Regular and Singular Perturbations in the Filtering of a Markov Chain

by

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Abstract

We obtain an asymptotic expansion of the conditional distribution for a homogeneous Markov chain in the filtering problem. In a first case, we consider a regular perturbation situation, and in a second case, a singular perturbation situation, according to the time scale of the observed process.

In all cases, we obtain an asymptotic expansion which the terms are calculated easier by decentralization and aggregation.

Key-words


1 Introduction

We consider the homogeneous Markov chain \((z_i^t; t \geq 0)\) with values in a finite state space \(E\).

Let \(B + \varepsilon A\) be its generator, where \(B\) and \(A\) are two \(n \times n\) matrices which themselves generate Markov chains, and \(0 < \varepsilon < 1\).

The matrix \(B\) is assumed block-diagonal; every block represents the strong interactions between states of the same group; and the matrix \(\varepsilon A\) represents the weak interactions or the slow transitions between the states belonging to different groups.

In section 3.1, we consider the Fokker-Planck equation

\[
\begin{align*}
\frac{dp_t^i}{dt} &= p_t^i(B + \varepsilon A)dt \\
p_t^i(0) &= p
\end{align*}
\]

(1)

where \(H\) is the observation matrix.

\(p_t^i\) is the \(n\)-dimensional probability row vector with components \(p_t^i(i)\), where \(p_t^i(i)\) is the probability of being in state \(i\) at time \(t\). The main result in this section will be to write \(p_t^i\) as a power series in \(\varepsilon^k\).

In section 3.2, we consider the filtering problem whose solution satisfies the Zakai equation.
Let \((y_t; t \geq 0)\) be the observation process modelled by

\[
dy_t = h(x_t^\varepsilon) dt + dw_t^\varepsilon
\]

\[
y_0 = 0
\]

then the Zakai equation is

\[
dp_t^\varepsilon = p_t^\varepsilon(B + \varepsilon A)dt + p_t^\varepsilon Hd\tilde{y}_t
\]

\[
p_0^\varepsilon = p
\]

where \(p_t^\varepsilon\) is now the unnormalized conditional probability distribution of \(x_t^\varepsilon\) given \(\mathcal{F}_t^\varepsilon = \sigma(y_s; 0 \leq s \leq t)\).

We shall write \(p_t^\varepsilon\) as a power series in \(\varepsilon^k\).

In section 3, we shall consider the regular perturbation case, where the limit equations (with \(\varepsilon = 0\)) are of the same nature as the perturbed equations (1) and (2).

We remark that the influence of the weak interactions, \(\varepsilon A\) in equations (1) and (2) will become significant after a long period of time \(t\).

Hence the \(t\) time scale is called "fast time". To see the influence of \(\varepsilon A\) "sooner", we introduce the "slow time" \(\tau = \varepsilon t\).

In the \(\tau\)-scale, the perturbations of equations (1) and (2) become singular, i.e. the limit equations are of a different nature (algebraic) from the perturbed equations:

In the \(\tau\)-scale, equation (1) is written

\[
dp_\tau^\varepsilon = p_\tau^\varepsilon(B_t^\varepsilon + A)d\tau
\]

\[
p_0^\varepsilon = p
\]

and, in section 4.1, we shall write an asymptotic expansion of this solution using the boundary layer method.

Finally, in section 4.2, we consider the filtering problem in the slow time scale, where the observation process is given by

\[
dy_\tau = h(x_\tau^\varepsilon)d\tau + dw_\tau^\varepsilon
\]

\[
y_0 = 0
\]

and the Zakai equation becomes
\[ dp_\gamma^* = p_\gamma^*(\frac{\partial}{\partial \gamma} + A) d\gamma + p_\gamma^* H dy, \]

\[ p_0^* = p. \]

We shall show that \( p^* \) admit an uniform asymptotic expansion.

In all cases, we shall reduce the dimension of the systems which define the terms of the expansions.

2 Preliminaries

2.1 Spectral projection and potential matrix

We consider an \( n \times n \) matrix \( B \), with eigenvalues \( (\lambda_k) \). Assume that \( \lambda_0 = 0 \) is an eigenvalue.

We define the operators

\[ R_\lambda = (\lambda I - B)^{-1} \quad \text{for } \lambda \neq \lambda_k. \]

\( R_\lambda \) is called the resolvent of \( B \) (see [4] and [6]).

We define the spectral projection \( P_{\lambda_k} \) and the operator \( H_{\lambda_k} \), associated with the eigenvalue \( \lambda_k \), by

\[ P_{\lambda_k} = \frac{1}{2\pi i} \int_{C_k} R_\lambda d\lambda \]

\[ H_{\lambda_k} = \frac{1}{2\pi i} \int_{C_k} (\lambda - \lambda_k)^{-1} R_\lambda d\lambda \]

where \( C_k \) is a positively oriented contour enclosing \( \lambda_k \), but no other eigenvalue of \( B \).

We have the following results for the eigenvalue \( \lambda_0 = 0 \)

\[ P_{\lambda_0}^2 = P_{\lambda_0} \quad P_{\lambda_0} B = BP_{\lambda_0} = 0 \]

\[ P_{\lambda_0} H_{\lambda_0} = H_{\lambda_0} P_{\lambda_0} = 0 \quad BH_{\lambda_0} = H_{\lambda_0} B = P_{\lambda_0} - I. \]

In fact, \( P_{\lambda_0} \) is the eigenprojection onto \( \ker(B) \) along \( R(B) \). \( H_{\lambda_0} \) is the inverse of \( B \) in the subspace \( R(P_{\lambda_0} - I) \).

\( H_{\lambda_0} \) is called the potential matrix, and we have

\[ H_{\lambda_0} = (P_{\lambda_0} - B)^{-1} - P_{\lambda_0}. \]

In the sequel, we shall write \( H_B \) and \( P_B \) instead of \( H_{\lambda_0} \) and \( P_{\lambda_0} \).
2.2 Canonical decomposition

We consider a homogeneous Markov chain \((x_t; t \geq 0)\) with values in a finite state space \(E\), with generator \(B\) where 0 is an eigenvalue of \(B\) of multiplicity \(r\). Here, \(r\) is the number of recurrent classes (see [2] and [4]).

Let \(\bar{E}\) be the set of these \(r\) classes with elements denoted \(\bar{x}\).

We denote by \(q_\bar{x}(y)\) the probability of ending in the class \(\bar{x}\) starting from state \(y\). Let \(T_\bar{x} = \inf\{t \geq 0; x_t \in \bar{x}\}\) then

\[
q_\bar{x}(y) = P_y(T_\bar{x} < \infty).
\]

The \(r\) functions (column vectors) \(q_\bar{x}(.), \bar{x} \in \bar{E}\), are solutions of the equation

\[
Bq_\bar{x} = 0
\]

and form a basis of the \(r\)-dimensional subspace \(\ker B = R(P_B)\).

