ABSTRACT

Title of dissertation: EINSTEIN VS. AETHER: CONSTRAINTS ON A CLASS OF LORENTZ-SYMMETRY-VIOLATING THEORIES OF GRAVITY

Brendan Zdon Foster, Doctor of Philosophy, 2007

Dissertation directed by: Professor Theodore Jacobson
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Violation of Lorentz invariance in nature is a possibility suggested by various candidate theories of quantum gravity and exotic extensions of the standard model, and by general curiosity. Lorentz-violating effects in particle interactions are strongly constrained, but effects involving gravity are not. Here, observational constraints on, and theoretical aspects of, a certain class of gravitational theories that violate Lorentz invariance are considered. "Einstein–aether" theory is a four-parameter class of theories in which gravity couples to a dynamical, timelike, unit-norm vector field: the "aether". This family provides a means for studying Lorentz violation in a generally covariant setting. Demonstrated first is the effect on the four parameters, of stretching the metric along the aether direction. Next, the Noether charge method for defining the Hamiltonian of a diffeomorphism invariant field theory is applied to obtain expressions for the total energy, momentum, and angular momentum of an Einstein-aether spacetime. The method is also used to discuss the mechanics of Einstein–aether black holes. Next, the computation of the theory's post-Newtonian parameters are reported. Constraints on their values are combined with other constraints concerning the properties of linearized wave modes and Einstein–aether cosmology. All of these constraints are satisfied by parameters in a large two-dimensional region in the theory's four-dimensional parameter space. Next, constraints from the motion of binary pulsar systems are considered. Derived to lowest post-Newtonian order are wave forms for the metric and aether far from a nearly Newtonian system and the rate of energy radiated by the system, in the limit that effects due to strong fields are neglected. There exists a one-parameter family of Einstein–aether theories for which the radiation rate expression is identical to that of general relativity to the order worked to here. Finally, strong field effects are included by treating the compact bodies as point particles with nonstandard, velocity dependent interactions. Precise constraints cannot be stated for general parameter values until the values of the coupling coefficients of the nonstandard interactions can be calculated for a given stellar source. It is argued, though, that if the parameters are smaller than roughly (0.1), then all current observational tests impose just the three conditions that guarantee agreement in the weak field limit. Thus, there exists a family of Einstein–aether theories, with one mildly bounded free parameter, that satisfy the collected constraints.

EINSTEIN VS. AETHER: CONSTRAINTS ON A CLASS OF LORENTZ-SYMMETRY-VIOLATING THEORIES OF GRAVITY

by

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DEDICATION

To my mother.

To my father.

To blue and green,

A beautiful day.

PREFACE

It took a long time, but it was worth it. I want to thank my family for loving kindness—dad, mom, Vern, Susan, Jennifer, Peter and Julie. I want to thank all my friends and colleagues, especially Chad Finley, Jonathan Fortney, Sam Gannaway, Mike Keller; Niall O'Murchadha, Edward Anderson, Chris Eling, Bryan Kelleher; and my mentor, hero, and friend Julian Barbour.

Finally, I want to give special thanks to my advisor Ted Jacobson for encouragement, ideas, time, and respect.

Portions of this dissertation have been published previously, in slightly altered form.

Chapter 2 as B. Z. Foster, "Metric redefinitions in Einstein-aether theory," Phys. Rev. D **72**, 044017 (2005).

Chapter 3 as B. Z. Foster, "Noether charges and black hole mechanics in Einstein-aether theory," Phys. Rev. D **73**, 024005 (2006).

Chapter 4 as B. Z. Foster and T. Jacobson, "Post-Newtonian parameters and constraints on Einstein-aether theory," Phys. Rev. D **73**, 064015 (2006).

Chapter 5 as B. Z. Foster, "Radiation damping in Einstein-aether theory," Phys. Rev. D **73**, 104012 (2006).

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Chapter 1

Introduction

1.1 Lorentz non-invariance

It is possible that Lorentz invariance is not a fundamental symmetry of nature. It is possible that it is fundamental, but is broken by some yet unknown effect. However it might happen, it is a valid question to ask, does nature exhibit perfect Lorentz symmetry? In recent years, this question has received increasing attention, and the purpose of this dissertation is to report on investigations into this question through the use of a particular class of theories that incorporate Lorentz violation into a gravitational setting.

The increased attention to this question is sourced largely by hints of Lorentz violation in popular candidate theories of quantum gravity and high energy extensions of the standard particle model. For example, a complete quantum field theory of strings may permit fields that obtain tensor valued vacuum expectation values [1]. These fields would then act as structures singling out preferred frames. In a complete loop quantum gravity theory, in a natural semiclassical ground state the effective spacetime may act as a dispersive medium that induces Lorentz-variant photon propagators [2]. And in noncommutative field theories, the noncommutativity of spacetime coordinates is intrinsically Lorentz violating [3]. (By contrast, in causal set theory, where spacetime is replaced by a discrete set of points, the random distribution of points and causal relations between them in a typical set should preclude preferred directions [4]. In this way, discreteness does not imply Lorentz variance.) The review [5] discusses various theoretical models that feature Lorentz-symmetry violating effects and observational searches for violations.

So far no conclusive sign of Lorentz variance has been seen, and very strong bounds exist on the size of couplings for Lorentz-violating effects in standard model extensions [5, 6]. I mention, however, the issue of the Greisen [7], Zatsepin and Kuzmin [8] (GZK) cutoff, as it is a strong motivator of interest in Lorentz violation. A sharp drop is expected in the number of cosmic ray protons observed in the Earth's atmosphere that have energies above the GZK cutoff—about 10^{20} eV. This drop is due to the surpassing of the threshold for pion production in interactions between protons and the cosmic microwave background, together with the absence of any known source for such ultra high energy cosmic rays that is both copious and close-by.

The AGASA cosmic ray detector group has reported [9] that it does not see the expected suppression of events above the cutoff. Their report has fueled enormous interest in Lorentz violation since it was realized that the absence of the suppression could be caused by an absence of the symmetry (there are also a variety of other explanations, see [10] for a review). Conversely, the HiRes experiment reports confirmation of the cutoff [11, 12]. There still exists a great deal of systematic and statistical uncertainty in the analysis of both groups. This is due to difficulty in determining the energy of a given cosmic ray, and the small number of detected events [13]: the AGASA report is based on just 11 events above the cutoff. Thus, the original report is not a reliable indication of Lorentz violation, but it has certainly stimulated research on it.

There is an additional, deep motivation for interest in this topic, which happens to be the main reason for my interest. The assumption of Lorentz invariance is part of the bedrock of modern physics. It is just exciting to challenge the massive authority of that assumption. And it is satisfying that we can do so with minimal change to familiar tools, as evidenced in this dissertation. Studying Lorentz violation is audacious, but tractable.

1.2 Lorentz violation and gravity

The effects of Lorentz violation in a gravitational context are not covered by the bounds from particle interactions. This dissertation will consider theoretical and observational aspects of a certain class of theories of gravity that contain a preferred frame, with an emphasis on how current experimental tests of gravity constrain the theory's free parameters. Incorporating Lorentz violation into a gravitational setting requires a mechanism that breaks this symmetry while preserving the distinct symmetry of diffeomorphism invariance. "Einstein–aether" theory—or "ae-theory" for short—is a classical metric theory of gravity that contains an additional dynamical vector field. The vector field "aether" is constrained to be timelike everywhere and of fixed norm. The aether can be thought of as a remnant of unknown, Planck-scale, Lorentz-violating physics. It defines a preferred frame, while its status as a dynamical field preserves diffeomorphism invariance. The condition on the vector norm, which can always be scaled to unity, ensures that the aether just picks out a preferred direction, and removes instabilities in the unconstrained theory [14].

1.2.1 A brief history of ae-theory

Vector-tensor theories were first studied in the early 70's, but without the unit-norm constraint; see for example [15, 16]. The theories were of interest primarily as toy examples for newly developed methods of probing the post-Newtonian regime of gravitational theories—vector-tensor was just the next simplest thing after scalar-tensor. Results concerning the weak field form and the speeds and polarizations of linearized vector-tensor plane waves were derived [17]. These theories suffered from the problem that some of the degrees of freedom were always associated with negative energies and thus instability, due to the fact that the vector field could become spacelike. The instability is thus removed by the imposition of the unit constraint. This constraint alters the theory's dynamics so that the mentioned results must be derived anew for ae-theory. Furthermore, expressions for the post-Newtonian equations of motion of compact bodies in the presence of strong fields, and the radiation damping rate were never determined.

In the 80's, Gasperini [18] wrote down the first example of a unit-vector–tensor theory of gravity, although in a tetrad formalism. The relationship between the parameters of Gasperini's action and those of ae-theory have not been worked out, because Gasperini's results are not useful for placing observational constraints on the parameter values. One phenomenological result worked out by Gasperini [19] was the fact that in a cosmological setting, certain values of the parameters lead to a negative value of the effective gravitational coupling constant G_{cosmo} and thus repulsive forces. This effect could then eliminate a big-bang singularity. Doing so would require time-dependent parameters, since G_{cosmo} is positive today. The presence of dynamical parameters introduces complexities—such as how to determine parameter dynamics—that I choose to leave for future work: in this dissertation, strictly constant parameters are assumed.

In 1989, the argument of Kostelecky and Samuel [1] that string theory might spontaneously violate Lorentz symmetry led them to investigate [20] an Einstein–Maxwell system with a potential for the vector field that would induce symmetry breaking. Their potential was a general function of the unit-norm condition that would be minimized when the condition was satisfied, but they did examine the special case in which the constraint was strictly enforced. Finally, the unit-vector–tensor theory with the general action considered below was presented by Jacobson and Mattingly [21] in 2000, following an unpublished idea of Jacobson and Dell.

1.2.2 Additional ae-theory studies

Aspects of ae-theory that have received notable study, but that I will not focus on in this dissertation include ae-theory cosmology, ae-theory stellar solutions, and violations of the second law of black hole thermodynamics in Lorentz-violating theories. I will summarize this work here.

The field equations for a homogeneous, isotropic case were written down by

Gasperini in the tetrad formalism [18], and by Carroll and Lim in the vector formalism [22]. They observed that the aether would have two effects: independent renormalizations of Newton's constant and the spatial curvature contribution to the Einstein equation. Bounds from these effects are discussed in Chapter 4. The spectrum of primordial density perturbations generated by inflation in ae-theory have been studied. Lim [23] and Kanno and Soda [24] have shown that vector modes do not grow in ae-theory without a non-aether source, such as an inflaton field. The studies of [23, 24] have not yet led to observational constraints on the theory. A remaining task is to determine the evolution of perturbations to the surface of last scattering. Zlosnik, Ferreira, and Starkman [25] have studied a more complicated version of unit-vector–tensor theories, in which the Lagrangian can be an arbitrary polynomial in the scalar that defines the aether portion of the Lagrangian used here—see Eqn. (2.2). They have shown that for the aether to act as dark matter in the flattening of galactic rotation curves requires a non-integer polynomial. Such an action is very unusual from an effective field theory standpoint.

Investigations of ae-theory black holes and stars have been carried out, largely by Eling, Jacobson, and coworkers. Eling and Jacobson [26, 27] have shown numerically the existence of static, spherically symmetric ae-theory black hole and stellar solutions. There exists a two-parameter family of such solutions that are asymptotically flat, and a one-parameter subset that is regular on the horizon of spin-0 gravity-aether waves (see Chap. 5 for more on ae-theory wave modes). With Garfinkle [28], they have demonstrated the formation of an ae-theory black hole by collapsing scalar matter. Their results support the conjecture [26, 27] that collapse will select the special subclass. With Miller [29], Eling and Jacobson have begun examining properties of non-rotating neutron stars in ae-theory with varying equations of state. A step that is necessary to reach conclusive constraints from binary pulsar systems is to examine the form of non-static solutions—specifically, stars translating with respect to the aether frame; this point is discussed in Chapter 6.

One property of vector-tensor theories and more general Lorentz-violating theories of gravity is the existence of linearized field modes with differing characteristic speeds. In ae-theory, this fact was first discovered by Jacobson and Mattingly [30], and is demonstrated in Chapter 5. Dubovsky and Sibiryakov [31] have argued that the differing propagation speeds imply that black holes in Lorentz-violating theories will radiate the various modes at different Hawking temperatures, in such a way that the generalized second law of black hole thermodynamics can be violated. Essentially, the black hole is used as a perfect heat pump. Further analysis by myself and colleagues [32] has lead to evidence in support of their findings, including an additional, classical process that violates the law. The conclusion appears to be that the generalized second law can only be preserved by the presence of higher order effects that destroy the notion of a causally separated black hole interior—thus removing black holes from the theory—or by the absence of Lorentz violation. It is not yet known whether this second law violation has observational consequences.

1.3 Outline of Thesis

In this dissertation, I will examine theoretical and observational aspects of ae-theory. These include the effect of a rescaling of the metric along the aether direction, the form of asymptotic quantities such as total energy and momentum, the first law of ae-theory black hole mechanics, and observational constraints from solar system experiments and binary pulsar systems.

To summarize the contents:

In Chapter 2, I demonstrate the effect on the ae-theory coupling constants, defined as $c_n, (n = 1, \ldots, 4)$, of rescaling the aether and the metric along the aether direction.

In Chapter 3, I derive expressions for the total energy, momentum, and angular momentum of an ae-theory spacetime via the Noether charge method. This work is crucial for Chapters 5 and 6. I also use the Noether charges to write down the first law of ae-theory black hole mechanics and discuss difficulties in giving the law a thermodynamic interpretation.

In Chapter 4, I examine a variety of observational constraints on ae-theory. These constraints include those that probe the post-Newtonian limit of the theory; for that purpose, I calculate the parametrized post-Newtonian, or PPN, parameters. I also consider constraints derived by other authors that follow from the nature of linearized wave modes and from ae-theory cosmology.

In Chapter 5, I begin the examination of the motion of binary pulsar systems in ae-theory, considering the limit in which effects due to strong internal fields of the compact bodies can be neglected. Treating the bodies as perfect fluid spheres, I calculate the radiation damping rate, or the rate at which a system of compact bodies loses energy due to gravity-aether radiation.

In Chapter 6, I include strong field effects by treating the compact bodies as point particles with nonstandard, velocity dependent interactions. The interactions are parametrized by dimensionless "sensitivities". I determine the effective post-Newtonian equations of motion for the bodies, and the radiation damping rate. Additional work to calculate sensitivities for a given source is required to obtain precise constraints for all values of the c_n. I am able to estimate how small the c_n must be for the strong field effects to be negligible given current observational errors in the measurement of pulsar systems. The class of ae-theories with "small-enough" c_n is then subject to just the PPN and weak field constraints.

In Chapter 7, I review my thesis and the results of the dissertation, and discuss directions for future research.

1.3.1 Conventions

Throughout the dissertation, I follow the conventions of Wald [33]. In particular, I use units in which the flat space speed-of-light $c = 1$, and I use metric signature $(-, +, +, +)$. This signature is opposite to that employed in the previously published versions of the dissertation chapters 2–5, but is in accord with nearly all of the work on ae-theory conducted outside the "Maryland camp", and with the canonical literature on post-Netownian expansions and binary pulsar tests. The translation between the results as reported in this dissertation and as reported in the published versions of the chapters is made by substituting $\{c_1, c_2, c_3\} \to -\{c_1, c_2, c_3\}, c_4 \to +c_4$.

The following shorthand conventions for combinations of the ae-theory parameters $c_n; n = (1, \ldots, 4)$ will be used:

$$c_{14} = c_1 - c_4, \tag{1.1}$$

$$c_{123} = c_1 + c_2 + c_3 \tag{1.2}$$

$$c_\pm = c_1 \pm c_3. \tag{1.3}$$

When covariant equations are expanded in Minkowskian coordinates, the following conventions are observed. Spatial indices will be indicated by lowercase Latin letters from the middle of the alphabet: $i, j, k, \ldots$. One exception is when the coefficients $c_{1,2,3,4}$ are referred to collectively as c_n, when no confusion should arise. Indices will be raised and lowered with the flat metric η_{ab}. Repeated spatial indices will be summed over, regardless of vertical position: $T_{ii} = \sum_{i=1\ldots3} T_{ii}$. The flat-space Laplacian will be denoted by $\triangle$: $\triangle f \equiv f_{,ii}$. Time indices will be indicated by a 0; time derivatives will be denoted by an overdot: $\dot{f} \equiv \partial_0 f$.

Chapter 2

Field Redefinitions

2.1 Introduction

In this chapter, I will introduce the four-parameter ae-theory action, and then demonstrate the effect on it of a field redefinition. The redefinition considered is of the form $g_{ab} \rightarrow g'_{ab} = A(g_{ab} + (1 - B)u_a u_b)$, $u^a \rightarrow u'^a = (1/\sqrt{AB})u^a$, where g_{ab} is a Lorentzian metric and u^a is the "aether". The action has the most general form that is generally covariant, second order in derivatives, and in which the unit-norm constraint is imposed. The redefinition preserves this most-general form, since it preserves covariance, does not introduce higher derivatives, and preserves the unit-norm constraint. The net effect is then a transformation of the coupling constants in the action. The study of ae-theory systems can be simplified in certain cases by invoking this transformation to give the couplings more convenient values; e.g. by setting one of the constants to zero.

This work generalizes a result of Barbero and Villaseñor [34] that shows equivalence between vacuum general relativity and an ae-theory system whose coupling constants satisfy certain relations. The four coupling constants must be specific functions of one free parameter for their result to apply. I consider here the general case in which the constants have arbitrary values. This work also uses a simpler parametrization of the redefinition than that of [34] and works with a now more common form of ae-theory action. The translation between this work and [34] will be given below.

2.2 Transformation of the Action

The conventional, second-order ae-theory action S is defined as the most general that is covariant, second-order in derivatives, and in which the constraint $u^a u_a = -1$ is enforced; thus:

$$S = \int d^4x \, \sqrt{|g|} \, \mathcal{L}, \tag{2.1}$$

with Lagrangian $\mathcal{L}$

$$\begin{aligned} \mathcal{L} = \frac{1}{16\pi G} \Big(& R + c_1 (\nabla_a u_b)(\nabla^a u^b) + c_2 (\nabla_a u^a)(\nabla_b u^b) \\ & + c_3 (\nabla_a u^b)(\nabla_b u^a) + c_4 (u^a \nabla_a u^c)(u^b \nabla_b u_c) \Big), \end{aligned} \tag{2.2}$$

where R is the scalar curvature of the metric g_{ab} and the c_n are dimensionless constants.

I will assume that the fields are on-shell with respect to the constraint, rather than incorporate it via a Lagrange multiplier. This approach is justified if one views two actions as equivalent if they lead to the same equations of motion. I obtain the same equations of motion either by subjecting the off-shell action with a multiplier term to general variations, then solving for the multiplier in terms of the other fields, or by subjecting the on-shell action to variations that preserve the constraint. It follows that two actions are equivalent if they agree on-shell. The redefinition given below preserves the constraint; thus, it preserves this sense of equivalence.

I will begin by considering unprimed variables: metric g_{ab} and aether u^a, satisfying $g_{ab}u^a u^b = -1$. I then define primed fields:

$$
\begin{aligned}
g'_{ab} &= A\big(g_{ab} + (1-B)u_a u_b\big) \\
u'^a &= \frac{1}{\sqrt{AB}} u^a
\end{aligned}
\tag{2.3}
$$

where A and B are positive constants. The sign of A merely changes the signature convention of the metric so is irrelevant. A negative value of B results in a primed metric of Euclidean signature. I restrict to positive B to ensure comparison of Lorentzian theories. The primed inverse-metric g'^{ab} and the primed aether one-form $u'_a \equiv g'_{ab}u'^b$ are then uniquely determined in terms of unprimed fields:

$$
\begin{aligned}
g'^{ab} &= \frac{1}{A}\Big(g^{ab} + \Big(1 - \frac{1}{B}\Big)u^a u^b\Big) \\
u'_a &= \sqrt{AB}\, u_a.
\end{aligned}
\tag{2.4}
$$

It follows that $u'^a g'_{ab} u'^b = -1$.

To demonstrate the effect of this redefinition on the action (2.1), I shall start with the primed action and express it in terms of unprimed variables. I will show that the form of the action is left invariant, with new parameters G, c_n given as functions of A, B, and the original G', c'_n. The calculation is straightforward but lengthy—the demonstration will be explicit to ease the checking of the final results.

I will begin by considering the role of the parameter A, whose net effect is a rescaling of the action. This occurs because A rescales the field variables in such a way that each term in the Lagrangian (2.2) acquires the same factor. Writing the Lagrangian in terms of primed variables, then invoking the substitutions (2.3) and (2.4) reveals that each term in the un-primed Lagrangian carries an overall factor of $1/A$. The ratio of primed-to-unprimed metric determinants will equal A^4, times a B-dependent factor given below. Thus, the un-primed action (2.1) will carry a net factor of A and will have no other A-dependence. This factor can be absorbed into a redefinition of G. Having thus accounted for the effect of A, I will set $A = 1$ in the calculations that follow.

The full relation between metric determinants g, g' can be deduced by evaluating them in a basis orthonormal with respect to g_{ab}, of which u^a is a member. In this basis, $g = -1$. From the expression $g_{ab} = -u_a u_b + h_{ab}$, with $h_{ab}u^a = 0$, we have $g'_{ab} = -u'_a u'_b + h_{ab}$. It follows that $g' = -(u^a u'_a)^2 = -B$ in this basis. Generalizing to an arbitrary basis leads to

$$g' = Bg. \tag{2.5}$$

The action then re-scales: $S' = \sqrt{B}S$. The above rescalings effect a redefinition of Newton's constant:

$$G = \frac{G'}{A\sqrt{B}}, \tag{2.6}$$

restoring A temporarily.

2.2.1 Curvature Term

I turn now to the curvature term in the Lagrangian (2.2). I will start by listing properties of the redefined connection coefficients Γ^a_{bc},

$$(\Gamma^a_{bc})' = \Gamma^a_{bc} + g^{ad}D_{dbc}, \tag{2.7}$$

where

$$D_{abc} = \frac{(1-B)}{2}\left(\delta^d_a + (1-1/B)u^d u_a\right)\left[\nabla_b(u_d u_c) + \nabla_c(u_d u_b) - \nabla_d(u_b u_c)\right]. \tag{2.8}$$

Define the following quantities:

$$
\begin{aligned}
S_{ab} &= \nabla_a u_b + \nabla_b u_a \\
F_{ab} &= \nabla_a u_b - \nabla_b u_a \\
\dot{u}^a &= u^b \nabla_b u^a .
\end{aligned}
\tag{2.9}
$$

D_{abc} can be organized as follows:

$$
D_{abc} = \frac{(1-B)}{2}\big(u_a X_{bc} + u_b F_{ca} + u_c F_{ba}\big), \tag{2.10}
$$

where

$$
X_{bc} = \frac{1}{B}\big(S_{bc} + (1-B)(\dot{u}_b u_c + u_b \dot{u}_c)\big), \tag{2.11}
$$

and the unit-constraint has been enforced.

I will now note some useful relations involving D_{abc}. To begin,

$$
u^a S_{ab} = u^a X_{ab} = u^a F_{ab} = \dot{u}_b . \tag{2.12}
$$

Then, contraction once with u^a gives

$$
\begin{aligned}
u^a D_{abc} &= -\frac{(1-B)}{2B}\big(S_{bc} + \dot{u}_b u_c + u_b \dot{u}_c\big), \\
u^c D_{abc} &= \frac{(1-B)}{2}\big(\dot{u}_a u_b + u_a \dot{u}_b + F_{ab}\big),
\end{aligned}
\tag{2.13}
$$

and contraction twice gives

$$
\begin{aligned}
u^b u^c D_{abc} &= -(1-B)\dot{u}_a , \\
u^a u^b D_{abc} &= 0 .
\end{aligned}
\tag{2.14}
$$

In addition,

$$
\begin{aligned}
X^{ab} u^c D_{abc} &= (1-B)\dot{u}_a \dot{u}^a , \\
F^{ab} u^c D_{abc} &= \frac{(1-B)}{2} F_{ab} F^{ab} .
\end{aligned}
\tag{2.15}
$$

As for the trace of D_{abc},

$$D^b{}_{bc} \equiv g^{ab} D_{abc} = 0. \tag{2.16}$$

I now demonstrate the transformation of the scalar curvature. A short calculation reveals that

$$(R_{abc}{}^d)' = R_{abc}{}^d + 2\nabla_{[b} D^d{}_{a]c} + 2D^e{}_{c[a} D^d{}_{b]e}, \tag{2.17}$$

so that

$$(R_{ab})' = R_{ab} + W_{ab}, \tag{2.18}$$

where

$$W_{ab} = \nabla_d D^d{}_{ab} - D^d{}_{ea} D^e{}_{db}. \tag{2.19}$$

The scalar curvature $R' = R'_{ab}\, g'^{ab}$ takes the form

$$R' = R_{ab} g^{ab} - \frac{(1-B)}{B} R_{ab} u^a u^b + W_{ab} \left(g^{ab} - \frac{(1-B)}{B} u^a u^b \right). \tag{2.20}$$

The second term on the right-hand-side can be re-expressed via the definition of the curvature tensor:

$$\begin{aligned} R_{ab} u^a u^b &= u^a \nabla_b \nabla_a u^b - u^a \nabla_a \nabla_b u^b \\ &= (\nabla_a u^a)(\nabla_b u^b) - (\nabla^a u^b)(\nabla_b u_a) + \upsilon, \end{aligned} \tag{2.21}$$

where υ represents a total divergence. This can be discarded with the same justification given above for taking the fields as on-shell. The symbol υ will continue to represent other total divergences that appear in the calculations below, but the specific form of the divergence will differ by equation. The third term on the right-hand-side of (2.20) has the form

$$
\begin{aligned}
W_{ab}g^{ab} &= -D^{cba}D_{abc} + \upsilon \\
&= -\frac{(1-B)}{2}\left(u^c X^{ab} + u^b F^{ac}\right) D_{abc} + \upsilon \\
&= \frac{-(1-B)^2}{2}\left(\dot{u}^a \dot{u}_a + \frac{1}{2}F_{ab}F^{ab}\right) + \upsilon.
\end{aligned} \tag{2.22}
$$

As for the last term in (2.20),

$$
\begin{aligned}
W_{ab}u^a u^b &= u^a u^b (\nabla^c D_{cab} - D_{cda}D^{dc}{}_b) \\
&= -D_{abc}u^c(2\nabla^a u^b + D^{bad}u_d) + \upsilon \\
&= -\frac{(1-B)}{2}\left(\dot{u}_a u_b + u_a \dot{u}_b + F_{ab}\right) \\
&\quad \times \left(S^{ab} + \frac{1-B}{2}(\dot{u}^a u^b + u^a \dot{u}^b) + \frac{(1+B)}{2}F^{ab}\right) + \upsilon \\
&= -\frac{(1-B^2)}{2}\left(\dot{u}^a \dot{u}_a + \frac{1}{2}F_{ab}F^{ab}\right) + \upsilon.
\end{aligned} \tag{2.23}
$$

Combining the above and suppressing a total divergence, the transformation of the scalar curvature can be expressed as

$$
\begin{aligned}
R' = R &- \frac{(1-B)}{B}\big((\nabla_a u^a)(\nabla_b u^b) - (\nabla^a u^b)(\nabla_b u_a)\big) + \frac{(1-B)^2}{2B}\left(\dot{u}^a \dot{u}_a + \frac{1}{2}F_{ab}F^{ab}\right) \\
= R &+ \frac{(1-B)}{2B}\Big\{(1-B)(\nabla_a u_b)(\nabla^a u^b) - 2(\nabla_a u^a)(\nabla_b u^b) \\
&\qquad + (1+B)(\nabla^a u^b)(\nabla_b u_a) + (1-B)(\dot{u}^a \dot{u}_a)\Big\}.
\end{aligned} \tag{2.24}
$$

Contributions a_n to the redefined c_n can be extracted from this expression:

$$
\begin{aligned}
a_1 &= \frac{(1-B)^2}{2B} \\
a_2 &= -\frac{(1-B)}{B} \\
a_3 &= \frac{(1-B^2)}{2B} \\
a_4 &= \frac{(1-B)^2}{2B}.
\end{aligned} \tag{2.25}
$$

The constants a_n are characterized by the relations

$$0 = a_1 - a_4 = a_1 + a_2 + a_3 = 2a_1 + a_1^2 - a_3^2 \tag{2.26}$$

and $a_1 > 0$. Therefore, if the c_n satisfy these conditions, the ae-theory system is equivalent to pure gravity via a field redefinition. The translation from this result in terms of A and B to that of [34] in terms of α and β is made by choosing $A = \sqrt{|\alpha(\alpha+\beta)|}/2$ and $B = -\alpha/(\alpha + 2\beta)$. (Compare the first line of (2.24) with Eqn. (6) of [34].)

