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Nonlinear HÄ Control with Delayed Measurements

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Abstract

This paper considers the nonlinear H_∞ control problem for systems subject to delayed measurements. Necessary and sufficient conditions for the solvability of the problem are presented. A key point of our approach is the extension of the information state concept. In particular, the information state is no longer the "worst case cost to come" function. We also present the certainty equivalence principle for such systems, and draw an analogy with the solution to the linear case. A simple example is also presented.

1   Introduction

In recent years, nonlinear H_∞ control has received a great deal of attention as a potentially viable methodology for designing controllers for nonlinear systems [5],[6],[9],[8],[11]. Although, a lot of work remains to be done, in particular on the computational aspects, the pieces are slowly falling into place. What has been conspicuously absent is a general framework for dealing with systems with delays. In this paper, we consider the case of systems with measurement delays. Such systems are widespread in the chemical process, and semiconductor industries, where either one takes samples to a laboratory for off-line measurements, or the sensors have a finite data processing delay. A typical class of such sensors are those responsible for composition measurements.

In a recent paper, [9], it was shown that the standard nonlinear H_∞ control problem is solvable provided one solves a filter equation, a dynamic programming equation, and satisfies a coupling condition. These results have an interpretation similar to the linear case, where one solves a pair of Riccati equations, and satisfies a coupling condition [3]. Furthermore, in the linear case with delayed measurements, one needs to solve [2] a control Riccati equation, a filter Riccati equation, satisfy the coupling condition, and an additional open loop Riccati equation whose initial conditions are determined by the solution of the control Riccati equation. This derivation involves certainty equivalence, which does not hold in the general nonlinear case. Hence, we would like to see whether one can draw any analogies between the solutions to the linear and nonlinear cases.

Our approach is based on identifying an appropriate *information state* for the delayed measurement problem. Such an approach leads to separation between estimation and control.

In addition one obtains both necessary and sufficient conditions for solvability. However, the controller so obtained maybe infinite dimensional in general, although, for the delay free case, there exist certain systems for which the controller is finite dimensional (for example bilinear systems [10]). In fact, as we shall see, if the delay free system has a finite dimensional controller, then the controller for the system subject to a finite measurement delay is also finite dimensional.

We begin in Section 2 by stating the problem and introduce some notation. In Section 3, we derive the information state for the problem at hand. The solution to the problem in terms of both necessary and sufficient conditions is presented in Section 4. Section 5 discusses the certainty equivalence principle, and an analogy is drawn with the solution to the linear case presented in [2]. We then present a simple example.

For purposes of brevity, we will concentrate on the discrete finite time case. The results can be extended to the infinite time case by invoking stationarity of the control dynamic programming equation, and by making a detectability assumption. We can also apply the ideas presented here to the continuous time case.

2 Statement of the Problem

The system under consideration is

$$
\Sigma \left\{
\begin{array}{rcl}
x_{k+1} & = & f(x_k, u_k, w_k) \\
y_{k+1} & = & g(x_{k-\tau}, u_{k-\tau}, w_{k-\tau}) \\
z_{k+1} & = & h(x_k, u_k, w_k).
\end{array}
\right.
\tag{1}
$$

Here, $x_k \in \mathbf{R}^n$ are the states, $y_k \in \mathbf{R}^t$ are the measurements. $u_k \in \mathbf{U} \subset \mathbf{R}^m$ are the control inputs, and $z_k \in \mathbf{R}^q$ are the regulated outputs. It is assumed that the origin is an equilibrium point for the system Σ, i.e. $f(0,0,0) = 0$, $g(0,0,0) = 0$, and $h(0,0,0) = 0$. Also, we assume that $\mathbf{U}$ is compact. Furthermore, the delay $\tau \geq 0$ is assumed to be fixed. It is clear that if $k \leq \tau$, then no measurements y_k are available. In general, one may have variable amounts of delay, in which case, one fixes τ to correspond to the largest possible delay.

We denote the space of output feedback policies as O. Hence, if $u \in O$ then $u_k = u(y_{\tau+1,k}, u_{0,k-1})$, where in general $s_{i,j}$ is the vector $[s_i \; s_{i+1} \ldots s_j]$. The finite time H_∞ control problem can now be stated as [9], given $K \geq 0$, and $\gamma > 0$, find $u \in O$, such that there exists a finite quantity $\beta_K^u(x) \geq 0$, with $\beta_K^u(0) = 0$, such that for each initial condition $x_0 \in \mathbf{R}^n$, we have

$$
\sum_{i=0}^{K-1} |z_{i+1}|^2 \leq \gamma^2 \sum_{i=0}^{K-1} |w_i|^2 + \beta_K^u(x_0).
\tag{2}
$$

This is also called the *finite gain* property, since it implies that if $x_0 = 0$, then

$$
\|\Sigma_0^u\|_{H_\infty} \triangleq \sup_{w \in l_2([0,K-1],\mathbf{R}^r), w \neq 0} \frac{\|z\|_{l_2([1,K],\mathbf{R}^q)}}{\|w\|_{l_2([0,K-1],\mathbf{R}^r)}} \leq \gamma.
$$

