \cap gQg^{-1} = \{1\}$ and all elements are stable. So

$$H^*(Qd(p);M)_{(q)} \cong H^*(Q;M)^{N_{Qd(p)}Q}.$$

By the same argument,

$$H^*(SL(2,p);M)_{(q)} \cong H^*(Q;M)^{N_{SL(2,p)}Q}.$$

Again looking at $Qd(p)$ as a semi-direct product, it is easy to see that

$$N_{Qd(p)}Q \cong N_{SL(2,p)}Q.$$

□

Therefore, by Theorem 9.1 in [40],

Proposition 1.7.3. *For $q \neq p$ a prime dividing the order of $Qd(p) = p^3(p^2-1)$, $H^*(Qd(p);\mathbb{F}_q) \cong \mathbb{Z}_q[\beta_4] \otimes \wedge(u_3)$, where u_3 is a 3-dimensional class and $\sigma(u_3) = \beta_4$, and $H^*(Qd(p);\mathbb{Z})_{(q)} \cong \mathbb{Z}_q[\beta_4]$.*

Proof: Of course, if q does not divide the order of $Qd(p)$, the cohomology is zero. Using the identity

$$H^*(G;M)_{(p)} \cong H^*(Syl_p(G);M)^{N_G(Syl_p(G))}$$

for a group G with $Syl_p(G)$ abelian and a list of conjugacy classes in $SL(2,p)$, Thomas verifies the integral case. Clearly, the argument for the mod-q cases follows the same reasoning.

1.8 The Case $p = 3$

We will denote the Sylow-3 subgroup of $Qd(3)$ by P_3 to avoid confusion. Our methods are the same as in the case $p \neq 3$, so some details will be omitted. The presentation for the cohomology ring of the Sylow-3 subgroup is different in this case, and, in fact, Milgram and Tezuka [30] show that the ring is detected by the restriction maps to proper subgroups, which is not true in the case $p \neq 3$ since the generators d_i and c_i, $4 \leq i < p-1$, are *essential*: they restrict to zero on every maximal subgroup.

Theorem 1.8.1. *(Leary, [25]) For the prime $p = 3$, $H^*(P_3;\mathbb{F}_3)$ is generated by elements y,y',x,x',Y,Y',X,X',z, with*

$$deg(y) = deg(y') = 1,\ deg(x) = deg(x') = deg(Y) = deg(Y') = 2,$$

$$deg(X) = deg(X') = 3,\ deg(z) = 6,$$

$$\beta(y) = x, \beta(y') = x', \beta(Y) = X, \beta(Y') = X',$$

subject to the following relations:

$$yy' = 0,\ xy' = x'y,\ yY = y'Y' = xy',\ yY' = y'Y,$$

$$YY' = xx',\ Y^2 = xY',\ Y'^2 = x'Y,$$

$$yX = xY - xx,\ y'X' = x'Y' - xx',$$

$$Xy' = x'Y - xY',\ X'y = xY' - x'Y,$$

$$XY = x'X,\ X'Y' = xX',\ XY' = -X'Y,\ xX' = -x'X,$$

$$XX' = 0,\ x(xY' + x'Y) = -xx'^2,\ x'(xY' + x'Y) = -x'x^2,$$

$$x^3y' - x'^3y = 0,\ x^3x' - x'^3x = 0,$$

$$x^3Y' + x'^3Y = -x^2x'^2,\ x^3X' + x'^3X = 0.$$

The elements y and y' are defined by cocycles $y(A^iB^jC^k) = i$ and $y'(A^iB^jC^k) = j$ as before, and $Y =< y,y,y' >$, $Y' =< y',y',y >$. That the relations in this group look somewhat different than in the case $p \neq 3$ can be explained by properties of Massey products, which imply that $x =< y,y,y >$ and $x' =< y',y',y' >$ (see [23] for background).

It is easy to show that an element of even-degree has the (non-unique) form

$$f_1 + f_2Y + f_3Y'$$

and an odd element has the form

$$f_1y + f_2y' + f_3X + f_4X' + f_5XY',$$

with the f_i polynomials in x, x', and z. The map Res is a monomorphism on polynomials $f_i(x,x',z) = f_i(0,x',z)$ by Lemma 1.4.6, so the above argument also shows that the inverse image of $H^*(H;\mathbb{F}_3)^{SL(2,3)}$ is generated by $z^2 + x'^6$, $x'z$, Y', X', $x'^2X' - zy'$ and $ker(Res)$, which has generators y, x, Y, and X over $H^*(P_3;\mathbb{F}_3)$. It remains to compute the fixed points under the action of the Weyl group, which can be done directly once we know the action:

Element	$\mathbf{W_{Qd(3)}(P_3)}$**-Image**	**Element**	$\mathbf{W_{Qd(3)}(P_3)}$**-Image**
y	$\alpha^2 y \equiv y$	y'	$\alpha^{-1}y' \equiv \alpha y'$
x	$\alpha^2 x \equiv x$	x'	$\alpha^{-1}x' \equiv \alpha x'$
Y	$\alpha^3 Y \equiv \alpha Y$	Y'	Y'
X	$\alpha^3 X \equiv \alpha X$	X'	X'
z	αz		

Table 1.3: Images under $W_{Qd(3)}(P_3)$-action

We can now prove

Theorem 1.8.2. *The ring $H^*(Qd(3);\mathbb{F}_3)$ is isomorphic to the subalgebra of $H^*(P_3;\mathbb{F}_3)$ generated by $z^2+x'^6$, $x'z$, Y', X', $x'^2X'-zy'$, y, x, zY, zX. Furthermore, $H^*(Qd(3);\mathbb{Z})_{(3)}$ is generated over $\mathbb{Z}$ by $z^2+x'^6$, $x'z$, X', x, zX, and $xY'+x'Y$, subject to the relations for the generators of the same names in $H^*(P_3;\mathbb{F}_3)$ and the additional relation $9z = 3x' = 3x = 3X = 3X' = 3Y = 3Y = 0$.*

Proof: The argument is the same as in the case $p \neq 3$, see Theorem 1.6.1 and Proposition 1.7.1.

Remark 1.8.1.

We now know that an arbitrary element of $H^*(Qd(3);\mathbb{F}_3)$ can be expressed as:

$$f_1 + f_2Y' + f_3zY$$

in even degrees and

$$f_1y + f_2(x'^2X' - zy') + f_3zX + f_4X'$$

in odd degrees, where the f_i are homogeneous polynomials in x, $x'z$, and $z^2+x'^6$.

1.9 The Restriction Maps from $H^*(Qd(p);\mathbb{F}_p)$

We would like to know whether a particular element of $H^*(Qd(p);\mathbb{F}_p)$ is contained in a given $\mathcal{A}_p$-invariant prime ideal in $H^*(Qd(p);\mathbb{F}_p)$. These ideals are in one-to-one correspondence with conjugacy classes of elementary abelian p-subgroups E of $Qd(p)$, and the corresponding prime ideals are

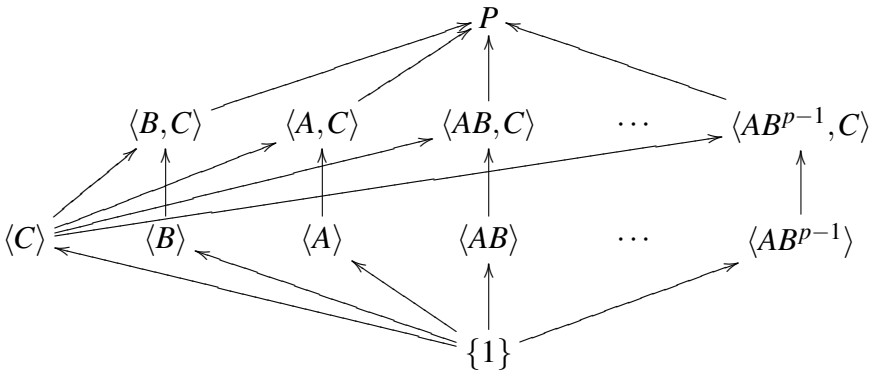
$$\sqrt{ker(Res_E^{Qd(p)})}$$

(see [6] Theorem 7.5.1 for the result of Quillen). Note that extensions and contractions of $\mathcal{A}_p$-invariant ideals are again $\mathcal{A}_p$-invariant, or, more simply, $Res_E^{Qd(p)} = Res_E^P Res_P^{Qd(p)}$, so we can find these by computing the

$$\sqrt{ker(Res_E^P)}$$

and applying the results to $H^*(Qd(p);\mathbb{F}_p)$. This is the approach we will take.

Proposition 1.9.1. *Up to conjugacy, the subgroup lattice of P is:*



Proof: Since the exponent of P is p, all of the subgroups of P are elementary abelian. Consider P as the extension

$$0 \to \langle C\rangle \xrightarrow{i} P \xrightarrow{\pi} \langle A\rangle \times \langle B\rangle \to 1.$$

Clearly P has rank 2, since $\langle A\rangle \times \langle B\rangle \cong (\mathbb{Z}_p)^2$ is contained in P and $P \ncong (\mathbb{Z}_p)^3$.

The subgroup $\langle A\rangle \times \langle B\rangle$ has order p^2 and $p^2-1 = (p-1)(p+1)$ nonzero elements. Each nonzero element has order p, so generates a cyclic subgroup of order p with $p-1$ nonzero elements. Hence there are exactly $p+1$ distinct rank-1 subgroups of $\langle A\rangle \times \langle B\rangle$: the subgroups $\langle A\rangle$, $\langle B\rangle$ and $\langle AB^k\rangle$, $0 <\leq p-1$, without loss of generality.