2.2.2 Aether Term

I now proceed to examine the transformation of the aether portion of the Lagrangian. From the form of the covariant derivative

$$(\nabla_a u^b)' = \frac{1}{\sqrt{B}}\big(\nabla_a u^b + D^b{}_{ac}u^c\big), \tag{2.27}$$

and the relations (2.16) and (2.14), the transformation of the c_2 and c_4 terms can be deduced: $(\nabla_a u^a)' = (1/\sqrt{B})(\nabla_a u^a)$, $(\dot{u}^a)' = \dot{u}^a$ and further $(\dot{u}_a)' = \dot{u}_a$. Thus,

$$\big((\nabla_a u^a)(\nabla_b u^b)\big)' = \frac{1}{B}\big((\nabla_a u^a)(\nabla_b u^b)\big), \tag{2.28}$$

and

$$\big(\dot{u}^a \dot{u}_a\big)' = \big(\dot{u}^a \dot{u}_a\big). \tag{2.29}$$

These results indicate contributions of c_2'/B to c_2 and c_4' to c_4.

It will be convenient to reorganize the c_1 and c_3 terms:

$$c_1(\nabla_a u_b)(\nabla^a u^b) + c_3(\nabla_a u_b)(\nabla^b u^a) = \frac{c_+}{4}S_{ab}S^{ab} + \frac{c_-}{4}F_{ab}F^{ab}, \tag{2.30}$$

where $c_\pm = c_1 \pm c_3$. Then the form of the covariant derivative of u'_a is

$$\begin{aligned}(\nabla_a u_b)' &= \sqrt{B}\big(\nabla_a u_b - D_{cab}u^c\big) \\ &= \frac{1}{2\sqrt{B}}\big(S_{ab} + (1-B)(\dot{u}_a u_b + u_a \dot{u}_b) + BF_{ab}\big).\end{aligned} \tag{2.31}$$

Raising an index on the symmetrized derivative,

$$(S_{cb}g^{ac})' = \frac{1}{\sqrt{B}}\big(S^a{}_b + (1-B)u_b\dot{u}^a\big), \tag{2.32}$$

there is indeed no $\dot{u}_b u^a$ term—leads to

$$(S_{ab}S^{ab})' = (S^a{}_b S^b{}_a)' = \frac{1}{B}\left(S_{ab}S^{ab} + 2(1-B)\dot{u}^a\dot{u}_a\right), \tag{2.33}$$

indicating contributions of c'_+/B to c_+ and $(1-B)c'_+/2B$ to c_4. Raising an index on the anti-symmetrized derivative,

$$(F_{cb}g^{ac})' = \sqrt{B}\left(F^a{}_b - \frac{(1-B)}{B}u^a\dot{u}_b\right), \tag{2.34}$$

leads to

$$(F_{ab}F^{ab})' = -(F^a{}_b F^b{}_a)' = B\left(F_{ab}F^{ab} - 2\frac{(1-B)}{B}\dot{u}^a\dot{u}_a\right), \tag{2.35}$$

indicating contributions of Bc'_- to c_- and $-(1-B)c'_-/2$ to c_4.

Collecting the above results reveals contributions b_n to the redefined c_n:

$$\begin{aligned}b_1 &= \frac{1}{2B}\big(c'_+ + B^2 c'_-\big) \\ &= \frac{1}{2B}\big((1+B^2)c'_1 + (1-B^2)c'_3\big) \\ b_2 &= \frac{c'_2}{B} \\ b_3 &= \frac{1}{2B}\big(c'_+ - B^2 c'_-\big) \\ &= \frac{1}{2B}\big((1-B^2)c'_1 + (1+B^2)c'_3\big) \\ b_4 &= c'_4 + \frac{1-B}{2B}\big(c'_+ - Bc'_-\big) \\ &= c'_4 + \frac{1-B}{2B}\big((1-B)c'_1 + (1+B)c'_3\big).\end{aligned} \tag{2.36}$$

The redefined c_n are given by the sum of a_n (2.25) and b_n (2.36):

$$
\begin{aligned}
c_1 &= \frac{1}{2B}\big(c'_+ + B^2 c'_- + (1-B)^2\big) \\
&= \frac{1}{2B}\big((1+B^2)c'_1 + (1-B^2)c'_3 + (1-B)^2\big) \\
c_2 &= \frac{1}{B}\big(c'_2 - 1 + B\big) \\
c_3 &= \frac{1}{2B}\big(c'_+ - B^2 c'_- + (1-B^2)\big) \\
&= \frac{1}{2B}\big((1-B^2)c'_1 + (1+B^2)c'_3 + (1-B^2)\big) \\
c_4 &= c'_4 + \frac{1-B}{2B}\big(c'_+ - Bc'_- + (1-B)\big) \\
&= c'_4 + \frac{1}{2B}\big((1-B)^2 c'_1 + (1-B^2)c'_3 + (1-B)^2\big).
\end{aligned}
\tag{2.37a}
$$

In addition,

$$
\begin{aligned}
c_+ &= \frac{1}{B}\big(c'_+ + 1 - B\big) \\
c_- &= Bc'_- - 1 + B
\end{aligned}
\tag{2.37b}
$$

2.3 Discussion

The redefinition (2.3) can simplify the problem of characterizing solutions for a specific set of c_n. This is done by transforming that set into one in which the c_n take on more convenient values. This has been done, for example, by Eling and Jacobson in their study of ae-theory black holes and stars [26, 27].

It was noted in [34] that a system with restricted values of the coefficients, equivalent to c_n that satisfy (2.26), can be transformed into aether-free general relativity. The current work extends this result by allowing for general values of the c_n. Using this result, different sets of c_n are seen to be equivalent. For example, it follows from the relations (2.37) that a set of c_n is equivalent to one in which one of c_+, c_-, or c_2 vanishes if the original values satisfy, respectively, $c'_+ > -1, c'_- > -1$, or $c'_2 < 1$.

Certain combinations transform in convenient ways. In particular, it follows from the relations (2.37) that

$$c_{14} = c'_{14}, \quad 1 + c_- = B(1 + c_-), \quad c_{123} = \frac{1}{B} c'_{123}, \quad 1 + c_+ = \frac{1}{B}(1 + c_+). \tag{2.38}$$

This fact is noteworthy as it gives the first hint in this dissertation that the above combinations of coefficients are somehow special. Many of the results that follow feature those combinations. Of course, any combination of the four c_n could be written in terms of four other independent combinations such as those above, but the directness with which they appear seems significant. The reasons for their status are not known to me.

An extra constant can be eliminated in the case of spherically symmetric configurations [35]. In this case, the hypersurface orthogonality and unit norm of the aether imply the vanishing of the twist $\omega_a = \epsilon_{abcd} u^b \nabla^c u^d$, so that

$$\omega_a \omega^a = \dot{u}^a \dot{u}_a + \frac{1}{2} F^{ab} F_{ab} = 0. \tag{2.39}$$

Redefinition of a particular configuration preserves any Killing symmetries shared by the metric and aether fields, so it preserves the relation (2.39). Then, for instance, c_+ can be eliminated by redefinition and c_4 by absorption into c_-. The Lagrangian reduces to the form

$$\mathcal{L} = \frac{1}{16\pi G} \Big(R + \frac{c_-}{4} F_{ab} F^{ab} + c_2 (\nabla_a u^a)^2 \Big). \tag{2.40}$$

This is considerably simpler than the general form (2.2).

Once non-aether matter is included, a metric redefinition not only changes the c_n coefficients, but also modifies the matter action. The fact that Lorentz-violating effects in non-gravitational physics are already highly constrained [5, 6] means that, to a very good approximation, there is a universal metric to which matter couples. Within the validity of this approximation, the field g_{ab} can be identified with this universal metric, thus excluding any aether dependence from the matter action. This identification then eliminates the freedom to redefine the metric.

Chapter 3

Noether Charges and Black Hole Mechanics

3.1 Introduction

Constraints on the acceptable values of the four c_n appearing in the ae-theory action are implied by observational evidence, but one can also argue for limits imposed by theoretical considerations. A possible requirement that motivates the work of this chapter is that the theory should satisfy some form of energy positivity. It may be that imposing positivity for all solutions is more restrictive than necessary, or perhaps that one should only require positivity in the rest frame of the aether. Whatever the argument, an expression for the energy is required to know how the c_n are constrained.

With this goal in mind, I give here an expression for the total energy of an asymptotically flat ae-theory spacetime, as well as expressions for the total momentum and angular momentum. These are generated via the "Noether charge" method [36, 37] of defining the value of the on-shell Hamiltonian for a diffeomorphism invariant field theory, directly from the theory's Lagrangian. The conventional ADM and Komar expressions [33], which have the form of integrals at spatial infinity, acquire aether-dependent corrections due to the nonvanishing of the aether at infinity. Constraints on the c_n are not discussed. The results here complement those of Eling [38], in which expressions for the total energy and the energy of linearized wave modes are derived via pseudotensor methods.

The Noether charge method also allows one to write down a differential identity that governs variations of stationary, axisymmetric black hole solutions. As shown by Wald [36] and Iyer and Wald [37], in a wide variety of theories this identity can be massaged into the familiar form of the "first law" of black hole mechanics and then interpreted as a law of thermodynamics. The discovery of ae-theory black hole solutions [26, 27] motivates the study of the first law for ae-theory black holes. Eling and Jacobson demonstrate existence of these solutions, but have not found analytic expressions for the fields; therefore, the form of the first law cannot be inferred directly from the solutions. One can, however, attempt to derive the law via the Noether charge method. Unfortunately, the algorithms of [36, 37] fail for ae-theory since the vector field cannot be regular on the bifurcation surface of the horizon, where a crucial calculation is performed. Below, a law resembling the first law is derived by less elegant means for static, spherically symmetric solutions, but a thermodynamic interpretation of this expression is not given. In particular, a definitive expression for the horizon entropy in ae-theory has not yet been found.

The Noether charge methodology is briefly reviewed in Section 3.2. The requisite differential forms for ae-theory are derived in Section 3.3. These are used to determine expressions for the total energy, momentum, and angular momentum of an asymptotically flat ae-theory spacetime in Section 3.4. The first law of ae-theory black holes is discussed in Section 3.5.

3.2 Noether charge methodology

I will summarize here the application of the Noether charge method [36, 37] to the definition of total energy, momentum, and angular momentum of an asymptotically flat spacetime. Given a diffeomorphism invariant field theory defined from an action principle, one can construct a phase space with symplectic structure from the space of field configurations and the theory's Lagrangian. For the case of an ae-theory system on a globally hyperbolic spacetime, the phase space structure permits a well defined Hamiltonian formulation. For every diffeomorphism on spacetime, generated by vector field ξ^a, there is a corresponding evolution in phase space, with Hamiltonian generator H_ξ. This generator is implicitly defined through Hamilton's equation, which takes the form [36, 37]

$$\delta H = \int_C (\delta \mathbf{J} - d(i_\xi \mathbf{\Theta})) \tag{3.1}$$

where $\mathbf{J}$ and $\mathbf{\Theta}$ are differential 3-forms that depend on the dynamical fields, a variation of the fields, and the vector field ξ^a; the surface of integration is a spacelike Cauchy surface C of the spacetime.

The forms $\mathbf{\Theta}$ and $\mathbf{J}$ are obtained from the theory's Lagrangian as follows. Let the Lagrangian $\mathbf{L}$ be a 4-form constructed locally out of the dynamical fields, denoted collectively by ψ. The 3-form $\mathbf{\Theta}$ is defined by the variation of $\mathbf{L}$ due to a variation of ψ:

$$\delta \mathbf{L} = \mathbf{E}[\psi] \cdot \delta\psi + d\mathbf{\Theta}[\delta\psi], \tag{3.2}$$

where $\mathbf{E}[\psi]$ are identified as the equations of motion for the fields, the dot representing contraction over appropriate indices.

This definition only determines $\mathbf{\Theta}$ up to the addition of a closed form, which must be exact by the result of [39]. The contribution to δH (3.1) from an asymptotic boundary is typically not effected by such ambiguity, though, since the falloff conditions on the dynamical fields that guarantee convergence of δH imply that any covariant, exact 3-form added to $\mathbf{\Theta}$ will give no asymptotic contribution to δH. Such is the case for ae-theory with the conditions chosen below. A contribution might arise given an inner boundary to the spacetime. Here (Sec. 3.5) I only consider stationary configurations on such spacetimes, and one can then show that the contribution to δH vanishes. I will therefore fix the definition of $\mathbf{\Theta}$ by taking the "most obvious" choice that emerges from variation of the Lagrangian.

To each vector field on spacetime ξ^a, associate the Noether current 3-form $\mathbf{J}[\xi]$,

$$\mathbf{J}[\xi] = \mathbf{\Theta}[\mathcal{L}_\xi \psi] - i_\xi \mathbf{L}. \tag{3.3}$$

This current is conserved, $d\mathbf{J} = 0$, for arbitrary ξ^a when ψ satisfies the equations of motion. This fact implies [39] that $\mathbf{J}$ can be expressed in the form

$$\mathbf{J}[\xi] = d\mathbf{Q}[\xi] \tag{3.4}$$

when $\mathbf{E}[\psi] = 0$. If in addition $\delta\psi$ is such that the equations of motion linearized about ψ are satisfied, then $\delta\mathbf{J} = d\delta\mathbf{Q}$, where here and below I choose $\delta\xi^a = 0$. $\mathbf{Q}$ is only defined up to addition of a closed, hence exact [39], 2-form, but this ambiguity does not effect δH. I will therefore fix the definition of $\mathbf{Q}$ by taking the "most obvious" choice.

An additional ambiguity can arise if one thinks of the Lagrangian $\mathbf{L}$ as defined only up to the addition of an exact 4-form, i.e. a boundary term. Adding such a form to $\mathbf{L}$ effects $\boldsymbol{\Theta}$, $\mathbf{J}$, and $\mathbf{Q}$ individually but leads to no net effect on δH. I will fix this ambiguity by again taking the "most obvious" choices for the forms.

The Hamiltonian differential evaluated on-shell—when the full and linearized equations of motion are satisfied—is thus a surface term

$$\delta H_\xi = \int_{\partial C} (\delta \mathbf{Q} - i_\xi \boldsymbol{\Theta}). \tag{3.5}$$

I will restrict attention to the case where C is asymptotically flat at spatial infinity. The boundary of C will consist of a surface "at infinity"—the limit of a two-sphere whose radius is taken to infinity—and a possible inner surface, such as a black hole horizon.

One can define a Hamiltonian function H_ξ if there exists a 2-form $\mathbf{B}$ such that

$$\int_{\partial C} \delta(i_\xi \mathbf{B}) = \int_{\partial C} i_\xi \boldsymbol{\Theta}. \tag{3.6}$$

The Hamiltonian is then defined as

$$H_\xi = \int_{\partial C} (\mathbf{Q} - i_\xi \mathbf{B}). \tag{3.7}$$

I will assume that the fall-off conditions on the fields are such that at infinity, $d(\mathbf{Q} - i_\xi \mathbf{B}) = 0$. It follows that the value of the contribution to H_ξ from the surface at infinity is conserved and can be interpreted as the conserved quantity associated with the symmetry generated by ξ^a.

The total energy $\mathcal{E}$ of the spacetime is defined to be the value of the asymptotic Hamiltonian for the case where ξ^a is a time translation $t^a = (\partial/\partial t)^a$ at infinity

$$\mathcal{E} = \int_\infty (\mathbf{Q}[t] - i_t \mathbf{B}). \tag{3.8}$$

Likewise, with $x_i^a = (\partial/\partial x_i)^a$ a constant, spatial translation at infinity, the total momentum in the x_i^a-direction $\mathcal{P}_i$ is defined as

$$\mathcal{P}_i = -\int_\infty (\mathbf{Q}[x_i] - i_{x_i}\mathbf{B}). \tag{3.9}$$

The total angular momentum $\mathcal{J}$ about a given axis is defined via a vector field φ^a that is a rotation around that axis at infinity, tangent to the bounding 2-sphere. The pull-back to the boundary of $i_\varphi \mathbf{B}$ vanishes, giving

$$\mathcal{J} = -\int_\infty \mathbf{Q}[\varphi]. \tag{3.10}$$

I note parenthetically that it follows from this definition that the total angular momentum must be zero for any axisymmetric configuration (one for which $\mathcal{L}_\varphi \psi = 0$), on C possessing no inner boundary. This follows from the vanishing of $\mathbf{J}[\varphi] = d\mathbf{Q}[\varphi]$, when evaluated on such a configuration and pulled back to C. This result does not appear to have been stated explicitly with this generality before, although an early application is found in the proof of Cohen and Wald [40] that there are no rotating, axisymmetric geons, in work that predates the precise formulation of the Noether charge method.

This result also provides a short proof that there can be no rotating, axisymmetric boson stars in general relativity. This generalizes the known result [41, 42] that there are no *stationary*, rotating, axisymmetric boson stars. Here, axisymmetry must include any complex argument of the scalar field, as well as its modulus; this is a stronger sense of “axisymmetric” than is common in the boson star literature. In the presence of an inner boundary, such as an event horizon, the vanishing of $d\mathbf{Q}$ implies that the total angular momentum, i.e. the integral of $\mathbf{Q}$ over the boundary at infinity, is equal to the integral of $\mathbf{Q}$ over the inner boundary. Consequently, this result is not in conflict with the existence of rotating, axisymmetric black holes.

3.3 Ae-theory forms

In this section, I will give the explicit expressions of the differential forms defined above, for ae-theory. The conventional, second order ae-theory Lagrangian 4-form $\mathbf{L}$ is

$$\begin{aligned}\mathbf{L} = \frac{1}{16\pi G}\big(&R + c_1(\nabla_a u_b)(\nabla^a u^b) + c_2(\nabla_a u^a)(\nabla_b u^b) \\ &+ c_3(\nabla_a u^b)(\nabla_b u^a) + c_4(u^a \nabla_a u^c)(u^b \nabla_b u_c)\big)\boldsymbol{\epsilon}\end{aligned} \tag{3.11}$$

where R is the scalar curvature of the metric g_{ab}, the $c_n; n = 1, \ldots, 4$ are dimensionless constants, and $\boldsymbol{\epsilon}$ is the canonical volume form associated with g_{ab}. The constraint can be accounted for by adding to $\mathbf{L}$ a term of the form

$$\lambda\big(u^a u^b g_{ab} + 1\big)\boldsymbol{\epsilon} \tag{3.12}$$

where λ is a Lagrange multiplier; such a term does not contribute to the forms sought.

Varying $\mathbf{L}$ gives $\boldsymbol{\Theta}$:

$$\begin{aligned}\boldsymbol{\Theta}_{abc} = \frac{1}{16\pi G}\boldsymbol{\epsilon}_{dabc}\Bigg[&g^{de}g^{fh}\left(\nabla_f \delta g_{eh} - \nabla_e \delta g_{fh}\right) \\ &+ \left(2\, K^d{}_e \delta u^e + \left(K^{ef}u^d + \left(K^{df} - K^{fd}\right)u^e\right)\delta g_{ef}\right)\Bigg],\end{aligned} \tag{3.13}$$

where

$$K^a{}_c = (c_1 g^{ab} g_{cd} + c_2 \delta^a_c \delta^b_d + c_3 \delta^a_d \delta^b_c + c^4 u^a u^b g_{cd})\nabla_b u^d. \tag{3.14}$$

From this follows **J** (3.3):

$$\mathbf{J}_{abc} = \frac{1}{16\pi G}\boldsymbol{\epsilon}_{dabc}\left(A^{def}{}_{h}\nabla_{(e}\nabla_{f)}\xi^{h} + B^{de}{}_{h}\nabla_{e}\xi^{h} + C^{d}{}_{h}\xi^{h}\right), \tag{3.15}$$

where

$$A^{def}{}_{h} = \big(g^{ef}\delta^{d}_{h} - g^{d(e}\delta^{f)}_{h}\big), \tag{3.16a}$$

$$B^{de}{}_{h} = -2\big(K^{[d}{}_{h}u^{e]} + K_{h}{}^{[d}u^{e]} - K^{[de]}u_{h}\big), \tag{3.16b}$$

$$C^{d}{}_{h} = \frac{3}{2}R^{d}{}_{h} + 2K^{d}{}_{e}\nabla_{h}u^{e} - \delta^{d}_{h}(R + K^{e}{}_{f}\nabla_{e}u^{f}). \tag{3.16c}$$

The Noether charge **Q** (3.4) can be extracted via an algorithm of Wald [39], yielding

$$\begin{aligned}\mathbf{Q}_{ab} &= \frac{1}{16\pi G}\,\boldsymbol{\epsilon}_{abcd}\Big[\frac{2}{3}A^{cdf}{}_{h}\nabla_{f}\xi^{h} + \frac{1}{2}B^{cd}{}_{h}\xi^{h}\Big] \\ &= -\frac{1}{16\pi G}\,\boldsymbol{\epsilon}_{abcd}\Big[\nabla^{c}\xi^{d} + \Big(\big(K^{c}{}_{h} + K_{h}{}^{c}\big)u^{d} - K^{cd}u_{h}\Big)\xi^{h}\Big].\end{aligned} \tag{3.17}$$

3.4 Conserved quantities

I now consider the expressions for the total energy, momentum, and angular momentum of an asymptotically flat spacetime in ae-theory. For the requisite integrals to be convergent, falloff conditions must be set for the fields and their variations. I will assume that at spatial infinity, there exists an asymptotic Cartesian coordinate basis, with respect to which the components of the metric and its derivatives are

$$g_{\mu\nu} = \eta_{\mu\nu} + O(1/r), \tag{3.18}$$

and

$$\frac{\partial g_{\mu\nu}}{\partial x^{\alpha}} = O(1/r^{2}), \tag{3.19}$$

where η_{ab} is the flat metric. The variations of the metric δg_{ab} must be $O(1/r)$. For the aether, I will require that

$$u^\mu = \bar{u}^\mu + O(1/r), \tag{3.20}$$

where asymptotically, $\nabla_a \bar{u}^b = 0$. The frame can always be chosen to be the aether frame, so that $\bar{u}^a = t^a$ at infinity. With respect to the asymptotic Cartesian basis,

$$\frac{\partial u^\mu}{\partial x^\alpha} = O(1/r^2). \tag{3.21}$$

The variation δu^a will be assumed to be $O(1/r)$.

I turn now to the total energy. For ae-theory with the above falloff conditions, $\mathbf{\Theta} = \mathbf{\Theta}_G + O(1/r^3)$ asymptotically, where $\mathbf{\Theta}_G$ is the form which arises for GR in vacuum. Hence, $\mathbf{B} = \mathbf{B}_G$, the vacuum GR form. The total energy can then be written as $\mathcal{E} = \mathcal{E}_G + \mathcal{E}_{AE}$, where $\mathcal{E}_G$ is the standard ADM mass [33]

$$\mathcal{E}_G = \frac{1}{16\pi G}\int_\infty dS\; r^i(\partial_j g_{ij} - \partial_i g_{jj}), \tag{3.22}$$

where dS is the spherical area element and $r^a = (\partial/\partial r)^a$. The aether portion $\mathcal{E}_{AE}$ is

$$\begin{aligned}\mathcal{E}_{AE} &= \frac{1}{16\pi G}\int_\infty dS\; 2\, t_{[c}r_{d]}t^e\Big(\big(K^c{}_e + K_e{}^c\big)\bar{u}^d - K^{cd}\bar{u}_e\Big)\\ &= -\frac{1}{8\pi G}\int_\infty dS\; (K^t{}_r\bar{u}^t + K^t{}_t\bar{u}^r).\end{aligned} \tag{3.23}$$

Setting $\bar{u}^a = t^a$ gives

$$\begin{aligned}\mathcal{E}_{AE} &= \frac{c_{14}}{8\pi G}\int_\infty dS\; t^a r_b \nabla_a u^b\\ &= \frac{c_{14}}{8\pi G}\int_\infty dS\; \big(\partial_t u^r - \partial_r u^t\big),\end{aligned} \tag{3.24}$$

where I have used the unit constraint, which requires that $t_a \nabla_b u^a = u_a \nabla_b u^a + O(1/r^3)$ and $\partial_\mu u^t = +(1/2)\partial_\mu g_{tt} + O(1/r^3)$.

