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Stationary Bifurcation Control for Systems with Uncontrollable Linearization

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Abstract

Stationary bifurcation control is studied under the assumption that the critical zero eigenvalue is uncontrollable for the linearized system. The development facilitates explicit construction of feedback control laws that render the bifurcation supercritical. Thus, the bifurcated equilibria in the controlled system are guaranteed stable. Both pitchfork bifurcation and transcritical bifurcation are addressed. The results obtained for pitchfork bifurcations apply to general nonlinear models smooth in the state and the control. For transcritical bifurcations, the results require the system to be affine in the control.

1 Introduction

In this paper, feedback control of stationary bifurcations is considered in the case that the critical mode is uncontrollable for the linearized system. The aim of the control design is described as follows Abed and Fu (1987). A nonlinear system is given for which a nominal equilibrium loses stability through a real eigenvalue crossing the imaginary axis at the origin. Under these circumstances, the system undergoes a stationary bifurcation. This can be a transcritical bifurcation or a pitchfork bifurcation. Figure 1(a) illustrates a transcritical bifurcation, and figure 1(b),(c) illustrate the two types of pitchfork bifurcation (subcritical and supercritical, respectively). The *direction* of a pitchfork bifurcation is its subcriticality or supercriticality. As discussed by Abed and Fu (1987), the supercritical pitchfork bifurcation (figure 1(c)) is preferable in practice since after the nominal solution has lost stability, new stable equilibria arise that provide a nearby operating condition. This observation motivates the search for feedback control laws that render a stationary bifurcation supercritical. This is the local stationary bifurcation control problem.

As shown in Abed and Fu (1987), nonlinear stabilizing controllers can be readily obtained if the zero eigenvalue (critical mode) is controllable for the linearized system at the bifurcation point. The situation was found to be significantly more difficult if the controllability condition fails, even though an analogous problem for Hopf bifurcation was addressed successfully (Abed and Fu 1986).

Applications arise in which a stationary bifurcation occurs with the critical mode uncontrollable and for which the design of stabilizing controllers is of significant importance. One such example is control of rotating stall in axial flow compressors (Liaw and Abed 1996).

There have been other investigations into design of bifurcation control laws for systems with an uncontrollable critical mode. For example, Fu and Abed (1993) consider the design of linear feedback control laws for nonlinear systems affine in the control. Kang (1998) also studies stabilization for systems affine in the control under the condition that the noncritical modes are controllable.

In the present paper, no assumption of controllability of the noncritical modes is made, and the general multi-input case is considered. Control of pitchfork bifurcations is considered for general nonlinear system models. For the case of transcritical bifurcations, the analysis is relegated to a narrow class of affine models. In both cases, the results permit explicit derivation of control laws.

The remainder of the paper proceeds as follows. In Section 2, basic results on bifurcation analysis and control are recalled. In Section 3.1, these results are used for control design of general systems undergoing pitchfork bifurcation. In Section 3.2, control of a class of affine systems undergoing transcritical bifurcation is considered. Conclusions are collected in Section 4.

2 Background

This section reviews background material on stationary bifurcations and their control from Abed and Fu (1987).

2.1 Bifurcation formulas for stationary bifurcation

Determination of whether a stationary bifurcation is supercritical, subcritical or transcritical can be achieved using so-called bifurcation coefficients. These are coefficients in Taylor series expansions of quantities, especially eigenvalues, of bifurcated solutions in a small neighborhood of the bifurcation point. Formulas for these coefficients are referred to as bifurcation formulas. Next, we recall bifurcation formulas that will be needed in the control design of this paper. References (Howard 1979, Iooss and Joseph 1980, Abed and Fu 1987) can be consulted for further details.

Consider a one-parameter family of nonlinear autonomous systems

$$\dot{x} = f(x, \mu) \tag{1}$$

where $x \in R^n$ is the state vector and μ is a real-valued parameter. Let $f(x, \mu)$ be sufficiently smooth in x and μ and let x^0_μ be the nominal equilibrium point of the system as a function of the parameter μ.

Suppose that the next hypothesis holds.