To each \(\bar{x} \in \bar{E}\) is also associated the invariant measure (row vector) \(m_\bar{x}\) of the recurrent subchain defined on the class \(\bar{x}\), by

\[
m_\bar{x}B = 0 \quad \text{and} \quad \sum_{y \in \bar{x}} m_\bar{x}(y) = (m_\bar{x}, 1) = 1.
\]

These \(r\) measures \(m_\bar{x}\) form a basis of \(\ker B^*\), dual to \(\{q_\bar{x}, \bar{x} \in \bar{E}\}\), such that

\[
(m_\bar{x}, q_\bar{x}) = 1 \quad \text{and} \quad (m_\bar{x}, q_{\bar{x'}}) = 0 \quad \text{if} \ \bar{x} \neq \bar{x'}.
\]

If \(Q\) is the \(n \times r\) matrix defined by \(Q(y, \bar{x}) = q_{\bar{x}}(y)\), and \(M\) is the \(r \times n\) matrix defined by \(M(\bar{x}, y) = m_\bar{x}(y)\), then

\[
BQ = 0 \quad MB = 0 \quad \text{and} \quad MQ = I.
\]

Finally the operator \(P_B\), spectral projection on \(\ker B = R(P_B)\) along \(R(B) = R(I - P_B)\), admits the canonical decomposition

\[
P_B = QM.
\]

We can deduce

\[
P_BQ = Q \quad MP_B = M.
\]
Using the Gershgorin theorem, we can show

**Proposition 2.1**

The nonzero eigenvalues of $B$ have strictly negative real parts.

**Proposition 2.2**

The spectral projection $P_B$ associated with the eigenvalue 0 of the matrix $B$ satisfies

$$\lim_{t \to \infty} e^{Bt} = P_B$$

(for the proof, see [1]).

## 2.3 Aggregate chain

Let $A$ and $B$ be two generators of homogeneous Markov chains with values in a finite state space $E$ (see [4] and [5]).

Let $P_A$ and $P_B$ be the two spectral projections associated with the eigenvalue 0 respectively for the matrices $A$ and $B$.

We decompose the operator $P_B$ as $P_B = QM$.

**Theorem 2.3**

The restriction of the operator $P_B A P_B$ to the subspace $R(P_B)$ is represented in the basis $\{q_{\tilde{z}}; \tilde{z} \in \tilde{E}\}$ by the matrix

$$A^a(\tilde{z}, \tilde{z}') = m_{\tilde{z}} A q_{\tilde{z}} = (MAQ)(\tilde{z}, \tilde{z}')$$

which is the generator of a Markov chain on $\tilde{E}$.

(for the proof, see [4])

## 3 Regular perturbation

We consider the homogeneous Markov chain $(x_t^\epsilon; t \geq 0)$, with values in a finite state space $E$.

Let $B + \epsilon A$ be its generator, where $B$ and $A$ are two $n \times n$ matrices, which themselves generate Markov chains, and $0 < \epsilon < 1$. 

To motivate this decomposition $B + \varepsilon A$, we must consider the following proposition (see [3])

**Proposition 3.1**

Let $G$ be the generator of a Markov chain $(x_t; t \geq 0)$ in discrete time, then there exists $0 \leq \varepsilon \leq 1$ such that $G = B + \varepsilon A$, where $B$ and $A$ are two $n \times n$ matrices, which generate two Markov chains and $B$ is block-diagonal.

**Proof**

If $N$ is the number of blocks in the matrix $B$, then we have $n = \sum_{I=1}^{N} n(I)$, where $n(I)$ is the dimension of the $I$th block of $B$. Since $A$ is a generator matrix, for each row $i_f$,

$$\sum_{k=1}^{n(I)} a_{i_f k} = - \sum_{J=1, J \neq I}^{N} \sum_{j=1}^{n(J)} a_{i_f j}.$$ 

Moreover, we choose $\varepsilon$ and $A$ such that for all rows $i_f$,

$$\varepsilon \sum_{J=1, J \neq I}^{N} \sum_{j=1}^{n(J)} a_{i_f j} = \sum_{J=1, J \neq I}^{N} \sum_{j=1}^{n(J)} g_{i_f j},$$

and

$$\varepsilon = \max_{i_f} \left( \sum_{J=1, J \neq I}^{N} \sum_{j=1}^{n(J)} g_{i_f j} \right).$$

**Example**

Let $G$ be the $4 \times 4$ matrix defined by

$$
\begin{pmatrix}
-0.1 & 0.01 & 0.09 & 0 \\
0.1 & -0.8 & 0.7 & 0 \\
0.05 & 0.05 & -0.1 & 0 \\
0.1 & 0 & 0 & -0.1 \\
\end{pmatrix}.
$$

Then we have, for $n(1) = 1$, $n(2) = 2$ and $n(3) = 1$,

$$B = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & -0.7 & 0.7 & 0 \\
0 & 0.05 & -0.05 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}, \quad A = \begin{pmatrix}
-1 & 0.1 & 0.9 & 0 \\
1 & -1 & 0 & 0 \\
0.5 & 0 & -0.5 & 0 \\
1 & 0 & 0 & -1 \\
\end{pmatrix},$$

and $\varepsilon = 0.1$. 

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3.1 Regular perturbation of the Fokker-Planck equation

We consider the homogeneous Markov chain \( x^0_t; \ t \geq 0 \), with values in a finite state space \( E \).

Let \( B \) be the generator of this chain; it is an \( n \times n \) matrix, block-diagonal, \( (N \) blocks of dimension \( n(I) \) such that \( \sum_{i=1}^{N} n(I) = n \) \) not irreducible \( (r \) ergodic classes, \( N \leq r < n \)).

The process \( x^0_t \) satisfies the transposed Fokker-Planck equation

\[
dp_0 = p_0 B dt; \quad p_0(0) = p,
\]

where \( p_0(t) \) is the \( n \)-dimensional row of probabilities \( P(x^0_t = i) \).

We perturb the chain \( x^0_t \) as follows; the perturbed generator is \( B + \varepsilon A \), where \( A \) is an \( n \times n \) matrix, and \( 0 < \varepsilon < 1 \).

The matrix \( \varepsilon A \) represents the transitions between the different blocks of \( B \). Let \( x^\varepsilon_t \) be the perturbed Markov chain, with Fokker-Planck equation

\[
dp^\varepsilon = p^\varepsilon (B + \varepsilon A) dt; \quad p^\varepsilon(0) = p.
\]

We shall show, in the following sections, that this vector admits a series expansion,

\[
p^\varepsilon(t) = \sum_{k=0}^{\infty} \varepsilon^k p_k(t),
\]

for every \( t \in [0, T] \). Also, every \( p_k \) is calculated by solving \( N \) systems, the \( I \)th system is of dimension \( n(I) \) respectively, plus one aggregated system of dimension \( r \).

3.1.1 Asymptotic expansion

Consider the equation (1), and assume \( p^\varepsilon(t) \) admits the expansion

\[
p^\varepsilon(t) = \sum_{k=0}^{\infty} \varepsilon^k p_k(t).
\]

We get formally

\[
(\varepsilon^0) \quad \hat{p}_0 = p_0 B; \quad p_0(0) = p,
\]

and for \( k \geq 1 \), we obtain

\[
(\varepsilon^k) \quad \hat{p}_k = p_k B + p_{k-1} A; \quad p_k(0) = 0.
\]
3.1.2 Convergence

Using the Gronwall-Bellman inequality, we can show

**Lemma 3.2**

For any $k \geq 1$ and any $t \in [0, T],$

$$|p_k(t)| \leq \frac{(Ce)^k}{k!},$$

where $C$ is a constant which depends on the norms of $A$ and $B.$

By the lemma 3.2, we deduce

**Proposition 3.3**

For any $0 < \varepsilon < 1$ and any $t \in [0, T],$

$$p^\varepsilon(t) = \sum_{k=0}^{\infty} \varepsilon^k p_k(t).$$

3.1.3 Decentralization and aggregation

We consider the equations

$$\dot{p}_0 = p_0B; \quad p_0(0) = p,$$

and, for $k \geq 1,$

$$\dot{p}_k = p_kB + p_{k-1}A; \quad p_k(0) = 0.$$

Let $P_B$ be the eigenprojection for the eigenvalue 0 of the matrix $B,$ such that

$$\ker B = R(P_B) \text{ and } P_B B = B P_B = 0.$$}

Therefore, all vector $q$ can be decomposed as

$$q = \tilde{q} + \bar{q} \text{ with } \tilde{q}B = 0 \text{ and } \tilde{q}P_B = \bar{q}.$$}

