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A Lyapunov Functional for the Cubic Nonlinearity Activator-Inhibitor Model Equation

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A Lyapunov functional for the cubic nonlinearity activator-inhibitor model equation

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Abstract

The cubic nonlinearity activator-inhibitor model equation is a simple example of a pattern-forming system for which strong mathematical results can be obtained. Basic properties of solutions and the derivation of a Lyapunov functional for the cubic nonlinearity model are presented. Potential applications include control of large MEMS actuator arrays.

1. Introduction

There are a variety of pattern-forming systems, and the properties and generation of patterns in various physical contexts, such as in chemical reactions, have been extensively studied [1]. Distributed control schemes based on pattern-forming-system dynamics might be useful for applications involving large numbers of actuators and sensors. For example, with MEMS technology it is now possible to realize a 1000 by 1000 array of torsional microflaps in a square inch (for digital micromirror chips), and similar actuator arrays could potentially be used to influence boundary layer fluid flow, micro-position small parts, or manipulate small amounts of substances for chemical reactions.

The cubic nonlinearity activator-inhibitor model equation is a simple example of a pattern-forming system for which strong mathematical results can be obtained. Besides spatially periodic patterns, interesting equilibria such as finite-amplitude spike solutions can also be excited. After discussing the general behavior of the system and citing the basic properties of solutions, a Lyapunov functional for the cubic nonlinearity model is derived. The Lyapunov functional derivation is based on a technique developed by Brayton and Moser for nonlinear circuit theory [2]. Finally, some generalizations of the dynamics and Lyapunov functional are briefly described.

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2. Activator-inhibitor equations

Activator-inhibitor equations are a special case of the generalized reaction-diffusion equation

$$\tau \partial_t \mathbf{u} = \mathbf{D} \Delta \mathbf{u} + \mathbf{f}(\mathbf{u}), \quad (1)$$

where \mathbf{D} and τ are constant matrices, and $\mathbf{u}(\mathbf{x}, t)$ represents the vector of concentrations of the reactants at each point $\mathbf{x} \in \Omega \subset \mathbf{R}^n$, Ω open and bounded, and at each time $t \geq 0$. The activator-inhibitor dynamics are

$$\begin{aligned} \tau_\theta \partial_t \theta &= l^2 \Delta \theta - q(\theta, \eta, C) \\ \tau_\eta \partial_t \eta &= L^2 \Delta \eta - Q(\theta, \eta, C), \end{aligned} \quad (2)$$

where θ is the activator concentration, η is the inhibitor concentration; τ_θ , τ_η , l , and L are positive constants setting the time and length scales for the variation of the activator and inhibitor; and C is the bifurcation (or control) parameter. The spatially uniform (or homogeneous) equilibrium state (θ_h, η_h) is determined from

$$q(\theta_h, \eta_h, C) = 0, \quad Q(\theta_h, \eta_h, C) = 0, \quad (3)$$

and we also have the following conditions on q and Q for the dynamics to be activator-inhibitor dynamics:

$$\begin{aligned} \partial_\theta q(\theta_h, \eta_h, C) &< 0, \quad \partial_\eta Q(\theta_h, \eta_h, C) > 0, \\ (\partial_\theta q(\theta_h, \eta_h, C))(\partial_\eta Q(\theta_h, \eta_h, C)) \\ &- (\partial_\eta q(\theta_h, \eta_h, C))(\partial_\theta Q(\theta_h, \eta_h, C)) > 0, \end{aligned} \quad (4)$$

for some range of values for C . To simplify the analysis, it is further assumed that in fact

$$\partial_\eta Q > 0, \quad (\partial_\theta q)(\partial_\eta Q) - (\partial_\eta q)(\partial_\theta Q) > 0 \quad \forall C, \quad (5)$$

where the arguments (θ_h, η_h, C) have been suppressed, and that as C passes through some critical value, $\partial_\theta q$ goes from being positive to negative. Furthermore, for $\partial_\theta q > 0$, the spatially uniform equilibrium state is stable, and for $\partial_\theta q < 0$, the spatially uniform equilibrium state is unstable so that patterns form. When $\partial_\theta q > 0$, spike solutions or other dissipative structures may be stable with some region of attraction, while simultaneously the spatially uniform equilibrium solution is also stable with a different region of attraction. If the system is bistable, there are three spatially uniform equilibrium solutions, two

of which are stable (satisfying $\partial_\theta q > 0$), and one of which is unstable (satisfying $\partial_\theta q < 0$). There are two important dimensionless quantities that can be used to classify activator-inhibitor equations based on the types of patterns they support beyond the bifurcation threshold. These same classifications also indicate what types of dissipative structures can exist before the bifurcation threshold is reached. The parameters are

$$\begin{aligned}\alpha &= \tau_\theta/\tau_\eta \text{ (ratio of time constants)} \\ \beta &= l/L \text{ (ratio of diffusion lengths)}.\end{aligned}\quad (6)$$