This expression can be evaluated more explicitly for a static, spherically symmetric solution using the results of Chapter 4 (the necessary ingredients were first reported in [35]). In isotropic coordinates, the line element has the form:

$$dS^2 = -N(r)dt^2 + B(r)(dr^2 + r^2 d\Omega) \tag{3.25}$$

and $\bar{u}^a = t^a$. In the generic case $c_{123} \neq 0$, to $O(1/r)$, $N = 1-(r_0/r)$, $B = 1+(r_0/r)$, and $u^t = 1 + (r_0/2r)$, for arbitrary constant r_0. The total energy is then

$$\mathcal{E} = \frac{r_0}{2G}(1 + \frac{c_{14}}{2}). \tag{3.26}$$

This result was previously found by Eling using pseudotensor methods [38]. The quantity

$$G_N = G(1 + \frac{c_{14}}{2})^{-1} \tag{3.27}$$

has been identified in studies of the ae-theory Newtonian limit [22](see also Chapter 4) as the value of Newton's constant that would be measured far from gravitating matter, assuming no direct interaction between aether and non-aether matter. A Newtonian gravitating mass $M = 2r_0/G_N$ can be defined, in which case

$$\mathcal{E} = M. \tag{3.28}$$

The total momentum in the x_i^a direction also has the form $(\mathcal{P}_G)_i + (\mathcal{P}_{AE})_i$, where $(\mathcal{P}_G)_i$ is the standard ADM momentum [33],

$$(\mathcal{P}_G)_i = -\frac{1}{16\pi G}\int dS\; r^j \big(\partial_0 g_{ji} - \partial_j g_{0i} - \delta_{ij}(\partial_0 g_{kk} - \partial_k g_{0k})\big). \tag{3.29}$$

The aether contribution, setting $\bar{u}^a = t^a$, is

$$(\mathcal{P}_{AE})_i = -\frac{1}{16\pi G}\int_\infty dS\,\Big(c_+(r^a\nabla_a u_i + r_a\nabla_i u^a) + 2c_2 r_i\nabla_b u^b\Big). \tag{3.30}$$

The total angular momentum takes the form $\mathcal{J}_G + \mathcal{J}_{AE}$, where $\mathcal{J}_G$ is the generalization to a non-axisymmetric spacetime of the conventional Komar expression for vacuum GR [33],

$$\mathcal{J}_G = \frac{1}{16\pi G}\int_\infty dS\; n_{ab}\nabla^a\varphi^b, \tag{3.31}$$

where n_{ab} is the binormal of the boundary of C. The aether contribution $\mathcal{J}_{AE}$ is

$$\mathcal{J}_{AE} = -\frac{c_+}{16\pi G}\int_\infty dS\; 2\,r_{(a}\phi_{b)}\nabla^a u^b, \tag{3.32}$$

having set $\bar{u}^a = t^a$.

3.5 First law of black hole mechanics

For a stationary black hole spacetime, the Noether charge formalism allows one to write down a differential identity that relates variations in the total energy and angular momentum to variations of integrals over a cross-section of the horizon. It has been shown [36, 37] that this identity becomes the "first law" of black hole mechanics/thermodynamics for a wide variety of generally covariant gravitational theories.

A one-parameter family, for a fixed set of c_n values, of static, spherically symmetric ae-theory black hole solutions has been shown to exist [26, 27]. The existence proof is based on numerical integration of the field equations, and analytic expressions for the fields are only known asymptotically. Thus, a first law cannot be obtained by directly examining the solutions. Instead, the Noether charge method can be applied and an attempt to massage the variational identity into a form resembling the familiar first law. The closest I will come here will be for the static, spherically symmetric case, for which I will show that the identity can be written in the form

$$\delta M = \frac{\kappa}{8\pi G}\big[(1+\phi N)\delta A + \phi A \delta O\big], \tag{3.33}$$

where M is the black hole mass (i.e. the total energy, c.f. (3.28)), A is the area of the horizon, N and O are quantities depending on the metric, aether, and the local geometry of the horizon, and κ and ϕ are parameters defined below. Although this expression resembles the familiar first law, it does not lead to an obvious thermodynamic interpretation; in particular, I do not obtain a definitive expression for the horizon entropy.

The variational identity of interest is derived via the Noether charge method by applying Hamilton's equation (3.1) to perturbations of an asymptotically flat, stationary, axisymmetric configuration containing a Killing horizon. A Killing horizon is a null hypersurface to which a Killing field is normal—I take it to define the black hole horizon. The Cauchy surface C is assigned a boundary consisting of the 2-sphere "at infinity" and the surface B where C meets the horizon $\mathcal{H}$. I will assume that B is compact. Choose ξ^a to be the horizon-normal Killing field χ^a, normalized as

$$\chi^a = t^a + \Omega\phi^a, \tag{3.34}$$

where t^a is the stationary Killing field with unit norm at infinity, and ϕ^a is the axisymmetric Killing field; the constant Ω defines the angular velocity of the horizon. As $\delta\mathbf{J}[\chi] - d(i_\chi\mathbf{\Theta})$ is linear in $\mathcal{L}_\chi\psi$, which vanishes, δH_χ also vanishes. From the definitions of the total energy (3.8) and angular momentum (3.10), the identity emerges:

$$\delta\mathcal{E} - \Omega\delta\mathcal{J} = \int_B \delta\mathbf{Q} - i_\chi\mathbf{\Theta}. \tag{3.35}$$

The vanishing of $\delta\mathbf{J}[\chi] - d(i_\chi\mathbf{\Theta})$ also implies that the choice of B is arbitrary.

There is no precise definition of a "first law form" of an expression; roughly speaking, however, by analogy with the conventional thermodynamic expression, a black hole first law should relate variations of "macroscopic" variables—global variables and other parameters that describe the black hole spacetime. The explicit form that the identity takes for ae-theory, where $\mathbf{\Theta}$ (3.13) and $\mathbf{Q}$ (3.17) are as defined above, suggests that further manipulation is required for the identity (3.35) to take a first law form.

An algorithm for massaging (3.35) into such a form and defining the entropy associated with the horizon was given by Wald [36] and improved upon by Iyer and Wald [37]. For the algorithm to apply, it is necessary that the stationary spacetime be extendible to one whose Killing horizon possesses a bifurcation surface—a cross-section on which the horizon-normal Killing field vanishes—on which all dynamical fields are regular. In that case, one can work with the extended spacetime and choose B to be the bifurcation surface. The algorithm relies on the universal behavior of χ^a in a neighborhood of the bifurcation surface and reduces the horizon terms to the form $(\kappa/2\pi G)\delta S$. Here, κ is the surface gravity of the horizon, defined by

$\kappa^2 = -\frac{1}{2}\nabla_a\chi_b\nabla^a\chi^b$, evaluated on the bifurcation surface, and

$$S = 2\pi \int_B E^{abcd} n_{ab} n_{cd}, \tag{3.36}$$

where n_{ab} is the binormal of B, and E^{abcd} is the functional derivative of the Lagrangian with respect to the Riemann tensor R_{abcd}, treating it as a field independent of the metric. General kinematical arguments [43] in the context of quantum field theory in curved spacetime indicate that the temperature due to thermal radiation associated with a Killing horizon is always $\kappa/2\pi$. The form of the horizon terms then suggest that S/G be identified as the thermodynamic entropy associated with the horizon.

Unfortunately, the above requirement cannot be met for any ae-theory configuration [44]. Racz and Wald [45] have shown that a spacetime containing a Killing horizon can be extended smoothly to one containing a bifurcation surface if the horizon has compact cross-sections and constant, non-vanishing surface gravity. Regular extensions of matter fields on that spacetime are not guaranteed. In fact, no such extension can exist for a vector field u^a that is invariant under the Killing flow and not tangent to a horizon cross-section. The Killing flow acts at the bifurcation surface as a radially directed Lorentz boost, under which only vectors tangent to the surface can be invariant. In particular, the aether cannot possess a regular extension, since it is constrained to be timelike, while a cross-section of a null surface must be spacelike.

Progress must be made by less elegant and less general means. I will now restrict attention to the case of a perturbation between spherically symmetric, static solutions, and show that in this case the variational identity (3.35) can be written in the form (3.33).

Consider a variation between static, spherically symmetric solutions, each containing a Killing horizon $\mathcal{H}$. Identify the solutions such that the horizons coincide, and so that the Killing orbits coincide in a neighborhood of the horizon. That this can be done follows from the construction of "Kruskal-like" coordinates in ref. [45]. These coordinates can be further defined so that the variations of the non-angular components of the metric vanish on $\mathcal{H}$: the line element near $\mathcal{H}$ takes the form

$$ds^2 = -GdUdV + R^2 d\Omega^2. \tag{3.37}$$

where G and R are functions of the quantity UV, and $\mathcal{H}$ is defined by $UV = 0$; then, U and V can be properly rescaled such that $G(0) = 1$ for each solution.

Another effect of this identification [46] is that near $\mathcal{H}$, the Killing vector χ^a with surface gravity κ_0 in the unperturbed solution coincides with the Killing vector with the same surface gravity κ_0 in the perturbed solution. From this fact, it follows that on $\mathcal{H}$, $\delta\mathbf{Q}[\chi] = \kappa\delta\mathbf{Q}[k]$, where $k^a = \kappa^{-1}\chi^a$ is the unit-surface-gravity Killing field near $\mathcal{H}$ for both configurations, and is held fixed in the variation of $\delta\mathbf{Q}[k]$.

I will consider the portion of $\mathcal{H}$ defined by $U = 0, V > 0$, and a cross-section B corresponding to some value of V. A null dyad on $\mathcal{H}$ can be defined consisting of k^a and $\bar{k}^a$, where $\bar{k}^a$ is the unique null vector normal to B such that $k_a\bar{k}^a = -1$. From the vanishing on $\mathcal{H}$ of the variations of k^a and the transverse components of the metric, it follows that $\bar{k}^a$ is the same vector field for both solutions; i.e. $\delta\bar{k}^a = 0$.

The metric $h_{ab} = g_{ab} + 2k_{(a}\bar{k}_{b)}$ induced on B has a variation

$$\delta h_{ab} = \frac{\delta A}{A} h_{ab} \tag{3.38}$$

where A is the area of B. The aether u^a can be decomposed with respect to this dyad:

$$u^a = \frac{1}{2\phi} k^a + \phi \bar{k}^a. \tag{3.39}$$

where $\phi = -u^a k_a$. Then on $\mathcal{H}$,

$$\delta u^a = -\frac{\delta\phi}{\phi}\left(\frac{1}{2\phi} k^a - \phi \bar{k}^a\right) \equiv -\frac{\delta\phi}{\phi} \bar{u}^a. \tag{3.40}$$

A non-null dyad normal to B consists of u^a and the orthogonal unit-vector $\bar{u}^a$. The bi-normal n_{ab} of B is defined as the natural volume element on the tangent space normal to B, normalized such that $T^a n_{ab} R^b > 0$ for any future-pointing timelike T^a and spacelike R^a directed towards infinity. It can be expressed in various ways:

$$n_{ab} = 2k_{[a}\bar{k}_{b]} = -2u_{[a}\bar{u}_{b]} = -\frac{2}{\phi} k_{[a}\bar{u}_{b]}. \tag{3.41}$$

Now, the algorithm cited above can be used to evaluate the aether-independent horizon terms, which give [36, 37] the standard contribution $(\kappa/8\pi G)\delta A$. Evaluating the aether-dependent portion of $\mathbf{Q}[k]$ pulled-back to B gives

$$\begin{aligned}
\mathbf{Q}_{AE}[k] &= \frac{1}{16\pi G} \boldsymbol{\epsilon} n_{ab} k^c \big(u^a (K^b{}_c + K_c{}^b) + u_c K^{ab}\big) \\
&= \frac{1}{8\pi G} \boldsymbol{\epsilon} k_a \bar{u}_b K^{ab} \\
&= \frac{1}{8\pi G} \boldsymbol{\epsilon} \phi \big(c_{14} u^a \bar{u}_b + c_+ \bar{u}^a \bar{u}_b + c_2 \delta^a_b\big) \nabla_a u^b \\
&= -\frac{1}{8\pi G} \boldsymbol{\epsilon} \phi \big(c_{14} n^a{}_b + c_+ h^a_b - c_{123} \delta^a_b\big) \nabla_a u^b,
\end{aligned} \tag{3.42}$$

where $\boldsymbol{\epsilon}$ is the volume element of B, $h^a_b = g^{ac} h_{cb}$.

Next, the δu^a-dependent portion of $i_k \mathbf{\Theta}_{AE}$ pulled-back to B gives

$$\frac{1}{8\pi G}\,\boldsymbol{\epsilon} n_{ab} k^b K^a{}_c \delta u^c = \frac{1}{8\pi G}\,\boldsymbol{\epsilon}\frac{\delta\phi}{\phi} k_a \bar{u}_b K^{ab} = \frac{\delta\phi}{\phi}\mathbf{Q}_{AE}[k]. \tag{3.43}$$

For the portion containing metric variations, noting that $\delta g_{ab} = \delta h_{ab} = \delta A/A h_{ab}$, so that $u^a \delta g_{ab} = 0$, leads to

$$\frac{1}{16\pi G}\,\boldsymbol{\epsilon} n_{ab} u^a k^b \delta h_{cd} K^{cd} = \frac{\phi}{16\pi G}\,\boldsymbol{\epsilon}\frac{\delta A}{A}\big(c_{13} h^a_b + 2c_2 \delta^a_b\big)\nabla_a u^b. \tag{3.44}$$

Thus,

$$\begin{aligned}\int_B (\delta \mathbf{Q}[\chi] - i_\chi \mathbf{\Theta}) = \frac{\kappa}{8\pi G}\bigg[\bigg(1 - \phi\Big(c_{14} n^a{}_b + c_+\big(\frac{3}{2}h^a_b - \delta^a_b\big)\Big)\nabla_a u^b\bigg)\delta A \\ - \phi A \delta\Big(\big(c_{14} n^a{}_b + c_+ h^a_b - c_{123}\delta^a_b\big)\nabla_a u^b\Big)\bigg],\end{aligned} \tag{3.45}$$

and the variational identity (3.35) can be written in the form (3.33).

Although this first law *form* has been obtained, a thermodynamic interpretation of it has not emerged. In particular, a definitive expression for the horizon entropy has not been found. For variations between members of a one-parameter family of solutions, the horizon terms (3.45) must be reducible to $(\alpha\kappa/2\pi G)\delta A$ for some dimensionless constant α. Even with the Noether charge approach, however, the value of α cannot be discerned, nor is it known whether $\alpha A/G$ acts as the entropy in the non-static case.

It is possible that this confusion is related to an obscurity in the notion of a black hole horizon in ae-theory. Linearized perturbations about a flat spacetime and constant aether background were investigated in [30] (see also Chapter 5 for the results). It was found that there exist five independent wave modes that travel at three different c_n-dependent speeds. These speeds generally differ from the "speed of light" defined by the flat metric, and exceed it for certain c_n values. The behavior of perturbations about a curved background is not known, but a similar result probably holds. If that is so, then a Killing horizon is not generally a causal horizon. On the other hand, the perturbations about the flat background do all propagate on the lightcones of the flat metric [30] in the special case $c_{13} = c_4 = 0$, $c_2 = c_1/(1 + 2c_1)$, but the expression (3.45) does not drastically simplify in this case.

Given the wide applicability of the principles of black hole thermodynamics in generally covariant theories of gravity, it would be surprising if they did not apply to ae-theory. Recent work [31, 32], however, provides further evidence that they do not, in the form of a Lorentz-violating, black hole perpetual motion machine. It is not yet clear whether a breakdown of the thermodynamic interpretation would have observational consequences that could be used to constrain the theory. An important theoretical implication, if the breakdown is unescapable, is that Lorentz violation and black hole thermodynamics cannot both be windows into quantum gravity, as has long been suspected. This is a topic for future work.

Chapter 4

PPN Parameters and Collected Constraints

4.1 Introduction

Alternative theories of gravity have been systematically ruled out or severely constrained as observations have improved [17, 47]. It is now time to subject ae-theory to this scrutiny. In this chapter, I will discuss the complete collection of currently available observational and theoretical constraints on ae-theory, excepting those that result from binary pulsar systems. I will combine my result for the ae-theory post-Newtonian parameters with previously established constraints. Surprisingly, all of these constraints are compatible with ranges of order unity for two coefficients in the Lagrangian. I am aware of no other theory that comes this close to so many predictions of general relativity and yet is fundamentally different.

The ae-theory action for the metric g_{ab} and aether u^a contains four independent terms parametrized by four dimensionless constants c_n. Observations have already severely constrained Lorentz-symmetry violation in the matter sector [5, 6], hence to a very good approximation matter must couple universally to one metric, which I take to be g_{ab}. My goal is to determine the observational and theoretical constraints on the c_n.

4.1.1 Summary of PPN results

A standard way of beginning to compare an alternative gravity theory to general relativity is to examine the first post-Newtonian corrections. For a general metric theory of gravity there are ten 'parametrized post-Newtonian' (PPN) parameters [17, 47] characterizing the lowest order effects in v^2 and dimensionless gravitational potential ($G_N M/r$). Five of these parameters, ζ_1,ζ_2, ζ_3, ζ_4, and α_3, vanish identically for any "semi-conservative" theory, i.e. one derived from a covariant action principle. Two others, known as the Eddington–Robertson–Schiff parameters β and γ, characterize respectively the nonlinearity and the spatial curvature produced by gravity. Of the remaining three PPN parameters, two, α_1, α_2, characterize preferred frame effects, and the third, ξ (sometimes called the Whitehead parameter), characterizes a peculiar sort of three-body interaction.

In the weak field, slow motion limit, ae-theory reduces to Newtonian gravity [22] with a value of Newton's constant G_N related to the constant G in the ae-theory action (4.11) by

$$G_N = G\Big(1 + \frac{c_{14}}{2}\Big)^{-1}. \tag{4.1}$$

It was previously shown in [35] that in ae-theory $\beta = \gamma = 1$, just as in general relativity. The parameter α_2 for ae-theory was computed in Ref. [48] to lowest nontrivial order in the c_n.

Below, I give a comprehensive computation of all the PPN parameters, which confirms the previous results and determines the exact values of α_2 and the previously unknown parameters α_1 and ξ. The results indicate that the "time-time" and "space-space" components of the metric are the same in ae-theory and GR to calculated post-Newtonian order, where I refer to a nearly globally Minkowskian coordinate system with the aether aligned with the time direction at zeroth-order. The "time-space" components of the metric g_{0i}, $i = 1, 2, 3$, differ as

$$(g_{0i})_{\rm ae} - (g_{0i})_{\rm GR} = \frac{\alpha_2 - \alpha_1}{2} V_i - \frac{\alpha_2}{2} W_i, \tag{4.2}$$

where $\alpha_{1,2}$ are the PPN parameters given explicitly below. The components of the aether are

$$u^0 = 1 + U \tag{4.3}$$

$$u^i = (1 + \frac{\alpha_1}{8})\big(\frac{c_-}{c_1}\big)\big(V_i + W_i\big) + \frac{3c_{123} - 2c_+ + c_{14}}{2c_{123}}\big(V_i - W_i\big). \tag{4.4}$$

The potentials U, V_i, and W_i are defined by

$$\left\{\begin{array}{c} U(t,\mathbf{x}) \\ \\ V_i(t,\mathbf{x}) \\ \\ W_i(t,\mathbf{x}) \end{array}\right\} = G_N \int d^3y \, \frac{\rho}{|\mathbf{x} - \mathbf{y}|} \left\{\begin{array}{c} 1 \\ \\ v^i \\ \\ \frac{v^j (x^j - y^j)(x^i - y^i)}{|\mathbf{x} - \mathbf{y}|^2} \end{array}\right\}, \tag{4.5}$$

where $\rho(t, \mathbf{y})$ is the rest-mass energy density of the fluid source, and $v^\mu(t, \mathbf{y})$ is the fluid four-velocity.

The components of the perturbed metric show that the ae-theory PPN parameters are given by

$$\gamma = \beta = 1 \tag{4.6}$$

$$\xi = \zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = \alpha_3 = 0 \tag{4.7}$$

$$\alpha_1 = \frac{8(c_3^2 - c_1 c_4)}{2c_1 + c_+ c_-} \tag{4.8}$$

$$\alpha_2 = \frac{\alpha_1}{2} + \frac{(2c_+ - c_{14})(3c_{123} - 2c_+ + c_{14})}{(2 + c_{14})c_{123}} \tag{4.9}$$

$$\tag{4.10}$$

The parameters α_1 and α_2 are both of linear order in c_n when the coefficients are small compared to unity and the ratios amongst them are of order unity.

I will now present the calculation of these results, followed in Sec. 4.3 by a discussion of constraints on the theory.

4.2 Calculation of ae-theory PPN parameters

The following discussion provides details of the calculation of the Parametrized Post-Newtonian (PPN) parameters, α_1,α_2,α_3,β,γ,ζ_1,ζ_2,ζ_3,ζ_4,ξ, for ae-theory. The PPN formalism is defined in a weak field, slow motion limit, and describes the next-to-Newtonian order gravitational effects in terms of a standardized set of potentials and these ten parameters. I will determine the PPN parameters by solving the field equations order-by-order with a perfect fluid source in a standard coordinate gauge. More detailed explanations of the procedure and the general PPN formalism can be found in the classic reference of Will [17].

The ae-theory field equations follow from the action

$$S = \frac{1}{16\pi G}\int d^4x\,\sqrt{|g|}\,\Big(R + K^{ab}{}_{cd}\nabla_a u^c \nabla_b u^d + \lambda(u^a u^b g_{ab} + 1)\Big), \tag{4.11}$$

where

$$K^{ab}{}_{cd} = \big(c_1 g^{ab} g_{cd} + c_2 \delta^a_c \delta^b_d + c_3 \delta^a_d \delta^b_c + c_4 u^a u^b g_{cd}\big), \tag{4.12}$$

with an additional perfect fluid source coupled in the standard way to the metric g_{ab} and uncoupled to the aether u^a. There are the Einstein equations, written here in nonstandard form

$$R_{ab} = \big(S_{cd} + 8\pi G T_{cd}\big)\big(\delta^c_a \delta^d_b - \frac{1}{2} g_{ab} g^{cd}\big), \tag{4.13}$$

where

$$\begin{aligned} S_{ab} =&\nabla_c\big(K_{(a}{}^{c} u_{b)} - K^{c}{}_{(a} u_{b)} - K_{(ab)} u^c\big) \\ &+ c_1\big(\nabla_c u_a \nabla^c u_b - \nabla_a u_c \nabla_b u^c\big) + c_4 u^c \nabla_c u_a u^d \nabla_d u_b \\ &+ \lambda u_a u_b + \frac{1}{2} g_{ab}(K^c{}_d \nabla_c u^d), \end{aligned} \tag{4.14}$$

with

$$K^a{}_c = K^{ab}{}_{cd} \nabla_b u^d. \tag{4.15}$$

Also

$$T^{ab} = (\rho + \rho\Pi + p) v^a v^b + p g^{ab}, \tag{4.16}$$

where v^a is the four-velocity, ρ the rest-mass-energy density, Π the internal energy density, and p the isotropic pressure of the fluid. There is the aether field equation

$$\nabla_a K^a{}_b = c_4 u^c \nabla_c u_a \nabla_b u^a + \lambda u_b; \tag{4.17}$$

and the constraint

$$g_{ab}u^a u^b = -1. \tag{4.18}$$

Eqn. (4.17) can be used to eliminate λ, giving

$$\lambda = -u^c \nabla_a K^a{}_c + c_4 u^c \nabla_c u^a u^d \nabla_d u_a. \tag{4.19}$$

I assume a nearly globally Minkowskian coordinate system and basis with respect to which, at zeroth order, the metric is the Minkowski metric η_{ab} and the aether is purely timelike. The fluid variables are assigned orders of $\rho \sim \Pi \sim p/\rho \sim (v^i)^2 \sim O(1)$. Taking the time-derivative of a quantity will effectively raise its order by one-half: $X \sim O(N) \to \partial X/\partial t \sim O(N+1/2)$. The components of the metric perturbations h_{ab} with respect to this basis will be assumed to be of orders

$$h_{00} \sim O(1) + O(2), \quad h_{ij} \sim O(1), \quad h_{0i} \sim O(1.5). \tag{4.20}$$

This assignment preserves the Newtonian limit while allowing one to determine just the first post-Newtonian corrections. The aether perturbations δu^a are assumed to be of orders

$$\delta u^0 \sim O(1), \qquad \delta u^i \sim O(1.5). \tag{4.21}$$

Lower orders are disallowed by the field equations, given the above orders of h_{ab}. I will assume that h_{ab} and δu^a satisfy boundary conditions such that they vanish at spatial infinity.

