An Experimental versus Numerical Shape Optimization Method

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Abstract

This paper describes and compares an experimental and a numerical method for shape optimization of continuum structures. The experimental method is based on a systematic application of photoelasticity. The numerical method is based on coupling of a finite element analysis and an optimization method. The paper demonstrates the importance of designer’s interaction during the shape optimization process. The discussion is made taking shape optimization of a hole in a tall beam as an example.

1. Introduction

Shape optimal design, whether experimental or numerical, is an evolutionary process in which a designer seeks to obtain the “best” or “optimum” shape within a set of requirements. It generally comprises a series of steps in which each of the successive steps is directed at obtaining a shape better than the previous one. The process proceeds toward the optimum until either the current shape cannot be further improved (experimentally) or certain convergence criteria is satisfied (numerically). A good approach in shape optimization requires, besides the application of the principles of science and technology, a systematic interaction from the designer. This interaction, which is often based on intuition and experience, allows the designer to retain control over the design process and accomplish changes and revisions rapidly. It also facilitates verification of intermediate results, and guides the design toward a reliable solution.

This paper examines and compares several issues, including designer’s interaction, within an experimental and a numerical shape optimization process. Experimentally, optimal shape is developed by systematic application of a method using photoelasticity. Numerically, optimal shape is developed using a finite element method coupled with an optimization technique. The discussion is made with reference to shape optimal design of an interior discontinuity of an in-plane loaded tall beam (Fig. 1). The design objective is to remove as much material from inside the beam as possible and is subjected to constraints on the maximum stress in the beam.

2. The Tall Beam

Numerous contributions can be found dealing with the analysis of tall beams subjected to in-plane loading. Conway et al. (1951) have proposed the superposition of two stress functions and have mentioned several previous contributions. Attention is also called to a discussion of this paper (Durelli, 1951). More references related to the tall beam and in particular concrete walls can also be found (Azarm et al., 1986; Cardenas et al., 1973).
The rectangular tall beam considered here (Fig. 1), is under uniform compressive loading at the top and is uniformly supported at each bottom-end over a distance of one-tenth of the total width. The height to width ratio is 1:0.75. The objective of the design is to remove as much material as possible from inside the beam, i.e., by changing the shape of a centrally located hole. The hole boundary is load-free. The top, bottom, sides and thickness of the beam have to be left unchanged. While removing material, the maximum tensile stress in the beam should not increase beyond the maximum tensile stress present in the solid beam before the hole is made in it. Thus the final design will have reduced weight but a maximum stress no greater than in the initial beam. The maximum tensile stress criterion used here, has been found to apply with reasonable accuracy to materials which fail under brittle fracture (Juvinall, 1967).

It is assumed that not only the geometry and loading are symmetric, but that the design constraints also require a final design which is symmetric. Thus it is sufficient to analyze only the half structure. Furthermore, there are geometric constraints that limit the extent of the hole shape, e.g., the hole cannot cross the top, bottom or sides of the beam.

3. Experimental Method

3.1. Methodology and Previous Work

Details of the photoelastic technique used to minimizing stress concentration around stress raisers are discussed by Durelli et al. (1978, 1979). Examples of application in the solution of central holes in finite and infinite plates subjected to in-plane loading can be found in Durelli and Rajaiah (1979, 1980a). Application to fillets and rings can be found in Durelli and Rajaiah (1980b).

The photoelastic technique requires a transparent two-dimensional model of the structural component, made of birefringent material. The model is loaded and viewed through a large-field diffused-light polariscope which allows direct viewing of the fringe patterns. All points on the same fringe have an identical value of maximum shear stress ($\tau_{max}$). The value is given by the fringe order of the particular fringe. Along load free boundaries, the normal stress component perpendicular to the boundary (principal stress) is zero while the tangential component (again a principal stress) is twice the magnitude of $\tau_{max}$. The fringes along load free boundaries therefore, give a direct measure of the tangential stress along the load free boundary, without need for further calculation. In other words, obtaining a uniform tangential stress along the free boundary is the same as making a single photoelastic fringe coincide with the boundary.

