Robustness under Uncertainty with Phase Information

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ROBUSTNESS UNDER UNCERTAINTY WITH PHASE INFORMATION

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Abstract. The framework of Doyle’s structured singular value is extended to take advantage of possibly available phase information on the dynamic uncertainty. A computable upper bound is obtained for this phase-sensitive structured singular value.

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1. Introduction and preliminaries

Let $RH_{\infty}$ denote the set of real rational stable proper scalar transfer functions and let $RH_{\infty}^{n \times n}$ be the set of $n \times n$ matrices with entries in $RH_{\infty}$. Given two nonnegative integers $r$ and $c$ and a list of integers (block-structure) $\mathcal{K} = (k_1, \cdots, k_r; k_{r+1}, \cdots, k_{r+c})$, with $\sum_{i=1}^{r+c} k_i = n$, consider the subspace $\mathcal{X}_\mathcal{K}$ of $RH_{\infty}^{n \times n}$ given by

$$\mathcal{X}_\mathcal{K} = \{\text{block diag}(\delta_1 I_{k_1}, \cdots, \delta_{r+c} I_{k_{r+c}}) : \delta_i \in \mathbb{R}, i = 1, \cdots, r; \delta_{r+i} \in RH_{\infty}, i = 1, \cdots, c\}.$$  

Many robust stability analysis issues can be addressed via the following question for the feedback system $S$ of Figure 1 below [1-3]: Given $P \in RH_{\infty}^{n \times n}$ and $\delta > 0$, is $S$ well-formed and internally stable for any $\Delta \in \mathcal{X}_\mathcal{K}$ satisfying $\|\Delta\|_{\infty} \leq \delta$, where $\| \cdot \|_{\infty}$ denotes the $H_{\infty}$ norm? Here $\delta_i$, $i = 1, \cdots, r$, correspond to real parametric uncertainties and $\delta_{r+i}$, $i = 1, \cdots, c$, to unmodeled dynamics. In [1,2] Doyle et al. showed that this question can be answered by means of the structured singular value (SSV) $\mu_\mathcal{K}$, defined for any complex $n \times n$ matrix $M$ by $\mu_\mathcal{K}(M) = 0$ if $\det(I - \Delta M) \neq 0$ for all $\Delta \in \mathcal{X}_\mathcal{K}$, and

$$\mu_\mathcal{K}(M) = \left( \min_{\Delta \in \mathcal{X}_\mathcal{K}} \{ \bar{\sigma}(\Delta) : \det(I - \Delta M) = 0 \} \right)^{-1}$$

otherwise, where the subspace $\mathcal{X}_\mathcal{K}$ of $\mathbb{C}^{n \times n}$ is given by

$$\mathcal{X}_\mathcal{K} = \{\text{block diag}(\delta_1 I_{k_1}, \cdots, \delta_{r+c} I_{k_{r+c}}) : \delta_i \in \mathbb{R}, i = 1, \cdots, r; \delta_{r+i} \in \mathbb{C}, i = 1, \cdots, c\}$$

and where $\bar{\sigma}(\cdot)$ denotes the largest singular value. Specifically, the “Small $\mu$ Theorem” [2] asserts that $S$ is well-formed and internally stable for any $\Delta \in \mathcal{X}_\mathcal{K}$, $\|\Delta\|_{\infty} \leq \delta$ if, and only if,

$$\sup_{\omega} \mu_\mathcal{K}(P(j\omega)) < 1/\delta.$$  

While exact numerical evaluation of the SSV appears to be generally computationally prohibitive when $r \neq 0$, a reasonably good upper bound (yielding a sufficient stability condition) can be obtained at moderate cost [4,5].