Now, the equations are

$$\dot{p}_0 = \tilde{p}_0B; \quad p_0(0) = p,$$

and, for $k \geq 1,$

$$\dot{p}_k = \tilde{p}_kB + \tilde{p}_{k-1}A; \quad p_k(0) = 0.$$
We apply the operator $P_B$ and then get
\[
\dot{p}_0 = 0; \quad p_0(0) = p,
\]
and, for $k \geq 1$,
\[
\dot{p}_k = p_{k-1}AP_B; \quad p_k(0) = 0.
\]
From the previous equations, we get
\[
\dot{p}_0 = \dot{p}_0 B; \quad \dot{p}_0(0) = \dot{p},
\]
and, for $k \geq 1$,
\[
\dot{p}_k = \dot{p}_k B + p_{k-1}A(I - P_B); \quad \dot{p}_k(0) = 0.
\]

Remark
As the matrices $B$ and $P_B$ have the same block-diagonal structure, we can solve the equations on every block of $B$; we decentralize these equations on every block of $B$.

We now consider the equation
\[
\dot{p}_k = p_{k-1}AP_B, \quad \dot{p}_k(0) = 0.
\]
We can write
\[
\dot{p}_k = (\dot{p}_{k-1} + \dot{p}_{k-1})AP_B,
\]
or, since all vector $\bar{q}$ are invariant under $P_B$,
\[
\dot{p}_k P_B = \bar{p}_{k-1}P_BAP_B + \bar{p}_{k-1}AP_B.
\]
Since $P_B$ admits the canonical decomposition $QM$, we can write
\[
\dot{p}_k QM = \bar{p}_{k-1}QMAM + \bar{p}_{k-1}AQM.
\]
Applying the operator $Q$ and using $MQ = I$, we obtain
\[
\dot{p}_k Q = \bar{p}_{k-1}Q(MAQ) + \bar{p}_{k-1}AQ.
\]
Remark

In the previous equation, note that $MAQ$ is the aggregate matrix of the Markov chain with generator $A$.

The aggregate matrix $MAQ$ is defined by

$$A^a(\bar{x}, \bar{x}') = (MAQ)(\bar{x}, \bar{x}') = m_s A q_{\bar{x}'}$$

where $\bar{x}$ is an ergodic class of the nonperturbed Markov chain $x^0_t$.

If $q^a = qQ$ is the aggregate vector, the aggregate equations are defined by

$$\dot{p}^a_0 = 0; \quad p^a_0(0) = p^a Q,$$

and, for $k \geq 1$,

$$\dot{p}^a_k = p^a_{k-1} A^a + p_{k-1} A Q; \quad p^a_k(0) = 0.$$

Remark

We solve the previous equations in $r$-dimensional space, where $r$ is the number of the ergodic classes of $B$.

3.2 Regular perturbation of the Zakai equation

We consider the Markov chain $(x^r_t; t \geq 0)$ as in section 3.1.

Let $(y_t; t \geq 0)$ be the observation process of the signal $(x^r_t; t \geq 0)$, defined by

$$dy_t = h^r(x^r_t) dt + dw^r_t; \quad y_0 = 0,$$

where the observation function $h^r$ is a mapping from $E$ into $\mathbb{R}^m$.

Let $p^r_t$ be the unnormalized conditional probability distribution of $x^r_t$ given $\mathcal{F}_t^r = \sigma(y_s; 0 \leq s \leq t)$, satisfying the Zakai equation

$$dp^r_t = p^r_t (B + \varepsilon A) dt + p^r_t H^r dy_t; \quad p^r(0) = 0,$$

where $H^r$ is the observation diagonal matrix.
We study two cases according to the form of the observation matrix $H^\varepsilon$:

— in the first case, $H^\varepsilon$ is assumed independent of $\varepsilon$; then we show that $p^\varepsilon(t)$ admits an expansion

$$p^\varepsilon(t) = \sum_{k=0}^{\infty} \varepsilon^k p_k(t)$$

— in the second case, $H^\varepsilon$ is written $H + \varepsilon F$; we show that $p^\varepsilon(t)$ admits an expansion as above.

In all cases, if $H$ is constant on every block of $B$, we can aggregate and decentralize the equations for $p_k$.

### 3.2.1 $H$ is independant of $\varepsilon$

Let $H$ be the diagonal matrix with diagonal $(H_1, H_2, \ldots, H_m)$, where $H_j$ is a diagonal matrix with diagonal $(h_j(x_1), \ldots, h_j(x_n))$.

- asymptotic expansion

  Consider equation (2). Write $p^\varepsilon(t)$ as

  $$p^\varepsilon(t) = \sum_{k=0}^{\infty} \varepsilon^k p_k(t).$$

  We obtain formally

  $$dp_0 = p_0 B dt + p_0 \sum_{j=1}^{m} H_j dy_j; \quad p_0(0) = p,$$

  and, for $k \geq 1$,

  $$dp_k = p_k B dt + p_{k-1} A dt + p_k \sum_{j=1}^{m} H_j dy_j; \quad p_k(0) = 0.$$

- convergence

  We can show the following lemmas (see [7])

**Lemma 3.4**

Let $P^\nu$ be a reference probability, where $(y_t; t \geq 0)$ is a standard Wiener process; then for any $k \geq 1$, we have

$$E^\nu \left( \sup_{s \leq t} |p_k(t)|^2 \right) \leq \frac{(Ct)^k}{k!},$$

where $C$ is a constant which depends on the norms of $A, B$ and $H_j$.  

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Lemma 3.5

For any \( t \in [0, T] \),
\[
\lim_{k \to +\infty} E^{\nu} \left( \sup_{s \leq t} |p_k(s)|^2 \right) = 0.
\]

From these two lemmas, we deduce the following proposition

Proposition 3.6

The series \( \sum_{k=0}^{\infty} \epsilon^k p_k(t) \) converges almost surely and uniformly in \( t \) on \( [0, T] \) to \( p^*(t) \).

- decentralization and aggregation

Assume here that \( H_j \) is constant on every block of \( B \); then \( H_j P_B = P_B H_j \).

We can show that the decentralized system is written
\[
d\tilde{p}_0 = \tilde{p}_0 B dt + \tilde{p}_0 \sum_{j=1}^{m} H_j dy_j; \quad \tilde{p}_0(0) = \tilde{p},
\]
and, for \( k \geq 1 \),
\[
d\tilde{p}_k = \tilde{p}_k B dt + p_{k-1} A(I - P_B) dt + \tilde{p}_k \sum_{j=1}^{m} H_j dy_j; \quad \tilde{p}_k(0) = 0.
\]

The aggregate system is written
\[
dp_0^a = p_0^a \sum_{j=1}^{m} H_j^a dy_j; \quad p_0^a(0) = p^a,
\]
and, for \( k \geq 1 \),
\[
dp_k^a = p_{k-1}^a A^a dt + p_{k-1} A P_B dt + p_k^a \sum_{j=1}^{m} H_j^a dy_j; \quad p_k^a(0) = 0.
\]

3.2.2 The observation matrix is \( H + \epsilon F \)

We assume that \( (y_t; t \geq 0) \) is a one dimensional process.