If E is a rank-2 subgroup of P, it must contain $\langle C\rangle$, else $P \cong (\mathbb{Z}_p)^3$. E maps to a rank-1 subgroup under π. Now, there exists a bijection between the set of subgroups of P containing $\langle C\rangle$ and subgroups of $\langle A\rangle \times \langle B\rangle$, so there are exactly $p+1$ non-conjugate rank-2 subgroups of P: $\langle A,C\rangle$, $\langle B,C\rangle$ and $\langle AB^k,C\rangle$, $0 < k \leq p-1$, without loss of generality. Furthermore, P contains exactly $p+1$ non-conjugate rank-1 subgroups not containing $\langle C\rangle$. $\square$

We computed Res^P_H in Proposition 1.3.1. It is possible to compute the restrictions to other subgroups using the commutativity of restriction maps and automorphisms of P. Note that the automorphisms behave differently for $p = 3$, due to properties of Massey products, so we will consider this case later. For now, let $p > 3$. We will need to know the induced action on cohomology of the following automorphisms of P, where $1 \leq k \leq p-1$:

$$\begin{array}{llll} \phi_1: & A \mapsto B & \phi_{2,k}: & A \mapsto AB^k \\ & B \mapsto A & & B \mapsto B \\ & C \mapsto C^{-1} & & C \mapsto C \end{array}$$

The induced actions of ϕ_1 and $\phi_{2,k}$ on $H^*(P;\mathbb{F}_p)$ (see Lemma 1.5.1) are:

Element	**Image Under $\phi_{\mathbf{1}}^*$**	**Image Under $\phi_{\mathbf{2,k}}^*$**
y	y'	y
y'	y	$y'+ky$
x	x'	x
x'	x	$x'+kx$
Y	Y'	Y
Y'	Y	$Y'-kY$
X	X'	X
X'	X	$X'-kX$
d_i	$(-1)^i d_i$	d_i
c_i	$(-1)^i c_i$	c_i
z	$-z$	z

We now define our notation for the cohomology of the rank-2 subgroups of P. As before, $H \cong \langle B,C\rangle$ with its familiar cohomology ring. Let a denote the cocycle $a(A^iC^j) = i$ and w_k the cocycle $w_k((AB^k)^iC^j) = i$. By an abuse of notation, writing v for the cocycles $v(A^iC^n) = n$, $v(B^jC^n) = n$, and $v((AB^k)^iC^n) = n$, and $\gamma = \sigma(v)$, denote

- $H^*(\langle B,C\rangle;\mathbb{F}_p) \cong \mathbb{F}_p[\beta,\gamma] \otimes \wedge_p(u,v)$, where $\beta = \sigma(u)$
- $H^*(\langle A,C\rangle;\mathbb{F}_p) \cong \mathbb{F}_p[\alpha,\gamma] \otimes \wedge_p(a,v)$, where $\alpha = \sigma(a)$
- $H^*(\langle AB^k,C\rangle;\mathbb{F}_p) \cong \mathbb{F}_p[\omega_k,\gamma] \otimes \wedge_p(w_k,v)$, where $\omega_k = \sigma(w_k)$.

The induced actions of ϕ_1 and $\phi_{2,k}$ on the cohomology of H and $\langle A,C\rangle$ are

$$\phi_1(u) = a \qquad \phi_{2,k}(u) = -kw_k$$

$$\phi_1(v) = -v \quad \phi_{2,k}(v) = v$$

$$\phi_1(a) = u \qquad \phi_{2,k}(a) = w_k$$

$$\phi_1(\beta) = \alpha \qquad \phi_{2,k}(\beta) = -k\omega_k$$

$$\phi_1(\gamma) = -\gamma \quad \phi_{2,k}(\gamma) = \gamma$$

$$\phi_1(\alpha) = \beta \qquad \phi_{2,k}(\alpha) = \omega_k.$$

Using the commutativity of the diagram

$$\begin{array}{ccc} H^*(P) & \xrightarrow{\phi^*} & H^*(P) \\ \downarrow{\scriptstyle Res} & & \downarrow{\scriptstyle Res} \\ H^*(E) & \xrightarrow{\phi^*} & H^*(E') \end{array}$$

for all subgroups E, E' of P (that is, commutativity of induced automorphisms on cohomology and restriction maps) and Lemma 1.5.1, we obtain the restrictions to all subgroups of P and we can therefore compute the restrictions of the generators of $H^*(Qd(p);\mathbb{F}_p)$. See the appendix for these tables.

We will not show all of the computations, but the following illustrates the strategy:

$$\begin{aligned} Res^P_{\langle AB^k,C\rangle}(Y') &= \phi_{2,k}(Res^P_{\langle A,C\rangle}(Y' + kY)) \\ &= \phi_{2,k}(-kav) \\ &= -kw_k v. \end{aligned}$$

Note: these results agree with those of [19] on integral cohomology.

Element	**Degree**	**Image under $\mathbf{Res}^{\mathbf{P}}_{\langle \mathbf{A},\mathbf{C} \rangle}$**
y	1	a
y'	1	0
x	2	α
x'	2	0
Y	2	$-av$
Y'	2	0
X	3	$-v\alpha + a\gamma$
X'	3	0
d_i	2i-1	$0, \qquad i < p-1$
d_{p-1}	2p-3	$-\alpha^{p-2}a$
d_p	2p-1	$-\alpha^{p-2}(v\alpha - a\gamma)$
c_i	2i	$0, \qquad i < p-1$
c_{p-1}	2p-2	$-\alpha^{p-1}$
z	2p	$-\gamma^p + \gamma\alpha^{p-1}$

Table 1.4: Images under the map $Res^{P}_{\langle A,C \rangle}$

Element	**Degree**	**Image under $\mathbf{Res}^{\mathbf{P}}_{\langle \mathbf{AB^k},\mathbf{C}\rangle}$**
y	1	w_k
y'	1	$-kw_k$
x	2	ω_k
x'	2	$-k\omega_k$
Y	2	$-w_k v$
Y'	2	$-kw_k v$
X	3	$-(v\omega_k - w_k\gamma)$
X'	3	$-k(v\omega_k - w_k\gamma)$
d_i	2i-1	$0, \qquad i < p-1$
d_{p-1}	2p-3	$-\omega_k^{p-2} w_k$
d_p	2p-1	$-\omega_k^{p-2}(v\omega_k - w_k\gamma)$
c_i	2i	$0, \qquad i < p-1$
c_{p-1}	2p-2	$-\omega_k^{p-1}$
z	2p	$\gamma^p - \gamma\omega_k^{p-1}$

Table 1.5: Images under the map $Res^{P}_{\langle AB^k,C\rangle}$

Generator	$\mathbf{Res}^{\mathbf{Qd(p)}}_{\langle \mathbf{B},\mathbf{C}\rangle}$	$\mathbf{Res}^{\mathbf{Qd(p)}}_{\langle \mathbf{A},\mathbf{C}\rangle}$
$z^{p-1}+x'^{p(p-1)}$	$\sum_{i=0}^{p-1}(\gamma^{p-1})^{p-i}(\beta^{p-1})^{i}$	$(\gamma^p-\gamma\alpha^{p-1})^{p-1}$
$x'z$	$\beta\gamma^p-\gamma\beta^p$	0
$c_{p-1}z^{p-1}$	$-(\beta\gamma^p-\gamma\beta^p)^{p-1}$	$-(\alpha\gamma^p-\gamma\alpha^p)^{p-1}$
$x^{\frac{p-1}{2}}$	0	$\alpha^{\frac{p-1}{2}}$
x'^2x	0	0
x^nz^{p-1-2n}	0	$\alpha^n(\gamma^p-\gamma\alpha^{p-1})^{p-1-2n}$
$x'^{p-1}+c_{p-1}$	0	$-\alpha^{p-1}$
Y'	uv	0
X'	$v\beta-u\gamma$	0
$x'^{p-1}X'-zy'$	$v\beta^p-u\gamma^p$	0
$d_{p-1}+x'^{p-2}y'$	0	$-\alpha^{p-2}a$
x'^2y	0	0
c_iz^{p-1-i}	0	0
d_iz^{p-1-i}	0	0
$x^nz^{p-3-2n}y$	0	$\alpha^n(\gamma^p-\gamma\alpha^{p-1})^{p-3-2n}a$
$z^{p-4}Y$	0	$-(\gamma^p-\gamma\alpha^{p-1})^{p-4}av$
$z^{p-4}X$	0	$-(\gamma^p-\gamma\alpha^{p-1})^{p-4}(v\alpha-a\gamma)$

Table 1.6: Images under the restriction maps to $\langle B,C\rangle$ and $\langle A,C\rangle$

Generator	$\mathbf{Res}^{\mathbf{Qd(p)}}_{\langle \mathbf{AB^k,C} \rangle}$
$z^{p-1} + x'^{p(p-1)}$	$(\gamma^p - \gamma\omega_k^{p-1})^{p-1} + \omega_k^{p(p-1)}$
$x'z$	$-k\omega_k(\gamma^p - \gamma\omega_k^{p-1})$
$c_{p-1}z^{p-1}$	$-(\omega_k\gamma^p - \gamma\omega_k^p)^{p-1}$
$x^{\frac{p-1}{2}}$	$\omega_k^{\frac{p-1}{2}}$
x'^2x	$k^2\omega_k^3$
$x^n z^{p-1-2n}$	$\omega_k^n(\gamma^p - \gamma\omega_k^{p-1})^{p-1-2n}$
$x'^{p-1} + c_{p-1}$	0
Y'	$-kw_kv$
X'	$-k(v\omega_k - w_k\gamma)$
$x'^{p-1}X' - zy'$	$k^{-1}(v\omega_k^p - w_k\gamma^p)$
$d_{p-1} + x'^{p-2}y'$	0
x'^2y	$k^2\omega_k w_k$
c_iz^{p-1-i}	0
d_iz^{p-1-i}	0
$x^n z^{p-3-2n}y$	$\omega_k^n(\gamma^p - \gamma\omega_k^{p-1})^{p-3-2n}w_k$
$z^{p-4}Y$	$(\gamma^p - \gamma\omega_k^{p-1})^{p-4}w_kv$
$z^{p-4}X$	$(\gamma^p - \gamma\omega_k^{p-1})^{p-4}(v\omega_k - w_k\gamma)$

