TECHNICAL RESEARCH REPORT

Analysis of a complex activator-inhibitor equation

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CDCSS T.R. 99-1
(ISR T.R. 99-13)

CENTER FOR DYNAMICS AND CONTROL OF SMART STRUCTURES

The Center for Dynamics and Control of Smart Structures (CDCSS) is a joint Harvard University, Boston University, University of Maryland center, supported by the Army Research Office under the ODDR&E MURI97 Program Grant No. DAAG55-97-1-0114 (through Harvard University). This document is a technical report in the CDCSS series originating at the University of Maryland.

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1999 American Control Conference

Analysis of a complex activator-inhibitor equation

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Abstract

Basic properties of solutions and a Lyapunov functional are presented for a complex activator-inhibitor equation with a cubic nonlinearity. Potential applications include control of coupled-oscillator arrays (for quasi-optical power combining and phased-array antennas), and control of MEMS actuator arrays (for micro-positioning small items).

1. Introduction

Pattern-forming systems of partial differential equations are used to model many diverse phenomena from biology, chemistry, and physics. There is a large body of experimental and mathematical work on pattern-forming systems; however, these ideas have not yet been adequately exploited in engineering, particularly in the area of control. One control context in which pattern-forming systems could have a useful role is in problems of controlling large arrays of (e.g., MEMS) micro-actuators [1]. Potential applications for large arrays of micro-actuators include adaptive optics (in particular, micromirror arrays), suppressing turbulence and vortices in fluid boundary-layers, micro-positioning small parts, and manipulating small quantities of chemical reactants.

The complex activator-inhibitor equation represents a continuum limit of a type of coupled oscillator network, coupled to a network of resonant circuits. Although the results we present here are for systems of partial differential equations, they can easily be carried over to the systems of ordinary differential equations that arise from spatially discretizing the PDE systems. It is actually the spatially discretized systems that would be implemented in the coupled-oscillator and micro-actuator array applications considered here.

The role that pattern-forming systems are envisioned to play in the control of oscillator and actuator arrays is to endow the array with some simple, characteristic response to "coarse" higher-level control inputs. When, for example, the number of actuators in an actuator array exceeds about 10^6, it becomes impractical to individually control in detail each actuator in a centralized fashion. With the pattern-forming system dynamics in place, the centralized controller can influence the response of the array through control parameters common to all the actuators, or else the centralized controller can still set parameters for each actuator individually, but with lower bandwidth than would be necessary to provide detailed control signals to each actuator.

Potential actuator-array application areas for the complex activator-inhibitor equation are in quasi-optical power combining, phased-array antennas, and micropositioning small items [2,3,4]. After introducing the complex activator-inhibitor dynamics, presenting its Lyapunov functional (the main contribution of this work), and outlining the analysis of equilibria, we will discuss these applications in more detail. We will then discuss the analysis of the modal equations.

2. Lyapunov functional

We consider the complex activator-inhibitor equation

$$\begin{aligned} \tau_\theta \partial_t \theta &= l^2 \Delta\theta - |\theta|^2\theta + \theta + \eta, \\ \tau_\eta \partial_t \eta &= L^2 \Delta\eta - \eta - \theta + C, \end{aligned} \tag{1}$$

for $\mathbf{x} \in \Omega \subset \mathbb{R}^n$, Ω open and bounded, where Δ denotes the Laplacian operator. The "activator" $\theta(\mathbf{x}, t)$, the "inhibitor" $\eta(\mathbf{x}, t)$, and the bifurcation parameter C are complex, whereas τ_θ, τ_η, l, and L are positive real constants determining the time constants and diffusion lengths associated with the activator and inhibitor. We define $\alpha = \tau_\theta/\tau_\eta$ as the ratio of time constants and $\beta = l/L$ as the ratio of length scales, because these ratios determine the pattern-forming properties of system (1). We will assume that $\alpha > 1$ and $\beta < 1$. The complex activator-inhibitor equation can be used, under suitable hypotheses, to model the amplitude and phase evolution in the continuum limit of a network of coupled van der Pol oscillators (represented by θ), coupled to a network of resonant circuits (represented by η), with an external oscillating input (represented by C). The resonant frequencies of the van der Pol oscillators and the resonant circuits are assumed to be identical, and also equal to the frequency of the external input C.