Figure 1

Note that functions in $\mathcal{X}_\mathcal{K}$ take values in $\mathcal{X}_\mathcal{K}$
The purpose of this paper is to extend the framework of the SSV to allow for the case when, besides structure- and magnitude information, phase information is available for the uncertainty (see [6,7] for other work on robustness with phase information). Specifically, we consider the situation in which given some function $\Theta : \mathbb{R} \to [0, \pi]^c$, $\Delta$ is known to lie in the set $X_\mathcal{K}^\Theta \subset RH_\infty^{n \times n}$ defined by

$$X_\mathcal{K}^\Theta = \{ \Delta \in X_\mathcal{K} : |\angle \delta_{r+i}(j\omega)| \leq \Theta_i(\omega), \ i = 1, \ldots, c \} \cup \{0\},$$

where given any $z \in \mathbb{C}\setminus\{0\}$, $\angle z$ denotes its phase in $(-\pi, \pi]$. We are naturally lead to define a “phase-sensitive” SSV.

In the sequel, for $\theta \in [0, \pi]^c$, we make use of the set $X_\mathcal{K}^\theta \subset C^{n \times n}$, defined by

$$X_\mathcal{K}^\theta = \{ \Delta \in X_\mathcal{K} : |\angle \delta_{r+i}| \leq \theta_i, \ i = 1, \ldots, c \} \cup \{0\}.$$

**Definition 1.** Given a block-structure $\mathcal{K}$ and a vector $\theta \in [0, \pi]^c$, the phase-sensitive structured singular value $\mu_{\mathcal{K}}^\theta(M)$ of $M$ with respect to block-structure $\mathcal{K}$ and phase $\theta$ is given by $\mu_{\mathcal{K}}^\theta(M) = 0$ if there is no $\Delta \in X_\mathcal{K}^\theta$ such that $\det(I - \Delta M) = 0$, and

$$\mu_{\mathcal{K}}^\theta(M) = \left( \min_{\Delta \in X_\mathcal{K}^\theta} \{ \bar{\sigma}(\Delta) : \det(I - \Delta M) = 0 \} \right)^{-1}$$

otherwise. □

Below, it is first shown that a natural extension of the Small $\mu$ Theorem holds in this case. A more tractable formula is then proposed for $\mu_{\mathcal{K}}^\theta(M)$, a direct extension of one obtained in [4] for $\mu_{\mathcal{K}}(M)$. Finally, an efficiently computable upper bound (yielding a sufficient condition of stability) is given, again directly related to that for $\mu_{\mathcal{K}}(M)$ derived in [4].

2. Extended Small $\mu$ Theorem

For any $\delta > 0$, let

$$X_\mathcal{K}^\Theta(\delta) = \{ \Delta \in X_\mathcal{K}^\Theta : \|\Delta\|_{\infty} \leq \delta \}; \quad X_\mathcal{K}^\theta(\delta) = \{ \Delta \in X_\mathcal{K}^\theta : \bar{\sigma}(\Delta) \leq \delta \}.$$

**Theorem 1.** Given $P \in RH_\infty^{n \times n}$, $\delta > 0$, and $\Theta : \mathbb{R} \to [0, \pi]^c$ continuous, the following two statements are equivalent: (i) the feedback system $S$ of Figure 1 is well-formed and internally stable for all $\Delta \in X_\mathcal{K}^\Theta(\delta)$; (ii)

$$\sup_{\omega} \mu_{\mathcal{K}}^\Theta(\omega)(P(j\omega)) < 1/\delta.$$

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3. The more natural situation in which $\Phi_i(\omega) \leq \angle \delta_{r+i}(j\omega) \leq \Psi_i(\omega)$, for some $\Phi, \Psi : \mathbb{R} \to (-\pi, \pi]^c$ can be easily reduced to the case considered here.

4. $\Theta_i(\omega) = \pi$ for all $\omega$ accounts for blocks with no phase information.

5. More generally the theorem holds with $X_\mathcal{K}^\Theta$ replaced by any subset $\mathcal{X}$ of $RH_\infty^{n \times n}$ containing the origin such that, for any $\delta > 0$, $\{ \Delta \in \mathcal{X} : \|\Delta\|_{\infty} \leq \delta \}$ is pathwise connected, and $\mu(M)$ defined accordingly.

