ABSTRACT

Title of Document:	ADVANCES IN MATHEMATICAL MODELS IN MARKETING.
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This dissertation comprises a series of three essays that relate advances made to both theoretical and empirical issues in marketing.

The first essay discusses the issue of endogeneity of market share and price in logit models and provides a theoretical procedure to solve this problem. The inseparability of demand and price make the possibility of drawing definite conclusions about either almost impossible. We employ a recently rediscovered mathematical function called the 'LambertW' to solve this problem of endogeneity and in turn yield logit models more conducive to theoretical study. We also employ this methodology to the problem studied by Basuroy and Nguyen (1998).

The second essay deals with the issue of pricing implicit bundling. Implicit bundles are products that are sold separately but provide an enhanced level of satisfaction if purchased together. We develop a model that would account for the possible relationships of the products across the different product lines. We show that

accounting for these relationships would decrease the amount of price competition in the market and also allow the Firm to enjoy higher profits. We also account for the endogeneity of price and market share when deriving the optimal solutions. We show that optimal prices first increase as the relationship between the firm's two products become stronger and then decrease as the two products become more exclusive to each other. Finally, we also find that a firm's prices increase as the competitor's contingent valuations increase.

The third essay helps improve the efficacy of CRM interventions by analyzing the latent psychological loyalty states of the customer. We use state space models to predict these latent loyalty states using observed data. We then use the predicted values of loyalty to derive the probability of repurchase of the customer. We also identify the types of CRM interventions that play a role in improving the loyalty of the customer to the firm and those interventions that have no effect. We compare our model's predictions to those derived from two other estimation methods. We find that our predictions are better than those computed from the other methods discussed.

ADVANCES IN MATHEMATICAL MODELS IN MARKETING

By

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Overview

My dissertation consists of three essays. All three essays detail and develop new methodologies that can be used by researchers to better understand the market.

In ‘Essay 1’ and ‘Essay 2’ I develop a new methodology to solve the issue of endogeneity of price and market share in logit models. Endogeneity in logit models prevent meaningful interpretation of marketing dynamics when data for the particular phenomenon or situation is absent. Hence in Essay 1 we solve this problem by developing a procedure that explicitly separates the price and demand variables by utilizing a special class of functions called the LambertW. The LambertW function helps solve several complex equations that involve either exponential or logarithmic functions. Using this procedure we analytically solve the endogeneity problem between price and demand in logit models. We do so without the need for instrumental variables that would otherwise have been used. We also show the ease with which these functions can be adapted to different simulation and estimation procedures. Finally this procedure is validated by employing it to explicitly derive the results obtained by the Basuroy and Ngyuen (1998).

Essay 2 adds to the dissertation by applying the procedure developed in Essay 1 to study a new problem. We study the market for implicit bundles. Implicit bundles refer to the group of products that are sold separately (also referred to as pure components) but could be perceived by the consumer as products that would provide an enhanced level of satisfaction if purchased together. In many cases, the implicit bundles are sold by the same firm, in the form of products sold across different product lines. The aim of this study is to address the issue of pricing these implicit bundles. We do so by developing a model that would account for the possible relationships, also known as contingent valuations, of the products across the different product lines. The degree of contingency determines the strength of the relationship between the two products. If the contingency level is zero then the products are independent. As the level increases the relationship between the products increases, thereby increasing the exclusivity between the products. We show that accounting for these relationships would decrease the amount of price competition in the market and also allow the firm to enjoy higher profits, because it would be able to charge the consumer for the additional surplus gained by purchasing both products from the same Firm. We also account for the endogeneity of price and market share when deriving the optimal solutions.

The analysis carried out in this essay helps us establish the following results:

1. The prices of the products of a firm (X) increase with increase in the contingent valuation of the products manufactured by the same firm (X). However if the contingent valuation is increased beyond a particular value, then the prices of firm (X) decrease with increase in the relationship. Hence the relationship between prices and the contingent valuation is initially monotonically increasing and then monotonically decreasing.

2. The prices of the products of the firm (X) increase with increase in the contingent valuation of the products manufactured by a competing firm (Y).

3. Firm X experiences a higher profit for the same market share, when its products are priced considering the contingent valuations than when they are priced independently of each other.

Consumers who purchase the products would have a lower level of consumer surplus if the contingent valuation is recognized by a firm.

CRM (Customer Relationship Management) interventions like direct mailings have long been used by firms to improve customer relationships. In the third essay we develop a method that will allow the firm to understand the effect of these interventions for customer loyalty. Loyalty is assumed to be unobserved and hence is modeled as a latent variable. We use a generalization of the 'Hidden Markov Model' (HMM) called the 'State Space Model' (SSM) to better predict a customer's loyalty function towards a particular firm or product. The SSM models are structurally different from HMM models and they offer three main advantages over HMM models. First they are continuous and are described across all possible relationship states of the customer, hence we avoid the problem of explicitly choosing the number of states; second, they can be used to model an infinite number of relationship states; and third they are better at modeling recursive behavior, which is necessary when modeling customer behavior that involves the effect of experience. We also predict the customer's probability of purchase given certain marketing actions and the predicted loyalty state using a hazard model. We combine the hazard model and the SSM to predict the customer's probability of purchase at a given loyalty state. We apply this model to data from a retailer of health and beauty aids, in order to help them better understand the effect of their CRM interventions on the customer's loyalty towards the firm and their repurchase intentions. We also point out the types of CRM interventions that play a role in improving the loyalty of the customer to the firm and those interventions that have no effect. This information can hence help the firm better organize its menu of CRM interventions. We can also compute the probability distributions across the loyalty states for each individual customer, thus providing the researcher with knowledge of each customer's loyalty state. Finally we introduce a new methodology to the literature on modeling relationships in marketing. The methodology improves upon existing methods by allowing for a more flexible and efficient estimation procedure. We also compare our model's predictions to those derived from two other estimation methods. We find that the predictions derived from our estimation procedure are better than those computed from the other methods discussed.

Chapter 1: Essay 1 - The LambertW Transformation As An Approach To Solving Share Equations In Logit Models

Summary

Logit models allow the expression of individual demand and supply equations. However, closed-form solutions for equilibrium shares and prices are highly nonlinear and cannot readily be derived. This hinders the employment of logit models in theoretical studies, and also makes it difficult to develop reduced-form expressions for share and price as a function of exogenous variables for use in empirical studies. In this paper we propose that a recently rediscovered mathematical function called the 'LambertW' be employed in solving logit models for equilibrium shares and prices. We demonstrate this methodology on the problem studied by Basuroy and Nguyen (1998).

1. Introduction

Discrete choice models have been extensively used in both the marketing and economics literature to study various aspects of consumer behavior using data on market share, price and other variables that affect demand. Logit models are widely used in the empirical literature (Abramson, Andrews, Currim and Jones 2000; Kamakura and Russell 1989; Guadagni and Little 1983; McFadden 1978). However, these models have seldom have been used to derive theoretical relationships between variables of interest. Some exceptions include Basuroy and Nguyen (1998), Carpenter and Lehmann (1985), Lillien and Kotler (1983) and Lillien and Ruzdic (1982). The complexity of the resultant expressions is often attributed to be the cause of this non-usage (Gruca, Kumar and Sudharshan 1992; Gruca and Sudharsdhan 1991; Karnani 1985).

In this paper, we propose that a recently rediscovered mathematical function (first studied by Euler 1779), termed 'LambertW', be employed in obtaining equilibrium solutions for share and price in logit models. Traditionally, the LambertW function has been used to solve several exponential equations (Corless, et al. 1996). The rest of the paper is organized as follows— in §2, we provide a brief overview of the 'LambertW' function. We develop the standard logit model in §3, and in §4 we present the solution and simplify the model for estimation purposes. We provide a theoretical application of this methodology in §5. We provide our final conclusions in §6.

2. A Brief Note on the 'LambertW' Function

LambertW is the inverse function associated with the equation,

$$We^{W} = x \qquad (1).$$

The LambertW function belongs to the family of exponential and logarithmic functions. The function given in (1) resembles the exponential function and the inverse of this function resembles the logarithmic function. Hence, the shape of the LambertW function closely follows that of shape of the exponential function and the logarithmic function. The LambertW function differs from the exponential to the left of point $x = 0$. The exponential is always positive, however the LambertW dips to a minimum of -1 at $x = -\frac{1}{e}$. Similarly, the LambertW function differs from the logarithmic function for values of $x \leq 0$ as while the logarithmic function is not defined for these values of x, the LambertW function continues to have a value till $x = -\frac{1}{e}$. A special case of the LambertW is the case when x lies in the range between $-\frac{1}{e}$ and 0. In this case, $W(x)$ has not just a single valued function but has *two* values. In the case of LambertW, a single valued function $W_0(x)$ is defined for values of $x = -\frac{1}{e}$ and $W(x) \geq -1$. $W_0(x)$ is also referred to as the principal branch of the LambertW function. The other branch satisfying $W(x) \leq -1$ is denoted by $W_{-1}(x)$. The shape of the LambertW function is shown in Figure 1.

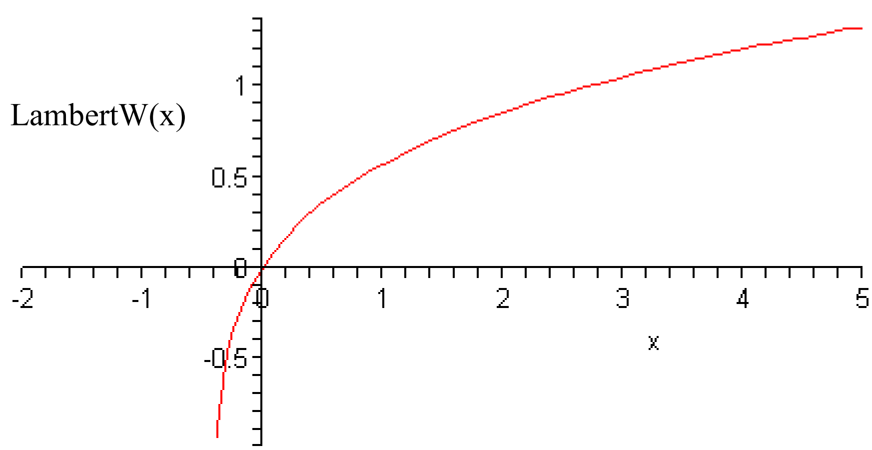

Figure 1: The LambertW function

At this point, let us examine and question the importance of the LambertW function. Similar to the exponential, logarithmic and square root functions, the LambertW is helpful in solving a series of previously unsolvable equations (readers may refer Corless, et al. 1996 for an illustration). The LambertW function has already seen widespread application in the fields of physics (Warburton and Wang 2004; Valluri, Jeffrey and Corless 2000) and applied mathematics (Corless, Jeffrey and Knuth 1997, Jeffrey, Hare and Corless 1996, Jeffrey, et al. 1995). Hence, there has been a movement to include the LambertW function in the core set of elementary functions that are used to solve equations (Hayes 2005, FOCUS 2000). Many equations involving exponentials can be solved using the LambertW function.

In this paper, our primary interest is to employ the LambertW function to solve a previously analytically unsolvable simultaneous equations problem. Before we proceed with our study, we will briefly outline a few important properties of the LambertW function. For a more detailed exposition on the properties of the LambertW function, the reader is advised to refer Corless, et al. (1996).

The LambertW function has the series expansion,

$$W_0(x)=\sum_{n=1}^{\infty}\frac{(-n)^{n-1}}{n!}x^n, \tag{2}$$

where n! is a factorial. This series however oscillates for values of x ≥ 0.4 and hence cannot always be used for practical numerical computation. However, there exists an asymptotic formulation, that is convergent for all *x*, which yields reasonably accurate real values. This is given by:

$$W(x)=L_1-L_2+\frac{L_2}{L_1}+\frac{L_2(-2+L_2)}{2L_1^2}+\frac{L_2(6-9L_2+L_2^2)}{6L_1^3}+\frac{L_2(-12+36L_2-22L_2^2+3L_2^3)}{12L_1^4}+\frac{L_2(60-300L_2+350L_2^2-125L_2^3+12L_2^4)}{60L_1^5}+O\left[\left(\frac{L_2}{L_1}\right)^6\right]$$

, (3a)

where $L_1 = \ln(x)$ and $L_2 = \ln(\ln(x))$.

A further simplification of this formulation, namely,

$$W(x)\approx 0.665[1+0.0195\ln(x+1)]\ln(x+1)+0.04\ , \tag{3b}$$

provides an accurate approximation to the LambertW function and is used primarily for numerical estimation purposes.

The derivative of LambertW is,

$$W'(x)=\frac{1}{[1+W(x)]\exp[W(x)]}=\frac{W(x)}{x[1+W(x)]} \text{ and} \tag{4}$$

the anti-derivative of $W(x)$ is,

$$\int W(x)dx=x[W(x)-1+\frac{1}{W(x)}]+C\ . \tag{5}$$

Also $W(0)=0$ and $W'(0)=1$.

Finally the logarithm of $W(x)$ is given by,

$$\log(W(x)) = \log(x)-W(x) \tag{6}$$

3. The Logit Model

3.1 Consumer's Demand Function

We consider the simplified scenario with two firms, namely firm *i* and firm *k*. Each of these firms sells a product. The products sold by these firms compete for market share. Let product *i* denote the product manufactured by firm *i* and product *k* denote the product manufactured by firm *k*. We conduct our analysis with respect to firm *i*. The scenario may be extended to include multiple firms.

As the aim of this paper is to develop techniques to analytically solve discrete choice problems, we proceed to define the probability of a consumer choosing a particular firm's product, i.e. we model the consumer's demand for each brand. If a consumer purchases product *i* then the utility to the consumer from the purchase is:

$$u_{ji} = \beta_{0i} - \beta_i P_i + \varepsilon_{ji} = U_i + \varepsilon_{ji}, \tag{7}$$

where u_{ji} is the utility obtained by consumer *j* from product *i*, β_{0i} is the brand or product specific parameter, β_i is the price response parameter for product *i*, P_i is the price of *i*, U_i is the deterministic component of utility that is assumed to be constant across consumers, and ε_{ji} is the random error term.

We wish to compute the share of *i* relative to the competing brand *k* and an outside good that represents non-purchase in the focal category. The utility of the outside good is normalized to 0; i.e. we assume $U_0 = 0$. Given that error terms for *i*, *k* and 0 follow an iid type 1 extreme value distribution, the choice share for *i* with respect to the outside product option, and with respect to the competitor *k*'s product is,

$$S_i = \frac{e^{U_i}}{1 + e^{U_i} + e^{U_k}}, \tag{8}$$

where S_i is the probability of choosing product *i*. The elasticities of demand for firm *i* with respect to the prices charged by firm *i* (own price) and firm *k* (competitor's price) are:

$$\begin{bmatrix} \dfrac{\partial S_i}{\partial P_i} \\ \dfrac{\partial S_i}{\partial P_k} \end{bmatrix} = \begin{bmatrix} -\beta_i S_i (1 - S_i) \\ \beta_k S_i S_k \end{bmatrix}. \tag{9}$$

3.2 Firm's Profit Function

The firms in the market are price setters. Let C_i be the marginal cost incurred by the firm to provide product *i*. Hence, the firm's profit from selling product *i* is:

$$\pi_i = (P_i - C_i)S_i, \tag{10}$$

where π_i is the profit firm *i* earns from selling the product at price P_i. Assuming the existence of a pure strategy interior equilibrium, the price vector satisfies the first order conditions. Hence differentiating (10) with respect to P_i, and setting the result as equivalent to zero, we derive the optimal price that the firm must charge so as to maximize profit as:

$$P_i^* = \frac{1 + \beta_i C_i (1 - S_i)}{\beta_i (1 - S_i)} \quad , \tag{11}$$

where P_i^* is the optimal price charged by the firm. This optimal price is a function of share, which, in turn, is a function of price.

4. The Analytical Solution

4.1 Solving the Firm Level Endogeneity Problem

In equilibrium, equations (8) and (11) must hold simultaneously. The solution to these simultaneous equations is highly nonlinear, and does not readily admit to a closed form solution. However, a closed-form solution can be provided in terms of the LambertW transformation. As demonstrated in the appendix, the following equations for price and share can be derived:

$$\mathrm{P_i^*} = \frac{1 + \mathrm{LambertW}\left(\dfrac{e^{\beta_{0i} - 1 - C_i \beta}}{1 + e^{\beta_{0k} - P_k \beta_k}}\right) + C_i \beta_i}{\beta_i} \quad , \tag{12}$$

and $$S_i = \frac{LambertW\left(\frac{e^{\beta_{0i}-1-C_i\beta_i}}{1+e^{\beta_{0k}-P_k\beta_k}}\right)}{1+LambertW\left(\frac{e^{\beta_{0i}-1-C_i\beta_i}}{1+e^{\beta_{0k}-P_k\beta_k}}\right)}. \tag{13}$$

This solution allows the price equation to be expressed in a form that allows the effects of market share and price to be separated. Hence the LambertW transformation allows us to rewrite the equation for market share and price in a manner in which they are independent of each other. Initially the market share of firm *i* was an integral part of the price equation (see equation 11) and hence we could not make any predictions about the price that the firm charges. Hence, through the LambertW transformation we can eliminate the econometric endogeneity problem that exists between P_i and S_i. The solution also allows for reduced-form expressions for price and share, which allows direct determination of changes in optimal price resulting from changes are made to the cost, other independent variables, or the competitor's price. For example if we assume that the prices of the competitors are given, then figures (2) and (3) show the variation of the market share of firm *i* (S_i) and the price charged by firm *i* (P_i) in relation to the price of the competitor's price (P_k) respectively.