This coupled system of parabolic PDEs enjoys exis-

This research was supported in part by grants from the National Science Foundation's Engineering Research Centers Program: NSFD CDR 8803012 and by the Army Research Office under the ODDR&E MURI97 Program Grant No. DAAG55-97-1-0114 to the Center for Dynamics and Control of Smart Structures (through Harvard University). Also, this work was partially supported by an Achievement Rewards for College Scientists (ARCS) scholarship.

tence and uniqueness of weak solutions for any finite time interval $[0, T]$, as well as a dissipativity property (the existence of an absorbing set and a compact, connected global attractor) [1,5]. When $\alpha > 1$, system (1) possess a Lyapunov functional

$$V^* = \int_\Omega \left[\frac{l^2}{2}|\nabla\theta|^2 + \frac{1}{4}|\theta|^4 - \frac{1}{2}|\theta|^2 + \frac{L^2}{2}|\nabla\eta|^2 + \frac{1}{2}|\eta|^2 \right. \\ -\mathrm{Re}\{\overline{C}\eta\} + \mathrm{Re}\{\overline{\theta}\eta\} \\ \left. -(\overline{\theta} - \overline{C}) \left[(-1 + L^2\Delta)^{-1}(\theta - C)\right] \right] d\mathbf{x}, \qquad (2)$$

where the overbar denotes complex conjugation. We can also express V^* in terms of the real variables θ_R, θ_I, η_R, and η_I, the real and imaginary parts of θ and η. The operator $(-1+L^2\Delta)^{-1}$ can be analyzed using Fourier methods, and is well-behaved [1,5]. The time derivative of V^* is then found to be

$$\dot{V}^* = -\int_\Omega \begin{bmatrix} \partial_t\theta_R \\ \partial_t\theta_I \\ \partial_t\eta_R \\ \partial_t\eta_I \end{bmatrix}^T \begin{bmatrix} \tau_\theta & 0 & W & 0 \\ 0 & \tau_\theta & 0 & W \\ 0 & 0 & \tau_\eta & 0 \\ 0 & 0 & 0 & \tau_\eta \end{bmatrix} \begin{bmatrix} \partial_t\theta_R \\ \partial_t\theta_I \\ \partial_t\eta_R \\ \partial_t\eta_I \end{bmatrix} d\mathbf{x}, \qquad (3)$$

where W represents the operator $-2\tau_\eta(-1+L^2\Delta)^{-1}$. When $\alpha = \tau_\theta/\tau_\eta > 1$, we can conclude that $\dot{V}^* \leq 0$, and $\dot{V}^* = 0$ only at equilibria [1,5]. Since V^* is radially unbounded, it serves as a Lyapunov functional provided $\alpha > 1$.

3. Equilibria

For the spatially discretized system of ODEs that give rise to the complex activator-inhibitor equation in the continuum limit, an analogous Lyapunov function exists, and LaSalle's invariance principle implies that trajectories converge to the set of equilibria. Therefore, we are interested in determining the equilibria of equation (1).

Consider first the equilibrium equations in a single space dimension:

$$\begin{aligned} 0 &= l^2\partial_{xx}\theta - |\theta|^2\theta + \theta + \eta, \\ 0 &= L^2\partial_{xx}\eta - \eta - \theta + C, \end{aligned} \qquad (4)$$

with $\beta < 1$ and periodic boundary conditions. For $C = 0$, there are stable spatially periodic pattern solutions, which in the one-dimensional case can be thought of as helical, since a three-dimensional (phasor) plot of the real and imaginary parts of either the activator or the inhibitor plotted along the single space dimension would trace out a helix. For $|C| > (1-\beta)^3$, there is a stable spatially uniform equilibrium solution. The intermediate values of $|C|$ can produce patterns in which the direction of the activator (and inhibitor) vectors oscillate between two

