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FUZZY PLANE GEOMETRY: TRIANGLES

Azriel Rosenfeld

Computer Vision Laboratory
Center for Automation Research
University of Maryland
College Park, MD 20742-3275

Abstract

A fuzzy triangle T (with a discrete-valued membership function) can be regarded as a nest of parallel-sided triangles T_i with successively higher membership values. Such a nest is determined by its max projections on any two of its "sides". The area (perimeter) of T is a weighted sum of the areas (perimeters) of the T_i's. The side lengths and altitudes of T can also be defined as weighted sums obtained from projections; using these definitions, the perimeter of T is the sum of the side lengths, and the side lengths are related to the vertex angles by the Law of Sines, but there is no simple relationship between the area of T and the products of the side lengths and altitudes.

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1 Introduction

For any direction θ in the plane, let (x_θ, y_θ) be Cartesian coordinates with x_θ measured along θ and y_θ measured perpendicular to θ. A fuzzy subset of the plane is called a *fuzzy halfplane* in direction θ [1] if $f(x_\theta, y_\theta)$ depends only on x_θ and is a monotonically nondecreasing function of x_θ. Evidently, a level set of a fuzzy halfplane in direction θ is either the entire plane, or a halfplane bounded by a line perpendicular to θ, or empty. [We recall that the level set f_t of a fuzzy set f is the set of points at which $f \geq t$.]

Proposition 1 *A fuzzy halfplane is a fuzzy convex set.*

Proof: We recall [2] that a fuzzy subset f of the plane is called fuzzy convex if for all points P, Q, R such that Q is on the line segment PR we have $f(Q) \geq \min[f(P), f(R)]$. For any direction θ, the x_θ-coordinates $P_\theta, Q_\theta, R_\theta$ of such a collinear triple P, Q, R must satisfy either $P_\theta \leq Q_\theta \leq R_\theta$ or $P_\theta \geq Q_\theta \geq R_\theta$; hence if f is a fuzzy halfplane, $\min[f(P), f(Q), f(R)]$ must be either $f(P)$ or $f(R)$. //

Fuzzy convex polygons of various types can be defined as infs of fuzzy halfplanes [1]. Note that such polygons must be fuzzy convex sets, since an inf of fuzzy convex sets is fuzzy convex. This note will be primarily concerned with fuzzy triangles, with emphasis on the case where the membership functions are discrete-valued.

2 Fuzzy triangles

Let α, β, γ be three directions in the plane which are not all contained in a halfplane. Let f, g, h be fuzzy halfplanes in directions α, β, γ, respectively. To avoid degenerate cases, we will assume that f, g, and h are all nonconstant and all take on the value 0. Then $f \wedge g \wedge h$ is called a *fuzzy triangle.*

Proposition 2 *Any nonempty level set of $f \wedge g \wedge h$ is a triangle with its sides perpendicular to α, β, and γ.*

Proof: The nonempty level sets of f are halfplanes bounded by lines perpendicular to α, and they lie on the sides of these lines in the direction of α (i.e., the direction of nondecreasing f); and similarly for the level sets of g and h. The level sets of $f \wedge g \wedge h$ are intersections of level sets of f, g, and h; indeed, $[f \wedge g \wedge h \geq t]$ iff $[f \geq t$ and $g \geq t$ and $h \geq t]$. Since α, β, and γ are not all contained in a halfplane, an intersection of level sets of f, g, and h is either empty or a triangle. //

Let f, g, and h be discrete-valued, and suppose that $f \wedge g \wedge h$ takes on the values $0 < t_1 < \cdots < t_n \leq 1$. Then we can specify T by defining a nest of triangles T_i each of which has its sides perpendicular to α, β, and γ. On the innermost nonempty triangle T_n, T has value t_n; on the remaining part of the triangle T_{n-1} immediately surrounding T_n, T takes on value $t_{n-1}; \ldots$; on the remaining part of the outermost triangle T_1, T takes on value t_1; and its value on the rest of the plane is zero. Note that the T's can be irregularly placed, as long as they are parallel-sided and nested; and note that the T_i's must all be similar. A simple example of a fuzzy triangle, involving only the membership values 0, $\frac{1}{2}$, and 1, is shown in Figure 1; as this example shows, some of the sides of the T_i's may coincide.

