# Speed and Accuracy Tests of the Variable-Step Störmer-Cowell Integrator


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# Overview

- Background
- Integrators
- Orbit Propagation Tests
- Orbit Determination Tests
- Conclusions and Recommendations

# Background

- US Space Command is tracking over 12,000 objects in orbit.
- Analytic methods (GP) no longer meet accuracy requirements, so numerical methods are used (SP).
- Numerical methods require much more computation time.
- Planned sensor upgrades to higher-frequency radar may increase the number of tracked objects to over 100,000.
- Need to find the fastest numerical integrator for each type of orbit.

# Integration Methods

| Method | Single / Multi | Fixed / Variable | Single / Double | Non-Summed / Summed |
|---|---|---|---|---|
| Runge-Kutta | Single | Fixed | Single | NA |
| Runge-Kutta-Fehlberg | Single | Variable | Single | NA |
| Adams (non-summed) | Multi | Fixed | Single | Non-Summed |
| Summed Adams | Multi | Fixed | Single | Summed |
| Shampine-Gordon | Multi | Variable | Single | Non-Summed |
| Störmer-Cowell | Multi | Fixed | Double | Non-Summed |
| Gauss-Jackson | Multi | Fixed | Double | Summed |
| New: var. S-C | Multi | Variable | Double | Non-Summed |

# Single / Multi-Step Integrators

- Single-Step Integrators
  - Integrate using information from only the current step.
  - The number of evaluations is dependent on the order.
- Multi-Step Integrators
  - Integrate forward using information from several *backpoints*.
  - Predictor-Corrector methods, with one or two evaluations per step.
  - Cannot integrate through a discontinuity.

# Single / Double Integration

- Single Integration
  - Gives velocity from acceleration.
  - Must integrate velocity to find position.
- Double Integration
  - Gives position directly from acceleration.
  - Used with a single integration method to find velocity.
  - Reduces round-off error (Herrick).
  - More stable than single integration, less evals per step required (for multi-step methods).

# Variable-Step Integration

- Fixed-step integrators take more steps than needed at apogee.
- Variable-step integrators change the step size to control local error.
- Variable-step integrators take fewer steps per orbit for elliptical orbits, for a given accuracy.
- To be more efficient, an integrator must have fewer **evaluations** per orbit than another – evals take 90% of run-time.

# Variable-Step Methods

- Shampine-Gordon
  - Single-integration method, two evaluations per step.
  - Step size only increased when it can be doubled.
  - Method is also variable-order, and self-starting.
- var. Störmer-Cowell
  - Double-integration method, one evaluation per step.
  - Step size increased whenever possible.
  - Method is not variable-order, except for starting.

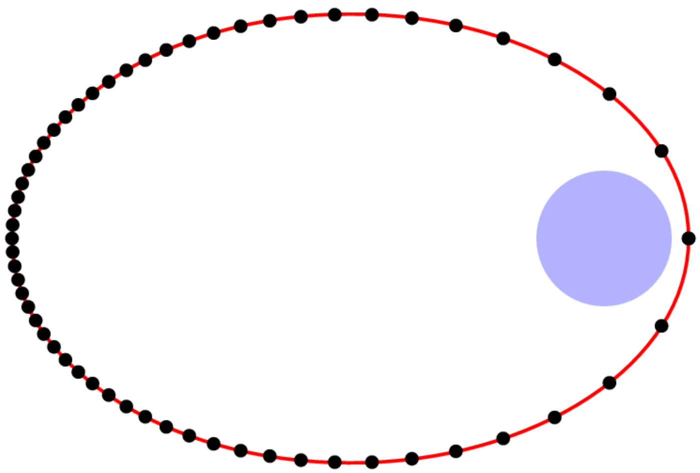
# $s$-Integration

- Another method of handling elliptical orbits is to change the independent variable from $t$ to $s$ with a Generalized Sundman transformation