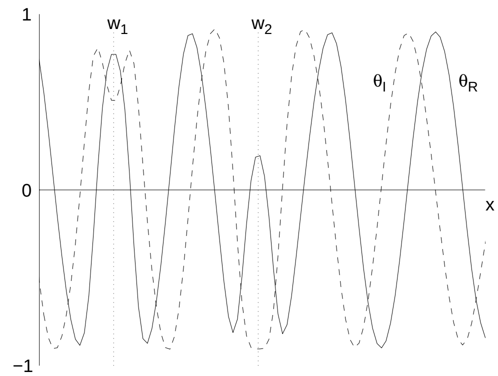

Figure 1: A one-dimensional equilibrium with two domain walls (labeled w_1 and w_2).

different directions as one moves along the space dimension (or, if $|C|$ is small enough, the helical solution is simply perturbed by higher spatial harmonics). The wave number for both the linear instability as $|C|$ is decreased through $(1-\beta)^3$ and the wave number for the minimum-energy (relative to the Lyapunov functional) helical equilibrium when $C = 0$, is found to be $\sqrt{1-\beta}/\sqrt{lL}$. If instead of thinking about the complex envelope, we instead return to the point of view of coupled oscillators, the helical solution corresponds to a traveling wave (or rotating wave), like the threads of a turning screw.

In two spatial dimensions, roll patterns can be stable for $C = 0$, where along the direction perpendicular to the rolls, the one-dimensional helical solution is obtained. However, if the initial conditions are random, there is no reason to expect that a regular pattern of rolls will emerge. In fact, even in the one-dimensional case, random initial conditions lead to regions of both right-handed and left-handed helixes, with domain walls in between, as shown in figure 1. In systems which are not highly homogeneous, such as an analog coupled oscillator array, these domain walls may persist indefinitely. For the phased-array antenna application, regular roll patterns are required to achieve the desired constructive and destructive interference patterns, so how the roll pattern is excited is important for that application. By constrast, for the micro-positioning application, we will see that having an irregular roll pattern is not a problem.

3.a Polar coordinate transformation

Since it is often natural to examine coupled oscillator equations in polar coordinates, we define

$$\theta = r_\theta e^{i\psi_\theta}, \quad \eta = r_\eta e^{i\psi_\eta}, \quad C = r_C e^{i\psi_C}. \qquad (5)$$

The dynamics in transformed coordinates are found to be

$$\begin{aligned} \tau_\theta\partial_t r_\theta &= l^2\left[\partial_{xx}r_\theta - r_\theta(\partial_x\psi_\theta)^2\right] - r_\theta^3 + r_\theta \\ &\quad + r_\eta\cos(\psi_\eta - \psi_\theta), \\ \tau_\eta\partial_t r_\eta &= L^2\left[\partial_{xx}r_\eta - r_\eta(\partial_x\psi_\eta)^2\right] - r_\eta - r_\theta\cos(\psi_\eta - \psi_\theta) \end{aligned}$$

$$
\begin{aligned}
&+r_C\cos(\psi_\eta-\psi_C),\\
\tau_\theta\partial_t\psi_\theta &= l^2\partial_{xx}\psi_\theta+\frac{2l^2}{r_\theta}(\partial_x r_\theta)(\partial_x\psi_\theta)+\frac{r_\eta}{r_\theta}\sin(\psi_\eta-\psi_\theta),\\
\tau_\eta\partial_t\psi_\eta &= L^2\partial_{xx}\psi_\eta+\frac{2L^2}{r_\eta}(\partial_x r_\eta)(\partial_x\psi_\eta)\\
&\quad+\frac{r_\theta}{r_\eta}\sin(\psi_\eta-\psi_\theta)-\frac{r_C}{r_\eta}\sin(\psi_\eta-\psi_C). \qquad (6)
\end{aligned}
$$