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Proof. We first show that (i) is equivalent to

$$\det(I - \Delta(j\omega)P(j\omega)) \neq 0 \quad \forall \Delta \in \mathcal{X}_K^\Theta(\delta), \quad \omega \in [-\infty, \infty].$$

(2)

Since $P \in RH_{n \times n}^\infty$ and $\mathcal{X}_K^\Theta(\delta) \subset RH_{n \times n}^\infty$, (i) holds if and only if, for all $\Delta \in \mathcal{X}_K^\Theta(\delta)$, $(I - \Delta(s)P(s))^{-1}$ is well defined and has no poles in $\mathbb{C}_+ \cup \{\infty\}$, i.e., $\det(I - \Delta(s)P(s)) \neq 0 \quad \forall s \in \mathbb{C}_+ \cup \{\infty\}$. Since $P$ and $\Delta$ are analytic and bounded in $\mathbb{C}_+$, in view of the Principle of the Argument, it follows that (i) holds if and only if for all $\Delta \in \mathcal{X}_K^\Theta(\delta)$ the closed curve

$$\Gamma := \{\det(I - \Delta(j\omega)P(j\omega)) : \omega \in [-\infty, \infty]\}$$

does not encircle or pass through the origin. Thus, in particular, (i) implies (2). Conversely suppose that (2) holds. Let $\Delta \in \mathcal{X}_K^\Theta(\delta)$ and let $\varphi : [0,1] \to [-\infty, \infty]$ be continuous and onto. In view of the definition of $\mathcal{X}_K^\Theta(\delta)$, $\alpha \Delta \in \mathcal{X}_K^\Theta(\delta)$ for any $\alpha \in [0,1]$. In view of (2), it follows that the continuous function $h : [0,1] \times [0,1] \to \mathbb{C}$ given by

$$h(\alpha, t) = \det(I - \alpha\Delta(j\varphi(t))P(j\varphi(t)))$$

defines a homotopy in the punctured plane $\mathbb{C}\setminus\{0\}$ between $h(0, \cdot) = 1$ and $h(1, \cdot) = \det(I - \Delta(j\varphi(\cdot))P(j\varphi(\cdot)))$. Thus the family of closed curves generated by $h(\alpha, \cdot)$, as $\alpha$ ranges over $[0,1]$, corresponds to a single element in the fundamental group of equivalence classes of closed curves in the punctured plane, i.e., all these curves encircle the origin the same number of times (see, e.g., [8, Section 8-5] for details). Thus the curve $\Gamma (= \{h(1, t) : t \in [0,1]\})$ does not encircle the origin, so that $S$ is well-formed and internally stable for the given $\Delta$. Thus (i) is equivalent to (2). To complete the proof, we now show that (2) is equivalent to (ii). First, note that, for given $\omega$, when $\Delta$ ranges over $\mathcal{X}_K^\Theta(\delta)$, its value $\Delta(j\omega)$ ranges over $\mathcal{X}_K^{\Theta(\omega)}(\delta)$. Thus (2) holds if, and only if,

$$\det(I - \Delta P(j\omega)) \neq 0 \quad \forall \Delta \in \mathcal{X}_K^{\Theta(\omega)}(\delta), \quad \forall \omega \in [-\infty, \infty]$$

or equivalently, since $\Theta$ is continuous,

$$\inf_{\omega} \left\{ \min_{\Delta \in \mathcal{X}_K^{\Theta(\omega)}(\delta)} \{\sigma(\Delta) : \det(I - \Delta P(j\omega)) = 0\} \right\} > \delta.$$

In view of Definition 1, the last statement is equivalent to (ii). \(\square\)

Note that the standard Small $\mu$ Theorem [2] is obtained as a particular case of Theorem 1, corresponding to $\Theta_i(\omega) = \pi$, $i = 1, \cdots, c$, for all $\omega$.