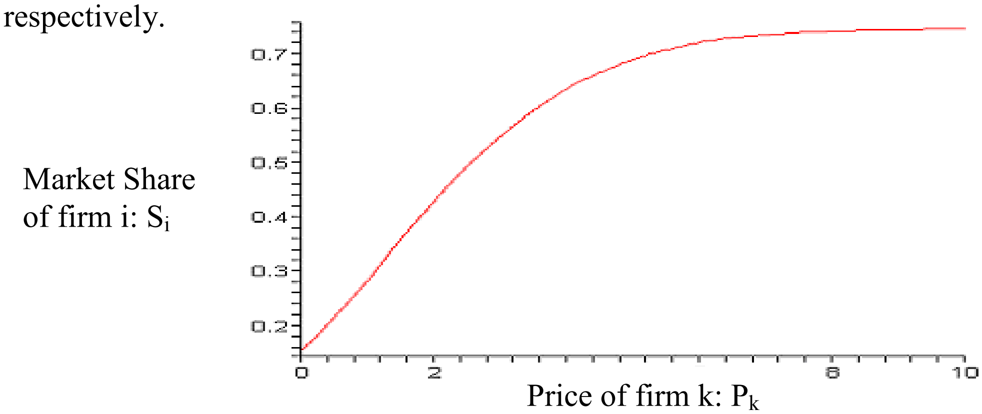

Figure 2: Relationship between market share of firm i and price of firm 'k'

Figure 2 describes the market share increase for firm ‘i’ when firm ‘k’ increases its prices. Figure 3 shows the amount by which firm *i* can increase its price when firm *k* increases its price, if firm *i* intends to retain the same level of market share.

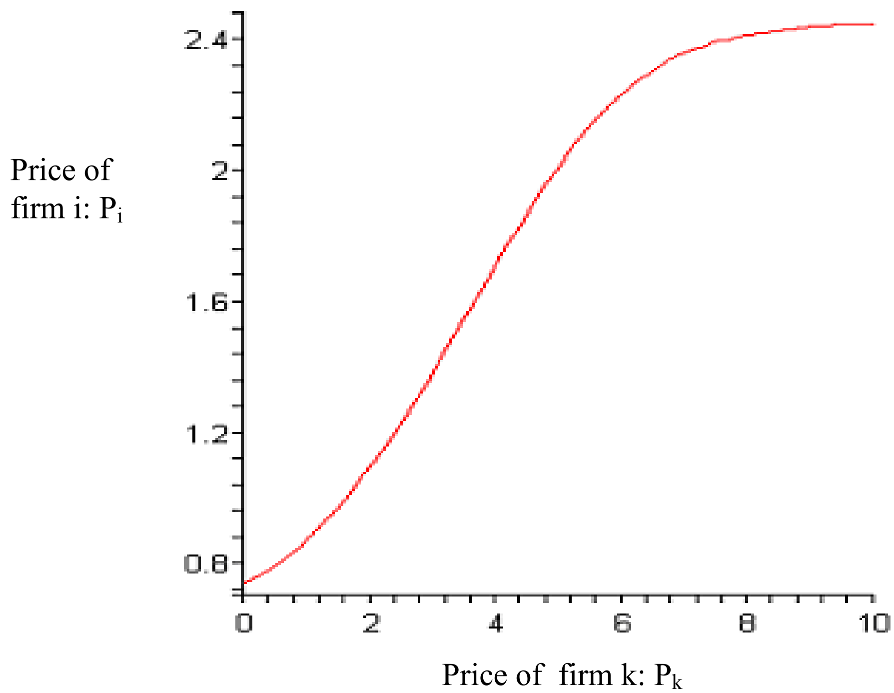

Figure 3: Relationship between the price of firm ‘i’ and price of firm ‘k’

While the above provides separate solutions for prices and shares of firm *i* by eliminating the econometric endogeneity between them, the price charged by firm *i* is still dependent on firm *k*’s price (see equation 12). Hence, the endogeneity arising from the fact that the prices of the two firms need to be jointly determined still remains, i.e. the problem of determining equilibrium market prices remains. Since P_i and P_k are symmetric, the price that firm *k* charges can be derived in the same way as the price of firm *i*, and can be expressed as:

$$P_k^* = \frac{1 + \mathrm{LambertW}\left(\dfrac{e^{\beta_{0k} - 1 - C_k \beta_k}}{1 + e^{\beta_{0i} - P_i \beta_i}}\right) + C_k \beta_k}{\beta_k} \qquad (14)$$

While equations 12 and 14 do not appear to have a readily expressible closed-form solution, Nash equilibrium prices can be determined as the point of intersection of plots of P_i given P_k and of P_k given P_i.

4.2 Estimation Procedures

Equation (3b) provides a formulation that accurately approximates the LambertW function for all x. Using the formulation in (3b) we can rewrite equation (13) in the following way,

$$P_i^* = \gamma_1(1.04) + \gamma_1(.665)\ln(\frac{e^{(\beta_{0i} - 1 - C_i \beta_i)}}{1 + e^{(\beta_{0k} - P_k \beta_k)}} + 1) + \gamma_1(.0129)[\ln(\frac{e^{(\beta_{0i} - 1 - C_i \beta_i)}}{1 + e^{(\beta_{0k} - P_k \beta_k)}} + 1)]^2 + C_i \ ,$$

(15)

where $\gamma_1 = 1/\beta_1$. As we are now interested in estimating equation (15), assume an unobservable term for the price equation denoted by ω_i, and that the price is linear in the unobservable term ω_i. Hence equation (15) can be rewritten as:

$$P_i^* = \gamma_1(1.04) + \gamma_1(.665)\ln(\frac{e^{(\beta_{0i} - 1 - C_i \beta_i)}}{1 + e^{(\beta_{0k} - P_k \beta_k)}} + 1) + \gamma_1(.0129)[\ln(\frac{e^{(\beta_{0i} - 1 - C_i \beta_i)}}{1 + e^{(\beta_{0k} - P_k \beta_k)}} + 1)]^2 + C_i + \omega_i$$

(16)

Equation (16) is now independent of the effect of the firm's own market share and hence can be estimated without the use of firm level instrumental variables. The estimation procedure would follow a standard nonlinear regression where the parameters to be estimated would include γ_1 and all the β's. It is important to note that if the market prices are set endogenously, the researcher would estimate (16) and a similar equation for the prices of firm *k* simultaneously, using suitable market level instrumental variables.

5. A More General Application Incorporating Marketing Expenditure

We apply this methodology to the multinomial logit problem studied by Basuroy and Nguyen (1998). The authors discuss the appropriateness of Multinomial Logit (MNL) market share models for equilibrium analysis. Their results show that a linear price response in conjunction with the typical concavity assumed in a large range of marketing response functions would yield an interior equilibrium solution. The authors then consider the optimal pricing and marketing expenditure reactions to entry and potential market expansion. In the context of the MNL models, they demonstrate that the entry of a new brand evokes a defensive reaction through a decrease in the equilibrium prices of the existing brands. They also note that while new entry into a fixed market triggers the incumbents to lower marketing expenditure, when faced with market expansion, firms tend to raise marketing activities. Consequently, there exist distinct possibilities that marketing efforts for the existing brands increase in view of entry in an expanding market.

In this section we will incorporate the characteristics outlined by the authors, namely—the linearity of price response and the concavity of market expenditure response functions into our MNL market share model. We then derive optimal solutions for price and market expenditure independent of market share by employing the methodology developed in the earlier part of this paper.

Consistent with Basuroy and Nguyen, let P_1 and m_1 be the price charged and marketing expenditure of Firm 1 for its product. Let P_2 and m_2 the price charged and marketing expenditure of Firm 2 for its product. We assume a linear price response function and a concave marketing response function. The market share function for Firm 1 is given by,

$$S_1 = \frac{e^{\beta_{11}P_1 + \beta_{21}(1-e^{-m_1})}}{e^{\beta_{11}P_1 + \beta_{21}(1-e^{-m_1})} + e^{\beta_{12}P_2 + \beta_{22}(1-e^{-m_2})}}, \tag{17}$$

where S_1 is the market share for Firm 1, β_{11} is the price response parameter that is assumed to be negative and β_{21} is the market response parameter that is assumed to be positive. We assume that Firm 2 has a market share function that is analogous to Firm 1. Given the market share function for Firm 1, the profit function for Firm 1 becomes,

$$\pi_1 = (P_1 - c_1)\cdot N \cdot S_1 - m_1 - FC_1, \tag{18}$$

where π_1 is the profit that Firm 1 gets from selling its product, c_1 is the marginal cost incurred by Firm 1 and FC_1 is the firm's fixed cost and N is the market size.

Assuming that the firm is a price setter and assuming the existence of a pure strategy interior equilibrium, the price vector satisfies the first order conditions. We can obtain the optimal price that Firm 1 should charge for its product by solving the first order conditions for P_1. The first derivative with respect to P_1 is given by,

$$\dot{\pi}_{P_1} = S_1 N - (P_1 - c_1)\cdot \beta_1 \cdot N \cdot S_1(1 - S_1) \tag{19}$$

Similarly, the first order conditions for marketing expenditure m_1 can be obtained by differentiating (2) with respect to m_1. This is given by:

$$\pi_{m_1} = (P_1 - c_1)\cdot\beta_{11}\cdot N\cdot e^{-m_1}\cdot S_1(1-S_1) - 1 \tag{20}$$

We can now use the LambertW functional form to solve equations (17), (19) and (20) and obtain the optimal solutions for price P_1 and marketing expenditure m_1, independent of the firm's market share. We solve the equations simultaneously and obtain the following solutions for the optimal price and optimal marketing expenditure:

$$P_1^* = \frac{-LambertW\left[\dfrac{e^{\left(\frac{\beta_{11}}{N}+\beta_{21}+\beta_{11}c_1-1\right)}}{1+e^{\left(\beta_{12}P_2+\beta_{22}-\frac{\beta_{22}}{e^{m_2}}\right)}}\right]-1}{\beta_{11}} + c_1\ , \tag{21}$$

where P_1^* is the optimal price charged by Firm 1 and,

$$m_1^* = -\ln\left(\frac{-\beta_{11}}{\beta_{21}N}\right), \tag{22}$$

where m_1^* is the optimal marketing expenditure. As β_{11} is assumed to have a negative value, both P_1 and m_1 from equations (21) and (22) will always be positive. Hence, the methodology developed in this paper allows us to derive closed form solutions to both the optimal price and the optimal marketing expenditure, where the optimal price depends on cost and competitor price.

Comparing our results with Basuroy and Nguyen (1998) we find, as shown in figure 4, that the optimal price charged by the Firm 1 would decrease upon the introduction of a new product in the market. This is consistent with the results obtained by the authors.

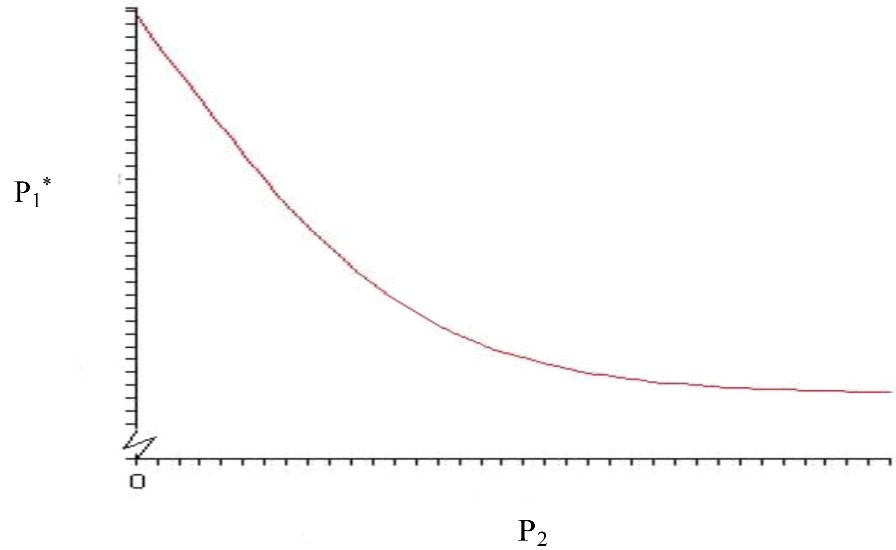

Figure 4: The decrease in P_1* with increase in P_2

It is relatively harder to analyze the marketing expenditure function as the functions only involve the price response parameter of Firm 1. Thus, we cannot make predictions about the change in marketing expenditure with respect to the entry of a competitor. However we can make predictions of the effect of an increasing market size on the marketing expenditure. Taking the first derivative of marketing expenditure function given in equation (22) we find that

$$\frac{\partial m_1^*}{\partial N} = \frac{1}{N} \qquad (23)$$

This implies that the marketing expenditure function shares a positive relationship with the market size. Hence as market size increases, the marketing expenditure of Firm 1 must also increase, albeit at a decreasing rate. This result is also consistent with the results obtained by Basuroy and Nguyen (1998).

6. Conclusion

In this paper we analytically solved the problem of econometric endogeneity in discrete choice models through the use of the LambertW function. The LambertW function lends itself to analytically solving exponential equations, and thereby facilitates the derivation of closed-form solutions for price and market share. The LambertW function also lends itself to easy estimation through a simple yet accurate approximation as explained in sections 2 and 4.2. This approximation can be used in estimating expressions for price and share that depend only on competitor actions, and that therefore do not require firm level instrumental variables. Market level instrumental variables, which are easier to obtain, suffice. The LambertW function is potentially useful whenever logit models are employed in theoretical or empirical work, or more generally, whenever solutions to exponential functions are required.

Chapter 2: Essay 2 - Pricing Related Products In A Competitive Environment – The Role Of Contingent Valuations Between Products

Summary

This paper extends the research by Venkatesh and Kamakura (2003) to consider the case of pure component pricing in a competitive setting. We develop an optimal pricing scheme for a Firm that has two products, when its products are sold as pure components. We show that the optimal price increases monotonically with respect to the contingent valuation of both the Firm's and its competitor's products. We also derive the consumer surplus when the products are priced independently and when they are priced by taking into account the contingent valuation. We conclude that consumer surplus is higher in the former case.

1. Introduction

Implicit bundles refer to the group of products that are sold separately (also referred to as pure components) but could be perceived by the consumer as products that would provide an enhanced level of satisfaction if purchased together. In many cases, the implicit bundles are sold by the same Firms. An example would be Microsoft selling both its Windows Operating System and Office Suite. It has been shown that even though Microsoft sells these products separately, they still price them as if they were selling them as a bundle, where Windows is priced lower than

Office, even though Windows costs almost twice as much to develop (Economides and Viard 2003). Most Firms market products in many different product lines, so it is very important for the Firms to develop optimal pricing strategies for their products in the different product lines. Many issues go into establishing optimal pricing schemes. The study of the effect of inter-category relationships between the products made by the same manufacturer on the pricing scheme is very important, because this could affect the consumption behavior of the consumer. This topic has been studied in some detail in the retail setting (Manchanda et al. 1999). However, with the exception of a handful of studies (Reibstein and Gatignon 1984; Urban 1969), the literature has generally ignored how manufacturers must treat this problem when pricing their products *across* the different product lines (Elrod et al. 2002).

The aim of this paper is to address the issue of pricing across different product lines. We do so by developing a model that would account for the possible relationships of the products across the different product lines. We hope to show that accounting for these relationships would decrease the amount of price competition in the market and also allow the Firm to enjoy higher profits, because it would be able to charge the consumer for the additional surplus gained by purchasing both products from the same Firm. We also account for the endogeneity of price and market share when deriving the optimal price.

The rest of the paper is organized as follows. We review the literature in §2. Model Development is carried out in §3. The conclusions and managerial implications are presented in §4. Directions for future research are provided in §5.

2. Literature Review

2.1 Product Relationships

Traditionally inter-category relationships between products have been studied in the form of substitutes or complements. However these relationships are far more complex than just substitutes or complements. Product relationships can no longer be viewed upon as a simple dichotomy (either complements or substitutes) but these relationships should be viewed in terms of degree of complementarity and substitutability, not absolute substitutability or complementarity.

Venkatesh and Kamakura (2003) explain these relationships in the form of contingent valuations. Many recent studies have indicated that buyers evaluate the components of a bundle by assigning certain valuations to the strength of the relationships in the bundle. (Jedidi et al. 2003, Yadav 1994). Contingent valuations measure this relationship between the components in the bundle. Contingent valuations may be defined as the degree of complementarity (or substitutability) between two products as perceived by the consumer, if the products were to be sold as a bundle (Venkatesh and Kamakura 2003). Contingent valuations can be either positive or negative. A positive relationship implies that the two products share a complementary relationship and a negative relationship implies substitutability. In this study we focus only on the positive values of the contingent valuations, i.e., we confine our analysis to only complementary products as we are interested in understanding how a firm can take advantage of a complementary relationship in pricing its products. More recently, Wang, Venkatesh and Chatterjee (2006) develop a new methodology to reformulate the the way reservation prices can be calculated. Their procedure named ICERANGE draws on literature on buyers' uncertainty in preference and product knowledge. Their results demonstrate that the ICERANGE method significantly outperforms previous models in terms of predictive validity.

For the purpose of this study we develop a pricing policy for two products that can be perceived by the consumer as having a relationship with each other. We then compare this pricing scheme to another pricing scheme where the products are priced independently of each other. In this study, we only consider implicit bundles, as we assume that it is not convenient for the manufacturer to explicitly bundle its products. The main difference between an explicit and implicit bundle would be that products in explicit bundles are sold together, while products in implicit bundles are sold separately. An example of an explicit bundle is the Microsoft Windows and Internet Explorer bundle. Here the consumer typically attains a higher level of satisfaction using the two products, but they are not sold separately. An example of an implicit bundle is the Windows-Office bundle, where the products are sold separately, and the consumer enjoys a higher level of satisfaction by owning both the products. We will explain both the economic and legal reasons for this later on in this section.