The metric components are to be expanded in terms of particular potential functions, thus defining the PPN parameters:

$$\begin{aligned}
g_{00} &= -1 + 2U - 2\beta U^2 - 2\xi\Phi_W + (2\gamma + 2 + \alpha_3 + \zeta_1 - 2\xi)\Phi_1 \\
&\quad + 2(3\gamma - 2\beta + 1 + \zeta_2 + \xi)\Phi_2 + 2(1 + \zeta_3)\Phi_3 \\
&\quad + 2(3\gamma + 3\zeta_4 - 2\xi)\Phi_4 - (\zeta_1 - 2\zeta)\mathcal{A}, \\
g_{ij} &= (1 + 2\gamma U)\delta_{ij}, \\
g_{0i} &= -2\big(\frac{1+\gamma}{2} + \frac{\alpha_1}{8}\big)(V_i + W_i) - \frac{1}{2}\big(1 + 2\gamma + \frac{\alpha_1}{2} - \alpha_2 + \zeta_1 + 2\xi\big)(V_i - W_i).
\end{aligned} \tag{4.22}$$

The potentials are all of the form

$$F(x) = G_N \int d^3y \frac{\rho(y) f}{|x-y|}, \tag{4.23}$$

where G_N is the current value of Newton's constant, which I determine below in terms of G and the c_n. The correspondences $F : f$ are given by

$$\begin{gathered}
U : 1 \qquad \Phi_1 : v^i v^i \qquad \Phi_2 : U \qquad \Phi_3 : \Pi \qquad \Phi_4 : p/\rho \\
\Phi_W : \int d^3z\, \rho(z) \frac{(x-y)_j}{|x-y|^2}\Big(\frac{(y-z)_j}{|x-z|} - \frac{(x-z)_j}{|y-z|}\Big) \qquad \mathcal{A} : \frac{(v_i(x-y)_i)^2}{|x-y|^2} \\
V_i : v^i \qquad W_i : \frac{v^j(x^j - y^j)(x^i - y^i)}{|x-y|^2}.
\end{gathered} \tag{4.24}$$

Note that for U, $\Phi_{1,2,3,4}$, and V_i,

$$\Delta F \equiv F_{,ii} = -4\pi G_N \rho f. \tag{4.25}$$

I will also make use of the "superpotential" χ:

$$\chi = -G_N \int d^3y\, \rho |x-y|, \tag{4.26}$$

which satisfies

$$\Delta\chi = -2U. \tag{4.27}$$

I also note the relation

$$\chi_{,i0} = V_i - W_i, \tag{4.28}$$

which follows from the formula

$$\frac{\partial}{\partial t}\int d^3y\,\rho(\mathbf{y},t) f(\mathbf{x},\mathbf{y}) = \int d^3y\,\rho(\mathbf{y},t) v^i(\mathbf{y},t)\frac{\partial f}{\partial y^i}[1+O(1)], \tag{4.29}$$

which follows from the continuity equation for the fluid

$$\rho_{,0} + (\rho v^i)_{,i} = 0, \tag{4.30}$$

assumed to hold to $O(1.5)$.

These potentials satisfy certain criteria of "reasonableness" and simplicity (see [17], Sec. (4.1) for details), and are general enough to describe all known viable theories of gravity. In particular, they suffice for ae-theory. The criteria permit g_{00} to depend also on the potential $\chi_{,00}$, and g_{ij} to depend on $\chi_{,ij}$. Such terms, however, can always be eliminated [17] by a suitable coordinate transformation that preserves the zeroth-order form of the components. The "standard PPN" gauge is thus defined as that post-Newtonian coordinate frame in which all dependence on $\chi_{,00}$ and $\chi_{,ij}$ has been removed from, respectively, g_{00} and g_{ij}. This fixing determines the coordinate frame up to necessary order so that the standard forms of the metric components are unambiguous.

In carrying out the calculations, I will impose the following gauge conditions:

$$h_{ij,j} = \frac{1}{2}(h_{jj,i} - h_{00,i}) \tag{4.31}$$

$$h_{0i,i} = 3U_{,0}\,. \tag{4.32}$$

These conditions are suggested by the standard conditions for general relativity. The conditions (4.31) suffice to put g_{ij} in standard form. The fourth condition (4.32) does not standardize g_{00} and must be adjusted at the end; it is convenient for calculation, however.

The solving procedure is as follows:

Step 1: Solve the constraint (4.18) for u^0 to $O(1)$;

Step 2: Solve the "time-time" component of the Einstein equation (4.13) for g_{00} to $O(1)$;

Step 3: Solve the "space-space" components of (4.13) for g_{ij} to $O(1)$;

Step 4: Solve the "space" components of the aether field equation (4.17) for u^i to $O(1.5)$;

Step 5: Solve the "time-space" components of (4.13) for g_{0i} to $O(1.5)$;

Step 6: Solve the "time-time" component of (4.13) for g_{00} to $O(2)$.

The cases in which $c_{123} = 0$, $c_{14} + 2 = 0$, or $2c_1 + c_+c_- = 0$ are special in that the found solutions diverge. Presumably the post-Newtonian approximation is not valid in these cases, and assume below that they do not hold. See the main text for more discussion of this point.

4.2.1 u^0 to $O(1)$

Solving the constraint (4.18) gives

$$u^0 = 1 + (1/2)h_{00} \tag{4.33}$$

to $O(1)$. The components of u_a are

$$u_0 \equiv u^a g_{a0} = -1 + \frac{1}{2}h_{00}, \tag{4.34}$$

and

$$u_i = u^a g_{ai} = u^i + h_{0i}. \tag{4.35}$$

I now express the covariant derivatives of u_a for later convenience. The constraint (4.18) implies that

$$\nabla_a u_0 = 0 \tag{4.36}$$

to $O(2)$. Also to $O(2)$,

$$\nabla_0 u_i = -\frac{1}{2}h_{00,i}(1 + \frac{1}{2}h_{00}) + h_{0i,0} + u^i_{,0}, \tag{4.37}$$

and

$$u^c \nabla_c u_i = u^0 \nabla_0 u_i = -\frac{1}{2}h_{00,i}(1 + h_{00}) + h_{0i,0} + u^i_{,0}. \tag{4.38}$$

To $O(1.5)$,

$$\nabla_j u_i = u^i_{,j} + \frac{1}{2}h_{ij,0} + h_{0[i,j]}. \tag{4.39}$$

4.2.2 g_{00} to $O(1)$

I now solve the "time-time" component of the Einstein equation (4.13) for g_{00} to $O(1)$. For the components of R_{00},

$$R_{00} = -\frac{1}{2}h_{00,ii} + \frac{1}{2}h_{ij}h_{00,ij} + \frac{1}{2}\left(h_{i0,i} - \frac{1}{2}h_{ii,0}\right)_{,0} - \frac{1}{4}h_{00,i}h_{00,i} + \frac{1}{4}h_{00,j}\left(2h_{ij,i} - h_{ii,j}\right) \tag{4.40}$$

to $O(2)$. At $O(1)$,

$$R_{00} = -\frac{1}{2}\Delta h_{00}, \tag{4.41}$$

$$T_{00} = \rho, \qquad T_{ij} = 0, \tag{4.42}$$

$$S_{00} = -K_0{}^c{}_{,c} - K_{00,0} = -K_{0i,i} = c_{14}(\nabla_0 u_i)_{,i} = \frac{c_{14}}{2}\Delta h_{00}, \tag{4.43}$$

$$S_{ij} = 0. \tag{4.44}$$

The field equation becomes

$$(1 + \frac{c_{14}}{2})\Delta h_{00} = -8\pi G\rho, \tag{4.45}$$

which gives h_{00} to O(1),

$$h_{00} = 2U, \tag{4.46}$$

with Newton's constant

$$G_N = \big(1 + \frac{c_{14}}{2}\big)^{-1} G. \tag{4.47}$$

4.2.3 g_{ij} to $O(1)$

I now solve the "space-space" components of (4.13) for g_{ij} to $O(1)$. To $O(1)$

$$\begin{aligned} R_{ij} &= -\frac{1}{2}(\Delta h_{ij} + h_{kk,ij} - 2h_{k(i,j)k} - h_{00,ij}) \\ &= -\frac{1}{2}\Delta h_{ij}, \end{aligned} \tag{4.48}$$

where I have imposed the gauge condition (4.31). Using (4.42), (4.43), and (4.44), the field equation becomes

$$\Delta h_{ij} = -8\pi G_N \rho\, \delta_{ij}, \tag{4.49}$$

giving

$$h_{ij} = 2U\delta_{ij}. \tag{4.50}$$

4.2.4 u^i to $O(1.5)$

I now solve the "space" components of the aether field equation (4.17) for u^i to $O(1.5)$, making use of the gauge condition (4.31) and the earlier results (4.34), (4.46), and (4.50). At $O(1.5)$ equation (4.17) has the form

$$K^a{}_{i,a} = -K_{0i,0} + K_{ji,j} = 0. \tag{4.51}$$

To $O(1.5)$,

$$K_{0i,0} = c_{14}(\nabla_0 u_i)_{,0} = -\frac{c_{14}}{2}h_{00,i0} = \frac{c_{14}}{2}\Delta\chi_{,0i}, \tag{4.52}$$

and

$$\begin{aligned} K_{ji,j} &= \big(c_1\nabla_j u_i + c_2\delta_{ij}\nabla_k u_k + c_3\nabla_i u_j\big) \\ &= c_1\Delta u^i + c_{23}u^j_{,ji} + \frac{1}{2}\big(2c_- h_{0[i,j]j} - (c_+ + 3c_2)\Delta\chi_{,0i}\big). \end{aligned} \tag{4.53}$$

The aether field equation can then be written

$$\Delta\Big(c_1 u^i + \frac{c_-}{2}h_{0i} - \frac{1}{2}(2c_1 + 3c_2 + c_3 - c_4)\chi_{,i0}\Big) - \big(\frac{c_-}{2}h_{0j,j} - c_{23}n_{j,j}\big)_{,i} = 0. \tag{4.54}$$

Taking the spatial divergence of the left-hand side gives the relation

$$\Delta u^i_{,i} = A\Delta\Delta\chi_{,0}, \tag{4.55}$$

where

$$A = \frac{2c_1 + 3c_2 + c_3 - c_4}{2c_{123}}, \tag{4.56}$$

which can be solved for $u^i_{,i}$. Substituting into (4.54), imposing the gauge condition (4.32), and using earlier results gives

$$u^i = -\frac{c_-}{2c_1}(h_{0i} + \frac{3}{2}\chi_{,0i}) + A\chi_{,0i}. \tag{4.57}$$

4.2.5 g_{0i} to $O(1.5)$

I now solve the "time-space" components of (4.13) for g_{0i} to $O(1.5)$, making use of the gauge conditions (4.31) and (4.32) and the earlier results (4.34), (4.46), (4.50), and (4.57). To $O(1.5)$

$$\begin{aligned} R_{0i} &= h_{0[j,i]j} + h_{j[i,j]0} \\ &= -\frac{1}{2}\Delta\big(h_{0i} - \frac{1}{2}\chi_{,0i}\big), \end{aligned} \tag{4.58}$$

Also to $O(1.5)$,

$$T_{0i} = -\rho v^i, \tag{4.59}$$

and

$$S_{0i} = -K_{(i0),0} - \frac{1}{2}K_i{}^c{}_{,c} = -\frac{1}{2}\big(K_{0i,0} + K_{ij,j}\big). \tag{4.60}$$

This gives

$$\begin{aligned} K_{ij,j} &= (c_1\nabla_i u_j + c_2\delta_{ij}\nabla_k u_k + c_3\nabla_j u_i)_{,j} \\ &= (c_{12}u^j_{,i} + c_3 u^i_{,j})_{,j} + \frac{1}{2}(2c_- h_{0[j,i]j} + c_+ h_{ij,j0} + c_2 h_{jj,i0}) \\ &= -\Delta\Big(\frac{c_-c_+}{2c_1}(h_{0i} + \frac{3}{2}\chi_{,0i}) - \frac{c_{14}}{2}\chi_{,0i}\Big). \end{aligned} \tag{4.61}$$

With (4.52), this gives

$$S_{0i} = \frac{1}{2}\Delta\Big(\frac{c_-c_+}{2c_1}(h_{0i} + \frac{3}{2}\chi_{,0i}) - c_{14}\chi_{,0i}\Big). \tag{4.62}$$

The field equation becomes

$$\begin{aligned} (1 + \frac{c_-c_+}{2c_1})\Delta(h_{0i} + \frac{3}{2}\chi_{,i0}) &= 16\pi G\rho v^i + (2 + c_{14})\Delta\chi_{,0i} \\ &= -(2 + c_{14})\Delta(V_i + W_i), \end{aligned} \tag{4.63}$$

which gives

$$h_{0i} = -2\left(1 + \frac{(c_3^2 - c_1c_4)}{(2c_1 + c_+c_-)}\right)(V_i + W_i) - \frac{3}{2}\chi_{,0i}. \tag{4.64}$$

4.2.6 g_{00} to $O(2)$

I now solve the "time-time" component of (4.13) for g_{00} to $O(2)$, making use of the gauge conditions (4.31) and (4.32) and the results (4.34), (4.46), (4.50), (4.57), and (4.64). Define $\tilde{h}_{00} = g_{00} + 1 - 2U$. Then, from eqn. (4.40),

$$R_{00} = -\frac{1}{2}\Delta\big(2U + \tilde{h}_{00} + 2U^2 - 8\Phi_2\big). \tag{4.65}$$

Also,

$$T_{00} = \rho(1 + v^2 - 2U + \Pi), \tag{4.66}$$

$$T_{ij} = \rho v^i v^j + p\delta_{ij}. \tag{4.67}$$

$$g_{00}(T_{ab}g^{ab}) = T_{00} - T_{ii}, \tag{4.68}$$

so that

$$\begin{aligned} T_{00} - \frac{1}{2}g_{00}(T_{ab}g^{ab}) &= \frac{1}{2}(T_{00} + T_{ii}) \\ &= \frac{1}{2}\rho(1 + 2v^2 - 2U + \Pi) + \frac{3}{2}p \\ &= \frac{-(2 + c_{14})}{16\pi G}\Delta\big(U + 2\Phi_1 - 2\Phi_2 + \Phi_3 + 3\Phi_4\big). \end{aligned} \tag{4.69}$$

To aid the reader in sorting through the terms appearing in S_{ab}, I note that to $O(2)$ in the chosen gauge,

$$\nabla_i u_i = (A - \frac{3}{2})\Delta\chi_{,0} \tag{4.70}$$

and

$$(\nabla_0 u_i)_{,i} = \frac{1}{2}\Delta\Big(2U + \tilde{h}_{00} + U^2 + 2\big(A - \frac{3}{2}\big)\chi_{,00}\Big). \tag{4.71}$$

Some bookkeeping leads to

$$S_{00} = \frac{c_{14}}{2}\Delta\big(2U + \tilde{h}_{00} + \frac{5}{2}U^2 - 9\Phi_2\big) - c_{14}(A - \frac{3}{2})\Delta\chi_{,00}, \tag{4.72}$$

and

$$S_{ii} = -\frac{c_{14}}{2}\Delta\big(\frac{1}{2}U^2 - \Phi_2\big) - (c_+ + 3c_2)(A - \frac{3}{2})\Delta\chi_{,00}. \tag{4.73}$$

Thus,

$$\begin{aligned} S_{00} - \frac{1}{2}g_{00}S_{ab}g^{ab} &= \frac{1}{2}(S_{00} + S_{ii}) \\ &= \frac{c_{14}}{4}\Delta\Big(2U + \tilde{h}_{00} + 2U^2 - 8\Phi_2\Big) \\ &\quad - c_{123}A(A - \frac{3}{2})\Delta\chi_{,00}. \end{aligned} \tag{4.74}$$

Combining (4.65),(4.69), and (4.74), and solving the field equation gives

$$\tilde{h}_{00} = -2U^2 + 4\Phi_1 + 4\Phi_2 + 2\Phi_3 + 6\Phi_4 + Q\chi_{,00}, \tag{4.75}$$

where

$$\begin{aligned} Q &= \frac{4c_{123}}{(2 + c_{14})}(A - \frac{3}{2})A \\ &= -\frac{(3c_{123} - 2c_+ + c_{14})(2c_+ - c_{14})}{(2 + c_{14})c_{123}}. \end{aligned} \tag{4.76}$$

Finally, the standard PPN gauge is obtained by subtracting $Q\chi_{,00}$ from h_{00}. The fields transform under a gauge transformation as

$$\delta h_{ab} = \xi_{a,b} + \xi_{b,a}, \quad \delta u^a = -\xi^a_{,0}. \tag{4.77}$$

Thus, the gauge is corrected by choosing $\xi_i = 0$, $\xi_0 = -(Q/2)\chi_{,0}$, which leads to $\delta h_{0i} = -(Q/2)\chi_{,0i}$, and $\delta u^a = 0$ to leading order. The standard PPN gauge condition that replaces (4.32) is thus

$$h_{0i,i} = (3 + Q)U_{,0}. \tag{4.78}$$

4.2.7 Summary

I now collect the results, eqns. (4.46), (4.50), (4.64), and (4.75) for the metric components, and (4.33) and (4.57) for the aether, imposing the gauge conditions (4.31) and (4.78), and using the relation (4.28). For the metric:

$$g_{00} = -1 + 2U - 2U^2 + 4\Phi_1 + 4\Phi_2 + 2\Phi_3 + 6\Phi_4 \tag{4.79}$$

$$g_{ij} = (1 + 2U)\delta_{ij} \tag{4.80}$$

$$g_{0i} = -2\Big(1 + \frac{(c_3^2 - c_1 c_4)}{(2c_1 + c_+ c_-)}\Big)(V_i + W_i) - \frac{1}{2}(3 + Q)(V_i - W_i). \tag{4.81}$$

The PPN parameters follow from comparison with the standard forms (4.22). They are

$$\gamma = \beta = 1 \tag{4.82}$$

$$\xi = \zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = \alpha_3 = 0 \tag{4.83}$$

$$\alpha_1 = \frac{8(c_3^2 - c_1 c_4)}{(2c_1 + c_+ c_-)} \tag{4.84}$$

$$\alpha_2 = \frac{\alpha_1}{2} + \frac{(3c_{123} - 2c_+ + c_{14})(2c_+ - c_{14})}{(2 + c_{14})c_{123}}. \tag{4.85}$$

For the aether:

$$u^0 = 1 + U \tag{4.86}$$

$$u^i = \big(1 + \frac{\alpha_1}{8}\big)\big(\frac{c_-}{c_1}\big)(V_i + W_i) + \frac{(3c_{123} - 2c_+ + c_{14})}{2c_{123}}(V_i - W_i). \tag{4.87}$$

The cases $c_{123} = 0$, $c_{14} + 2 = 0$, and $2c_1 + c_+ c_- = 0$ are special, since α_1 and/or α_2 diverges. Presumably the post-Newtonian approximation is not valid when the coefficients are close to these values. The expressions for the wave speeds (4.91) below indicate that the spin-0 speed vanishes in either of the first two cases and the spin-1 speed vanishes in the last case. This corresponds to the absence of spatial gradient terms in the action [48]. The case $c_{123} = 0$ corresponds [35] to the vector-tensor theory of Hellings and Nordtvedt [15] if the unit constraint on the aether is dropped. This theory was shown by Will [17] to be dynamically over-determined and thus observationally unacceptable.

4.3 Discussion of constraints

The current best constraints [47] on the preferred frame PPN parameters are $|\alpha_1| \lesssim 10^{-4}$ from an orbital polarization effect bounded by lunar laser ranging and binary pulsar observations, and $|\alpha_2| \lesssim 4\times10^{-7}$ from a spin precession effect bounded by the alignment of the solar spin with the ecliptic. These two conditions can be met with two unrestricted parameters to spare, having begun with four free c_n, by imposing $\alpha_1 = \alpha_2 = 0$. The condition $\alpha_1 = 0$ implies $c_4 = c_3^2/c_1$. Having put $\alpha_1 = 0$ in this way, α_2 can be put to zero in two ways. One is with $c_+ = 0$, which then implies $c_{14} = 0$. This case is degenerate, and is briefly discussed at the end of the chapter. The other way is to determine c_2 and c_4 in terms of c_1 and c_3 as

$$\begin{aligned} c_2 &= (-2c_1^2 - c_1c_3 + c_3^2)/3c_1 \\ c_4 &= c_3^2/c_1 \end{aligned} \tag{4.88}$$

Thus there is a two-parameter family of ae-theory Lagrangians for which all the PPN parameters are identical to those of GR.

I now consider other constraints on ae-theory. In alternate gravity theories including Brans–Dicke theory, the Newton constant G_N need not be constant in time. Observational bounds on $\dot{G}/G$ then constrain the theory. In the case of ae-theory, there is the relation (4.1), hence G_N is always constant.

Another constraint arises from the possible discrepancy between Newton's constant and the gravitational constant occurring in the equation for the dynamics of the cosmological scale factor. In GR, the scale factor satisfies the Friedman equation, which involves Newton's constant. In ae-theory, when the metric has the standard cosmological form (Robertson-Walker symmetry) and the aether is aligned with the cosmological rest frame, the aether stress tensor can be constructed purely from the spacetime metric with two derivatives, and must be identically divergence free. It must therefore be a linear combination of the Einstein tensor G_{ab} and a tensor constructed with the spatial curvature scalar ${}^{(3)}R$, which turns out to be [22, 49]:

$$T_{ab}^{\text{aether}} = \frac{c_+ + 3c_2}{2}\Big[G_{ab} + \frac{1}{6}{}^{(3)}R(g_{ab} - 2u_a u_b)\Big]. \tag{4.89}$$

The effect of the cosmological aether is thus to renormalize the gravitational constant and to add a stress tensor of perfect fluid type that in effect renormalizes the spatial curvature contribution to the field equations. The renormalized, cosmological gravitational constant is given by [22]

$$G_{\text{cosmo}} = G\Big(1 + \frac{2c_+ - 3c_{123}}{2}\Big)^{-1}. \tag{4.90}$$

Since this is not the same as G_N (4.1), the expansion rate of the universe differs from what would have been expected in GR with the same matter content. The ratio is constrained by the observed primordial ^{4}He abundance to satisfy $|G_{\text{cosmo}}/G_N - 1| < 1/8$, which imposes a constraint on the constants c_n [22]. Remarkably, if the constants are restricted by (4.88) so that $\alpha_{1,2} = 0$, then $G_N = G_{\rm cosmo}$. Primordial nucleosynthesis then imposes no additional constraint.

Even when the two gravitational constants coincide, the "curvature fluid" term in (4.89) represents a deviation from the Friedman equation in GR if the universe has non-zero spatial curvature. Observations have shown that the spatial curvature must be very small today, and it would have been even less important in the past when the relative contribution of matter and radiation would have been even more important. It thus seems unlikely that an interesting constraint can be obtained from this term. Another potential source of cosmological constraint is the modification of the primordial fluctuation spectrum [23], but this has not yet been worked out in full detail.

A further constraint on ae-theory comes from the possibility that the gravity and aether waves travel at less than the "speed of light"—that is, less than the limiting speed determined by the metric g_{ab} governing the propagation of matter fields. In this case, high energy matter moving inertially through the vacuum would produce vacuum Čerenkov radiation of gravitational and aether shock waves. A detailed analysis of this process and the corresponding observational constraints from ultra-high-energy cosmic ray observations was carried out in Ref. [14]. The constraints are characterized by very small numbers, ranging between 10^{-15} and 10^{-31}, depending on the wave mode type and emission process. These are all one-sided constraints, since they apply only when the wave speeds are smaller than the speed of light. To a first approximation then, the constraints imply that the wave speeds must be greater than the speed of light.

Some authors [14, 22, 23] have suggested that superluminal propagation be excluded *a priori* on the grounds that ae-theory should be viewed as an effective description of an underlying Lorentz invariant theory in a configuration with broken Lorentz symmetry. However, this is a theoretical bias, with no observational basis that I can see. Moreover, superluminal propagation does not threaten causality, as long as there is a limiting speed in at least one given reference frame, as there is in ae-theory. I thus adopt a phenomenological stance, allowing for superluminal propagation unless-and-until it is observationally ruled out.

There are five gravitational and aether wave modes in ae-theory: two correspond to the usual spin-2 modes, two are a transverse spin-1 mode, and one is a longitudinal spin-0 mode. The squared speeds of these modes are determined by the constants c_n, and are given by [30]

$$\begin{array}{ll} \text{spin-2} & 1/(1+c_+) \\ \text{spin-1} & (2c_1+c_+c_-)/2c_{14}(1+c_+) \\ \text{spin-0} & c_{123}(2+c_{14})/(2-c_+-3c_2)(1+c_+)c_{14}. \end{array} \tag{4.91}$$

Imposing the $\alpha_{1,2}=0$ conditions (4.88), the Čerenkov constraint that the spin-2 and spin-0 wave speeds be superluminal restricts c_1 and c_3 to the region

$$\begin{gathered} -1 < c_+ < 0 \\ c_+/3(1+c_+) < c_- < 0. \end{gathered} \tag{4.92}$$

These conditions also ensure that the spin-1 wave speed is superluminal.

In addition to observational constraints, there are two theoretical constraints that come from the requirement that the wave modes be stable—i.e. have real frequencies—and that the energy of the modes be positive. The first requirement is already guaranteed by the condition that the speeds be greater than unity. The signs of the energy densities of the wave modes averaged over a cycle are given by [38]

$$\begin{array}{ll} \text{spin-2} & 1 \\ \text{spin-1} & -(2c_1 + c_+c_-)/(1+c_+) \\ \text{spin-0} & -c_{14}(2+c_{14}). \end{array} \tag{4.93}$$

The spin-2 modes always carry positive energy. If the $\alpha_{1,2} = 0$ conditions (4.88) and the superluminal conditions (4.92) are satisfied, then the spin-1 and spin-0 modes also carry positive energy. By contrast, if the speeds are subluminal, then the latter two modes carry negative energy. Thus, both the Čerenkov constraints and the positive energy requirement excludes the case of subluminal wave speeds.

I earlier pointed out that an alternate way to set $\alpha_1 = \alpha_2 = 0$ is if $c_+ = c_{14} = 0$. In this case $G_N/G_{\text{cosmo}} = (1 - 3c_2/2)$, so nucleosynthesis would impose a constraint on c_2. The spin-0 and spin-1 wave speeds (4.91) diverge in this case, because there are no time derivative terms in the aether field equation [30]—equivalently, those modes are non-propagating. The theory then contains only two spin-2 "gravitons", which propagate along the light cones of the metric. Furthermore as shown in Chapter 5, the rate at which a system of weakly self-gravitating compact bodies loses energy in gravity-aether radiation is identical to that of GR. There are differences for strongly self-gravitating bodies, though. Further observational signatures of this class of ae-theories have not been worked out.