Table 1.7: Images under the map $Res^{Qd(p)}_{\langle AB^k,C\rangle}$

Generator	$\mathbf{Res}^{\mathbf{Qd(p)}}_{\langle \mathbf{C} \rangle}$	$\mathbf{Res}^{\mathbf{Qd(p)}}_{\langle \mathbf{B} \rangle}$	$\mathbf{Res}^{\mathbf{Qd(p)}}_{\langle \mathbf{A} \rangle}$	$\mathbf{Res}^{\mathbf{Qd(p)}}_{\langle \mathbf{AB^k} \rangle}$
$z^{p-1} + x'^{p(p-1)}$	$\gamma^{p(p-1)}$	$\beta^{p(p-1)}$	0	$\omega_k^{p(p-1)}$
$x'z$	0	0	0	0
$c_{p-1}z^{p-1}$	0	0	0	0
$x^{\frac{p-1}{2}}$	0	0	$\alpha^{\frac{p-1}{2}}$	$\omega_k^{\frac{p-1}{2}}$
x'^2x	0	0	0	$k^2\omega_k^3$
$x^n z^{p-1-2n}$	0	0	0	0
$x'^{p-1} + c_{p-1}$	0	0	$-\alpha^{p-1}$	0
Y'	0	0	0	0
X'	0	0	0	0
$x'^{p-1}X' - zy'$	0	0	0	0
$d_{p-1} + x'^{p-2}y'$	0	0	$-\alpha^{p-2}a$	$-\omega_k^{p-2}w_k$
x'^2y	0	0	0	$k^2\omega_k w_k$
$c_i z^{p-1-i}$	0	0	0	0
$d_i z^{p-1-i}$	0	0	0	0
$x^n z^{p-3-2n}y$	0	0	0	0
$z^{p-4}Y$	0	0	0	0
$z^{p-4}X$	0	0	0	0

Table 1.8: Images under restriction maps to rank-1 subgroups

1.9.1 The Case $p = 3$

Recall from Section 1.8 that $< y, y, y >= x$ and $< y', y', y' >= x'$ in $H^*(P_3; \mathbb{F}_3)$. This affects the automorphisms of $H^*(P_3; \mathbb{F}_3)$ induced by automorphisms of P_3, but only on the generators Y and Y'. The induced automorphisms have the following affect:

Element	**Image Under ϕ_1^***	**Image Under $\phi_{2,\mathbf{k}}^*$**
y	y'	y
y'	y	$y' + ky$
x	x'	x
x'	x	$x' + kx$
Y	Y'	$Y + kx$
Y'	Y	$Y' - kY + k^2x$
X	X'	X
X'	X	$X' - kX$
z	$-z$	z

Therefore, the restrictions are:

Element	**Degree**	**Image under $\mathbf{Res}^{\mathbf{P_3}}_{\langle \mathbf{A},\mathbf{C}\rangle}$**
y	1	a
y'	1	0
x	2	α
x'	2	0
Y	2	$-av$
Y'	2	0
X	3	$-v\alpha + a\gamma$
X'	3	0
z	2p	$-\gamma^3 + \gamma\alpha^2$

Table 1.9: Images under the map $Res^{P_3}_{\langle A,C\rangle}$

Element	**Degree**	**Image under $\mathbf{Res}^{\mathbf{P_3}}_{\langle \mathbf{AB^k},\mathbf{C}\rangle}$**
y	1	w_k
y'	1	$-kw_k$
x	2	ω_k
x'	2	$-k\omega_k$
Y	2	$-k\omega_k + w_k v$
Y'	2	$kw_k v - 2k^2\omega_k$
X	3	$v\omega_k - w_k\gamma$
X'	3	$k(v\omega_k - w_k\gamma)$
z	2p	$\gamma^3 - \gamma\omega_k^2$

Table 1.10: Images under the map $Res^{P_3}_{\langle AB^k,C\rangle}$

Generator	$\mathbf{Res}^{\mathbf{Qd(3)}}_{\langle \mathbf{B},\mathbf{C}\rangle}$	$\mathbf{Res}^{\mathbf{Qd(3)}}_{\langle \mathbf{A},\mathbf{C}\rangle}$
$z^2+x'^6$	$\sum_{i=0}^{3}\gamma^{2(3-i)}\beta^{2i}$	$(\gamma^3-\gamma\alpha^2)^2$
$x'z$	$\beta\gamma^3-\gamma\beta^3$	0
y	0	a
x	0	α
Y'	uv	0
X'	$v\beta-u\gamma$	0
$x'^2X'-zy'$	$v\beta^3-u\gamma^3$	0
zY	0	$-(\gamma^3-\gamma\alpha^2)av$
zX	0	$-(\gamma^3-\gamma\alpha^2)(v\alpha-a\gamma)$

Table 1.11: Images under the restriction maps to $\langle B,C\rangle$ and $\langle A,C\rangle$

Generator	$\mathbf{Res}^{\mathbf{Qd(3)}}_{\langle \mathbf{AB^k},\mathbf{C}\rangle}$
$z^2+x'^6$	$(\gamma^3-\gamma\omega_k^2)^2+\omega_k^6$
$x'z$	$-k\omega_k(\gamma^3-\gamma\omega_k^2)$
y	w_k
x	ω_k
Y'	$kw_kv-2k^2\omega_k$
X'	$k(v\omega_k-w_k\gamma)$
$x'^2X'-zy'$	$v\omega_k^3-w_k\gamma^3$
zY	$(\gamma^3-\gamma\omega_k^2)(-k\omega_k+w_kv)$
zX	$(\gamma^3-\gamma\omega_k^2)(v\omega_k-w_k\gamma)$

Table 1.12: Images under the map $Res^{Qd(3)}_{\langle AB^k,C\rangle}$

Generator	$\mathbf{Res}^{\mathbf{Qd(3)}}_{\langle \mathbf{C} \rangle}$	$\mathbf{Res}^{\mathbf{Qd(3)}}_{\langle \mathbf{B} \rangle}$	$\mathbf{Res}^{\mathbf{Qd(3)}}_{\langle \mathbf{A} \rangle}$	$\mathbf{Res}^{\mathbf{Qd(3)}}_{\langle \mathbf{AB^k} \rangle}$
$z^2 + x'^6$	γ^6	β^6	0	ω_k^6
$x'z$	0	0	0	0
y	0	0	a	w_k
x	0	0	α	ω_k
Y'	0	0	0	$-2k^2\omega_k$
X'	0	0	0	0
$x'^2X' - zy'$	0	0	0	0
zY	0	0	0	0
zX	0	0	0	0

Table 1.13: Images under restriction maps to rank-1 subgroups

Chapter 2

The Cohomology of $PSL(3,p)$

The group $PSL(3,p)$ is the quotient of $SL(3,p)$ by the central elements

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix},$$

where $\lambda \in \mathbb{F}_p^\times$ with $\lambda^3 = 1$. Equivalently, we may view elements of $PSL(3,p)$ as equivalence classes of elements in $SL(3,p)$, with two elements being equivalent if they differ by multiplication by an element of the center. Throughout this chapter, we will represent an equivalence class or coset by a matrix.

The group $PSL(3,p)$ has order

$$\frac{p^3(p^3-1)(p^2-1)}{gcd(p-1,3)}$$

and is simple of rank 2. We would like to know the cohomology of this rank-2 group in order to gain some insight into whether or not it can act freely on a finite complex having the homotopy type of a product of two spheres. Recall that groups without effective Euler classes, exactly those which p'-involve the group $Qd(p)$ (proved in [22]), do not yield to Adem and Smith's construction in [4]. Therefore, a new approach is necessary to construct an action, if one exists, and we hope that the cohomology of $PSL(3,p)$ will suggest some avenues.

We compute the cohomology of $PSL(3,p)$ in much the same way as the cohomol-

ogy of $Qd(p)$ was computed. Indeed, knowing $H^*(Qd(p);\mathbb{F}_p)$ will simplify the computations since $Qd(p)$ embeds in $PSL(3,p)$. In particular, these groups have the same Sylow-p subgroup P, which gives the isomorphisms

$$H^*(PSL(3,p);\mathbb{F}_p) \overset{Res}{\cong} H^*(Qd(p);\mathbb{F}_p)^{stab(PSL(3,p))}$$

and

$$H^*(PSL(3,p);\mathbb{F}_p) \overset{Res}{\cong} H^*(P);\mathbb{F}_p)^{stab(PSL(3,p))}$$

by Theorem 1.1.2. We will use both of these isomorphisms in our computations and we will regard $H^*(Qd(p);\mathbb{F}_p)$ as a subring of $H^*(P;\mathbb{F}_p)$ whenever it is convenient to do so.