The shape optimization process consists of the systematic removal of material from
the load-free boundary of a discontinuity (hole), to yield a uniform stress distribution along the boundary. Material is removed manually from low stress regions of the boundary while viewing the changing fringe patterns through a portable analyzer. A portable router and hand files are used for the material removal. Such material removal decreases the fringe order in high stress regions and increases it in low stress regions. Ultimately, this results in a shape having a single fringe coincident with the boundary.

If the discontinuities have stress reversals, the tensile and compressive segments of the boundary may have to be both optimized. The optimal shape, in this case, will have constant stress along the tensile segment, a steep slope at the point of zero stress, and a constant stress along the compressive segment. The degree of optimization can be evaluated by computing a coefficient of efficiency (Durelli et al., 1978)

\[ k_{eff} = \frac{1}{S_2 - S_0} \left( \frac{\int_{S_n} \sigma_t^+ ds}{\sigma_{all}} + \frac{\int_{S_1} \sigma_t^- ds}{\sigma_{all}} \right) \]

where \( \sigma_t^+ \) and \( \sigma_t^- \) are the tangential stresses along the tensile and compressive segments of the boundary, \( \sigma_{all}^+ \) and \( \sigma_{all}^- \) are the maximum allowable tensile and compressive stresses, and \( S_n \) \( (n=0,1,2) \) is the limiting point of the boundary segments subjected to tension or compression. The coefficient indicates how efficiently the material is used at the boundary for the given stress field. When stress levels are constants on both tensile and compressive segments, \( k_{eff} \) equals one. Coefficient \( k_{eff} \) also indicates how far the modified geometry is from the optimal shape.

3.2. Solution process

In all the cases discussed in the references of previous section, an interior discontinuity exists in the object and material removal starts at the zero stress point on the boundary of the discontinuity. This zero stress point is identified by the zero isochromatic (fringe having a zero fringe order). The tall beam differs from these problems in that there is no interior discontinuity to start with. Therefore, if a zero isochromatic point is present inside the field, as is the case here on the vertical axis (Fig. 2(a)), the maximum shear stress at the point is zero but the individual principal stress may be high. Making a hole at this point would increase the stress by a factor of two. Nevertheless, the optimization process is started at the location of the two zero-order isochromatics on the axis of symmetry. It so happens that the two equal stresses on the axis are relatively small and removal of material affects the stress only in a small neighborhood of the points.
Fig. 3(a) shows the fringes after the hole is machined to an approximately elliptic hole. It is seen that the level of stress about the hole boundary is quite low. The maximum tensile stress in the beam, both before and after the hole is made, occurs on the bottom boundary near the support. The value of this stress also remains the same as before. Thus the constraints of the design are satisfied. Also, the bottom part of the hole has a smaller stress than the rest of the hole boundary. This suggests that in the next step, material be removed from the bottom of the hole. The result of this operation, Fig. 3(b), produces an almost uniform stress distribution along the top of the hole but still keeps the bottom of the hole understressed. At the next step in the process, more material is removed, by enlarging the hole as shown in Fig. 3(c). This has not yet raised the stress along the bottom of the hole. In the fourth step, Fig. 3(d), the hole has a triangular shape and the stresses along the hole are practically uniform at 1) all points of the base with fringe order of 0.84, and 2) the two lateral parts with fringe order of 0.5. The hole corresponding to the fourth step, Fig. 3(d), results in about 14% saving in weight over the beam without any hole. The material removal from the hole may be continued, however, the purpose of the paper is more to emphasize the optimization process.

In summary, essentially the process may be continued to obtain a constant stress distribution along the hole boundary while keeping the maximum tensile stress within stipulated bounds. It is observed that every step of the optimization produced a better design than the previous one in terms of efficient use of material. At the same time, the maximum tensile stress has not increased and occurs on the bottom edge of the beam near the support. The stresses in the neighborhood of the support depend on friction and rigidity of the support and are not considered in the analysis. It is believed that the support reaction is localized and the shape of optimum hole is not affected by this local stress distribution. It should be mentioned that it may also be possible to have more than one discontinuity in the beam, as shown in Fig. 4, unless physical or geometric constraints in the design rule this out.