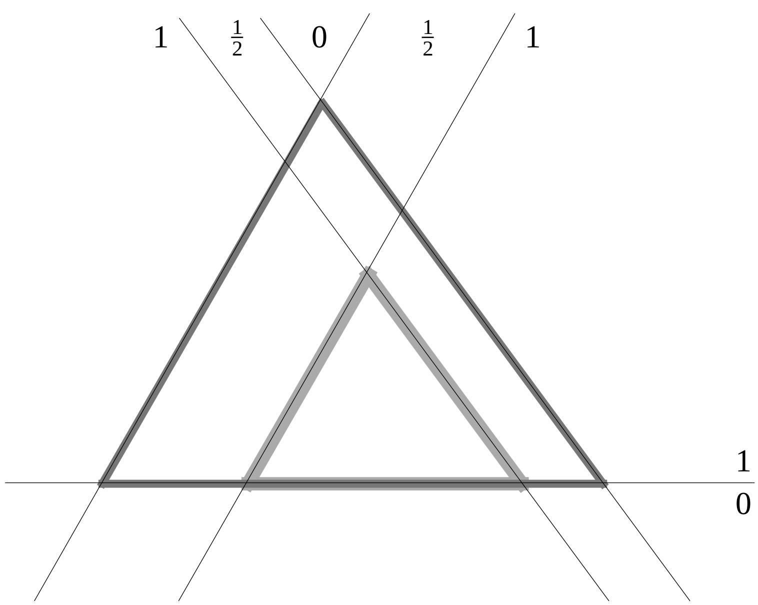

Figure 1: A simple example of a fuzzy triangle.

Projections

We recall [3; see also 4] that the sup projection of a fuzzy set f onto a line L is a fuzzy subset of L whose value at $P \in L$ is the sup of the values of f on the line perpendicular to L at P. Evidently, the projection of T onto the line L_α perpendicular to α is a "wedding cake" function whose outermost (nonzero) layer has height t_1 and length equal to the side of T_1 perpendicular to α; the successive inner layers have lengths and positions along L_α equal to the lengths and positions (in the direction along L_α) of the corresponding sides of the successive T_i's, $2 \leq i \leq n$. [Note that since sides of T_i's may coincide, some of the step "widths" of the wedding cake may be zero—in other words, some of the step heights may be differences between nonconsecutive t_i's.] Evidently, we have

Proposition 3 *A fuzzy triangle is completely determined by its sup projections on lines perpendicular to any two of the directions α, β, γ.* //

These lines are parallel to the sides of the T_i's; we can think of them as defining the "directions of the sides" of T.

3 Area, perimeter, and side lengths

Let the areas of $T_1, \ldots, T_n$ be $A_1, \ldots, A_n$, let their perimeters be $P_1, \ldots, P_n$, and let $\delta_i = t_i - t_{i-1}$ (where $t_0 \equiv 0$). Then we have

Proposition 4 *The area of T is $S = \sum_{i=1}^{n} \delta_i S_i$.* [This sum counts the area S_1 of T_1 with weight t_1, and counts the area S_i of each successive inner T_i with additional weight δ_i.] *The perimeter of T is $P = \sum_{i=1}^{n} \delta_i P_i$* [5; see also 4]. //

Let the side lengths of T_i perpendicular to α, β, and γ be a_i, b_i, an c_i, respectively; then we can define the "side lengths" of T as $a = \sum_{i=1}^{n} \delta_i a_i$, $b = \sum_{i=1}^{n} \delta_i b_i$, and $c = \sum_{i=1}^{n} \delta_i c_i$. Evidently we have $a + b + c = P$. Note that since the T_i's are parallel-sided, they all have the same vertex angles, say A, B, C; we can regard these as the "vertex angles" of T. Note that by the Law of Sines, we have for each T_i

$$\frac{a_i}{\sin A} = \frac{b_i}{\sin B} = \frac{c_i}{\sin C}.$$

If we multiply by δ_i and sum over i, this gives us

Proposition 5 $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$.