Recall that $Qd(p)$ is given by the extension

$$0 \longrightarrow (\mathbb{Z}_p)^2 \longrightarrow Qd(p) \rightleftarrows SL(2,p) \longrightarrow 1$$

with $SL(2,p)$ acting on $(\mathbb{Z}_p)^2$ by matrix multiplication on the left, having Sylow-p subgroup

$$P = \langle A,B,C | A^p = B^p = C^p = 1, C = [A,B], AC = CA, BC = CB \rangle,$$

where $C = [A,B] = A^{-1}B^{-1}AB$. P embeds in $Qd(p)$ as:

$$A = \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right), B = \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right), C = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right).$$

There is an embedding of $Qd(p)$ into $PSL(3,p)$ given by

$$\left(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) \mapsto \begin{pmatrix} a & b & x \\ c & d & y \\ 0 & 0 & 1 \end{pmatrix}.$$

Denote the image of M under this map by $\bar{M}$; we will sometimes use these interchangeably. Note that the image is, in fact, contained in $PSL(3,p)$ because of the entry 1 in the lower right. Furthermore, since the elements $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ generate $SL(2,p)$ (see [5]), $Qd(p)$ has generators $\bar{A}$, $\bar{B}$, $\bar{C}$, and $\bar{D} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

In this chapter, we will compute $H^*(PSL(3,p);\mathbb{F}_p)$ and $H^*(PSL(3,p);\mathbb{Z})_{(p)}$ for $p \geq 3$. See [3] for some properties of the mod-2 cohomology of $PSL(3,p)$.

2.1 Computation of Stable Elements

We want to find the stable elements of $H^*(P;\mathbb{F}_p)$, that is, the elements making the following diagram commute for all $g \in PSL(3,p)$:

$$\begin{array}{ccc} H^*(P;\mathbb{F}_p) & \xrightarrow{\quad c_g^* \quad} & H^*(gPg^{-1};\mathbb{F}_p) \\ & {\scriptstyle res} \searrow \quad \swarrow {\scriptstyle res} & \\ & H^*(P \cap gPg^{-1};\mathbb{F}_p) & \end{array}$$

If $g \in N_{PSL(3,p)}(Qd(p))$, $P \cap gPg^{-1} = P$, so the stable elements must be contained in the set of fixed points under the action of $W_{PSL(3,p)}(Qd(p))$. We compute these now.

2.1.1 The Weyl Groups of P and $Qd(p)$ in $PSL(3,p)$ and Their Induced Actions on Cohomology

The two Weyl groups are related, as we now show.

Claim 2.1.1. *The Weyl group* $W_{PSL(3,p)}(Qd(p)) = N_{PSL(3,p)}(Qd(p))/Qd(p)$ *is generated by the image of* $\begin{pmatrix} 1 & 0 & 0 \\ 0 & i^{-1} & 0 \\ 0 & 0 & i \end{pmatrix}$ *in* $PSL(3,p)$*, where* $i \in \mathbb{F}_p^\times$ *is a primitive root, and* $W_{PSL(3,p)}(P)$ *is generated by the images of* $\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ *and* $\begin{pmatrix} 1 & 0 & 0 \\ 0 & i^{-1} & 0 \\ 0 & 0 & i \end{pmatrix}$ *in* $PSL(3,p)$*, where* i *and* α *are primitive roots in* $\mathbb{F}_p^\times$.

Proof: Recall that an arbitrary element of $Qd(p)$ as a subgroup of $PSL(3,p)$ is given by $\begin{pmatrix} \alpha & \beta & x \\ \gamma & \delta & y \\ 0 & 0 & 1 \end{pmatrix}$. If an element $M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ of $SL(3,p)$ normalizes $Qd(p)$, it is straightforward to check that g and h must be 0. In fact, there are no further restrictions on normalizing elements, so $N_{PSL(3,p)}(Qd(p))$ is generated by the images of

$$\left\{ \begin{pmatrix} a & b & c \\ d & e & f \\ 0 & 0 & i \end{pmatrix} : a,b,c,d,e,f,i \in \mathbb{F}_p, i(ae-bd) = 1 \right\}$$

in $PSL(3,p)$. To generate $\begin{pmatrix} a & b & c \\ d & e & f \\ 0 & 0 & i \end{pmatrix}$, take $\begin{pmatrix} a & bi & ci^{-1} \\ d & ei & fi^{-1} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & i^{-1} & 0 \\ 0 & 0 & i \end{pmatrix}$.

Note that $\begin{pmatrix} a & bi & ci^{-1} \\ d & ei & fi^{-1} \\ 0 & 0 & 1 \end{pmatrix} \in Qd(p)$ since $i(ae-bd) = 1$ by assumption. Further-

more, two matrices $\begin{pmatrix} 1 & 0 & 0 \\ 0 & i^{-1} & 0 \\ 0 & 0 & i \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 0 \\ 0 & j^{-1} & 0 \\ 0 & 0 & j \end{pmatrix}$ are equivalent in $PSL(3,p)$ if and only if $i = j$, so that $W_{PSL(3,p)}(Qd(p)) \cong \mathbb{F}_p^\times \cong \mathbb{Z}_{p-1}$, generated by the matrix $\bar{i} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & i^{-1} & 0 \\ 0 & 0 & i \end{pmatrix}$ with i a primitive root.

An arbitrary element of $P \leq PSL(3,p)$ is given by $\begin{pmatrix} 1 & \beta & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$. Again, it is straightforward to verify directly that if M normalizes P, it must have $d,g,h = 0$, and therefore $aei = 1$. Hence, M can be written as the product

$$\begin{pmatrix} 1 & bia & ci^{-1} \\ 0 & 1 & fi^{-1} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 & 0 \\ 0 & a^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & i^{-1} & 0 \\ 0 & 0 & i \end{pmatrix},$$

since $a^{-1}i^{-1} = e$.

Recall that the Weyl group of P in $Qd(p)$ is $\left\langle \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle$ (see Section 1.5).

□

As before, the action of the Weyl group $W_{PSL(3,p)}(P)$ induces an automorphism of P, and it is easily verified that

$$A \mapsto A^i, B \mapsto B^{i^{-2}}, C \mapsto C^{i^{-1}}.$$

The effect on the cohomology of P and $Qd(p)$ can easily be computed:

Generator	**$\bar{\mathbf{i}}$-Image**	**Generator**	**$\bar{\mathbf{i}}$-Image**
y	iy	y'	$i^{-2}y'$
x	ix	x'	$i^{-2}x'$
Y	Y	Y'	$i^{-3}Y'$
X	X	X'	$i^{-3}X'$
d_j	$i^{-j}d_j$	c_j	$i^{-j}c_j$
z	$i^{-1}z$		
$x'z$	$i^{-3}x'z$		
$z^{p-1}+x'^{p(p-1)}$	$z^{p-1}+x'^{p(p-1)}$		
$x'^{p-1}+c_{p-1}$	$x'^{p-1}+c_{p-1}$		
x'^2x	$i^{-3}x'^2x$		
x^kz^{p-1-2k}	$i^{k-(p-1-2k)}x^kz^{p-1-2k}=i^{3k}x^kz^{p-1-2k}$		
$x'^{p-1}X'-zy'$	$i^{-3}x'^{p-1}X'-zy'$		
$x'^{p-2}y'+d_{p-1}$	$x'^{p-2}y'+d_{p-1}$		
x'^2y	$i^{-3}x'^2y$		
d_jz^{p-1-j}	d_jz^{p-1-j}		
c_jz^{p-1-j}	c_jz^{p-1-j}		
$x^nz^{p-3-2n}y$	$i^{3n+3}x^nz^{p-3-2n}y$		
$z^{p-4}Y$	$i^{-(p-4)}z^{p-4}Y$		
$z^{p-4}X$	$i^{-(p-4)}z^{p-4}X$.		

Table 2.1: Images of Generators of $\mathbf{H}^*(\mathbf{Qd}(\mathbf{p});\mathbb{F}_\mathbf{p})$ Under the $\bar{\mathbf{i}}$-Action

Notice that the value of $gcd(p-1,3)$ will affect the fixed points. As in the computation of $H^*(Qd(p);\mathbb{F}_p)$, the action of the Weyl group takes each generator to itself times a power of i, so a linear combination of generators is fixed if and only if each constituent monomial is fixed. Recalling the forms of arbitrary elements in $H^*(Qd(p);\mathbb{F}_p)$ (see 1.6.1), we can easily show that

Proposition 2.1.2. *$H^*(Qd(p);\mathbb{F}_p)^{W_{PSL(3,p)}(Qd(p))}$ is generated over $\mathbb{F}_p$ by $z^{p-1}+x'^{p(p-1)}$, x^{p-1}, $x'^{p-1}+c_{p-1}$, $x'^{p-2}y'+d_{p-1}$, d_iz^{p-1-i}, c_iz^{p-1-i}, $z^{p-4}XY'$, $z^{p-4}XX'$, $z^{p-4}Y(x'^2x)$, $z^{p-4}X(x'^2x)$, $z^{p-4}Y(x'z)$, $z^{p-4}X(x'z)$, $x^mz^{p-3-2m}y(x'^2x)^{m+1-m'}(x'z)^{m'}$, $x^kz^{p-1-2k}(x'^2x)^{k-t}(x'z)^t$, $x^kz^{p-1-2k}(x'^2x)^{k-1-t'}(x'z)^{t'}\left\{\begin{array}{c} Y' \\ X' \\ x'^2y \\ x'^{p-1}X'-zy' \end{array}\right\}$, and the following generators, which depend on $gcd(p-1,3)$: if $gcd(p-1,3)=1$, we have: $(x'^2x)^{p-1-s}(x'z)^s$, and $(x'^2x)^{p-2-s'}(x'z)^{s'}\left\{\begin{array}{c} Y' \\ X' \\ x'^2y \\ x'^{p-1}X'-zy' \end{array}\right\}$. If $3n=p-1$, we have $(x'^2x)^{n-h}(x'z)^h$, $(x'^2x)^{n-1-h'}(x'z)^{h'}\left\{\begin{array}{c} Y' \\ X' \\ x'^2y \\ x'^{p-1}X'-zy' \end{array}\right\}$, x^nz^{p-1-2n}, $x^{\frac{p-1}{2}}x^{\frac{n}{2}}z^{p-1-n}$ with the following conditions on the exponents and indices:*

- $4 \le i < p-1$

- $0 \leq m \leq \frac{p-3}{2}$, $0 \leq m' \leq m+1$;
- $0 < k \leq \frac{p-1}{2}$
- $0 \leq t \leq k$, $0 \leq t' \leq k-1$;
- $0 \leq s \leq p-1$, $0 \leq s' \leq p-2$;
- $0 \leq h \leq n$, $0 \leq h' \leq n-1$.