(S) The Jacobian matrix of system (1) at the equilibrium x^0_μ has a simple eigenvalue $\lambda_1(\mu)$ such that $\lambda_1(0) = 0$ and $\lambda_1'(0) \neq 0$, and the remaining eigenvalues lie in the open left half of the complex plane for $\mu = 0$.

The stationary bifurcation theorem (Chow and Hale 1982, Guckenheimer and Holmes 1983) asserts that hypothesis (S) implies a stationary bifurcation from x^0_0 at $\mu = 0$ for (1). That is, a new equilibrium point bifurcates from x^0_0 at $\mu = 0$. Near the point $(x^0_0, 0)$ of the $(n+1)$-dimensional (x, μ)-space, there is a parameter ϵ and a locally unique curve of critical points $(x(\epsilon), \mu(\epsilon))$, distinct from x^0_μ and passing through $(x^0_0, 0)$, such that for all sufficiently small $|\epsilon|$, $x(\epsilon)$ is an equilibrium point of (1) when $\mu = \mu(\epsilon)$.

The parameter ϵ may be chosen such that $x(\epsilon), \mu(\epsilon)$ are smooth. The series expansions of $x(\epsilon), \mu(\epsilon)$ can be written as

$$\mu(\epsilon) = \mu_1\epsilon + \mu_2\epsilon^2 + \ldots \tag{2}$$

$$x(\epsilon) = x^0_\mu + x_1\epsilon + x_2\epsilon^2 + \ldots \tag{3}$$

If $\mu_1 \neq 0$, the system undergoes a transcritical bifurcation from x^0_μ at $\mu = 0$. That is, there is a second equilibrium point besides x^0_μ for both positive and negative values of μ with $|\mu|$ small (see figure 1 (a)). The stability of the bifurcated equilibria in the case of transcritical bifurcation is as depicted in the figure. If $\mu_1 = 0$ and $\mu_2 \neq 0$, the system undergoes a pitchfork bifurcation for $|\mu|$ sufficiently small. That is, there are two new equilibrium points existing simultaneously, either for positive or for negative values of μ with $|\mu|$ small (see figure 1 (b),(c)). The bifurcated equilibrium points have an eigenvalue $\beta(\epsilon)$ determining their stability which vanishes at $\epsilon = 0$. The series expansion $\beta(\epsilon)$ is given by

$$\beta(\epsilon) = \beta_1\epsilon + \beta_2\epsilon^2 + \ldots \tag{4}$$

with

$$\beta_1 = -\mu_1\lambda'(0) \tag{5}$$

and, in case $\beta_1 = 0$, β_2 is given by

$$\beta_2 = -2\mu_2\lambda'(0) \tag{6}$$

The stability coefficients β_1 and β_2 can be determined solely from eigenvector computations and the coefficients of the series expansion of the vector field. System (1) can be written in the series form

$$\begin{aligned}
\dot{\tilde{x}} &= L_\mu \tilde{x} + Q_\mu(\tilde{x}, \tilde{x}) + C_\mu(\tilde{x}, \tilde{x}, \tilde{x}) + \dots \\
&= L_0\tilde{x} + \mu L_1\tilde{x} + \mu^2 L_2\tilde{x} + \dots \\
&\quad + Q_0(\tilde{x}, \tilde{x}) + \mu Q_1(\tilde{x}, \tilde{x}) + \dots \\
&\quad + C_0(\tilde{x}, \tilde{x}, \tilde{x}) + \dots
\end{aligned} \tag{7}$$

where $\tilde{x} = x - x_0^0, L_\mu, L_1, L_2$ are $n \times n$ matrices, $Q_\mu(x, x), Q_0(x, x), Q_1(x, x)$ are vector-valued quadratic forms generated by symmetric bilinear forms, and $C_\mu(x, x, x), C_0(x, x, x)$ are vector-valued cubic forms generated by symmetric trilinear forms.

By hypothesis (S), the Jacobian matrix L_0 has only one simple zero eigenvalue with the remaining eigenvalues stable. Denote by l and r the left (row) and right (column) eigenvectors of the matrix L_0 associated with the simple zero eigenvalue, respectively, where first component of r is set to be 1 and the left eigenvector l is chosen such that $lr = 1$. It is well known that

$$\lambda'(0) = lL_1 r \tag{8}$$

A stability result for the bifurcated equilibria of system (7) is given in the following lemma.