We can write the Lyapunov functional in transformed coordinates as well. The ideal helical equilibrium in the $C=0$ case takes a simple form,

$$
\begin{array}{rcccl}
\partial_x\psi_\theta &=& \partial_x\psi_\eta &\equiv \ \phi \ =& \text{constant},\\
\psi_\theta &=& \psi_\eta+\pi, &&\\
r_\theta &\equiv& R_\theta &=& \text{constant},\\
r_\eta &\equiv& R_\eta &=& \text{constant},
\end{array} \qquad (7)
$$

as does the Lyapunov functional in transformed coordinates at this equilibrium:

$$
V_e^* = -\int_\Omega \frac{1}{4}r_\theta^4\,dx. \qquad (8)
$$

Despite the simple form that the ideal equilibrium solutions and equilibrium energy take in polar coordinates, it turns out to be easier to analyze the stability of the ideal solutions using a different change of coordinates: one that retains the complex character of the dynamics, but rotates with the helical solution along the (single) space dimension.

3.b Stability analysis of equilibria

We will now outline the technique used for analyzing the stability of the helical equilibrium in the $C=0$ case in one space dimension. The goal of the analysis is to show that these equilibria are stable with respect to linear perturbations. The feature of the ideal helical solutions that makes this analysis tractable is that these solutions involve pure sinusoids with a single spatial frequency (or wave number), ϕ, which (without loss of generality) we will assume to be positive. Let the ideal helical equilibrium solution of interest have the form

$$
\theta = R_\theta e^{i\phi x},\ \ \eta = R_\eta e^{i\phi x}. \qquad (9)
$$

We define the new coordinates θ_H and η_H such that θ_H is always aligned with the ideal helical equilibrium solution:

$$
\theta_H = \theta e^{-i\phi x},\ \ \eta_H = \eta e^{-i\phi x}. \qquad (10)
$$

We now calculate the dynamics for θ_H and η_H:

$$
\begin{aligned}
\tau_\theta\partial_t\theta_H &= l^2(\partial_{xx}\theta_H + i2\phi\partial_x\theta_H - \phi^2\theta_H)\\
&\quad -|\theta_H|^2\theta_H + \theta_H + \eta_H, \qquad (11)\\
\tau_\eta\partial_t\eta_H &= L^2(\partial_{xx}\eta_H + i2\phi\partial_x\eta_H - \phi^2\eta_H) - \theta_H - \eta_H.
\end{aligned}
$$

The ideal helical equilibrium solution of interest is

$$
\begin{array}{rcccl}
\theta_H &=& R_\theta &=& \text{constant},\\
\eta_H &=& R_\eta &=& \text{constant}.
\end{array} \qquad (12)
$$

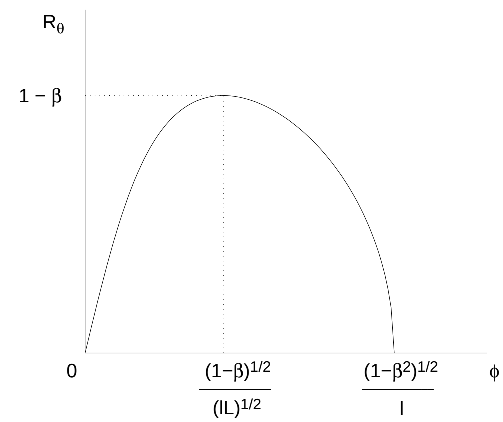

Figure 2: Plot of R_θ versus ϕ.

Solving for the constant values R_θ and R_η, we obtain

$$
\begin{array}{rcl}
R_\eta &=& -\dfrac{1}{1+L^2\phi^2}R_\theta,\\
R_\theta^2 &=& 1-l^2\phi^2-\dfrac{1}{1+L^2\phi^2}.
\end{array} \qquad (13)
$$

Note that $\max_\phi R_\theta = (1-\beta)$, and is attained for $\phi = \sqrt{1-\beta}/\sqrt{lL}$. Thus, the magnitude of the ideal helical solution is maximized for a value of ϕ near the reciprocal of the geometric mean of l and L, the activator and inhibitor length scales. (Keep in mind that we know from the form of the original Lyapunov functional on equilibria, V_e^*, that a larger R_θ corresponds to a lower energy V_e^*.) Figure 2 shows R_θ as a function of ϕ over the range of values $\phi > 0$ for which R_θ is well-defined.