Implicit bundles can be composed of both complements and substitutes. For the purpose of this study, we only consider complements. Hicks (1939) was one of the first researchers to define complementary relationships. He argued that complementarity could be determined through a modification to the cross price elasticity test, using observable variables like price and demand. Hicks (1939), Schultz (1938) and Hicks and Allen (1934) suggested that instead of using the sign of the cross price elasticity term as a measure of complementarity, it would be better to use a compensated price change to measure complementarity. The compensated price change term considers the effect of not only the change in price, but also the effect of income. A negative value of the compensated price change implied a complementary relationship. (Deaton and Muellbauer 1980). Samuelson (1972) also developed a measure for complementarity called the money metric. He proposed that if we consider the von Neumann utility function (as a money metric, with diminishing returns to the marginal utility of income), then two products are complements if $\frac{\partial^2 V[.]}{\partial q_i \partial q_j} > 0$, where V[.] is the von Neumann utility function and q_i is demand for product i and q_j is demand for product j.

In marketing, Guilitinan (1987) provides an excellent overview of how complementarity can arise between any two products. Two products become complements in the following situations:

- Savings in Search Economies: A motorist prefers going to a dealer who does both an engine tune up and an oil change as opposed to two dealers separately offering only one of the services each. The savings in time and effort gained by the consumer, by going to a single dealer creates a degree of complementarity between the services offered by the dealer.
- Enhance Customer Satisfaction: Guilitinan (1987) offers the example of a ski lodge that provides both ski lessons and rentals. The combination of services offered would enhance the satisfaction that a beginner gets from the ski lodge.

- Improved Total Image: He provides the example of a Firm that offers both lawn care as well as shrub care services, thereby enhancing the image of the Firm.

Shocker et al. (2004) also attempt to define complementary products. They identify three types of complementary relationships, perfect complements (e.g., video cassette recorder and video cassette), augmenting complements (products that add new benefits not present in an already existing one, e.g., washing machine and dryer) and enhancing complements (new products that improves the sales of an already existing product, e.g., clipart and presentation software).

Inspite of there being a lot of research on bundling of products (Hanson and Martin 1990, Bakos and Brynjolfsson 1999, Venkatesh and Mahajan 1993, Chung and Rao 2003), only Venkatesh and Kamakura (2003) have looked at the relationships between complementary products using contingent valuations. They derive optimal bundling and pricing strategies taking into account the contingent valuations of the two products of a monopolist. They compare the derived pricing schemes to the pricing scheme if these products were priced independently of each other. They show that the price increases monotonically as a function of the contingent valuation.

The aim of this study is to extend the paper by Venkatesh and Kamakura (2003) to incorporate the effects that competition would have on the optimal pricing strategy. This condition has not been explored in their study. We propose that the price will depend on both the manufacturer's and competitor's contingent valuations.

We also propose that the price that accounts for these valuations, will always be higher than if they were priced independently (when market share is the same), provided the valuations always have a positive value (in the case of substitutes these valuations could be negative).

We assume that the firm finds it is disadvantageous to offer an explicit bundle of its products in the market. This situation can arise under two conditions:

1. Competition: Anderson and Leruth (1993), show that in a duopoly environment, only pure components pricing may be offered in equilibrium since Firms fear the extra degree of competition inherent in offering the option of a bundle (in the case of complements). Our study differs from Anderson and Leruth (1993) in that they do not consider contingent valuations or try to formally develop a pricing scheme. Our assumption is further supported by Matutes and Regibeau (1992) who also show that in a competitive setting the pure components strategy is dominant.
2. Conditions for Legality: Stremersch and Tellis (2002) raise the issue of legality of introducing product bundles in a competitive market. U.S. Law has two rules which determine the legality of any bundle, the 'per se rule' and 'rule of reason'. The per se rule says that bundling is illegal when it involves pure bundling of separate products by a Firm with market power and when a substantial amount of commerce is at stake. The rule of reason says that bundling is illegal when it involves pure bundling of separate products by a Firm with market power,

involving a substantial amount of commerce, which poses a threat that the bundling Firm will acquire additional market power over at least one of the products that is bundled with the tying products and no plausible consumer benefits offset the potential damage to competition. Details of the above rules are provided in the Stremersch and Tellis (2002) paper.

Hence, under such conditions, as explicit bundling is either disadvantageous or restricted due to legal reasons, it is important for a Firm that manufactures two different yet related complementary products to develop an optimal pricing scheme that accounts for the relationship of the two products. We restrict the analysis in this paper to this scenario.

2.2 Complementary Pricing

Liao et al. (2002a, 2002b) study the effect of pure component pricing on the equilibrium of the market. They show that if Firms are restricted to pure component pricing scenario, then the market attains a stable equilibrium only when the two products made by the Firm are incompatible with the products made by the competing Firms. However, if the products are compatible, then the market attains a stable pure strategy equilibrium only if the bundle pricing of the complementary products is allowed.

In the marketing literature, Reibstein and Gatignon (1984) develop a model for optimally pricing related products in a product line by extending the mathematical model developed by Urban (1969). They develop a model that includes cross-elasticities of the various products made by the manufacturer, and use the method of seemingly unrelated regression to estimate the model.

Complementarity for the most part has been largely ignored by the marketing literature (Elrod et al. 2002), however there do exist some models of pricing summarized by Tellis (1986), which are presented below:

1. Captive Pricing: Captive pricing occurs when the manufacturer charges a low price for the base good and then charges a higher monopoly price for the accessory product (the base good is assumed to work only with the accessory). This is successful because the consumers might not view the basic product they purchased as a sunk cost and hence may try and recover their 'money' by purchasing the accessories and using it ('Sunk Cost Effect', Thaler 1980, 1985). The consumer may also use this product more than expected and hence buy more of the accessory. This has led researchers to label this scheme as the 'captive pricing scheme'.

   The main restraint on captive pricing is that, if the manufacturer charges a high price for the accessory, it could lead to the entry of competing Firms in the market for the accessory, thereby reducing the Firm's net profit.

2. Two-Part Pricing: The price here is broken into a fixed fee and a usage fee. Two part pricing is a type of captive pricing scheme that is adapted to services (Oi 1972). The usage fee helps Firms exploit the

heterogeneity in demand for the products. Hence the heavy user would pay more to use the service than the light user. Oi (1972) illustrates this by applying this model to the case of Disneyland where the customer would pay an entry fee to enter the park and then a usage fee, i.e., spend on rides and other items in the park based upon their demands for these items.

Sales from both captive and two-part pricing are classified as types of tie-in sales. Apart from the legal difficulties mentioned earlier, tie-in sales suffer from some other legal difficulties also. For example the procedure of tying the buyer to purchase the supplies from the same Firm that manufactured the base product may be considered to be illegal under the Sherman Act of 1890 or the Clayton act of 1914 (Burstein 1960, Mathewson and Winer 1997). This paper thus aims to develop a pricing scheme, when tie-in sales are not an option, by considering the contingent valuations of the products.

3. Model Formulation

3.1 Defining the Market Structure

First we describe the market structure. We consider two multi-product Firms (i = 1, 2). Each Firm has two products, one in category A and one in category B. The Firms compete with each other for market share in both markets A and B. Firm i manufactures products A_i and B_i and competes with the products made by the other Firm. This structure is illustrated in figure 5.

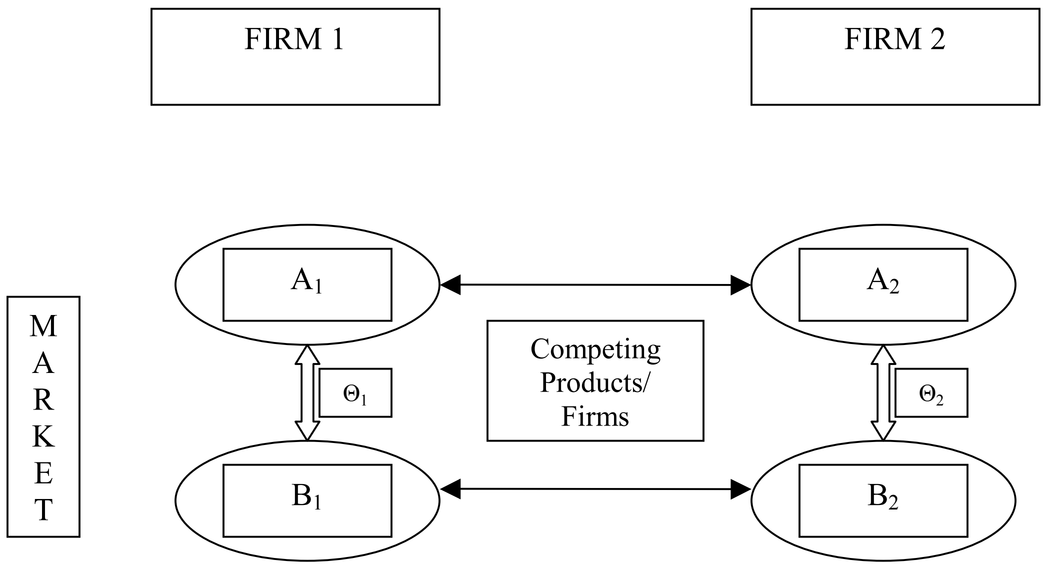

Figure 5: The Market Structure

Both products A_i and B_i share a complementary relationship which is parameterized by θ_i. θ_i measures the degree of complementarity between the two products within the Firm i. Thus θ_i is the contingent valuation of relationship between the two products.

The notion of contingent valuations arises when a consumer perceives the two products as if they were a bundle, complementing each other in some way. This perception of complementarity could be because the products belong to categories that could be consumed together. Schmalansee (1982) indicated that the consumer's reservation price for the bundle of A_i and B_i would be higher than the sum of the stand alone reservation prices for the products when the products are complements. The contingent valuation θ_i is defined as:

$$\theta_i = \frac{RP_{Ai+Bi} - (RP_{Ai} + RP_{Bi})}{(RP_{Ai} + RP_{Bi})} \quad \ldots\ldots\ldots\ldots..(1)$$

where:

RP_{Ai+Bi} = Consumer's reservation price for the perceived bundle of products A_i and B_i

RP_{Ai} = Consumer's reservation price for the product A_i

RP_{Bi} = Consumer's reservation price for the product B_i

θ_i lies in the range (0,1). When $\theta_i = 0$, the products do not share a relationship and thus are independent of each other. When $\theta_i = 1$, the products are perfect complements and hence cannot be used independently. An example of this case would be a video cassette player and a video cassette. When $0 < \theta_i < 1$, the products can be used independently, however the consumer obtains a higher utility buying both the products (e.g., MS Windows and MS Office). When $\theta_i < 0$, the products are substitutes. We however do not consider this case for the purpose of this study.

When $\theta_i > 0$, it is assumed that the consumer will prefer to buy both products from Firm i.

3.2 Consumer Demand Formulation

3.2.1 Contingent Valuation Condition

The next step in the analysis would be to model the consumer's demand for the two products. For convenience we perform our entire analysis with respect to

Firm 1, i.e., we provide optimal pricing schemes for Firm 1 when it competes with Firm 2.

Therefore we must first define the utility that each consumer would derive from the each product marketed by Firm 1.

If a consumer buys only product A_1, then the utility that the consumer derives owning A_1 is:

$$u_{j,A_1,t} = \beta_0 - \beta_1 PA_1 + \psi_{A_1,t} + \varepsilon_{j,A_1,t} \quad \ldots\ldots\ldots\ldots\ldots(2a)$$

where

$u_{j,A1,t}$ = utility obtained by consumer j from product A_1 at time t

β_0 = brand / product - specific parameter

β_1 = price response parameter for product A_1

PA_1 = price of A_1

$\Psi_{A1,t}$ = unobserved component of utility derived from product A_1

$\varepsilon_{j,A1,t}$ = the random error term, which follows an iid type 1 extreme value distribution

Similarly the utility for product B_1 in period t for consumer j is given by:

$$u_{j,B_1,t} = \gamma_0 - \gamma_1 PB_1 + \psi_{B1,t} + \varepsilon_{j,B1,t} \quad \ldots\ldots\ldots\ldots\ldots(2b)$$

where

$u_{j,B1,t}$ = utility obtained by consumer j from product B_1 at time t

γ_0 = brand / product specific parameter

γ_1 = price response parameter for product B_1

PB_1 = price of B_1

$\Psi_{B1,t}$ = unobserved component of utility derived from product B_1

$\varepsilon_{j,B1,t}$ = the random error term, which follows an iid type 1 extreme value distribution

We are interested in determining a pricing scheme for when the consumer might be interested in purchasing both the products. For this we will modify the Venkatesh and Kamakura (2003) definition slightly to make the value of one item depend on whether or not the companion item is purchased.

Let $V_{A1} = \beta_0 - \beta_1 PA_1 + \psi_{A_1,t}$ …………(3a) and

$V_{B1} = \gamma_0 - \gamma_1 PB_1 + \psi_{B1,t}$ ………………(3b),

where V_A and V_B are the values that the consumer attaches to the products A and B respectively.

Therefore the value of A_1 if B_1 is purchased is:

$$V_{A1|B1} = V_{A1} + \eta V_{B1} \text{……..(4a)}$$

where η is the contingent value between the two products.

Conversely the value of A_1 if B_1 is not purchased becomes:

$$V_{A1|NB1} = V_{A1} \text{………(4b)}$$

The expected value of product A_1 thus depends on whether or not the companion item is purchased.

$$V_{A1}^* = P_{B1}\, V_{A1|B1} + (1-P_{B1})V_{A1|NB1}$$

$$= V_{A1} + P_B\, \eta_1 V_{B1} \text{…………(5)}$$

where P_B is the probability that the companion item B_1 will be purchased. This probability depends on the price of B_1: PB_1, as well as the value of B_1. Hence the expected value of owning A_1 can be expressed as:

$$V_{A1}^* = V_{A1} + E\,[f\,(V_{B1}, PB_1)] \text{……………(6a),}$$

where 'E' is the expectation and 'f' represents a function. For simplicity we will write:

$E\,[f\,(V_{B1}, PB_1)] = \gamma_0 + \gamma_1 PB_1 + \psi_{B1}$ ………….(6b)

Thus $V_{A1}^{*} = V_{A1} + \gamma_0 + \gamma_1 PB_1 + \psi_{B1} + \varepsilon$…………(7)

Thus the consumer's utility function for A_1 would now depend on whether B_1 is purchased or not. Therefore the *consumer's* utility for product A_1 in period t for consumer j, given that consumer j expects to purchase B_1 also, is given by:

$$u_{j,A1,t} = \beta_0 + \beta_1 PA_1 + \eta_1(\gamma_0 + \gamma_1 PB_1 + \psi_{B1,t}) + \psi_{A1,t} + \varepsilon_{j,A1,t} \quad \text{…………(8a)}$$

Similarly utility of B_1 given A_1 is

$$u_{j,B1,t} = \gamma_0 + \gamma_1 PB_1 + \eta_1(\beta_0 + \beta_1 PA_1 + \psi_{A1,t}) + \psi_{B1,t} + \varepsilon_{j,B1,t} \quad \text{………….(8b)}$$

The next step would be to model the net demand for the products A_1 and B_1. To model demand we consider the logit demand model. The logit model has been used extensively in the marketing literature for modeling both household level data (Guadagni and Little 1983) as well as aggregate market share data (Allenby 1989). Here we model the aggregate market share using the logit formulation. The utility can be rewritten as a function of the deterministic part and the random component:

$u_{j,A1,t} = U_{A1,t} + \varepsilon_{j,A1,t}$ …………(9a)

$u_{j,B1,t} = U_{B1,t} + \varepsilon_{j,B1,t}$ …………(9b)

$U_{A1,t}$ and $U_{B1,t}$ represents the deterministic part, which is the aggregate utility obtained from A_1 and B_1. $\varepsilon_{j,A1,t}$ and $\varepsilon_{j,B1,t}$ represent the heterogeneity in consumer preferences for the two products, and they are assumed to be distributed iid type-1 extreme value.

To allow for the market share of the brand to expand and contract over the different periods with the choice of the marketing mix we allow for non-purchases (eg: an outside good, or choosing not to purchase), this is denoted as product 0. The utility of the outside product is normalized to 0 across periods; i.e. we assume $U_{0,t} = 0$.

We now derive the choice shares for products A_1 and B_1 with respect to the option of the outside product 0 and the competitor's products A_2 and B_2 respectively. Thus the choice share for A_1 is:

$$S_{A1,t} = \frac{e^{U_{A1,t}}}{1 + e^{U_{A1,t}} + e^{U_{A2,t}}} \text{.............}(10)$$

where $S_{A1,t}$ is the probability of the choice of product A_1. Hence the choice share for product A_1 is dependent on the utility of A_2, which is a function of the relationship between A_2 and B_2, which is given by η_2. Thus the choice share for B_1 is hence analogous to Equation (10) and $S_{B1,t}$ is the probability of the choice of product B_1.

For the next step in our analysis we derive the elasticities of the market share of the products A_1 and B_1 with respect to the prices of A_1, B_1, A_2 and B_2.

Therefore the first derivatives of $S_{A1,t}$ and $S_{B1,t}$ with respect to prices of A_1 and B_1 are:

$$\begin{bmatrix} \dfrac{\partial S_{A1,t}}{\partial PA_1} & \dfrac{\partial S_{B1,t}}{\partial PA_1} \\ \dfrac{\partial S_{A1,t}}{\partial PB_1} & \dfrac{\partial S_{B1,t}}{\partial PB_1} \end{bmatrix} = \begin{bmatrix} \beta_1 S_{A1,t}(1 - S_{A1,t}) & \eta_1 \beta_1 S_{B1,t}(1 - S_{B1,t}) \\ \eta_1 \gamma_1 S_{A1,t}(1 - S_{A1,t}) & \gamma_1 S_{B1,t}(1 - S_{B1,t}) \end{bmatrix} \text{..............}(11a)$$

The first derivatives of $S_{A1,t}$ and $S_{B1,t}$ with respect to A_2 and B_2 are:

$$\begin{bmatrix} \frac{\partial S_{A1,t}}{\partial PA_2} & \frac{\partial S_{B1,t}}{\partial PA_2} \\ \frac{\partial S_{A1,t}}{\partial PB_2} & \frac{\partial S_{B1,t}}{\partial PB_2} \end{bmatrix} = \begin{bmatrix} -\beta_2 S_{A1,t} S_{A2,t} & -\eta_2 \beta_2 S_{B2,t} S_{B2,t} \\ -\eta_2 \gamma_2 S_{A1,t} S_{A2,t} & -\gamma_2 S_{B1,t} S_{B2,t} \end{bmatrix} \ldots\ldots\ldots\ldots\ldots(11b)$$

3.2.2 No Contingent Valuation Condition

We also derive the case when manufacturers are unaware of the existence of the contingent valuations of the two products, and price the two products accordingly. This no contingent valuation condition is labeled 'nc'. This step is carried out to facilitate comparison and to elucidate the importance of identifying the various contingent effects.