Lemma 1 *If $\beta_1 = 0$ and $\beta_2 \neq 0$, then Eq. (1) undergoes a pitchfork bifurcation at $\mu = 0$. If $\beta_1 \neq 0$, then Eq. (1) undergoes a transcritical bifurcation. In the former case, the pitchfork bifurcation is supercritical if $\beta_2 < 0$, but is subcritical if $\beta_2 > 0$. Here,*

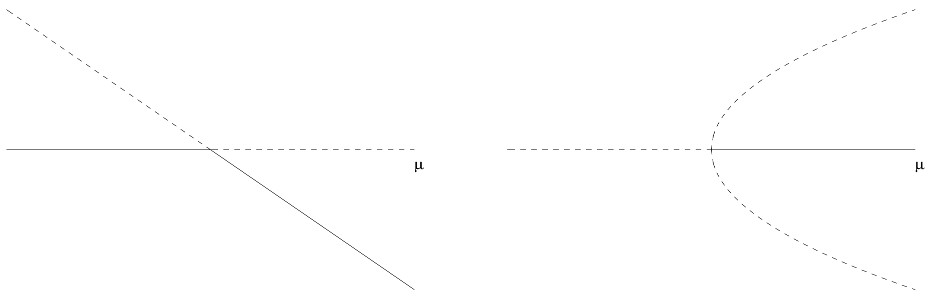
$$\beta_1 = lQ_0(r, r) \tag{9}$$

$$\beta_2 = 2l\{2Q_0(r, x_2) + C_0(r, r, r)\} \tag{10}$$

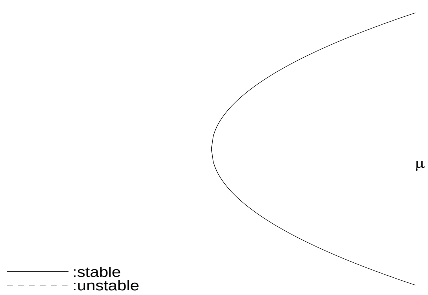
where

$$x_2 = -(R^T R)^{-1} R^T \begin{bmatrix} Q_0(r, r) \\ 0 \end{bmatrix}$$

$$R = \begin{bmatrix} L_0 \\ l \end{bmatrix} \tag{11}$$



(a) Transcritical bifurcation (b) Subcritical pitchfork bifurcation



(c) Supercritical pitchfork bifurcation

Figure 1: Stationary bifurcation diagrams. The variable in the vertical direction represents equilibrium values for a system in normal form.

2.2 Bifurcation control in the case of a controllable critical mode

Next, we recall the results of Abed and Fu (1987) which give sufficient conditions for local stabilizability of an equilibrium point at criticality and for local stabilizability of bifurcated equilibria under assumption (S). These conditions involve assumptions on the controllability of the critical mode of the linearized system.

Consider a system

$$\dot{x} = f(x, \mu, u) \tag{12}$$

where $x \in R^n$ is the state vector, $u \in R^m$ is input vector, and $\mu \in R$ is the bifurcation parameter. Expanding the vector field of (12) in x, μ and u, we have

$$\begin{aligned} \dot{x} &= f(x, \mu, u) \\ &= L_0 x + \mu L_1 x + \sum_{i=1}^{m} u_i \tilde{L}_1^i x + \sum_{i=1}^{m} b^i u_i + Q_0(x, x) \\ &\quad + \mu^2 L_2 x + \mu Q_1(x, x) + \sum_{i=1}^{m} u_i \tilde{Q}_i(x, x) \\ &\quad + C_0(x, x, x) + \dots \end{aligned} \tag{13}$$