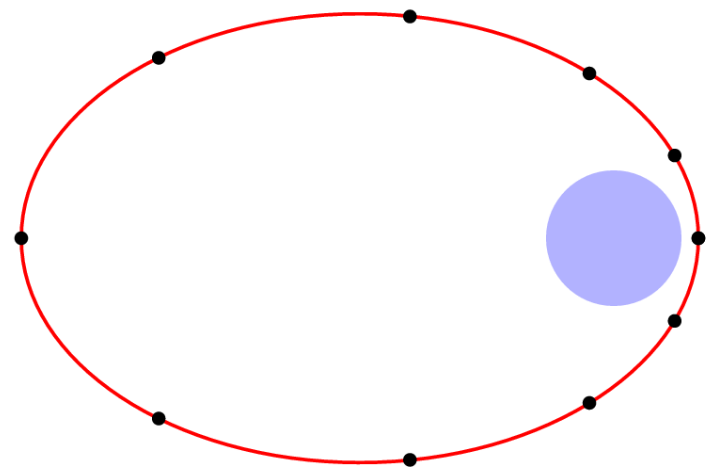
$$dt = cr^n ds$$

- Still a fixed-step method - no local error control.
- Must integrate to find time - leads to in-track error.
- Unstable with only one evaluation per step (PEC).
- Can use a PEC$\tilde{\text{E}}$C implementation - only re-evaluate two-body force on second evaluation.

# $s$-Integration



(a) $t$-integration with 58 steps.



(b) $s$-integration with 10 steps.

$$e = 0.75$$

# Integration Methods

| Method | Single / Multi | Fixed / Variable | Single / Double | Non-Summed / Summed |
|---|---|---|---|---|
| Runge-Kutta | Single | Fixed | Single | NA |
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# Testing Accuracy - Error Ratio

- Compare computed numerical integration to some reference.
- Define an error ratio (Merson):

$$\rho_r = \frac{1}{r_A N_{\text{orbits}}} \sqrt{\frac{1}{n} \sum_{i=1}^{n} (\Delta r_i)^2}$$

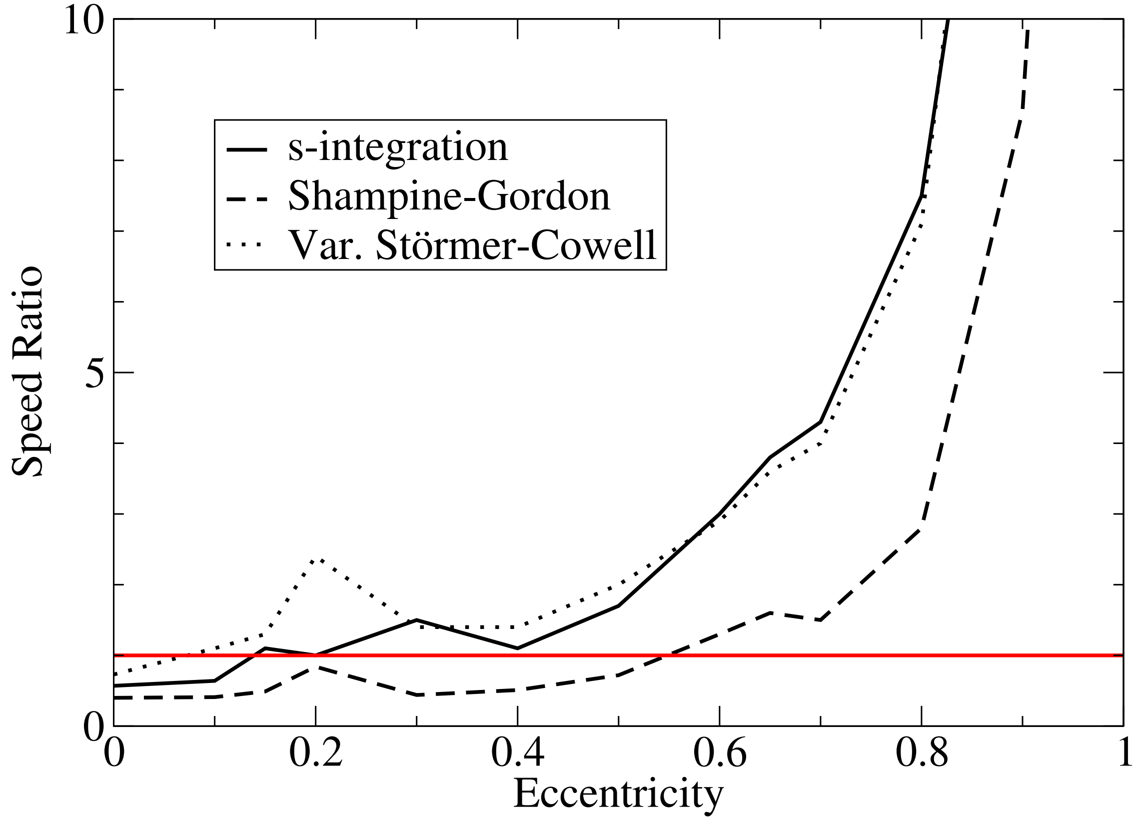
where $\Delta r = |r_{\text{computed}} - r_{\text{ref}}|$.