It is nontrivial that the PPN parameters are identical to those of GR and that the vacuum Čerenkov, nucleosynthesis, stability, and positive energy constraints are all satisfied in a large two-dimensional region (4.88,4.92) in the four-dimensional c_n parameter space. To further constrain the parameters one should look to strong field effects or radiative processes. This point is examined in the following chapters.

A strong field effect that I will not focus on is the existence and nature of black hole solutions to the vacuum field equations. Some alternate theories of gravity whose PPN parameters are equal or close to those of GR do not admit regular black hole solutions [17]. In these theories, astrophysical collapse would produce something other than a black hole—perhaps a naked singularity, or a bounce—which may not be difficult to rule out observationally. It is known [26, 27], however, that ae-theory does admit regular black hole solutions.

Chapter 5

Radiation Damping in Binary Systems

5.1 Introduction

In the previous chapter, I discussed the form of ae-theory's post-Newtonian expansion and the nature of linear, source-free wave phenomena. I demonstrated that the theory remains healthy with respect to the corresponding observational constraints for a large range of c_n values. In this chapter, I continue the examination of constraints on ae-theory by considering observations of binary pulsar systems. The central focus is the calculation of the generation of gravity-aether radiation by a nearly Newtonian source and the subsequent energy loss, or radiation damping, of the source. A formula is derived for the rate of change of energy:

$$\frac{d\mathcal{E}}{dt} = -G_N\left\langle \frac{\mathcal{A}}{5}\left(\frac{d^3Q}{dt^3}\right)^2 + \mathcal{B}\left(\frac{d^3I}{dt^3}\right)^2 + \mathcal{C}\left(\frac{d\Sigma}{dt}\right)^2\right\rangle, \tag{5.1}$$

where Q_{ij} is the trace-free quadrupole moment of the source, I is the trace of the second moment, Σ_i is a dipolar quantity defined below, and $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$ are dimensionless combinations of the c_n; G_N is the value of Newton's constant that one would measure far from an external gravitating source; the angular brackets indicate a time average over a period of the system's motion. This formula generalizes the "quadrupole" formula of standard general relativity, which predicts a similar expression but with $\mathcal{A} = 1$, $\mathcal{B} = \mathcal{C} = 0$.

In the case of a system of two compact bodies, this expression takes the form

$$\frac{d\mathcal{E}}{dt} = -G_N \Big\langle \Big(\frac{G_N \mu m}{r^2}\Big)^2 \Big(\frac{8}{15}\mathcal{A}\big(12v^2 - 11(\frac{dr}{dt})^2\big) + 4\mathcal{B}\big(\frac{dr}{dt}\big)^2 + \mathcal{C}'\mathcal{D}^2\Big)\Big\rangle, \tag{5.2}$$

where μ is the reduced mass of the system, m the total mass, v the relative velocity of the bodies, and r their orbital separation, which I assume is much larger than the size d of the bodies; $\mathcal{D}$ is the difference in self-gravitational binding energy per unit mass of the bodies, and the coefficient $\mathcal{C}'$ is another dimensionless combination of the c_n.

This expression gives the lowest-order effects in a post-Newtonian (PN), or weak-field–slow-motion, expansion. Aside from the $(G_N\mu m/r^2)^2$ prefactor, the first two terms are $O(G_N m/r)$ and the last is $O((G_N m/d)^2)$. It does not take into account strong field effects that may be important when the fields are not weak inside a given body. The strength of the field of a compact body can be characterized by the quantity $(G_N m/d)$; this is "small" for the sun ($\sim 10^{-6}$) or a typical white dwarf ($\sim 10^{-3}$), and "large" for a typical neutron star ($\sim 10^{-1}$) or black hole (~ 1). Thus, the field should be strong within the systems actually used to measure the damping rate. Strong field effects on the damping rate of a compact body can be associated with a dependence of the body's gravitating mass on the ambient non-metric fields—that is, with a violation of the strong equivalence principle [47]. These effects are not present in GR at lower post-Newtonian orders. Their presence in ae-theory is examined in Chapter 6. Comments on the validity of the leading order approximation applied to the binary pulsar systems actually observed are best made in Chapter 6; see the concluding Sec. 6.6.

The damping rate can be tested by observing the rate of change of the orbital period P of various binary systems, since $(dP/dt)/P = -(3/2)(d\mathcal{E}/dt)/\mathcal{E}$, equating the energy radiated to minus the change in mechanical energy of the system. In practice, this test is conjoined with tests of other "post-Keplerian" (PK) parameters [47, 50], in particular the rate of advance of periastron (the point at which the two objects are closest to each other) and the redshift or time delay due to the gravitational field of the system. These "quasi-static" [50] parameters are determined by the post-Newtonian forms of the fields and the effective equations of motion for the compact bodies. The conjoint technique is necessary, because the expressions for the PK parameters depend on the unknown masses of the systems' bodies. The expressions for the parameters will depend on the two masses, other measurable parameters, and a given theory's free parameters. Measurement of three mass dependent parameters, for fixed values of the theory parameters, gives three bands with widths due to errors in the two-dimensional space of mass values. The theory is consistent for those values of the free parameters if the bands overlap. The predictions of GR have been validated in this way using data from various binary systems containing pulsars, whose regular pulsing provides an accurate measuring device; see the review [50] for details.

For ae-theory, I find that if one assumes the strong field effects are negligible so that the results of this chapter are adequate, then there exists a one-parameter family of theories that satisfy all of the constraints summarized in Chapter 4, and whose predictions for the PK parameters match those of GR to the order worked to here. This can be seen as follows. To lowest PN order and neglecting strong field effects, the quasi-static parameters can be determined within the PPN framework [17]. Consequently, when α_1 and α_2 are set to zero, so that all of the ae-theory PPN parameters match those of GR, the two theories will make the same predictions for the quasi-static parameters. In this case, the c_n can be constrained by requiring that the damping rate equal that of GR. In fact, when α_1 and α_2 are set to zero, the radiated fields contain only quadrupolar contributions. The damping rates then coincide when $\mathcal{A}$ is set to one, which can be done by imposing one condition on the two remaining free c_n. To be consistent with the observational tests summarized in Chapter 4, this curve of theories must intersect with the allowed two-parameter family demarcated there (and below in Sec. 5.5). This is the case all along the curve, as long as $c_-, c_+ \leq 0$.

The calculation will proceed as follows. In Sec. 5.2, a weak field expansion of the field equations is performed. The perturbations are shown to satisfy the wave equation, with matter terms and nonlinear terms acting as sources. In Sec. 5.3, these equations are solved via integration of the sources with Green's functions. The source integrals are approximated in terms of time derivatives of moments of the sources, and evaluated to order of interest using the PPN expansion of the fields. In Sec. 5.4, an expression for the rate of change of energy contained within a volume of space is defined, and evaluated in terms of the wave forms. I will conclude with a discussion of the constraints on the c_n implied by observations from binary pulsars, in Sec. 5.5.

5.2 Field equations

In this section, I will expand the ae-theory field equations about a flat metric, constant aether background, obtaining a set of wave equations with matter terms and nonlinear terms as sources.

5.2.1 Exact equations

I begin with the standard four-parameter ae-theory action S

$$S = \frac{1}{16\pi G}\int d^4x\,\sqrt{|g|}\,\Big(R + K^{ab}{}_{cd}\nabla_a u^c \nabla_b u^d + \lambda(u^a u^b g_{ab} + 1)\Big), \tag{5.3}$$

where

$$K^{ab}{}_{cd} = \big(c_1 g^{ab} g_{cd} + c_2 \delta^a_c \delta^b_d + c_3 \delta^a_d \delta^b_c + c_4 u^a u^b g_{cd}\big), \tag{5.4}$$

with an additional aether-independent matter action. The matter can be assumed to couple universally to some metric since Lorentz-violating effects in nongravitational interactions are already highly constrained [5, 6]. Aether couplings are then excluded from the matter action and the field g_{ab} is identified as this universal metric.

The resulting equations of motion consist of the Einstein equations

$$G_{ab} - S_{ab} = 8\pi G T_{ab}, \tag{5.5}$$

where

$$G_{ab} = R_{ab} - \frac{1}{2} R g_{ab}, \tag{5.6}$$

$$\begin{aligned} S_{ab} =& \nabla_c\big(K_{(a}{}^c u_{b)} - K^c{}_{(a} u_{b)} - K_{(ab)} u^c\big) \\ &+ c_1\big(\nabla_c u_a \nabla^c u_b - \nabla_a u_c \nabla_b u^c\big) + c_4(u^c \nabla_c u_a)(u^d \nabla_d u_b) \\ &+ \lambda u_a u_b + \frac{1}{2} g_{ab}(K^c{}_d \nabla_c u^d), \end{aligned} \tag{5.7}$$

with

$$K^a{}_c = K^{ab}{}_{cd}\nabla_b u^d, \tag{5.8}$$

and T_{ab} is the matter stress tensor. There are also the aether field equations,

$$\nabla_b K^b{}_a = \lambda u_a + c_4(u^c\nabla_c u_b)\nabla_a u^b, \tag{5.9}$$

and the constraint

$$g_{ab}u^a u^b = -1. \tag{5.10}$$

Eqn. (5.9) can be used to eliminate λ, giving

$$\lambda = -u^c\nabla_a K^a{}_c + c_4(u^c\nabla_c u^a)(u^d\nabla_d u_a). \tag{5.11}$$

5.2.2 Linear-order variables

I will now expand the exact equations about a flat background. I assume a Minkowskian coordinate system and basis with respect to which, at zeroth order, the metric is the Minkowski metric η_{ab} and the aether is purely timelike. I then define variables h_{ab} and w^a, with

$$h_{ab} = g_{ab} - \eta_{ab}, \qquad w^0 = u^0 - 1, \qquad w^i = u^i. \tag{5.12}$$

I assume that h_{ab} and w^a fall off at spatial infinity like $1/r$.

I will further define variables by decomposing the above into irreducible transverse, or "divergence-free", and longitudinal, or "curl-free", pieces. The decomposition is unique and well-defined in Euclidean space, having imposed the above boundary conditions (one is essentially solving Laplace's equation—see [51] for more discussion). First, consider the spatial vectors w_i and h_{0i}, and define the following variables:

$$h_{0i} = \gamma_i + \gamma_{,i} \quad w^i = \nu^i + \nu^{,i}, \tag{5.13}$$

with $\gamma_{i,i} = \nu^i_{,i} = 0$. Next, consider the spatial components of the metric h_{ij}. A symmetric, 2-index tensor on Euclidean space can be uniquely decomposed into a transverse-trace-free tensor, a transverse vector, and two scalar quantities representing the transverse and longitudinal traces:

$$h_{ij} = \phi_{ij} + \frac{1}{2}P_{ij}[f] + 2\phi_{(i,j)} + \phi_{,ij}, \tag{5.14}$$

where

$$0 = \phi_{ij,j} = \phi_{jj} = \phi_{i,i}, \tag{5.15}$$

and

$$P_{ij}[f] = \delta_{ij}\triangle f - f_{,ij}; \tag{5.16}$$

hence, $P_{ij}[f]_{,j} = 0$, and $h_{ii} = \triangle(f + \phi)$. The list of variables then consists of a transverse-traceless spin-2 tensor ϕ_{ij}, transverse spin-1 vectors γ_i, ν_i, ϕ_i, and spin-0 scalars $h_{00}, w^0, \gamma, \nu, f, \phi$.

I will impose coordinate gauge conditions below, after expressing the field equations in unfixed form. The standard gauge to impose when performing the analogous calculation in conventional GR is the "harmonic" or "Lorentz" gauge, $2h_{ab}{}^{,b} = \eta^{cd}h_{cd,a}$, as this happens to reduce the field equations to a simple form when they are expressed covariantly. Some variant of this condition has a similar effect in several other alternative theories of gravity, as seen in [52]. Here, the increased complexity of the equations and the noncovariant decomposition of them and the field variables means that no obvious extension of the harmonic gauge has such a utility. Instead, the gauge will be chosen somewhat arbitrarily so as to eliminate certain variables:

$$0 = w_{i,i} = h_{0i,i} = h_{i[j,k]i}, \tag{5.17}$$

or equivalently,

$$0 = \nu = \gamma = \phi_i. \tag{5.18}$$

An infinitesimal coordinate gauge transformation has the linear-order form

$$\delta h_{ab} = \xi_{a,b} + \xi_{b,a} \quad \delta w^a = -\dot{\xi}^a. \tag{5.19}$$

The conditions (5.18) can be realized while in an arbitrary gauge (a prime denotes that the variables are evaluated in the original gauge) by choosing $\xi_0 = -(\gamma' + \nu')$ and the transverse part of ξ_i as $-\phi_i'$, and by solving for the longitudinal part ξ of ξ_i via $\dot{\xi} = \nu'$. One constraint on the choice of gauge is that it must be a valid PPN gauge, as defined in Chapter 4, so that the integrals of Sec. 5.3.3 can be evaluated by expressing the variables in terms of their PPN expansion. The above is a valid, albeit nonstandard, PPN gauge (in contrast to the gauge chosen in [30, 38]).

5.2.3 Linearized equations

I now express the field equations (5.5) and (5.9) in terms of the above variables, and arrange them in the form

$$\bar{G}_{ab} - \bar{S}_{ab} = 8\pi G\big(T_{ab} + t_{ab}\big), \tag{5.20}$$

$$\bar{K}^{b}{}_{a,b} = 8\pi G\sigma_a, \tag{5.21}$$

where the overbar denotes the portion of the tensor linear in h_{ab} and w^a, and the nonlinear source terms t_{ab} and σ_a are defined to a given order in the variables by asserting that the above equations equal the exact equations to that order. It will prove convenient to combine the equations in the form

$$\bar{G}_{ab} - \bar{S}_{ab} - \delta^0_{[a}\bar{K}^{c}{}_{b],c} = 8\pi G\tau_{ab}, \tag{5.22}$$

thus defining the source $\tau_{ab} = T_{ab}+t_{ab}-\delta^0_{[a}\sigma_{b]}$. Identities satisfied by the linear-order terms will imply conservation of τ_{ab}.

The constraint (5.10) to linear order is

$$w^0 = \frac{1}{2}h_{00}. \tag{5.23}$$

I will use this result to eliminate w^0. The form of nonlinear terms will not be needed as explained in Sec. 5.3.3.

Now,

$$\bar{G}_{ab} = -\frac{1}{2}(\triangle h_{ab} - \ddot{h}_{ab}) - \frac{1}{2}h_{,ab} + h_{c(a,b)}{}^{c} + \frac{1}{2}\eta_{ab}(\triangle h - \ddot{h} - h_{cd}{}^{,cd}), \tag{5.24}$$

where $h = \eta^{ab}h_{ab}$. Hence,

$$\bar{G}_{ij} = -\frac{1}{2}\big[\triangle\phi_{ij} - \ddot{\phi}_{ij}\big] + \big[\ddot{\phi}_{(i,j)} - \dot{\gamma}_{(i,j)}\big] + \frac{1}{4}P_{ij}[\triangle f - \ddot{f} - 2h_{00} - 2\ddot{\phi} + 4\dot{\gamma}] - \frac{1}{2}\ddot{f}_{,ij}, \tag{5.25}$$

$$\bar{G}_{0i} = -\frac{1}{2}\triangle(\gamma_i - \dot{\phi}_i) - \frac{1}{2}(\dot{F})_{,i}, \tag{5.26}$$

$$\bar{G}_{00} = -\frac{1}{2}\triangle F, \tag{5.27}$$

where $F = \triangle f$.

The linear-order forms of the covariant derivatives of u_a are

$$\begin{aligned}\overline{\nabla_0 u^i} &= \dot{w}^i + \dot{h}_{0i} - \frac{1}{2}h_{00,i} \\ &= \dot{\nu}^i + \dot{\gamma}_i + (\dot{\nu} + \dot{\gamma} - \frac{1}{2}h_{00})_{,i},\end{aligned} \tag{5.28}$$

$$\begin{aligned}\overline{\nabla_i u^j} &= w^j_{,i} + h_{0[j,i]} + \frac{1}{2}\dot{h}_{ij} \\ &= \frac{1}{2}\dot{\phi}_{ij} + \nu^j_{,i} + \gamma_{[j,i]} + \dot{\phi}_{(j,i)} + \frac{1}{4}P_{ij}[\dot{f}] + (\nu + \frac{1}{2}\dot{\phi})_{,ij},\end{aligned} \tag{5.29}$$

and $\overline{\nabla_a u^0} = 0$.

From

$$\bar{S}_{ab} = -\dot{\bar{K}}_{(ab)} + \delta^0_{(a}\bar{K}_{b)}{}^c{}_{,c}, \tag{5.30}$$

follows

$$\begin{aligned}\bar{S}_{ij} &= -\partial_0(c_+\overline{\nabla^{(i}u^{j)}} + c_2\delta_{ij}\overline{\nabla_k u^k}) \\ &= -\frac{c_+}{2}\ddot{\phi}_{ij} - c_+(\dot{\nu}^{(i,j)} + \ddot{\phi}_{(i,j)}) - \frac{1}{2}P_{ij}[c_2(2\dot{\nu} + \ddot{\phi} + \ddot{f}) + \frac{c_+}{2}\ddot{f}] \\ &\quad - \frac{1}{2}\big((c_2 + c_+)(2\dot{\nu} + \ddot{\phi}) + c_2\ddot{f}\big)_{,ij},\end{aligned} \tag{5.31}$$

$$\begin{aligned}\bar{S}_{0i} - \frac{1}{2}\bar{K}_{ai}{}^{,a} &= -\bar{K}_{(ij),j} = -c_+\partial_j(\overline{\nabla^{(i}u^{j)}}) \\ &= -\frac{1}{2}\triangle\Big(c_+(\nu^i + \dot{\phi}_i) + \big((c_+ + c_2)(2\nu + \dot{\phi}) + c_2\dot{f}\big)_{,i}\Big),\end{aligned} \tag{5.32}$$

$$\begin{aligned}\bar{S}_{0i} + \frac{1}{2}\bar{K}_{ai}{}^{,a} &= -\dot{\bar{K}}_{0i} - \bar{K}_{[ij],j} = -c_{14}\partial_0(\overline{\nabla_0 u^i}) - c_-\partial_j(\overline{\nabla^{[i}u^{j]}}) \\ &= -c_{14}(\ddot{\nu}^i + \ddot{\gamma}_i) + \frac{c_-}{2}\triangle(\nu^i + \gamma_i) - c_{14}\big(\ddot{\nu} + \ddot{\gamma} - \frac{1}{2}\dot{h}_{00}\big)_{,i}\end{aligned} \tag{5.33}$$

and

$$\bar{S}_{00} = -c_{14}\partial_j(\overline{\nabla_0 u^j}) = -\triangle\Big(c_{14}(\dot{\nu} + \dot{\gamma} - \frac{1}{2}h_{00})\Big). \tag{5.34}$$

The above expressions indicate that the linear-order terms satisfy the identity

$$\left(\bar{G}_{ab} - \bar{S}_{ab} - \delta^0_{[a} \bar{K}^c{}_{b],c}\right)^{,b} = 0. \tag{5.35}$$

This implies that the source τ_{ab} obeys a conservation law

$$\tau_{ab}{}^{,b} = \tau_{ai,i} - \dot{\tau}_{a0} = 0. \tag{5.36}$$

5.2.4 Wave equations

The above equations can now be decomposed as the variables. The field equations for variables of different spin will separate. I will impose below the gauge conditions (5.18). The following results are equivalent to those of [30] expressed in a different gauge when $\tau_{ab} = 0$; in particular, the count of independent plane wave modes—two spin-2, two spin-1, and one spin-0—and the expressions for the wave speeds are recovered.

I now consider the different spins in turn. Define the set of operators

$$\Box_i \psi \equiv \triangle \psi - (s_i)^{-2} \ddot{\psi}, \tag{5.37}$$

for $i = 0, 1, 2$.

Spin-2

The transverse-traceless part of the space-space components of (5.22) gives

$$\Box_2 \phi_{ij} = -16\pi G \tau^{\rm TT}_{ij}, \tag{5.38}$$

with

$$(s_2)^2 = \frac{1}{1 + c_+}, \tag{5.39}$$

and where TT signifies the transverse-traceless projection.

Spin-1

Now the spin-1 variables. The transverse parts of (5.22) give

$$\triangle(c_+\nu^i - \gamma_i) = 16\pi G\tau_{i0}^{\mathrm{T}}, \tag{5.40}$$

and

$$c_{14}(\ddot{\nu}^i + \ddot{\gamma}_i) - \frac{1}{2}\triangle(c_-\nu^i + (1+c_-)\gamma_i) = 8\pi G\tau_{0i}^{\mathrm{T}}. \tag{5.41}$$

where the T signifies the transverse projection. These relations imply

$$\Box_1(\nu^i + \gamma_i) = \frac{-16\pi G}{2c_1 + c_+c_-}\big(c_+\tau_{i0} - (1+c_+)\sigma^i\big)^{\mathrm{T}}, \tag{5.42}$$

with

$$(s_1)^2 = \frac{2c_1 + c_+c_-}{2(1+c_+)c_{14}}. \tag{5.43}$$

Spin-0

Now consider the spin-0 variables. The transverse-trace and longitudinal-trace portions of the space-space components of (5.22) give

$$(1 - 2c_2 - c_+)\ddot{F} - \triangle\big(F - 2h_{00} - 2(1-c_2)\ddot{\phi}\big) = -16\pi G\tau_{ii}^{\mathrm{T}}, \tag{5.44}$$

and

$$(1 - c_2)\ddot{F} - c_{123}\triangle\ddot{\phi} = -16\pi G\tau_{ii}^{\mathrm{L}}, \tag{5.45}$$

where $\tau_{ij}^{\mathrm{L}} = \tau_{ij} - \tau_{ij}^{\mathrm{T}}$. The time-time component of (5.22) gives

$$\triangle(F + c_{14}h_{00}) = -16\pi G\tau_{00}, \tag{5.46}$$

and the longitudinal space-time component of (5.22) gives

$$\triangle\big((1-c_2)\dot{f} - c_{123}\dot{\phi}\big)_{,i} = -16\pi G\tau^{\rm L}_{i0}, \tag{5.47}$$

where $\tau^{\rm L}_{i0} = \tau_{i0} - \tau^{\rm T}_{i0}$. These equations imply

$$\Box_0 F = \frac{16\pi G c_{14}}{2+c_{14}}(\tau_{ii} + \frac{2-3c_2-c_+}{c_{123}}\tau^{\rm L}_{ii} - \frac{2}{c_{14}}\tau_{00}), \tag{5.48}$$

with

$$(s_0)^2 = \frac{(2+c_{14})c_{123}}{(2-3c_2-c_+)(1+c_+)c_{14}}. \tag{5.49}$$

Further implied by these and the untraced, transverse-trace part of (5.22) is the equation

$$\Box_0 f_{,ij} = \tau'_{ij}. \tag{5.50}$$

The form of the source τ'_{ij} is unimportant; only the fact that $f_{,ij}$ satisfies a sourced wave equation is needed so that later eqn. (5.95) can be applied when evaluating the damping rate expression in Sec. 5.4.

5.3 Evaluation of source integrals

The above equations can be formally solved via integration of the sources with the appropriate Green's function, and the resulting integrals approximated in terms of time derivatives of moments of the source. Upon doing so, the nonstatic contributions to the fields to desired accuracy at points far from the material source depend on two integral quantities, the second mass moment of the material source

$$I_{ij} = \int d^3x \rho\, x_i x_j, \tag{5.51}$$

where $\rho = T_{00}$ to lowest order, and the integral

$$\Sigma_i = \int d^3x\, \sigma_i, \tag{5.52}$$

where σ_i are the quadratic terms from the aether field equation (5.21).

5.3.1 Approximation of source integrals

Equations of the form

$$\triangle\psi - (s)^{-2}\ddot{\psi} = -16\pi\tau, \tag{5.53}$$

can be solved with outward-going disturbances at infinity by writing

$$\psi(t, \mathbf{x}) = 4\int d^3x' \frac{\tau(t - z/s, \mathbf{x}')}{z}, \tag{5.54}$$

where $z = |\mathbf{x} - \mathbf{x}'|$.

The source integral can be simplified with a standard approximation [17]. As indicated by the energy loss rate expression (5.98), only the portion of the fields that fall off as $(1/r)$ are of interest. A weak field, slow motion assumption will be made: the material source should be described by a mass m, a size L, and a time-scale T such that $(G_N m/L)$ and (L/Ts) are small quantities. Then only terms of interesting order are retained in the following expansion:

$$\psi(t, \mathbf{x}) \approx \frac{4}{R}\Big(\sum_{m=0}^{\infty} \frac{1}{m! s^m} \frac{\partial^m}{\partial t^m} \int \tau(t - R/s, \mathbf{x}')\big(x'^i \hat{x}^i\big)^m\Big), \tag{5.55}$$

where $R = |\mathbf{x}|$ and $\hat{\mathbf{x}} = \mathbf{x}/R$, and $R \gg L$.

Now, the following sleight of hand justifies solving the decomposed ae-theory equations by first approximating the integral on the right side of (5.54) using the full

τ and *then* taking the projection. Introduce the notation $[[\tau(t,\mathbf{x}')]] \equiv \tau(t - z/s, \mathbf{x}')$. Because the quantity on the right side of (5.54) depends on $\mathbf{x}$ only through z, it follows that

$$\frac{\partial}{\partial x^i}\int \frac{[[\tau(t,\mathbf{x}')]]}{z} = -\int \frac{\partial}{\partial x'^i}(\frac{[[\tau(t,\mathbf{x}')]]}{z}) + \int \frac{[[\partial_i{}'\tau(t,\mathbf{x}')]]}{z}. \tag{5.56}$$

It follows from this that, e.g.,

$$\int \frac{[[\tau^{\mathrm{T}}_{ij}(t,\mathbf{x}')]]}{z} = \Big(\int \frac{[[\tau_{ij}(t,\mathbf{x}')]]}{z}\Big)^{\mathrm{T}}, \tag{5.57}$$

after discarding integrals of total derivatives, where T on the left side signifies transverse with respect to $\mathbf{x}'$, and on the right side transverse with respect to $\mathbf{x}$. As a further convenience, it follows that to $O(1/R)$, the transverse projection is equal to the algebraic projection in the direction orthogonal to $\hat{\mathbf{x}}$.