The types of patterns which appear beyond the bifurcation threshold are

- $\alpha > 1$ but $\beta \ll 1$ leads to a spatially periodic patterns which are stationary in time,
- $\alpha \ll 1$ but $\beta > 1$ leads to spatially uniform but time-periodic patterns, and
- $\alpha \ll 1$ and $\beta \ll 1$ leads to patterns which are both spatially periodic and time periodic.

In this work we are mainly concerned with the $\alpha > 1$ but $\beta \ll 1$ case, because static spike solutions can be stable below threshold in such systems. Although static spike solutions can also be stable below threshold when $\alpha \ll 1$ and $\beta \ll 1$, traveling spike solutions and other more complicated solutions are also possible in that case, so the analysis is more difficult [3,4,5].

For the cubic nonlinearity model,

$$\begin{aligned}q &= \theta^3 - \theta - \eta \\ Q &= \theta + \eta - C.\end{aligned}\quad (7)$$

The spatially uniform equilibrium solution is easily found to be $(\theta_h, \eta_h) = (C^{1/3}, C - C^{1/3})$, and we have

$$\begin{aligned}\partial_\eta Q(\theta, \eta) &= 1 > 0 \\ \partial_\theta Q(\theta, \eta) &= 1 > 0 \\ \partial_\eta q(\theta, \eta) &= -1 < 0, \quad \forall(\theta, \eta).\end{aligned}\quad (8)$$

Furthermore, $\partial_\theta q(\theta_h, \eta_h) = 3C^{2/3} - 1$, so that

$$\partial_\theta q(\theta_h, \eta_h) < 0 \text{ for } -\frac{1}{3\sqrt{3}} < C < \frac{1}{3\sqrt{3}}, \quad (9)$$

and

$$(\partial_\theta q)(\partial_\eta Q) - (\partial_\eta q)(\partial_\theta Q) \Big|_{(\theta_h, \eta_h)} = 3C^{2/3} > 0. \quad (10)$$

3. Basic properties of solutions

The purpose of this section is to present mathematically rigorous statements concerning the cubic nonlinearity model. For convenience, we restate the coupled

system of PDEs:

$$\begin{aligned}\tau_\theta \partial_t \theta &= l^2 \Delta \theta - \theta^3 + \theta + \eta \\ \tau_\eta \partial_t \eta &= L^2 \Delta \eta - \theta - \eta + C,\end{aligned}\quad (11)$$

where τ_θ , τ_η , l , and L are positive constants, and C is also a constant. It turns out that for none of the results presented below does it matter whether the spatially uniform equilibrium state of the system is stable. Also, it doesn't matter what any of the constants are, until finally in the last subsection we will see that indeed $\alpha = \tau_\theta/\tau_\eta > 1$ is the sufficient condition for static solutions to be stable.

Existence, uniqueness, and regularity

The cubic nonlinearity model belongs to a general class of models

$$\begin{aligned}\tau_\theta \partial_t \theta + L_\theta \theta + f_\theta(\theta) &= \eta \\ \tau_\eta \partial_t \eta + L_\eta \eta + f_\eta(\eta) &= -\theta,\end{aligned}\quad (12)$$

defined on an open bounded subset $\Omega \subset \mathbb{R}^n$, where L_θ and L_η are uniformly parabolic operators, and $f_\theta(\theta)$ and $f_\eta(\eta)$ are odd-order polynomials with positive leading coefficients. Suppose also that the boundary conditions are one of the three basic types:

1. Dirichlet: $\theta(\mathbf{x}, t) = 0, \eta(\mathbf{x}, t) = 0$ on $\partial\Omega$,
2. Neumann: $\nabla_{\mathbf{x}} \theta \cdot \mathbf{n} = 0, \nabla_{\mathbf{x}} \eta \cdot \mathbf{n} = 0$ on $\partial\Omega$ where \mathbf{n} is normal to $\partial\Omega$, or
3. periodic boundary conditions.