Therefore in this case *the Firm expects* that the consumer's utility function would be given by:

$$u_{j,A1,t,nc} = \beta_{0,nc} + \beta_{1,nc} PA_{1,nc} + \psi_{A1,t,nc} + \varepsilon_{j,A1,t,nc} \quad \ldots\ldots\ldots(12a)$$

$$u_{j,B1,t,nc} = \gamma_{0,nc} + \gamma_{1,nc} PB_{1,nc} + \psi_{B1,t,nc} + \varepsilon_{j,B1,t,nc} \quad \ldots\ldots\ldots..(12b)$$

Hence we again split the model into the deterministic component and random component as follows:

$u_{i,A1,t,nc} = V_{A1t} + \varepsilon_{iA1t,nc}$………..(13a)

$u_{i,B1,t,nc} = V_{B1t} + \varepsilon_{iB1t,nc}$………..(13b)

where:

$V_{A1,t}$ = deterministic component of utility for product A_1

$V_{B1,t}$ = deterministic component of utility for product B_1

Thus market share of product A_1 at time t is given by:

$$S_{A1,t,nc} = \frac{e^{V_{A1t}}}{1 + e^{V_{A1t}} + e^{V_{A2t}}} \quad \text{...........(14)}$$

Thus market share of product B_1 at time t is analogous to Equation (14).

For the next step in our analysis we derive the elasticities of the market share of the products A_1 and B_1 with respect to the prices of A_1, B_1, A_2 and B_2.

Therefore the first derivatives of $S_{A1,t,nc}$ and $S_{B1,t,nc}$ with respect to A_1 and B_1 are:

$$\begin{bmatrix} \frac{\partial S_{A1,t,nc}}{\partial PA_{1,nc}} & \frac{\partial S_{B1,t,nc}}{\partial PA_{1,nc}} \\ \frac{\partial S_{A1,t,nc}}{\partial PB_{1,nc}} & \frac{\partial S_{B1,t,nc}}{\partial PB_{1,nc}} \end{bmatrix} = \begin{bmatrix} \beta_{1,nc} S_{A1,t,nc}(1 - S_{A1,t,nc}) & 0 \\ 0 & \gamma_{1,nc} S_{B1,t,nc}(1 - S_{B1,t,nc}) \end{bmatrix} \text{...........(15a)}$$

The first derivatives of $S_{A1,t,nc}$ and $S_{B1,t,nc}$ with respect to A_2 and B_2 are:

$$\begin{bmatrix} \frac{\partial S_{A1,t,nc}}{\partial PA_{2,nc}} & \frac{\partial S_{B1,t,nc}}{\partial PA_{2,nc}} \\ \frac{\partial S_{A1,t,nc}}{\partial PB_{2,nc}} & \frac{\partial S_{B1,t,nc}}{\partial PB_{2,nc}} \end{bmatrix} = \begin{bmatrix} -\beta_{2,nc} S_{A1,t,nc} S_{A2,t,nc} & 0 \\ 0 & -\gamma_{2,nc} S_{B1,t,nc} S_{B2,t,nc} \end{bmatrix} \text{..............(15b)}$$

In the above equations 15a and 15b, as we assume the 'nc' condition the off-diagonal terms are zero in contrast to equations 11a and 11b. Specifically there is no effect of the change of price of A_2 on the market share of B_1 or of the change of price of B_2 on the market share of A_1. It is also evident that the change in price of A_1 has to effect on the market share of B_1 and vice versa.

3.3 Manufacturer's Profit Function

We now proceed to derive the manufacturer's profit function for the two situations, namely, the contingent valuation condition and the no contingent valuation condition.

3.3.1 Pricing with no contingent valuation

We first consider the case when the manufacturer prices not realizing the presence of a relationship between the two products it manufactures.

The manufacturer profit function $\pi_{t,nc}$ in the nc condition is given by:

$$\pi_{t,nc} = (PA_{1,nc} - CA_1)N \cdot S_{A1,t,nc} + (PB_{1,nc} - CB_1)N \cdot S_{B1,t,nc} \quad \text{……………..(16)}$$

where:

CA_1: marginal cost of product A_1

CB_1: marginal cost of product B_1

N: Total number of consumers in the market for the products.

Maximizing the profit equation with respect to $PA_{1,nc}$ and $PB_{1,nc}$ we get:

$$PA^*_{1,nc} = \frac{1+\beta_1 CA_1(1-S_{A1,t,nc})}{\beta_1(1-S_{A1,t,nc})} \quad \text{………………(17a)}$$

$$PB^*_{1,nc} = \frac{1+\gamma_1 CB_1(1-S_{B1,t,nc})}{\gamma_1(1-S_{B1,t,nc})} \quad \text{………………(17b)}$$

Equations (17a) and (17b) represent the optimal prices charged in the 'nc' condition.

3.3.2 Pricing with Contingent Valuations

The next step in our formulation of the optimal prices is to define the manufacturer's profit function when contingent valuations are recognized by the Firm. As we account for the market share of each product in each period, the manufacturer's profit function π_t can be formulated as:

$$\pi_t = (PA_1 - CA_1) \cdot N \cdot S_{A1,t} + (PB_1 - CB_1) \cdot N \cdot S_{B1,t} \quad \text{…………..(18)}$$

Firm 1's aim is thus to maximize profit with respect to PA_1 and PB_1 in each period. Therefore we differentiate the profit function with respect to PA_1 and PB_1, and obtain the following equations:

$$\frac{\partial \pi_t}{\partial PA_1} = (S_{A1,t} + (PA_1 - CA_1)\cdot \beta_1 \cdot S_{A1,t}(1 - S_{A1,t}) + (PB_1 - CB_1)\cdot \eta_1 \cdot \beta_1 \cdot S_{B1,t}(1 - S_{B1,t}))\cdot N$$

..(19a)

and

$$\frac{\partial \pi_t}{\partial PB_1} = (S_{B1,t} + (PA_1 - CA_1)\cdot \eta_1 \cdot \gamma_1 \cdot S_{A1,t}(1 - S_{A1,t}) + (PB_1 - CB_1)\cdot \gamma_1 \cdot S_{B1,t}(1 - S_{B1,t}))\cdot N$$

..(19b)

We then solve equations 19a and 19b to get the optimal prices PA_1 and PB_1 for Firm 1's products A_1 and B_1. Thus the optimal prices are:

$$PA_1^* = \frac{\eta_1 \beta_1 S_{B1,t} - \gamma_1 S_{A1,t}}{\gamma_1 \beta_1 S_{A1,t}(1 - S_{A1,t})} + CA_1 \quad \ldots\ldots\ldots\ldots(20a)$$

$$PB_1^* = \frac{\eta_1 \gamma_1 S_{A1,t} - \beta_1 S_{B1,t}}{\gamma_1 \beta_1 S_{B1,t}(1 - S_{B1,t})} + CB_1 \quad \ldots\ldots\ldots\ldots(20b)$$

This leads us to our first proposition.

Proposition 1

The price charged by the manufacturer when contingent valuations are considered is always greater than the price charged by the manufacturer in the no contingent valuation condition for all positive values of η_1.

Proof

The Equations (20a) and (20b) for the price of the products when contingent valuations are considered are given below:

$$PA_1^* = \frac{\eta_1 \beta_1 S_{B1,t} - \gamma_1 S_{A1,t}}{\gamma_1 \beta_1 S_{A1,t}(1 - S_{A1,t})} + CA_1 \ldots\ldots\ldots\ldots(20a)$$

$$PB_1^* = \frac{\eta_1 \gamma_1 S_{A1,t} - \beta_1 S_{B1,t}}{\gamma_1 \beta_1 S_{B1,t}(1 - S_{B1,t})} + CB_1 \ldots\ldots\ldots\ldots(20b)$$

The equations 17a and 17b when the manufacturer prices in the 'nc' condition are given below:

$$PA^*_{1,nc} = \frac{1}{\beta_1(1 - S_{A1,t,nc})} + CA_1 \ldots\ldots\ldots\ldots\ldots\ldots(17a)$$

$$PB^*_{1,nc} = \frac{1}{\gamma_1(1 - S_{B1,t,nc})} + CB_1 \ldots\ldots\ldots\ldots\ldots\ldots(17b)$$

Comparing the equation (17a) with (20a) we can see that equation (20a) ≥ (17a) for all $\eta_1 \geq 0$, when the market shares are assumed to be equal. Also (20b) ≥ (17b) for all $\eta_1 \geq 0$. We can then rewrite equations (20a) and (20b) as functions of the price charged in the 'nc' condition (17a and 17b) and the premium charged when contingent valuations are considered to express this result:

$$PA_1^* = PA_{1,nc}^* + \frac{\eta_1 S_{B1,t}}{\gamma_1 S_{A1,t}(1 - S_{A1,t})} \quad \text{..........(21a)}$$

$$PB_1^* = PB_{1,nc}^* + \frac{\eta_1 S_{A1,t}}{\beta_1 S_{B1,t}(1 - S_{B1,t})} \quad \text{..........(21b)}$$

Thus the premium charged when contingent valuations are considered is an increasing function of the contingent valuations.

Result 1

For a given market share, net profit will always be greater when the manufacturer prices the products utilizing the symbiotic relationship between them.

Proof

From the above analysis, we can conclude that, when the market shares are the same, the profit obtained under the 'nc' condition will be less than the profit obtained when the contingent valuations are considered. Thus $\pi_t \geq \pi_{t,nc}$.

One of the main issues of using the logit model in empirical analysis is the endogeneity of price with market share (Berry 1994). For example, if we consider product A1, its price PA_1* and market share $S_{A1,t}$ will be correlated. It is thus difficult to predict whether market share drives the price or vice versa. Berry (1994) suggests that when estimating the equations with data, the researcher should make use of appropriate instrumental variables so as to overcome this endogeneity problem. However, as we are deriving an analytical model for the optimal prices for the Firm, we follow the procedure employed by Aydin and Ryan (Working Paper). They simultaneously solve the market share and price equation, to derive the expression for price, which is independent of the market share drivers. Following this approach we solve for PA_1* and $S_{A1,t}$, so as to make the price equation of product A_1 dependent on only the market share of product B_1 and η_1. Similarly we solve for PB_1* and $S_{B1,t}$ so as to make the price equation of B_1 dependent on only the market share of A_1 and η_1. Solving for PA_1^* and PB_1^* we get the following expressions:[1]

$$PA_1^* = \frac{\left[LambertW\left[\frac{(1-\eta_1 S_{B1,t})\beta_1 e^{\left(\frac{\eta_1 S_{B1,t}\beta_1}{\gamma_1} - (1-CA_1)\beta_1 + \eta_1\right)}}{\left(1+e^{(\beta_2 PA_2 \eta_2)}\right)\gamma_1} \right] - \left[\frac{\eta_1 S_{B1,t}\beta_1}{\gamma_1}\right] + CA_1 - 1 \right]}{\beta_1}$$

….(22a)[2, 3]

$$PB_1^* = \frac{\left[LambertW\left[\frac{(1-\eta_1 S_{A1,t})\gamma_1 e^{\left(\frac{\eta_1 S_{A1,t}\gamma_1}{\beta_1} - (1-CB_1)\gamma_1 + \eta_1\right)}}{\left(1+e^{(\gamma_2 PB_2 \eta_2)}\right)\beta_1} \right] - \left[\frac{\eta_1 S_{A1,t}\gamma_1}{\beta_1}\right] + CB_1 - 1 \right]}{\gamma_1}$$

……(22b)[2,3]

---

[1] Solving 21a and 21b simultaneously eliminates the endogeneity between the firm's own price and own market share. However endogeneity between the competing firms' prices still remains.

[2] The Lambert W function is the inverse of the function given by $f(x) = xe^x$, where W is the function that satisfies $W(x)e^{W(x)} = x$ for all real values of x. The Lambert W function has a concave shape (Chapeau-Blondeau, F. and Monir, A 2002; Corless et.al. 1996).

[3] Equation (22a) and (22b) are now independent of the effects of the 'own market share' of the product. Hence it is possible to isolate the effects of the contingent valuations on the price without the possibility of price endogeneity bias due to the relation between price and market share, i.e., changes in price due to changes in η_1 will have no effect on the own market share of the product. In Equation (20a) and (20b) it would not have been possible to isolate the effect of the contingent valuation on price alone due to the presence of the market share of the product.

RESULT 2

Concluding from proposition 1 we know that PA_1* and PB_1* will always be greater than $PA^*_{1,nc}$ and $PB^*_{1,nc}$ for all $\eta_1 > 0$. Thus when $\eta_1 > 0$, the premium charged above the 'nc' price will increase monotonically with respect to the contingent valuation η_1, for values between $0 \leq \eta_1 \leq 0.45$. For values between $0.45 < \eta_1 \leq 1$, the premiums decrease monotonically with respect to the contingent valuation η_1. Hence premiums always increase for A_1 are B_1 when $0 \leq \eta_1 \leq 0.45$ but decrease slightly when $0.45 < \eta_1 \leq 1$.

COMMENT

The variation of PA_1* with respect to η_1 is given in Figure 6. It is evident from the graph that the price of A_1 varies with respect to the contingent valuation. This result is also applicable to the relationship between the price of B_1 and η_1.

For values $0 \leq \eta_1 \leq 0.45$, valuation of a pair of complements ($\eta_1 > 0$) exceed those for independently valued products ($\eta_1 = 0$), the seller thus gains more by charging higher prices while stimulating the consumers to buy both the products. Venkatesh and Kamakura (2003) also obtained the same results. However as seen in our results (Figure 2a), when η_1 lies between (0.45,1) the optimal price decreases slightly. This result is different from the results obtained by Venkatesh and Kamakura (2003). The main reason for this difference is that they did not consider interdependent utilities and competitive effects.

When η_1 increases from 0.45 to 1, the Firm's products become tied to each other and hence they cannot be used individually. Thus if η_2 is low, and η_1 increases the consumers will be locked to Firm 1. When the consumer is locked to a Firm, all three marketing strategies (pure components, pure bundling and mixed bundling) are equivalent to pure bundling (Matutes and Regibeau 1992). Hence the price charged for each individual item would be equivalent to price charged if the products were sold as a bundle. As the price charged for a bundle is usually lower than when the two products are sold separately, we would expect a decline in the premium that would be charged.A very high degree of complementarity between the two products manufactured by the same Firm implies that the products cannot be used individually and hence the consumer gets no utility from purchasing just one of the products. This might explain the reason why we see that the optimal price starts decreasing as $\eta_1 \rightarrow (0.45, 1)$. However it is important to note that even thought the premium decreases slightly, it is still higher than if the products were priced in the 'nc' condition.

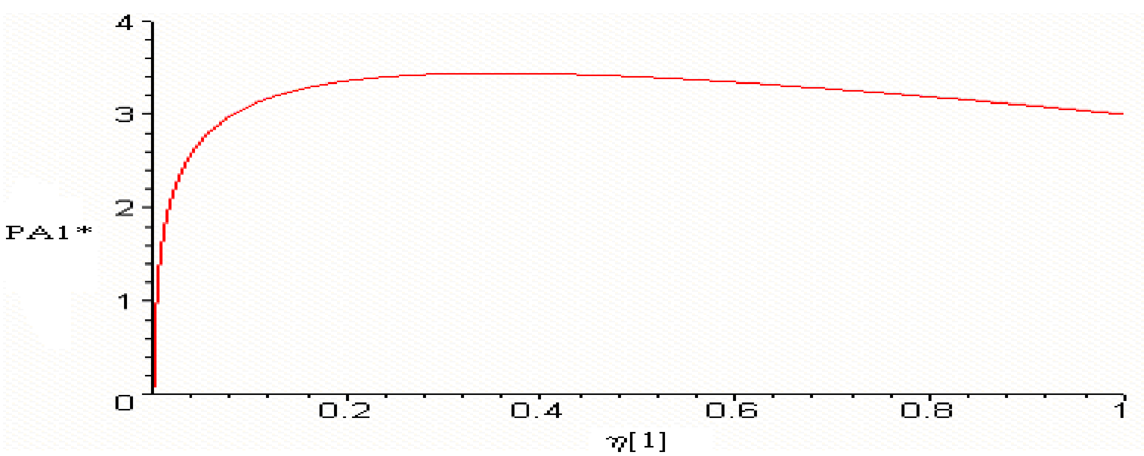

Figure 6: Variation of PA_1 with respect to η_1.

RESULT 3

PA_1* and PB_1* increase monotonically with respect to the contingent valuation η_2.

COMMENT

The variation of PA_1* with respect to η_2 is given in Figure 7. It is evident from the graph that the price of A_1 increases monotonically with the contingent valuation of its competitor's products. This result is also applicable to the relationship between the price of B_1 and η_2. As η_2 increases the competitor can afford to increase its products prices without having any effect on its market share. As the competitor increases her prices, price competition between the two firms decreases, and hence Firm 1 can also increase her price slightly.

