MARYLAND

TECHNICAL RESEARCH REPORT

SRC TR 88-109

Convexity of the Largest Singular Value of $e^{D}Me^{-D}$ - a Convexity Lemma

by

N.-K. Tsing

SYSTEMS RESEARCH CENTER

UNIVERSITY OF MARYLAND

COLLEGE PARK, MARYLAND 20742

Convexity of the Largest Singular Value of $e^D M e^{-D}$ — a Convexity Lemma

Nam-Kiu Tsing

Systems Research Center and
Electrical Engineering Department
University of Maryland
College Park, Maryland 20742

Abstract

A rigorous proof is given for a convexity lemma used by Chu and Doyle to prove the convexity of the largest singular value of $e^D M e^{-D}$ on a commuting, convex subset of matrices.

The structured singular value (SSV), which was originated by Doyle [1], is a useful tool for the analysis and synthesis of feedback systems with structured uncertainties (e.g., see [2,3]). While the exact value of the SSV is, in general, difficult to compute, one can always bound it from above by the numerical value

$$\inf_{D \in \mathcal{D}} \bar{\sigma}(e^D M e^{-D}) \ ,$$

where $M \in \mathbb{C}^{n \times n}$ is fixed, $\mathcal{D} \subset \mathbb{R}^{n \times n}$ a certain (convex) subset of diagonal matrices, and $\bar{\sigma}(M)$ denotes the largest singular value of the matrix M. As the real-valued function $\phi : \mathcal{D} \to \mathbb{R}$ defined by

$$\phi(D) = \bar{\sigma}(e^D M e^{-D})$$

is convex on $\mathcal{D}$, the optimization problem (1) is tractable. A rather long proof of the convexity of ϕ can be found in [4]. A much shorter proof was later proposed in [5] (see also [3]), based on the following convexity lemma.

Lemma. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous and that for all $s \in \mathbb{R}$, there exists $g_s : \mathbb{R} \to \mathbb{R}$ twice differentiable such that $f(s) = g_s(s)$, $f(t) \geq g_s(t)$ for all t, and

$$\frac{d^2}{dt^2} g_s(t) \Big|_{t=s} \geq 0 \ .$$

Then f is convex.

Although the lemma seems intuitive, to our knowledge it has nowhere been rigorously proved. The purpose of this note is to provide such a proof.

Proof of Lemma. Let $f : \mathbb{R} \to \mathbb{R}$ satisfy the hypotheses of the lemma. For any closed interval $[a, b]$ (with $a < b$), let Γ be the graph of f on $[a, b]$, i.e.,

$$\Gamma = \{(t, f(t)) \in \mathbb{R}^2 : t \in [a, b]\} \ ,$$

and let K be the convex hull of Γ. Since Γ is compact, K is also compact. Define $\psi : [a, b] \to \mathbb{R}$ by

$$\psi(t) = \max\{u : (t, u) \in K\} \quad \forall t \in [a, b] \ ,$$

(i.e., the graph of ψ is the upper boundary of K). Then ψ is continuous on $[a, b]$. For any $s \in (a, b)$, we consider two cases. First, suppose that $(s, \psi(s))$ is not an extreme point of K. Then ψ is linear on a neighborhood of s. Hence ψ is infinitely differentiable at s and, in particular, $\psi''(s) = 0$. Second, suppose that $(s, \psi(s))$ is an extreme point of K. Then $f(s) = \psi(s)$. Let $h(t) = c_0 + c_1(t - s)$ $(t \in \mathbb{R})$ be the equation of a supporting line to K at $(s, \psi(s))$. Then

$$c_0 = h(s) = \psi(s) = f(s) = g_s(s) \ , \tag{2}$$

and

$$h(t) \geq \psi(t) \geq f(t) \geq g_s(t) \quad \forall t \in [a, b] \ . \tag{3}$$

Since both h and g_s are differentiable at s, by (2) and (3) we must have

$$g'(s) = h'(s) = c_1 \ .$$

Since g_s is twice differentiable at s, we can write

$$g_s(t) = c_0 + c_1(t - s) + c_2(t - s)^2 + \epsilon(t - s)$$

where

$$\lim_{t \to s} \frac{\epsilon(t - s)}{(t - s)^2} = 0 \ .$$

Now, since

$$c_0 + c_1(t - s) = h(t) \geq g_s(t) = c_0 + c_1(t - s) + c_2(t - s)^2 + \epsilon(t - s) \quad \forall t \in [a, b] \ ,$$

we must have $c_2 \leq 0$. However, $c_2 = \frac{1}{2} g_s''(s) \geq 0$ by assumption. Thus $c_2 = 0$. By (3) again,

$$c_0 + c_1(t - s) = h(t) \geq \psi(t) \geq g_s(t) = c_0 + c_1(t - s) + \epsilon(t - s) \quad \forall t \in [a, b] \ .$$