2

Before proceeding further, we introduce the spaces

$$\mathcal{E} \triangleq \{p : \mathbf{R}^n \to \mathbf{R}^*\}$$

and

$$\mathcal{U}^k \triangleq \{u : u = u_{i,j}, u_t \in \mathbf{U}, i \le t \le j, 0 \le j - i \le k, \text{ or } u = \phi\}.$$

Now consider their direct sum

$$\mathcal{D} \triangleq \mathcal{E} \oplus \mathcal{U}^{\tau-1} = \left\{ \begin{bmatrix} p \\ u \end{bmatrix} : p \in \mathcal{E}, u \in \mathcal{U}^{\tau-1} \right\}$$

and define the operators $\pi_1 : \mathcal{D} \to \mathcal{E}$, and $\pi_2 : \mathcal{D} \to \mathcal{U}^{\tau-1}$ as

$$\pi_1 \left(\begin{bmatrix} p \\ u \end{bmatrix} \right) = p \text{ and, } \pi_2 \left(\begin{bmatrix} p \\ u \end{bmatrix} \right) = u.$$

Also, we associate with a sequence $u_{i,j}$ its length given by $l(u) = j - i + 1$. Here, we use the convention that $l(\phi) = 0$. We also define the "sup-pairing",

$$(p, q) \triangleq \sup_{x \in \mathbf{R}^n} \{p(x) + q(x)\}.$$

We now consider the functional

$$L_{p,k}(u) \triangleq \sup_{w \in l_2([0,k-1], \mathbf{R}^r)} \sup_{x_0 \in \mathbf{R}^n} \left\{ p(x_0) + \sum_{i=0}^{k-1} |z_{i+1}|^2 - \gamma^2 |w_i|^2 \right\}.$$

Then, we have the following result

Lemma 1 *(i)* Σ^u *is finite gain on* $[0, K]$ *if and only if there exists a finite quantity* $\beta_K^u(\cdot) \ge 0$, *such that*

$$L_{-\beta_K^u, K}(u) \le 0.$$

(ii) If each map $\Sigma_{x_0}^u$ *is finite gain on* $[0, K]$, *then*

$$(p, 0) \le L_{p,K}(u) \le (p, \beta_K^u).$$

The robust control problem can now be expressed as, find $u^* \in O_{0,k-1}$, such that

$$L_{p,k}(u^*) = \inf_{u \in O_{0,k-1}} L_{p,k}(u)$$

3 The Information State

For a fixed $u_{0,k-1} \in l_2([0, k-1], \mathbf{U})$, $y_{\tau+1,k} \in l_2([\tau+1, k], \mathbf{R}^t)$, we define the *cost to come* function $p_k \in \mathcal{E}$ as

$$p_k(x) \triangleq \sup_{w \in l_2([0,k-1], \mathbf{R}^r)} \sup_{x_0 \in \mathbf{R}^n} \{p_0(x_0) + \sum_{i=0}^{k-1} |z_{i+1}|^2 - \gamma^2 |w_i|^2 : x_k = x, \tag{3}$$

$$y_{i+1} = g(x_{i-\tau}, u_{i-\tau}, w_{i-\tau}), \tau \le i \le k - 1,$$
$$x_{i+1} = f(x_i, u_i, w_i), 0 \le i \le k - 1\}$$

3

We would like to express p_k as a dynamical equation. For this purpose, define $H(p, u, y) \in \mathcal{E}$ by

$$H(p_k, u, y)(x) \triangleq \begin{cases} \sup_{\xi \in \boldsymbol{R}^n} \{p_k(\xi) + B(\xi, x, u, y)\} & \text{if } k \geq \tau \\ p_k(x) & \text{else} \end{cases}$$

where the extended real valued function B is defined by

$$B(\xi, x, u, y) \triangleq \sup_{w \in \boldsymbol{R}^r} \left\{ |h(\xi, u, w)|^2 - \gamma^2 |w|^2 : f(\xi, u, w) = x, g(\xi, u, w) = y \right\}.$$

Here, we use the convention that the supremum over an empty set equals $-\infty$.

Let $\hat{p} \in \mathcal{D}$, and define the shift/pad operation $\eta : \mathcal{U}^{\tau-1} \times \boldsymbol{U} \to \mathcal{U}^{\tau-1}$ by

$$\eta(u_{i,j}, u_{j+1}) \triangleq \begin{cases} u_{i,j+1} & \text{if } j - i < \tau - 1 \\ u_{i+1,j+1} & \text{else} \end{cases}$$

