TECHNICAL RESEARCH REPORT

Network Tomography

by Franklin Gavilanez, Carlos Berenstein and John Baras

TR 2005-70

NETWORK TOMOGRAPHY[1]

CARLOS BERENSTEIN[2,3], FRANKLIN GAVILÁNEZ[2] , and JOHN BARAS[3]

January 24, 2005

Abstract

While conventional tomography is associated to the Radon transform in Euclidean spaces, electrical impedance tomography or EIT is associated to the Radon transform in the hyperbolic plane. We discuss some recent work on network tomography that can be associated to a problem similar to EIT on graphs and indicate how in some sense it may be also associated to the Radon transform on trees.

1 Introduction

As communication networks have become an essential part of everyday life, disruptions may have very serious consequences. Thus, the need to prevent or, at least, detect them early on, has become very important. In order to do that we discuss two models of the problem, one based on weighted graphs and the second based on trees. The first one is the discrete equivalent of the inverse conductivity problem, that is, of Electrical Impedance Tomography. The second model was mentioned recently by E. Jonckheere and his collaborators [29].

In both cases, the data we collect are obtained by monitoring traffic only at distinguished subsets of the network. We think about this subset as being the periphery of the network.

2 The weighted graph model

In this case we model our network in the following way. We have a collection of nodes and edges between the nodes in a finite planar connected graph G. We denote by V the set of nodes of G and by E the set of edges of G. Usually, the graph G is denoted by $G(E, V)$. A particular subset of this graph G is denoted by ∂G and called the boundary

[1] The present is an extended version of the lecture given by Prof. Berenstein at the special session on tomography, 2004 AMS meeting, Rider University. The authors acknowledge partial support on this research from grants ARO-DAAD-190110494 and NSF-DMS-0400698.
[3] Department of Mathematics, University of Maryland, College Park, Maryland, 20742, USA
[3] Institute for Systems Research, University of Maryland, College Park, Maryland, 20742, USA.

of G. In our context these are the nodes accessible to whoever is trying to monitor the traffic in G. The boundary edges are those links whose two endpoints are in ∂G. We assume that G remains connected even if we remove the boundary edges. For our present purposes, the boundary edges play no role, thus we may as well assume that there are none. We also assume that ∂G is not empty.

Furthermore, we assume that to every edge in E we have an associated non-negative number $\omega(x, y)$ which corresponds to the traffic between the endpoints x and y of the edge. Note that this is a static model and we are really thinking that the graph is a planar graph, although this is not used anywhere in the reasoning. We define the degree $d_\omega x$ of a node x in the weighted graph G with weight ω by

$$d_\omega x = \sum_{y \in V} \omega(x, y)$$

the Laplacian operator corresponding to this weight ω is defined by

$$\Delta_\omega f(x) = \sum_{y \in V} [f(y) - f(x)] \cdot \frac{\omega(x, y)}{d_\omega x}, \ x \in V$$

A graph $S = S(V', E')$ is said to be a *subgraph* of $G(E, V)$ if $V\prime \subset V$ and $E\prime \subset E$. In this case, we call G a *host graph* of S. The integration of a function $f : G \to \mathbb{R}$ on a graph $G = G(V, E)$ is defined by

$$\int_G f = \sum_{x \in V} f(x) d_\omega x \text{ or simply } \int_G f d_\omega$$