2.1.2 Stability Under Elements of $PSL(3,p)$ Which Do Not Normalize P

Finding the subring of stable elements of $H^*(P;\mathbb{F}_p)$ under $PSL(3,p)$ will not be as easy as in the case of $Qd(p)$, since $PSL(3,p)$ is simple and therefore Lemma 1.1.1 does not apply. First, we find the elements of $PSL(3,p)$ which do not normalize P. Since nonsingular matrices over a field have an LU-decomposition, an arbitrary matrix in $PSL(3,p)$ has a preimage $N \in SL(3,p)$ which can be written as $N = LU$, where L is a lower-triangular matrix of the form

$$L = \begin{pmatrix} 1 & 0 & 0 \\ c & 1 & 0 \\ a & b & 1 \end{pmatrix}$$

and U is an upper-triangular matrix; note that $U \in N_{PSL(3,p)}(P)$ since the normalizer is the subgroup of upper-triangular matrices. Since $c_N = c_L c_U$,

$$P \cap NPN^{-1} = P \cap LPL^{-1}.$$

We need to know these intersections in order to compute the stable elements. Let M be a matrix of the form

$$M = \begin{pmatrix} 1 & 0 & 0 \\ c & 1 & 0 \\ a & b & 1 \end{pmatrix}.$$

By direct inspection, we find that

- If $M = \begin{pmatrix} 1 & 0 & 0 \\ c & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $P \cap MPM^{-1} = \langle B, C \rangle$;
- If $M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & b & 1 \end{pmatrix}$, $P \cap MPM^{-1} = \langle A, C \rangle$;
- Otherwise, $P \cap MPM^{-1} = \{1\}$.

Note that

$$M_A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

generates all matrices of the form $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & b & 1 \end{pmatrix}$. Likewise, $M_B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ generates all matrices of the form $\begin{pmatrix} 1 & 0 & 0 \\ c & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. These matrices induce the following auto-

morphisms of $\langle A, C\rangle$ and H,

$$M_A: \quad A \mapsto A \qquad M_B: \quad B \mapsto BC$$
$$C \mapsto A^{-1}C \qquad\qquad C \mapsto C$$

i.e., M_A normalizes $\langle A, C\rangle$ and M_B normalizes H. (We should expect the latter since $M_B \in Qd(p)$ and H is normal there.) All of $H^*(P;\mathbb{F}_p)$ is stable under the M for which $P \cap MPM^{-1} = \{1\}$. Furthermore, since

$$Res_P^{PSL(3,p)}(w) = Res_P^{Qd(p)}(Res_{Qd(p)}^{PSL(3,p)}(w))$$

for all $w \in H^*(PSL(3,p);\mathbb{F}_p)$ and both maps are monomorphisms (see Theorem 1.1.2), it suffices to find the stable elements in $H^*(Qd(p);\mathbb{F}_p)$. Since $M_B \in Qd(p)$, clearly all of $H^*(Qd(p);\mathbb{F}_p)$ is stable under this element.

Therefore, all that remains is to find the stable elements of $H^*(Qd(p);\mathbb{F}_p)$ under M_A. We claim that these are exactly the inverse image of the fixed points $H^*(\langle A, C\rangle;\mathbb{F}_p)^{M_A}$ under the restriction map $Res_{\langle A,C\rangle}^{Qd(p)}$. For, $M_A \in N_{PSL(3,p)}(\langle A, C\rangle)$ acts on $H^*(\langle A, C\rangle)$ and the following diagram commutes:

$$\begin{array}{ccc} H^*(P) & \xrightarrow{M_A^*} & H^*(M_A P M_A^{-1}) \\ \downarrow Res & & \downarrow Res \\ H^*(\langle A, C\rangle) & \xrightarrow{M_A^*} & H^*(\langle A, C\rangle) \end{array}$$

Finally, we will use an argument similar to the one given in the proof of Theorem 1.6.1 to compute $H^*(PSL(3,p);\mathbb{F}_p)$.

The induced action of $\langle M_A\rangle$ on

$$H^*(\langle A, C\rangle;\mathbb{F}_p) \cong \mathbb{F}_p[\alpha,\gamma] \otimes \wedge_p(a, v)$$

(where α is the image of a under the Bockstein and γ is the image of v) is

$$a \mapsto a - v \qquad v \mapsto v$$

$$\alpha \mapsto \alpha - \gamma \qquad \gamma \mapsto \gamma$$

This corresponds to the action of the matrix $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ on $\mathbb{F}_p[\alpha,\gamma] \otimes \wedge_p(a,v)$. Therefore, the results of H. Mui of invariants of $GL(2,p)$ apply (see [32]). In the course of his proof of Theorem 1.4.3, Mui proves:

Lemma 2.1.3 (Mui, [32]). . *The fixed points of* $\mathbb{F}_p[\alpha,\gamma] \otimes \wedge(a,v)$ *under the action of* $\langle M_A \rangle$ *are generated over* $\mathbb{F}_p$ *by* v, γ, av, $v\alpha - a\gamma$, $v\alpha^p - a\gamma^p$, *and* $\alpha^p - \alpha\gamma^{p-1}$.

(In fact, Mui's lemma gives the fixed points for the general case $H^*((\mathbb{Z}_p)^k;\mathbb{F}_p)$.)

It now remains to find the preimage of these elements under $Res^{Qd(p)}_{\langle A,C\rangle}$. We will proceed as in the computation of $H^*(Qd(p);\mathbb{F}_p)$ and will skip some verifications. We know the form of an arbitrary even-degree element of $H^*(Qd(p);\mathbb{F}_p)$ (see Remark 1.6.1):

$$f_1 + f_2 z^{p-4} Y + f_3 Y' + \left\{ \begin{array}{c} z^{r_1(p-1)+p-4} XX' \\ c_i z^{r_i(p-1)+p-1-i} \\ (x'^{p-1} + c_{p-1}) z^{r(p-1)} \end{array} \right\}.$$

The restriction is

$$\bar{f}_1 - \bar{f}_2(\gamma^p - \gamma\alpha^{p-1})^{p-4} av + 0 + \left\{ \begin{array}{c} 0 \\ 0 \\ -\alpha^{p-1}(\gamma^p - \gamma\alpha^{p-1})^{r(p-1)} \end{array} \right\}.$$

This element is fixed if and only if $\bar{f}_1 - \alpha^{p-1}(\gamma^p - \gamma\alpha^{p-1})^{r(p-1)}$ and $\bar{f}_2(\gamma^p - \gamma\alpha^{p-1})^{p-4}$ are fixed.

Recalling the tables of restrictions in Subsection 1.9, we can check that the following generators of $H^*(Qd(p);\mathbb{F}_p)$ over $\mathbb{F}_p$ are in the kernel of $Res^{Qd(p)}_{\langle A,C\rangle}$:

$$x'z,\ x'^2x,\ Y',\ X',\ x'^{p-1}X'-zy',\ x'^2y,\ d_iz^{p-1-i},\ c_iz^{p-1-i}.$$

In particular, we have $Res^P_{\langle A,C\rangle}(x')=0$, $Res^P_{\langle A,C\rangle}(x)=\alpha$, and $Res^P_{\langle A,C\rangle}(z)=\gamma^p-\gamma\alpha$. By an argument exactly as in the proof of Lemma 1.4.6, we can show

Lemma 2.1.4. *The map* $\mathbb{F}_p[x,z]\xrightarrow{g}\mathbb{F}_p[\alpha,\gamma]$ *given by*

$$g:\begin{cases} x\mapsto\alpha \\ z\mapsto\gamma^p-\gamma\alpha^{p-1}\end{cases}$$

is a monomorphism on homogeneous elements.

Furthermore, we can show that a homogeneous polynomial in the elements x^kz^{p-1-2k} has the form

$$bx^{n(\frac{p-1}{2})}z^{m(p-1)}x^kz^{p-1-2k}$$

for some $0<k<\frac{p-1}{2}$. Thus, $\bar{f}_t$ is a homogeneous polynomial in $\sum_{j=0}^{p}\gamma^{(p-1)(p-j)}\alpha^{(p-1)j}$ and $\alpha^{n(\frac{p-1}{2})+k}(\gamma^p-\gamma\alpha^{p-1})^{m(p-1)+p-1-2k}$ for some $0<k<\frac{p-1}{2}$. These generators are fixed under the action of $\begin{pmatrix}1&1\\0&1\end{pmatrix}$:

$$\alpha\mapsto\alpha \qquad \gamma\mapsto\alpha+\gamma.$$

The fixed points must also be fixed under the action of $\begin{pmatrix}1&0\\-1&1\end{pmatrix}$ and we can show that these two matrices generate all of $SL(2,p)$. Hence we are looking for the inverse image of $\sum_{j=0}^{p}\gamma^{(p-1)(p-j)}\alpha^{(p-1)j}$ and $\alpha\gamma^p-\gamma\alpha^p$ under the map $Res^{Qd(p)}_{\langle A,C\rangle}$. By the above

lemma (Lemma 2.1.4), and using the same argument as in the argument for $H^*(Qd(p);\mathbb{F}_p)$, the preimage (modulo the kernel of $Res^{Qd(p)}_{\langle A,C\rangle}$) is generated by $z^{p-1}+x'^{p(p-1)}+x^{p(p-1)}$, $(xz)^{p-1}$, and $(x'^{p-1}+c_{p-1})z^{p-1}$. (By a degree argument, we can show that no multiple of $z^{p-4}Y$ can restrict to a fixed point.)