Next, we perturb the dynamics using

$$
\theta_H = R_\theta + \delta\theta_H,\ \ \eta_H = R_\eta + \delta\eta_H. \qquad (14)
$$

We can write down a Lyapunov functional for the $\delta\theta_H$ and $\delta\eta_H$ dynamics, and then drop all but the lowest-order (i.e., quadratic) terms, which yields

$$
\begin{aligned}
V_q^* = \int_\Omega \Bigg[&\frac{l^2}{2}|\partial_x\delta\theta_H + i\phi\delta\theta_H|^2 - \frac{1}{2}(1-R_\theta^2)|\delta\theta_H|^2\\
&+\frac{L^2}{2}|\partial_x\delta\eta_H + i\phi\delta\eta_H|^2 + \frac{1}{2}|\delta\eta_H|^2 + \mathrm{Re}\{\overline{\delta\theta_H}\delta\eta_H\}\\
&-(\overline{\delta\theta_H})\left[(-1+L^2(\partial_x+i\phi)^2)^{-1}(\delta\theta_H)\right]\\
&+R_\theta^2(\mathrm{Re}\{\delta\theta_H\})^2\Bigg]dx, \qquad (15)
\end{aligned}
$$

At this point, we consider the Fourier series expansions

$$
\delta\theta_H = \sum_k \delta\theta_k e^{ikx},\ \ \delta\eta_H = \sum_k \delta\eta_k e^{ikx}. \qquad (16)
$$

To complete the analysis, we plug the Fourier series representations into V_q^*, and determine conditions on ϕ under which $V_q^* > 0$ for all nonzero perturbations (16). We find that For $\phi \in [\phi_{min}, \phi_{max}]$, where $\phi_{min} = \sqrt{1-\beta}/\sqrt{lL}$ and where ϕ_{max} is a constant with $\phi_{max} - \phi_{min} > 0$ sufficiently small, the ideal helical equilibrium solution is stable with respect to linear perturbations; i.e., $V_q^* > 0$ for all small perturbations (16) of the ideal helical equilibrium solution (except for purely translational perturbations).

4. Applications

Consider an endfire phased-array antenna. For an endfire array, we are interested in the antenna pattern due to constructive and destructive interference (some distance away) in the same plane as the array. Our goal is to control the mainlobe direction by exciting an ideal roll pattern oriented appropriately. (We do not care whether the helical solution along the dimension perpendicular to the rolls is left-handed or right-handed, but we do not want both present.) Figure 3 shows the initial in-phase excitation of a line of oscillators, with the other oscillators turned off. Figure 4 shows the resulting pattern that emerges when the entire oscillator array is turned on.

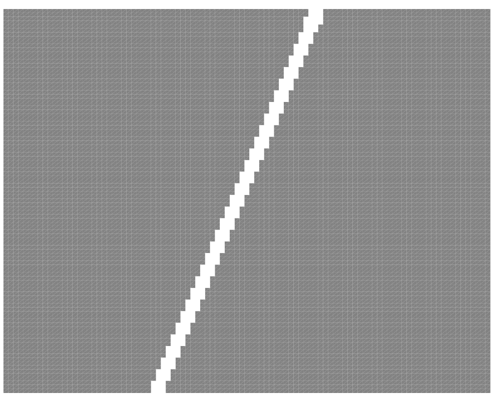

Figure 3: Initial excitation of a two-dimensional oscillator array.

In figure 4, there are perturbations of the ideal pattern at the edges of the array, but toward the middle of the array, the ideal pattern with the desired orientation is evident. To change the mainlobe direction of the endfire array, we would turn off the entire array of oscillators, excite a different line of oscillators to oscillate in-phase, and once again turn on the entire array and let it evolve toward the new equilibrium roll state with the orientation determined by the initial condition.