Also, suppose the initial data $(\theta(0), \eta(0)) \in L^2(\Omega) \times L^2(\Omega)$. Using the standard techniques for proving existence and uniqueness for parabolic PDEs, one can prove that the above system of PDEs has a unique weak solution $(\theta(\mathbf{x}, t), \eta(\mathbf{x}, t))$, with

$$\begin{aligned}\theta &\in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H(\Omega)) \\ &\quad \cap L^{2p_\theta}(0, T; L^{2p_\theta}(\Omega)) \\ \eta &\in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H(\Omega)) \\ &\quad \cap L^{2p_\eta}(0, T; L^{2p_\eta}(\Omega)),\end{aligned}\quad (13)$$

where $2p_\theta - 1$ is the degree of $f_\theta(\theta)$, $2p_\eta - 1$ is the degree of $f_\eta(\eta)$, and $H(\Omega)$ is the appropriate Sobolev space corresponding to the boundary conditions (e.g., $H(\Omega) = H_0^1(\Omega)$ for Dirichlet boundary conditions) [6,7]. In addition to existence and uniqueness of solutions, the solutions also depend continuously on the initial data.

If we have the further assumptions that the boundary $\partial\Omega$ is C^2 and that $(\theta(0), \eta(0)) \in L^{2p_\theta}(\Omega) \times L^{2p_\eta}(\Omega)$, then we can show further that

$$\begin{aligned}D^2 \theta &\in L^2(0, T; L^2(\Omega)) \\ D^2 \eta &\in L^2(0, T; L^2(\Omega)),\end{aligned}\quad (14)$$

which implies

$$\begin{aligned}\theta &\in L^2(0, T; H^2(\Omega)) \\ \eta &\in L^2(0, T; H^2(\Omega)),\end{aligned}\quad (15)$$

with $H^2(\Omega)$ corresponding to $H(\Omega)$ defined in the appropriate way. These bounds on the second partial derivatives of θ and η are what we mean by ‘‘regularity.’’ We will require this amount of regularity in the calculations that follow. The restriction that the coupling between the θ and η equations is linear is used in the proofs of existence, uniqueness, and regularity.

Dissipativity property

For finite-dimensional systems, the physical notion of dissipativity can be tied to the mathematical concept of the existence of an absorbing set. For infinite-dimensional systems, it is not so clear how dissipativity should be precisely defined, since there are systems which are considered ‘‘dissipative,’’ but for which the existence of absorbing sets has not been established [7]. However, if for an infinite-dimensional system we can prove the existence of an absorbing set, we can certainly label the system dissipative, and the cubic nonlinearity model does possess an absorbing set. The energy bounds required to show the existence of an absorbing set are stronger than those required to show existence, uniqueness, and regularity of solutions.

Let $\mathbf{u}(t) = (\theta(t), \eta(t))$ denote the solution for the cubic nonlinearity model, let $\mathbf{u}_0 = \mathbf{u}(0)$, and let $\mathbf{L} = L^2(\Omega) \times L^2(\Omega)$. Then the semigroup $\{S(t)\}_{t \geq 0}$ defined by

$$\begin{aligned}S(t) : \mathbf{L} &\rightarrow \mathbf{L} \\ \mathbf{u}_0 &\rightarrow \mathbf{u}(t)\end{aligned}\quad (16)$$

is well-defined $\forall t \in [0, T]$ for T arbitrarily large. Note that $\mathbf{H} = (H(\Omega) \cap L^{2p_\theta}(\Omega)) \times (H(\Omega) \cap L^{2p_\eta}(\Omega)) \subset \mathbf{L}$ is the Hilbert space in which $\mathbf{u}(t)$ lies for almost every t . However, writing $S(t) : \mathbf{L} \rightarrow \mathbf{L}$ reflects the fact that our initial conditions only need to be in \mathbf{L} for the existence and uniqueness theory to hold.

The semigroup $\{S(t)\}_{t \geq 0}$ satisfies the basic semigroup properties,

$$\begin{aligned}S(t+s) &= S(t) \cdot S(s) \quad \forall s, t \geq 0 \\ S(0) &= I \text{ (the identity)} \\ \mathbf{u}(t+s) &= S(t)\mathbf{u}(s) = S(s)\mathbf{u}(t),\end{aligned}\quad (17)$$

and in addition, because of the continuous dependence of solutions on initial data, we have that $S(t)$ is a continuous operator $\forall t \geq 0$. A set $\mathcal{B} \subset \mathbf{U}$, where \mathbf{U} is an open set in \mathbf{L} , is called an absorbing set in \mathbf{U} if the orbit of any bounded set of \mathbf{U} enters into \mathcal{B} after a certain time (which may depend on the set); i.e., $\forall \mathcal{B}_0 \subset \mathbf{U}$, \mathcal{B}_0 bounded, $\exists t_1(\mathcal{B}_0)$ such that $S(t)\mathcal{B}_0 \subset \mathcal{B} \quad \forall t \geq t_1(\mathcal{B}_0)$. An attractor is a set $\mathcal{A} \subset \mathbf{L}$ such that