Additionally, there are the Poissonnian equations (5.40), (5.46), and (5.47) of the form

$$\triangle\psi = -16\pi\tau. \tag{5.58}$$

Solving via Green's function and expressing to $O(1/R)$ far from the source gives

$$\psi(t,\mathbf{x}) \approx \frac{4}{R}\int d^3x'\tau(t,\mathbf{x}'). \tag{5.59}$$

The integrals of the sources in these particular equations happen to be conserved quantities. Thus, ignoring static terms in the wave forms, the equations are effectively *un*sourced, and simply

$$\psi = 0. \tag{5.60}$$

5.3.2 Sorting of source integrals

To evaluate the integrals indicated by Eqn. (5.55), I will express the sources in terms of their post-Newtonian expansions. I will further assume that the system is composed of compact bodies of individual size $d \ll L$ that exert negligible tidal forces on each other. The system will then have an orbital velocity $v \sim \sqrt{G_N m/L}$. Following the discussion in [17], the leading-order terms in the fields will be $O(G_N m^2/L)$, which give the quadrupolar and monopolar contributions, and $O((G_N m^2/d)v)$, giving the dipolar contribution. Terms of these orders can only result from integrals of terms that are, respectively, 2PN and 2.5PN order. Integrals of interest can be identified by noting that since the rate of change of the system is governed by its velocity, assumed to be .5PN order, taking the time derivative of a quantity effectively multiplies it by a factor of v and raises it by .5PN orders. Also, only nonstatic, or non-conserved, terms are of interest as only the time derivatives of the fields will appear in the expression for the energy loss.

I begin by considering the moments of τ_{ij}. First, the conservation law implies

$$\int \tau_{ij} = \frac{1}{2}\int \ddot{\tau}_{00} x'_i x'_j - \int \dot{\sigma}_{(i} x'_{j)} = \frac{1}{2}\int \ddot{T}_{00} x'_i x'_j = \frac{1}{2}\ddot{I}_{ij}, \tag{5.61}$$

where the last two equalities hold to desired order and for the last I have used the Eulerian continuity equation for the fluid

$$\dot{\rho} + (\rho v^i)_{,i} = 0, \tag{5.62}$$

assumed to hold at $O(1.5)$. Then,

$$\int \dot{\tau}_{ij} x'_k = -\frac{1}{2}\int \left(\ddot{\tau}_{i0} x_j x_k + \ddot{\tau}_{j0} x_k x_i - \ddot{\tau}_{k0} x_i x_j\right), \tag{5.63}$$

which is of uninteresting order, as are remaining moments.

I now consider the moments of τ_{i0}. The integral of τ_{i0} is conserved, so I ignore it. Next,

$$\int \dot{\tau}_{i0} x_j = -\int \tau_{ij} = -\frac{1}{2}\ddot{I}_{ij}. \tag{5.64}$$

The other moments are of uninteresting orders.

I now consider moments of τ_{00}. First, the integral of τ_{00} is conserved, so I ignore it. Then,

$$\int \dot{\tau}_{00} x_i = -\int \tau_{0i} = \Sigma_i, \tag{5.65}$$

where the second equality ignores the static integral of τ_{i0}. Finally,

$$\int \ddot{\tau}_{00} x_i x_j = \ddot{I}_{ij}, \tag{5.66}$$

to desired order.

5.3.3 Evaluating Σ_i

I now consider the moments of σ_i. The terms in σ_i are at least 2.5PN order. The only integral of interest is thus $\Sigma_i = \int \sigma_i$, which is $O((G_N m^2/d)v)$. At this point, I can explain why the nonlinear terms in the unit constraint (5.10) can be ignored. The previous subsection makes clear that we only their appearance in σ_i need be considered. As follows from the post-Newtonian forms given in Chapter 4, the nonlinear constraint terms are integer PN orders starting with 2PN and have no free indices, and there are no field variables that are .5PN order. It follows that any nonlinear constraint terms appearing at 2.5PN order in σ_i must do so in the form $(terms)_{,0i}$. Total derivatives do not contribute to Σ_i, so these terms can be ignored.

I will evaluate Σ_i explicitly by expressing the fields in terms of the PPN expansion, but in the nonstandard coordinate gauge (5.18). The PPN forms in the standard PPN coordinate gauge are reported in Chapter 4 as

$$\phi_{ij} = 0 \tag{5.67}$$

$$\gamma_i = -\frac{2c_1}{c_-}\nu^i = -\frac{8+\alpha_1}{4}(V_i + W_i), \qquad \phi_i = 0 \tag{5.68}$$

$$h_{00} = -\triangle\chi, \quad f = 2\phi = -2\chi, \tag{5.69}$$

$$\gamma = -\frac{1}{4}(6+\alpha_1 - 2\alpha_2)\dot{\chi}, \quad \nu = \frac{2c_1 + 3c_2 + c_3 - c_4}{2c_{123}}\dot{\chi}, \tag{5.70}$$

where

$$V_i(x) = G_N \int d^3x' \frac{\rho(x')v^i}{z}, \qquad W_i(x) = G_N \int d^3x' \frac{\rho(x)v_j z_j z^i}{z^3}, \tag{5.71}$$

$$\chi(x) = V_i(x) - W_i(x) = -G_N \int d^3x'\, \rho(x')\, z, \tag{5.72}$$

with $z^i = x^i - x'^i$, and

$$G_N = \frac{G}{1 + (c_{14}/2)}, \tag{5.73}$$

$$\alpha_1 = \frac{8(c_3^2 - c_1 c_4)}{2c_1 + c_+ c_-}, \tag{5.74}$$

$$\alpha_2 = \frac{\alpha_1}{2} + \frac{(c_1 + 2c_3 + c_4)(2c_1 + 3c_2 + c_3 - c_4)}{(2 + c_{14})c_{123}}. \tag{5.75}$$

Adjustment from the standard to the nonstandard gauge is done by defining a gauge parameter ξ_a with $\xi_0 = -(\gamma' + \nu')$, $\dot{\xi}_i = \nu'_{,i}$, where γ', ν' are the standard-gauge values. Then in the nonstandard gauge, the variables are as above except $\nu = \gamma = 0$ and

$$\phi = \frac{(c_1 + 2c_2 - c_4)}{c_{123}}\chi. \tag{5.76}$$

With these forms, Σ_i can be evaluated, and after some algebra gives

$$\Sigma_i = \frac{c_{14}}{2}\int \rho\big((\alpha_2 - \alpha_1)V_i - \alpha_2 W_i\big). \tag{5.77}$$

I will later consider the special cases of a single, compact, spherically symmetric body and of a pair of compact bodies that are static and spherically symmetric in their own rest frames. In the first case, spherical symmetry implies that Σ_i vanishes—otherwise it would define a symmetry-breaking spatial vector. In the second case, or more generally with n such bodies, Σ_i can be simplified via the following observations. First, the Newtonian potential U felt at a given body contains an $O(G_N m/L)$ contribution from the presence of the other bodies, plus an $O(G_N m/d)$ self-contribution $\bar{U}$,

$$\bar{U}_a(x_a) = G_N \int_a d^3x' \frac{\rho}{|\mathbf{x_a} - \mathbf{x}'|}, \tag{5.78}$$

where the integral extends just over the "a-th" body. Second, spherical symmetry of each body implies that $(\Omega_a)_{ij} = (1/3)\Omega_a \delta_{ij}$, where

$$(\Omega_a)_{ij} \equiv -\frac{1}{2} G_N \int_a d^3x\, d^3x' \frac{\rho(x)\rho(x') z_i z_j}{z^3}, \tag{5.79}$$

and

$$\Omega_a = (\Omega_a)_{ii} = -\frac{1}{2} \int_a d^3x \rho \bar{U}_a. \tag{5.80}$$

Third, staticity of each body implies that $V_i = \sum_a (v_a)^i U$, and similarly for W_i. These facts imply that

$$\int \rho V_i = 3 \int \rho W_i = -2 \sum_a (v_a)^i \Omega_a, \tag{5.81}$$

plus terms of $O(G_N m^2 v/L)$. Therefore, to interesting order,

$$\Sigma_i = c_{14}(\alpha_1 - \frac{2}{3}\alpha_2) \sum_a (v_a)^i \Omega_a. \tag{5.82}$$

5.3.4 Wave forms

I can now express the nonstatic, radiation-zone fields, to desired accuracy. For spin-2,

$$\phi_{ij} = \frac{2G}{R}(\ddot{Q}_{ij})^{\mathrm{TT}}. \tag{5.83}$$

where

$$Q_{ij} = I_{ij} - \frac{1}{3}I, \qquad I = I_{ii}. \tag{5.84}$$

For spin-1,

$$\nu^i = \frac{-2G}{R}\frac{1}{2c_1 + c_+c_-}\left(\frac{c_+}{(1+c_+)s_1}\ddot{Q}_{ij}\hat{x}_j + 2\Sigma_i\right)^{\mathrm{T}}, \tag{5.85}$$

$$\gamma_i = c_+\nu^i, \tag{5.86}$$

For spin-0,

$$F = \frac{-2G}{R}\frac{c_{14}}{2+c_{14}}\left(3(Z-1)\hat{x}^i\ddot{Q}_{ij}\hat{x}^j + Z\ddot{I} - \frac{4}{c_{14}s_0}\Sigma_i\hat{x}^i\right), \tag{5.87}$$

$$h_{00} = -\frac{1}{c_{14}}F, \tag{5.88}$$

$$\dot{\phi}_{,i} = \frac{(1-c_2)}{c_{123}}\dot{f}_{,i}, \tag{5.89}$$

where

$$Z = \frac{2(2\alpha_2 - \alpha_1)(1+c_+)}{3(2c_+ - c_{14})}. \tag{5.90}$$

5.4 Energy loss formulas

I now turn to the expression for the rate of change of energy contained within a volume of space. Such an expression can be derived via the Noether charge method for defining the total energy of an asymptotically-flat space-time [37], using the ae-theory Noether current derived in Chapter 3. One can equivalently work in terms of pseudotensors, using the results of [38].

Following the discussion in the appendix of [37], an expression for the total energy $\mathcal{E}$ contained in a volume of space, for a theory linearized about a flat background, is given by the integral over that volume of a certain differential 3-form $\mathbf{J}_{abc} \equiv J^d \epsilon_{dabc}$. $\mathbf{J}$ can be obtained from the quadratic-order ae-theory Lagrangian modulo a boundary term. It will depend on the metric, aether, and an arbitrary background vector field. To define the energy, choose the background vector field as $t^a = (\partial/\partial t)^a$. Choose the volume V to be that contained within a sphere of coordinate radius R. Then,

$$\mathcal{E} \equiv \int_V \mathbf{J}[t] = \int d^3x J^0[t], \tag{5.91}$$

$$\dot{\mathcal{E}} \equiv \int \mathcal{L}_t \mathbf{J}[t] = \int d(t \cdot \mathbf{J}[t]) = -\int_R d\Omega R^2 \hat{x}^i J^i[t], \tag{5.92}$$

where in the second line I have used the formula $\mathcal{L}_t \mathbf{J} = d(t \cdot \mathbf{J}) + t \cdot d\mathbf{J}$ and the fact that $d\mathbf{J} = 0$ when the dynamical fields satisfy the equations of motion [37].

I will define $\mathbf{J}$ with respect to the ae-theory Lagrangian $\mathcal{L}$ modulo a total derivative:

$$\begin{aligned} \mathcal{L}' \equiv & \mathcal{L} - \frac{1}{16\pi G}\Big(\sqrt{|g|}\big(\Gamma^c_{ab} g^{ab} - \Gamma^b_{ab} g^{ac}\big)\Big)_{,c} \\ & = \frac{\sqrt{|g|}}{16\pi G}\Big(g^{ab}\big(\Gamma^c_{ad}\Gamma^d_{cb} - \Gamma^c_{cd}\Gamma^d_{ab}\big) + K^a{}_b \nabla_a u^b\Big). \end{aligned} \tag{5.93}$$

The procedure of [37] gives:

$$\begin{aligned} J^a = & \frac{1}{16\pi G}\Big[\dot{h}_{bc}\Big(h^{ab,c} - \frac{1}{2}h^{bc,a} + \frac{1}{2}\eta^{bc}\big(h^{d,a}_d - h^{ad}{}_{,d}\big) - \frac{1}{2}g^{d,b}_d \eta^{ca} \\ & + u^a K^{bc} + 2K^{[ab]}u^c\Big) + 2\dot{u}^b K^a{}_b\Big] - t^a \mathcal{L}', \end{aligned} \tag{5.94}$$

where all indices on h_{ab} are raised with the flat metric η^{ab}.

I will presume that only the time average of the damping rate need be determined. It is then crucial to note that the damping rate is calculated to lowest nonvanishing PN order by treating the system as exactly Newtonian. The motion of the system can then be decomposed into a uniform translation of the center of mass–recall the conservation of $\int \tau_{i0}$–and a fixed Keplerian orbit in the center-of-mass frame. As indicated below, the field wave forms do not depend on the center-of-mass motion. It then follows that if the time average is taken over an orbital period, total time derivatives in (5.94) do not contribute.

It is also useful to note that approximation of the source integrals (5.55) implies that to $O(1/R)$,

$$\psi_{,j} = -(1/s)\dot{\psi}\hat{x}^j, \tag{5.95}$$

for field ψ satisfying the sourced wave equation (5.53). This relation then implies that

$$0 = \hat{x}^i\dot{\phi}_{ij}(x) = \hat{x}^i\dot{\nu}_i(x) = \hat{x}^i P_{ij}[\dot{f}(x)]. \tag{5.96}$$

These facts permit manipulation of terms within the integral, e.g.:

$$\int < \dot{h}_{jk}\phi_{,jki} >= \int < -\frac{1}{s_0}\dot{\phi}_{,jk}\dot{\phi}_{,ji}\hat{x}_k >= \int < -\frac{1}{s_0}\triangle\dot{\phi}\triangle\dot{\phi}\hat{x}_i >, \tag{5.97}$$

where the angular brackets denote the time average.

The energy loss rate then evaluates to

$$\dot{\mathcal{E}} = \frac{-1}{16\pi G}\int_R d\Omega R^2 \Big\langle \frac{1}{2s_2}\dot{\phi}_{jk}\dot{\phi}_{jk} - \frac{(2c_1 + c_+c_-)(1+c_+)}{s_1}\dot{\nu}^j\dot{\nu}^j - \frac{2+c_{14}}{4c_{14}s_0}\dot{F}\dot{F}\Big\rangle. \tag{5.98}$$

The sign of the coefficient of the term for each spin is opposite to the sign of the energy density associated with linearized plane waves, as found in [38]. Thus, a positive-energy mode implies energy loss due to radiation of that mode.

The energy loss rate can be further evaluated by substituting in the expressions for the fields given in Sec. 5.3.4, and performing the angular integral. With the results:

$$\frac{1}{4\pi}\int d\Omega\, \dot{\phi}_{ij}\dot{\phi}_{ij} = \frac{8G^2}{5R^2}(\dddot{Q}_{ij}\dddot{Q}_{ij}), \tag{5.99}$$

$$\frac{1}{4\pi}\int d\Omega \dot{\nu}_i\dot{\nu}_i = \frac{4G^2}{R^2}\frac{1}{(2c_1+c_+c_-)^2}\left(\frac{1}{5}\Big(\frac{c_+}{(1+c_+)s_1}\Big)^2\Big(\dddot{Q}_{ij}\dddot{Q}_{ij}\Big)+\frac{8}{3}\Big(\dot{\Sigma}_i\dot{\Sigma}_i\Big)\right), \tag{5.100}$$

$$\frac{1}{4\pi}\int d\Omega \dot{F}\dot{F} = \frac{4G^2}{R^2}\Big(\frac{c_{14}}{2+c_{14}}\Big)^2\Big(\frac{6(Z-1)^2}{5}(\dddot{Q}_{ij}\dddot{Q}_{ij})+Z^2(\dddot{I}\dddot{I})+\frac{16}{3(c_{14}s_0)^2}(\dot{\Sigma}_i\dot{\Sigma}_i)\Big), \tag{5.101}$$

substituted into expression (5.98) gives

$$\dot{\mathcal{E}} = -G_N\left\langle\frac{\mathcal{A}}{5}\Big(\dddot{Q}_{ij}\dddot{Q}_{ij}\Big)+\mathcal{B}\Big(\dddot{I}\dddot{I}\Big)+\mathcal{C}\Big(\dot{\Sigma}_i\dot{\Sigma}_i\Big)\right\rangle, \tag{5.102}$$

where

$$\mathcal{A} = \Big(1+\frac{c_{14}}{2}\Big)\Big(\frac{1}{s_2}-\frac{2c_{14}c_+^2}{(2c_1+c_+c_-)^2}\frac{1}{s_1}-\frac{3(Z-1)^2c_{14}}{2(2+c_{14})}\frac{1}{s_0}\Big), \tag{5.103}$$

$$\mathcal{B} = -\frac{Z^2c_{14}}{8}\frac{1}{s_0}, \tag{5.104}$$

$$\mathcal{C} = -\frac{2}{3c_{14}}\Big(\frac{2+c_{14}}{s_1^3}+\frac{1}{s_0^3}\Big), \tag{5.105}$$

where Z is given in (5.90). This constitutes the generalization to ae-theory of the quadrupole formula of general relativity, and contains additional contributions from monopolar and dipolar sources.

The presence of the monopolar term means that a spherically symmetric source, such as a spherically pulsating star, can radiate at this lowest nontrivial

PN order in the presence of the aether, whereas it would not in pure GR. In this case, $I_{ij} = (1/3)\delta_{ij} I$, and as observed above, $\Sigma_i = 0$. Only the spin-0 radiation fields are nonvanishing, and the energy loss rate is $\dot{\mathcal{E}} = -G_N \mathcal{B}\Big\langle(\dddot{I})^2\Big\rangle$. Bounds discussed in Sec. 5.5, however, require that the PPN parameters α_1 (5.74) and α_2 (5.75), hence $\mathcal{B}$ (5.104), vanish for observationally viable ae-theories to sufficient accuracy that the monopole term is negligible. Thus, for instance, there should be no detectable influence on the slowing of axial spin ("slow down rate") of millisecond pulsars.

For a binary system, treating the two bodies as static and spherically symmetric in their own rest frames leads to

$$\dddot{I}_{ij} = -\frac{2G_N \mu m}{r^2}(4\hat{r}_{(i}v_{j)} - 3\hat{r}_i\hat{r}_j\dot{r}), \tag{5.106}$$

where $m = m_1 + m_2$, $\mu = m_1 m_2/m$, $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ is the relative separation and $\mathbf{v} = \dot{\mathbf{r}}$. Then,

$$\dddot{Q}_{ij}\dddot{Q}_{ij} = \frac{8}{3}(\frac{G_N \mu m}{r^2})^2(12v^2 - 11\dot{r}^2). \tag{5.107}$$

Also,

$$\dot{\Sigma}_i = -c_{14}(\alpha_1 - \frac{2}{3}\alpha_2)\frac{G_N \mu m}{r^2}\mathcal{D}\hat{x}_i, \tag{5.108}$$

where $\mathcal{D}$ is the difference in binding energy per unit rest mass:

$$\mathcal{D} = \frac{\Omega_a}{m_a} - \frac{\Omega_b}{m_b}. \tag{5.109}$$

Therefore,

$$\dot{\mathcal{E}} = -G_N\Big\langle\Big(\frac{G_N \mu m}{r^2}\Big)^2\Big(\frac{8}{15}\mathcal{A}\big(12v^2 - 11(\dot{r})^2\big) + 4\mathcal{B}\big(\dot{r}\big)^2 + c_{14}^2(\alpha_1 - \frac{2}{3}\alpha_2)^2\mathcal{C}\mathcal{D}^2\Big)\Big\rangle. \tag{5.110}$$

5.5 Parameter constraints

I will now discuss bounds on the c_n that can be derived by imposing the observational constraints summarized in Chapter 4 and by comparing the damping rate prediction (5.110) with measurements of binary pulsar systems. The four c_n can be reduced to one free parameter by requiring that the PPN parameters α_1 and α_2 vanish, and that the damping rate coincide with that of GR to lowest order. The theory will then satisfy all solar system based tests, but it is not correct to say that it would pass the binary pulsar test. This is because the fields inside a neutron star pulsar or black hole companion are not weak, and strong field corrections to the quasi-static parameters may arise. Nevertheless, the weak field results are adequate for small enough c_n, as discussed in Chapter 6. Therefore, it is useful to check whether this curve of ae-theories intersects the region allowed by positive-energy, real-frequency, vacuum-Čerenkov, and nucleosynthesis constraints (Chapter 4).

The PPN parameters α_1 (5.74) and α_2 (5.75) for ae-theory were determined in Chapter 4. It was shown that they can be set to zero, so that all of the ae-theory PPN parameters coincide with those of GR, with the choices

$$c_2 = -\frac{2c_1^2 + c_1 c_3 - c_3^2}{3c_1}, \qquad c_4 = \frac{c_3^2}{c_1}. \tag{5.111}$$

The positive-energy, real-frequency, vacuum-Čerenkov, and nucleosynthesis constraints can then be satisfied if c_1 and c_3 lie within the region

$$-1 < c_+ < 0, \qquad \frac{c_+}{3(1+c_+)} < c_- < 0 \tag{5.112}$$

When α_1 and α_2 vanish, so does Z (5.90) hence $\mathcal{B}$ (5.104), and Σ_i. The fields then contain only a quadrupole contribution, and the ae-theory damping rate (5.110) will match that of GR when $\mathcal{A} = 1$. Solving numerically shows that a solution curve exists in (c_+, c_-) space that intersects the allowed region (5.112) for all negative values of c_-. Thus, there exists a one-parameter family of ae-theories which satisfy all of the constraints summarized in Chapter 4, and which predict a damping rate identical in the weak field limit to that of GR.

Observational error allows this curve to be widened into a band. The standard method of measuring radiation damping is to observe the rate of change of orbital period $\dot{P}$ of a binary system [47, 50], which will be proportional to $\dot{\mathcal{E}}$; some details and subtleties are mentioned in Sec. 5.1. The smallest relative observational error in $\dot{P}$, which equates with the relative uncertainty in $\dot{\mathcal{E}}$, is of order 0.1% for the Hulse-Taylor binary B1913+16 [47, 50]. This error permits the band $|\mathcal{A} - 1| \sim 10^{-3}$. Numerical results indicate that at least for small $c_\pm$, this band corresponds roughly to $c_\pm$ within about 10^{-3} of the $\mathcal{A} = 1$ curve.

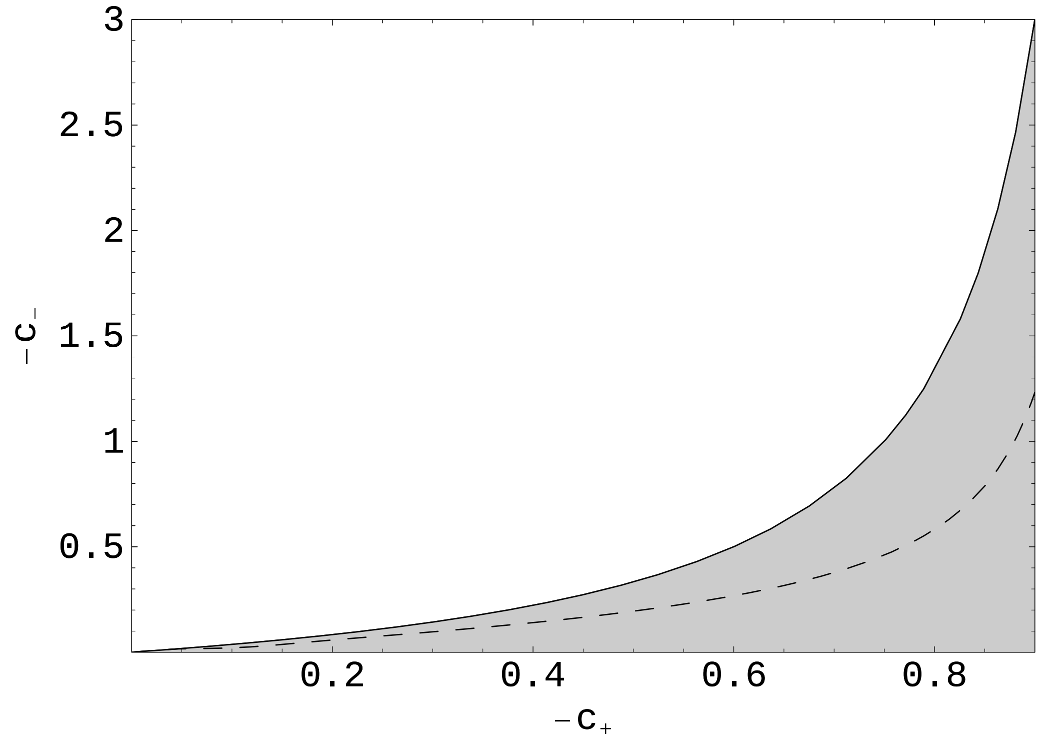

Figure 5.1: Class of allowed ae-theories, if strong field effects in binary pulsar systems can be ignored (for $c_+ \gtrsim -(0.9)$). The four-dimensional c_n space has been restricted to the (c_+, c_-) plane by setting the PPN parameters α_1 and α_2 to zero via the conditions (5.111). The shaded region is the region allowed by collected non-binary constraints considered in Chapter 4, demarcated in (5.112). The dashed curve is the curve along which binary pulsar tests will be satisfied, assuming ae-theory weak field expressions. Specifically, it is the curve along which the damping rate (5.102) is identical to the quadrupole formula of GR—that is, along which $\mathcal{A} = 1$ (5.103) in the $\alpha_{1,2} = 0$ case. Along both this curve and the boundary of the allowed region, $c_- \to -\infty$ as $c_+ \to -1$. The curve remains within the allowed region for all c_+ between -1 and 0. As explained in Chapter 6, strong field effects may lead to system-dependent corrections to the binary pulsar curve for large c_n; however, all such curves will coincide with the weak field curve for $|c_n| \lesssim (0.1)$ given current observational errors (see Chapter 6).