and the functional $J : \mathcal{D} \to \boldsymbol{R}^*$ by

$$\begin{aligned} J(\hat{p}_k)(x) \quad \triangleq \quad & \sup_{w \in l_2([0, l(\pi_2(\hat{p}_k)) - 1], \boldsymbol{R}^r)} \sup_{x_0 \in \boldsymbol{R}^n} \left\{ \pi_1(\hat{p}_k)(x_0) + \sum_{i=0}^{l(\pi_2(\hat{p}_k)) - 1} |z_{i+1}|^2 - \gamma^2 |w_i|^2 : \right. \\ & \left. x_{l(\pi_2(\hat{p}_k))} = x, x_{i+1} = f(x_i, \pi_2(\hat{p}_k)_i, w_i), 0 \leq i \leq l(\pi_2(\hat{p}_k)) - 1 \right\} \end{aligned} \tag{4}$$

where $\pi_2(\hat{p}_k)_i$ denotes the ith element of $\pi_2(\hat{p}_k)$, assuming that the indexing starts from 0. In particular, if $\pi_2(\hat{p}_k) = \phi$, then $J(\hat{p}_k)(x) = \pi_1(\hat{p}_k)(x)$. We now define the functional $F \in \mathcal{D}$ by

$$F(\hat{p}_k, u_k, y_{k+1})(x) \triangleq \begin{bmatrix} H(\pi_1(\hat{p}_k), \pi_2(\hat{p}_k)_0, y_{k+1})(x) \\ \eta(\pi_2(\hat{p}_k), u_k) \end{bmatrix}.$$

We can now express the cost to come function recursively as follows:

Lemma 2 *The cost to come function (p_k) is the solution to the following recursion*

$$\Xi \begin{cases} \hat{p}_{k+1} &= F(\hat{p}_k, u_k, y_{k+1}), k \in [0, K-1] \\ p_{k+1} &= J(\hat{p}_{k+1}) \end{cases} \tag{5}$$

for any $\hat{p}_0 \in \mathcal{D}$ of the form $\begin{bmatrix} p_0 \\ \phi \end{bmatrix}$, with $p_0 \in \mathcal{E}$.

Proof:

Given the initial condition of the form $\hat{p}_0 = \begin{bmatrix} p_0 \\ \phi \end{bmatrix}$, with $p_0 \in \mathcal{E}$, we have

$$\hat{p}_{k+1}(x) = \begin{bmatrix} \pi_1(\hat{p}_{k+1})(x) \\ \pi_2(\hat{p}_{k+1}) \end{bmatrix} = \begin{bmatrix} H(\pi_1(\hat{p}_k), \pi_2(\hat{p}_k)_0, y_{k+1})(x) \\ \eta(\pi_2(\hat{p}_k), u_k) \end{bmatrix}.$$

4

By the definition of η it is clear that

$$\pi_2(\hat{p}_{k+1}) = \begin{cases} u_{0,k} & \text{if } k < \tau \\ u_{k-\tau+1,k} & \text{if } k \geq \tau. \end{cases}$$

Also, by definition,

$$H(\pi_1(\hat{p}_k), \pi_2(\hat{p}_k)_0, y_{k+1})(x) = p_0(x) \text{ if } k < \tau$$

else, if $k \geq \tau$, we have

$$\begin{aligned} H(\pi_1(\hat{p}_k), \pi_2(\hat{p}_k)_0, y_{k+1})(x) &= H(\pi_1(\hat{p}_k), u_{k-\tau}, y_{k+1})(x) \\ &= \sup_{\xi \in \mathbf{R}^n} \{\pi_1(\hat{p}_k)(\xi) + B(\xi, x, u_{k-\tau}, y_{k+1})\} \\ &= \sup_{\xi \in \mathbf{R}^n} \{\pi_1(\hat{p}_k)(\xi) + \sup_{w \in \mathbf{R}^r} |h(\xi, u_{k-\tau}, w)|^2 - \gamma^2 |w|^2 : \\ & \qquad x = f(\xi, u_{k-\tau}, w), y_{k+1} = g(\xi, u_{k-\tau}, w)\}. \end{aligned}$$

Which implies that

$$\begin{aligned} \pi_1(\hat{p}_{k+1})(x) = \ &\sup_{w \in l_2([0,k-\tau], \mathbf{R}^r)} \sup_{x_0 \in \mathbf{R}^n} \{p_0(x_0) + \textstyle\sum_{i=0}^{k-\tau} |z_{i+1}|^2 - \gamma^2 |w_i|^2 : \quad (6) \\ & x_{k-\tau+1} = x, \\ & y_{i+1+\tau} = g(x_i, u_i, w_i), x_{i+1} = f(x_i, u_i, w_i), 0 \leq i \leq k - \tau\} \end{aligned}$$

for $k \geq \tau$.

Recalling the definition of J (equation (4)), and assuming $k < \tau$, we have

$$J(\hat{p}_{k+1})(x) = \sup_{w \in l_2([0,k], \mathbf{R}^r)} \sup_{\xi \in \mathbf{R}^n} \{p_0(\xi) + \sum_{i=0}^{k} |z_{i+1}|^2 - \gamma^2 |w_i|^2 : x_{k+1} = x\}$$

which equals p_{k+1} by definition.

