An Approximate Linear Approach for the Fundamental Matrix Computation

Gutemberg Guerra Filho and Yiannis Aloimonos

Department of Computer Science,
University of Maryland,
A. V. Williams Building,
College Park, MD
{guerra@cs, yiannis@cfar}.umd.edu

Abstract. We introduce a new robust approach for the computation of the fundamental matrix taking into account the intrinsic errors (uncertainty) involved in the discretization process. The problem is modeled as an approximate equation system and reduced to a linear programming form. This approach is able to compute the solution set instead of trying to compute only a single vertex of the solution polyhedron as in previous approaches. Therefore, our algorithm is a robust generalization of the eight-point algorithm. The exact solution computation feasibility is proved for some pure translation motions, depending only on the distribution of the discretization errors. However, a single exact solution for the fundamental matrix is not feasible for pure rotation cases. The feasibility of an exact solution is decided according to an error distance between a nontrivial exact solution and the faces of the solution set.

1 Introduction

Recovering a three-dimensional model from a sequence of images may require the estimation of the *fundamental matrix* [2]. The fundamental matrix F encapsulates the projective geometry between two images, depending only on the internal parameters of the camera and on the relative pose. If a scene point P is projected as p in one image and p' in another, then the image points satisfy the equation $p'^T F p = 0$, where F is a 3×3 matrix of rank 2.

A set of point correspondences between two images is given for computing the fundamental matrix. Each point match gives rise to one linear equation in the entries of F known as the *epipolar constraint* [5]. Let p be the point $(x, y, 1)^T$ and p' be $(x', y', 1)^T$, the equation corresponding to a pair of points p and p' is

$$x'xF_{11} + x'yF_{12} + x'F_{13} + y'xF_{21} + y'yF_{22} + y'F_{23} + xF_{31} + yF_{32} + F_{33} = 0.$$

The set of point matches corresponds to a homogeneous system of linear algebraic equations. Given a sufficient number of matches, a solution for the system is feasible and the unknown matrix F is computed by the eight-point algorithm [4] or a robust method such as RANSAC [8].

An *image* is a continuous two-dimensional function of brightness corresponding to the projection of a three-dimensional scene onto a plane. For computer vision applications, spatial discretization of the image is required to obtain a finite number of picture cells (pixels). Each *pixel* is associated with the average irradiance over a small sampling area in the image plane. This way, the image is represented as a rectangular array of pixels.

Pixel coordinates of points in the epipolar constraint are not exact. The correspondence problem or stereo matching is an ill-posed problem and difficulties arise from noise in the image, the aperture problem in textureless regions, among others. Even if the correspondence problem was solved exactly, the discretization process intrinsically inserts some amount of error which sub-pixel precision computation is not able to overcome completely. Since real models of computation are finite, the discretization process is unavoidable and, consequently, there are intrinsic errors embedded in image point coordinates.

If any of the coefficients and right-hand constants in a system of equations is not known exactly, then the system is called *approximate*. The study of the effects of uncertainties in the coefficients and constants on the solution of an approximate system is called *approximate equation analysis* [6].

In this paper, we discuss the effects of discretization errors in the computation of the fundamental matrix. We investigate the exact solution computation feasibility for some particular rigid motions. In some pure translation cases, the exact computation is possible and depends on the distribution of the errors. On the other hand, pure rotation cases have a solution set characterization such that a single exact solution for the fundamental matrix is not feasible.

The feasibility of a nontrivial exact solution for the fundamental matrix is decided according to an error distance $\Delta(f, \Gamma)$ from an exact solution f to the faces Γ of a polyhedron. This polyhedron represents the solution set of an approximate system modeling the epipolar constraints. This error distance also describes the goodness of a particular epipolar constraint related to the solution process. Hence, the distance $\Delta(f, \Gamma)$ is an attribute that characterizes point matches with respect to its contribution in the solution of the approximate system.

In this paper, we introduce a new robust approach for the computation of the fundamental matrix taking into account the intrinsic errors involved in the image acquisition process. The problem is modeled as an approximate equation system and reduced to a linear programming form [1]. A linear programming module is used in the algorithm that works as a search in solution space. The algorithm computes nine intervals of uncertainty that contains admissible solutions for the elements of the fundamental matrix. Therefore, this approach is able to compute the polyhedron representing the solution set while previous approaches only tries to compute a single vertex of this polyhedron. This vertex is at the center of a ball with some positive radius intercepting the most number of faces corresponding to point matches.

This paper is organized as follows. In Section 2, we discuss approximate equation analysis and the admissible solution conditions. In Section 3, a solution set characterization is performed for some particular rigid motions showing the computation feasibility of an exact fundamental matrix. We present an approximate algorithm for the fundamental matrix computation in Section 4. Finally, Section 5 highlights our results and future work.