- Comparisons are over 3 days with and w/o perturbations.
- Perturbations include 36 $\times$ 36 WGS-84 geopotential, Jacchia 70 drag model, and lunar/solar forces.

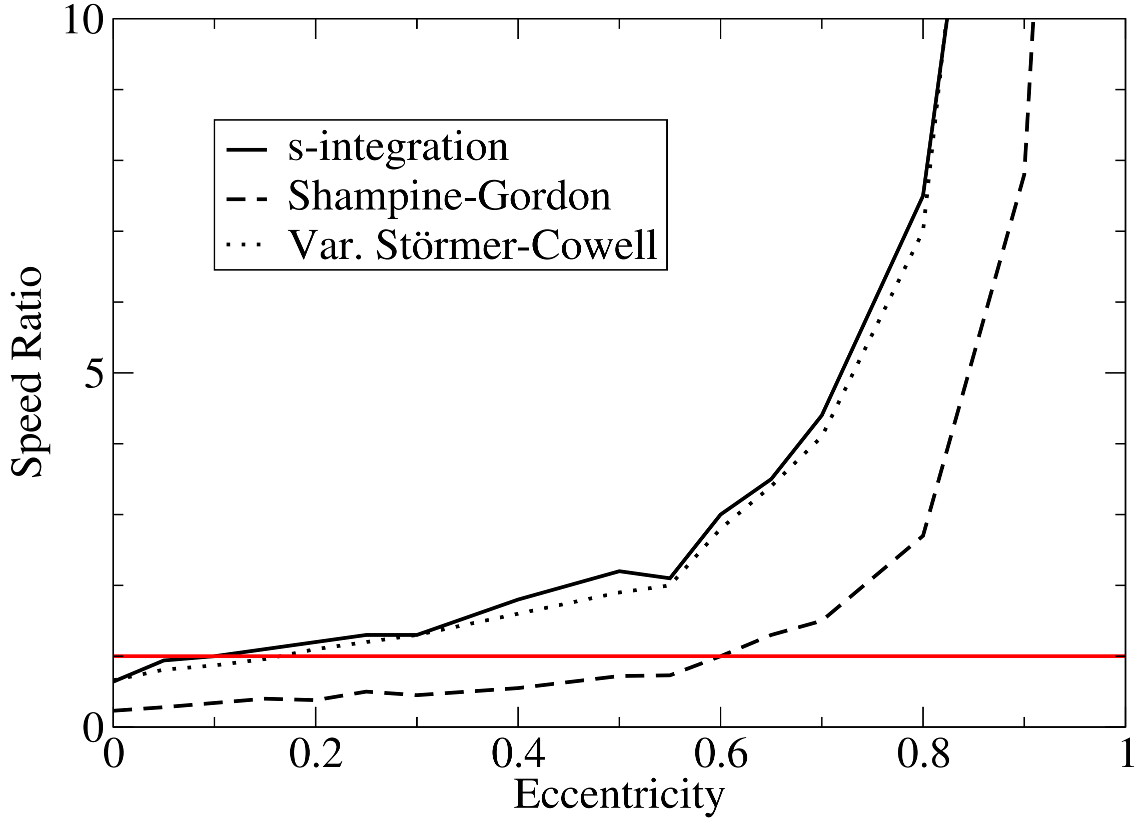
# Speed Testing

- Compare methods using speed tests with equivalent accuracy.
- Step sizes found for GJ-8 with $t$- and $s$-integration which give error ratios of $1 \times 10^{-9}$.
- Tolerance found for Shampine-Gordon and Var. Störmer-Cowell which gives an error ratio of $1 \times 10^{-9}$.
- Time found to run for 30 days with perturbations using this step size or tolerance, for various eccentricities and perigee heights.
- Speed ratio is the time of the variable-step method over the time of the fixed-step method.