As η_2 increases the competitor's products become increasingly dependent on each other. Thus if η_1 is low, Firm 1 can capitalize on the exclusivity of the competitors products and charge a higher price for its own line of products. This could be the reason why we do not see a decrease in the premium that Firm 1 charges, even though the premium charged by Firm 2 decreases when η_2 lies in the region (0.45, 1).

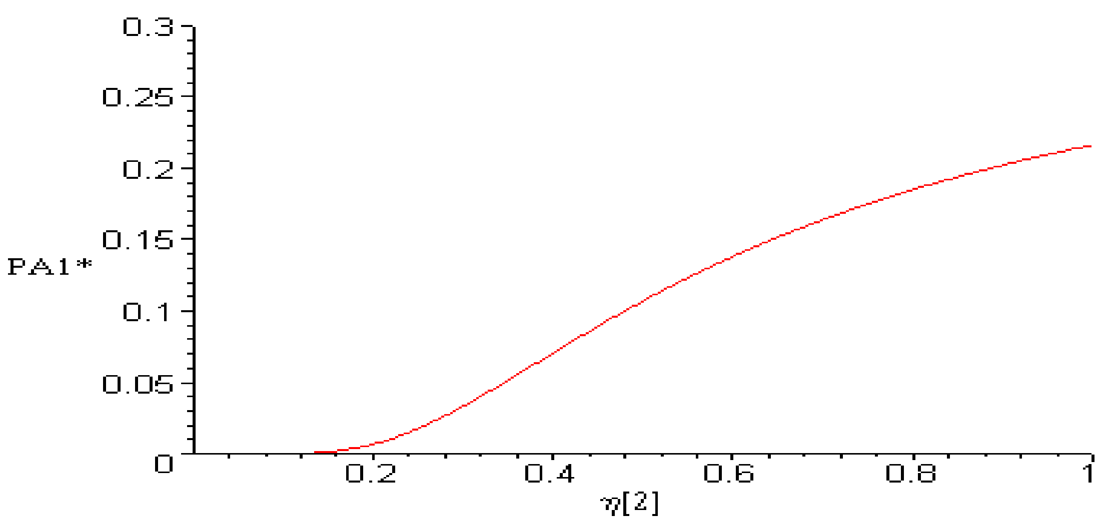

Figure 7: Variation of PA_1 with respect to η_2.

Figure 8 shows the variation of profit for Firm 1 with respect to both η_1 and η_2. We see that profit is maximum for high values of η_2 and low values of η_1. This result follows directly from result 2 and result 3.

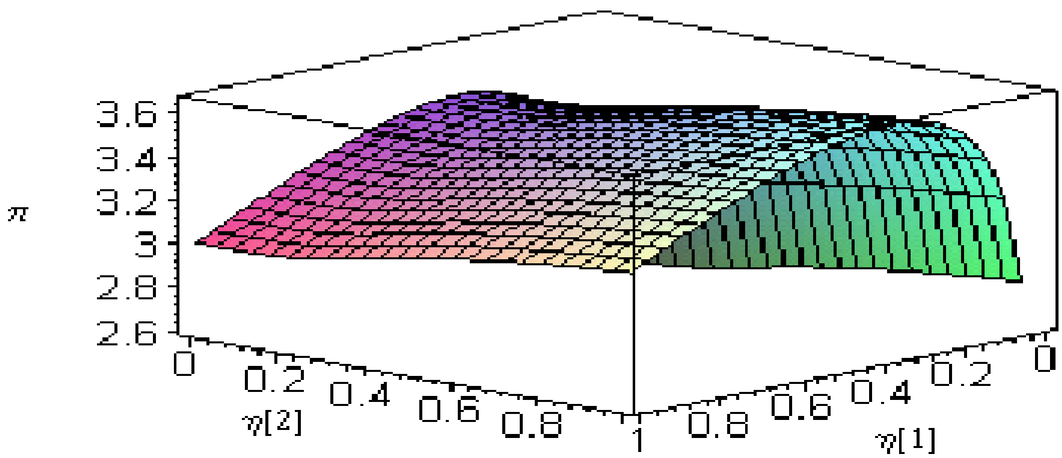

Figure 8: Variation of Profit With Respect to η_1 and η_2

3.4 Consumer Surplus

Proposition 2

Consumer surplus is lower when the Firm prices in the contingent valuation condition than when the Firm prices in the 'nc' condition.

Proof:

We derive the consumer surplus for consumer j. Assuming utility has a dollar value, the consumer surplus, when the Firm recognizes the complementary effects and charges prices accordingly, is given by,

$CS_{j,t} = u_{j,A1,t} + u_{j,B1,t} - PA_1^* - PB_1^*$............(23a)

When the Firm prices in the 'nc' condition, the consumer surplus is,

$$CS_{j,t,nc} = u_{j,A1,t} + u_{j,B1,t} - PA_{1,nc}^* - PB_{1,nc}^* \ldots\ldots\ldots\ldots(23b)$$

In equation 23b, the Firm is unaware of the complementary relationships between its two products, however the consumer enjoys the effects of complementarity between the two products. Therefore s/he is charged the price derived in the 'nc' condition. Hence (s)he would enjoy a higher utility with a lower price, thereby higher surplus. It is straightforward to see $CS_{j,t} \leq CS_{j,t,nc}$ as $PA_1^* > PA_{1,nc}^*$ and $PB_1^* > PB_{1,nc}^*$. This can also be observed pictorially in Figure 9. Consumer surplus when the manufacturer prices the product independently is given by the areas numbered (5) + (2) + (4). When the products are priced by considering the contingent value, the consumer surplus is given by the area numbered (5). Thus consumer surplus is higher when products are priced in the 'nc' condition. Thus, it is evident that consumer surplus decreases when the consumer purchases both products from the same manufacturer, who prices the products taking into account the contingent valuations.

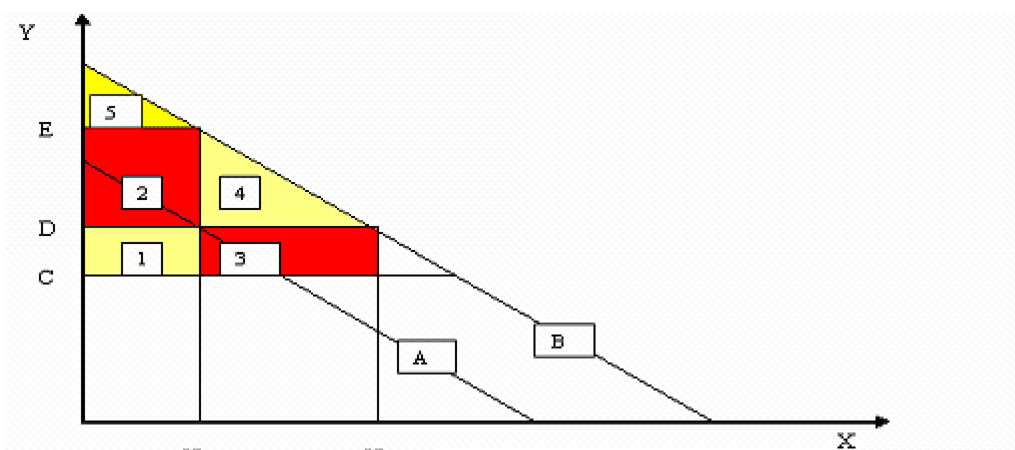

Figure 9: Consumer Surplus

X Axis: Quantity Purchased

Y Axis: Net Price Paid

A: $u_{i,A1,t,nc} + u_{i,B1,t,nc}$

B: $u_{i,A1,t} + u_{i,B1,t}$

C: Marginal cost of producing A_1 and B_1 (CA_1+CB_1)

D: Net Price Paid when the products are charged independently ($PA_{1,nc}* + PB_{1,nc}*$)

E: Net Price Paid when the products are charged considering the contingent effects $PA_1* + PB_1*$

1: Expected Profit at $PA_{1,nc}* + PB_{1,nc}*$

2: Expected Profit at $PA_1* + PB_1*$. It is also the lost profit when products are not priced as if they are related.

1.+ 3.: Net profit achievable at $PA_{1,nc}* + PB_{1,nc}*$, given that the entire market is not yet satisfied.

4: Deadweight loss when the product are increased in price from $PA_{1,nc}* + PB_{1,nc}*$ to $PA_1* + PB_1*$.

5: Consumer Surplus after pricing at $PA_1* + PB_1*$

5+2+4: Consumer Surplus when priced at $PA_{1,nc}* + PB_{1,nc}*$.

4. Conclusion And Managerial Implications

Main Contribution

This study contributes to the pricing literature by extending the paper by Venkatesh and Kamakura (2003) study to the case of competition. The study adds to the literature in the following ways:

1. Venkatesh and Kamakura (2003) show that the optimal price charged increases linearly and monotonically with respect to contingent valuations when products are sold as pure components. We show that this is not necessarily the case, while the optimal price does increase with respect to the contingent valuation the relationship is not linear or monotonically increasing.

The price first increases rapidly for small values of η and then increases at a slower rate till $\eta = 0.45$. When $0.45 < \eta < 1$, we see that the optimal price actually decreases. Thus locking the consumer to the product reduce the optimal price that the manufacturer can charge and hence would reduce the optimal profit.

2. We show that the optimal price will also increase with respect to the competitor's contingent valuation, albeit at a slower rate.

3. We also derive the surplus that the consumer would enjoy in each of the cases and show that the Firm can charge a higher price based on the contingent valuation mainly because the creation of a contingent valuation causes an increase in the overall surplus that the consumer could enjoy. The firm can thus take advantage of this increase and charge a higher price to the consumer.

4. We isolate the effect of η on the price of the product by accounting for the endogeneity of market share and price. This helps us predict the variation of price with respect to η, without the price having any effect on its own market share. This is significant because we can predict changes in price with respect to η, without the effect of market share of the product.

The analysis carried out in the previous sections have helped us establish the following results:

4. The prices of the products of Firm 1 increase with increase in the contingent valuation of the products manufactured by Firm 1.

5. The prices of the products of Firm 1 increase with increase in the contingent valuation of the products manufactured by Firm 2.

6. Firm 1 experiences a higher profit for the same market share, when its products are priced considering the contingent valuations than when they are priced independently of each other.

7. Consumers who purchase the products would have a lower level of consumer surplus if the contingent valuation is recognized by the Firm.

Managerial Implications

We have established that the Firm can charge a premium for the products. It is important for a manager to decide how to split the net premium that can be charged between the two products. Should the net premium be split equally between the two products or applied solely to one of the products?

Economides and Viard (2003) answer this question on the basis of the network externality of the products. They consider two products MS Windows and MS Office, both made by the same manufacturer. They propose that an optimal pricing scheme would be to charge high for Office and low for Windows (even though it costs Microsoft twice as much to develop Windows than it does to develop Office). They show that the network externality generated through the sale of Windows would increase the value of Office, and hence the profit lost by Windows can be recouped by Office. This is however only true of markets with strong network externality effects.

For other markets, we could use the axioms of prospect theory (Kahneman and Tversky 1979) and mental accounting (Thaler 1980,1985) to understand how to distribute the premium among the products. For example, when the consumer has to purchase a base product (e.g., printer) and then make repeated purchases of an accessory (it is assumed that the accessory is not exclusive) (e.g., ink) to use the base product, it might be better to charge a higher price for the base product. This is because the consumer would mostly relegate the price paid for the base product to the level of a sunk cost (decoupling the costs and benefits, Soman and Gourville 2001), and look to only the future price of the accessory as the basis for his/her choice of the accessory product.

Another important example is the razor and blade pricing strategy. This is a form of captive pricing, since the consumer is locked to the Firm ($\eta = 1$) once they purchase the razor.

Purchase behavior may also be affected by heterogeneity in the willingness to pay for the base product. For a light user/low end segment it might be better to lower the cost of the base product, so that the segment enters the market, and increase the cost of the accessory as they would not use as much of the accessory. This is akin to the captive pricing scenario talked about earlier. It suffers from an obvious weakness, the entry of competitors in the accessory market would cause the low (light) end (user) segment to choose the competitor. In such a situation, for the low end segment, it is better to split the premium equally between the two products.

Implications for Marketing Management

The analysis in this paper provides the following managerial conclusions:

1. Building contingent valuations is important as it helps in limiting the role of price competition in the market.

2. If the Firm accounts for the existence of contingent valuations when setting the prices of the different products, the Firm would get a higher profit as it would get some of the extra surplus that the consumer accrues from owning the two products.

Therefore it is important for Firms to ensure that its new product lines have a positive relationship to its existing range of products. To better understand how a Firm could build contingent valuations into its product line we consider the example of Apple Computer. In October 2001, Apple introduced its portable MP3 player iPod. The iPod allowed for ease of portability of each customers MP3 music collection and was considered a better option to the available MP3 players at that time due to the fact that it could hold significantly more songs that the average MP3 player and also because it had a more sophisticated design. Although MP3 music was easy to download, the practice of downloading music suffered from a multitude of legal difficulties. This did not permit iPod sales to reach their full potential.

In the beginning of 2003, several new competitors like Dell began entering the market and selling their own MP3 players at significantly lower prices. Thus Apple now faced two challenges, one from the legal problems of downloaded MP3 music, and the second challenge was a price war with the new competitors.

As a means to overcome this problem, Apple introduced its iTunes online music download service. This service allowed consumers to download music both legally and at a very low price. However Apple ensured exclusivity of the downloaded music, as it was not in the MP3 format (Kanellos 2004, CNET News). This ensured that the downloaded music could be played only on the iPod (which was played both the MP3 and the AAC format, the AAc format was copyrighted by Apple) and not on competing MP3 players. As the iTunes service allowed for easy and legal downloads, consumers increasingly turned to iTunes as a source of music. This simultaneously increased demand for the iPod, as it allowed for portability of the consumer's new music collection. Similarly consumers who purchased the iPod found iTunes to be a convenient source for music for their iPod player. Thus consumers derived a higher level of satisfaction when they purchased both the iPod and the iTunes service as opposed to purchasing only one of them. This allowed Apple to charge a higher price for its iPod player. Apple also escaped a future price competition that would have ensued with the introduction of new MP3 players by its competitors like Dell (Dalrymple, Technology Business Research 2003). Hence the introduction of the iTunes service created a contingent valuation between iTunes and iPod and this allowed Apple to gain higher market share without sacrificing profits or bundling the two products together. After introducing iTunes, iPod sales were up 235% (Hasseldahl, Forbes 2004).

5. Future Research

The procedure employed in this paper accounts for and solves the econometric endogeneity that is present between each firm's own price and own marketshare. However, there is an extra level of endogeneity, namely between the focal firm's price and the competitor's price that needs to be considered when deriving equilibrium prices.

To keep the model tractable we did not consider heterogeneity of the contingent valuation. In reality, the contingent valuations could vary across consumers, hence it is important to account for it. However there would be no closed form solution when accounting for the heterogeneity and hence one must develop simulation methods when trying to derive optimal pricing schemes accounting for differences in η_i. We could also consider the case when $\eta_{A,B} \neq \eta_{A,B}$, however we do not expect the results to be qualitatively different.

Chapter 3: Essay 3 - Modeling Loyalty For Better Customer Relationship Management

Summary

CRM (Customer Relationship Management) interventions like direct mailings have long been used by firms to improve customer relationships. In this study we develop a method that will allow the firm to understand the effect of these interventions on customer loyalty. Loyalty is assumed to be unobserved and hence is modeled as a latent variable. We use an adaptation of a generalization of the 'Hidden Markov Model' (HMM) called the 'State Space Model' (SSM) to better predict a customer's loyalty function towards a particular firm or product. The SSM models are structurally different from HMM models and they offer three main advantages over HMM models. First they are continuous and are described across all possible relationship states of the customer, hence we avoid the problem of explicitly choosing the number of states; second, they can be used to model an infinite number of relationship states; and third they are better at modeling recursive behavior, which is necessary when modeling customer behavior that involves the effect of experience. We adapt the SSM model to our study by combining the generic model with a set of covariates that we use to better understand customer loyalty. We call this model the SSMC (State Space Model with Covariates). We also predict the customer's probability of purchase given certain marketing actions and the predicted loyalty state using a hazard model. We combine the hazard model and the SSMC to predict the customer's probability of purchase at a given loyalty state. We apply this model to data from a retailer of health and beauty aids, to help them better understand the effect of their CRM interventions on the customer's loyalty towards the firm and their repurchase intentions. We also point out the types of CRM interventions that play a role in improving the loyalty of the customer to the firm and those interventions that have no effect. This information can hence help the firm better organize its menu of CRM interventions. We can also compute the probability distributions across the loyalty states for each individual customer, thus providing the researcher with knowledge of each customer's loyalty state. Finally we introduce a new methodology to the literature on modeling relationships in marketing. The methodology improves upon existing methods by allowing for a more flexible and efficient estimation procedure. We also compare our model's predictions to those derived from two other estimation methods. We find that the predictions derived from our estimation procedure are better than those computed from the other methods discussed.

1. Introduction

Customer relationship management (CRM) is a firm-wide approach to understanding and influencing customer acquisition, customer retention and customer value through interactive and relevant information exchange between the firm and the customer. The past few years have seen a multitude of research in the field of CRM, exploring several important facets such as selecting the right candidates from a mailing list and then selecting the appropriate communication or marketing intervention for each of the selected candidates (Elsner, Krafft and Huchzermeier 2004; Venkatesan and Kumar 2004; Rust and Verhoef 2005; Gonul and Shi 1998; Bitran and Mondschein 1996; Bult and Wansbeck 1995; Banslaben 1992; Roberts and Berger 1989; Kass 1976; Sonquist 1970), studying the link between satisfaction and commercial success (Kamakura et al. 2002), the link between customer loyalty and profitability (Rust, Lemon and Zeithaml 2004; Reinartz and Kumar 2000), customer profitability heterogeneity (Niraj, Gupta and Narasimhan 2001), customer loyalty programs (Verhoef 2003) and towards establishing a sound construct of the CRM process (Reinartz, Krafft and Hoyer 2004). In this essay we develop a methodology that actually estimates a customer's loyalty state and allows one to predict the effect of a customer's loyalty on their intention to repurchase.