2 Approximate Equation Analysis

An approximate system of n linear algebraic equations in m unknowns $Af = b$ is a system in which the coefficients A_{ij} and the right-hand constants b_i are intervals. A vector f is an *admissible solution* of the approximate system of equations if $A^*f = b^*$ for some $A^* \in A$ and some $b^* \in b$. The *solution set* S is the union of all admissible solutions.

The minimum and maximum values of each solution component defines the *intervals of uncertainty* $U_j = [min(f_j), max(f_j)]$, for all $f \in S$ and $j = 1, ..., m$. An admissible solution is not necessarily one whose components fall within the intervals of uncertainty, since the solution set S is a subset of the set defined by the intervals of uncertainty. Furthermore, specifying one component of a solution vector restricts the intervals in which the other components may lie.

If $A_{ij} = [\alpha_{ij} - \epsilon_{ij}, \alpha_{ij} + \epsilon_{ij}]$ and $b_i = [\beta_i - \epsilon_i, \beta_i + \epsilon_i]$, a necessary and sufficient set of conditions for a solution f to be an admissible solution of $Af = b$ are

$$\sum_{j=1}^{m}(\alpha_{ij} f_j + \epsilon_{ij}|f_j|) \geq \beta_i - \epsilon_i \text{ and } \sum_{j=1}^{m}(\alpha_{ij} f_j - \epsilon_{ij}|f_j|) \leq \beta_i + \epsilon_i$$

for $i = 1, ..., n$. The uncertainties ϵ_{ij} and ϵ_i are nonnegative. The presence of the terms involving $|f_j|$ (absolute value) in these conditions cause a nonlinear behavior in the conditions. Therefore, the set of linear constraints is different for each of the 2^m orthants of the solution space (see Fig. 1).

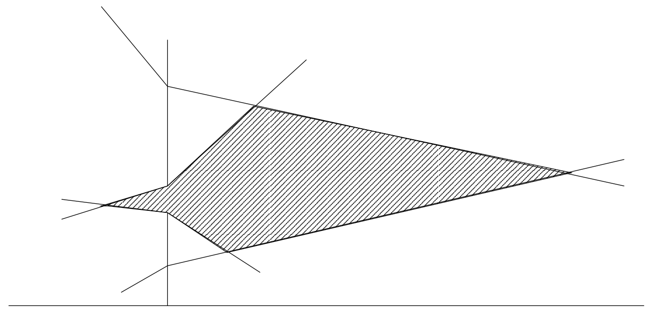


Fig. 1. Region of admissible solutions.

Solution sets of approximate systems of equations are not necessarily convex and possibly non-convex. A system of equations is *critically ill-conditioned* if there exists a singular coefficient matrix within the limits of the uncertainties in the coefficients. The solution set of a non-critically ill-conditioned approximate system of equations is bounded and not disjoint. A critically ill-conditioned system has the interval of uncertainty for at least one unknown unbounded and may be disjoint.

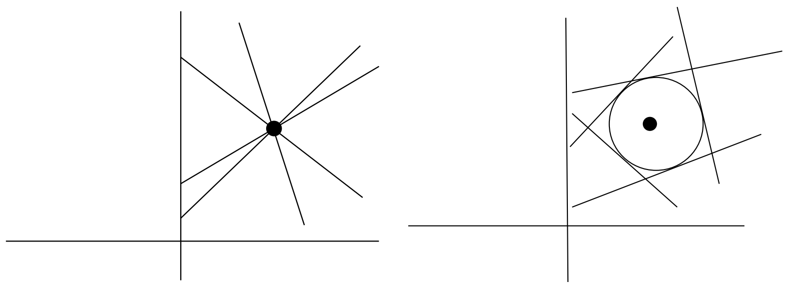
3 Solution Set Characterization

Since the discrete coordinates of points in an image are not exact, there is an error associated with each spatial measurement. Due to the errors, the epipolar constraint used to compute the fundamental matrix becomes an approximate equation. The solution set for a homogeneous approximate system is a polyhedron that contains the ray representing the exact solution, since the fundamental matrix can be determined only up to a scale.

We analyze how this polyhedron approximates an exact solution. The analysis consists in computing an error distance $\Delta(f, \Gamma)$ from a nontrivial exact fundamental matrix f to the faces Γ of the polyhedron: $\Delta(f, \Gamma) = \min_{h \in \Gamma} |h - f|$. Let f be a vector in the solution space and h be a face (hyper-plane) defined by $av = 0$ corresponding to an admissible solution condition. The distance $|h - f|$ from f to h is the inner product $f \bullet a$. If the distance is zero, then h contains the vector f.

A solution set characterization is found to be based on the errors associated with each point match when the distance $\Delta(f, \Gamma)$ to the faces is minimum. The characterization consists in either finding conditions on error distribution for an exact solution computation ($\Delta(f, \Gamma) = 0$) or proving that the computation of a single exact solution for the fundamental matrix is not feasible ($\Delta(f, \Gamma) > 0$) due to discretization errors in the spatial measurements (see Fig. 2).