4. Numerical Method

4.1. Methodology and Previous Work

The numerical approach to shape optimization combines the finite element method with a numerical optimization technique. The basic approach can be summarized as follows. The structure is first discretized by a finite element mesh for analysis purposes with the hole boundary being modelled mathematically by a curve. The shape of the curve is governed by the values of certain variables (shape variables). Sensitivity derivatives, are calculated for each shape variable. These give quantitative information on how the
structural response is affected by a change in a shape variable, i.e., the change in stress, displacement, etc., due to a change in the shape of the hole. Subsequently the optimizer utilizes the sensitivity derivatives to change the values of the shape variables, in accordance with a design objective function and constraints. The resulting change in the boundary shape necessitates remeshing of the structure before further analysis can be performed. After remeshing, the optimizer takes over again. This iterative process is then repeated several times until a satisfactory optimum shape is obtained.

Shape optimization techniques pose special problems in practical implementation, primarily because of the continuously changing boundary shape. Some of the more important problems include the need for robust mesh regeneration techniques, accurate evaluation of sensitivity derivatives and proper selection of design variables. These and other problems are discussed by Haftka and Grandhi (1986) in an extensive review of recent literature (with 139 references). Bennett and Botkin (1985), use regional mesh refinement schemes in addition to automatic mesh regeneration, for improving the accuracy of finite element analysis. For sensitivity calculations two broad approaches are available. In one, finite difference methods are used to obtain the derivatives after finite element discretization (Haftka and Kamat, 1987). The other approach is based on differentiating the continuum equations before finite element discretization (Haug et al., 1986). There are also a number of ways of selecting the shape variables. Improper selection of variables can lead to a jagged boundary shape as shown in the next section. A robust method used by Braibant and Fleury (1985) utilizes a design element in which the boundaries are described by B-Splines and Beizer curves. Finally, it should be noted that boundary elements can be used to model the boundary instead of using finite elements. Some work in this area has been done by Mota Soares et al. (1984).

4.2. Solution process

4.2.1. Boundary Representation

The hole boundary shape is modelled by a B-spline defined by the position of eight control points (Fig. 5). Note that the control points are different from the boundary nodes in the finite element model. The position of the control points are determined by polar coordinates $(r, \theta)$. For each control point the angle $\theta$ is fixed but the radius $r$ is a variable. The shape of the hole boundary can therefore be changed by varying $r$. These then are the so-called shape variables. The shape representation used here is similar to the one followed by Fleury (1987).

An important property of the B-spline representation is that it passes through only the first and last control points. The intermediate points serve as a kind of weight for the
curve. In the early stages of the solution process, the hole boundary was represented by a series of straight lines connected at the junction points between two lines. The shape variables were the \((x, y)\) coordinates of the junction points, which were also boundary nodes in the finite element model. The straight lines would pass through each and every control node resulting in a jagged boundary shape (Fig. 6) in some stages of the design. This was due to a one to one correspondence between the finite element model and the design variables (see also, Haftka and Kamat, 1986). The choice of the design variables as the \((x, y)\) coordinates of the nodes also presented problems in maintaining mesh integrity. A domain of variation had to be imposed on each control node to prevent them from overlapping each other.

These problems are eliminated by the B-spline representation. The cubic B-spline used here is composed of piecewise third-order polynomials and prevents oscillatory boundary shapes due to its variation diminishing property. This would not have been possible had a higher-order polynomial been used for the entire hole boundary. Also, since the angle \(\theta\) is fixed for each control point, maintaining mesh integrity becomes much easier since the control points cannot overlap each other. From Fig. 5 it is evident that only an upper limit need be imposed on \(r\) to prevent the hole boundary from crossing the top, bottom or sides of the beam.

4.2.2. Finite Element Discretization

Eight-node isoparametric quadrilateral elements are used for discretizing the structure. The commercial software PATRAN 2.0, developed by PDA Engineering (1984), is used for pre-processing (mesh generation). After finite element analysis, the results (stresses, displacements, etc.) are fed back into PATRAN for postprocessing operations (stress contour plots, etc.). The stresses are calculated at 2x2 Gauss integration points and are extrapolated to the element node points (Cook, 1981).