For a subgraph S of a graph $G = G(V, E)$ the (node) *boundary* ∂S of S is defined to be the set of all nodes $z \in V$ not in S but adjacent to some node in S, i.e.,

$$\partial S = \{z \in V \mid z \sim y \text{ for some } y \in S\}$$

and the *inner boundary* $\overset{\circ}{\partial} S$ by

$$\overset{\circ}{\partial} S = \{z \in S \mid y \sim z \text{ for some } y \in \partial S\}$$

where $z \sim y$ means that the two nodes z and y are connected by an edge in E. Also, by $\overline{S}$ we denote a graph whose nodes and edges are in $S \cup \partial S$. The (outward) normal derivative $\frac{\partial f}{\partial n_\omega}(z)$ at $z \in \partial S$ is defined to be

$$\frac{\partial f}{\partial \, n_\omega}(z) = \sum_{y \in S} [f(z) - f(y)] \cdot \frac{\omega(z, y)}{d'_\omega \, z},$$

where $d'_{\omega} z = \sum_{y \in S} \omega(z, y)$

In this model, there are two kinds of disruptions of traffic data that could arise. In one of them, disruptions occurs when an edge "ceases" to exist, in this case the "topology" of the graph has changed, and we refer to the important work of Fan Chung and her collaborators which offers crucial insights into this question. (See, for instance [16], [17] and [18].). In the other, the weights change because of "increase" of traffic, that is, the network configuration remains the same but the weights have either increased or remained the same. In this second situation, we can appeal to the following theorem

Theorem 1 [11] *Let ω_1 and ω_2 be weights with $\omega_1 \leq \omega_2$ on $\overline{S} \times \overline{S}$, G a graph and $f_1, f_2 : \overline{S} \to \mathbb{R}$ be functions satisfying that for $j = 1, 2,$*

$$\begin{cases} \Delta_{\omega_j} f_j(x) = 0, \ x \in S \\ \frac{\partial f_j}{\partial n_{\omega_j}}(z) = \Phi(z), \ z \in \partial S \\ \int_S f_j d_{\omega_j} = K \end{cases}$$

for any given function $\Phi : \partial S \to \mathbb{R}$ with $\int_{\partial S} \Phi = 0$, and for a suitably chosen number $K > 0$. If we assume that

$$\begin{array}{l} (i)\, \omega_1(z, y) = \omega_2(z, y) \ on \ \partial S \times \overset{\circ}{\partial S} \\ (ii)\, f_1|_{\partial S} = f_2|_{\partial S}, \end{array}$$

then we have

$$f_1 = f_2 \ on \ \overline{S}$$

and

$$\omega_1 = \omega_2 \ on \ \overline{S} \times \overline{S}$$

whenever $f_1(x) \neq f_1(y)$ and $f_2(x) \neq f_2(y)$.

We conclude that the data distinguishes the two cases. That is, we can decide whether there is an increase of traffic somewhere in the network or not. While this is only a uniqueness theorem, nevertheless, we can effectively compute the actual weights from the knowledge of the Dirichlet data for convenient choices of the input Neumann data in a way similar to that done in [21] and [23] for lattices. Similarly, the Green function of this Neumann boundary value problem can be represented by an explicit matrix.

What we want to discuss now is the relationship between the above results to the problem of understanding a large network like the internet. One way to make more concrete this problem was discussed by T. Munzner in [32] and [33] on visualizing the internet. It implies that the natural domain might be a hyperbolic space of dimension higher than 2. One can see that Munzner's suggestion leads to a question closely resembling EIT, and it is natural to consider it a problem in hyperbolic tomography [7], [8]. On the other hand, we have just obtained a significant result on the inversion of the Neumann-Dirichlet problem by studying it directly on "weighted" graphs [11]. Similarly, the Radon transform in the hyperbolic plane has been studied in [7], [8], and [27].

In addition, in a recent lecture, E. Jonckheere [29] indicated that at least locally internet traffic could see modelled as being part of a tree and therefore it can be visualized using 2-dimensional hyperbolic geometry. As a consequence, a different way to study locally this kind of networks can be done using the Radon transform on trees. As it turns out, inversion formula for the Radon transform on trees is already known and it can be found in [9].

For the sake of completeness, we will describe here a simplified version of the Radon transform on trees and its inversion formula. As explained below, this seems to be enough to deal with the network problems we are interested in.