We consider a normalized projective camera model with an unit focal length. The first camera is centered at the origin of the three-dimensional space. The



(a) $\Delta(f, \Gamma) = 0$.   (b) $\Delta(f, \Gamma) > 0$.

Fig. 2. Exact solution computation feasibility.

principal axis is coincident with the z-axis facing the positive half-space $Z > 0$. The second camera is related to the first one according to a rigid motion. Let P be a three-dimensional point $(X, Y, Z, 1)^T$, the point P is projected onto points p and p' with pixel coordinates $(x, y, 1)^T = (\frac{X}{Z}, \frac{Y}{Z}, 1)^T$ in the first image and $(x', y', 1)^T$ in the second image, respectively.

The discretization process inserts errors into the coordinates of the points p and p'. Hence, the discrete pixel coordinates of p and p' are defined as $(x_\diamond, y_\diamond, 1)^T = (x + e_x, y + e_y, 1)^T$ and $(x'_\diamond, y'_\diamond, 1)^T = (x' + e_{x'}, y' + e_{y'}, 1)^T$, respectively, where e_x, e_y, $e_{x'}$, $e_{y'}$ are independent spatial errors whose absolute values are less than a constant uncertainty $\epsilon > 0$ (see Fig. 3). The continuous case is defined by the special condition $\epsilon = 0$.

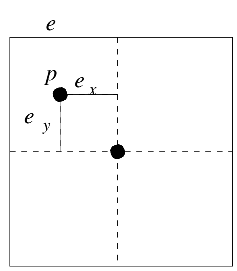


Fig. 3. Discretization process.

Once a point is projected onto an image, the exact coordinates can only be defined as belonging to intervals centered at the pixels with radius equal to the uncertainty ϵ. Hence, the epipolar constraint corresponding to the pair of points

p and p' is an approximate equation[1]

$$\widehat{x'}\widehat{x}f_{11} + \widehat{x'}\widehat{y}f_{12} + \widehat{x'}f_{13} + \widehat{y'}\widehat{x}f_{21} + \widehat{y'}\widehat{y}f_{22} + \widehat{y'}f_{23} + \widehat{x}f_{31} + \widehat{y}f_{32} + f_{33} = 0,$$

where $\widehat{x} = [x_\diamond - \epsilon, x_\diamond + \epsilon]$, $\widehat{y} = [y_\diamond - \epsilon, y_\diamond + \epsilon]$, $\widehat{x'} = [x'_\diamond - \epsilon, x'_\diamond + \epsilon]$, and $\widehat{y'} = [y'_\diamond - \epsilon, y'_\diamond + \epsilon]$.

According to interval arithmetic [7], the product of two intervals $\widehat{i} = [\underline{i}, \overline{i}]$ and $\widehat{j} = [\underline{j}, \overline{j}]$ is an interval defined as $\widehat{k} = [min\{\underline{ij}, \underline{i}\overline{j}, \overline{i}\underline{j}, \overline{ij}\}, max\{\underline{ij}, \underline{i}\overline{j}, \overline{i}\underline{j}, \overline{ij}\}]$. However, if $\underline{i} \geq 0$ and $\underline{j} \geq 0$, the product interval $\widehat{k}$ becomes $[\underline{ij}, \overline{ij}]$. Therefore, making these nonnegative assumptions ($x_\diamond \geq \epsilon$, $y_\diamond \geq \epsilon$, $x'_\diamond \geq \epsilon$, and $y'_\diamond \geq \epsilon$) without loss of generality, the approximate epipolar constraint becomes

$$\begin{aligned}
[(x'_\diamond - \epsilon)(x_\diamond - \epsilon), (x'_\diamond + \epsilon)(x_\diamond + \epsilon)]\ f_{11} + \\
+ [(x'_\diamond - \epsilon)(y_\diamond - \epsilon), (x'_\diamond + \epsilon)(y_\diamond + \epsilon)]\ f_{12} + \\
+ [x'_\diamond - \epsilon, x'_\diamond + \epsilon]\ f_{13} + \\
+ [(y'_\diamond - \epsilon)(x_\diamond - \epsilon), (y'_\diamond + \epsilon)(x_\diamond + \epsilon)]\ f_{21} + \\
+ [(y'_\diamond - \epsilon)(y_\diamond - \epsilon), (y'_\diamond + \epsilon)(y_\diamond + \epsilon)]\ f_{22} + \\
+ [y'_\diamond - \epsilon, y'_\diamond + \epsilon]\ f_{23} + \\
+ [x_\diamond - \epsilon, x_\diamond + \epsilon]\ f_{31} + \\
+ [y_\diamond - \epsilon, y_\diamond + \epsilon]\ f_{32} + \\
+\ f_{33} = 0.
\end{aligned}$$