- (i) \mathcal{A} is invariant; i.e., $S(t)\mathcal{A} = \mathcal{A} \quad \forall t \geq 0$, and

- (ii) $\exists \mathcal{U}$, open, such that $\forall \mathbf{u}_0 \in \mathcal{U}$, $S(t)\mathbf{u}_0 \rightarrow \mathcal{A}$ as $t \rightarrow \infty$; i.e., $\text{dist}(S(t)\mathbf{u}_0, \mathcal{A}) \rightarrow 0$ as $t \rightarrow \infty$.

The largest such \mathcal{U} is the basin of attraction of \mathcal{A} . If the basin of attraction of \mathcal{A} is all of \mathbf{L} , then \mathcal{A} is a global attractor for $\{S(t)\}_{t \geq 0}$.

The existence of a global attractor implies the existence of an absorbing set. For the cubic nonlinearity model, the existence of an absorbing set implies the existence of an attractor, due to the following theorem:

Theorem: Suppose \mathbf{L} is a Banach space and that the operators $S(t)$ satisfy the semigroup properties and are continuous operators from \mathbf{L} into itself $\forall t > 0$. Suppose that there exists an open set \mathcal{U} and a bounded set \mathcal{B} of $\mathcal{U} \cap \mathbf{H}$ such that $\mathcal{B} \subset \mathbf{H} \subset \mathbf{L}$ and \mathcal{B} is absorbing in \mathcal{U} . Then the ω -limit set of \mathcal{B} , $\mathcal{A} = \omega(\mathcal{B})$, is a compact attractor which attracts the bounded sets of \mathcal{U} . It is the maximal bounded attractor. Furthermore, if \mathcal{U} is convex and connected, then \mathcal{A} is connected, too.

Remark: By $\mathbf{H} \subset \mathbf{L}$, for \mathbf{H} and \mathbf{L} Banach spaces, we mean \mathbf{H} is compactly embedded in \mathbf{L} . For our \mathbf{H} and \mathbf{L} , $\mathbf{H} \subset \mathbf{L}$ follows from standard compactness theory [6].

So to demonstrate the dissipativity of the cubic nonlinearity model, we need to exhibit an absorbing set in \mathbf{H} . Such an absorbing set does exist for the cubic nonlinearity model. In fact, this absorbing set absorbs all the bounded sets of \mathbf{H} . The existence of this absorbing set then implies the existence of a global attractor by the above theorem.

4. Lyapunov functional derivation

The energy estimates of the previous subsections were essentially $L^2(\Omega)$ -norm bounds on solutions and their derivatives. Now we consider energy calculations relative to energy functionals. The initial observation is that for the cubic nonlinearity model, there is an energy functional

$$\begin{aligned}V &= \int_{\Omega} \left[\frac{l^2}{2} |\nabla \theta|^2 + \frac{1}{4} \theta^4 - \frac{1}{2} \theta^2 - \theta \eta - \frac{l^2}{2} |\nabla \eta|^2 \right. \\ &\quad \left. - \frac{1}{2} \eta^2 + C \eta \right] d\mathbf{x},\end{aligned}\quad (18)$$

such that

$$\begin{aligned}\dot{V} &= \frac{\delta V}{\delta \theta} \cdot (\partial_t \theta) + \frac{\delta V}{\delta \eta} \cdot (\partial_t \eta) \\ &= - \int_{\Omega} [\tau_\theta (\partial_t \theta)^2 - \tau_\eta (\partial_t \eta)^2] d\mathbf{x} \\ &= - \left(\begin{bmatrix} \partial_t \theta \\ \partial_t \eta \end{bmatrix}, \begin{bmatrix} \tau_\theta & 0 \\ 0 & -\tau_\eta \end{bmatrix} \begin{bmatrix} \partial_t \theta \\ \partial_t \eta \end{bmatrix} \right).\end{aligned}\quad (19)$$

An equivalent way of expressing this is

$$-J \begin{bmatrix} \partial_t \theta \\ \partial_t \eta \end{bmatrix} = \nabla V, \quad J = \begin{bmatrix} \tau_\theta & 0 \\ 0 & -\tau_\eta \end{bmatrix}.\quad (20)$$

so that

$$\dot{V} = \left(\nabla V, \begin{bmatrix} \partial_t \theta \\ \partial_t \eta \end{bmatrix} \right) = - \left(\begin{bmatrix} \partial_t \theta \\ \partial_t \eta \end{bmatrix}, J \begin{bmatrix} \partial_t \theta \\ \partial_t \eta \end{bmatrix} \right) \quad (21)$$