Returning to the odd-degree elements in $H^*(Qd(p);\mathbb{F}_p)$, we see that the restriction of

$$f_1\begin{Bmatrix} x'^2y \\ x^nz^{p-3-2n}y \end{Bmatrix}+f_2\begin{Bmatrix} x'^{p-1}X'-zy' \\ x'^{p-2}y'+d_{p-1} \end{Bmatrix}+f_3z^{p-4}X+f_4X'+\begin{Bmatrix} z^{p-4}XY'g_s \\ d_iz^{p-1-i}g_i \end{Bmatrix}$$

is

$$\bar{f}_1\begin{Bmatrix} 0 \\ \alpha^n(\gamma-\gamma\alpha^{p-1})^{p-3-2n}a \end{Bmatrix}+\bar{f}_2\begin{Bmatrix} 0 \\ -\alpha^{p-1}a \end{Bmatrix}-\bar{f}_3(\gamma-\gamma\alpha^{p-1})^{p-4}(v\alpha-a\gamma)+0+0.$$

Since a and $\bar{f}_3(\gamma-\gamma\alpha^{p-1})^{p-4}$ are never fixed (the latter by the same argument as presented in the even-degree case), it remains to determine when

$$\bar{f}_1\alpha^n(\gamma^p-\gamma\alpha^{p-1})^{p-3-2n}-\bar{f}_2\alpha^{p-1}=0.$$

Observing that $\frac{p-3}{2}+\frac{p-1}{2}=p-2$, we see that $x^{\frac{p-1}{2}}x^{\frac{p-3}{2}}y=x^{p-2}y$ and $f_1x^nz^{p-3-2n}y+f_2(x'^{p-2}y'+d_{p-1})$ must be a multiple of $x^{p-2}y+x'^{p-2}y'+d_{p-1}$. Since $x^{p-1}+x'^{p-1}+c_{p-1}$ is the Bockstein of $x^{p-2}y+x'^{p-2}y'+d_{p-1}$, it must also be in the kernel.

Altogether, we have shown:

Proposition 2.1.5. *The stable elements of $H^*(P;\mathbb{F}_p)$ under the action of the lower-triangular matrices in $PSL(3,p)$ consist of the $\mathbb{F}_p$-subalgebra generated by $z^{p-1}+x'^{p(p-1)}+x^{p(p-1)}$ and $(xz)^{p-1}$ and the kernel of $Res^{Qd(p)}_{\langle A,C\rangle}$, which is the ideal in $H^*(Qd(p);\mathbb{F}_p)$ generated*

by $x'z$, x'^2x, Y', X', $x'^{p-1}X'-zy'$, x'^2y, d_iz^{p-1-i}, c_iz^{p-1-i}, $x^{p-2}y+x'^{p-2}y'+d_{p-1}$, *and* $x^{p-1}+x'^{p-1}+c_{p-1}$.

2.2 The Ring $H^*(PSL(3,p);\mathbb{F}_p)$

All that remains is to compute the intersection of the elements given in Proposition 2.1.5 and $H^*(P;\mathbb{F}_p)^{W_{PSL(3,p)}(P)}$. Recall that the value of $gcd(3,p-1)$ determines the order of $PSL(3,p)$, so it also affects the cohomology ring.

Theorem 2.2.1. *The ring* $H^*(PSL(3,p);\mathbb{F}_p)$ *is isomorphic to the* $\mathbb{F}_p$ *subalgebra of* $H^*(P;\mathbb{F}_p)$ *generated by* $z^{p-1}+x'^{p(p-1)}+x^{p(p-1)}$, $(xz)^{p-1}$, $x^{p-2}y+x'^{p-2}y'+d_{p-1}$, $x^{p-1}+x'^{p-1}+c_{p-1}$, d_iz^{p-1-i}, c_iz^{p-1-i}, $z^{p-4}XY'$, $z^{p-4}XX'$, $z^{p-4}Y(x'^2x)$, $z^{p-4}X(x'^2x)$, $z^{p-4}Y(x'z)$, $z^{p-4}X(x'z)$, $x^mz^{p-3-2m}y(x'^2x)^{m+1-m'}(x'z)^{m'}$, $x^kz^{p-1-2k}(x'^2x)^{k-t}(x'z)^t$,

$$x^kz^{p-1-2k}(x'^2x)^{k-1-t'}(x'z)^{t'}\left\{\begin{array}{c} Y' \\ X' \\ x'^2y \\ x'^{p-1}X'-zy' \end{array}\right\},$$

and the following generators, which depend on $gcd(p-1,3)$*:*

If $gcd(p-1,3)=1$, *we have:* $(x'^2x)^{p-1-s}(x'z)^s$ *and* $(x'^2x)^{p-2-s'}(x'z)^{s'}\left\{\begin{array}{c} Y' \\ X' \\ x'^2y \\ x'^{p-1}X'-zy' \end{array}\right\}$.

If $3n = p-1$, *we have* $(x'^2x)^{n-h}(x'z)^h$, $(x'^2x)^{n-1-h'}(x'z)^{h'}\left\{\begin{array}{c} Y' \\ X' \\ x'^2y \\ x'^{p-1}X' - zy' \end{array}\right\}$, $x^n z^{p-1-2n}$, *and* $x^{\frac{p-1}{2}}x^{\frac{n}{2}}z^{p-1-n}$ *with the following conditions on the exponents and indices:*

- $4 \leq i < p-1$;
- $0 \leq m \leq \frac{p-3}{2}$, $0 \leq m' \leq m+1$;
- $0 < k \leq \frac{p-1}{2}$;
- $0 \leq t \leq k$, $0 \leq t' \leq k-1$;
- $0 \leq s \leq p-1$, $0 \leq s' \leq p-2$;
- $0 \leq h \leq n$, $0 \leq h' \leq n-1$.

Proof: An argument similar to the one presented in Theorem 1.6.1 shows that these generators in fact give all of

$$[Res^P_{\langle A,C\rangle}]^{-1}(H^*(\langle A,C\rangle;\mathbb{F}_p)^{M_A}) \cap H^*(Qd(p);\mathbb{F}_p)^{\bar{i}}.$$

□

2.3 The Integral Cohomology of $PSL(3,p)$

We will only compute the p-primary part,

$$H^*(PSL(3,p);\mathbb{Z})_{(p)}.$$

As in the case of $H^*(Qd(p);\mathbb{Z})_{(p)}$, we look for the elements in $H^*(PSL(3,p);\mathbb{F}_p)$ which lift to integral cohomology. In Proposition 1.7.1, we determined which classes in $H^*(Qd(p);\mathbb{F}_p)$ lift to integral cohomology and, since $H^*(PSL(3,p);\mathbb{F}_p)$ embeds in $H^*(Qd(p);\mathbb{F}_p)$, this proposition is all we need to show:

Theorem 2.3.1. *The ring $H^*(PSL(3,p);\mathbb{Z})_{(p)}$ is generated as an algebra over $\mathbb{Z}$ by $z^{p-1}+x'^{p(p-1)}+x^{p(p-1)}$, $(xz)^{p-1}$, $x^{p-1}+x'^{p-1}+c_{p-1}c_iz^{p-1-i}$, $z^{p-4}XX'$, $z^{p-4}X(x'^2x)$, $z^{p-4}X(x'z)$, $x^kz^{p-1-2k}(x'^2x)^{k-t}(x'z)^t$, $z^{p-3}(x'Y+xY')$, $z^{p-3}(x'^pY+x^pY')$, $x^kz^{p-1-2k}(x'^2x)^{k-1-t'}(x'z)^{t'}X'$, and the following generators, which depend on $gcd(p-1,3)$:*

If $gcd(p-1,3)=1$, we have: $(x'^2x)^{p-1-s}(x'z)^s$ and $(x'^2x)^{p-2-s'}(x'z)^{s'}X'$.

If $3n=p-1$, we have $(x'^2x)^{n-h}(x'z)^h$, $(x'^2x)^{n-1-h'}(x'z)^{h'}X'$, x^nz^{p-1-2n}, and $x^{\frac{p-1}{2}}x^{\frac{n}{2}}z^{p-1-n}$.

These generators satisfy the relations in $H^*(P;\mathbb{F}_p)$ for the generators of the same names, the additional relation

$$px=px'=pX'=pX=pc_i=p(x'Y+xY')=p^2z=0,$$

and the following conditions on the exponents and indices:

- $4\le i<p-1$;
- $0<k\le\frac{p-1}{2}$;
- $0\le t\le k$, $0\le t'\le k-1$;
- $0\le s\le p-1$, $0\le s'\le p-2$;
- $0\le h\le n$, $0\le h'\le n-1$.