A fundamental matrix is represented by a ray cf in the solution space, where c is a scalar and f is a vector $(f_{11}, f_{12}, f_{13}, f_{21}, f_{22}, f_{23}, f_{31}, f_{f32}, f_{33})$. The vector f specifies an orthant according to the signs of f_{ij} for $i, j = 1, ..., 3$. This orthant contains a nontrivial ($c \neq 0$) exact solution f. The admissible solution conditions for the approximate equation in an orthant have the form

$$\begin{aligned}
(x'_\diamond x_\diamond \pm \epsilon(x'_\diamond + x_\diamond) + \epsilon^2)f_{11} + (x'_\diamond y_\diamond \pm \epsilon(x'_\diamond + y_\diamond) + \epsilon^2)f_{12} + (x'_\diamond \pm \epsilon)f_{13} + \\
+(y'_\diamond x_\diamond \pm \epsilon(y'_\diamond + x_\diamond) + \epsilon^2)f_{21} + (y'_\diamond y_\diamond \pm \epsilon(y'_\diamond + y_\diamond) + \epsilon^2)f_{22} + (y'_\diamond \pm \epsilon)f_{23} + \\
+(x_\diamond \pm \epsilon)f_{31} + (y_\diamond \pm \epsilon)f_{32} + f_{33} \geq 0
\end{aligned}$$

and

$$\begin{aligned}
(x'_\diamond x_\diamond \mp \epsilon(x'_\diamond + x_\diamond) + \epsilon^2)f_{11} + (x'_\diamond y_\diamond \mp \epsilon(x'_\diamond + y_\diamond) + \epsilon^2)f_{12} + (x'_\diamond \mp \epsilon)f_{13} + \\
+(y'_\diamond x_\diamond \mp \epsilon(y'_\diamond + x_\diamond) + \epsilon^2)f_{21} + (y'_\diamond y_\diamond \mp \epsilon(y'_\diamond + y_\diamond) + \epsilon^2)f_{22} + (y'_\diamond \mp \epsilon)f_{23} + \\
+(x_\diamond \mp \epsilon)f_{31} + (y_\diamond \mp \epsilon)f_{32} + f_{33} \leq 0.
\end{aligned}$$

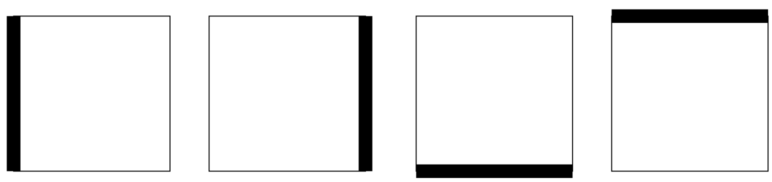
3.1 Pure Translation Cases

The first rigid motion considered is a pure translation t_x parallel to the x-axis. In this case, the point P is projected onto point $(x', y', 1)^T = (\frac{X+t_x}{Z}, \frac{Y}{Z}, 1)^T$ in

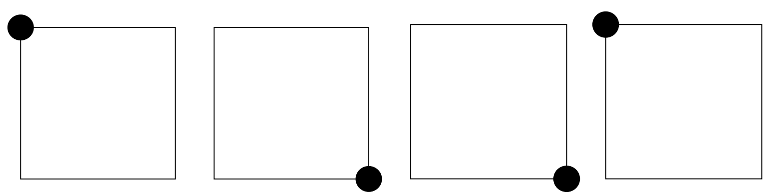
[1] Note that the intervals corresponding to unknowns in the approximate equation are not independent. However, the assumption that these intervals are independent just implies in a solution set which contains the original set which considers dependent intervals.

the second image. This translation corresponds to a fundamental matrix represented by the vector $f_{t_x} = (0, 0, 0, 0, 0, -t_x, 0, t_x, 0)$ in the solution space. Hence, assuming without loss of generality that $t_x > 0$, an orthant containing an exact solution is defined by $f_{23} < 0$ and $f_{32} > 0$. The error distance Δ_{t_x} from vector f_{t_x} to the admissible solution conditions is $(2\epsilon + (e_y - e_{y'}))t_x$. Similarly, a translation t_y parallel to the y-axis results in the error distance $\Delta_{t_y} = (2\epsilon + (e_x - e_{x'}))t_y$.

When the error distance is zero, an approximate equation becomes a constraint that helps to find an exact solution for the fundamental matrix. If $t_x > 0$, the distance Δ_{t_x} is zero when $\langle e_y = -\epsilon,\ e_{y'} = \epsilon \rangle$ (see Fig. 4a). Similarly, if $t_y > 0$, the distance $\Delta_{t_y} = 0$ when $\langle e_x = -\epsilon,\ e_{x'} = \epsilon \rangle$ (see Fig. 4b).