- asymptotic expansion

We consider the same equation and write \( p^*(t) \) as
\[
p^*(t) = \sum_{k=0}^{\infty} \epsilon^k p_k(t).
\]

We obtain formally
\[
dp_0 = p_0 B dt + p_0 H dy; \quad p_0(0) = p,
\]

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and, for \( k \geq 1 \),

\[
dp_k = p_k Bdt + p_{k-1} A dt + p_k Hdy + p_{k-1} Fdy; \quad p_k(0) = 0.
\]

- convergence

We can show the following lemma

**Lemma 3.7**

Let \( P^u \) be a reference probability where \( (y_i; t \geq 0) \) is a standard Wiener process, then for any \( k \geq 1 \), we have

\[
E^u \left( \sup_{s \leq t} |p_k(t)|^2 \right) \leq \frac{(Ct)^k}{k!},
\]

where \( C \) is a constant which depends on the norms of \( A, B, H, \) and \( F \).

With this lemma, we can prove

**Proposition 3.8**

The series \( \sum_{k=0}^{\infty} e^k p_k(t) \) converges almost surely and uniformly in \( t \) on \( [0, T] \) to \( p^*(t) \).

- decentralization and aggregation

We assume that the matrix \( H \) is constant on every block of \( B \); then \( H \) and \( P_B \) commute.

Also, we obtain the decentralized system defined by

\[
d\tilde{p}_0 = \tilde{p}_0 Bdt + \tilde{p}_0 Hdy; \quad \tilde{p}_0(0) = \tilde{p},
\]

and, for \( k \geq 1 \),

\[
d\tilde{p}_k = \tilde{p}_k Bdt + p_{k-1} A(I - P_B)dt + \tilde{p}_k Hdy + p_{k-1} F(I - P_B)dy; \quad \tilde{p}_k(0) = 0,
\]

and, the aggregate system is

\[
dp_0^a = p_0^a H^ady; \quad p_0^a(0) = p^a,
\]

and, for \( k \geq 1 \),

\[
dp_k^a = p_{k-1}^a A^ady + \tilde{p}_{k-1} AQdt + p_k^a H^ady + p_{k-1} F^ady + \tilde{p}_{k-1} F^Qdy; \quad p_k^a(0) = 0.
\]
4 Singular perturbation

We consider the homogeneous Markov chain \( (x_t^x, t \geq 0) \), with values in a finite state space \( E \).
Let \( \frac{B}{\varepsilon} + A \) be its generator.

4.1 Singular perturbation of the Fokker-Planck equation

If \( p_t^x \) is the \( n \)-dimensional row of probabilities \( p_t^x(i) \) to be in state \( i \) at time \( t \), then \( p_t^x \) satisfies the tranposed Fokker-Planck equation

\[
\frac{dp_t^x}{dt} = p_t^x \left( \frac{B}{\varepsilon} + A \right) dt; \quad p^x(0) = p.
\]

In section 4.1.1, we study the limit equation (for \( \varepsilon = 0 \)).
In sections 4.1.2 and 4.1.3, we show that \( p^x(t) \) admits a uniform expansion on \([0, T]\)

\[
p^x(t) = p_0(t) + P_0 \left( \frac{t}{\varepsilon} \right) + \varepsilon \left( p_1(t) + P_1 \left( \frac{t}{\varepsilon} \right) \right) + \ldots + \varepsilon^m \left( p_m(t) + P_m \left( \frac{t}{\varepsilon} \right) \right) + O(\varepsilon^{m+1})
\]

Finally, in section 4.1.4, by decentralization and aggregation, we show that the dimension of the equations for \( p_k \) and \( P_k \) may be reduced.

4.1.1 The limit problem

We consider the equation

\[
\varepsilon \dot{p}^x = p^x(B + \varepsilon A); \quad p^x(0) = p.
\]

Setting \( \varepsilon = 0 \),

\[0 = p_0 B.\]

And, we have the following theorem

**Theorem 4.1**

If \( P_B \) is the eigenprojection of the eigenvalue 0 of the matrix \( B \), then, for any \( t \in ]0, T[ \)

\[
\lim_{\varepsilon \to 0} \varepsilon^t (\frac{B}{\varepsilon} + A) = e^{tP_B} P_B = P_B e^{tAP_B}
\]

(for the proof, see [6]).
Therefore \( p_0(t) \) belongs to \( \ker B \), and \( p_0(t) \) is the \( n \)-dimensional row vector, probability law of the Markov chain \( x_t^0 \) with generator \( PB \).

**Remark**

The initial condition \( p_0(0) \) belongs to \( \ker B \); the initial condition \( p^\varepsilon(0) = p \) is arbitrary, and we must introduce in the asymptotic expansion the correction terms \( P_k(\frac{t}{\varepsilon}) \), called boundary layer terms at \( t = 0 \).

### 4.1.2 Asymptotic expansion

We consider the equation and we write \( p^\varepsilon \) in the form:

\[
p^\varepsilon(t) = p_0(t) + P_0(\frac{t}{\varepsilon}) + \varepsilon(p_1(t) + P_1(\frac{t}{\varepsilon})) + \ldots
\]

we obtain formally

\[
(\varepsilon^{-1}) \quad \dot{P}_0(\frac{t}{\varepsilon}) = (p_0(t) + P_0(\frac{t}{\varepsilon}))B
\]

\[
(\varepsilon^0) \quad \dot{p}_0(t) + \dot{P}_1(\frac{t}{\varepsilon}) = (p_0(t) + P_0(\frac{t}{\varepsilon}))A + (p_1(t) + P_1(\frac{t}{\varepsilon}))B
\]

and, for \( k \geq 1 \),

\[
(\varepsilon^k) \quad \dot{p}_k(t) + \dot{P}_{k+1}(\frac{t}{\varepsilon}) = (p_k(t) + P_k(\frac{t}{\varepsilon}))A + (p_{k+1}(t) + P_{k+1}(\frac{t}{\varepsilon}))B
\]

**Remarks**

- we shall denote \( p_k \) instead of \( p_k(t) \), \( P_k \) for \( P_k(\frac{t}{\varepsilon}) \), in the same way as the derivatives.
- we put \( \tau = \frac{t}{\varepsilon} \).

Now, we assume that, for \( k \geq 0 \),

\[
(H_k) \quad P_k(\infty) = 0 \quad \text{and} \quad \dot{P}_k(\infty) = 0.
\]

**Remark**

The influence of the correction terms \( P_k(\tau) \) vanishes when \( \tau \) goes to \( +\infty \).

With the assumptions \((H_k)\), we can get the systems of equations

\[
(S_{-1}) \quad 0 = p_0B
\]
\[ p_0(0) + P_0(0) = p \]

\((S_0)\)
\[
\dot{p}_0 = p_0A + p_1B \\
\dot{P}_0 = P_0B
\]

and, for \( k \geq 1, \)

\[ p_k(0) + P_k(0) = 0 \]

\((S_k)\)
\[
\dot{p}_k = p_kA + p_{k+1}B \\
\dot{P}_k = P_kB + P_{k-1}A
\]

Using the decomposition on \( \ker B \) and \( R(B) \),

\[ q = \bar{q} + \tilde{q} \] with \( \bar{q}B = 0 \) and \( qP_B = \bar{q}P_B = \tilde{q} \),

we get finally

\((S_{-1})\)
\[ 0 = p_0B \]

\[ p_0(0) + P_0(0) = 0 \]

\((S_0)\)
\[
\dot{p}_0 = p_0A + \tilde{p}_1B \\
\dot{P}_0 = \tilde{P}_0B
\]

and, for \( k \geq 1, \)

\[ p_k(0) + P_k(0) = 0 \]

\((S_k)\)
\[
\dot{p}_k = p_kA + \tilde{p}_{k+1}B \\
\dot{P}_k = \tilde{P}_kB + P_{k-1}A
\]

**Remark**

The vector \( \tilde{p}_k \) is obtained by using the inverse of \( B \) with respect of \( R(P_B - I) \) in the equation

\[ \dot{p}_{k-1} = p_{k-1}A + p_kB \]

of system \((S_{k-1})\).