Note that the linearized system is

$$\dot{x} = L_0 x + \sum_{i=1}^{m} b^i u_i \tag{14}$$

At the critical parameter $\mu := \mu_c$, L_0 has one zero eigenvalue. A feedback control consisting of quadratic and cubic terms is considered:

$$u(x) = \sum_{i=1}^{m} x^T Q_{u_i} x + C_{u_i}(x, x, x) \tag{15}$$

Following (Abed and Fu 1987) use an asterisk to denote quantities for the closed-loop system. Then the bifurcation coefficients taking into account the state feedback (15) are (details in (Abed and Fu 1987)):

$$\beta_1^* = l\{Q_0(r, r) + \sum_{i=1}^{m} (r^T Q_{u_i} r) b^i\} \tag{16}$$

and, if $\beta_1^* = 0$,

$$\begin{aligned} \beta_2^* &= 2l\{2Q_0(r, x_2^*) + 2\sum_{i=1}^{m} (r^T Q_{u_i} x_2^*) b^i \\ &\quad + \sum_{i=1}^{m} (r^T Q_{u_i} r \tilde{L}_1^i r) + C_0(r, r, r) + \sum_{i=1}^{m} C_{u_i}(r, r, r) b^i\} \end{aligned} \tag{17}$$

Here, x_2^* is given by

$$x_2^* = -(R^T R)^{-1} R^T \begin{bmatrix} Q_0(r, r) + \sum_{i=1}^{m} (r^T Q_{u_i} r) b^i \\ 0 \end{bmatrix}$$

$$= \quad x_2 - \sum_{i=1}^{m}(r^T Q_{u_i} r)(R^T R)^{-1} R^T \begin{bmatrix} b^i \\ 0 \end{bmatrix} \tag{18}$$

where x_2 is the value for $u \equiv 0$.

By employing these formulas, the following two results were obtained in (Abed and Fu 1987).

Theorem 1 *Let hypothesis (S) hold and assume* $lb^i \neq 0$ *for some* $i \in 1, \ldots, m$*, that is, the critical zero eigenvalue is controllable for the linearized version of Eq. (12) near the origin. Then there is a smooth feedback control* $u = u(x)$ *with* $u(0) = 0$*, containing only quadratic and cubic terms in* x*, which solves the local stationary bifurcation problem for (12). Moreover, the quadratic terms in* $u(x)$ *can be used to ensure that* $\beta_1 = 0$ *for the controlled system, and the cubic terms can then be used to ensure that* $\beta_2 < 0$.

Theorem 2 *Let hypothesis (S) hold and assume* $lb^i = 0$ *for all* $i = 1, \ldots, m$*, that is, the critical zero eigenvalue is uncontrollable for the linearized version of Eq. (12) near the origin. Then if* $\beta_1 \neq 0$ *for with* $u(x) \equiv 0$*, the local stationary problem for (12) is not solvable by a smooth feedback control with vanishing linear part.*

3 Stationary Bifurcation Control with Uncontrollable Linearization

In this section, we investigate the design of feedback control laws for stabilizing stationary bifurcation for systems with an uncontrollable critical mode. We first consider the case of pitchfork bifurcation and then transcritical bifurcation.

3.1 Pitchfork bifurcation control

Suppose that the system (12) undergoes a pitchfork bifurcation and that the critical (zero) eigenvalue is uncontrollable for the linearized system at bifurcation. This implies that the bifurcation coefficient $\beta_1 = 0$ for the open-loop system.

To facilitate the search for a stabilizing control law, perform a linear coordinate transformation

$$\begin{aligned} z &= Tx \\ &= \begin{bmatrix} r & \xi^2 & \dots & \xi^n \end{bmatrix} x \end{aligned} \tag{19}$$

where $r, \xi^2, \dots, \xi^n$ are orthogonal column vectors in R^n. Thus $det\ T \neq 0$. In z coordinates, the system becomes

$$\begin{aligned} \dot{z} &= f(z, \mu, u) \\ &= L_0 z + \mu L_1 z + \sum_{i=1}^{m} u_i \tilde{L}_1^i z + \sum_{i=1}^{m} b^i u_i + Q_0(z,z) \\ &\quad + \mu^2 L_2 z + \mu Q_1(z,z) + \sum_{i=1}^{m} u_i \tilde{Q}_1(z,z) \\ &\quad + C_0(z,z,z) + \dots \end{aligned} \tag{20}$$

For simplicity, we do not change the notation for the quantities $L_0, L_1, Q_0, \dots$, even though they are affected by the transformation. Because of the orthogonality assumption above, L_0 takes the form

$$L_0 = \begin{bmatrix} 0 & 0 \\ 0 & A_1 \end{bmatrix} \tag{21}$$

where A_1 is an $(n-1)\times(n-1)$ invertible matrix. Take the right and left eigenvectors corresponding to the zero eigenvalue of L_0 (21) to be $r = [1, 0, \dots, 0]^T$ and $l = [1, 0, \dots, 0]$, respectively.