2.4 The Case $p = 3$

Proceeding as in the case $p \neq 3$ above, we will skip most verifications. Notice that the action of the Weyl group is as follows:

Generator	**$\bar{\mathbf{i}}$-Image**	**Generator**	**$\bar{\mathbf{i}}$-Image**
y	iy	y'	$i^{-2}y' \equiv y'$
x	ix	x'	$i^{-2}x' \equiv x'$
Y	Y	Y'	$i^{-3}Y' \equiv iY'$
X	X	X'	$i^{-3}X' \equiv iX'$
z	$i^{-1}z \equiv iz$		

Table 2.2: Images of Generators of $H^*(P_3;\mathbb{F}_3)$ Under the $\bar{\mathbf{i}}$-Action

Using Remark 1.8.1, we know that an arbitrary element of $H^*(Qd(3);\mathbb{F}_3)$ can be expressed as:

$$f_1 + f_2Y' + f_3zY$$

in even degrees and

$$f_1y + f_2(x'^2X' - zy') + f_3zX + f_4X'$$

in odd degrees, where the f_i are homogeneous polynomials in x, $x'z$, and $z^2 + x'^6$. Hence, we have:

Theorem 2.4.1. *$H^*(PSL(3,3);\mathbb{F}_3)$ is isomorphic to the subalgebra of $H^*(P(3);\mathbb{F}_3)$ gen-*

erated by $z^2 + x'^6$ *and*

$$\left\{ \begin{array}{c} x'z \\ x \end{array} \right\} \left\{ \begin{array}{c} x'z \\ Y' \\ X' \\ x'^2X' - zy' \\ y \\ x \\ zY \\ zX \end{array} \right\},$$

with braces indicating possible choices. $H^*(PSL(3,3);\mathbb{Z})_{(3)}$ *is generated as an algebra over* $\mathbb{Z}$ *by* $z^2 + x'^6$, $xY' + x'Y$ *and*

$$\left\{ \begin{array}{c} x'z \\ x \end{array} \right\} \left\{ \begin{array}{c} x'z \\ X' \\ x'^2X' - zy' \\ x \\ zX \end{array} \right\},$$

subject to the relations in $H^*(P(3);\mathbb{F}_3)$ *for the generators of the same names and the additional relation* $9z = 3x' = 3x = 3X = 3X' = 3Y = 3Y = 0$.

Chapter 3

Depth and Dimension

One familiar invariant of a ring A is its Krull dimension, $dim(A)$, which is equal to the length of the longest chain of prime ideals in A. Another invariant is the *depth* of a ring, which can be defined in terms of a *regular sequence* in A:

Definition 3.0.1. A **regular sequence** in a ring A is an ordered set of elements $(a_1, \ldots, a_k)$ such that a_1 is not a zero-divisor and, for each $1 < i \leq k$, a_i is not a zero-divisor in $A/(a_1, \ldots, a_{i-1})$. The **depth** of A is the largest possible value of k. A ring whose depth equals its dimension is called **Cohen-Macaulay**.

We are interested in cohomology rings of finite groups. In [35], Quillen shows that

$$dim(H^*(G;\mathbb{F}_p)) = r_p(G)$$

for a finite group G. In order for the mod-p cohomology ring of a finite group to be Cohen-Macaulay, it is necessary that all of the maximal abelian p-subgroups have equal rank:

Proposition 3.0.2 ([10], Prop. 1.2)**.** *Let G be a finite group and k a field of characteristic p. If G has maximal elementary abelian p-subgroups of different ranks, then $H^*(G;k)$ is not Cohen-Macaulay.*

Proof: By Quillen's Dimension Theorem in [35], the irreducible components of the maximal ideal spectrum of $H^*(G;k)$ are in one-to-one correspondence with conju-

gacy classes of maximal elementary abelian p-subgroups of G, and, furthermore, the dimension of the component is equal to the rank of the corresponding subgroup. But all irreducible components of the maximal ideal spectrum of a Cohen-Macaulay ring have equal dimension (see [28] Theorem 17.3). □

The condition is not sufficient, as evidenced by the groups P, $Qd(p)$, and $PSL(3,p)$; these groups each have every maximal elementary abelian subgroup of rank 2, but are not Cohen-Macaulay:

Claim 3.0.3. *The rings $H^*(P;\mathbb{F}_p)$, $H^*(Qd(p);\mathbb{F}_p)$, and $H^*(PSL(3,p);\mathbb{F}_p)$ are not Cohen-Macaulay for $p > 3$.*

Proof: We will use the following theorem of Carlson:

Theorem 3.0.4 ([12])**.** *Let G be a finite group and k a field of characteristic p. Suppose that $H^*(G;k)$ has a nonzero element ζ which restricts to zero on $C_G(E)$ for each elementary abelian p-subgroup of $E \leq G$ of rank s. Then $H^*(G;k)$ has an associated prime q such that the Krull dimension of $H^*(G;k)/q$ is strictly less than s. In particular, the depth of $H^*(G;k)$ is strictly less than s.*

(Recall that a prime ideal $\mathfrak{q}$ in a ring A is *associated* if one of the following equivalent conditions hold (see [27]):

1. there exists an element $w \in A$ with $Ann(w) = \mathfrak{q}$
2. A contains a subring isomorphic to $A/\mathfrak{q}$.)

Since each of these groups has p-rank equal to 2, Quillen's Dimension Theorem (see [35]) shows that the Krull dimension of each of the mod-p cohomology rings of

these groups is 2. In order to show that these rings are not Cohen-Macaulay, we want to show that their depth is strictly less than 2, so it suffices to exhibit a nonzero element ζ in each ring restricting to zero on the centralizer of every rank-2 elementary abelian subgroup.

For the ring $H^*(P;\mathbb{F}_p)$:

- Proof 1. In Section 1.3, we showed that the image of d_i and c_i, $4 \leq i < p-1$, under Res^P_H was zero. Using the maps on cohomology induced by automorphisms of P (computed in Subsection 1.9), we can show that the d_i and c_i restrict to zero on every rank-2 subgroup of P. Since P has order p^3 and is not abelian, every rank-2 subgroup of P must be its own centralizer.

- Proof 2. It suffices to exhibit a rank-2 polynomial subalgebra over which the ring is finitely generated but not free (see also Proposition 2.5.1 in [6]). By the Noether Normalization theorem, there exists a rank-2 polynomial subalgebra R over which $H^*(P;\mathbb{F}_p)$ is finitely generated. If $H^*(P;\mathbb{F}_p)$ were free over R, clearly we could find a regular sequence of length 2 in $H^*(P;\mathbb{F}_p)$. By Macaulay's theorem (cite), it suffices to exhibit one such subalgebra.

  In $H^*(P;\mathbb{F}_p)$, all generators except x, x', c_{p-1}, and z square to zero. The generators x and x' are not algebraically independent, in view of the relation $x^p x' - x'^p x = 0$. In view of the relations $xc_{p-1} = -x^p$ and $x'c_{p-1} = -x'^p$ and the form of elements given in Fact 1.2.1, the ring $H^*(P;\mathbb{F}_p)$ is finitely generated over the subalgebra $\mathbb{F}_p[c_{p-1}, z]$. But since c_{p-1} is a zero-divisor ($c_{p-1}c_i = 0$ for all $4 \leq i < p-1$), $H^*(P;\mathbb{F}_p)$ is not free over $\mathbb{F}_p[c_{p-1}, z]$.

Note that the method of Proof 2 is not as easily implemented in the other cases.

For the rings $H^*(Qd(p);\mathbb{F}_p)$ and $H^*(PSL(3,p);\mathbb{F}_p)$, we utilize

Proposition 3.0.5 ([1] Prop. 2.4)**.** *Let $Ess(P)$ denote the essential cohomology of a p-group P. The elements $Ess(P)^{Out(P)}$ are universally stable, i.e., are contained in any $H^*(G;M)_{(p)}$ for a finite group G with $Syl_p(G) = P$.*

Thus, it suffices to exhibit a nontrivial element in $Ess(P)^{Out(P)}$.

Claim 3.0.6. *For $4 \leq i < p-1$, the elements $c_i z^{p-1-i}$ and $d_i z^{p-1-i}$ fixed under $Out(P)$.*

Proof: Since compositions of the automorphisms ϕ_1 and $\phi_{2,k}$ (see Subsection 1.9) give all possible permutations of conjugacy classes of elementary abelian subgroups of P, it is clear that they generate $Out(P)$. By inspection of the table 1.9, it is clear that $c_i z^{p-1-i}$ and $d_i z^{p-1-i}$ are fixed under ϕ_1 and $\phi 2,k$ for all $4 \leq i < p-1$.

Remark 3.0.1.

Note that, by Duflot's theorem (see [18]): for any finite group G,

$$depth(H^*(G;\mathbb{F}_p)) \geq r_p(Z(Syl_p(G))),$$

each of $H^*(P;\mathbb{F}_p)$, $H^*(Qd(p);\mathbb{F}_p)$, and $H^*(PSL(3,p);\mathbb{F}_p)$ have depth 1.

Theorem 3.0.4 represents a partial solution to the following conjecture:

Conjecture 3.0.7 (see [13])**.** *The depth of $A = H^*(G;k)$ is equal to $dim(A/\mathfrak{q})$ for some associated prime ideal $\mathfrak{q}$.*

We know that the associated primes in $H^*(Qd(p);\mathbb{F}_p)$ are $\sqrt{ker(Res_E^{Qd(p)})}$ (see [6] Theorem 7.5.1). But even when the required associated prime exists, it is not immediately

clear which associated prime realizes the depth. We will now exhibit an associated prime $\mathfrak{q}$ for which $dim(H^*(Qd(p);\mathbb{F}_p))/\mathfrak{q} = 1$.

Proposition 3.0.8. *The annihilator of $d_i z^{p-1-i}$ and $c_i z^{p-1-i}$, $4 \leq i < p-1$, in $H^*(Qd(p);\mathbb{F}_p)$ is the $\mathcal{A}_p$-invariant prime ideal $\sqrt{ker(Res^{Qd(p)}_{\langle C\rangle})} = \sqrt{ker(Res^{Qd(p)}_{\langle B\rangle})}$. The dimension of the quotient by this ideal is* 1.