Previous marketing literature defines loyalty as being either psychological or behavioral in nature. Psychological loyalty considers the underlying motivation of the consumer to repurchase the same brand. It is based on the attitudes of the customer to the brand in question (Jacoby and Chestnut 1978). Behavioral loyalty on the other hand depends on the purchase patterns of the customer. It is hence defined by the revealed behavior of the customer (Fader and Hardie 1996). Our model derives estimates of loyalty that indicate a tendency to engage in behavior that results in a purchase. This is similar to psychological loyalty as described in the literature. We then predict the probability of repurchase, thereby linking psychological loyalty to the revealed behavior of the customer, i.e. the customer's behavioral loyalty. We carry out the tests in the following way. First we test whether a firm's CRM interventions have an effect on the customer's psychological loyalty towards the firm's products and estimate the customer's loyalty states. We then check whether loyalty affects the probability that a customer will repurchase a product from the firm.

We conduct the above tests using an adaptation of state space models. We combine the state space model with a set of covariates that are used to predict loyalty. These models, continuous generalizations of hidden Markov models, allow one to develop a continuous loyalty function that can help the firm better understand the customer and also allow the firm to understand the efficacy of their marketing interventions. State space models supersede hidden Markov models on two counts:

- In the hidden Markov model the researcher would have to make assumptions about the possible number of loyalty states, while in the state space model no such assumptions have to be made due to the continuous nature of the function over all possible states
- The second advantage lies in the fact that the hidden Markov model assumes loyalty to be a discrete variable, which is a simplifying approximation. The state space model allows us to relax this assumption.

Hence using state space models, the firm will learn which CRM interventions actually have an effect on customer loyalty and hence can better organize their interventions to take advantage of this new insight. The other advantage of this methodology lies in the fact that one can even use discrete data to obtain a continuous loyalty function.

The rest of the paper is organized as follows. We review the existing CRM models in §2 and point out how our paper improves upon the current literature. Model development is carried out in §3. We describe the data and estimation procedure in §4 and present results in §5. Finally in §6 we provide our conclusions.

2. CRM Models

CRM has become increasingly important in marketing. There are now many different ways to manage customer relationships in marketing. Direct mail, which was once synonymous with customer relationship management studies, is now complemented by a host of other techniques like email marketing and other advanced procedures for customer targeting etc., which are now increasingly used by retailers and manufacturers to manage customer relationships. Hence the efficient management of these customer interventions forms the essence of customer relationship management. Towards the goal of efficient management, there are many factors that play an important role, for example the timing of the marketing intervention, the frequency of purchase, the monetary value of the purchase, the customer characteristics etc. (Nash 1984). There are many models proposed in marketing to help firms manage customer relationships by choosing both the right customers and the right market intervention scheme.

One of the first CRM models was the RFM model (recency, frequency and monetary value) (Bitran and Mondschein 1996; Roberts and Berger 1989). Recency refers to the time since last purchase or the number of mailings since last purchase. Frequency refers to the number of purchases in a given period of time and monetary value refers to the monetary value of all purchases in a given period of time. Gonul and Shi (1998) extend the RFM to include a dynamic component and analyze the key determinants of an optimal mailing policy, while simultaneously maximizing both customer utility and firm profit. The authors discuss that due to its dynamic nature the model outperforms its single period counterparts. The model proposed in this paper incorporates the RFM variables and also includes a dynamic component.

There are several other models that exist in the literature with an aim of improving the CRM intervention policy. These include AID and CH-AID (Sonquist 1970, Kass 1976); the ‘Gains Chart Analysis’ (Banslaben 1992); finite mixture models (Bult and Wansbeck 1995); the DMLM Procedure (Elsner, Krafft and Huchzermeier 2004) which determines the optimal frequency size and customer segmentation of direct marketing activities; customer lifetime value models (Dwyer 1989, Blattberg and Deighton 1996, Berger and Nasr 1998) which used the customer’s lifetime value as a guide towards developing a better CRM intervention schedule.

More recently Rust, Lemon and Zeithaml (2004) develop an approach that allows firms to achieve financial accountability by considering the effect of strategic marketing expenditures and by relating the improvements in customer equity to the expenditure required to achieve it. This thus allows firms to distinguish between customers and hence concentrate on its most profitable customers. Venkatesan and Kumar (2004) also use a CLV metric for customer selection and marketing resource allocation by developing a dynamic framework that enables managers to maintain and improve customer relationships proactively through marketing contacts across various channels and maximize CLV simultaneously. The authors show that customers who are selected on the basis of their lifetime value provide higher profits in the future than do customers selected on the basis of several other customer based metrics.

Apart from the methods discussed above there are several other procedures that allow firms to better manage customer relationships. For example Ansari and Mela (2003) develop a model to show that (in the case of an email manufacturer) it is better to customize the email marketing interventions to each individual customer's taste as this would increase the number of click throughs and hence increase revenue for the firm. As previously discussed the customer-centric approach in marketing lays great emphasis on the calculation of customer lifetime value (CLV) which is defined as the value of future cash flows associated with a customer (Pfeifer, Haskins and Conroy 2005). CLV measures focus on the future and not on the past. In contrast, the direct marketing literature use measures of customer’s prior behavior to predict their future behavior. This is best summarized through the RFM (Recency Frequency Monetary value) models.

Fader, Hardie and Lee (2005a) by the means of a stochastic model integrate the RFM paradigm with CLV. They use ‘iso-value’ curves to illustrate the interactions between RFM measures and CLV and discuss the practical application of the model. Their approach proposes using the RFM variables as sufficient variables for an individual customer’s purchasing history and eliminating the need for additional data to calculate CLV. Additionally Fader, Hardie and Lee (2005b) develop the beta-geometric/NBD (BG/NBD) as an alternative model to the Pareto/NBD model developed by Schmittlein et al. (1987). The BG/NBD model allows for easier estimation of parameters as compared to the Pareto/NBD model.

The model is used in cases where predicting the future purchase of the customer is important to managers as they are interested in estimating the customer's lifetime value to the firm.

In our study we are interested in learning about the psychological loyalty of the customer to the firm. As we do not have any data on the measures of psychological loyalty of the customer, we need to incorporate loyalty as a latent variable. Additionally we also model the effect of the CRM interventions on the transition of the customer across the different loyalty states. Past research (Rust, Zeithaml and Lemon 2000) used Markov Chain Models to study customer relationships, migration and retention scenarios. More recently Netzer, Lattin and Srinivasan (2005) model the dynamics of customer relationships using transaction data using Hidden Markov models (HMM). Their model of relationship dynamics incorporates the idea that customer encounters may have an enduring impact by shifting the customer from one unobservable relationship state to another. The hidden Markov model (HMM) allows for transitions among latent relationship states and effects on buying behavior. The dynamics of customer relationships with firms are more or less continuous, owing not just to purchase encounters with the firm, but also with competing firms, CRM interventions etc. Hence, we extend this methodology by employing a state space model, a method that allows us to model relationships as a continuous function, as a more efficient approach towards studying customer relationships. We elaborate on the advantages of our method over the HMM in the next section.

Finally, Rust and Verhoef (2005) propose a hierarchical model to manage customer relationships that individualize rather than segmentize the population of customers. Their results show that customers are highly heterogeneous in their responses to marketing interventions. As we conduct our analysis at the level of the individual customer, we use a hierarchical model to study the customer relationships and interactions between the customer and the firm.

3. Model Development

We will provide a short description of the scenario that will be considered in this paper. The case is of a multi-product customer goods company that wants to sell products to its customers through its own line of stores. The company sends out a variety of marketing interventions (email, snail mail, etc.) periodically. The aim of our study is to help the firm understand which CRM interventions strengthen the loyalty of the customer towards the brand and in turn study the impact of loyalty on a customer's intention to repurchase from the firm. We do so by predicting the customer's probability of purchase at a given point in time, their loyalty state at a given time and the likelihood that a customer will make a purchase at a given loyalty state.

The model described here is one of individual level buying behavior. We consider a panel of customers and their repeated interactions with the firm. The data that we use is the typical transaction data that is commonly used in various models of customer choice. Hence the manufacturer observes not only each customer's choice history but also the marketing environment at that time. This information will help the manufacturer better understand the relationship between the customer and the firm. The relationship is comprised of a longitudinal sequence of encounters, each of which contain information including, but not limited, to purchases made, whether mailings were sent, the recency and frequency of purchases etc.

Additionally we define a series of hidden or latent loyalty states for each customer. These loyalty states are indicative of a tendency of the customer to engage in behavior that results in a purchase. The transitions between these states are probabilistically determined and are affected by each relationship encounter. To date hidden Markov models (HMM) have been used to identify these states and predict transitions between these states. Hidden Markov models are discrete models. They have a finite number of different internal states that produce different kinds of outputs. Typically there are a couple of states for each encounter or a pair of encounters. The whole dynamical process of producing a relationship function is thus modeled by discrete transitions between the states corresponding to the different encounters.

Prior research, as mentioned above, has looked at psychological loyalty as comprising of discrete states that the customers transition between. While it is useful to consider loyalty as a discrete variable, it is a simplifying approximation. The approach we develop in this paper enables us to relax this approximation in a computationally feasible way. Psychological loyalty is modeled in this paper as a variable that is continuous across all possible loyalty states. We elaborate on this in the next section.

Additionally, the dynamics of customer relationships with firms are more or less continuous, owing to not just purchase encounters with the firm, but also with competing firms, CRM interventions etc. This is of importance to a manager since customer interactions with the firm are no longer confined to just purchases. They now also include the firm's own CRM interventions as well as their competitors' interventions, which subsequently play a significant role in determining a customer's future purchases. For example, an individual who receives a CRM intervention after making a purchase might be influenced positively by the mailing even if it does not result in an immediate purchase. Therefore, it is important to understand how these loyalty or relationship states evolve continuously over time with respect to the CRM interventions. It would be more beneficial to model the data with a continuous model, as the frequency with which CRM interventions are sent out is typically much higher than the frequency of purchase. Loyalty to a firm is hence assumed to be a continuous function and a good candidate for the task of modeling these loyalty transitions is a state space model (SSM).

The SSM can be described as the continuous counterpart of the HMM. SSMs are a general method for the probabilistic modeling of sequences and time-series. They take the form of iterated maps on continuous state-spaces, and can have either discrete or continuous valued output functions. They are basically generalizations of the better known state-space models such as Hidden Markov models (HMMs). A SSM is, however, more powerful than a HMM. For example, a SSM can represent infinitely many distinct states as a consequence of their real-valued state-spaces (Moore 1990, Olivier Bournez 1996). By contrast, HMMs have a finite number of states and hence can be no more powerful than strictly finite-automata. Finite-automata cannot model many of the recursive structures found in human behavior, especially instances where learning or the effect of experience is involved. This occurs because a researcher is forced into organizing such behavior into a predetermined finite number of states, a step that is not necessary when using a SSM. A SSM can be expressed across all possible real valued states and transitions between these states are not confined to a predetermined number. Additionally the estimation of a hidden Markov model becomes much more inefficient as the number of states becomes very large (a situation encountered when modeling recursive behavior), a problem that is avoided in state space models due to their continuous nature.

If we assume x(t) is the observed data, and s(t) is the collection of internal hidden states of the dynamical system then a standard/generic SSM can be expressed as follows:

$$\begin{aligned} s(t+1) &= g(s(t)) + m(t) \\ x(t) &= f(s(t)) + n(t) \end{aligned} \tag{1}$$

Both vectors m(t) and n(t) are the noise components of the two equations with m(t) being the process noise and n(t) the observation noise. Functions f and g are the linear or nonlinear mappings, with f being the observation mapping and g the process mapping. Figure 10 provides a visual interpretation of the process involved in a standard SSM.

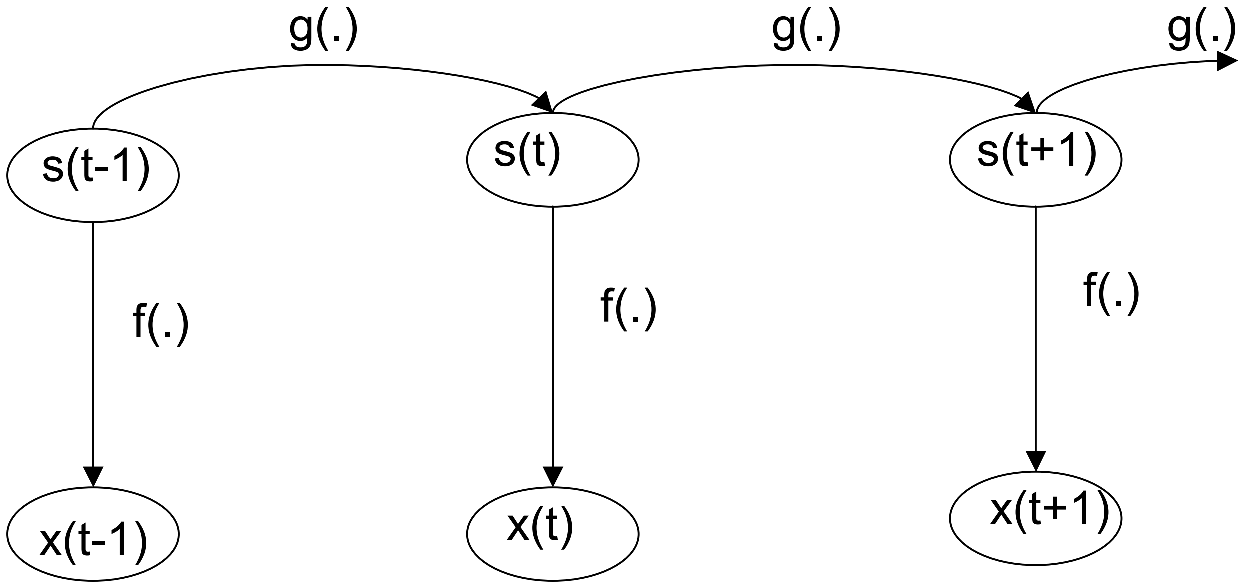

Figure 10: A standard State Space Model

3.1 Modeling the Customer's Probability of Purchase

To model customer loyalty and likelihood of purchase we construct a model to predict the customer's probability of re-purchase. We are interested in examining the effect of mailings and other variables on the probability of repurchase. At the same time we need to control for the time between two consecutive purchases made by the same household. A model that incorporates both these features (i.e. intrinsic purchase patterns over time and the effect of marketing variables) is the hazard model. The hazard model allows us to model the survival of the customer through the subsequent periods of interest, at the same time controlling for the effects of the marketing variables that are important to manufacturers, like the effect of CRM interventions on repurchase.(Gonul and Srinivasan 1993, Vilcassim & Jain 1991). The customer is assumed to survive as long as s/he does not make a purchase and the customer dies when s/he repurchases from the firm in a future period. The hazard model captures this intrinsic propensity of the customer to repurchase in a computationally feasible way.

Before defining the probability of purchase we first model the loyalty states of the customer. There are two approaches to loyalty considered in the literature. The first being a psychological approach towards loyalty, as was proposed by Jacoby and Chestnut (1978). The approach distinguishes itself from the behavioral approach (i.e. repeat purchases of a particular product in a given period of time) by including the attitudinal aspects of loyalty like the cognitive (the brand is preferable to competitive offerings), affective (preferential attitude for the brand) and conative (higher intention to buy the brand as compared to the alternatives) elements (Oliver 1999). For example loyalty could be viewed as a favorable set of stated beliefs towards the brand purchased. These attitudes can be gauged by asking how much people like the brand, feel attached to it, will recommend it to others, and have positive beliefs and feelings about it – relative to some other competing brands (Reichheld 1996, Dick and Basu, 1994). This then translates into behaviors that result in repurchase. The tendency to repeatedly purchase the same brand leads to behavioral loyalty. Thus behavioral loyalty is the propensity to repurchase from the same firm in future time periods and is defined mainly on the pattern of past purchases. (Fader and Hardie 1996, Ehrenberg and Scriven 1999).

The approach we develop in our study estimates the hidden loyalty states that indicate a tendency to engage in behavior that results in a purchase. Hence the loyalty states we derive are similar to the psychological loyalty as described in the literature. Our study relates the psychological loyalty of the customer to the behavioral outcomes explained above. We construct a model to estimate the psychological loyalty of the customer to the firm and then relate this estimated psychological loyalty of the customer to the behavioral loyalty that the customer exhibits in the form of a repurchase. Thus the framework developed in this paper relates the two loyalties, one observed (behavioral loyalty) and the other unobserved (psychological loyalty).

We assume that there is a set of latent psychological loyalty states that influence behavior. The loyalty states can range from the customer being completely disloyal to the brand to one in which the customer is completely loyal to the brand. At a given point in time, each customer is assumed to occupy a particular loyalty state. A customer's loyalty state can be affected by several variables like satisfaction, the purchase experience, affinity to a competitor's product, marketing activities of the focal firm, quantity of the brand purchased etc. If the impact is positive it can cause the customer to transition to a loyalty state that is more loyal than the one s/he was previously in. On the other hand a negative impact can cause the customer to transition to a less loyal state.

We divide the time a customer is in each loyalty state into intervals of 3 months. Each quarter is represented by τ, where $\tau = 1$ in quarter 1. At time τ, the loyalty state for each individual 'i' is represented by $l_i(\tau)$, and the distribution of these loyalty states within each individual, is represented by $p_i(l_i(\tau))$. As we are interested in estimating $p_i(l_i(\tau))$ using a Bayesian approach, we assume a prior distribution across each individual's set of loyalty states at $\tau = 1$, denoted by $p_i(l_i(1))$. The distribution for loyalty varies along the real number line from $-\infty$ to $+\infty$. If the loyalty state has a value of $-\infty$ then the customer would be completely disloyal. At the value of 0, the customer is neither loyal nor disloyal to the brand and is more likely to engage in switching behavior, as (s)he is indifferent to the experience of using the brand. At this stage the customer behaves similar to a switcher. At $+\infty$ the customer would be completely psychologically loyal to the brand.