(a) x-axis translation. (b) y-axis translation.



(c) z-axis. (d) x-axis and y-axis.

Fig. 4. Geometric interpretation of conditions for pure translation exact computation.

In the case where the translation t_z is parallel to the z-axis, the point p' is $(x', y', 1)^T = (\frac{X}{Z+t_z}, \frac{Y}{Z+t_z}, 1)^T$. This translation corresponds to a fundamental matrix represented by the vector $f_{t_z} = (0, -t_z, 0, t_z, 0, 0, 0, 0, 0)$ in the solution space and, assuming without loss of generality that $t_z > 0$, an orthant containing an exact solution is defined by $f_{12} < 0$ and $f_{21} > 0$. Hence, the inner product between f_{t_z} and the hyper-planes representing admissible solution conditions is the error distance Δ_{t_z} defined as $((\epsilon + e_{y'})x + (\epsilon - e_{x'})y + (\epsilon - e_y)x' + (\epsilon + e_x)y' + \epsilon(e_x + e_y + e_{x'} + e_{y'}) + e_x e_{y'} - e_y e_{x'})t_z$. The necessary conditions for a zero error distance in the translation parallel to the z-axis are $\langle e_x = -\epsilon,\ e_y = \epsilon,\ e_{x'} = \epsilon,\ e_{y'} = -\epsilon \rangle$ (see Fig. 4c).

A translation with two components t_x and t_y corresponds to a fundamental matrix represented by the vector $f_{t_{xy}} = (0, 0, t_y, 0, 0, -t_x, -t_y, t_x, 0)$ in the solution space. Hence, the error distance $\Delta_{t_{xy}}$ between $f_{t_{xy}}$ and the vectors representing admissible solution conditions is defined as $(2\epsilon + (e_y - e_{y'}))t_x + (2\epsilon - (e_x + e_{x'}))t_y$. If $t_x > 0$ and $t_y > 0$, a zero error distance for this translation case is achieved when $\langle e_x = \epsilon,\ e_y = -\epsilon,\ e_{x'} = -\epsilon,\ e_{y'} = \epsilon \rangle$ (see Fig. 4d).

Considering a pure translation with three components t_x, t_y, t_z, the point p' is $(x', y', 1)^T = (\frac{X+t_x}{Z+t_z}, \frac{Y+t_y}{Z+t_z}, 1)^T$. This translation corresponds to a fundamental matrix represented by the vector $f_{t_{xyz}} = (0, -t_z, t_y, t_z, 0, -t_x, -t_y, t_x, 0)$ in the solution space. Therefore, the error distance $\Delta_{t_{xyz}}$ is $(\delta_x X + \delta_y Y + \delta_z Z + \delta_{xz} XZ + \delta_{yz} YZ + \delta_{z^2} Z^2)/(Z(Z+t_z))$, where

$$\begin{aligned}
\delta_x &= (\epsilon + e_{y'})t_z^2, \\
\delta_y &= (\epsilon - e_{x'})t_z^2, \\
\delta_z &= (3\epsilon - e_{y'})t_x t_z + (3\epsilon + e_{x'})t_y t_z + \delta t_z^2, \\
\delta_{xz} &= (2\epsilon - e_y + e_{y'})t_z, \\
\delta_{yz} &= (2\epsilon + e_x - e_{x'})t_z, \\
\delta_{z^2} &= (2\epsilon + e_y - e_{y'})t_x + (2\epsilon - e_x + e_{x'})t_y + \delta t_z, \text{ and} \\
\delta &= \epsilon e_x + \epsilon e_y + \epsilon e_{x'} + \epsilon e_{y'} + e_x e_{y'} - e_{x'} e_y.
\end{aligned}$$

Note that $\delta_x = \delta_y = \delta_{xz} = \delta_{yz} = \delta = 0$ when $\langle e_x = -\epsilon,\ e_y = \epsilon,\ e_{x'} = \epsilon,\ e_{y'} = -\epsilon \rangle$. On the other hand, $\delta_{z^2} = \delta = 0$ when $\langle e_x = \epsilon,\ e_y = -\epsilon,\ e_{x'} = -\epsilon,\ e_{y'} = \epsilon \rangle$. Even assuming these conditions are not contradictory, $\delta_z \neq 0$ and, consequently, the absolute distance Δ_{xyz} is greater than zero. The necessary conditions for $\Delta_{xyz} = 0$ are the conditions for $\delta_{z^2} = 0$ and $t_z = 0$. Hence, the exact solution is feasible in a two-component pure translation case only with t_x and t_y and not possible with three components.

Eight independent approximate equations satisfying zero distance conditions will give rise to a feasible linear system of equations with an exact solution for the fundamental matrix. Therefore, the computation of an exact solution for pure translation cases depends only on the distribution of the discretization errors.