Now, assuming that $k \geq \tau$, we obtain

$$\begin{aligned} J(\hat{p}_{k+1})(x) &= \sup_{w \in l_2([k-\tau+1,k], \mathbf{R}^r)} \sup_{\xi \in \mathbf{R}^n} \{\pi_1(\hat{p}_{k+1})(\xi) + \sum_{i=k-\tau+1}^{k} |z_{i+1}|^2 - \gamma^2 |w_i|^2 : \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x_{k+1} = x\} \\ &= p_{k+1} \text{ by substituting } \pi_1(\hat{p}_{k+1}) \text{ from equation (6).} \end{aligned}$$

$\square$

Remark: Note that we could have expressed the *cost to come* function p_k as

$$\begin{aligned} p_k(x) = \ &\sup_{\xi \in l_2([0,k], \mathbf{R}^n)} \{p_0(\xi_0) + \textstyle\sum_{i=0}^{k-\tau-1} B(\xi_i, \xi_{i+1}, u_i, y_{i+1+\tau}) + \\ & \sup_{w \in l_2([i=\underline{k-\tau},k-1], \mathbf{R}^r)} \textstyle\sum_{k-\tau}^{k-1} \{|h(\xi_i, u_i, w_i)|^2 - \gamma^2 |w_i|^2 : \xi_k = x, \xi_{i+1} = \\ & f(\xi_i, u_i, w_i), i \in [\underline{k-\tau}, k-1]\}\} \end{aligned}$$

where $\underline{k-\tau}$ equals $k - \tau$ if $k \geq \tau$, or else equals 0.

Theorem 1 *For $u \in O_{0,k-1}$, $p \in \mathcal{E}$, such that $L_{p,k}(u)$ is finite, we have*

$$L_{p,k}(u) = \begin{cases} \sup_{y \in l_2([\tau+1,k],\boldsymbol{R}^t)} \{(p_k, 0) : p_0 = p\}, \ k \in [\tau+1, K] \\ \{(p_k, 0) : p_0 = p\}, \ k \le \tau. \end{cases}$$

Proof:

In particular, we have for $k \le \tau$,

$$\begin{aligned} \{(p_k, 0) : p_0 = p\} &= \\ &= \sup_{w \in l_2([0,k-1],\boldsymbol{R}^r)} \sup_{x_0 \in \boldsymbol{R}^n} \{p(x_0) + \textstyle\sum_{i=0}^{k-1} |z_{i+1}|^2 - \gamma^2 |w_i|^2\} \\ &= L_{p,k}(u). \end{aligned}$$

Now if $k > \tau$, we obtain

$$\begin{aligned} &\sup_{y \in l_2([\tau+1,k],\boldsymbol{R}^t)} \{(p_k, 0) : p_0 = p\} \\ &= \sup_{y \in l_2([\tau+1,k],\boldsymbol{R}^t)} \sup_{\xi \in l_2([0,k],\boldsymbol{R}^n)} \sup_{w \in l_2([0,k-1],\boldsymbol{R}^r)} \{p(\xi_0) + \\ &\qquad\qquad \textstyle\sum_{i=0}^{k-\tau-1} B(\xi_i, \xi_{i+1}, u_i, y_{i+1+\tau}) + \sum_{i=k-\tau}^{k-1} |z_{i+1}|^2 - \gamma^2 |w_i|^2\} \\ &= \sup_{w \in l_2([0,k-1],\boldsymbol{R}^r)} \sup_{x_0 \in \boldsymbol{R}^n} \{p(x_0) + \textstyle\sum_{i=0}^{k-1} |z_{i+1}|^2 - \gamma^2 |w_i|^2\} \\ &= L_{p,k}(u). \end{aligned}$$

$\square$

This immediately yields the following corollary.

Corollary 1 *For any output feedback controller $u \in O_{0,K-1}$, the closed-loop system Σ^u is finite gain if and only if $\hat{p}_k$ satisfies*

$$0 \ge \begin{cases} \sup_{y \in l_2([\tau+1,k],\boldsymbol{R}^t)} \left\{ (J(\hat{p}_k), 0) : \hat{p}_0 = \begin{bmatrix} -\beta_K^u \\ \phi \end{bmatrix} \right\}, \ \forall k \in [\tau+1, K] \\ \left\{ (J(\hat{p}_k), 0) : \hat{p}_0 = \begin{bmatrix} -\beta_K^u \\ \phi \end{bmatrix} \right\}, \ \forall k \in [0, \tau] \end{cases} \tag{7}$$

for some finite $\beta_K^u(x) \ge 0$, $\beta_K^u(0) = 0$.

In fact, the above result yields a separation principle, in the sense that $\hat{p}_k \in \mathcal{D}$ contains all the relevant information required to solve the problem. This justifies naming $\hat{p}_k \in \mathcal{D}$ obtained via dynamics (5), with initial conditions of the form $\begin{bmatrix} p_0 \\ \phi \end{bmatrix}$, $p_0 \in \mathcal{E}$ the *information state*. In particular, we have transformed the problem into one with full information, with a new (infinite dimensional) system Ξ, whose states are $\hat{p}_k$, and the disturbance are the measurements y_k. The cost is now given by (7).