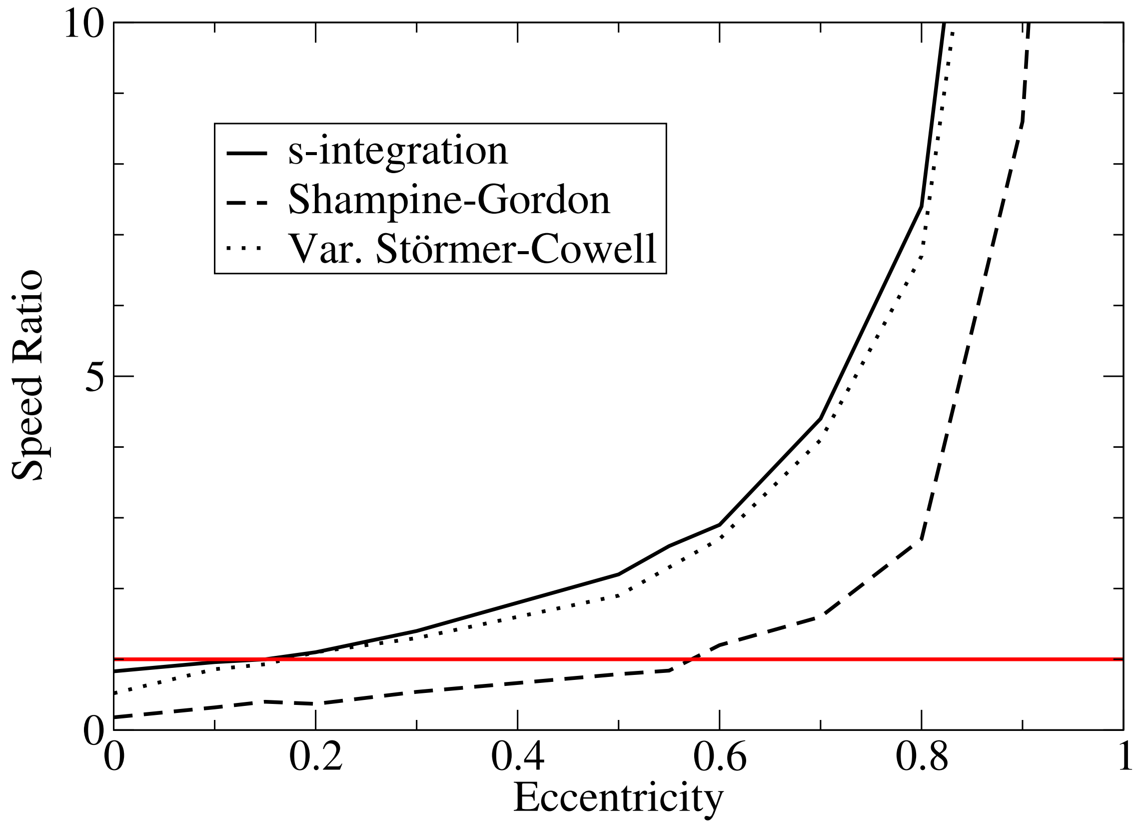
# Speed Ratios at 300 km Perigee



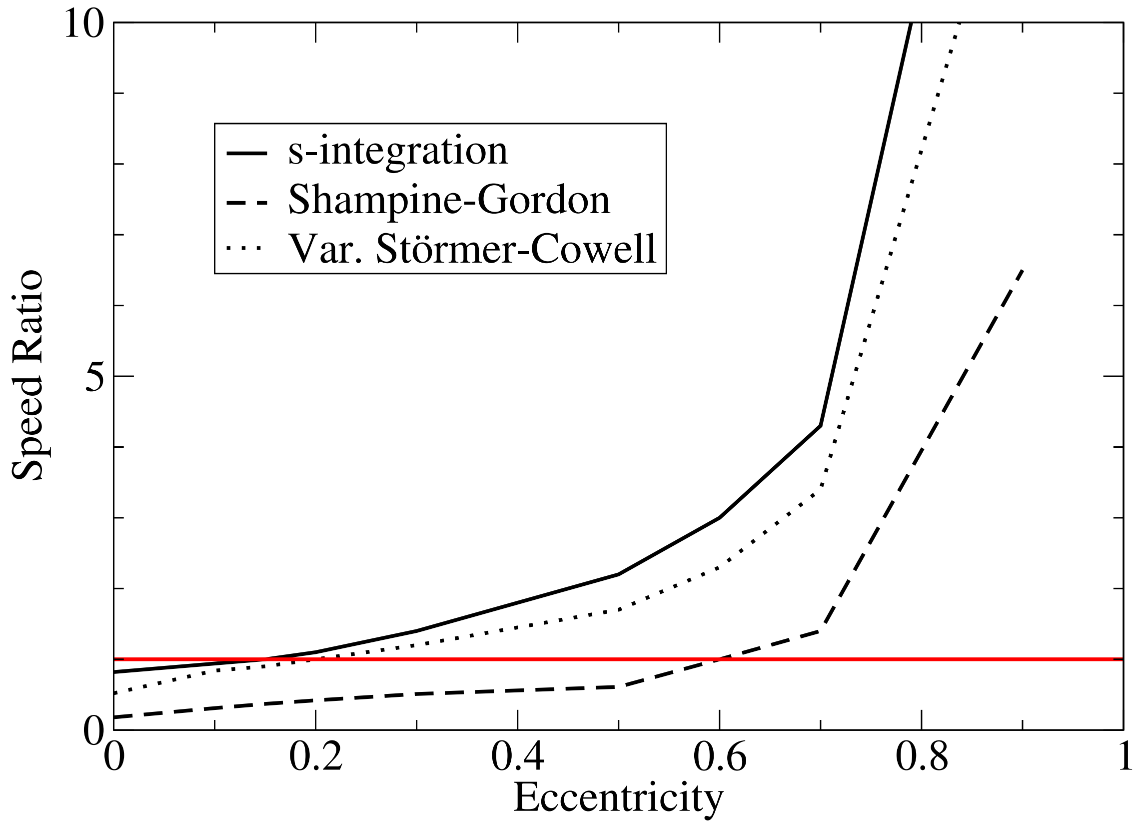
# Speed Ratios at 400 km Perigee



# Speed Ratios at 500 km Perigee



# Speed Ratios at 1000 km Perigee



# Orbit Determination Testing

- Test performed on set of cataloged objects from 1999-09-29.
- 8003 objects in catalog, 1000 randomly selected for test.
- Perform 3 tests:
  - Time all 1000 objects with GJ-8 using $t$-integration.
  - Use $t$-integration, $s$-integration, and var. Störmer-Cowell on objects with $e > 0.15$.
  - Use both $t$-integration and Shampine-Gordon on objects with $e > 0.60$.

# Orbit Determination Results

- Takes 11.2 hrs to fit 1000 objects.
- Var. Störmer-Cowell is 1.65 hours faster than $t$-integration. 14.7% improvement.
- $s$-integration has a 14.6% improvement over $t$-integration.
- Shampine-Gordon has a 7.0% improvement over $t$-integration.
- $s$-integration and Shampine-Gordon give comparable results to Gauss-Jackson.
- Var. Störmer-Cowell is more robust than Gauss-Jackson
  - Updates 3 more objects.

# Summary

- Local error control gives var. Störmer-Cowell an advantage over $s$-integration for low-perigee orbits.
- var. Störmer-Cowell is more than twice as fast as Shampine-Gordon because there are fewer restrictions on the step size.
- var. Störmer-Cowell updates more objects in OD than fixed-step methods.

# Recommendations

- A variable-step method should be used for objects with eccentricities over 0.15.
- $s$-integration can be used in regions where drag is less significant.
- var. S-C method with local error control should be used in regions with high drag.
- A study combining $s$-integration with var. S-C method could show how to improve $s$-integration results with drag.