3.2 Pure Rotation Cases

In the pure rotation cases, we show that the computation of an exact fundamental matrix is not possible considering the approximate equation system. We prove that the error distance from an exact solution to any face of the polyhedron representing the solution set is greater than zero. Hence, there exists a closed ball with radius greater than zero contained in the solution set which contains a nontrivial exact solution. This way, a nontrivial exact solution is an interior point of the solution polyhedron and, consequently, can not be determined exactly by the approximate equation system.

A rotation about the z-axis is through an angle θ_z. In this case, the point p' is $(x', y', 1)^T = (\frac{c_z X - s_z Y}{Z}, \frac{s_z X + c_z Y}{Z}, 1)^T$, where $s_z = \sin\theta_z$ and $c_z = \cos\theta_z$. The fundamental matrix associated with this rotation corresponds to the vector $f_{\theta_z} = (s_z, c_z, -1, -c_z, s_z, -1, c_z + s_z, c_z - s_z, 0)$ in the solution space. In order to specify an unique orthant, we assume without loss of generality $\theta_z \in (0, \frac{\pi}{4}]$.

The cases where $\theta_z > \frac{\pi}{4}$ are similar. Hence, since s_z, c_z, $c_z - s_z$ are nonnegative, an orthant containing an exact solution is defined by $f_{13}, f_{21}, f_{23} < 0$ and $f_{11}, f_{12}, f_{22}, f_{31}, f_{32} > 0$. Therefore, the error distance Δ_{θ_z} from f_{θ_z} to the admissible solution conditions is defined as $\delta_x \frac{X}{Z} + \delta_y \frac{Y}{Z} + \delta_s s_z + \delta_c c_z + \delta_{\theta_z}$, where

$$\begin{aligned}
\delta_x &= s_z(\epsilon + e_{x'}) + c_z(\epsilon - e_{y'}) + 2\epsilon s_z c_z + \epsilon + e_y,\\
\delta_y &= s_z(\epsilon + e_{y'}) + c_z(\epsilon + e_{x'}) + 2\epsilon {c_z}^2 - \epsilon - e_x,\\
\delta_s &= e_x e_{x'} + e_{y'} e_y + \epsilon(e_x + e_y + e_{x'} + e_{y'}) + e_x - e_y + 2\epsilon^2,\\
\delta_c &= e_{x'} e_y - e_{y'} e_x + \epsilon(e_x + e_y + e_{x'} + e_{y'}) + e_x + e_y + 2\epsilon, \text{ and}\\
\delta_{\theta_z} &= 2\epsilon - e_{x'} - e_{y'}.
\end{aligned}$$

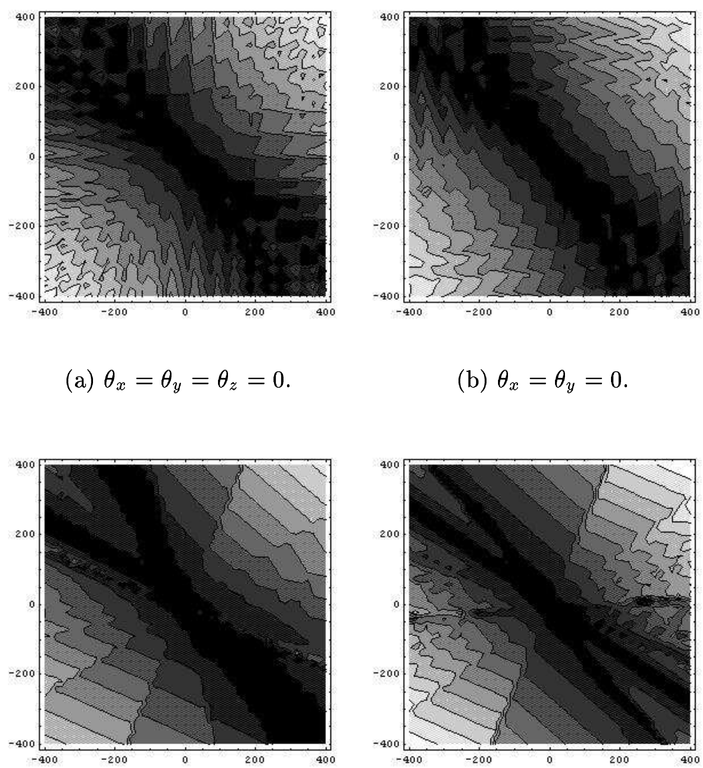
Note that the minimum value of δ_x is $2\epsilon s_z c_z$, when $\langle e_{x'} = -\epsilon, e_{y'} = \epsilon, e_y = -\epsilon\rangle$. Analogously, the minimum value of δ_y is $2\epsilon {c_z}^2$, when $\langle e_{y'} = -\epsilon, e_{x'} = -\epsilon, e_x = \epsilon\rangle$. On the other hand, $\delta_s s_z + \delta_c c_z + \delta_{\theta_z}$ is zero when $\langle e_x = -\epsilon, e_y = -\epsilon, e_{x'} = \epsilon, e_{y'} = \epsilon\rangle$. Therefore, even assuming these minimum conditions are not contradictory, the minimum distance Δ_{θ_z} is $2\epsilon s_z c_z \frac{X}{Z} + 2\epsilon {c_z}^2 \frac{Y}{Z}$. According to the nonnegative assumptions about the approximate equation intervals, $X \geq 2\epsilon$ and $Y \geq 2\epsilon$ when $\delta_s s_z + \delta_c c_z + \delta_{\theta_z} = 0$. Therefore, considering a rotation about the z-axis, the distance from an exact solution to any face of the solution set is greater than zero.