We thus have a gradient system with respect to an indefinite metric. What Brayton and Moser showed was that for systems of ODEs which are gradient with respect to an indefinite metric, it is sometimes possible to find a radially unbounded Lyapunov function V^* such that $\dot{V}^* \leq 0$ and $\dot{V}^* = 0$ only at equilibrium points [2]. The conclusion is then that all system trajectories converge to the set of equilibrium points. For the cubic nonlinearity model, the linearized analyses of Kerner and Osipov indicate that the stable solutions are static in time if $\alpha = \tau_\theta / \tau_\eta$ is sufficiently large (i.e., $\alpha > 1$), and the stable solutions oscillate, or pulsate, in time if α is sufficiently small (i.e., $\alpha \ll 1$). The non-oscillating case is the one for which we could hope to find a radially unbounded energy function which decreases along system trajectories.

To see how the technique of Brayton and Moser works for ODE systems, consider the simplest discretization of the cubic nonlinearity model in one spatial dimension with periodic boundary conditions,

$$\begin{aligned} \tau_\theta \dot{\theta}_k &= l^2 \left(\frac{\theta_{k-1} - 2\theta_k + \theta_{k+1}}{\delta^2} \right) - \theta_k^3 + \theta_k + \eta_k \\ \tau_\eta \dot{\eta}_k &= L^2 \left(\frac{\eta_{k-1} - 2\eta_k + \eta_{k+1}}{\delta^2} \right) - \theta_k - \eta_k + C, \end{aligned} \quad (22)$$

where δ is the distance between the discretized points along the x -axis where we are evaluating θ_k and η_k , and the indices k are taken to be mod N where $2N$ is the total number of ODEs.

As for the PDE system, we can write this discretized system as a gradient system with respect to an indefinite metric: let

$$\begin{aligned} V &= \frac{l^2}{\delta^2} \left(\sum_k \theta_k^2 - \sum_k \theta_k \theta_{k+1} \right) + \frac{1}{4} \sum_k \theta_k^4 - \frac{1}{2} \sum_k \theta_k^2 \\ &\quad - \frac{L^2}{\delta^2} \left(\sum_k \eta_k^2 - \sum_k \eta_k \eta_{k+1} \right) - \frac{1}{2} \sum_k \eta_k^2 + C \sum_k \eta_k \\ &\quad - \sum_k \theta_k \eta_k, \end{aligned} \quad (23)$$

so that

$$\begin{aligned} \dot{V} &= \sum_k \frac{\partial V}{\partial \theta_k} \dot{\theta}_k + \sum_k \frac{\partial V}{\partial \eta_k} \dot{\eta}_k \\ &= - \sum_k \left[\tau_\theta (\dot{\theta}_k)^2 - \tau_\eta (\dot{\eta}_k)^2 \right] \\ &= - [\dot{\theta}^T \ \dot{\eta}^T] \begin{bmatrix} \tau_\theta I & 0 \\ 0 & -\tau_\eta I \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \dot{\eta} \end{bmatrix}, \end{aligned} \quad (24)$$

where $\theta = (\theta_1, \dots, \theta_N)$, $\eta = (\eta_1, \dots, \eta_N)$, and I denotes the $N \times N$ identity matrix. We define ∇V and J in analogy with equations (20) and (21).

The technique of Brayton and Moser involves first computing D^2V , which looks like

$$D^2V = \begin{bmatrix} P & -I \\ -I & Q \end{bmatrix}, \quad (25)$$

where each block is $N \times N$. It turns out not to be necessary to compute P , but it is necessary to write down Q and show that Q is invertible. A simple calculation gives

$$Q = - \left(I + \frac{L^2}{\delta^2} R \right), \quad (26)$$

where

$$R = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & & \vdots \\ \vdots & 0 & & \ddots & & 0 \\ 0 & \vdots & & & 2 & -1 \\ -1 & 0 & \cdots & 0 & -1 & 2 \end{bmatrix}. \quad (27)$$

Since R is positive semidefinite, Q is invertible, and we can define

$$\begin{aligned} M &= \begin{bmatrix} 0 & 0 \\ 0 & -2Q^{-1} \end{bmatrix} \\ J^* &= J + (D^2V)MJ = \begin{bmatrix} \tau_\theta I & -2\tau_\eta Q^{-1} \\ 0 & \tau_\eta I \end{bmatrix}. \end{aligned} \quad (28)$$

Corresponding to this J^* , there is a

$$V^* = V + \frac{1}{2} (\nabla V)^T M \nabla V, \quad (29)$$

such that

$$-J^* \begin{bmatrix} \dot{\theta} \\ \dot{\eta} \end{bmatrix} = \nabla V^*, \quad (30)$$

as can be easily seen by taking the gradient of V^* .