Remark: Note that the *information state* is no longer the cost to come, as it was in the case of no measurement delay [9]. However, in case we have $\tau = 0$, the two definitions coincide.

Remark: Furthermore, note that we could have taken the supremum in equation (7) over $y \in l_2([1,k], \boldsymbol{R}^t)$, since the cost is independent of y_k, for $k \in [0, \tau]$.

Remark: It is also clear, that if the delay-free case yields a finite dimensional information state, then the information state for the delayed measurement case is also finite dimensional, provided that the delay (τ) is finite.

4 Solution to the Finite Time Delayed Measurement Problem

We employ dynamic programming to solve the problem. Define

$$M_k(\hat{p}) \overset{\triangle}{=} \inf_{u \in O_{0,k-1}} \sup_{y \in l_2([1,k], \boldsymbol{R}^t)} \left\{ (J(\hat{p}_k), 0) : \hat{p}_0 = \hat{p} \right\}. \tag{8}$$

For a function $M : \mathcal{D} \to \boldsymbol{R}^*$, we write

$$dom\ M = \{\hat{p} \in \mathcal{D} : M(\hat{p}) \text{ is finite}\}$$

and, we also write

$$dom\ L_{\cdot,k}(u) = \{p \in \mathcal{E} : L_{p,k} \text{ is finite}\}.$$

Now consider the following dynamic programming equation.

$$\begin{aligned}
W_k(\hat{p}) &= \inf_{u \in \boldsymbol{U}} \sup_{y \in \boldsymbol{R}^t} \{W_{k-1}(F(\hat{p}, u, y))\} \\
&\quad \hat{p} \in dom\ W_k,\ k \in [1, K] \\
W_0(\hat{p}) &= (\pi_1(\hat{p}), Q_0^{\pi_2(\hat{p})})
\end{aligned} \tag{9}$$

where $Q_0^{\pi_2(\hat{p})}$ is obtained via the following open-loop dynamic programming equation

$$\begin{aligned}
Q_k^{\pi_2(\hat{p})}(x) &= \sup_{w \in \boldsymbol{R}^r} \{|h(x, \pi_2(\hat{p})_k, w)|^2 - \gamma^2 |w|^2 + Q_{k+1}^{\pi_2(\hat{p})}(f(x, \pi_2(\hat{p})_k, w))\} \\
&\quad x \in \boldsymbol{R}^n,\ k = 0, \ldots, l(\pi_2(\hat{p})) - 1 \\
Q_{l(\pi_2(\hat{p}))}^{\pi_2(\hat{p})}(x) &= 0.
\end{aligned} \tag{10}$$

Lemma 3 *Let $\hat{p} \in \mathcal{D}$, and let $Q_0^{\pi_2(\hat{p})}$ be obtained as a solution to the open-loop dynamic programming equation (10). Then*

$$(J(\hat{p}), 0) = (\pi_1(\hat{p}), Q_0^{\pi_2(\hat{p})}).$$

Proof:

Dynamic programming arguments imply that

$$Q_0^{\pi_2(\hat{p})}(x) = \sup_{w \in l_2([0, l(\pi_2(\hat{p}))-1], \boldsymbol{R}^r)} \left\{ \sum_{i=0}^{l(\pi_2(\hat{p}))-1} |z_{i+1}|^2 - \gamma^2 |w_i|^2 : x_0 = x \right\}.$$

7

Which in turn implies that

$$
\begin{aligned}
(\pi_1(\hat{p}), Q_0^{\pi_2(\hat{p})}) &= \sup_{\xi \in \boldsymbol{R}^n} \{\pi_1(\hat{p})(\xi) + \sup_{w \in l_2([0, l(\pi_2(\hat{p}))-1], \boldsymbol{R}^r)} \sum_{i=0}^{l(\pi_2(\hat{p}))-1} |z_{i+1}|^2 - \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \gamma^2 |w_i|^2 : x_0 = \xi\} \\
&= \sup_{x \in \boldsymbol{R}^n} J(\hat{p})(x) \\
&= (J(\hat{p}), 0).
\end{aligned}
$$

$\square$

Theorem 2 *Let W be the solution of the dynamic programming equation (9), initialized via (10). Then $W = M$.*

Proof:

Note that $M_0(\hat{p}) = (J(\hat{p}), 0) = W_0(\hat{p})$. We now establish that M satisfies (9). We use induction. Let this be true for k. Then we have

$$
\begin{aligned}
M_{k+1}(\hat{p}) &= \inf_{u \in O_{0,k}} \sup_{y \in l_2([1, k+1], \boldsymbol{R}^t)} \{(J(\hat{p}_{k+1}), 0) : \hat{p}_0 = \hat{p}\} \\
&= \inf_{u_0 \in \boldsymbol{U}} \sup_{y_1 \in \boldsymbol{R}^t} \inf_{u \in O_{1,k}} \sup_{y \in l_2([2, k+1], \boldsymbol{R}^t)} \{(J(\hat{p}_{k+1}), 0) : \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \hat{p}_1 = H(\hat{p}, u, y)\} \\
&\qquad \text{(where we interchange the minimization over } u_{1,k} \text{ and maximization} \\
&\qquad \text{over } y_1, \text{ since } u_{1,k} \text{ is a function of } y_1.) \\
&= \inf_{u \in \boldsymbol{U}} \sup_{y \in \boldsymbol{R}^t} \inf_{u \in O_{0,k-1}} \sup_{y \in l_2([1, k], \boldsymbol{R}^t)} \{(J(\hat{p}_k), 0) : \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \hat{p}_0 = \quad H(\hat{p}, u, y)\} \\
&\qquad \text{(due to time invariance.)} \\
&= \inf_{u \in \boldsymbol{U}} \sup_{y \in \boldsymbol{R}^t} M_k(H(\hat{p}, u, y)).
\end{aligned}
$$

Hence, since $M_0 = W_0$, an induction argument also establishes that $M_k = W_k$, $k \in [0, K]$.

$\square$

We now state the necessary and sufficient conditions for the solvability of the finite time robust control problem.

Theorem 3 *(Necessity) Assume that $u^o \in O_{0,K-1}$ solves the finite time output feedback problem subject to a constant measurement delay of $\tau \geq 0$. Then there exists a solution M to the dynamic programming equation (9), such that $dom\ L_{\cdot,K}(u^o) \subset \pi_1(dom\ M_k)$,
$M_k\left(\begin{bmatrix} -\beta_K^{u^o} \\ \phi \end{bmatrix}\right) = 0$, $M_k(\hat{p}) \geq (J(\hat{p}), 0)$, $\hat{p} \in dom\ M_k$, $k \in [0, K]$.*

Proof: We first establish that $M_k(\hat{p}) \geq (J(\hat{p}), 0)$. Let $\hat{p} \in dom\ M_k$. We can write $M_k(\hat{p})$ as

$$
\begin{aligned}
M_k(\hat{p}) &= \inf_{u \in O_{0,k-1}} \sup_{w \in l_2([0, k-1], \boldsymbol{R}^r)} \sup_{x_0 \in \boldsymbol{R}^n} \{J(\hat{p})(x_0) + \sum_{i=0}^{k-1} |z_{i+1}|^2 - \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \gamma^2 |w_i|^2\} \\
&\geq (J(\hat{p}), 0).
\end{aligned}
$$

8

Let $p \in dom\ L_{\cdot,K}(u^o)$, and set $\hat{p} = \begin{bmatrix} p \\ \phi \end{bmatrix}$. Now by (8)

$$
\begin{aligned}
M_k\left(\begin{bmatrix} p \\ \phi \end{bmatrix}\right) &= \inf_{u \in O_{0,k-1}} L_{p,k}(u) \\
&\leq L_{p,k}(u^o) \\
&\leq (p, \beta_K^{u^o}).
\end{aligned}
$$

Thus, $dom\ L_{\cdot,K}(u^o) \subset dom\ M_k$. Since, $\beta_K^{u^o}(x) \geq 0$, $\beta_K^{u^o}(0) = 0$, we have

$$
J\left(\begin{bmatrix} -\beta_K^{u^o} \\ \phi \end{bmatrix}\right) = (-\beta_K^{u^o}, 0) = 0
$$

This implies that $M_k\left(\begin{bmatrix} -\beta_K^{u^o} \\ \phi \end{bmatrix}\right) = 0$. Also Theorem 2 establishes that M is the unique solution to the dynamic programming equation (9).

$\square$

Theorem 4 *(Sufficiency) Assume there exists a solution M to the dynamic programming equation (9) on some non-empty domain $dom\ M_k$, such that $\begin{bmatrix} -\beta \\ \phi \end{bmatrix} \in dom\ M_k$, $M_k\left(\begin{bmatrix} -\beta \\ \phi \end{bmatrix}\right) = 0$, for some $\beta \geq 0$, $\beta(0) = 0$ and also that $M_k(\hat{p}) \geq (J(\hat{p}), 0)$, for all $\hat{p} \in dom\ M_k$, $k \in [0, K]$. Let $\bar{u}_k(\hat{p})$ achieve the minimum in (9) for each $\hat{p} \in dom\ M_k$, $k \in [1, K]$. Let u^* be a policy such that $u_k^* = \bar{u}_{K-k}(\hat{p}_k)$, where $\hat{p}_k$ is the corresponding trajectory with initial conditions $\hat{p}_0 = \begin{bmatrix} -\beta \\ \phi \end{bmatrix}$, assuming $\hat{p}_k \in dom\ M_{K-k}$, $k \in [0, K]$. Then u^* solves the finite-time output feedback problem subject to a constant delay of $\tau \geq 0$.*

Proof: Observe that

$$
M_k\left(\begin{bmatrix} p \\ \phi \end{bmatrix}\right) = L_{p,k}(u^*) \leq L_{p,k}(u)
$$

for all $u \in O_{0,k-1}$, $\begin{bmatrix} p \\ \phi \end{bmatrix} \in dom\ M_k$. Hence,

$$
\sup_{y \in l_2([1,k], \mathbf{R}^r)} \left\{ (J(\hat{p}_k), 0) : \hat{p}_0 = \begin{bmatrix} -\beta \\ \phi \end{bmatrix}, u = u^* \right\} \leq M_k(-\beta) = 0
$$

which implies by Corollary 1 that Σ^{u^*} is finite gain, and thus u^* solves the finite time output feedback problem.