Since $\beta_1 = 0$ in the absence of control, we limit our search to feedback laws with a vanishing linear part. This ensures that β_1^*, the value of β_1 after feedback, is also 0. (As in the preceding section, an asterisk indicates a quantity for the closed-loop system.) Thus, controls are sought in the form

$$u_i(z) = z^T Q_{u_i} z + C_{u_i}(z,z,z) \qquad \text{for } i = 1, \dots, m \tag{22}$$

The coefficient β_2^*, which determines whether the pitchfork bifurcation is supercritical or subcritical for the closed loop system, is given by (17). Using $l = [1, 0, \dots, 0]$ and the structure of L_0, we find that

$$(R^T R)^{-1} R^T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & A_1^{-1} & 0 \end{bmatrix} \tag{23}$$

In addition, since only the first element of r is nonzero,

$$r^T Q_{u_i} r = c_u^i \tag{24}$$

where c_u^i denotes the coefficient of z_1^2 in the formula (22) for the input u_i. Thus, we can conclude that only the z_1^2 terms in an input of the form (22) affect the direction of the pitchfork bifurcation. Eq. (18) for x_2^* now gives

$$x_2^* = x_2 - \sum_{i=1}^{m} c_u^i \zeta^i, \tag{25}$$

where the vectors ζ^i, $i = 1, \ldots, m$, are defined as

$$\zeta^i = \begin{bmatrix} 0 \\ \bar{\zeta}^i \end{bmatrix} = \begin{bmatrix} 0 \\ A_1^{-1}\bar{b}^i \end{bmatrix} \tag{26}$$

and where the $\bar{b}^i$ are $(n-1)$-dimensional vectors containing all but the first (0) component of the vectors b^i; i.e.,

$$b^i = \begin{bmatrix} 0 \\ \bar{b}^i \end{bmatrix} \tag{27}$$

Note that $lb^i = 0$ and $l = [1, 0, \ldots, 0]$.

Again, using the fact $lb^i = 0$ for $i = 1, \ldots, m$, Eq. (17) simplifies to

$$\beta_2^* = 2l\{2Q_0(r, x_2^*) + \sum_{i=1}^{m} c_u^i \tilde{L}_1^i r + C_0(r, r, r)\} \tag{28}$$

Note that the eigenvectors l and r are unaffected by the feedback since the control law does not contain linear terms; thus the asterisk notation is not needed for these vectors. This will change below in our discussion of control of transcritical bifurcations. Note also that the cubic terms in the input do not have any effect on β_2^*. Moreover, since only the first elements of r and l are nonzero and the z_1^2 term in $\dot{z}_1$ vanishes (since $\beta_1 = lQ_0(r, r) = 0$), Eq. (28) gives

$$\begin{aligned} \beta_2^* &= 4\{\sum_{i=2}^{n} c_{1i} x_2^i - \sum_{j=1}^{m} c_u^j \sum_{i=2}^{n} c_{1i} k_i^j\} + 2(\sum_{j=1}^{m} c_u^j \tilde{L}_1^{j1}) + 2c_3 \\ &= 2\{2\sum_{i=2}^{n} c_{1i} x_2^i + 2c_3\} + 2\sum_{i=1}^{m} c_u^i \{\sum_{j=2}^{n} c_{1j} \zeta_j^i + \tilde{L}_1^{i1}\} \end{aligned} \tag{29}$$

where c_{1i} and c_3 are the coefficient of $z_1 z_i$ and the coefficient of z_1^3 in $\dot{z}_1$, respectively. Also, $\tilde{L}_1^{i1}$ denotes the $(1, 1)$ element of the matrix $\tilde{L}_1^i$.

The following assumption is now introduced.