If J^* were symmetric, it would be a metric, and the dynamics would be gradient dynamics. However, J^* is not symmetric. But if J^* is positive definite, V^* will still be decreasing along trajectories. If we can further show that V^* is radially unbounded and $\dot{V}^* = 0$ if and only if $\dot{\theta} = \dot{\eta} = 0$, we will be able to conclude that the trajectories of the system converge to the set of equilibrium points.

As for the positive definiteness of J^* , we have

$$\begin{aligned} [\dot{\theta}^T \ \dot{\eta}^T] \begin{bmatrix} \tau_\theta I & -2\tau_\eta Q^{-1} \\ 0 & \tau_\eta I \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \dot{\eta} \end{bmatrix} &= \tau_\theta |\dot{\theta}|^2 + \tau_\eta |\dot{\eta}|^2 - 2\tau_\eta \dot{\theta}^T Q^{-1} \dot{\eta} \\ &= \left| \sqrt{\tau_\theta} \dot{\theta} - \frac{1}{\sqrt{\alpha}} Q^{-1} (\sqrt{\tau_\eta} \dot{\eta}) \right|^2 + |\sqrt{\tau_\eta} \dot{\eta}|^2 - \\ &\quad \frac{1}{\alpha} |Q^{-1} (\sqrt{\tau_\eta} \dot{\eta})|^2, \end{aligned} \quad (31)$$

and if

$$\frac{1}{\sqrt{\alpha}} \|Q^{-1}\| < 1, \quad (32)$$

then we see that J^* is positive definite. Furthermore, we can calculate $\|Q^{-1}\| = 1$, and arrive at the condition that J^* is positive definite if

$$\alpha > 1. \quad (33)$$

Because

$$\dot{V}^* = -[\dot{\theta}^T \quad \dot{\eta}^T] J^* \begin{bmatrix} \dot{\theta} \\ \dot{\eta} \end{bmatrix}, \quad (34)$$

we also see that if J^* is positive definite, then $\dot{V}^* \leq 0$ and $\dot{V}^* = 0$ if and only if $\dot{\theta} = \dot{\eta} = 0$.

We compute V^* and verify that it is radially unbounded:

$$\begin{aligned} V^* &= \frac{l^2}{\delta^2} \left(\sum_k \theta_k^2 - \sum_k \theta_k \theta_{k+1} \right) + \frac{1}{4} \sum_k \theta_k^4 - \frac{1}{2} \sum_k \theta_k^2 \\ &+ \frac{L^2}{\delta^2} \left(\sum_k \eta_k^2 - \sum_k \eta_k \eta_{k+1} \right) + \frac{1}{2} \sum_k \eta_k^2 - C \sum_k \eta_k \\ &+ \sum_k \theta_k \eta_k - (\theta - C\gamma)^T Q^{-1} (\theta - C\gamma), \end{aligned} \quad (35)$$

where $\gamma = [1 \ 1 \ \dots \ 1]^T$.

We thus arrive at the conclusion that for the discretized one-dimensional system with periodic boundary conditions, regardless of the fineness of the discretization (N and δ), as long as $\alpha > 1$, we can find a radially unbounded Lyapunov function V^* such that $\dot{V}^* \leq 0$ and $\dot{V}^* = 0$ if and only if $\dot{\theta} = \dot{\eta} = 0$. We can therefore conclude that all trajectories must converge to the set of equilibrium points, provided $\alpha > 1$. (Even with Dirichlet or Neumann boundary conditions, or with multiple spatial dimensions, we still have the same conclusion provided $\alpha > 1$.)

In fact, we can adapt the same technique just used to the original system of PDEs, which is in fact the main contribution of this work. The first step is to calculate the second derivative of the energy functional V . We have

$$V : X \rightarrow \mathbb{R}, \quad (36)$$

where X represents the space in which the (θ, η) lie. At each $\mathbf{p} \in X$, there is a derivative map,

$$\begin{aligned} DV_{\mathbf{p}} : X &\rightarrow \mathbb{R} \\ \mathbf{u} &\mapsto DV_{\mathbf{p}} \cdot \mathbf{u} = \left. \frac{d}{d\epsilon} V(\mathbf{p} + \epsilon \mathbf{u}) \right|_{\epsilon=0}, \end{aligned} \quad (37)$$