We will now proceed to model the customer's probability of purchase. The inter-purchase times in the equation for probability of purchase is given by the number of days since purchase. Here time is indicated by 't' and is calibrated at the daily level. The probability that customer 'i' chooses to purchase product j at time t is given by $h^{i}_{j,t}$. Let $\gamma_{it}(l_i(\tau))$ be the baseline hazard function for customer i at time t and loyalty state $l_i(\tau)$. β_{ij} is a vector of response parameters for customer i at time t for product j. We assume a prior distribution for β_{ij} given by $p_i(\beta_i)$. X_{ijt} is a row vector of covariates, which include customer attitudes and behaviors towards the brand that could have an impact on the probability of purchase. Some examples of such variables that affect the utility of a purchase and hence the probability of purchase include satisfaction, quantity purchased, price sensitivity, number of repeat purchases, distance to the store, interaction with frontline employees, trust and the variety of product offerings (Agustin and Singh 2005; Sirdeshmukh, Singh and Sabol 2002; Oliver 1999, 1997). The covariates also include marketing variables that are under the researcher's control.

Loyalty is also a covariate in the equation for the probability of purchase and is represented by $l_i(\tau)$. Oliver (1999 & 1997) highlights the significant role of loyalty in understanding purchase behavior. He states that the psychological elements, as defined earlier that constitute our definition of loyalty can lead the customer to repurchase the brand. At a higher loyalty state a customer's intention to make a purchase becomes stronger. Hence, it is important to include loyalty as a covariate in the purchase equation. α_{ij} is a measure of how much loyalty affects the probability of purchase. We assume a prior for α_{ij} given by $p_i(\alpha_i)$.

The utility that the customer will derive from purchasing the particular brand directly impacts the probability of purchase. Brands with a higher utility will have a higher probability of purchase. As both X_{ijt} and $l_i(\tau)$ impact the probability of purchase and hence the utility, we include them as elements in the utility function (Heilman, Bowman and Wright 2000, Krishnamurthi and Raj 1991). We assume that the utility that a customer i obtains from purchasing product j is given by

$$U_{ijt} = X_{ijt}\beta_{ij} + l_i(\tau)\alpha_{ij} + \varepsilon_{ijt} \tag{2}$$

ε_{ijt} is assumed to follow an iid type-1 extreme value distribution.

Therefore, the probability that customer 'i' purchases the product 'j' at time 't' in a particular loyalty state is given by:

$$h^i_{j,t}(l_i(\tau), X_{ijt}, \beta_{ij}, \alpha_{ij}) = \frac{e^{\gamma_{it}(l_i(\tau))} \cdot e^{(X_{ijt}\beta_{ij} + l_i(\tau)\alpha_{ij})}}{1 + e^{\gamma_{it}(l_i(\tau))} \cdot e^{(X_{ijt}\beta_{ij} + l_i(\tau)\alpha_{ij})}} \tag{3}$$

We assume a general functional form for the baseline hazard as given by (Vilcassim & Jain 1991):

$$\gamma_{it}(l_i(\tau)) = \gamma_{it}^0(l_i(\tau)) + \gamma_{it}^1(l_i(\tau)) \times \Delta t + \gamma_{it}^2(l_i(\tau)) \times \Delta t^2 + \gamma_{it}^3(l_i(\tau)) \times \ln \Delta t \quad (4)$$

where Δt is the time elapsed since the product was purchased. The baseline hazard function is a very general specification that nests most of the commonly used probability distributions for inter-purchase times. Vilcassim & Jain (1991) show that setting certain γ's to zero allows the function to assume various distributions ranging from the Weibull, Exponential and the second order power series approximation of an Erlang-2 type distribution.

We now proceed to model the likelihood function for the hazard model, which is given by:

$$L_{iH}(Y_{ijt} \mid X_{ijt}, l_i(\tau), \beta_{ij}, \alpha_{ij}) = \prod_t (h_{j,t}^i(X_{ijt}, l_i(\tau), \beta_{ij}, \alpha_{ij}))^{\delta_v} (1 - h_{j,t}^i(X_{ijt}, l_i(\tau), \beta_{ij}, \alpha_{ij}))^{1-\delta_v}$$

(5)

where Y_{ijt} is the choice made by the customer and $\delta_v = 1$ if a purchase occurred at time t, and 0 otherwise.

3.2 Modeling Loyalty Transitions

The second step in our analysis is to model the transitions of customer loyalty. The transitions are allowed to occur in either direction, i.e. in the direction of increasing or decreasing loyalty. In the state space equation $l_i(\tau+1)$ is the loyalty state in quarter τ +1, $l_i(\tau)$ is the loyalty state in quarter τ, and $\eta_i(\tau)$ is the error term, assumed to be distributed N(0,Ω), where Ω is the variance of the prior distribution. We know from previous research (Johnson, Herrmann and Huber 2006, Yi and Jeon 2003 and Oliver 1997) that the loyalty function is also dependant on customer attitudes and behaviors towards the brand as well as firm level marketing variables, given by $X_{ij\tau}$, which could have an impact on the probability of purchase. ψ_i measures the effect of $X_{ij\tau}$ on the transition of loyalty from a state at time t to a state at time τ +1. We also assume a prior distribution for ψ_i given by $p_i(\psi_i)$. To capture the evolution of loyalty from one time period to another, we need to establish a link between loyalties across adjacent time periods. Hence we use a state space equation that allows us to model this link across adjacent time periods. We adapt the SSM model to our study by combining the generic model with the above mentioned set of covariates that we use to better understand customer loyalty. We call this model the SSMC (State Space Model with Covariates).

The state space equation is then defined as follows:

$$l_i(\tau+1) = l_i(\tau) + X_{ij\tau}\psi_i + \eta_i(\tau) \qquad (6),$$

The likelihood function can then be written in the following manner,

$$L_i(l_i(1),\ l_i(2),\ldots..,l_i(\mathrm{T})|\ \psi_i, X_{ij\tau}) = P_i(l_i(1))\prod_{\tau=1}^{T-1} P_i(l_i(\tau+1)\,|\,l_i(\tau),\psi_i, X_{ij\tau}) \qquad (7)$$

Hence the combined likelihood of the system of equations to predict both the probability of purchase and the loyalty state is given by multiplying equations (5) and (7):

$$L_{i,l_i(\tau),h} = L_{iH}(Y_{ijt}\,|\,X_{ijt}, l_i(\tau), \beta_{ij}, \alpha_{ij}) \cdot L_i(l_i(1), l_i(2),\ldots..,l_i(T)\,|\,\psi_i, X_{ij\tau}) \qquad (8)$$

The likelihood function given in equation (7) has three components, namely, the initial state distribution, the transitions and the state dependent choice.

3.3 The Bayesian Framework

Combining the likelihood function given by equation (7) with the priors on β_i given by $p_i(\beta_i)$, the prior on α_i given by $p_i(\alpha_i)$, the prior on ψ_i given by $p_i(\psi_i)$ and the prior on $l_i(\tau)$, the loyalty state, given by $p_i(l_i(\tau))$, we obtain, by Bayes rule the joint posterior distribution for the hazard model, which is given by:

$$p_i(\beta_i, l_i(\tau), \alpha_i, \psi_i \mid \text{data}) \propto p_i(\beta_i) \cdot p_i(\alpha_i) p_i(\psi_i) \cdot p_i(l_i(\tau)) \cdot L_{i, l_i(\tau), h} \quad (9)$$

The priors for β_i, α_i, ψ_i and $l_i(\tau)$ vary across customers as follows:

$$p_i(\beta_i) \sim MVN(\mu, \zeta); \quad (10)$$

$$p_i(\alpha_i) \sim N(\omega, \chi); \quad (11)$$

$$p_i(\psi_i) \sim MVN(\pi, \xi); \quad (12)$$

$$\mu \sim MVN(0,1.0); \ \zeta \sim Gamma(1.0,1.0); \quad (13)$$

$$\omega \sim N(0,1.0); \ \chi \sim Gamma(1.0,1.0); \quad (14)$$

$$\pi \sim MVN(0,1.0); \ \xi \sim Gamma(1.0,1.0); \quad (15)$$

We draw the initial state prior for loyalty from a population wide distribution of loyalty. We assume that the mean of this distribution is 0 and the variance is 100I (where 'I' is the identity matrix) as we have no specific information about the variance of the distribution, implying an uninformative prior (uninformative due to the high numerical value of the variance).

$$p_i(l_i(1)) \sim N(0,100I) \quad (16)$$

4. Data and Estimation Procedure

We obtained data from a leading manufacturer of beauty products. The products manufactured by the firm are sold through its own retail outlets. The dataset is divided into two parts: a calibration dataset (also referred to simply as the dataset) and a validation dataset. The calibration dataset consists of purchases of 250,717 customers over a period of 18 months. The validation dataset consists of purchase data on the same panel of customers who made purchases over the next 6 months. The firm has provided data on several variables including the date the customer was first included in the database, the dates the purchases were made, the amount spent per purchase, the different types of mailings that were sent and whether the customer's email address and physical address was in the system. The company had provided data on 15 different types of mailings that were sent to the customers in the dataset. If a mailing was sent to the customer the value of the particular mailing variable was 1 and if a mailing was not sent then the value of the mailing variable was 0. We also calculate a cumulative mailings variable which is the sum total of all the different types of mailings sent to the customer. The average number of mailings sent out by the firm to a customer over the 18 months was 2.61. We then calculate the recency of purchase (the time since last purchase), the frequency with which purchases were made (the number of times the customer made a purchase at the store) and the monetary value of each purchase for each customer (the total dollar value of purchases made on each purchase occasion). Finally, we divide the data into three month intervals at $\tau=0$, $\tau=1$, $\tau=2$, $\tau=3$, $\tau=4$, $\tau=5$, and $\tau=6$.

Estimation is carried out using the MCMC procedure based on a Gibbs sampling scheme (Geman and Geman 1984). We approximate the posterior distribution as described in (9) by sampling from the full conditional distributions. We use equations (10)-(16) to fully specify the model. We derive the estimates of the loyalty states for each quarter and then we derive the estimates of the probability of purchase at the daily level. We ran 50000 iterations using the WinBUGS software package, where the first 40000 iterations were used for burn-in and the last 10000 were used for estimation. We controlled for autocorrelation by thinning the observations-only every 4^{th} observation was used for our estimation procedure. Finally we also checked for convergence by running two chains simultaneously and monitoring the Brooks-Gelman-Rubin (BGR) convergence diagnostic, where convergence is said to have been achieved if the BGR statistic for each chain approaches the value of unity.

We estimate the two equations (3) and (6) using the following covariates. The repurchase equation (3) included the following covariates: monetary value of the purchase, whether the customer's email address is available, whether the customer's name and physical address are available, frequency of purchase, whether the purchase was made on a national holiday, the total number of mailings sent mailings, the recency of purchase, whether the visit was to return a previous purchase, and finally the predicted loyalty of the customer. The equation for predicted loyalty (6) includes the following covariates: the recency of purchase, monetary value of the purchase, frequency of purchase, and 15 variables that indicate whether the particular type of mailing was sent or not. These variables have a value of 1 if the mailing was sent and 0 if they were not. The mailing variables included are: product mail1, product mail2, product mail3, product mail4, product mail5, product mail6, product mail6, product mail7, relationship mailing1, relationship mailing2, relationship mailing3, action mailing1, action mailing2, action mailing3 and action mailing4.

5. Results and Discussion

Table 1a and 1b present the descriptive statistics of the variables in the dataset. Table 2 reports the posterior means and posterior standard deviations of the estimation procedure outlined in the preceding section. In figure 11 we see the distribution of the population wide loyalty function. The function is centered close to a mean of zero on the x-axis. An implication of this result is that on average customers are neither loyal nor disloyal to the firm's products.

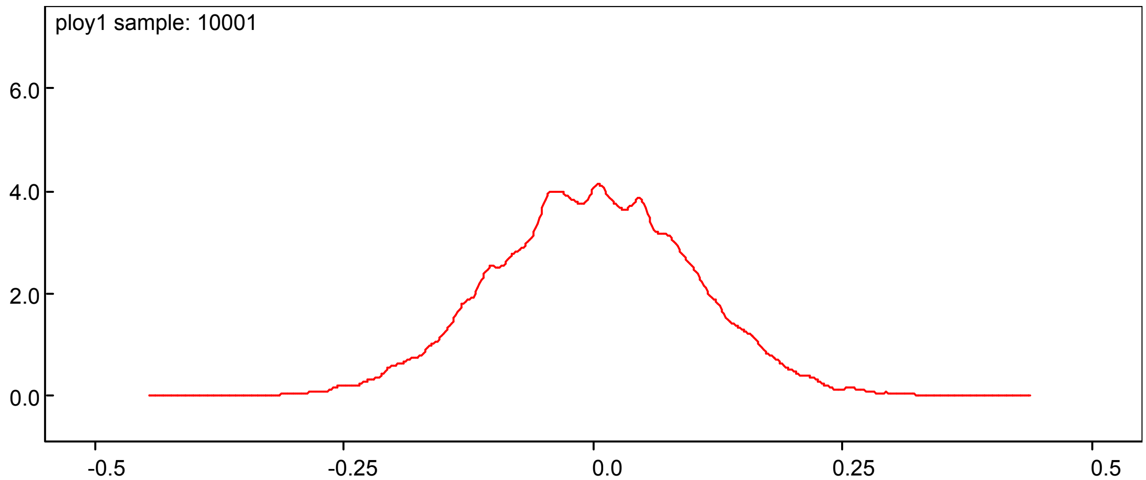

Figure 11: Population-wide distribution of the individual means of the loyalty function

Variable	Mean	Std Dev
Monetary Value	11.0881	15.63892
Recency	543.9177	136.3432
Frequency	3.001363	1.184601
Mailings	2.612951	1.800003

Table 1a: Descriptive Statistics – Mean and Standard Deviation

Variable	Value	Frequency (%)
Customer Email Address	1	24.86
Customer Name & Address	1	79.87
Product mail 1	1	4.02
Product mail 2	1	43.78
Product mail 3	1	5.87
Product mail 4	1	11.13
Product mail 5	1	2.36
Product mail 6	1	7.05
Product mail 7	1	4.59
Product mail 8	1	4.41
Relationship mailings 1	1	2.62
Relationship mailings 2	1	4.87
Relationship mailings 3	1	5.74
Action mailings 1	1	7.63
Action mailings 2	1	0.91
Action mailings 3	1	3.53
Action mailings 4	1	3.11
Holiday	1	0.66

Table 1b: Descriptive Statistics – Frequency

5.1 The Loyalty Equation

The variables included in the analysis are presented in table 2. We see from the results that frequency of visits has a significant effect on loyalty of the customer towards the firm. We find that frequency has a positive effect on the customer's loyalty, indicating that customers who visit the company store more often tend to be more loyal to the company's products. Neither the recency of purchase nor the monetary value of purchase has a significant effect on the customer's loyalty towards the firm. This result is of interest to firms because it implies that the amount a customer spends at a store on a given purchase occasion does not necessarily relate to how loyal the customer is to the firm's products.

We now examine the effect of the firm's CRM interventions on the predicted loyalty of the customer. We segment the CRM interventions in the following way:

1. Interventions that provide information about a specific product type or offering are classified as product mails. The variables in this category include: Product mailings 1 through 8. From the results shown in table 2, we find that a majority of the mailings that fall into this category have a significant effect on the loyalty of the customer towards the firm. Product mailings 1, 2, 3, 4, and 7 have significant positive effects on the loyalty of the customer to the firm. Mailings that are in this category provide information about the specific product, thereby arming the customer with knowledge about the firm's product offerings. This helps reduce the information uncertainty that the customer has with respect to the firm's products. This reduction in uncertainty also reduces the risk that the customer accepts when purchasing the firm's products. We hypothesize that this reduction in risk and increase in information available to customer help them make a more informed decision about the firm's products and hence have a positive effect on the loyalty of the customer to the firm. Hence such mailings help in improving the loyalty of customer to the firm.

2. Interventions that provide information about the entire suite of products offered by the firm in the form of a catalog can be classified as relationship mails. The variables that fall under this classification are: Relationship

mailing 1 through 3. Relationship mailings provide more social benefits (i.e., information on the product lines and lifestyle information) and focus on both relationship building and the creation of additional sales. These mailings tend to be more involved as they provide more information than just simple product mailings, and also include information on a much larger variety of product offerings. As they also involve greater customer participation, these mailings add towards strengthening the relationship between the customer and the firm. Hence we find that all the relationship mailings, 1, 2 and 3 are significant and have a positive effect on the loyalty of the customer to the firm.

3. Finally, interventions sent at specific times of the year that are meant to prompt the customer to visit the store are referred to as action oriented mails. These mails are generally devoid of any information about a particular product or other offerings. The main purpose of such mailings is to remind the customer of the firm's products and induce them to make a purchase. The mailings that fall into this category are: action mailings 1 through 4. We find that of these mailings only action mailing 2 plays a significant towards improving the loyalty of the customer to the firm. These mailings are generally less involved than the other mailings described above, and their main purpose as mentioned before is to provide a quick increase in sales. Hence to a large extent these mailings would inspire feelings of opportunism in the customer rather than feelings of loyalty, as the amount of investment

that the customer makes in using these mailings is less than both relationship mailings and product mails.