Denoting this inverse \( H_B \), we obtain

\[ \dot{p}_{k-1}H_B = p_{k-1}AH_B + \tilde{p}_kBH_B. \]

Therefore

\[ BH_B = H_BB = P_B - I, \]
and thus the component \( \tilde{p}_k \) is such that

\[
\tilde{p}_k = p_{k-1}A H_B - \hat{p}_{k-1}H_B.
\]

The initial conditions of the vectors \( P_k \) must be chosen to satisfy the assumptions \( (H_k) \).

For this, we need the following proposition

**Proposition 4.2**

The row vector \( P_0 \) belongs to \( R(B) \).

**Proof**

The row vector \( P_0 \) satisfies the following equations

\[
p_0(0) + P_0(0) = p \\
\dot{P}_0 = P_0B
\]

Applying the operator \( P_B \), we obtain

\[
\tilde{p}_0(0) + \tilde{P}_0(0) = \tilde{p} \\
\dot{\tilde{P}}_0 = 0
\]

Yet, by the theorem 4.1, \( \tilde{p}_0(0) = \tilde{p} \), thus the vector \( \tilde{P}_0(0) \) is equal to zero; then \( \tilde{P}_0(\tau) \) is equal to zero for every \( \tau \).

Thus

\[
\dot{\tilde{P}}_0 = \tilde{P}_0B,
\]

and, for every \( \tau \),

\[
\tilde{P}_0(\tau) = \tilde{p}e^{rB}.
\]

We choose, for \( k \geq 1 \),

\[
\tilde{P}_k(0) = -\int_0^\infty P_{k-1}(s)AP_Bds.
\]

since we have

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Proposition 4.3

There exists $\lambda < 0$ such that, for $k \geq 1$ and for any positive real $\tau$,

$$|\bar{P}_k(\tau)| \leq \bar{C}_k(1 + \tau)e^{\lambda\tau}$$

$$|\bar{P}_k(\tau)| \leq \bar{C}_k(1 + \tau)e^{\lambda\tau}$$

$$|P_k(\tau)| \leq C_k(1 + \tau)e^{\lambda\tau}$$

where $\bar{C}_k, \bar{C}_k$ and $C_k$ are positive constants.

To prove this, we need the following lemma

Lemma 4.4

Let $\lambda$ be the supremum of the real parts of the nonzero eigenvalues ($\lambda_i$) of the matrix $B$. Then, for any $t$, we have

$$\|(I - P_B)e^{tB}\| \leq Ce^{\lambda t}$$

where $C$ is a positive constant, and $\lambda < 0$.

Proof of the proposition 4.3

The components of the vector $P_1$ are

$$\bar{P}_1(\tau) = \bar{P}_1(0) + \int_0^\tau \bar{P}_0(s)AP_Bds,$$

and

$$\bar{P}_1(\tau) = \left(\bar{P}_1(0) + \int_0^\tau \bar{P}_0(s)A(I - P_B)e^{-sB}ds\right)e^{\tau B}.$$

We write $\bar{P}_1(\tau)$ as

$$\bar{P}_1(\tau) = \bar{P}_1(0) - \bar{B}H_B(e^{\tau B} - I)AP_B,$$

and we then put

$$\bar{P}_1(0) = -\bar{B}H_BAP_B.$$

Thus

$$\bar{P}_1(\tau) = -\bar{B}H_Be^{\tau B}AP_B.$$

We remark that for any vector $\bar{q}$ in $R(B)$, $\bar{q} = \bar{q}(I - P_B)$, therefore, the components are

$$\bar{P}_1(\tau) = -\bar{B}H_B(I - P_B)e^{\tau B}AP_B,$$

$$\bar{P}_1(\tau) = \bar{P}_1(0)(I - P_B)e^{\tau B} + \bar{B}B \int_0^\tau (I - P_B)e^{sB}A(I - P_B)e^{(\tau - s)B}ds.$$
We deduce from lemma 4.4, the following bounds

$$|\tilde{P}_1(\tau)| \leq \tilde{C}_1 e^{\lambda \tau},$$

and

$$|\tilde{P}_1(\tau)| \leq \tilde{C}_1 (1 + \tau) e^{\lambda \tau}.$$ 

Now, we suppose that the result is true until the $k$-th rank, then, as

$$P_{k+1}(\tau) = \left( P_{k+1}(0) + \int_0^\tau P_k(s) e^{-sB} ds \right) e^{\tau B},$$

we get, using the decomposition,

$$\tilde{P}_{k+1}(\tau) = \tilde{P}_{k+1}(0) + \int_0^\tau P_k(s) A P_B ds,$$

and

$$\tilde{P}_{k+1}(\tau) = \left( \tilde{P}_{k+1}(0) + \int_0^\tau P_k(s) A (I - P_B) e^{-sB} ds \right) e^{\tau B}.$$ 

We note that

$$|\int_0^\tau P_k(s) ds| \leq \int_0^\tau |P_k(s)| ds,$$

$$\leq C \int_0^\tau (1 + s) e^{\lambda s} ds.$$ 

Since $\lambda$ is strictly negative, the integral $\int_0^\infty (1 + s) e^{\lambda s} ds$ is well defined; thus, the integral $\int_0^\infty P_k(s) ds$ is well defined and we can put

$$\tilde{P}_{k+1}(0) = -\int_0^\infty P_k(s) ds A P_B.$$ 

Then

$$\tilde{P}_{k+1}(\tau) = -\int_\tau^\infty P_k(s) A P_B ds,$$

so that

$$|\tilde{P}_{k+1}(\tau)| \leq C \int_\tau^\infty (1 + s) e^{\lambda s} ds.$$ 

Since $\int_\tau^\infty (1 + s) e^{\lambda s} ds \leq D (1 + \tau) e^{\lambda \tau}$, we obtain

$$|\tilde{P}_{k+1}(\tau)| \leq \tilde{C}_{k+1} (1 + \tau) e^{\lambda \tau}.$$ 

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By lemma 4.4, we get
\[ |\tilde{P}_{k+1}(\tau)| \leq \tilde{C}_{k+1}(1 + \tau)\varepsilon^r, \]

Using the above bounds, we get
\[ |P_{k+1}(\tau)| \leq C_{k+1}(1 + \tau)\varepsilon^r. \]

Remark

Using the proposition 4.3, we can deduce the same result for \(|\dot{P}_{k+1}(\tau)|\).