3. Computation of $\mu_K^\Theta(M)$

Computing $\mu_K^\Theta(M)$ by solving optimization problem (1) is impractical as this problem may have many local minimizers that are not global. Such local minima yield lower bounds to $\mu_K^\Theta(M)$, and thus sufficient conditions for instability may be tested. Short of computing $\mu_K^\Theta(M)$ exactly, of more interest would be an upper bound to it, allowing a sufficient condition for stability to be checked. As a first step toward this goal, we express $\mu_K^\Theta(M)$ as the optimal value of a smooth constrained maximization problem (note that
problem (1) is nonsmooth). In this section, we sacrifice generality for clarity and restrict ourselves to the structure $\mathcal{K} = (\cdot, 1, 1)$, i.e., to two scalar dynamic uncertainty blocks, and we assume that no phase information is available concerning the first block, i.e., $\theta_1 = \pi$. For notational simplicity, $\theta_2$ is renamed $\theta$ and all $\mathcal{K}$ subscripts are dropped. Extension to more general structures presents no conceptual difficulties.

For any $\beta \in \mathbb{R}$, let $G_\beta \in \mathbb{C}^{2 \times 2}$ be given by

$$G_\beta = \begin{bmatrix} 0 & 0 \\ 0 & 1 + j\beta \end{bmatrix}.$$ 

Also let $P_1 = \text{diag}(1, 0)$, and $P_2 = \text{diag}(0, 1)$. Finally, we denote by $\partial B$ the unit Euclidean ball in $\mathbb{C}^n$, i.e., $\partial B = \{ x \in \mathbb{C}^n : \| x \|_2 \leq 1 \}$ and superscript $H$ indicates conjugate transpose. The phase sensitive SSV can be expressed as the optimal solution of a smooth constrained optimization problem as follows.

**Theorem 2.**

$$\mu^\theta(M) = \begin{cases} 0 \\
\max_{\gamma \in \mathcal{S}^\theta(M)} \{ \gamma : \| P_i M x \|_2 \geq \gamma \| P_i x \|_2, \quad i = 1, 2 \} 
\end{cases}$$ 

if $\mathcal{S}^\theta(M) = \emptyset$; otherwise.

where $\mathcal{S}^\theta(M)$ is defined as follows:

(i) $\mathcal{S}^0(M) = \{ x \in \partial B : x^H (M^H P_2 - P_2 M) x = 0, \quad x^H (M^H P_2 + P_2 M) x \geq 0 \}$

(ii) for $\theta \in (0, \frac{\pi}{2}]$,

$$\mathcal{S}^\theta(M) = \{ x \in \partial B : x^H (M^H G_\beta + G_\beta^H M) x \geq 0 \quad \forall \beta \in \{ \pm \cot \theta \} \}$$

(iii) for $\theta \in (\frac{\pi}{2}, \pi)$

$$\mathcal{S}^\theta(M) = \{ x \in \partial B : x^H (M^H G_\beta + G_\beta^H M) x \geq 0, \quad \beta = -\cot \theta, \quad j x^H (M^H P_2 - P_2 M) x \geq 0 \}$$

$$\cup \{ x \in \partial B : x^H (M^H G_\beta + G_\beta^H M) x \geq 0, \quad \beta = \cot \theta, \quad j x^H (M^H P_2 - P_2 M) x \leq 0 \}.$$ 

Note that the second inequalities in each of the components of $\mathcal{S}^\theta(M)$ for the case $\theta \in (\frac{\pi}{2}, \pi)$ correspond to the limit of inequality

$$x^H (M^H G_\beta + G_\beta^H M) x \geq 0$$

as $\beta$ tends to $+\infty$ and $-\infty$, respectively. This leads to the next step, which is to observe that, as (3) is affine in $\beta$, for $\theta \in (0, \pi)$, the set $\mathcal{S}^\theta(M)$ can be equivalently written as