$\square$

9

Remark: We see that the solvability of the delayed measurement case requires: (i) existence of a solution $\hat{p}_k$ to (5), (ii) existence of a solution $Q^{\pi_2(\hat{p})}$ to (10), (iii) existence of a solution M to (9), and (iv) a coupling condition, viz. $\hat{p}_k \in dom\ M_{K-k}$.

5 Certainty Equivalence

In practice, solving the problem is computationally hard. The reason for this is the infinite dimensional dynamic programming equation (9). There is a tremendous reduction in complexity if one uses the certainty equivalence controller. However, certainty equivalence controllers are in general non-optimal [7]. Identifying $J(\hat{p}_k)$ as the "past stress", and V_k as the "future stress", where V_k is the upper value function of the state feedback dynamic game obtained via

$$
\begin{aligned}
V_k(x) &= \inf_{u \in \boldsymbol{U}} \sup_{w \in \boldsymbol{R}^r} \{|h(x, u, w)|^2 - \gamma^2 |w|^2 + V_{k+1}(f(x, u, w))\} \\
&\quad k = 0, \ldots, K-1, \quad x \in \boldsymbol{R}^n \\
V_K(x) &= 0
\end{aligned}
$$

and u_F is the corresponding minimizing control policy. Then, we estimate

$$
\hat{x}_k \in \arg \max_{x \in \boldsymbol{R}^n} \{J(\hat{p}_k)(x) + V_k(x)\} \tag{11}
$$

and use $u_k(\hat{p}_k) = u_F(\hat{x}_k)$ as the control value. The condition for certainty equivalence to hold stated in [7] can be extended to the delayed measurement case as well, and can be stated as

$$
M_k(\hat{p}_k) = (J(\hat{p}_k), V_k), \ k = 0, \ldots, K-1
$$

or, if we want to avoid reference to the infinite dimensional value function M as, [1]

$$
(J(\hat{p}_k), V_k) = \inf_{u \in \boldsymbol{U}} \sup_{x \in \boldsymbol{R}^n} \{J(\hat{p}_k)(x) + \sup_{w \in \boldsymbol{R}^r} |h(x, u, w)|^2 - \gamma^2 |w|^2 + V_{k+1}(f(x, u, w))\}
$$

for $k = 0, \ldots, K-1$.

Note that we could have expressed the RHS of (11) as

$$
\left(\pi_1(\hat{p}_k), P_0^{\pi_2(\hat{p}_k)}\right)
$$

where $P^{\pi_2(\hat{p}_k)}$ is the solution of the following open-loop dynamic programming equation

$$
\begin{aligned}
P_i^{\pi_2(\hat{p}_k)}(x) &= \sup_{w \in \boldsymbol{R}^r} \{|h(x, \pi_2(\hat{p}_k)_i, w)|^2 - \gamma^2 |w|^2 + P_{i+1}^{\pi_2(\hat{p}_k)}(f(x, \pi_2(\hat{p}_k)_i, w))\} \\
&\quad 0 \le i \le l(\pi_2(\hat{p}_k)) - 1 \\
P_{l(\pi_2(\hat{p}_k))}^{\pi_2(\hat{p}_k)}(x) &= V_k(x)
\end{aligned}
\tag{12}
$$

Remark: Equation (12) is analogous to the third Riccati equation encountered in the linear case, whose initial conditions depend on the solution to the state feedback Riccati equation [2]. In fact, it is simply equation (10) with a different initial condition.

6　Example

We now present a simple example to illustrate the advantages of delay compensation. The example is based on a simple system presented in [4], and is described by

$$
\begin{aligned}
\frac{dx}{dt} &= u - x - \beta \frac{1}{1 + \frac{K_1}{x} + \frac{x}{K_2}} + w \\
y &= x + v
\end{aligned}
\tag{13}
$$

Here, y is the measured reactant concentration, x is the true reactant concentration, t is the dimensionless time, u is the feed reactant concentration, K_1 and K_2 are kinetic constants, β is a constant, w is the disturbance in the input concentration, and v is the sensor noise. In [4] it is mentioned that the model for the single enzyme-catalyzed reaction with substrate-inhibited kinetics, as well as the model for the ethylene hydrogenation in an isothermal CSTR are of the above form. The reactant concentration is controlled by manipulating the feed reactant concentration u, based on the measured concentration y. The constants are fixed as $K_1 = 0.01$, $K_2 = 0.1$, and $\beta = 2.0$. We pick the operating point for this reactor to correspond to an unstable steady state at $x = 0.125$, and $u = 0.9834$. The objective of the controller design is to reject the influence of the disturbances on the regulated output z, given by

$$
z = \xi^2 + 0.0001(u - 0.9834)^2
\tag{14}
$$

Here, ξ represents the filtered error given by

$$
\frac{d\xi}{dt} = -0.2\xi + 10(x - 0.125)
\tag{15}
$$

and the control effort is weighted to prevent large values of the control.