4.1.3 Asymptotic result

In this section, we give an approximation result for the \((m+1)\)th order. Towards this end, we consider the following system of equations

\((S_{-1})\)
\[ 0 = p_0 B \]
\[ p_0(0) + P_0(0) = p \]

\((S_0)\)
\[ \dot{p}_0 = p_0 A + p_1 B \]
\[ \dot{P}_0 = P_0 B \]

and, for \(1 \leq k \leq m\),
\[ p_k(0) + P_k(0) = 0 \]

\((S_k)\)
\[ \dot{p}_k = p_k A + p_{k+1} B \]
\[ \dot{P}_k = P_k B + P_{k-1} A \]

We choose \(p_{m+1}\) such that
\[ p_{m+1}(0) + P_{m+1}(0) = 0 \]
\[ \dot{p}_{m+1} = p_{m+1} A \]
\[ \dot{P}_{m+1} = P_{m+1} B + P_m A \]

Proposition 4.5

For any \(t \in [0, T]\), we have
\[ |p^x(t) - \sum_{k=0}^{m} \varepsilon^k (p_k(t) + P_k(\frac{t}{\varepsilon}))| = 0(\varepsilon^{m+1}) \]

Proof

We define, for any \(t \in [0, T]\),
\[ r^x(t) = p^x(t) - \sum_{k=0}^{m} \varepsilon^k (p_k(t) + P_k(\frac{t}{\varepsilon})) - \varepsilon^{m+1} (p_{m+1}(t) + P_{m+1}(\frac{t}{\varepsilon})) \]

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It can be show that $r^\varepsilon$ is solution of the following equation

$$
\dot{r}^\varepsilon = r^\varepsilon(\frac{B}{\varepsilon} + A) + \varepsilon^{m+1}P_{m+1}A
$$
$$
r^\varepsilon(0) = 0
$$

Thus we have

$$
|r^\varepsilon(t)|^2 = 2\int_0^t (r^\varepsilon(s)(\frac{B}{\varepsilon} + A), r^\varepsilon(s))ds + 2\varepsilon^{m+1}\int_0^t (P_{m+1}(s)A, r^\varepsilon(s))ds
$$

But, for every $q$, we have $(q(\frac{B}{\varepsilon} + A), q) \leq 0$, so we obtain

$$
|r^\varepsilon(t)|^2 \leq \varepsilon^{2(m+1)}\int_0^t |P_{m+1}(s)A|^2ds + \int_0^t |r^\varepsilon(s)|^2ds
$$

using a bound of the norm of $P_{m+1}(s)$, and then the Gronwall-Bellman inequality, we obtain, for any $t \in [0, T]$

$$
|r^\varepsilon(t)| \leq \varepsilon^{m+1}c
$$

and we obtain the desired result.

4.1.4 Decentralization and aggregation

We consider the systems $(S_0)$ and $(S_k)$

$$
p_0(0) + P_0(0) = p
$$

$(S_0)$

$$
\dot{p}_0 = p_0A + p_1B
$$
$$
\dot{P}_0 = P_0B
$$

and, for $k \geq 1$,

$$
p_k(0) + P_k(0) = 0
$$

$(S_k)$

$$
\dot{p}_k = p_kA + \bar{p}_{k+1}B
$$
$$
\dot{P}_k = P_kB + P_{k-1}A
$$

If we apply the operator $P_B$ to the different equations of the systems $(S_0)$ and $(S_k)$, we get

$(S_0)$

$$
\dot{\bar{p}}_0 = \bar{p}
$$

and, for $k \geq 1$,

$$
\dot{\bar{p}}_k = p_kAP_B
$$
$$
\dot{\bar{P}}_k = P_{k-1}AP_B
$$

and

$$
\bar{p}_k(0) + \bar{P}_k(0) = 0
$$

$(S_k)$

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By subtraction, we have

\[
\begin{align*}
\tilde{S}_0 & \quad \tilde{P}_0(0) = \tilde{p} \\
& \quad \dot{\tilde{P}}_0 = \tilde{P}_0 B
\end{align*}
\]

and, for \( k \geq 1 \),

\[
\begin{align*}
\tilde{p}_k(0) + \tilde{P}_k(0) &= 0 \\
\dot{\tilde{p}}_k &= \tilde{p}_k A(I - P_B) + \tilde{p}_{k+1} B \\
\dot{\tilde{P}}_k &= \tilde{P}_k B + P_{k-1} A(I - P_B).
\end{align*}
\]

Thus, by the block-diagonal structure of \( B \), \( \tilde{p}_k \) and \( \tilde{P}_k \) may be calculated by solving all equations on every block of \( B \). The dimension of these systems is reduced.

Now, consider the systems \((\tilde{S}_0)\) and \((\tilde{S}_k)\)

\[
\begin{align*}
\tilde{S}_0 & \quad \tilde{p}_0(0) = \tilde{p} \\
& \quad \dot{\tilde{p}}_0 = \tilde{p}_0 A P_B \\
\end{align*}
\]

and, for \( k \geq 1 \),

\[
\begin{align*}
\tilde{p}_k(0) + \tilde{P}_k(0) &= 0 \\
\dot{\tilde{p}}_k &= \tilde{p}_k A P_B \\
\dot{\tilde{P}}_k &= \tilde{P}_k - 1 A P_B
\end{align*}
\]

or,

\[
\begin{align*}
\tilde{p}_k(0) + \tilde{P}_k(0) &= 0 \\
\dot{\tilde{p}}_k &= \tilde{p}_k A P_B + \tilde{p}_k A P_B \\
\dot{\tilde{P}}_k &= \tilde{P}_k - 1 A P_B + \tilde{P}_k - 1 A P_B
\end{align*}
\]

Put \( p_k^a = p_k Q, P_k^a = P_k Q \) and \( A^a = M A Q \). Then we obtain the aggregate systems

\[
\begin{align*}
\tilde{S}_0^a & \quad p_0^a(0) = p^a \\
& \quad \dot{p}^a = p^a A^a \\
\end{align*}
\]

and, for \( k \geq 1 \),

\[
\begin{align*}
p_k^a(0) + P_k^a(0) &= 0 \\
\dot{p}_k^a &= p_k^a A^a + \tilde{p}_k A Q \\
\dot{P}_k^a &= P_{k-1} A^a + \tilde{P}_{k-1} A Q
\end{align*}
\]

**Remark**

There are \( r \) different equations are of dimension \( r \), where \( r \) is the number of the ergodic sets of the matrix \( B \).
4.2 Singular perturbation of the Zakai equation

We consider the Markov chain \((x_t^i)\), with values in a finite state space \(E = \{x_1, \ldots, x_n\}\) and infinitesimal generator \(B^\varepsilon + A\). Here \(B\) and \(A\) are two \(n \times n\) matrices which also generate Markov chains, and \(0 < \varepsilon < 1\).

\(B\) is supposed block-diagonal (\(s\) blocks), with \(r\) recurrent classes (\(s \leq r \leq n\)).

We consider the observation \(y_t\) of this chain

\[ dy_t = h(x_t^i)dt + dw_t^i; \quad y_0 = 0, \]

where \(h: E \rightarrow \mathbb{R}\), is bounded function, and \(w_t^i\) is a standard Wiener process, independent of \((x_t^i; t \geq 0)\).

We denote by \(p_t^i\) the \(\mathbb{R}^n\)-valued row vector solution of

\[ dp_t^i = p_t^i(\varepsilon B + A)dt + p_t^i H dy_t, \]

\[ p_0^i = p \]

where \(H\) is the \(n \times n\) diagonal matrix with diagonal \((h(x_1), \ldots, h(x_n))\).

The solution \(p_t^i\) of equation (4) equals, up to a normalization factor, the conditional law of \(X_t^i\), given \(\sigma(y_t; 0 \leq s \leq t)\).