which corresponds to the first variation of V evaluated at a particular (θ, η) . By ∇V , we mean

$$\nabla V = \begin{bmatrix} -l^2 \Delta \theta + \theta^3 - \theta - \eta \\ L^2 \Delta \eta - \eta - \theta + C \end{bmatrix}, \quad (38)$$

for then

$$DV_{(\theta, \eta)} \cdot \begin{bmatrix} \delta \theta \\ \delta \eta \end{bmatrix} = \int_{\Omega} \nabla V \cdot \begin{bmatrix} \delta \theta \\ \delta \eta \end{bmatrix} d\mathbf{x}, \quad (39)$$

where $(\mathbf{u}, \mathbf{v}) = \int_{\Omega} \mathbf{u} \cdot \mathbf{v} d\mathbf{x}$ is our inner product. We can then define the second-derivative map at a point $\mathbf{p} \in X$ as

$$\begin{aligned} D^2V_{\mathbf{p}} : X \times X &\rightarrow \mathbb{R} \\ (\mathbf{u}, \mathbf{v}) &\mapsto \left. \frac{d^2}{d\epsilon d\xi} V(\mathbf{p} + \epsilon \mathbf{u} + \xi \mathbf{v}) \right|_{\epsilon=0, \xi=0}. \end{aligned} \quad (40)$$

We can define the second-derivative matrix D^2V by

$$\begin{aligned} D^2V_{(\theta, \eta)} \cdot \left(\begin{bmatrix} \delta \theta_1 \\ \delta \eta_1 \end{bmatrix}, \begin{bmatrix} \delta \theta_2 \\ \delta \eta_2 \end{bmatrix} \right) \\ = \int_{\Omega} [\delta \theta_1 \quad \delta \eta_1] D^2V \begin{bmatrix} \delta \theta_2 \\ \delta \eta_2 \end{bmatrix} d\mathbf{x}. \end{aligned} \quad (41)$$

The second-derivative matrix D^2V is computed to be

$$D^2V = \begin{bmatrix} (3\theta^2 - 1 - l^2 \Delta) & -1 \\ -1 & (-1 + L^2 \Delta) \end{bmatrix}. \quad (42)$$

We thus see that the quantity that plays the role of the matrix Q in the discretized system is the operator $-1 + L^2 \Delta$. Therefore, we need to address the issue of finding an inverse for $-1 + L^2 \Delta$.

Suppose first that we have periodic boundary conditions. Since the functions $(\delta \theta, \delta \eta)$ we are working with are in $L^2(\Omega)$, their Fourier series are well-defined (in the distributional sense):

$$\begin{aligned} u(\mathbf{x}) &= \sum_{\mathbf{k}} u_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}} \\ u_{\mathbf{k}} &= \frac{1}{|\Omega|} \int_{\Omega} u(\mathbf{x}) e^{-i\mathbf{k} \cdot \mathbf{x}} d\mathbf{x}, \\ u(\mathbf{x}) &\in L^2(\Omega), \quad \sum_{\mathbf{k}} |u_{\mathbf{k}}|^2 < \infty. \end{aligned} \quad (43)$$

(Here we are thinking of \mathbf{k} as a vector containing indices which are not necessarily integer. For example, in the one-dimensional case, we would have $k = 2\pi m / \mathcal{L}$ where m is an integer and $\mathcal{L} = |\Omega|$ is the length of the interval Ω .) Then

$$(-1 + L^2 \Delta) u(\mathbf{x}) = - \sum_{\mathbf{k}} (1 + L^2 |\mathbf{k}|^2) u_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}}, \quad (44)$$

so the inverse operator for $-1 + L^2 \Delta$ has the form

$$(-1 + L^2 \Delta)^{-1} v(\mathbf{x}) = w(\mathbf{x}) * v(\mathbf{x}) = \int_{\Omega} w(\mathbf{x} - \mathbf{y}) v(\mathbf{y}) d\mathbf{y}, \quad (45)$$

where $w(\mathbf{x})$ can be represented as

$$w(\mathbf{x}) = \sum_{\mathbf{k}} \frac{-1}{1 + L^2 |\mathbf{k}|^2} e^{i\mathbf{k} \cdot \mathbf{x}}. \quad (46)$$

Before we can conclude that we have an appropriate inverse, however, we need to verify, since $-1 + L^2\Delta$ takes functions in $H^2(\Omega)$ to $L^2(\Omega)$, that $w(\mathbf{x}) * \cdot$ takes functions in $L^2(\Omega)$ to $H^2(\Omega)$. But this is in fact the case, since an equivalent norm to the $H^2(\Omega)$ norm is

$$\|u\|_2 = \left(\sum_{\mathbf{k}} |u_{\mathbf{k}}|^2 (1 + |\mathbf{k}|^2)^2 \right)^{1/2}. \quad (47)$$

Thus, from the form of $w(\mathbf{x}) * \cdot$, it is clear that $w(\mathbf{x}) * v(\mathbf{x}) \in H^2(\Omega)$ if $v(\mathbf{x}) \in L^2(\Omega)$. Thus, we have verified (at least for periodic boundary conditions) that $(-1 + L^2\Delta)^{-1} : L^2(\Omega) \rightarrow H^2(\Omega)$ is a well-defined operator.