3 The Radon transform on homogeneous trees

Let us now remind the reader what do we mean by a tree T. A tree T is a finite or countable collection V of vertices $\{v_j, j = 0, 1, ...\}$ and a collection E of edges $e_{jk} = (v_j, v_k)$, in other words, pairs of vertices. We orient the edge e_{jk} by thinking that v_j is the first node and v_k the second node. We always include the edges e_{kj} in this collection, which have the reverse orientation. Given two vertices u and v,we say they are neighbors if (u, v) is an edge and write $u \backsim v$ in this case. A geodesic γ from u_0 to u_l is a collection $u_0, u_1, ..., u_{l-1}, u_l$ of pairwise distinct vertices such that $u_0 \backsim u_1$, $u_1 \backsim u_2, ..., u_{l-1} \backsim u_l$. It turns out that $u_0 \backsim u_l$ then we consider the closed geodesic path $\bar{\gamma}$ by adding the edge (u_l, u_0) to γ. Unless explicitly mentioned, our geodesics will not be closed. To simplify the notation, for any geodesic $\gamma = u_0 \backsim u_1 \backsim u_1 \backsim u_2 \backsim .... \backsim u_{l-1} \backsim u_l$ open or closed, we denote by $-\gamma$ the geodesic with the opposite orientation , i.e., -$\gamma = u_l \backsim u_{l-1} \backsim .... \backsim u_0$. The collection of all (open) geodesics is denoted by Γ. If T is infinite, then a complex valued function $f \in L^1(T)$ if .

$$\sum_{v \in V} |f(v)| < \infty$$

the Radon transform R of a function $f \in L^1(T)$ is simply the bounded function Rf on Γ defined by

$$Rf(\gamma) = \sum_{v \in \gamma} f(v)$$

Given a node v we denote by $v(\nu)$ the number of edges that contain v as an endpoint.

This number is sometimes called the degree of the node. We will assume throughout that we always have $v(\nu) \geq 3$ to ensure that the Radon transform in injective. (In our applications this is only needed for nodes v that lie in supp(f). In the terminology of [9] we are assuming there are neither black holes nor flat points in T. Under these conditions, the Radon transform in a tree is invertible. In fact, the explicit inversion formula resembles that of the inversion for the Radon transform in the Euclidean plane [10], [12], [13], and [27]. Unfortunately, even in this case, we need to introduce a significant amount of auxiliary notation. For the purpose of illustration we describe the inversion formula here only for the case of homogeneous trees.

4 Inversion of the Radon transform in homogeneous trees

Consider a homogeneous tree T in which each vertex touches $q+1$ edges with $q \geq 2$. If n is a nonnegative integer, let $v_{(n)}$ the number of vertices of T at distance n from a fixed vertex of T. It follows that

$$\begin{cases} 1 & \text{if } n = 0 \\ (q+1)q^{n-1} & \text{if } n \geq 1 \end{cases}$$

We give the following definitions. Let v, w two vertices in T that are connected by a path $(v = v_0, ..., v_m = w)$, then the *distance* between v and w is the nonnegative integer $|v, w| = m$. Also, for $f \in L^1(T)$, let μ_n the average operator defined by

$$\mu_n f(v) = \frac{1}{v_{(n)}} \sum_{|v,w|=n} f(w), \quad \text{for } v \in T$$

It can be seen that μ_n is basically a convolution with radial kernel

$$h_n(v, w) = \begin{cases} \frac{1}{v_{(n)}} & \text{if } \ |v, w| = n \\ 0 & \text{if } \ |v, w| \neq n \end{cases}$$

Let $\beta = q/(2(q+1))$ and R^* be the dual Radon transform defined for $\Phi \in L^\infty(\Gamma)$ by

$$R^*\Phi(v) = \int_{\Gamma_v} \Phi(\gamma) d\rho_v(\gamma) \text{ for each vertex } v \in T,$$

with respect to a suitable family $\{\rho_v : v \in T\}$ of measures on Γ where Γ_v is the set of all of the geodesics containing the vertex v.