We present a new expansion for the vector \(p_t^i\) where all terms can be calculated (see [6]); so we write \(p_t^i\) in the form

\[ p^i(t) = p_0(t) + P_0(t)(I - P_B)e^{B_t^i} + \varepsilon(p_1(t) + P_1(t)(I - P_B)e^{B_t^i}) + \ldots \]

where \(P_B\) is the eigenprojection for the eigenvalue 0 of the matrix \(B\), and \(e^{B_t}\) is the semigroup associated with \(B\).

Remarks

- \(P_k(t)(I - P_B) = \tilde{P}_k(t)\) belongs to \(R(B)\).

- \(P_k(t)(I - P_B)e^{tB}\) represents the \(k\) th-order boundary layer term; indeed, we have

\[ \|(I - P_B)e^{tB}\| \leq e^{-\lambda t} \]

where \(\lambda\) is positive and \(-\lambda = \max(\lambda_i)\) and \((\lambda_i)\) are the nonzero real part of eigenvalues of the matrix \(B\).
Notation

We denote $C_t^\varepsilon = (I - P_B) e^{\varepsilon B}$, $C_t^{*-1} = (I - P_B) e^{-\frac{1}{\varepsilon} B}$ and $\tilde{A} = A - \frac{1}{2} H^2$.

4.2.1 Asymptotic expansion

We write (4) in the Stratanovich form

$$\dot{p}_t^\varepsilon = p_t^\varepsilon \left( \frac{B}{\varepsilon} + A \right) - \frac{1}{2} p_t^\varepsilon H^2 + p_t^\varepsilon H \dot{y}_t,$$

or, by using the above notation

$$\dot{p}_t^\varepsilon = p_t^\varepsilon \left( \frac{B}{\varepsilon} + \tilde{A} \right) + p_t^\varepsilon H \dot{y}_t.$$

Consider the expansion of $p^\varepsilon(t)$; we obtain formally

$$\dot{p}_0(t) + \dot{P}_0(t) C^\varepsilon(t) + P_0(t) C^\varepsilon(t) \frac{B}{\varepsilon} + \varepsilon (p_1(t) + \dot{P}_1(t) C^\varepsilon(t) + P_1(t) C^\varepsilon(t) \frac{B}{\varepsilon}) + \ldots$$

$$= (p_0(t) + P_0(t) C^\varepsilon(t) + \varepsilon (p_1(t) + P_1(t) C^\varepsilon(t)) + \ldots) \left( \frac{B}{\varepsilon} + \tilde{A} \right)$$

$$+ (p_0(t) + P_0(t) C^\varepsilon(t) + \varepsilon (p_1(t) + P_1(t) C^\varepsilon(t)) + \ldots) H \dot{y}(t)$$

Separating terms, we have

$$(\varepsilon^{-1}) \quad P_0 C^\varepsilon B = (p_0 + P_0 C^\varepsilon) B$$

$$(\varepsilon^0) \quad \dot{p}_0 + \dot{P}_0 C^\varepsilon + P_1 C^\varepsilon B = (p_1 + P_1 C^\varepsilon) B + (p_0 + P_0 C^\varepsilon) \tilde{A} + (p_0 + P_0 C^\varepsilon) H \dot{y}$$

and, for $k \geq 1$,

$$(\varepsilon^k) \quad \dot{p}_k + \dot{P}_k C^\varepsilon + P_{k+1} C^\varepsilon B = (p_{k+1} + P_{k+1} C^\varepsilon) B + (p_k + P_k C^\varepsilon) \tilde{A} + (p_k + P_k C^\varepsilon) H \dot{y}$$

By taking the limit when $\varepsilon$ tends to 0, we obtain systems of equations defined by

$$(S_{-1}) \quad 0 = p_0 B$$

$$p_0(0) + P_0(0) = p$$

$$(S_0) \quad \dot{p}_0 = p_1 B + p_0 \tilde{A} + p_0 H \dot{y}$$

$$\dot{P}_0 C^\varepsilon = P_0 C^\varepsilon \tilde{A} + P_0 C^\varepsilon H \dot{y}$$

and, for $k \geq 1$,

$$p_k(0) + P_k(0) = 0$$

$$(S_k) \quad \dot{p}_k = p_{k+1} B + p_k \tilde{A} + p_k H \dot{y}$$

$$\dot{P}_k C^\varepsilon = P_k C^\varepsilon \tilde{A} + P_k C^\varepsilon H \dot{y}.$$
As $P_B$ is the eigenprojection associated with $B$, we can decompose the vector $p$ as
\[ q = \bar{q} + \tilde{q} \text{ where } \bar{q}P_B = \bar{q} \text{ and } qB = \tilde{q}B, \]
so we can write the systems
\[
(S_{-1}) \quad 0 = \bar{p}_0 B \\
\bar{p}_0(0) = \bar{p} \\
\tilde{p}_0(0) = \tilde{p}
\]
\[
(S_0) \quad \begin{align*}
\dot{\bar{p}}_0 &= \bar{p}_1 B + p_0 \bar{A} + p_0 H \dot{y} \\
\dot{\tilde{p}}_0 &= \tilde{p}_0 C^\epsilon \bar{A} C^{\epsilon^*} + \tilde{p}_0 C^\epsilon H C^{\epsilon^*} \dot{y}
\end{align*}
\]
and, for $k \geq 1$,
\[ p_k(0) + P_k(0) = 0 \]
\[
(S_k) \quad \begin{align*}
\dot{\bar{p}}_k &= \bar{p}_{k+1} B + p_k \bar{A} + p_k H \dot{y} \\
\dot{\tilde{p}}_k &= \tilde{p}_k C^\epsilon \bar{A} C^{\epsilon^*} + \tilde{p}_k C^\epsilon H C^{\epsilon^*} \dot{y}
\end{align*}
\]

Remark

As $\bar{P}_k(t)$ belongs to $R(B)$, then the vectors $\bar{p}_k(0)$ are such that
\[ \bar{p}_0(0) = \bar{p} \]
and, for $k \geq 1$,
\[ \bar{p}_k(0) = 0. \]

We next write all systems in Itô-form. In first, consider the system $(S_{-1})$; and, we have, for every $t$
\[ p_0(t) = \bar{p}_0(t) \]
and,
\[ d\bar{p}_0 = \bar{p}_1 B dt + \bar{p}_0 A dt + \bar{p}_0 H d\dot{y}; \quad \bar{p}_0(0) = \bar{p} \]
and, if we apply the operator $P_B$, we obtain
\[ d\bar{p}_0 = \bar{p}_0 A P_B dt + \bar{p}_0 H P_B d\dot{y}; \quad \bar{p}_0(0) = \bar{p}. \]
Now, by the unicity of the decomposition of semimartingales, we have
\[ \bar{p}_0 H = \bar{p}_0 H P_B. \]
This means that we must verify the following equality
\[ P_B H = P_B H P_B \quad \text{or} \quad M H = M H P_B. \]
For instance, this equality is verified if the matrix $H$ is diagonal constant on every block of the matrix $B$: we observe every block of $B$. As $P_B$ and $B$ have the same block-structure, $P_B$ and $H$ commute.