Chapter 6

Strong field effects on binary pulsar systems

6.1 Introduction

In Chapter 4, I showed how the four free ae-theory parameters c_n are cut down to two by the requirement that the PPN parameters match those of GR. In Chapter 5, I demonstrated how observations of binary pulsar systems would lead to one additional constraint on the c_n, if there were justification for ignoring effects of strong fields inside the binary bodies. This assumption is a dangerous one, since the fields inside neutron star pulsars are expected to be very strong. That justification requires an unclear assumption on the values of the c_n.

In this chapter, I will incorporate strong field effects on binary pulsar systems, by calculating the post-Newtonian (PN) equations of motion and the rate of radiation damping of a system of strongly gravitating bodies. I will do this via an effective approach in which the compact bodies are treated as point particles whose action contains couplings that depend on the velocity of the particles in the preferred frame. This approach introduces new dimensionless coefficients—"sensitivities"—that parametrize the nonstandard couplings and can be calculated for a given stellar source by matching the effective theory onto the exact, perfect fluid theory. This work does not render Chapter 5 redundant; in fact, comparing the weak field limit of the point particle approach with the perfect fluid approach serves as a check of the results, and reveals how the sensitivities scale in the weak field limit.

The expressions obtained can be used in principle to obtain precise bounds on the allowed class of ae-theories, but taking this step will require work beyond the scope of this dissertation. Specifically, precise bounds will require a methodology for dealing with dependence on the system's unknown center-of-mass velocity, and a formula for the values of the sensitivities of a given source. The center-of-mass velocity plays a role, because the theory is not Lorentz invariant at PN order when strong field effects are included. The sensitivities describe perturbations of static stellar solutions, about which very little is known at this time.

For the time being, a few comments can be made, which will be defended in the text. A crucial piece of information learned by comparing the weak field limit with the expressions of Chapter 5 is that the sensitivities will be "small"—at least as small as $(G_N m/d)^2$, where m is the body's mass and d its size, times a c_n dependent coefficient that must scale at least quadratically with c_n in the small c_n limit. For neutron stars in GR, $(G_N m/d) \sim (.1 - .3)$, and it is reasonable to expect similar for ae-theory. Then, bounds on the magnitude of violations of the strong equivalence principle [50] constrain the c_n dependent factor to be less than roughly (0.01). Also, if the three weak field conditions are imposed on the c_n, then the strong field corrections fall below the level of current observational error when $c_n \lesssim (0.1)$. In that case, *all current tests from binary pulsar systems will be passed.*

I will now present the strong field formulas. First, the effective particle action is constructed, and the exact field equations are defined in Sec. 6.2. The PN expansions of the metric and aether fields are then determined in Sec. 6.3, and used to express the PN equations of motion for a binary system in Sec. 6.4. The rate of radiation damping is then determined in Sec. 6.5. Comments on dealing with center-of-mass velocity and sensitivity dependence are given in Sec. 6.6. Finally, an appendix repeats the definitions of various quantities used throughout the paper.

6.2 Effective action and field equations

6.2.1 Particle action

I wish to treat within ae-theory a system of compact bodies that potentially possess strong internal gravitational fields. I will deal with the complicated internal workings of the bodies via an effective approach, since I am only interested in the bulk motion of the bodies and in the fields far from them. I will thus assume that a given body can be treated to sufficient accuracy as a point particle with the composition dependent effects encapsulated in nonstandard couplings in the particle action. Such a method was pioneered by Eardley [53] within the Brans–Dicke scalar-tensor theory, and Will and Eardley [54] within Rosen's "Bi-metric" theory. More recently, Damour and Esposito–Farese [55] have applied it to a general class of scalar-tensor models, and Goldberger and Rothstein [56] have used it to determine higher PN order and spin dependent corrections in pure GR.

The action must be invariant under general diffeomorphisms and individual reparametrizations of the particle worldlines. The one-particle action S_A will thus have the form $S_A = -\tilde{m} \int dt \mathcal{O}$, where $\tilde{m}$ has dimensions of mass, and $\mathcal{O}$ is a sum of dimensionless local operators. The fundamental theory contains only one dimensionful parameter G; the particle theory will contain in addition the size d of the underlying finite-sized bodies. I neglect the spin of the bodies. I will only consider here operators that are not suppressed by powers of d; this excludes derivative couplings, for example[1]. I will assume that the action reduces to the standard free particle action if the particle is comoving with the local aether.

I thus arrive at the following one-particle action:

$$S_A = -\tilde{m}_A \int d\tau_A \big(1 + \sigma_A(u^a v_a + 1) + \frac{\sigma'_A}{2}(u^a v_a + 1)^2 + \cdots\big). \tag{6.1}$$

where A labels the body, τ_A is the proper time along the body's curve, v^a is the body's unit four-velocity, and u^a is the aether. The quantity $u^a v_a$ expressed in a PN expansion with the aether purely timelike at lowest order, will be of order v^2, the square of the velocity of the body in the aether frame, which is assumed to be first PN order (1PN). In what follows, I will only be interested in terms that follow from the part of the action that is $(m_A \times 2\text{PN})$, so I retain only the terms in S_A written above. For a system of N particles, the action is given by the sum of N copies of S_A.

This action can be thought of as a Taylor expansion of the standard worldline action, but with a mass that is a function of $\gamma \equiv -u^a v_a$:

$$S_A = -\int d\tau\, \tilde{m}_A[\gamma]. \tag{6.2}$$

The expansion is made about $\gamma = 1$. The parameters σ, σ' are then defined as

$$\sigma_A = -\frac{d \ln \tilde{m}_A}{d \ln \gamma}|_{\gamma=1} \quad \sigma'_A = \sigma_A + \sigma_A^2 + \bar{\sigma}_A \quad \bar{\sigma}_A = \frac{d^2 \ln \tilde{m}_A}{d(\ln \gamma)^2}|_{\gamma=1}. \tag{6.3}$$

[1] I thank I. Rothstein for clarifying this point.

This form of S_A suggests that that $\sigma_A, \bar{\sigma}_A$ can be determined by considering asymptotic properties of perturbations of static stellar solutions. Little work has been done on such perturbations; hence little can be said about the values of $\sigma_A, \bar{\sigma}_A$. One thing that is known and explained below is that they must scale as some combination of the c_n times $(G_N\tilde{m}/d)^2$.

6.2.2 Field equations

The full action is the four-parameter ae-theory action S

$$S = \frac{1}{16\pi G}\int d^4x\,\sqrt{|g|}\,\Big(R + K^{ab}{}_{cd}\nabla_a u^c \nabla_b u^d + \lambda(u^a u^b g_{ab} + 1)\Big), \tag{6.4}$$

where

$$K^{ab}{}_{cd} = \big(c_1 g^{ab} g_{cd} + c_2 \delta^a_c \delta^b_d + c_3 \delta^a_d \delta^b_c + c_4 u^a u^b g_{cd}\big), \tag{6.5}$$

plus the sum of N copies of S_A (6.1); only the terms written in (6.1) are retained in S_A. The field equations are then as follows. There are the Einstein equations

$$G_{ab} - S_{ab} = 8\pi G T_{ab}, \tag{6.6}$$

where

$$G_{ab} = R_{ab} - \frac{1}{2} R g_{ab}, \tag{6.7}$$

$$\begin{aligned} S_{ab} =& \nabla_c\big(K_{(a}{}^c u_{b)} - K^c{}_{(a} u_{b)} - K_{(ab)} u^c\big) \\ &+ c_1\big(\nabla_c u_a \nabla^c u_b - \nabla_a u_c \nabla_b u^c\big) + c_4 (u^c \nabla_c u_a)(u^d \nabla_d u_b) \\ &+ \lambda u_a u_b + \frac{1}{2} g_{ab} (K^c{}_d \nabla_c u^d), \end{aligned} \tag{6.8}$$

with

$$K^a{}_c = K^{ab}{}_{cd} \nabla_b u^d, \tag{6.9}$$

and T^{ab} is the particle stress tensor

$$T^{ab} = \sum_A \tilde{m}_A \tilde{\delta}_A \left[A_{1A} v_A^a v_A^b + 2A_{2A} u^{(a} v_A^{b)}\right], \tag{6.10}$$

with the covariant delta-function

$$\tilde{\delta}_A = \frac{\delta^3(\vec{x} - \vec{x}_A)}{v_A^0 \sqrt{|g|}}, \tag{6.11}$$

and

$$A_{1A} = 1 + \sigma_A - \frac{\sigma'_A}{2}((u_c v_A^c)^2 - 1) \tag{6.12}$$

$$A_{2A} = -\sigma_A - \sigma'_A(u_c v_A^c + 1). \tag{6.13}$$

The aether field equation is

$$\nabla_b K^{ba} = c_4(u^c \nabla_c u_b) \nabla^a u^b + \lambda u^a + 8\pi G \sigma^a, \tag{6.14}$$

where

$$\sigma^a = \sum_A \tilde{m}_A \tilde{\delta}_A \, A_{2A} v_A^a. \tag{6.15}$$

Varying λ gives the constraint $g_{ab} u^a u^b = -1$. Eqn. (6.14) can be used to eliminate λ, giving

$$\lambda = -u^a \big(\nabla_b K^b{}_a - c_4(\nabla_a u^b)(u^c \nabla_c u_b) - 8\pi G \sigma_a\big). \tag{6.16}$$

The covariant equation of motion for a single particle has the form

$$\nabla_b T_A^{ab} - \nabla_b((\sigma_A)^a u^b) - (\sigma_A)_b \nabla^a u^b = 0, \tag{6.17}$$

where T_A^{ab} and $(\sigma_A)^a$ are the one-particle summands in (6.10) and (6.15). This can be written more explicitly as

$$v_A^b \nabla_b (A_{1A} v_A^a + A_{2A} u^a) - A_{2A} v_{Ab} \nabla^a u^b = 0. \tag{6.18}$$

6.3 Post-Newtonian fields

The PN expansion of the fields can be determined by iteratively solving the field equations in a weak field, slow motion approximation, following the method of Chapter 4. I assume a background in which to lowest order, the metric is flat and the aether purely timelike. I find:

$$\begin{aligned} g_{00} =& -1 + 2\sum_A \frac{G_N \tilde{m}_A}{r_A} - 2\sum_{A,B} \frac{G_N^2 \tilde{m}_A \tilde{m}_B}{r_A r_B} - 2\sum_{A,B\neq A} \frac{G_N^2 \tilde{m}_A \tilde{m}_B}{r_A r_{AB}} \\ &+ 3\sum_A \frac{G_N \tilde{m}_A}{r_A} v_A^2 (1+\sigma_A) \\ g_{ij} =& \delta_{ij}(1 + 2\sum_A \frac{G_N \tilde{m}_A}{r_A}) \\ g_{0i} =& \sum_A \left(B_{1A} + B_{2A}\right) \frac{G_N \tilde{m}_A}{r_A} v_A^i + \sum_A \left(B_{1A} - B_{2A}\right) \frac{G_N \tilde{m}_A}{r_A^3} (v_A^j r_A^j) r_A^i \end{aligned} \tag{6.19}$$

where $r_A^i = x^i - x_A^i$, $r_{AB}^i = x_A^i - x_B^i$,

$$G_N = \frac{2}{2 + c_{14}} G \tag{6.20}$$

$$\alpha_1 = \frac{8(c_3^2 - c_1 c_4)}{2c_1 + c_+ c_-} \tag{6.21}$$

$$\alpha_2 = \frac{\alpha_1}{2} + \frac{(c_1 + 2c_3 + c_4)(2c_1 + 3c_2 + c_3 - c_4)}{(2 + c_{14}) c_{123}}, \tag{6.22}$$

and

$$B_{1A} = -\frac{1}{4}(8 + \alpha_1)(1 + \frac{c_-}{2c_1}\sigma_A) \tag{6.23}$$

$$B_{2A} = -\frac{3}{2} - \frac{1}{4}(\alpha_1 - 2\alpha_2)(1 - \frac{2 + c_{14}}{2c_+ - c_{14}}\sigma_A). \tag{6.24}$$

The values of α_1 and α_2 are constrained to be very small by weak-field experiments, via analysis that allows for a possible lack of Lorentz symmetry in the underlying theory [47]. There are two independent pairs of conditions on the c_n that will set them to zero. One choice is

$$c_2 = -\frac{2c_1^2 + c_1 c_3 - c_3^2}{3c_1}, \qquad c_4 = \frac{c_3^2}{c_1}. \tag{6.25}$$

The other is $c_{14} = c_+ = 0$. From the results of Sec. 6.5, this choice gives a theory with no propagating vector and scalar degrees of freedom in the linearized limit—the theory contains just the two spin-2 "graviton" degrees of freedom, and these travel at exactly the background speed-of-light. The weak field radiation damping rate is identical to that of GR, but there are still strong field corrections to the N-body equations of motion, as indicated below.

The aether to order of interest is

$$\begin{aligned} u^0 =& 1 + \sum_A \frac{G_N \tilde{m}_A}{r_A} \\ u^i =& \sum_A \left(B_{3A} + B_{4A}\right) \frac{G_N \tilde{m}_A}{r_A} (v_A)^i + \sum_A \left(B_{3A} - B_{4A}\right) \frac{G_N \tilde{m}_A}{r_A^3} (v_A^j r_A^j) r_A^i \end{aligned} \tag{6.26}$$

where

$$B_{3A} = (\frac{8+\alpha_1}{8}) \frac{1}{c_1} (c_- + (1 + c_-)\sigma_A) \tag{6.27}$$

$$B_{4A} = \frac{1}{2c_{123}} ((2c_1 + 3c_2 + c_3 - c_4) + (2 + c_{14})\sigma_A) \tag{6.28}$$

The expressions above are equivalent to the weak-field expressions obtained in Chapter 4 when σ_A is set to zero.

6.4 Post-Newtonian equations of motion

The equations of motion for the compact bodies follow by expressing the exact result (6.17) in a PN expansion using the forms of the fields given above. The Newtonian limit determines the relation between G and the effective two-body coupling

$\mathcal{G}$, and between $\tilde{m}$ and the "active" gravitational mass m. It is

$$\dot{v}^i_A = \sum_{B\neq A} \frac{-G_N \tilde{m}_B}{(1+\sigma_A) r^3_{AB}} r^i_{AB} \equiv \sum_{B\neq A} \frac{-\mathcal{G}_{AB} m_B}{r^3_{AB}} r^i_{AB}, \tag{6.29}$$

with the two-body coupling

$$\mathcal{G}_{AB} = \frac{G_N}{(1+\sigma_A)(1+\sigma_B)} \tag{6.30}$$

and the active gravitational mass

$$m_B = (1+\sigma_B)\tilde{m}_B. \tag{6.31}$$

One can continue on to determine the 1PN order terms, making use of the Newtonian result, then work backwards to determine the Einstein–Infeld–Hoffman [57] form of the Lagrangian—that is, the Lagrangian expressed purely in terms of particle quantities—that gives rise to those equations of motion. I will give the result here just for the case of a two body system. The Lagrangian is

$$\begin{aligned} L = &-(m_1+m_2) + \frac{1}{2}(m_1 v_1^2 + m_2 v_2^2) \\ &+ \frac{1}{8}((1 - \frac{\sigma_1'}{1+\sigma_1}) v_1^4 + (1 - \frac{\sigma_2'}{1+\sigma_2}) v_2^4) \\ &+ \frac{\mathcal{G} m_1 m_2}{r} \big[1 + \frac{3}{2}((1+\sigma_1) v_1^2 + (1+\sigma_2) v_2^2) \\ &- \frac{1}{2}(\frac{\mathcal{G} m_1}{r}(1+\sigma_2) + \frac{\mathcal{G} m_2}{r}(1+\sigma_1)) \\ &+ C_1(v_1^j v_2^j) + C_2(v_1^j \hat{r}^j v_2^k \hat{r}^k)\big], \end{aligned} \tag{6.32}$$

which leads to

$$
\begin{aligned}
\dot{v}_1^i =& \frac{\mathcal{G}m_2}{r^2}\hat{r}^i[-1 + 4\frac{\tilde{m}_2}{r} + (1 - \frac{2}{1+\sigma_2}C_1)\frac{\tilde{m}_1}{r} \\
& - \frac{1}{2}(2 + 3\sigma_1 + \frac{\sigma_1'}{1+\sigma_1})v_1^2 - (\frac{3}{2}(1+\sigma_2) + (C_2 - C_1))v_2^2 \\
& - 2C_1 v_1^j v_2^j + 3(C_2 - C_1)(v_2^j \hat{r}^j)^2] \\
& + \frac{\mathcal{G}m_2}{r^2}[v_1^i(v_1^j \hat{r}^j(4 + 3\sigma_1 - \frac{\sigma_1'}{1+\sigma_1}) - 3(1+\sigma_1)v_2^j \hat{r}^j) \\
& + v_2^i(2C_1 v_1^j \hat{r}^j - 2C_2 v_2^j \hat{r}^j)]
\end{aligned}
\tag{6.33}
$$

where $\mathcal{G} = \mathcal{G}_{12}$, $r^i = r_1^i - r_2^i$, and the coefficients C_1 and C_2 are

$$C_1 = B_{12} - \sigma_1 B_{32} = B_{11} - \sigma_2 B_{31} \tag{6.34}$$

$$C_2 = B_{22} - \sigma_1 B_{42} = B_{21} - \sigma_2 B_{41}. \tag{6.35}$$

The expression for $\dot{v}_2^i$ is obtained by exchanging all body-1 quantities and body-2 quantities (which includes the switch $r^i \to -r^i$).

The above Lagrangian is not Lorentz invariant unless $\sigma_A = \sigma_A' = 0$. This follows from the analysis of Will [17] and the list of criteria therein. In particular, the action and the equations of motion depend on the velocity of the system's center of mass in the aether frame.

6.5 Radiation damping rate

I now turn to the radiation damping rate—the rate at which the particle system loses energy via gravity-aether radiation. This can be determined by adapting the methods applied to the case of weakly gravitating perfect fluid bodies in Chapter 5. I will work solely with the "untilded" mass m_A (6.31). It is also convenient to introduce the parameter

$$s_A = \sigma_A/(1+\sigma_A). \tag{6.36}$$

To begin, I assume a Minkowskian coordinate system and basis with respect to which, at zeroth order, the metric is the Minkowski metric η_{ab} and the aether is purely timelike. I then decompose these variables into irreducible transverse and longitudinal pieces, as in Chapter 5. For convenience, I repeat the decomposition here. The spatial vectors u^i and h_{0i} are written as:

$$h_{0i} = \gamma_i + \gamma_{,i} \quad u^i = \nu^i + \nu_{,i}, \tag{6.37}$$

with $\gamma_{i,i} = \nu^i_{,i} = 0$. The spatial metric h_{ij} is decomposed into a transverse, trace-free tensor, a transverse vector, and two scalar quantities giving the transverse and longitudinal traces:

$$h_{ij} = \phi_{ij} + \frac{1}{2}P_{ij}[f] + 2\phi_{(i,j)} + \phi_{,ij}, \tag{6.38}$$

where

$$0 = \phi_{ij,j} = \phi_{jj} = \phi_{i,i}, \tag{6.39}$$

and

$$P_{ij}[f] = \delta_{ij} f_{,kk} - f_{,ij}; \tag{6.40}$$

hence, $P_{ij}[f]_{,j} = 0$, and $h_{ii} = (f+\phi)_{,ii}$. I further define

$$F = f_{,jj}. \tag{6.41}$$

The list of variables then consists of a transverse-traceless spin-2 tensor ϕ_{ij}, transverse spin-1 vectors γ_i, ν^i, ϕ_i, and spin-0 scalars $\gamma, \nu, F, \phi, h_{00}$, and u^0. I will impose the gauge conditions

$$0 = u^i_{,i} = h_{0i,i} = h_{i[j,k]i}, \tag{6.42}$$

or equivalently,

$$0 = \nu = \gamma = \phi_i. \tag{6.43}$$

The field equations can then be linearized and expressed in terms of the above variables, and sorted to obtain a set of wave equations with matter terms and nonlinear terms as sources. Having done this, it is relatively easy to note that the linearized portions of the field equations satisfy a conservation law, thus implying the existence of a conserved source τ^{ab} constructed from the matter sources and non-linear terms:

$$\tau^{ab} = T^{ab} - \sigma^a \delta^b_0 + \tilde{\tau}^{ab} \tag{6.44}$$

where T^{ab} and σ^a are as defined above and $\tilde{\tau}^{ab}$ is constructed from nonlinear terms—its precise form will not be needed. As defined, τ^{ab} satisfies the conservation law with respect to the right-index only: $\tau^{ab}{}_{,b} = 0$. The corresponding conserved total energy E and momentum P^i to lowest PN order are

$$E = \int d^3x\, \tau^{00} = \sum_A \tilde{m}_A = \sum_A (1 - s_A) m_A, \tag{6.45}$$

$$P^i = \int d^3x\, \tau^{i0} = \sum_A m_A v^i_A. \tag{6.46}$$

Conservation of P^i means that the system center of mass X^i defined via m_A

$$X^i = \frac{1}{m} \sum_A m_A x^i_A, \tag{6.47}$$

where $m = \sum_A m_a$, is unaccelerated to lowest order.

The field equations reduce to the following. For spin-2,

$$\frac{1}{w_2^2}\ddot{\phi}_{ij} - \phi_{ij,kk} = 16\pi G\tau_{ij}^{\mathrm{TT}}, \tag{6.48}$$

where TT signifies the transverse, trace-free components, and

$$w_2^2 = \frac{1}{1+c_+}. \tag{6.49}$$

For spin-1,

$$\frac{1}{w_1^2}(\ddot{\nu}^i + \ddot{\gamma}_i) = \frac{16\pi G}{2c_1 + c_+c_-}(c_+\tau_{i0} - (1+c_+)\sigma^i)^{\mathrm{T}} \tag{6.50}$$

$$(c_+\nu^i - \gamma_i)_{,kk} = 16\pi G\tau_{i0}^{\mathrm{T}}, \tag{6.51}$$

where T signifies the transverse components, and

$$w_1^2 = \frac{2c_1 + c_+c_-}{2(1+c_+)c_{14}}. \tag{6.52}$$

For the spin-0 variables, the constraint gives to linear order

$$u^0 = 1 + \frac{1}{2}h_{00}. \tag{6.53}$$

Non-linear terms are ultimately uninteresting, as explained in more detail in Chapter 5. For the rest,

$$\frac{1}{w_0^2}\ddot{F} - F_{,kk} = \frac{-16\pi G c_{14}}{2+c_{14}}(\tau_{kk} + \frac{2-3c_2-c_+}{c_{123}}\tau_{kk}^{\mathrm{L}} - \frac{2}{c_{14}}\tau_{00}) \tag{6.54}$$

$$(F + c_{14}h_{00})_{,kk} = -16\pi G\tau_{00} \tag{6.55}$$

$$(1-c_2)\dot{F}_{,i} - c_{123}\dot{\phi}_{,kki} = -16\pi G\tau_{i0}^{\mathrm{L}} \tag{6.56}$$

where L signifies the longitudinal component, and

$$w_0^2 = \frac{(2+c_{14})c_{123}}{(2-3c_2-c_+)(1+c_+)c_{14}}. \tag{6.57}$$

These equations can be solved formally via Greens function methods, and the resulting integrals expanded in a far-field, slow-motion approximation. The expressions can be further simplified using the conservation of τ^{ab}. A result that holds within the approximation scheme is that for a field ψ satisfying a wave equation with speed w evaluated at field point $x^i \equiv |x|\hat{n}^i$,

$$w\psi_{,i} = -\dot{\psi}\hat{n}^i. \tag{6.58}$$

Also, differentially transverse becomes equivalent to geometrically transverse to $\hat{n}^i$.

The results to lowest PN order and ignoring static contributions are as follows. For spin-2,

$$\phi_{ij}(t, x^i) = \frac{2G}{|x|}\ddot{Q}_{ij}^{\mathrm{TT}} \tag{6.59}$$

where I assume that the system is near the origin, the right-hand side is evaluated at time $(t - |x|/w_2)$, and the quadrupole moment Q_{ij} is the trace-free part of the system's second mass moment I_{ij}

$$I_{ij} = \sum_A m_A x_A^i x_A^j. \tag{6.60}$$

For spin-1 variables,

$$\nu^i(t, x^i) = \frac{-2G}{|x|}\frac{1}{2c_1 + c_+c_-}\left(\frac{\hat{n}^j}{w_1}\left(\frac{c_+}{1+c_+}\ddot{Q}_{ij} - \ddot{\mathcal{Q}}_{ij}\right) + 2\Sigma^i\right)^{\mathrm{T}} \tag{6.61}$$

$$\gamma_i = c_+\nu^i, \tag{6.62}$$

where the right-hand side of the first equation is evaluated at time $(t - |x|/w_1)$, $\mathcal{Q}_{ij}$ is the trace-free part of the rescaled mass moment $\mathcal{I}_{ij}$:

$$\mathcal{I}_{ij} = \sum_A s_A m_A x_A^i x_A^j, \tag{6.63}$$

and

$$\Sigma^i = -\sum_A s_A m_A v_A^i. \tag{6.64}$$

For spin-0 variables,

$$F(t, x^i) = \frac{-2G}{|x|}\frac{c_{14}}{2+c_{14}}(\hat{n}^i\hat{n}^j((Z-3)\ddot{Q} + \frac{2}{w_0^2 c_{14}}\dddot{\mathcal{Q}}) + Z\ddot{I} + \frac{2}{3w_0^2 c_{14}}\dddot{\mathcal{I}} - \frac{4}{w_0 c_{14}}\hat{n}^i\Sigma^i) \tag{6.65}$$

$$h_{00} = \frac{-1}{c_{14}}F \tag{6.66}$$

$$\dot{\phi}_{,i} = \frac{1-c_2}{c_{123}}\dot{f}_{,i}, \tag{6.67}$$

where the right-hand side of the first equation is evaluated at time $(t - |x|/w_0)$,

$$Z = \frac{2c_1 + 3c_2 + c_3 - c_4}{(2+c_{14})c_{123}} = \frac{2\alpha_2 - \alpha_1}{2(2c_+ - c_{14})} \tag{6.68}$$

and $I = I_{ii}$, $\mathcal{I} = \mathcal{I}_{ii}$.