In order to obtain the inversion of R we observe that R^*R acts as a convolution operator given by the radial kernel $h = \beta h_0 + \sum_{n=1}^{\infty} 2\beta h_v$.

Proposition 2 *The identity*

$$R^*R = \beta\mu_0 + \sum_{n=1}^{\infty} 2\beta\mu_n \ \text{ on } L^1(T),$$

holds in $L^1(T)$, *where the series is absolutely convergent in the convolution operator norm on* $L^2(T)$, *thus providing a bounded extension of* R^*R *to* $L^2(T)$.

Theorem 3 *The unique bounded extension to* $L^2(T)$ *of the operator* R^*R *is invertible on* $L^2(T)$, *and its inverse is the operator*

$$E = \frac{2(q+1)^3}{q(q-1)^2} \left[\mu_0 + \sum_{n=1}^{\infty} (-1)^n 2\mu_n \right]$$

which acts as the convolution with the radial kernel $\frac{2(q+1)^3}{q(q-1)^2} \left[h_0 + \sum_{n=1}^{\infty} (-1)^n 2h_n\right]$. *As before, this series converges absolutely in the convolution operator norm on* $L^2(T)$; *in particular,* E *is bounded.*

Corollary 4 *The Radon transform* $R : L^1(T) \to L^\infty(\Gamma)$ *is inverted by*

$$ER^*Rf = f.$$

References

[1] J. Baras, C. A. Berenstein and F. Gavilánez, Local monitoring of the internet network. Available at http://techreports.isr.umd.edu/TechReports/ISR/2003/TR_2003-7/TR_2003-7.phtml

[2] J. Baras, C. A. Berenstein and F. Gavilánez, Discrete and continuous inverse conductivity problems, AMS, Contemporary Math, Vol. 362, 2004.

[3] D. C. Barber and B. H. Brown, Inverse problems in partial differential equations (Arcata, CA, 1989), 151-164, SIAM, Philadelphia, PA, 1990.

[4] A. Bensoussan and J. L. Menaldi, Difference Equations on weighted graphs, 2003

[5] R. M. Brown, and R. Torres, Uniqueness in the inverse conductivity problem for conductivities with $\frac{3}{2}$-derivatives in L^p, $p > 2n$, JFAA, 9 (2003), 563-574.

[6] R. W. Brown, E. Haacke, M. Thompson, R. Venkatesan, Magnetic Resonance Imaging: Physical Principles and Sequence Design. Wiley, May 1999.

[7] C. A. Berenstein and E. Casadio Tarabusi, The inverse conductivity problem and the hyperbolic Radon transform, "75 years of Radon Transform", S. Gindikin and P. Michor, editors. International Press, 1994.

[8] C. A. Berenstein and E. Casadio Tarabusi, Integral geometry in hyperbolic spaces and electrical impedance tomography, SIAM J. Appl. Math. 56 (1996), 755-764.

[9] C. A. Berenstein et al, Integral Geometry on Trees, American Journal of Mathematics 113 (1991), 441-470.

[10] C. A. Berenstein, Radon transforms, wavelets, an applications, Lecture notes in Mathematics. Berlin: Springer, Vol. 1684, 1998, pp.1-33.

[11] C. A. Berenstein and S-Y. Chung, $\omega-$Harmonic functions and inverse conductivity problems on networks. Available at http://techreports.isr.umd.edu/TechReports/ISR/2003/TR_2003-16/TR_2003-16.phtml. To appear in SIAM J. Appl. Math.

[12] C. A. Berenstein, Local tomography and related problems, AMS Contemporary Mathematics, Vol. 278, 2001.

[13] C. A. Berenstein, and D. Walnut, Local inversion of the Radon transform in even dimensions using wavelets, "75 years of Radon transform", S. Gindikin and P. Michor, editors, International Press, 1994, 45-69.