Proceeding by analogy with the discretized case, we can compute

$$V^* = V - (\nabla_{\eta} V, (-1 + L^2\Delta)^{-1} \nabla_{\eta} V), \quad (48)$$

where

$$\nabla_{\eta} V = L^2\Delta\eta - \theta - \eta + C = (-1 + L^2\Delta)\eta - \theta + C, \quad (49)$$

obtaining

$$V^* = \int_{\Omega} \left[\frac{l^2}{2} |\nabla\theta|^2 + \frac{1}{4}\theta^4 - \frac{1}{2}\theta^2 + \frac{L^2}{2} |\nabla\eta|^2 + \frac{1}{2}\eta^2 - C\eta + \theta\eta - (\theta - C)[(-1 + L^2\Delta)^{-1}(\theta - C)] \right] d\mathbf{x}. \quad (50)$$

From this expression for V^* , it is apparent that V^* is radially unbounded. In analogy with the discretized case, we obtain

$$\begin{aligned} \nabla V^* &= - \begin{bmatrix} \tau_{\theta} & -2\tau_{\eta}(-1 + L^2\Delta)^{-1} \\ 0 & \tau_{\eta} \end{bmatrix} \begin{bmatrix} \partial_t\theta \\ \partial_t\eta \end{bmatrix} \\ \dot{V}^* &= - \int_{\Omega} [\partial_t\theta \quad \partial_t\eta] \begin{bmatrix} \tau_{\theta} & -2\tau_{\eta}(-1 + L^2\Delta)^{-1} \\ 0 & \tau_{\eta} \end{bmatrix} \begin{bmatrix} \partial_t\theta \\ \partial_t\eta \end{bmatrix} d\mathbf{x} \end{aligned} \quad (51)$$

For periodic boundary conditions, the operator norm of $(-1 + L^2\Delta)^{-1}$ turns out to be one, just like the matrix norm of Q^{-1} in the discretized case. This leads to the same conclusion as before, that if $\alpha > 1$ then there is a radially unbounded Lyapunov functional V^* such that $\dot{V}^* \leq 0$ and $\dot{V}^* = 0$ only at equilibrium points of the dynamics.

For Dirichlet or Neumann boundary conditions, we need to assume that the boundary of Ω is C^2 (so that we have the necessary second-derivative bounds required for calculating the variations of V and V^*), and we need to use Fourier transform techniques instead of Fourier series techniques. The same Lyapunov functional is computed and the same conclusions apply as for periodic boundary conditions.

5. Generalizations and future work

The existence of a Lyapunov functional for the cubic nonlinearity model is a starting point for further analysis in several directions. Since (for periodic boundary conditions) the Lyapunov functional takes a simple form at equilibria,

$$V_e^* = \int_{\Omega} -\frac{1}{4}\theta^4 - \frac{1}{2}C\theta + \frac{1}{2}C^2 d\mathbf{x}, \quad (52)$$

the Lyapunov functional may be useful for analyzing the stability of equilibria. Also, there are several generalizations of the basic cubic nonlinearity model that respect generalizations of the Lyapunov functional, one of which is the complex activator-inhibitor equation,

$$\begin{aligned} \tau_{\theta} \partial_t \theta &= l^2 \Delta \theta - |\theta|^2 \theta + \theta + \eta \\ \tau_{\eta} \partial_t \eta &= L^2 \Delta \eta - \theta - \eta + C, \end{aligned} \quad (53)$$

where θ , η , and C are complex. The complex activator-inhibitor equation can be used, under suitable hypotheses, to model the amplitude and phase evolution in the continuum limit of a network of coupled van der Pol oscillators (represented by θ), coupled to a network of resonant circuits (represented by η), with an external oscillating input (represented by C), all with the same natural frequency. Another generalization of the basic cubic nonlinearity model for which the Lyapunov functional can also be adapted is the addition of symmetric long-range coupling to the θ dynamics. Long-range coupling can be used to select ideal patterns from among competing patterns.

6. References

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