(A1) At least one among the coefficients ρ_i, $i \in 1, \ldots, m$, does not vanish, where the ρ_i are defined as follows:

$$\rho_i = -4 \sum_{j=2}^{n} c_{1j} \zeta_j^i + 2\tilde{L}_1^{i1} \tag{30}$$

Theorem 3 *Let the system (12) with input $u \equiv 0$ undergo a pitchfork bifurcation from the origin at $\mu = 0$. Also, assume $lb^i = 0$ for all $i = 1, \ldots, m$; that is, the critical zero eigenvalue is uncontrollable for the linearized version of (12). Moreover, suppose that (A1) holds. Then, there exists a smooth feedback control containing only quadratic terms in z_1, which solves the local stationary bifurcation control problem for Eq. (12). Moreover, the nonvanishing of at least one of the ρ_i is necessary and sufficient for stabilization using feedback of the form (22).*

3.2 Transcritical bifurcation control

Next, we consider the case in which the system (12) undergoes a transcritical bifurcation, that is, (S) holds and $\beta_1 \neq 0$. As in the foregoing, the critical zero eigenvalue is assumed uncontrollable for the linearized version of Eq. (12). We proceed in two steps. First, we use *linear* state feedback to transform the transcritical bifurcation into a pitchfork bifurcation. Then, we add *quadratic* (plus higher order if desired) feedback to stabilize the achieved pitchfork bifurcation.

For simplicity, we again consider the system after the linear transformation (19) has been performed. From Theorem 2, a feedback consisting of only quadratic and cubic terms cannot transform the transcritical bifurcation into a pitchfork bifurcation. It is for this reason that we seek a feedback control that also includes a linear term, taking the form

$$u_i(z) = K_i z + z^T Q_{u_i} z + C_{u_i}(z, z, z) \qquad \text{for } i = 1, \ldots, m \tag{31}$$

Unfortunately, a linear term in the state feedback significantly complicates analysis of the bifurcation. A linear feedback can affect the eigenvalues and eigenvectors of the linearization, making determination of its effect on the bifurcation formulas (9), (10) very difficult.

To mitigate this problem, we focus on the following special class of nonlinear systems:

$$\dot{z} = f(z, \mu) + \sum_{i=1}^{m} b^i u_i \tag{32}$$

Note that this model is affine in the inputs, and that the inputs enter the dynamics through multiplication by constant vectors.

For the system (32) to undergo a transcritical bifurcation for open-loop system, bifurcation coefficient β_1 must be nonzero. The bifurcation coefficient β_1 for the open-loop system (32) can be calculated by using Eq. (9). Recall that only the first element of the left and right eigenvectors corresponding to critical zero eigenvalue are nonzero. The coefficient β_1 is given by

$$\begin{aligned}\beta_1 &= lQ_0(r,r) = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} r^T \Xi_1 r \\ r^T \Xi_2 r \\ \vdots \\ r^T \Xi_n r \end{bmatrix} \\ &= r^T \Xi_1 r = c_{z_1^2} \neq 0 \end{aligned} \tag{33}$$

where the Ξ_i are symmetric matrices that are easy to compute given Q_0 (the bilinear form $Q_0(x,y)$).

Since the first element of each of the b^i vanishes (27), the controls u_i do not change the right side of the equation for $\dot{z}_1$ in (32). Thus, Ξ_1 is not affected by feedback. It should also be noted that linear state feedback does not change l (the left eigenvector associated with the critical zero eigenvalue) due to the structure of the Jacobian matrix (21) and the uncontrollability of the critical mode. Thus, the only way to transform the transcritical bifurcation into a pitchfork bifurcation (i.e., to make $\beta_1 = 0$) is to change the right eigenvector r through linear state feedback.

Here, for simplicity we limit the linear state feedback to be a function of z_1 alone. Thus, we consider state feedbacks of the form

$$u_i(z) = k_i z_1 \text{ for } i = 1, \dots, m \tag{34}$$

where $k_i \in R$ for $i = 1, \dots, m$. (Note that the i-th scalar k_i is the first component of the row vector K_i of Eq. (31), for $i = 1, \dots, m$.) Once appropriate gains k_i are found, nonlinear terms will be re-inserted in the feedback.

In Theorem 4 below, we will require at least one of the following two assumptions to hold.