Proof: The relations in $H^*(P;\mathbb{F}_p)$ imply that the product of d_i or c_i, $4 \leq i < p-1$, with any element other than z is zero. Hence, all generators for $H^*(Qd(p);\mathbb{F}_p)$ over $\mathbb{F}_p$ except for $z^{p-1} + x'^{p(p-1)}$ annihilate $d_i z^{p-1-i}$ and $c_i z^{p-1-i}$, $4 \leq i < p-1$; the ideal they generate is $\mathfrak{q} = ker(Res^{Qd(p)}_{\langle C\rangle})$, as can be seen by inspection of the tables of restrictions in Subsection 1.9. In fact, this ideal is a radical ideal ($\mathfrak{q} = \sqrt{\mathfrak{q}}$) since $Res^{Qd(p)}_{\langle C\rangle}(z^{p-1} + x'^{p(p-1)} = \gamma^{p(p-1)}$, a non-zero divisor, and the restriction map $Res^{Qd(p)}_{\langle C\rangle}$ factors as follows:

$$\begin{array}{ccccc} H^*(Qd(p);\mathbb{F}_p) & & \xrightarrow{\quad Res \quad} & & \mathbb{F}_p[\gamma^{p(p-1)}] \hookrightarrow H^*(\langle C\rangle;\mathbb{F}_p) \\ & \searrow & & \nearrow_{\simeq} & \\ & & H^*(Qd(p);\mathbb{F}_p)/\mathfrak{q} & & \end{array}$$

which implies that $H^*(Qd(p);\mathbb{F}_p)/\mathfrak{q}$ is an integral domain. This also clearly shows that $dim(H^*(Qd(p);\mathbb{F}_p)/\mathfrak{q}) = 1$. □

An analogous argument also shows:

Proposition 3.0.9. *For $p > 3$ and odd prime,*

1. *The annihilator of d_i and c_i, $4 \leq i < p-1$, in $H^*(P;\mathbb{F}_p)$ is the $\mathcal{A}_p$-invariant prime ideal $\sqrt{ker(Res^{P}_{\langle C\rangle})}$.*

2. *The annihilator of* $d_i z^{p-1-i}$ *and* $c_i z^{p-1-i}$, $4 \leq i < p-1$, *in* $H^*(PSL(3,p);\mathbb{F}_p)$ *is the* $\mathcal{A}_p$*-invariant prime ideal* $\sqrt{ker(Res^{PSL(3,p)}_{\langle C \rangle})}$.

In both cases, the dimension of the quotient by this ideal is 1.

3.0.1 The Case $p = 3$

Note that $H^*(P_3;\mathbb{F}_3)$ is Cohen-Macaulay (see [30]) and has no essential cohomology. This implies, by Proposition 6.8 in [10], that $H^*(Qd(3);\mathbb{F}_3)$ is also Cohen-Macaulay.

Bibliography

[1] A. Adem and D. Karagueuzian, *Essential cohomology of finite groups*, Comment. Math. Helv. **72** (1997), 101–109.

[2] A. Adem and R. J. Milgram, *Cohomology of finite groups*, first ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], no. 309, Springer-Verlag, Berlin, 1994.

[3] _____, *The mod* 2 *cohomology rings of rank*-3 *simple groups are Cohen-Macaulay*, Prospects in Topology (Princeton, NJ, 1994), Ann. of Math. Stud., no. 138, Princeton Univ. Press, Princeton, NJ, 1995, pp. 3–12.

[4] Alejandro Adem and Jeff H. Smith, *Periodic complexes and group actions*, Ann. of Math. (2) **154** (2001), no. 2, 407–435.

[5] H. Behr and J. Mennicke, *A presentation of the groups PSL(2,p)*, Canad. J. Math. **20** (1968), 1432–1438.

[6] Dave Benson, *Trends in commutative algebra*, Math. Sci. Res. Inst. Publ., no. 51, ch. "Commutative algebra in the cohomology of groups", pp. 1–50, Cambridge Univ. Press, Cambridge, 2004.

[7] David J. Benson and Jon F. Carlson, *Complexity and multiple complexes*, Math. Z. **195** (1987), no. 2, 221–238.

[8] _____, *The cohomology of extraspecial groups*, Bull. London Math. Soc. **24** (1992), no. 3, 209–235.

[9] _____, *Erratum to "The cohomology of extraspecial groups"*, Bull. London Math. Soc. **25** (1993), 498.

[10] _____, *Projective resolutions and Poincaré duality complexes*, Trans. Amer. Math. Soc. **432** (1995), no. 2, 447–488.

[11] William Browder, *Cohomology and group actions*, Invent. Math. **71** (1983), 559–607.

[12] Jon F. Carlson, *Depth and transfer maps in the cohomology of groups*, Math. Z. **218** (1995), no. 3, 461–468.

[13] _____, *Problems in the calculation of group cohomology*, Computational methods for representations of groups and algebras (Essen, 1997) (Basel) (G. O. Michler P. Dräxler and C. M. Ringel., eds.), Progr. Math., no. 173, Birkhäuser, 1999, pp. 107–120.

[14] Gunnar Carlsson, *On the rank of abelian groups acting freely on* $(S^n)^k$, Invent. Math **69** (1982), 393–400.

[15] ———, *Lecture notes in mathematics*, vol. 1217, ch. Free $(\mathbb{Z}_2)^k$ actions and a problem in commutative algebra, pp. 79–83, Springer, 1985.

[16] H. Cartan and S. Eilenberg, *Homological algebra*, Princeton University Press, Princeton, New Jersey, 1956.

[17] J.F. Davis and R.J. Milgram, *A survey of the spherical space form problem*, Mathematical Reports, vol. 2, Harwood Academic Publishers, 1985.

[18] J. Duflot, *Depth and equivariant cohomology*, Comment. Math. Helv. **56** (1981), no. 4, 627–637.

[19] David John Green and Pham Anh Minh, *Transfer and chern classes for extraspecial p-groups*, Group representations: Cohomology, group actions and topology (Seattle, WA, 1996) (Providence), Proc. Sympos. Pure Math., no. 63, Amer. Math. Soc., 1998, pp. 245–255.

[20] Allen Hatcher, *Algebraic topology*, Cambridge University Press, 2002.

[21] H. Hopf, *Zum Clifford-Kleinschen raumproblem*, Math. Ann. **95** (1925), 313–329.

[22] Michael A. Jackson, *Qd(p)-free rank two finite groups act freely on a homotopy product of two spheres*, J. Pure Appl. Algebra (2007), no. 3, 821–831.

[23] D. Kraines, *Massey higher products*, Trans. Amer. Math. Soc. **124** (1966), 431–449.

[24] I.J. Leary, *The integral cohomology rings of some p-groups*, Math. Proc. Cambridge Philos. Soc. **110** (1991), no. 1, 25–32.

[25] ———, *The mod-p cohomology rings of some p-groups*, Math. Proc. Cambridge Philos. Soc. **112** (1992), no. 1, 63–75.

[26] Gene Lewis, *The integral cohomology rings of groups of order* p^3, Trans. Amer. Math. Soc. **132** (1968), no. 2, 501–529.

[27] Hideyuki Matsumura, *Commutative algebra*, W. A. Benjamin, Inc., New York, 1970.

[28] ———, *Commutative ring theory*, Cambridge Studies in Advanced Mathematics, no. 8, Cambridge University Press, Cambridge, 1989.

[29] John McCleary, *A user's guide to spectral sequences*, Cambridge Studies in Advanced Mathematics, no. 58, Cambridge University Press, Cambridge, 2001.

[30] R.J. Milgram and M. Tezuka, *The geometry and cohomology of* M_{12}*. II.*, Bol. Soc. Mat. Mexicana (3) **1** (1995), no. 2, 91–108.

[31] J. Milnor, *Groups which act on* S^n *without fixed points*, Amer. J. Math. **79** (1957), 623–630.

[32] Huýnh Mui, *Modular invariant theory and cohomology algebras of symmetric groups*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **22** (1975), no. 3, 319–369.

[33] M.D. Neusel and L. Smith, *Invariant theory of finite groups*, Mathematical Surveys and Monographs, vol. 94, American Mathematical Society, 2002.

[34] Robert Oliver, *Free compact group actions on products of spheres*, Algebraic topology, Aarhus 1978 (Proc. Sympos., Univ. Aarhus, Aarhus, 1978), Lecture Notes in Math., no. 763, Springer, Berlin, 1979, pp. 539–548.

[35] Daniel Quillen, *The spectrum of an equivariant cohomology ring. I and II*, Ann. of Math. (2) **94** (1971), 549–572; ibid. 573–602.

[36] U. Ray, *Free linear actions of finite groups on products of spheres*, J. Algebra **147** (1992), 456–490.

[37] Justin R. Smith, *Topological realizations of chain complexes. I. The general theory*, Topology Appl. **22** (1986), 301–313.

[38] P.A. Smith, *Permutable period transformations*, Proc. Nat. Acad. Sci. USA **30** (1944), 105–108.

[39] R. Swan, *Periodic resolutions for finite groups*, Ann. of Math. **72** (1960), 267–291.

[40] C.B. Thomas, *Characteristic classes and the cohomology of finite groups*, Cambridge Studies in Advanced Mathematics, no. 9, Cambridge University Press, Cambridge, 1986.

[41] I. Madsen. C.B. Thomas and C.T.C. Wall, *The topological spherical space form problem II*, Topology **15** (1976), 375–382.

[42] Clarence Wilkerson, *A primer on the Dickson invariants*, Proceedings of the Northwestern Homotopy Theory Conference (Evanston, Ill., 1982) (Haynes R. Miller and Stewart B. Priddy, eds.), Contemporary Mathematics, vol. 19, American Mathematical Society, Providence, R.I., 1983, pp. 421–434.