The system ((13)-(15)) is discretized with a sampling period of 0.02, and the state feedback problem is solved with $\gamma = 1.0$ and a time horizon (K) of 100 steps. We then implement the certainty equivalence controller (11), with a moving horizon control (obtained by replacing V_k in (11), with V_0 at every time step k). The information state is initialized as $p_0(x, \xi) = 0$ if $x = 0.125, \xi = 0$, or equals $-\infty$ else. The system is initialized to start from equilibrium. For purposes of simulation, a zero order hold was employed. The measurement noise (v) is modeled as zero mean Gaussian with a standard deviation of $2e^{-6}$. The response of the system with no delay, and a delay of 0.2 (corresponding to a delay of 10 samples), to a sinusoidal disturbance with magnitude 0.05, and frequency 0.2 rad/time in the feed concentration (w) is illustrated in Figure 1. One observes that the performance of the system with delay deteriorates. However, stability is still maintained. On the other hand, if no compensation were employed the system goes unstable, and oscillates as shown in Figure 2. In fact, even a delay of 0.02 (corresponding to one sample) results in instability.

7　Conclusion

This paper establishes a general framework for solving the nonlinear H_∞ control problem for systems subject to measurement delays. In particular, our approach yields both necessary and sufficient conditions for the solution to exist. The information state employed
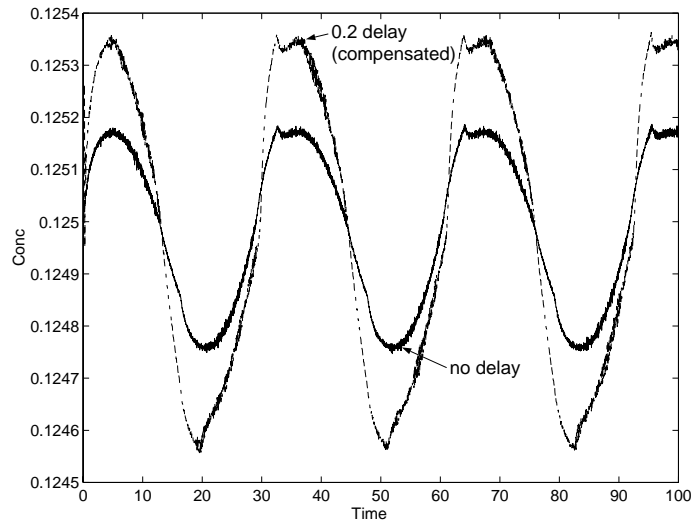
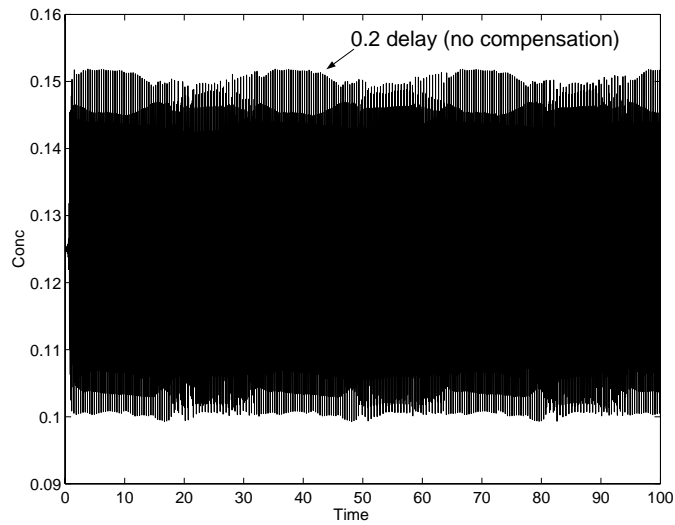Figure 1: Closed-loop response to feed disturbance with and without measurement delays.

Figure 2: Closed-loop response to feed disturbance with a measurement delay of 0.2 employing the controller corresponding to the delay-free system.

12

to solve the problem is no longer the "cost to come" function. The conditions for solvability require solutions to two dynamic programming equations, a filter equation for the information state, and satisfaction of a coupling condition. We also discussed the certainty equivalence principle for such systems and draw parallels with the solution for linear systems. An example was presented to illustrate the ideas. One of the most pressing issues is regarding good approximations (in particular, finite dimensional approximations to the information state), and computationally efficient solutions to the nonlinear H_∞ problem. This is currently being worked upon.

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