Corollary 6 *If two vertex angles of T are equal, their opposite side lengths must be equal, and conversely.*

4 Some concepts that don't generalize

Many properties of ordinary triangles do not generalize to arbitrary fuzzy triangles. For example, let the side lengths of a fuzzy right triangle be $a = \sum_{i=1}^{n} \delta_i a_i$, $b = \sum_{i=1}^{n} \delta_i b_i$, and $c = \sum_{i=1}^{n} \delta_i c_i$. Since the T_i's are all right triangles, we have $a_i^2 + b_i^2 = c_i^2$ for each i; hence we cannot (in general) have $a^2 + b^2 = c^2$. Some other generalization failures are described in the following paragraphs.

Altitudes

The projections of T onto the lines parallel to α, β, and γ are also "wedding cake" functions. In this case, the outermost step is the altitude of T_1 in the given direction. Let the altitudes of T_i in directions α, β, and γ be u_i, v_i, and w_i, respectively; then we can define the "altitudes" of T as $\sum_{i=1}^{n} \delta_i u_i$, $\sum_{i=1}^{n} \delta_i v_i$, and $\sum_{i=1}^{n} \delta_i w_i$. Unfortunately, there is no simple relationship between the area of T and the products of its side lengths and corresponding altitudes, even if we define the projections as in [4].

Perpendicular bisectors

Let P_α be the point on L_α that "bisects" the projection of T onto L_α (i.e., such that the integrals of the projections on the two half-lines terminating at P_α are equal). We call the line through P_α in direction α a *perpendicular bisector* of T; and similarly for directions β and γ. In the crisp case, the perpendicular bisectors of (the sides of) a triangle T all meet at a point which is equidistant from all three vertices of T; but this property does not hold in general for fuzzy triangles (see, however, the next paragraph).

Circumcircle and incircle

If the T_i's are placed so the centers of their circumscribed circles coincide, these circles define a fuzzy disk which we can call the circumscribed fuzzy disk of T; evidently it is the minimal fuzzy disk whose membership function is not less than that of T, and its center is equidistant from all three vertices of each T_i. Similarly, if the T_i's are placed so the centers of their inscribed circles coincide, these circles define a fuzzy disk which we can call the inscribed fuzzy disk of T; evidently it is the maximal fuzzy disk whose membership function does not exceed that of T, and its center is equidistant from all three sides of each T_i. Unfortunately, neither of these properties holds for general fuzzy triangles.

5 Concluding remarks

We have seen (at least in the discrete-valued case) that some properties of ordinary triangles (e.g., the Law of Sines) generalize to arbitrary fuzzy triangles, but that other properties generalize only to fuzzy triangles that are suitably "symmetric". It would be of interest to determine necessary conditions for the validity of these properties.

References

[1] A. Rosenfeld, Fuzzy rectangles, *Pattern Recognition Letters* **11**, 1990, 677–679.

[2] L. Zadeh, Fuzzy sets, *Information and Control* **8**, 1965, 338–353.

[3] A. Rosenfeld, The diameter of a fuzzy set, *Fuzzy Sets and Systems* **13**, 1984, 241–246.

[4] A. Bogomolny, On the perimeter and area of fuzzy sets, *Fuzzy Sets and Systems* **23**, 1987, 257–269.

[5] A. Rosenfeld and S. Haber, The perimeter of a fuzzy set, *Pattern Recognition* **18**, 1985, 125–130.