$$\mathcal{S}^\theta(M) = \{ x \in \partial B : x^H (M^H G + G^H M) x \geq 0 \quad \forall G \in \mathcal{G}^\theta \}$$
for \( \theta \in [0, \frac{\pi}{2}] \), and
\[
S^\theta(M) = \{ x \in \partial B : x^H(M^H G + G^H M)x \geq 0 \quad \forall G \in \mathcal{G}^\theta \}
\cup \{ x \in \partial B : x^H(M^H G + G^H M)x \geq 0 \quad \forall G \in \mathcal{G}^\theta \}
\]
for \( \theta \in (\frac{\pi}{2}, \pi) \), with
\[
\mathcal{G}^0 = \{ \alpha G_\beta : \alpha \geq 0, \beta \in \mathbb{R} \},
\]
\[
\mathcal{G}^\theta = \{ \alpha G_\beta : \alpha \geq 0, |\beta| \leq \cot \theta \}, \quad \theta \in (0, \frac{\pi}{2}],
\]
\[
\mathcal{G}^\theta_+ = \{ \alpha G_\beta : \alpha \geq 0, \beta \geq -\cot \theta \},
\]
\[
\mathcal{G}^\theta_- = \{ \alpha G_\beta : \alpha \geq 0, \beta \leq \cot \theta \}.
\]

Using an argument similar to that employed in [4] one can then prove the following.

**Theorem 3.**
\[
\mu^\theta(M) \leq \nu^\theta(M) \leq \bar{\sigma}(M) \quad \forall \theta \in [0, \pi)
\]

where
\[
\nu^\theta(M)^2 = \begin{cases} 
\max\{0, \inf_{G \in \mathcal{G}^\theta} \bar{\lambda}(M^H M + M^H G + G^H M)\} & \theta \in [0, \frac{\pi}{2}] \\
\max\{0, \inf_{G \in \mathcal{G}^\theta_+} \bar{\lambda}(M^H M + M^H G + G^H M), \inf_{G \in \mathcal{G}^\theta_-} \bar{\lambda}(M^H M + M^H G + G^H M)\} & \theta \in (\frac{\pi}{2}, \pi)
\end{cases}
\]

\[\Box\]

It is readily checked that, like \( \mu(M) \), \( \mu^\theta(M) \) satisfies \( \mu^\theta(DMD^{-1}) = \mu^\theta(M) \) for any nonsingular diagonal matrix \( D \) (with our current assumption of scalar uncertainty blocks).

It follows that
\[
\mu^\theta(M) \leq \hat{\mu}^\theta(M) := \inf\{ \nu^\theta(DMD^{-1}) : D = \text{diag}(d_1, d_2), d_1 \neq 0 \neq d_2 \}
\]

It turns out that the value of the infimum is unchanged if \( d_1 \) and \( d_2 \) are constrained to be real positive. The algorithm proposed in [5] can be modified to compute \( \hat{\mu}^\theta(M) \).

**4. Discussion**

As mentioned above, the results of Section 3 can be readily extended to more general structures. It should be noted however that if, say, \( k \) of the components of \( \theta \) lie in \((\pi/2, \pi)\), the expression for \( \nu^\theta(M) \) will involve \( 2^k \) optimization problems instead of 2. If this is computationally prohibitive, an alternate upper bound to \( \mu^\theta_k(M) \) can be obtained as follows. Rewrite the constraint \( |\Delta \delta_i| \leq \theta \), with \( \theta \in [\pi/2, \pi) \), as \( \delta_i = \delta_i^1 \delta_i^2 \), with \( |\Delta \delta_i^1| \leq \pi/2 \), \( |\Delta \delta_i^2| \leq \theta - \pi/2 \). The magnitude constraint \( |\delta_i| \leq \delta \) can be expressed, e.g., as \( |\delta_i^1| \leq \delta \), \( |\delta_i^2| \leq 1 \) and, by elementary block diagram transformations, the system can be represented as in Figure 1 with \( \delta_i^1 \) and \( \delta_i^2 \) corresponding to distinct diagonal blocks in \( \Delta \). The results of Section 3 can be extended to such structure.
Preliminary numerical tests have been carried out based on (4) and (5), and on the algorithm proposed in [5]. The results are promising in that the computed upper bound is typically lower (yielding a less conservative sufficient stability test) than when the phase information is not taken into account.

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References