(B1) There exists a $v = \begin{bmatrix} v_1 \\ \bar{v} \end{bmatrix}$ with $v_1 \neq 0$ $v_1 \in R$, that belongs to the null space of Q_1 and $\bar{v}$ belongs to the linear space spanned by the $\bar{\zeta}^i$ for $i = 1, \dots, m$ (26).

(B2) Q_1 has a positive eigenvalue (λ_p) and the negative eigenvalue (denote λ_n) with corresponding eigenvector p and n such that following three conditions hold.

(B2-1) At least one of the eigenvectors (n and p) has a nonzero first element.

(B2-2) Writing $n = \begin{bmatrix} n_1 \\ \bar{n} \end{bmatrix}$ and $p = \begin{bmatrix} p_1 \\ \bar{p} \end{bmatrix}$ where n_1 and p_1 are scalars, $\bar{n}$ are $\bar{p}$ belong to the linear space spanned by the $\bar{\zeta}^i$ for $i = 1, \dots, m$ (26).

(B2-3) The following pair of equations is valid:

$$\begin{aligned} n_1 + p_1 &= 1 \\ \lambda_p \sum_{i=1}^{n} p_i^2 + \lambda_n \sum_{i=1}^{n} n_i^2 &= 0 \end{aligned} \tag{35}$$

where p_i and n_i denote the i-th element of p and n, respectively.

The condition (B2-3) fixes a scaling on the eigenvectors n and p. Also, note that if only one of n_1 and p_1 is nonzero, then Eq. (35) can always be achieved.

Theorem 4 *Let the system (32) with input $u \equiv 0$ undergo a transcritical bifurcation from the origin at $\mu = 0$. Also, assume $lb^i = 0$ for all $i = 1, \ldots, m$; that is, the critical zero eigenvalue is uncontrollable for the linearized version of (32). Moreover, suppose that either assumption (B1) or (B2) holds. Then, the transcritical bifurcation of system (32) is transformable to a pitchfork bifurcation by means of linear state feedback of the form (34).*

Proof: The Jacobian matrix after applying a linear state feedback of the form (34) is given by

$$L_0^* = \begin{bmatrix} 0 & 0 \\ \sum_{i=1}^{m} k_i \bar{\zeta}_i & A_1 \end{bmatrix} \tag{36}$$

The right eigenvector r corresponding to the zero eigenvalue becomes $\begin{bmatrix} 1 \\ \bar{r} \end{bmatrix}$, where the first element of r is 1 since $lr = 1$ and $l = [1, 0, \ldots, 0]$, and where the subvector $\bar{r}$ solves the following equation:

$$\sum_{i=1}^{m} k_i \bar{b}^i + A_1 \bar{r} = 0 \tag{37}$$

Since A_1 is an $(n-1) \times (n-1)$ full rank matrix, we have

$$\begin{aligned} \bar{r} &= \sum_{i=1}^{m} k_i A^{-1} \bar{b}^i \\ &= \sum_{i=1}^{m} k_i \bar{\zeta}^i \end{aligned} \tag{38}$$

It is easy to see that if either assumption (B1) or (B2) is satisfied, then we can set the right eigenvector corresponding to the critical eigenvalue to a value which results in $\beta_1 = 0$. ■

Note that including other states besides z_1 in the linear state feedback will change the A_1 matrix. Thus, it is possible to design a linear state feedback that will maintain stability of matrix A_1 and result in a desired set of vectors $\bar{\zeta}^i = A_1^{-1}\bar{b}^i$ which satisfies the conditions of Theorem 4.

The following theorem follows directly from these observations.

Theorem 5 *Let the system (32) with input $u \equiv 0$ undergo a transcritical bifurcation from the origin at $\mu = 0$. Assume $lb^i = 0$ for all $i = 1, \ldots, m$; that is, the critical zero eigenvalue is uncontrollable for the linearized version of (32). Moreover, suppose that Ξ_1 is either positive definite or negative definite. Then, bifurcation control of the system (32) cannot be achieved by means of state feedback.*