[14] C. A. Berenstein, and D. Walnut, Wavelets and local tomography, "Wavelets in Medicine and Biology", A. Aldroubi and M. Unser, editors, CRC Press, 1996, 231-261.

[15] A. P. Calderon, On an inverse boundary value problem, Seminar on numerical analysis and its applications to continuum physics, Soc. Brz. Math. 1980, 65-73.

[16] F. Chung, and R. Ellis, A chip-f iring game and Dirichlet eigenvalues, U. of California.

[17] F. Chung, Spectral Graph Theory,AMS, 1997.

[18] F. Chung, and K. Oden, Weighted Graph Laplacians and Isoperimetric Inequalities, U. of California-Harvard University.1999.

[19] M. Coates, A. Hero III, R. Nowak, and B. Yu, Internet tomography, IEEE Signal processing magazine, may 2002.

[20] E. B. Curtis, T. Ingerman, and J. A. Morrow, Circular planar graphs and resistors networks, Linear algebra and its applications, 283 (1998), 115-150.

[21] E. B. Curtis, and J. A. Morrow, Determining the resistors in a network, SIAM J. Appl. Math., Vol. 50, No. 3, pp. 918-930, June 1990.

[22] E. B. Curtis, and J. A. Morrow, Inverse problems for electrical networks, Series on Applied Mathematics, Vol. 13, 2000.

[23] E. B. Curtis, and J. A. Morrow, The Dirichlet to Neumann map for a resistor network, AMS, 1990.

[24] T. Daubechies, Ten lectures in wavelets. SIAM, 1992.

[25] Evans T. and D. Walnut, Wavelets in hyperbolic geometry. MS dissertation, Department of Mathematics, George Mason University, Fairfax: 1999, pp.1-51.

[26] F. Gavilánez, Multiresolution Analysis on the Hyperbolic plane, Master's thesis, University of Maryland, College Park, 2002.

[27] S. Helgason, The Radon transform. Boston: Birkhauser, 1999.

[28] A. I. Katsevich and A. G. Ramm, The Radon transform and local tomography. Boca Raton: CRC Press, 1996.

[29] E. Jonckheere, Dept. of Electrical Engineering & Mathematics University of Southern California, Los Angeles, CA 90089. Lecture given at Institute for Systems Research, University of Maryland, College Park, October 2004.

[30] S. Lissianoi S. and I. Ponomarev, On the inversion of the geodesic Radon transform on the hyperbolic plane. Journal of Inverse problems, vol. 13, IOP Publishing Ltd.

1997, pp. 1053-1062.

[31] S. Mallat, A wavelet tour of signal processing. London: Academic press, 2001.

[32] T. Munzner, Exploring large graphs in 3D hyperbolic space, IEEE Computer Graphics and Applications, July/August 1998.

[33] T. Munzner and P. Burchard, Visualizing the structure of the world wide web in 3D hyperbolic space, Proceedings of VRML '95, (San Diego, California, December 1995), special issue of Computer Graphics, ACM SIGGRAPH, New York, 1995, pp.33-38.

[34] A.I. Nachman, Global uniqueness for a two-dimensional inverse boundary problem, Annals Math. 143 (1996), 71-96.

[35] F. Natterer, and F. Wubbeling, Mathematical methods in image reconstruction. SIAM 2001.

[36] F. Natterer, The mathematics of computerized tomography. SIAM 2001.

[37] F. Santosa, and M. Vogelius, A backprojection algorithm for electrical impedance imaging. SIAM J. Appl. Math. 50 (1990), no. 1, 216-243.

[38] J. Sylvester, and G. Uhlmann, A global uniqueness theorem for an inverse boundary value problem, Ann. of Math. (2) 125 (1987), no. 1, 153-169.

[39] G. Uhlmann, Developments in inverse problems since Calderon's foundational paper, in Harmonic analysis and partial differential equations, U. Chicago Press, 1999, 295-345.