Although in principle, the array of resonant circuits required for the inhibitor equation could be implemented electronically like the oscillator array, MEMS microwave resonators are currently being developed to achieve higher quality factors in less integrated circuit area [6].

The micropositioning application involves MEMS more directly. It has been shown that arrays of asym-

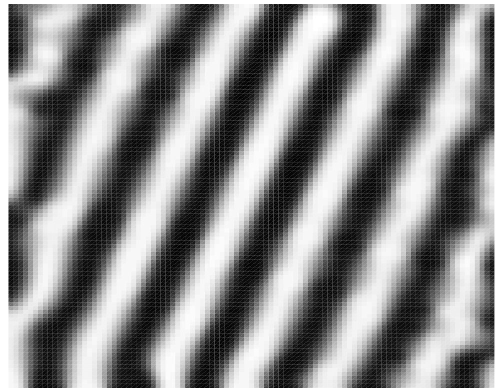

Figure 4: The resulting oscillator phases in the two-dimensional array.

metrical oscillating torsional microflaps can translate objects on top of the array [4]. Figure 5 shows an alternative structure based on torsional microflaps that produces rotory motion to translate a small object (which is large compared to the size of a single actuator) atop the array. We could drive the torsional microflaps using electrical oscillator circuits coupled so as to implement the complex activator-inhibitor equation (1).

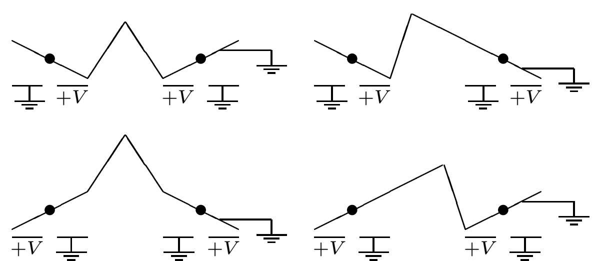

Figure 5: Schematic of a MEMS structure for converting torsional microflap actuation into rotory motion.

Although an ideal pattern of rolls for the complex activator-inhibitor equation would work for conveying an object atop the array, all we really need for inducing motion is that approximately equal numbers of actutators be supporting the object at each instant of time, which is expected to be the case when $C = 0$ regardless of whether the roll pattern is ideal or not. Whether the helical solutions are left- or right-handed also makes no difference in this application, because the direction of rotation of each actuator is the same regardless of what the phase differences are between adjacent actuators. So all we really need to control for the micropositioning application is where in the array we excite the pattern solution (with the alternative to exciting the pattern solution being turning the oscillators off). Furthermore, if we had (at least) three interlaced arrays of micro-actuators with their directions of motion distributed evenly around the unit circle, we could position an object by separately controlling the three interlaced arrays using a complex activator-inhibitor equation for each.

5. Modal dynamics

The fact that for $C = 0$ in the complex activator-inhibitor equation there is a stable helical solution is particularly suggestive that finite modal approximations of ideal spatially periodic pattern equilibria when $C \neq 0$ may be useful. For the modal analysis, we will assume (in addition to one spatial dimension and periodic boundary conditions) that $\Omega \subset \mathbb{R}$ is an interval with length equal to an integer number of periods of the ideal solution we are considering.

The first step is to rescale x by the wave number of the ideal solution, which we will assume to be $\sqrt{1-\beta}/\sqrt{lL}$ (since this is both the wave number of the minimum energy helical equilibrium when $C = 0$ and the wave number of the linear instability when $|C|$ is decreased through $(1-\beta)^3$). Next, we obtain the dynamical equations for the modal coefficients. Then we present a Lyapunov function for any finite number of modes. Finally, we present a bound on the higher-order modes so that we can justify retaining only the lower-order modes to obtain a good approximation of the equilibrium of interest. (It turns out that the same approach that works for the complex activator-inhibitor equation can also be applied to the modal dynamics for the real cubic nonlinearity activator-inhibitor equation, i.e., equation (1) where θ, η, and C are real [1].)