Thus, the zero order terms are in the Itô-form

$\tilde{p}_0 = \tilde{p}$

$\tilde{P}_0 = \tilde{p}$

(S0)

$$d\tilde{p}_0 = p_0 A P_B dt + \tilde{p}_0 H P_B dy$$

$$d\tilde{P}_0 = \tilde{P}_0 C^* A C^* dt + \tilde{P}_0 C^* H C^* dy$$

and, for $k \geq 1$,

$$p_k + P_k = 0$$

(Sc)

$$dp_k = \tilde{p}_{k+1} B dt + p_k A dt + p_k H dy$$

$$d\tilde{P}_k = \tilde{P}_k C^* A C^* - dt + \tilde{P}_k C^* H C^* - dy$$

Remarks

- The component $\tilde{p}_k$ is obtained by solving the Poisson-equation

$$\tilde{p}_k B = \frac{d}{dt} \left( p_{k-1} - \int_0^t p_{k-1}(s) H dy(s) \right) - p_{k-1} A$$

Let $H_B$ be the inverse of $B$ in the subspace $R(P_B - I)$:

$$H_B B = B H_B = P_B - I \quad \text{and} \quad H_B P_B = P_B H_B = 0.$$ 

Then

$$\tilde{p}_k = -\frac{d}{dt} \left( p_{k-1} - \int_0^t p_{k-1}(s) H dy(s) \right) H_B + p_{k-1} A H_B.$$ 

- The component $\tilde{p}_k$ satisfies the following equation

$$d\tilde{p}_k = p_k A P_B dt + p_k H P_B dy,$$

obtained by applying the operator $P_B$.

4.2.2 Approximation result

We consider the following systems

(S-1)

$$0 = pB$$

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\[ p_0(0) + P_0(0) = p \]

\((S_0)\)

\[ dp_0 = p_1 B dt + p_0 Adt + p_0 Hdy \]
\[ dP_0 = P_0 C^* AC^* dt + P_0 C^* HC^* dy \]

and, for \(1 \leq k \leq m\),

\[ p_k(0) + P_k(0) = 0 \]

\((S_k)\)

\[ dp_k = p_{k+1} B dt + p_k Adt + p_k Hdy \]
\[ dP_k = P_k C^* AC^* dt + P_k C^* HC^* dy \]

We choose \(p_{m+1}\) such that

\[ p_{m+1}(0) = c \]
\[ dp_{m+1} = p_{m+1} Adt + p_{m+1} Hdy \]

**Proposition 4.6**

For all \(t \in [0, T]\), we have

\[ E^v \left( |p^*(t) - (p_0(t) + P_0(t)C_i^* + \ldots + \varepsilon^m (p_m(t) + P_m(t)C_i^*))|^2 \right) = 0(\varepsilon^{2(m+1)}), \]

where \(E^v\) is the expectation with respect to \(P^v\), under which \((y_t; t \geq 0)\) is a standard Wiener process.

**Proof**

We put

\[ r^*(t) = p^*(t) - (p_0(t) + P_0(t)C_i^* + \ldots + \varepsilon^m (p_m(t) + P_m(t)C_i^*)) - \varepsilon^{m+1} p_{m+1}(t) \]

After some calculation, we obtain

\[ dr^* = r^* \left( \frac{B}{\varepsilon} + A \right) dt + r^* H dy \]
\[ r^*(0) = -\varepsilon^{m+1} c \]

Then

\[ |r^*(t)|^2 = \varepsilon^{2(m+1)} c + 2 \int_0^t \langle r^*(s) \left( \frac{B}{\varepsilon} + A \right), r^*(s) \rangle ds + 2 \int_0^t \langle r^*(s) H, r^*(s) \rangle dy \]
\[ + \int_0^t |r^*(s) H|^2 ds. \]
After taking expectations, we obtain
\[ E^\nu(\|r^\nu(t)\|^2) = \varepsilon^{2(m+1)}c + 2 \int_0^t E^\nu(\|r^\nu(s)\|^{B/\varepsilon} + A, r^\nu(s))\,ds + \int_0^t E^\nu(\|r^\nu(s)H\|^2)\,ds. \]

For every vector \( q \), we have \( (q(\frac{B}{\varepsilon} + A), q) \leq 0 \), thus
\[ E^\nu(\|r^\nu(t)\|^2) \leq \varepsilon^{2(m+1)} + h^2 \int_0^t E^\nu(\|r^\nu(s)\|^2)\,ds, \]
where
\[ h = \max_{i \in [1,n]} |h(x_i)|. \]

Applying the Gronwall-Bellman lemma, we see that
\[ E(\|r^\nu(t)\|^2) \leq \varepsilon^{2(m+1)}c \]
and obtain the desired result.

### 4.2.3 Decentralization and aggregation

We consider the systems \((S_k)\) defined by

\[ p_0(0) + P_0(0) = p \]
\[
(S_0) \quad dp_0 = \tilde{p}_0Bdt + p_0Adt + p_0Hdy \]
\[
\quad dP_0 = P_0C^\nu AC^\nu dt + P_0C^\nu HC^\nu dy \]

and, for \(k \geq 1\),

\[ p_k(0) + P_k(0) = 0 \]
\[
(S_k) \quad dp_k = \tilde{p}_{k+1}Bdt + p_kAdt + p_kHdy \]
\[
\quad dP_k = P_kC^\nu AC^\nu dt + P_kC^\nu HC^\nu dy \]

If we apply the operator \(P_B\) to the different equations, we get

\[ \bar{p}_0(0) = \bar{p} \]
\[
(S_0) \quad d\bar{p}_0 = \bar{p}_0AP_Bdt + \bar{p}_0HP_Bdy \]

and, for \(k \geq 1\),

\[ \bar{p}_k(0) = 0 \]
\[
(S_k) \quad d\bar{p}_k = p_kAP_Bdt + p_kHP_Bdy \]
and, by subtraction, the decentralized systems are

$$\dot{\tilde{P}}_0(0) = \tilde{p}$$
$$d\dot{\tilde{P}}_0 = \tilde{P}_0 C^e A C^e - dt + \tilde{P}_0 C^e H C^e - dy$$

and, for \(k \geq 1\),

$$\tilde{p}_k(0) + \dot{\tilde{P}}_k(0) = 0$$

$$d\tilde{p}_k = p_k A (I - P_B) dt + \tilde{p}_{k+1} B dt + p_k H (I - P_B) dy$$

$$d\dot{\tilde{P}}_k = \tilde{P}_k C^e A C^e - dt + \tilde{P}_k C^e H C^e - dy$$

The block-diagonal structure of \(B\) (also \(P_B\) and \(H_B\)) permits us to decentralize the calculation: indeed, we can solve the equations by restriction on every block of the matrix \(B\).

Consider the systems \((\tilde{S}_0)\) and \((\tilde{S}_k)\).

$$\bar{p}_0(0) = \bar{p}$$
$$d\bar{p}_0 = \bar{p}_0 A P_B dt + \bar{p}_0 H P_B dy$$

and, for \(k \geq 1\),

$$\bar{p}_k(0) = 0$$

$$d\bar{p}_k = \bar{p}_k A P_B dt + \bar{p}_k A P_B dt + \bar{p}_k H P_B dy + \bar{p}_k H P_B dy$$

Put \(p_k^e = \bar{p}_k Q A^e = M A Q \) and \(H^e = M H Q\). Then, we obtain the aggregate systems

$$p_0^e(0) = p^e$$
$$dp_0^e = p_0^e A^e dt + p_0^e H^e dy$$

and, for \(k \geq 1\),

$$p_k^e(0) = 0$$

$$dp_k^e = p_k^e A^e dt + \bar{p}_k A Q dt + p_k^e H^e dy + \bar{p}_k H Q dy$$

Remark

The vector \(p_k^e\) belongs to \(r\) dimensional space, where \(r\) is the number of recurrent classes of the matrix \(B\).

Remark

In this section 4.2, we obtain an expansion of \(p_k^e\) where all terms can be calculated easily, whereas in the author's thesis [7], only the zero and first order terms could be calculated.
References