Proof: For a pitchfork bifurcation to occur in the closed-loop system, β_1^* must vanish. For the system (32), β_1^* is given by

$$\beta_1^* = r^{*T}\Xi_1 r^* \tag{39}$$

where r^* is the right eigenvector of L_0^* (the closed loop Jacobian matrix) associated with the eigenvalue 0. Since Ξ_1 is either positive definite or negative definite, there is no nonzero $r^* \in R^n$ for which β_1^* vanishes. ■

Now, we want to add a quadratic term and/or cubic term to the linear feedback such that the pitchfork bifurcation that system (32) will undergo is guaranteed to be supercritical. To determine direction of pitchfork bifurcation, we calculate β_2. Recall that the left eigenvector corresponding to the critical zero eigenvalue is unaffected by linear state feedback. Also, recall that only the first element of the left eigenvector is nonzero and it takes the value 1. Using these facts and Eq. (17), we find

$$\beta_2^* = 4r^{*T}\Xi_1 x_2^* + 2\sum_{i=1}^{m} r^{*T} Q_{u_i} r^* \tilde{L}_1^{1i} r^* + 2C_{01}(r^*, r^*, r^*) \tag{40}$$

where C_{01} represents the cubic terms in $\dot{z}_1$ and $\tilde{L}_1^{1i}$ denotes first row of matrix $\tilde{L}_1^i$. Using Eq. (18), (40) becomes

$$\begin{aligned} \beta_2^* &= 4r^{*T}\Xi_1 x_2 - 4r^{*T}\Xi_1 \sum_{i=1}^{m} r^{*T} Q_{u_i} r^* \delta_i + 2\sum_{i=1}^{m} r^T Q_{u_i} r \tilde{L}_1^{1i} r^* + 2C_{01}(r^*, r^*, r^*) \\ &= 4r^{*T}\Xi_1 x_2 + 2C_{01}(r^*, r^*, r^*) + \sum_{i=1}^{m} r^{*T} Q_{u_i} r^* \{-4r^{*T}\Xi_1 \delta_i + 2\tilde{L}_1^{1i} r^*\} \end{aligned} \tag{41}$$

where $\delta_i = (R^T R)^{-1} R^T \begin{bmatrix} b^i \\ 0 \end{bmatrix}$. The next theorem follows from the preceding discussion. The theorem make use of the following assumption.

(B3) At least one among the coefficients ν_i, $i \in 1, \ldots, m$ does not vanish, where the ν_i are defined as follows:

$$\nu_i := -4r^T \Xi_1 \delta_i + 2\tilde{L}_1^{1i} r^* \tag{42}$$

Theorem 6 *Let the system (32) with $u \equiv 0$ undergo a transcritical bifurcation from the origin at $\mu = 0$. Assume $lb^i = 0$ for all $i \in 1, \ldots, m$, that is, the critical zero eigenvalue is uncontrollable for the linearized version of (32). Moreover, suppose that (B3) holds and also that at least one of (B1) or (B2) holds. Then, there exists a smooth feedback control in the form of (31), which solves the local stationary bifurcation control problem for Eq. (32).*

Proof: Let $\hat{i}$ be an index for which $\nu_{\hat{i}} \neq 0$. Then, letting the control $u_{\hat{i}}$ of (31) be $u_{\hat{i}}(z) = K_{\hat{i}} z + z^T Q_{u_{\hat{i}}} z = k_{\hat{i}} z_1 + z^T Q_{u_{\hat{i}}} z$ we can set β_2^* (41) to any desired value. One such example, which is independent of r^*, is $z^T Q_{u_o} z = c z_1^2$ since the first element of r^* is always 1. Here, c is a real constant chosen to ensure that β_2^* is negative. ■

4 Conclusion

In this paper, we studied stationary bifurcation control for systems with uncontrollable linearization. For systems undergoing pitchfork bifurcation, purely nonlinear state feedback was used to achieve a supercritical bifurcation in the closed-loop system. This was done under general conditions for a large class of real-analytic models. Analogous results for control of transcritical bifurcation were obtained for a smaller class of systems using a two-step approach. In the first step, linear feedback was used to transform the bifurcation into a pitchfork bifurcation. In the second step, quadratic feedback was employed to ensure supercriticality of the achieved pitchfork bifurcation. In both cases, the designed feedback laws would represent the first phase of an actual design, which could be followed with inclusion of other feedback terms to optimize system performance.

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