The rescaled dynamics are

$$\begin{aligned} \tau_\theta \partial_t \theta &= \beta(1-\beta)\partial_{xx}\theta - |\theta|^2\theta + \theta + \eta, \\ \tau_\eta \partial_t \eta &= \frac{1-\beta}{\beta}\partial_{xx}\eta - \eta - \theta + C. \end{aligned} \tag{17}$$

Plugging the Fourier series expansions

$$\theta = \sum_k \theta_k e^{ikx}, \quad \eta = \sum_k \eta_k e^{ikx}, \tag{18}$$

where because of the rescaling, k takes integer values, gives

$$\begin{aligned} \tau_\theta \dot{\theta}_m &= \left(1 - m^2\beta(1-\beta)\right)\theta_m - \Bigg[|\theta_m|^2\theta_m + 2\sum_{j\neq m} |\theta_j|^2 \theta_m \\ &\quad + \sum_{k\neq m} \overline{\theta}_{2k-m}\theta_k^2 + \sum_j \sum_{k\neq j,m,(m+j)/2} \overline{\theta}_j \theta_k \theta_{m+j-k} \Bigg] + \eta_m, \\ \tau_\eta \dot{\eta}_m &= -\left(1 + m^2\frac{1-\beta}{\beta}\right)\eta_m - \theta_m + C\delta_{m0}, \end{aligned} \tag{19}$$

where $\delta_{jk} = 1$ for $j = k$ and $\delta_{jk} = 0$ otherwise. Let

$$\begin{aligned} V &= -\frac{1}{2}\sum_m \left(1 - m^2\beta(1-\beta)\right)|\theta_m|^2 + \frac{1}{4}\sum_m |\theta_m|^4 \\ &+ \frac{1}{2}\sum_m \sum_{j\neq m} |\theta_j|^2 |\theta_m|^2 + \mathrm{Re}\left\{ \frac{1}{2}\sum_m \sum_{k\neq m} \overline{\theta}_{2k-m}\theta_k^2 \overline{\theta}_m \right\} \\ &+ \mathrm{Re}\left\{ \frac{1}{4}\sum_m \sum_{j\neq m} \sum_{k\neq j,m,(m+j)/2} \overline{\theta}_j \theta_k \theta_{m+j-k}\overline{\theta}_m \right\} \\ &+ \frac{1}{2}\sum_m \left(1 + m^2\frac{1-\beta}{\beta}\right)|\eta_m|^2 - \mathrm{Re}\{C\overline{\eta}_0\} \\ &+ \sum_m \mathrm{Re}\{\overline{\theta_m}\eta_m\} + \sum_m \left(1 + m^2\frac{1-\beta}{\beta}\right)^{-1} |\theta_m - C\delta_{m0}|^2. \end{aligned} \tag{20}$$

Then the function V given by equation (20), appropriately truncated, is a radially unbounded Lyapunov function for the dynamics given by equation (19) for any finite number of modes (as can be proved by direct differentiation). We find that if $\alpha = \tau_\theta/\tau_\eta > 1$, then $\dot{V} \leq 0$, with $\dot{V} = 0$ only for equilibria (for any finite number of modes).

We can obtain the following bound for the error in approximating the exact equilibrium solution θ with its finite modal approximation $\sum_{k=-N}^{N} \theta_k e^{ikx}$:

$$\left| \theta - \sum_{k=-N}^{N} \theta_k e^{ikx} \right| \leq \frac{1}{N}\left[\frac{4(1+|C|^2)}{\beta(1-\beta)} \right], \tag{21}$$

for $N^2 \geq 2/(\beta(1-\beta))$ [1]. Thus, the error in approximating an exact periodic equilibrium solution θ of equation (19) using a finite number of modes approaches zero as the number of modes used becomes large. Furthermore, the smaller β is (for $\beta < 1/2$), the more terms are needed to achieve a given error tolerance, in accord with what one would expect, considering β represents the ratio of the two length scales present in the dynamics.

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