Automatic regeneration of the finite element mesh for each new shape of the hole hinges on the way the basic beam geometry is described by PATRAN. The beam is divided into quadrangular areas called patches, each defined by four corner points called grids. The grid points are the first entities that are defined while creating the finite element model. Subsequently the basic geometry defined by the grids, is divided into nodes and elements by further PATRAN directives. All the directives used for creating the mesh are recorded on a file. The important fact that is to be noted here is that to remesh a new geometry, only the grid coordinates have to be changed in the file, to conform to the new geometry. The other directives remain the same. Automatic mesh regeneration for a new hole boundary shape is accomplished by simply reading in the new values of the grid point coordinates
and running the file again. The new coordinate values are provided by the optimizer and the file editing is done automatically. The user needs to define the basic mesh only once at the start of the program. The mesh integrity is automatically maintained except when a patch becomes too small in one direction, resulting in elements with bad aspect ratio. In that case the mesh has to be redefined by the user.

4.2.3. The Optimizer

The optimizer is a nonlinear programming algorithm for solving a problem of the form

Minimize $f(x)$

Subject to:

$$g_j(x) \leq 0 \quad j = 1, m$$

where $x$, $f$, and $g_j$ represent design variables, objective function, and constraint function, respectively. To solve the above problem, a sequential quadratic programming algorithm of the type suggested by Powell (1977) was used. In the case of the tall beam, the design variables are the position of the control nodes (radius $r$). The objective function may be selected, for example, to minimize the area (or weight) of the beam. This can be calculated as the sum of all the areas of finite elements. The objective may also be selected, as in the experimental method, to maximize the coefficient of efficiency, $k_{eff}$. The design constraint is that the maximum tensile stress in the beam at any point should not at any stage exceed the maximum tensile stress in the original beam without the hole. The stresses used here are the ones calculated at the 2x2 Gauss integration points since they are the most accurate stresses within the element. Thus, there are four constraints in each element, one at each Gauss point. The inequality constraints can now be written as

$$g_j(X) = \sigma_j - \sigma_{max} \leq 0 \quad j = 1, m$$

where $\sigma_j$ is the tensile stress at each Gauss integration point $j$, and $\sigma_{max}$ is the maximum tensile stress in the initial condition of the beam. The tensile stress used here is the principal stress.

The entire design process can be represented in flow-chart form as shown in Fig. 7. Sensitivity derivatives are calculated using a finite difference method (Ricketts and Zienkiewicz, 1984).

4.3. Results

Fig. 2(b) shows the contour plot of the maximum principal stress ($\tau_{max}$) in the solid tall beam. The contours match rather closely the photoelastic fringes in Fig. 2(a) (note
that, experimentally, $\tau_{max} = 0.376N$, where $N$ is the fringe-order given on Fig. 2(a), the corresponding $\tau_{max}$ values for the numerical method is given on the right-hand side of Fig. 2(b)). The stress contours does not match in the neighborhood of the support. This is because the reaction forces occurring at the support in the photoelastic experimental set-up cannot be modelled exactly in the numerical approach. However as this only affects the stress field locally, we can go ahead with the numerical optimization. The numerical optimization process, for minimum weight design objective, started with an initial hole shape, Fig. 8, where some intermediate iterations are also shown. The final hole shape (also shown in Fig. 9(a)) represents a saving in weight of about 38% over the beam without any hole. The final hole shape for a rather high degree of optimization ($k_{eff}$) is shown in Fig. 9(b). Note that the hole shape for minimum weight design has stress concentration on the boundary (Fig. 9(a)). This stress concentration has been eliminated for a hole shape design having a high degree of optimization (Fig. 9(b)).

5. Manual Interaction

One of the main purposes of this paper is to describe various means of "interaction" which could facilitate obtaining the experimental or numerical results. Evidently, the experimental method relies exclusively on the interaction, in that the operator changes the shape of the hole with a hand file, as he or she watches the corresponding transformation of the photoelastic stress patterns through the analyzer. The observation of the patterns permits the experienced operator to properly remove material at each step until the stress distribution is the desired one.