Thus

$$|\psi(t) - c_0 - c_1(t - s)| \leq |\epsilon(t - s)| \quad \forall t \in [a, b] \ ,$$

which shows that ψ is twice differentiable at s and that $\psi''(s) = 0$. Combining the two cases, we see that

$$\psi''(t) = 0 \quad \forall t \in (a, b) .$$

As a result, ψ is linear on $[a, b]$ and thus the graph of f on $[a, b]$ is below the line segment joining $(a, f(a))$ and $(b, f(b))$. Since this is true for any $a < b$, f is convex. □

To conclude this note, and for the sake of ease of reference, we now reproduce the proof of the convexity of ϕ given in [5]. This proof does not make use of the specific definition of $\mathcal{D}$ but only relies on the fact that $\mathcal{D}$ is a commuting convex set (i.e., $X, Y \in \mathcal{D}$ and $0 < t < 1$ will imply $XY = YX$ and $tX + (1-t)Y \in \mathcal{D}$).

Proof of convexity of ϕ. Since $\mathcal{D}$ is commuting, if $X, Y \in \mathcal{D}$ and $0 \le t \le 1$, then

$$e^{tX+(1-t)Y} M e^{-(tX+(1-t)Y)} = e^{t(X-Y)}(e^Y M e^{-Y})e^{-t(X-Y)} .$$

Thus it suffices to prove convexity of $f(t) = \bar{\sigma}(e^{tD} M e^{-tD})$ over $\mathbb{R}$ for arbitrary $D, M \in \mathbb{C}^{n \times n}$. Define $M_s = e^{sD} M e^{-sD}$ and let u and v be any singular vectors such that

$$f(s) = \bar{\sigma}(M_s) = u^H M_s v .$$

Define

$$g_s(t) = Re(u^H e^{tD} M e^{-tD} v) .$$

Since $f(t) \ge g_s(t)$ and

$$\begin{aligned}
\frac{d^2}{dt^2} g_s(t) \Big|_{t=s} &= Re(u^H(D^2 M_s - 2DM_s D + M_s D^2)v) \\
&= f(s)(u^H D^2 u + v^H D^2 v) - 2Re(u^H DM_s Dv) \\
&= [u^H D v^H D] \begin{bmatrix} f(s)I & -M_s \\ -M_s & f(s)I \end{bmatrix} \begin{bmatrix} Du \\ Dv \end{bmatrix} \ge 0 ,
\end{aligned}$$

the result follows from the Lemma. □

Remark. Very recently, another proof of the convexity of ϕ, not based on the convexity lemma, has been proposed by Sezginer and Overton [6].

Acknowledgement. The author wishes to thank Drs. André Tits and Michael Fan for introducing this problem to him and their helpful suggestions and comments. This work was supported in part by the National Science Foundation's Engineering Research Centers Program: NSF CDR 8803012 and by the National Science Foundation under Grant DMC-84-51515.

References

[1] J.C. Doyle, "Analysis of Feedback Systems with Structured Uncertainties," *Proc. IEE-D* 129 (1982), 242–250.

[2] M.K.H. Fan & A.L. Tits, "Characterization and Efficient Computation of the Structured Singular Value," *IEEE Trans. Automat. Control* AC-31 (1986), 734–743.

[3] A.K. Packard, "What's New with μ: Structured Uncertainty in Multivariable Control," University of California, PhD Thesis, Berkeley, California, 1988.

[4] M.G. Safonov & J.C. Doyle, "Minimizing Conservativeness of Robustness Singular Values," in *Multivariable Control*, S.G. Tzafestas, ed., D. Reidel Publishing Company, 1984, 197–207.

[5] J. C. Doyle & C. C. Chu, "Robust Control of Multivariable and Large Scale Systems," Final Technical Report for AFOSR, Contract No. F49620-84-C-0088, March 1986.

[6] R. Sezginer & M. Overton, "The Largest Singular Value of $e^X A_0 e^{-X}$ is Convex on Convex Sets of Commuting Matrices," Courant Inst. of Math. Sc., New York Univ., Tech. Rep. 403, October 1988.