At this point, I can explain the expected smallness of the sensitivities. I do this by taking the weak-field ($s_A \to$ "small") limit of the above wave forms and comparing them with the perfect fluid wave forms determined in Chapter 5. The only s_A-dependence that remains potentially leading-order is in Σ_i. Comparing with Σ_i in Chapter 5, Eqn. (5.82) indicates that in the small s_A limit,

$$s_A = -c_{14}(\alpha_1 - \frac{2}{3}\alpha_2)\frac{\Omega_A}{m_A} + \mathcal{O}(\frac{G_N m}{d})^2, \tag{6.69}$$

where Ω_A is the binding energy of the body—i.e. $\Omega/m \sim (G_N m/d)$, where d is the characteristic size of the body. The implication is that when $\alpha_1, \alpha_2 = 0$, s must scale as $(G_N m/d)^2$, times a c_n dependent coefficient. This coefficient should scale in the small c_n limit as c_n^2 to avoid divergences in the wave forms; see for example eqn. (6.61).

Next, the wave forms are inserted into an expression for the rate of change of energy. This expression can be derived via the Noether charge method as in Chap. 5, with the result

$$\dot{\mathcal{E}} = \frac{-1}{16\pi G}\int d\Omega R^2 \Big(\frac{1}{2w_2}\dot{\phi}_{ij}\dot{\phi}_{ij} - \frac{(2c_1 + c_+c_-)(1+c_+)}{w_1}\dot{\nu}^i\dot{\nu}^i - \frac{2+c_{14}}{4w_0c_{14}}\dot{F}\dot{F}\Big) + \dot{O} \tag{6.70}$$

where $\dot{O}$ is a total time-derivative, which will be argued away in a moment.

Using the above results for the wave forms, performing the angular integral, and ignoring $\dot{O}$ gives

$$\dot{\mathcal{E}} = -G_N\Big(\frac{\mathcal{A}_1}{5}\dddot{Q}_{ij}\dddot{Q}_{ij} + \frac{\mathcal{A}_2}{5}\dddot{Q}_{ij}\ddot{\mathcal{Q}}_{ij} + \frac{\mathcal{A}_3}{5}\dddot{\mathcal{Q}}_{ij}\ddot{\mathcal{Q}}_{ij} + \mathcal{B}_1\dddot{I}\dddot{I} + \mathcal{B}_2\dddot{I}\dddot{\mathcal{I}} + \mathcal{B}_3\dddot{\mathcal{I}}\dddot{\mathcal{I}} + \mathcal{C}\dot{\Sigma}^i\dot{\Sigma}^i\Big) \tag{6.71}$$

where

$$\mathcal{A}_1 = \Big(1 + \frac{c_{14}}{2}\Big)\Big(\frac{1}{w_2} - \frac{2c_{14}c_+^2}{(2c_1 + c_+c_-)^2}\frac{1}{w_1} - \frac{(3-Z)^2c_{14}}{6(2+c_{14})}\frac{1}{w_0}\Big) \tag{6.72}$$

$$\mathcal{A}_2 = \Big(\frac{(2+c_{14})c_+}{2c_1 + c_+c_-}\frac{1}{w_1^3} + \frac{(3-Z)}{3}\frac{1}{w_0^3}\Big) \tag{6.73}$$

$$\mathcal{A}_3 = -\frac{1}{c_{14}}\Big(\frac{2+c_{14}}{4}\frac{1}{w_1^5} - \frac{1}{3}\frac{1}{w_0^5}\Big) \tag{6.74}$$

$$\mathcal{B}_1 = \frac{-Z^2c_{14}}{72}\frac{1}{w_0} \tag{6.75}$$

$$\mathcal{B}_2 = \frac{-Z}{6}\frac{1}{w_0^3} \tag{6.76}$$

$$\mathcal{B}_3 = \frac{-1}{6c_{14}}\frac{1}{w_0^5} \tag{6.77}$$

$$\mathcal{C} = -\frac{2}{3c_{14}}\Big(\frac{2+c_{14}}{w_1^3} + \frac{1}{w_0^3}\Big), \tag{6.78}$$

and Z is given in (6.68). The coefficient $\mathcal{A}_1$ is identical to $\mathcal{A}$ of Chap. 5, $\mathcal{B}_1$ identical to $\mathcal{B}$, and $\mathcal{C}$ identical to $\mathcal{C}$.

It is now crucial to note that the damping rate is calculated to lowest non-vanishing PN order by treating the system as exactly Newtonian. The motion of the system can then be decomposed into a uniform center-of-mass motion—recall the conservation of P^i—and a fixed Keplerian orbit in the center-of-mass frame. Since the motion is steady-state, the damping rate must have no secular time dependence. This observation implies that secular terms arising from $\dddot{\mathcal{I}}_{ij}$, see below, must cancel with secular terms in $\dot{O}$. I will presume only the time average of the damping rate over an orbital period need be calculated. I can then dispose of $\dot{O}$, as remaining, non-secular terms average to zero.

Thus restricting attention to a binary system, and taking a time average over an orbital period, the expression reduces as follows. First, I define the quantities

$$m = m_1 + m_2, \quad \mu_A = m_A/m, \quad \mu = m_1 m_2/m, \tag{6.79}$$

and the vectors

$$r^i = x_1^i - x_2^i, \quad v^i = \dot{r}^i, \tag{6.80}$$

$$X^i = \mu_1 x_1^i + \mu_2 x_2^i, \quad V^i = \dot{X}^i. \tag{6.81}$$

To Newtonian order, $\dot{v}^i = -(\mathcal{G}m/r^2)\hat{r}^i$, and $\dot{V}^i = 0$. I_{ij} can be diagonlized:

$$I_{ij} = \mu r^i r^j + m X^i X^j, \tag{6.82}$$

hence

$$\dddot{I}_{ij} = \frac{2\mathcal{G}\mu m}{r^2}(3\hat{r}^i\hat{r}^j\dot{r} - 4v^{(i}\hat{r}^{j)}). \tag{6.83}$$

As for $\mathcal{I}_{ij}$,

$$\mathcal{I}_{ij} = \mu(s_1\mu_2 + s_2\mu_1)r^i r^j + m(s_1\mu_1 + s_2\mu_2)X^i X^j + 2\mu(s_1 - s_2)r^{(i}X^{j)}, \tag{6.84}$$

and

$$\dddot{\mathcal{I}}_{ij} = S\dddot{I}_{ij} - 6V^{(i}\dot{\Sigma}^{j)} + 2\mu(s_1 - s_2)\dddot{r}^{(i}X^{j)}, \tag{6.85}$$

where

$$\mathcal{S} = s_1\mu_2 + s_2\mu_1, \tag{6.86}$$

and

$$\dot{\Sigma}_i = (s_1 - s_2)\frac{\mathcal{G}\mu m}{r^3}r^i. \tag{6.87}$$

Terms in $\dddot{\mathcal{I}}_{ij}$ with $X^i \equiv (X_0^i + V^i t)$ are secular; following the discussion above, they can be discarded.

Substituting into Eqn. (6.71) and imposing the time average gives the final expression

$$\begin{aligned}\dot{\mathcal{E}} = -G_N\Big\langle\big(\frac{\mathcal{G}\mu m}{r^2}\big)^2 \\ &\times \Big[\frac{8}{15}(\mathcal{A}_1 + \mathcal{S}\mathcal{A}_2 + \mathcal{S}^2\mathcal{A}_3)(12v^2 - 11\dot{r}^2) \\ &+ 4(\mathcal{B}_1 + \mathcal{S}\mathcal{B}_2 + \mathcal{S}^2\mathcal{B}_3)\dot{r}^2 \\ &+ (s_1 - s_2)^2\Big(\mathcal{C} + \frac{6}{5}(3\mathcal{A}_3V^2 + (\mathcal{A}_3 + 30\mathcal{B}_3)(V^i\hat{r}^i)^2)\Big) \\ &+ (s_1 - s_2)\Big(\frac{8}{5}(\mathcal{A}_2 + 2\mathcal{S}\mathcal{A}_3)(3v^iV^i - 2V^i\hat{r}^iv^j\hat{r}^j) + 12(\mathcal{B}_2 + 2\mathcal{S}\mathcal{B}_3)V^i\hat{r}^iv^j\hat{r}^j\Big)\Big]\Big\rangle.\end{aligned} \tag{6.88}$$

Taking the weak field limit corresponds to retaining only the three terms with coefficients $\mathcal{A}_1$, $\mathcal{B}_1$, and $\mathcal{C}$, and invoking the relation (6.69) for s_A in the $\mathcal{C}$ term. In the case that $\alpha_1 = \alpha_2 = 0$, Z and thus $\mathcal{B}_1$, and s_A vanish. The damping rate then contains only a quadrupole contribution and is identical to the GR rate when $\mathcal{A}_1 = 1$. This remaining curve of c_n values lies entirely within the range of values allowed by collected constraints considered in Chapter 4. Thus, if the weak-field results were exact, there would exist a one parameter family of viable ae-theories.

6.6 Discussion

6.6.1 Velocity and sensitivity dependence

The above formulas can be used to obtain constraints on the values of c_n by comparing with observations of binary pulsar systems. Stating a precise constraint requires additional work beyond this dissertation, though. Specifically, what is needed are methods of dealing with the center-of-mass velocity dependence and of calculating the sensitivities s_A and σ'_A for a given body.

Dependence on the center-of-mass velocity V should actually be beneficial, since constraints on the theory can arise from a failure to observe V dependent effects, such as a precession of the orbital plane of a binary system. Furthermore, it may be possible to formulate such constraints without having to define the physical frame, as in the manner of bounds on the PPN parameter α_2. The presence of alignment between the sun's spin axis and the ecliptic plane signals the absence of frame dependent effects, and leads to a strong bound of $|\alpha_2| < 4 \times 10^{-7}$ [47]. This argument does require the assumption that the component of the preferred frame in the sun's rest frame is not conveniently aligned with the sun's spin axis; such an assumption may generally be required for similar arguments. For example, the binary's orbital plane will not precess if V^i happens to be normal to it.

An assumption on the order of magnitude of V is necessary to justify the use of just the leading PN order expressions for the PK parameters when applied to observed binary systems. The validity of the expressions depends on whether corrections of relative order v^2 and (V^4/v^2) are smaller than observational error. Terms of order v^2 are negligible for all observed systems, for now, although the "double pulsar" is pushing this limit. For all but the double pulsar, $v^2 \sim 10^{-6}$, and errors are at least 1000 times this [50]. The double pulsar [58] PSR J0737-3039A/B is the so-far unique binary containing two pulsars. The orbital velocity is high, $v^2 \sim 10^{-5}$, and the presence of two pulsars happens to make measurement of system parameters much easier and thus more precise—the smallest relative error is 10^{-4} on the rate of periastron advance. The v^2 corrections are therefore small enough for now, but it is expected that precision will increase to probe the next PN order within the next 10-20 years [50].

The V dependent terms must feature c_n dependent factors, since it is known that there is no V dependence at next PN order in pure GR [47]. Ignoring those factors for the moment, validity of leading PN order for the double pulsar requires that $(V^4/v^2) \lesssim 10^{-4}$, giving $V^2 \lesssim 10^{-4.5}$, or $(V^2/v^2) \lesssim 10^{0.5} \approx 3$. For other systems, given errors ranging from $(10^{-1} \sim 10^{-3})$, the conditions are $(V^4/v^2) \lesssim (10^{-1} \sim 10^{-3})$, giving $V^2 \lesssim (10^{-3.5} \sim 10^{-2.5})$, or $(V^2/v^2) \lesssim (10^{2.5} \sim 10^{1.5}) \approx (300 \sim 30)$. Presumably, the c_n dependent factor actually goes to zero as some positive power of c_n, so V can be larger in the small c_n limit. A reasonable first guess for the aether frame is the rest frame of the cosmic microwave background. A typical velocity for compact objects in our galaxy, in this frame, is $V^2 \sim 10^{-6}$, so the required assumption on V is reasonable.

As for the sensitivities s and σ', a formula for the sensitivities for a general source should be obtainable by comparing the strong field results of this chapter with the exact perfect fluid theory of Chapter 5. Higher order terms in the exact theory must be calculated, though, since the leading order results only give the $O(G_N m/d)$ part of s expressed in (6.69). The calculation can be done in the quasi-static case, carrying on the iterative procedure used to determine the PPN parameters in Chapter 4. The process may be lengthy, but straightforward.

6.6.2 Constraints

For now, two preliminary comments can be made. I have shown that the sensitivity s will scale like $f[c_n](G_N m/d)^2$, where f is some c_n-dependent coefficient that scales like c_n^2 in the small c_n limit. Then, a constraint can be roughly stated: $f \lesssim (1 \sim 0.1)$. And, the theory will pass all current constraints if the c_n are less than roughly (0.1) and the three weak field conditions are imposed. For in that case, the strong field corrections will be smaller than current observational error. These statements can be derived as follows.

First, that $f \lesssim (.1 \sim .3)$. This condition follows from constraints [50] on the magnitude of violations of the strong equivalence principle (SEP)—that is, that a body's acceleration is independent of its composition. A violation would lead to a polarization of the orbit of pulsar systems due to unequal acceleration of the binary bodies in the gravitational field of the galaxy. The observed lack of polarization in neutron star-white dwarf systems leads to a constraint that can be stated here as $s < (0.01)$, where here s is the sensitivity of the neutron star in the considered pulsars. Assuming that $(G_N m/d) \approx (.1 \sim .3)$ for the pulsar, as it is in GR, the constraint on the size of f arises. It is possible that when the three weak field conditions are imposed, f will automatically satisfy the above inequality; certainly it will in the small c_n regime when $|c_n| < (1 \sim 0.3)$.

Now for the statement that current tests will be satisfied if the three weak field conditions are imposed and the remaining degree of c_n freedom satisfies $|c_n| \lesssim (0.1)$. First consider tests that probe only the quasi-static PK parameters, i.e. all but the damping rate. The size of the strong field corrections relative to the PN GR corrections is simply s_A. The tightest quasi-static test comes from the double pulsar [58]. The prediction of GR has been confirmed to within a relative observational error of 0.05%. Then, roughly enforcing $s \lesssim (0.1)$ and assuming that $(G_N m/d) \approx (.1 \sim .3)$ for the pulsars, the condition $|c_n| \lesssim (0.1)$ arises. Given this and the two conditions that set the PPN parameters α_1 and α_2 to zero, all current quasi-static tests will be passed.

Tests that incorporate the damping rate will also be satisfied by the above condition and the weak field conditions. I note first that for systems in which the damping rate is probed, error on its measurement dominates errors on quasi-static parameters [47, 50]. Thus, it is conventional to use the measurements of the quasi-static parameters to solve for the mass values of the binary bodies. When $\alpha_{1,2} = 0$, and $|c_n| \lesssim (0.1)$, so that the expressions for the quasi-static parameters are close to those of GR, the predicted mass values will also be close.

In general, the dominant contribution to the damping rate comes from the dipole term

$$\dot{\mathcal{E}}_{Dipole} = -G_N \langle (\frac{\mathcal{G}\mu m}{r^2})^2 \rangle \mathcal{C}(s_1 - s_2)^2 \qquad (6.89)$$

which is order $(\mathcal{C}s^2/10v^2)$ compared to the quadrupole and monopole contribution, if the difference in the binary bodies' sensitivities is not too small. Constraints have been derived [50] on the magnitude of dipole radiation from neutron star–white dwarf binaries PSR B0655+64 and PSR J1012+5307 by requiring that the dipole radiation rate be of the order of the observed rate. The analysis carries over here, and roughly leads to the condition $1/\mathcal{C} \gtrsim (s^2/10^{-4})$, where s is the sensitivity of the neutron star, and $1/\mathcal{C} \approx |c_{14}|$. In the small c_n regime, this translates again to the condition $|c_n| \lesssim (0.1)$.

For double neutron star binaries, the dipole rate is further suppressed by the similarities of the sensitivities, and the quadrupole and monopole contributions become dominant. The tightest test involving radiation is associated with the Hulse–Taylor binary PSR1913+16, with a relative error of (0.2)% [47, 50]. In the small c_n regime, the condition $A_1 = 1$ matches the leading order damping rate to that of GR. The strong field corrections are of relative order $s \sim c_n^2(G_N m/d)^2$. To be smaller than the error requires $c_n \lesssim (0.1)$.

This upper limit will decrease as observational errors decrease. The most promising candidate for lowering the limit is the double pulsar [58]—2PN-order and spin-dependent effects should be observable within the next ten or twenty years. Another type of system, yet undetected, for which high levels of accuracy could be obtained is a neutron star–black hole binary, as the structureless black hole would decrease noise due to finite-size effects and mass transfer between the bodies. I wish to emphasize, though, that the large c_n values are not yet excluded—rather, there is no conclusion on them as yet.

6.A Definitions

This appendix collects the definitions of various quantities used throughout Chapter 6.

Metric and aether variables:

$$h_{ab} = g_{ab} - \eta_{ab} \qquad w^0 = u^0 - 1 \qquad w^i = u^i \tag{6.90}$$

$$h_{0i} = \gamma_i + \gamma_{,i} \quad w_i = \nu_i + \nu_{,i} \tag{6.91}$$

$$h_{ij} = \phi_{ij} + \frac{1}{2}P_{ij}[f] + 2\phi_{(i,j)} + \phi_{,ij} \tag{6.92}$$

$$P_{ij}[f] = \delta_{ij}\triangle f - f_{,ij} \tag{6.93}$$

$$0 = \gamma_{i,i} = \nu_{i,i} = \phi_{ij,j} = \phi_{jj} = \phi_{i,i} \tag{6.94}$$

Wave speeds:

$$(w_2)^2 = \frac{1}{1+c_+} \tag{6.95}$$

$$(w_1)^2 = \frac{2c_1 + c_+c_-}{2(1+c_+)c_{14}} \tag{6.96}$$

$$(w_0)^2 = \frac{(2+c_{14})c_{123}}{(2-3c_2-c_+)(1+c_+)c_{14}} \tag{6.97}$$

Coefficients:

$$G_N = \frac{2G}{2 + c_{14}} \tag{6.98}$$

$$\alpha_1 = \frac{8(c_3^2 - c_1 c_4)}{2c_1 + c_+ c_-} \tag{6.99}$$

$$\alpha_2 = \frac{\alpha_1}{2} + \frac{(c_1 + 2c_3 + c_4)(2c_1 + 3c_2 + c_3 - c_4)}{(2 + c_{14})c_{123}} \tag{6.100}$$

$$s_A = \frac{\sigma_A}{(1 + \sigma_A)} \tag{6.101}$$

$$m_B = (1 + \sigma_B)\tilde{m}_B = \frac{\tilde{m}_B}{(1 - s_B)} \tag{6.102}$$

$$\mathcal{G}_{AB} = \frac{G_N}{(1 + \sigma_A)(1 + \sigma_B)} = G_N(1 - s_A)(1 - s_B) \tag{6.103}$$

$$\mathcal{S} = s_1 \mu_2 + s_2 \mu_1 \tag{6.104}$$

$$A_{1A} = 1 + \sigma_A - \frac{\sigma'_A}{2}((u_c v_A^c)^2 - 1) \tag{6.105}$$

$$A_{2A} = -\sigma_A - \sigma'_A(u_c v_A^c + 1) \tag{6.106}$$

$$B_{1A} = -\frac{1}{4}(8 + \alpha_1)(1 + \frac{c_-}{2c_1}\sigma_A) \tag{6.107}$$

$$B_{2A} = -\frac{3}{2} - \frac{1}{4}(\alpha_1 - 2\alpha_2)(1 - \frac{2 + c_{14}}{2c_+ - c_{14}}\sigma_A) \tag{6.108}$$

$$B_{3A} = (\frac{8 + \alpha_1}{8})\frac{1}{c_1}(c_- + (1 + c_-)\sigma_A) \tag{6.109}$$

$$B_{4A} = \frac{1}{2c_{123}}((2c_1 + 3c_2 + c_3 - c_4) + (2 + c_{14})\sigma_A) \tag{6.110}$$

$$C_1 = B_{12} - \sigma_1 B_{32} = B_{11} - \sigma_2 B_{31} \tag{6.111}$$

$$C_2 = B_{22} - \sigma_1 B_{42} = B_{21} - \sigma_2 B_{41} \tag{6.112}$$

$$Z = \frac{2c_1 + 3c_2 + c_3 - c_4}{(2 + c_{14})c_{123}} = \frac{2\alpha_2 - \alpha_1}{2(2c_+ - c_{14})} \tag{6.113}$$

Chapter 7

Conclusion

7.1 Summary of results

I have shown that a one-parameter family of ae-theories whose coupling constants c_n satisfy a mild bound will satisfy all collected constraints. Given present observational errors in the measurement of binary pulsar systems, the mild bound is $|c_n| \lesssim (0.1)$. For the class of theories that does not satisfy this bound, the results are inconclusive. Constraints on this class will follow from the above results once more work has been done to determine the values of the sensitivities for a given matter source.

To summarize the results of the individual chapters:

In Chapter 2, I demonstrated the effect on the c_n of rescaling the aether and the metric along the aether direction. I showed how one can use this redefinition to set one c_n to zero. Doing this can simplify study of solutions and has been put to use by Eling and Jacobson [26, 27] in their work on stars and black holes in ae-theory.

In Chapter 3, I derived expressions for the total energy, momentum, and angular momentum of an ae-theory spacetime. Because the aether does not vanish at infinity, the canonical expressions of GR receive aether dependent corrections. This work permits future study of conditions under which positivity of total energy holds.

Also in Chapter 3, I used the Noether charges to write down the first law of ae-theory black hole mechanics. I encountered difficulties in giving the law a thermodynamic interpretation. In particular, the algorithm of Wald and Iyer [37] for defining the entropy of the horizon in a diffeomorphism invariant field theory fails here due to the singular behavior of the aether on the horizon's bifurcation surface. Perhaps related to this difficulty is the apparent incompatibility of Lorentz violation and the second law of black hole thermodynamics, examined in [32].

In Chapter 4, I examined a variety of observational constraints on ae-theory and showed that the combined constraints are satisfied by a two-parameter family of theories. To accommodate constraints that probe the post-Newtonian limit of the theory, I calculated the "parametrized post-Newtonian" parameters. I showed that all but two of the ten PPN parameters, α_1 and α_2, differ from the GR values, and that these two can be set to the GR values by imposing two conditions on the c_n. I also considered constraints that follow from the nature of linearized wave modes and from ae-theory cosmology and showed how they restrict this family, but still permit an infinite region of c_n space.

In Chapter 5, I considered the motion of binary pulsar systems in ae-theory in the limit in which effects due to the strong internal fields of the compact bodies can be neglected. Treating the bodies as perfect fluid spheres, I calculated the rate at which a system of compact bodies loses energy to gravity-aether radiation. The effective N-body equations of motion follow from the PPN results in this limit. It follows that observational constraints from binary pulsars could be satisfied by matching the PPN parameters and the damping rate to those of GR, *if* strong field effects can be neglected. There would be just three simply stated conditions on the four c_n, leaving a one-parameter family of ae-theories that also happens to satisfy the additional constraints considered in Chapter 4.

In Chapter 6, I included strong field effects by treating the compact bodies as point particles with nonstandard, velocity dependent interactions parametrized by dimensionless "sensitivities". I determined the effective post-Newtonian equations of motion for the bodies and the radiation damping rate. More work is needed to calculate the numerical value of the sensitivities for a given stellar model, so as to be able to state precise constraints on the c_n. However, taking the weak field limit and comparing with the perfect fluid calculation of Chapter 5 reveals how the sensitivities scale with a body's mass and size, and regularity of the field equations implies the scaling with the c_n. I was then able to estimate that if the c_n are less than roughly (0.1) and the three weak field conditions are imposed, then the strong field effects will be negligible given current observational errors in the measurement of pulsar systems. This remaining one-parameter family of ae-theories passes all current observational constraints.

7.2 Future directions

The primary goal for future research is to find a way to rule out this theory by overconstraining the c_n. One important task left to do is to calculate the sensitivities for a given stellar source. This should be doable by comparison of the strong field results of Chapter 6 and higher order calculations in the perfect fluid theory of

Chapter 5. It would be helpful, though, to think of a more efficient method than just grinding out the next order terms. Whatever method used, it would permit determining how observations constrain the large c_n regime. As errors on observational constraints improve, the size of c_n such that the strong field terms are negligible will decrease—“large” c_n will signify a larger portion of the parameter space—so the need to determine the sensitivities will grow. Very precise constraints will also require accounting for effects due to the spin and tidal deformations (“finite-size” effects) of the binary bodies.

It would be useful to consider other methods of constraining the theory, especially since the binary pulsar constraints will always be beatable for small enough c_n. Given that the work here covers the range of standard tests of gravitational theories, finding new constraints indeed means heading into new territory. One method that should be viable in the future is the use of gravitational wave detectors: constraints could come from failure to observe spin-1 or spin-0 wave modes, from measurement of the speed of gravity waves, or from absence of anomalous effects on the phase of detected waves [1]. Constraints could also come by restricting parameter values that violate positivity of total energy using the definition of energy found in Chapter 3; this method requires formulating a positive-energy theorem for ae-theory. Perhaps it would be useful to consider further cosmological effects of the aether—polarization of the cosmic microwave background, for example. And perhaps constraints can be generated from arguments [31, 32] that the second law of black hole thermodynamics is violated in ae-theory.

[1] I thank A. Buonanno for pointing out this third possibility.

The theory suggests many questions. Why are the combinations of c_n that beat the constraints so special? Is the aether good-for-something, such as a source of dark matter (when suitably sourced by non-aether matter [25]) or dark energy, or a solution to the problem of time in quantum gravity [59]? And finally, where might the aether come from—what kind of quantum gravity theory would be so clever as to have ae-theory as a classical limit, with just the right combination of parameters? If anyone can tell me, I would like to know.

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