Numerically, interaction can facilitate the post-processing stage of analysis (Fig. 7). The designer can either use interaction to supply data to the optimization process or can allow the process to continue automatically without interfering. Interaction therefore consists of two separate parts. The first is a visual representation of the results of the optimization which enables the designer to interpret results in physical terms. It also helps the designer to verify whether the optimization process is proceeding in the right direction. The interactive module (second part) then allows the designer to act upon the model based on conclusions drawn from the first part.

The visual representation of results may be in the form of contour plot of stresses, intermediate hole shapes for comparison purposes. Numerically, it may also include iteration history (plot of objective function values versus iteration number), and recalling and simultaneous plots of previous hole shapes.

Numerically, changing the values of shape variables gives the designer direct control over the shape of the hole boundary. This can be done for recovering from errors (e.g., when
the hole boundary takes on physically impossible shapes) or to explore several different paths taken by the optimization process starting from an intermediate hole boundary shape. In both cases the designer essentially inputs a new shape for the hole and then allows the optimization process to take over again. In case of a fatal error, as when the hole boundary crosses the sides of the beam, the finite element analysis will blow up. In such cases the interaction capability can be used to continue the process from a previous iteration after modifying some parameters. If it were not for this, all information would be lost and the the optimization process would have to be started from the very beginning.

Fig. 10, shows the $\tau_{\text{max}}$ contours obtained numerically using manual interaction, and it corresponds to the hole shapes obtained in the experimental approach (Fig. 3). There is a rather close correlation between the stress patterns in the experimental and numerical cases which indicates that the type of finite element discretization used was adequate for the problem. It is also seen that, numerically, the maximum tensile stress does indeed occur on the bottom boundary of the beam, as obtained experimentally.

6. Conclusions

This paper has demonstrated several issues which might be of general interest:

1) Experimentally, the optimization process started at the location of the two isotropic points (zero-order fringe) on the axis of the beam. Then, every step of the process produced a better design than the original one (and a better design than the one developed in the previous step). Numerically, the optimization process started from an arbitrary initial hole shape. As it is evident in Fig. 8, not every step of numerical optimization produced a design better than the previous one. This can be resolved either by using a numerical optimization method which has both descent and feasible direction properties, or as demonstrated in Fig. 10, using manual interaction.

2) The concept of "degree of optimization", defined in the experimental method, corresponds to the boundary that has all or close to all points subjected to the same stress (zero gradient along the boundary). This concept could be applied to every step of the perforated tall beam, as was demonstrated experimentally in Fig. 3 and numerically using the manual interaction in Fig. 10. But there is no guarantee that any of the perforated shapes so obtained, is the minimum weight under all conditions.

3) Experimentally, the ease of application of the photoelastic method is based on the fact that $\tau_{\text{max}}$ fringes give a direct measure of the tangential stresses on free boundaries, and on the assumption that the maximum stress occurs on a load free boundary (i.e, on the bottom free boundary or on the hole boundary). However, if the constraints
require complete knowledge of stresses in the interior of the beam, then extensive calculations may be required. In such cases it may not be practical to apply the photoelastic method.

4) Numerically, the accuracy of the finite element approach depends on the fineness of the finite element mesh and on the type of element used. As the number of elements are increased, the number of constraints are also increased and both the finite element analysis and the optimization become costlier. The same problem results, if the number of shape variables is increased in an attempt to improve the accuracy of the boundary representation.

5) Exploring multiple discontinuities (Fig. 4) is easier with the experimental method. Maintaining mesh integrity in the general case makes this more difficult if the numerical approach is used. A question could be raised here if the final design obtained is the one giving the global optimum, i.e., whether the same design would result if material removal is started from a different point. The answer is that there is no guarantee that the design obtained following the experimental or numerical method is the true global optimum. There could be an alternative optimum design which has a different shape or even multiple discontinuities.

7. Acknowledgement

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\[ \beta = \frac{H}{W} = 0.75 \]

\[ \frac{\tau_{\text{max}}}{p} = \text{KN} \]

\[ t = 0.25 \text{ in} \]
\[ H = 3.0 \text{ in} \]
\[ W = 4.0 \text{ in} \]
\[ A = 1.0 \text{ in}^2 \]

\( \text{Fig. 1} \)