In the case where the rotation θ_x is about the x-axis, the point p' is $(x', y', 1)^T = (\frac{X}{s_x Y + c_x Z}, \frac{c_x Y - s_x Z}{s_x Y + c_x Z}, 1)^T$, where $s_x = \sin\theta_x$ and $c_x = \cos\theta_x$. In order to simplify this analysis, we assume $\theta_x = \frac{\pi}{2}$ and, consequently, this rotation corresponds to a fundamental matrix represented by the vector $f_{\theta_x} = (0, -1, 1, 1, 1, 0, 1, 0, 1)$ in the solution space. In this case, an orthant containing an exact solution is defined by $f_{12} < 0$ and $f_{13}, f_{21}, f_{22}, f_{31}, f_{33} > 0$. Therefore, the distance Δ_{θ_x} is $\delta_{xy} \frac{X}{Y} + \delta_{xz} \frac{X}{Z} + \delta_{yz} \frac{Y}{Z} - \delta_{zy} \frac{Z}{Y} + \delta_{\theta_x}$, where

$$\begin{aligned}
\delta_{xy} &= \epsilon - e_y,\\
\delta_{xz} &= \epsilon + e_{y'},\\
\delta_{yz} &= 2\epsilon - e_{x'} + e_{y'},\\
\delta_{zy} &= 2\epsilon + e_x + e_y, \text{ and}\\
\delta_{\theta_x} &= (\epsilon + 1)(e_x + e_{x'}) + 2\epsilon(e_y + e_{y'} + 1) + \epsilon^2.
\end{aligned}$$

Note that $\delta_{xy} = 0$ when $\langle e_y = \epsilon\rangle$, $\delta_{xz} = 0$ when $\langle e_{y'} = -\epsilon\rangle$, $\delta_{yz} = 0$ when $\langle e_{x'} = \epsilon, e_{y'} = -\epsilon\rangle$, and $\delta_{zy} = 0$ when $\langle e_x = -\epsilon, e_y = -\epsilon\rangle$. The conditions for $\delta_{xy} = 0$ and $\delta_{zy} = 0$ are contradictory with respect to e_y. Therefore, even assuming the conditions for $\delta_{\theta_x} = 0$ do not represent further contradictions, the distance Δ_{θ_x} is either $2\epsilon \frac{X}{Y}$ or $-2\epsilon \frac{Z}{Y}$. Therefore, considering a rotation about the x-axis through $\theta_x = \frac{\pi}{2}$, the absolute error distance from an exact solution to any face of the solution set is greater than zero. A similar analysis apply to the case of a pure rotation about the y-axis through an angle θ_y.

The error distance $\Delta(f, \Gamma)$ measures the contribution of a point match in approximating the exact solution. Therefore, a point p in the first image represents a better approximation when the error distance is minimized. Although each rigid motion corresponds to a different error distance and, consequently, a different minimum global valley, this valley always passes through the point at the center of the image (see Fig. 5). Therefore, a heuristic for choosing a starting point match in the computation of the fundamental matrix is to choose the pair which p is closest to the center of the image.



(a) $\theta_x = \theta_y = \theta_z = 0$. (b) $\theta_x = \theta_y = 0$.

(c) $\theta_x = 0$. (d) General motion.

Fig. 5. Minimum global valleys of the error distance.

4 An Approximate Algorithm for the Fundamental Matrix

The true intervals of uncertainty of an approximate system can be found by using the necessary and sufficient conditions for admissible solutions and applying Linear Programming (LP) techniques. If one of the unknowns is specified then LP can be used to find the restricted intervals in which the remaining unknowns lie. In order to obtain the intervals of uncertainty in the unknowns of an approximate system of equations, the approximate equation problem is reduced to a standard LP form [6].

In order to satisfy the non-negativity condition, the variables are changed according to a particular orthant. The variables $f'_j \geq 0$, for $j = 1, ..., m$ are introduced such that $f'_j = f_j$ if $f_j \geq 0$ and $f'_j = -f_j$ otherwise.