5.2 The Repurchase Equation

The next step of our analysis involved estimating the repurchase equation. The parameter estimates are also given in table 2. We first analyze the effect of the three RFM variables, namely recency, frequency and monetary value, on the customer's probability to repurchase. The estimates indicate that monetary value of purchase and frequency or purchase visits are most likely to have a significant effect on the probability that the customer comes back to the store to make a repeat purchase. The monetary value of a purchase has a negative effect on the likelihood of repurchase, i.e. the higher the monetary value of purchase, lower the probability of the customer coming back to the store. We also find that the frequency with which a customer makes a purchase visit has a positive effect on the likelihood of the customer repurchasing from the firm.

Next we examine the effect of CRM interventions on the probability of repurchase. We model this variable as the cumulative total of all the different types of mailings received by the customer. We find that the total number of mailings received by the customer does in fact have a very strong positive effect on the probability of repurchase. Hence, the efforts undertaken by the firm to send out their CRM interventions do in fact bear fruit. The analysis suggests that the firm, given the appropriate profit constraints, might even find it in its interest to increase the total number of CRM interventions sent out.

Repurchase Equation				
	node	**Mean**	**sd**	**mean/s.d**
Monetary Value	μ_1	-0.05	0.007243	-6.90322
Customer Email Address	μ_2	-0.02047	1.023	-0.02001
Frequency	μ_3	0.3999	0.007179	55.73365
Holiday	μ_4	2.14E-04	0.00817	0.026144
Total Number of Mailings	μ_5	0.5002	0.007173	69.70584
Recency	μ_6	-6.15E-05	0.00829	-0.00741
Returns	μ_7	-2.30E-05	0.008258	-0.00279
Customer Physical Address	μ_8	8.39E-05	0.008263	0.010156
Predicted Loyalty	μ_9	0.02084	0.00702	2.968661
Loyalty Equation				
	node	**Mean**	**sd**	**mean/s.d**
Frequency	Φ_1	4.67E-04	1.59E-04	2.939547
Monetary Value	Φ_2	-9.05E-05	0.01098	-0.00824
Product mail 1	Φ_{3a}	.990	0.1012	9.881423
Product mail 2	Φ_{3c}	1.001	0.1084	9.234317
Product mail 3	Φ_{3f}	1.09	0.09905	10.18677
Product mail 4	Φ_{3g}	1.02	0.09441	10.82512
Product mail 5	Φ_{3h}	0.007276	0.1048	0.069427
Product mail 6	Φ_{3i}	0.008962	0.09926	0.090288
Product mail 7	Φ_{3k}	1.006	0.09742	10.32642
Product mail 8	Φ_{3n}	1.658	2.149	0.771522
Relationship mailings 1	Φ_{3j}	1.112	0.1078	10.3154
Relationship mailings 2	Φ_{3l}	0.998	0.1004	9.940239
Relationship mailings 3	Φ_{3m}	1.007	0.1008	9.990079
Action mailings 1	Φ_{3b}	-0.00332	0.09832	-0.0338
Action mailings 2	Φ_{3d}	1.021	0.09019	11.09879
Action mailings 3	Φ_{3e}	0.8881	0.5952	1.492103
Action mailings 4	Φ_{3o}	1.632	2.142	0.761905
Recency	Φ_4	2.99E-05	0.00824	0.003625

Table 2: Model Parameter Estimates

We also find that predicted loyalty has a significant effect on the probability of repurchase. This effect is positive, implying that an increase in loyalty does in fact increase the probability of the customer coming back to repurchase from the firm.

Hence, it would bode well for the firm to work towards increasing the loyalty of the customer towards its products. We have also shown in section 5.5.1 that certain CRM interventions like product mails and relationship mailings help improve the customer's loyalty towards the firm. Thus the firm would do well to orient its CRM intervention schedule to include more of these types of mailings as they have been shown to improve loyalty and loyalty has in turn been shown to increase the probability that the customer would repurchase from the firm.

Through our analysis we have shown that it is possible to infer a customer's latent loyalty towards the firm and examine the effects of a firm's CRM schedule on this derived loyalty function. We have also shown that the derived loyalty does in fact have a strong positive effect on the probability of the customer repurchasing from the same firm. Hence a firm can use these results to better design their CRM schedule.

5.3 Model Validation

We validated our model's results by comparing the prediction ability of our model to two additional models that are used in the literature to study relationship dynamics. We used the validation dataset to compare our predictions to those of the benchmark models that are described below. The first model estimated was the hidden Markov model (Netzer, Lattin and Srinivasan 2005) and the second model estimated was the loyalty model (Guadagni and Little 1983). Netzer, Lattin and Srinivasan (2005) show that the HMM is useful at estimating the relationship states of the customer and then predicting each customer's probability of purchase. Therefore we decided to compare our model to the HMM model as described by them.

Additionally, the authors also state that the Guadagni and Little (1983) model was closest in terms of predictive ability to the results obtained from their model. Hence we chose the loyalty model as described by Guadagni and Little (1983) as our second benchmark model. The models are summarized below:

Model 1: The first model we compared our predictions to was the hidden Markov model. We assumed that the customer transitions between three possible loyalty states namely: loyal, switcher and disloyal. The probability of transition between these loyalty states is modeled using a logit framework. Similar to the methodology adopted in our model, we assume that the mailing variables have an impact on the probability of transitioning from one state to another or remaining in the same state. Hence the transition probabilities are defined using the following function, where *s* is the state at time t, *s'* is the state at time t+1 and ι_{its} is the effect of the relationship encounter on the probability of transition between loyalty states and $\upsilon_{ss'}$ is the state specific threshold. The threshold is the value that the cumulative impact of the encounters has to pass in order for a transition to occur:

$$l_{it}(s,s') = \frac{e^{(\vartheta_{ss'} + \iota_{its} X_{ijt})}}{1 + e^{(\vartheta_{ss'} + \iota_{its} X_{ijt})}} \tag{19}$$

Model 2: In the second model, loyalty is modeled using the same formulation as that devised by Guadagni and Little (1983), where ρ is the decay parameter and $purchase_{it-1} = 1$ if customer 'i' made a purchase occurred in the previous period t-1.

$$l_{it} = \rho l_{it-1} + (1-\rho)(purchase)_{it-1} \tag{20}$$

The probability of choice is modeled similar to the model shown in (3).

Following the procedure employed by Netzer, Lattin and Srinivasan (2005) we used the RMSPE (Root Mean Squared Predicted Error) and validation log-likelihood procedures to compare the predictive powers of the three models. The RMSPE (Root Mean Squared Predicted Error) measures the error between the predicted purchase probabilities and the actual purchases across customers and time. The validation log likelihood compares the predictive performance of the models. The results are shown in table 3. We can see from the results that the state space model results outperform those obtained from model 1 and 2.

We also compared the differences in means between the repurchase predictions made by the three models. We checked for differences using a pairwise t-test, the Bonferroni test and the Tukey test. The tests show that all the means are significantly different from each other. The results of the analysis are also presented in table 3.

The SSMC model was also superior to the HMM in terms of computational efficiency. The HMM model took significantly longer time (almost twice the time) to achieve convergence than the SSMC. Additionally the SSMC spared the researcher from making any assumptions about the number of states that the customer could have. The continuous nature of the SSMC hence provided the researcher with a significant advantage over the HMM in terms of both the efficiency of the estimation procedure as well as the ability to avoid making assumptions about the relationship state of the customer.

Model	*SSMC*	*HMM*	*Loyalty Model*
RMSPE	0.3571	0.3703	0.3827
Log Likelihood	80788	83434	85542

Means Comparison	*t-value*	*Significance*
Pairwise t-test of SSMC and HMM	-79.181	<.0001
Pairwise t-test of SSMC and Loyalty Model	-84.733	<.0001

Means Comparison		*HMM*	*Loyalty Model*
Bonferroni Test	***SSMC***	<.0001	<.0001
Tukey Test	***SSMC***	<.0001	<.0001

Table 3: Model Comparison Table – Comparing the Predictive Ability of the State Space Model with the Hidden Markov Model and the Loyalty Model
Legend: SSMC – State Space Model with Covariates; HMM – Hidden Markov Model

Model 2, while incorporating the importance of state dependence, only includes the effects of lagged loyalty on the loyalty state of the customer in current time period. Hence our model, which incorporates customer-firm interactions, allows additional insights on the effects of different CRM interventions and other variables of importance to the researcher on the current relationship state of the customer.

Hence from the validation tests carried out above we can see that the SSMC provides advantages over previous methods in terms of efficiency, better predictive ability and it also frees the researcher from making any assumptions about the number of relationship states that the customer might be in.

6. Conclusion

In this paper we used standard transaction data on customer purchase behavior to predict the relationship dynamics that exist between a customer and the firm.

Previous marketing literature solved problems of this type either by developing models that incorporate only state dependence, or using hidden Markov models. We develop a state space model to estimate the dynamics of relationships between the customer and the firm. The SSMC was estimated using a hierarchical Bayes MCMC procedure to account for both observed and unobserved heterogeneity.

The main contribution of this research is the introduction of a new methodology to study the relationship dynamics between the customer and the firm, which helps manufacturers infer the underlying structure of relationship states. The researcher can dynamically classify customers into the relationship states, and assess the dynamic effect of interactions between the customer and the firm on the customer's relationship state and consequent buying behavior. While the HMM can also achieve similar results, we show in section 5.3.2 that our model outperforms the HMM both in terms of its predictive ability as well as the efficiency with which convergence can be achieved. The number of states of the HMM determines the number of parameters that need to be estimated, as the researcher has to estimate a set of parameters for each individual state. On the other hand, the SSMC provides a major advantage in computational efficiency by allowing the researcher to derive a continuous probability distribution for the parameters across the states of the individual customer. Hence the number of parameters to be estimated no longer increases due to the continuous nature of the state space distribution.

An additional advantage of the SSMC over the HMM is that the researcher no longer needs to make an assumption about the number of states that the customer might have. The flexibility of the SSMC lies in the fact that the researcher can predict the most likely relationship state of the customer at a given point in time by simply studying the probability distribution of the customer at that time across all the possible states.

The empirical application of the SSMC model is demonstrated in sections 4 and 5. From the results we can see the usefulness of the model in studying dynamic relationships. The results indicate that CRM interventions do play a role in shifting the customer from a state of lower loyalty to one of a higher loyalty. Specifically, we find that CRM interventions that belong to the product mail category and the relationship mailings category tend to have more of an impact on increasing the loyalty of the customer than interventions that are more action–oriented, i.e., those that incentivize the customer into making a purchase immediately through means of a coupon or sale.

We also find that loyalty plays a positive and significant role in affecting the customer's probability of repurchase. The higher the loyalty state of the customer, the greater is the probability of the customer coming back to the store to make a purchase. Hence using the SSMC we can test long term impact of the customer-firm interactions on the relationship between the customer and the firm.

There are two main limitations of this study. Our dataset is comprised of only those customers who made a purchase in the time period we consider. Customers in the database who never purchased in the two year span were not included in the dataset. The exclusion of customers who never make a purchase from the dataset leads to a selection bias. Such biases can be controlled for by using an indicator variable that equals 1 if the customer is a purchaser and 0 otherwise. However the lack of data on non-purchasers prevents us from doing so. Hence, due to the limitation of the dataset, the results derived must be weighted taking the selection bias into consideration.

The second limitation arises because of the presence of endogeneity in the model. The endogeneity exists due to the fact that relationship mailings were sent only to customers who spent >$70 at the store. The problem therefore is that the marginal distribution of the relationship mailing variable is not independent of the conditional distribution of the loyalty variable given the relationship mailings. Hence the relationship mailings variable is endogenous, and this might lead to a bias in the estimates of the effect of relationship mailings on the loyalty of customer.

The main purpose of this paper is to demonstrate the use of a state space model to study customer – firm relationship dynamics and illustrate the advantage of using the state space model over methods prescribed in the previous literature. Hence our effort was mainly concentrated on using this method to study the customer relationship characteristics in order to deduce their loyalty states, taking into account various customer-firm interactions. To keep the model parsimonious we made simplifying assumptions with respect to the model parameters. A richer dataset, which includes survey data on variables like customer satisfaction levels, would provide a better insight into variables that drive loyalty and provide an additional insight into the effect of these CRM interventions. The use of longitudinal survey data would be extremely beneficial in determining and shaping a customer's loyalty state. Data on channels of purchase would also play an important role in shaping the loyalty function of the customer.

Appendices

Appendix 1: Solving for price and market share in logit demand models.

Solving for price P_i

From (12) in the paper we have, $P_i^* = \dfrac{1+\beta_i C_i(1-S_{it})}{\beta_i(1-S_{it})}$. (a)

Simplifying and rewriting (12) we get, $P_i^* = \dfrac{1}{\beta_i(1-S_{it})} + C_i$ (b)

Substituting (9) from the paper in (12) we get, $P_i^* = \dfrac{1}{\beta_i(1-\dfrac{e^{U_{it}}}{1+e^{U_{it}}+e^{U_{kt}}})} + C_i$, (c)

which can be simplified as $P_i^* = \dfrac{1}{\beta_i} + \dfrac{e^{\beta_{0i}-\beta_i P_i}}{\beta_i(1+e^{U_{kt}})} + C_i$. (d)

If we let $(1+\mathrm{e}^{\mathrm{U_{kt}}}) = \alpha$, then substituting this in (d) we get, $P_i^* = \dfrac{1}{\beta_i} + \dfrac{e^{\beta_{0i}-\beta_i P_i}}{\beta_i \alpha} + C_i$.(e)

Multiplying (e) by β_i and then subtracting β_0 from both sides, we get

$\beta_i P_i^* - \beta_0 = \dfrac{e^{\beta_{0i}-\beta_i P_i}}{\alpha} + 1 + \beta_i C_i - \beta_0$. (f)

Rewriting (f) we have, $\dfrac{e^{\beta_{0i}-\beta_i P_i}}{\alpha} - \beta_i P_i^* + \beta_0 = -1 - \beta_i C_i + \beta_0$. (f1)

Taking exponentials on both sides of (f1) and then dividing both sides by α we have,

$e^{\frac{e^{\beta_{0i}-\beta_i P_i}}{\alpha}} \dfrac{e^{-\beta_i P_i^* + \beta_0}}{\alpha} = \dfrac{e^{-1-\beta_i C_i + \beta_0}}{\alpha}$. (g)

Let $\dfrac{e^{-\beta_i P_i^* + \beta_0}}{\alpha} = W$, then rewriting (g) we have $We^W = \dfrac{e^{-1-\beta_i C_i + \beta_0}}{\alpha}$. (h)

It is hence easy to see that (h) is similar to equation (1) in the paper, hence the solution to (h) is given by $W = LambertW(\dfrac{e^{-1-\beta_i C_i + \beta_0}}{\alpha})$. (i.1)

Substituting for W, we have $\dfrac{e^{-\beta_i P_i^* + \beta_0}}{\alpha} = LambertW(\dfrac{e^{-1-\beta_i C_i + \beta_0}}{\alpha})$. (i.2)

Taking the natural logarithms on both sides of (i1) and using the result from equation (6) we have $-\beta_i P_i^* + \beta_0 - \ln(\alpha) = -1 - \beta_i C_i + \beta_0 - \ln(\alpha) - LambertW(\dfrac{e^{-1-\beta_i C_i + \beta_0}}{\alpha})$. (j)

Equation (j) further simplifies to $P_i^* = \dfrac{1 + \beta_i C_i + LambertW(\dfrac{e^{-1-\beta_i C_i + \beta_0}}{\alpha})}{\beta_i}$, (k) which is the closed form solution of P_i independent of the effect of its own market share. Substituting for α, we get the equation (13), which is,

$$P_i^* = \frac{1 + LambertW\left(\frac{e^{(\beta_{0i} - 1 - C_i\beta_i)}}{1 + e^{(\beta_{0k} - P_k\beta_k)}}\right) + C_i\beta_i}{\beta_i}. \qquad \text{(k')}$$

Solving for market share S_{it}

To simplify notations, we replace $1+\beta_i C_i$ with 'δ', β_0 with 'Δ' and $e^{-1-\beta_i C_i+\beta_0}$ with 'Ξ'. As previously noted, we still maintain the notation $(1+e^{U_{kt}}) = \alpha$. Thus substituting this into equation (9) from the paper we have,

$$S_{it} = \frac{e^{\Delta - \beta_i P_i^*}}{\alpha + e^{\Delta - \beta_i P_i^*}}. \qquad \text{(m)}$$

Rewriting (m) we have, $S_{it} = \dfrac{e^{\Delta}}{\alpha e^{\beta_i P_i^*} + e^{\Delta}}$. (m1)

Substituting (k) in (m) we have, $S_{it} = \dfrac{e^{\Delta}}{\alpha e^{\delta} e^{LambertW\left(\frac{\Xi}{\alpha}\right)} + e^{\Delta}}$. (n)

Using the result in equation (1), we can rewrite (n) as

$$S_{it} = \frac{e^{\Delta}}{\alpha e^{\delta} \dfrac{\Xi}{\alpha LambertW\left(\frac{\Xi}{\alpha}\right)} + e^{\Delta}}. \qquad \text{(o)}$$

It is evident that $\Xi = e^{-\delta}e^{\Delta}$; therefore the (o) becomes,

$$S_{it} = \frac{1}{\dfrac{1}{LambertW\left(\frac{\Xi}{\alpha}\right)} + 1}, \text{ which can be rewritten as } S_{it} = \frac{LambertW\left(\frac{\Xi}{\alpha}\right)}{1 + LambertW\left(\frac{\Xi}{\alpha}\right)}. \qquad \text{(p)}$$

The equation in (p) is analogous to the result obtained in equation (14). Simply substituting for the values of Ξ and α we get,

$$S_{it} = \frac{LambertW\left(\frac{e^{(\beta_{0i} - 1 - C_i\beta_i)}}{1 + e^{(\beta_{0k} - P_k\beta_k)}}\right)}{1 + LambertW\left(\frac{e^{(\beta_{0i} - 1 - C_i\beta_i)}}{1 + e^{(\beta_{0k} - P_k\beta_k)}}\right)}. \qquad \text{(p')}$$

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