The inequality conditions for admissible solutions are expressed in terms of equations by introducing *slack variables*. A constraint $\sum_{j=1}^{m} a_{ij} f'_j \leq b_i$ becomes $\sum_{j=1}^{m} (a_{ij} f'_j) + f'_{m+1} = b_i$ and a constraint $\sum_{j=1}^{m} a_{ij} f'_j \geq b_i$ becomes $\sum_{j=1}^{m} (a_{ij} f'_j) - f'_{m+1} = b_i$, where the slack variable f'_{m+1} is nonnegative. This way, $2n$ slack variables are introduced to convert the $2n$ inequalities to equalities. The $2n$ equations are multiplied by -1 when $b_i < 0$ to satisfy $b_i \geq 0$, for $i = 1, ..., n$.

Since solving the LP problem leads to the minimum value of the objective function, the minimum and maximum values of the kth component of f' is found by taking the objective function as f'_k and $-f'_k$, respectively. Therefore, the m intervals of uncertainty are obtained by solving $2m$ LP problems in each orthant. For each orthant, the set of solutions to an LP problem is a closed convex set. Hence, the solution set of an approximate system consist of the union of the convex subsets in all 2^m orthants.

An *admissible orthant* contains at least one admissible solution and an *empty orthant* contains no admissible solution. The solution set S is the union of the solution subsets in the admissible orthants. The intervals of uncertainty can be found in all 2^m orthants, but empty orthants should be avoided.

The orthant to be considered first is the one known to contain at least one admissible solution x^*. If x^* has q zero components, the search for a solution set must start in the 2^q orthants which x^* lies. If all values found for the endpoints of the orthant intervals of uncertainty are finite and none are zero, then an insular subset of the solution set exists in the orthant concerned and this insular subset is the entire solution set. If any of the endpoints are not bounded, then the system is critically ill-conditioned.

However, if q intervals of uncertainty in the orthant have endpoints equal to zero or are unbounded, then the solution may extend into other $2^q - 1$ adjacent orthants. An *adjacent orthant* corresponds to a sequence of q components each of which may be either positive or negative. The solution set in the adjacent orthants have to be investigated. If endpoints are found to be zero in any of the adjacent orthant intervals of uncertainty for components of f other than the q components in the current orthant, then further orthants must be searched.

5 Conclusions and Future Work

We proved the exact solution computation feasibility for some pure translation motions, depending only on the distribution of the discretization errors. However, a single exact solution for the fundamental matrix is not feasible for pure rotation cases. The feasibility of an exact solution is decided according to the error distance $\Delta(f, \Gamma)$.

We introduce a new robust approach for the computation of the fundamental matrix taking into account the intrinsic errors involved in the discretization process. The problem is modeled as an approximate equation system and reduced to a linear programming form. This approach is able to compute the solution set instead of trying to compute only a single vertex of the solution polyhedron as in previous approaches [4, 8].

Outliers are considered as sample point matches whose errors are much bigger than the expected uncertainty ϵ. Outlier analysis and experiments in synthetic and real images are future work. This way, we will be able to compare results obtained by our new approach with other robust techniques [3].

We intend to consider the discretization effects in other camera models, multiple view geometry (tensors), and further aspects of the model reconstruction process. This way, the quest is for a computer vision system that is able to compute the exact motion even considering the errors associated with discretization. Error effects in the camera calibration is another issue that we may consider using the same methodology developed in this paper.

The polyhedron associated with the solution set for fundamental matrices may be used as a framework to measure the hardness involved in recovering a particular rigid motion and scene. This framework may also be used as a tool in the investigation of the relation between motion and the structure of a scene.

References

1. G. Dantzig. *Linear programming and extesnsions.* Princeton University Press, Princeton, New Jersey, 1963.
2. O. Faugeras. What can be seen in three dimensions with an uncalibrated stereo rig? In G. Sandini, editor, *ECCV '92, Proceedings of the 2nd European Conference on Computer Vision*, volume 588 of *Lecture Notes in Computer Science*, pages 563–578, Santa Margherita Ligure, Italy, May 1992. Springer-Verlag.
3. M. Fischler and R. Bolles. Random sample consensus: a paradigm for model fitting with application to image analysis and automated cartography. *Communications of the ACM*, 24:381–395, 1981.
4. R. Hartley. In defence of the eight-point algorithm. *IEEE Trans. Pattern Analysis and Machine Intelligence*, 19(6):580–593, 1997.
5. R. Hartley and A. Zisserman. *Multiple view geometry in computer vision.* Cambridge University Press, June 2000.
6. I. Kuperman. *Approximate linear algebraic equations.* Van Nostrand Reinhold Company, London, 1971.
7. A. Neumaier. *Interval methods for systems of equations.* Cambridge University Press, 1990.
8. P. Torr and D. Murray. The development and comparison of robust methods for estimating the fundamental matrix. *International Journal of Computer Vision*, 24(3):271–300, 1997.