The Generalized Sundman Transformation for Propagation of High-Eccentricity Elliptical Orbits

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Overview

- Background

- Generalized Sundman Transformation

- Implementation

- Regularization

- Accuracy / Speed Comparisons

- Conclusions

Background - Special-K

- Special perturbations catalog system used by Naval Space Command.

- Uses a Gauss-Jackson numerical integrator.

- Used to maintain approximately 1300 satellites.

- Will be used to maintain 50,000 - 100,000 objects within 10 years.

- Computation time is a critical factor.

Generalized Sundman Transformation

- Sundman (1912) developed a time transformation to attempt to solve the three body problem,

$$dt = crds,$$

  where c is a 2 body constant.

- This regularizes and linearizes the equations of motions.

- Generalized form:

$$dt = cr^n ds.$$

- $n = 1$, $c = \sqrt{a/\mu}$, s is the eccentric anomaly.

- $n = 2$, $c = 1/\sqrt{\mu a(1 - e^2)}$, s is the true anomaly.

Intermediate Anomaly, $n = 3/2$

- Merson (1975)

  - concludes that n = 3/2 equally distributes the integration error around an orbit.

  - provides speed and accuracy results for $n = 3/2$ compared to other integrators, for 2 body force only.

- Nacozy (1977)

  - Expresses s in terms of the true anomaly for $n = 3/2$, and $c = 1/\sqrt{\mu}$.

  - Calls the angle the intermediate anomaly, though it is not an *orbit angle* - one orbit is not 0 to 2π in s.

Integration Steps
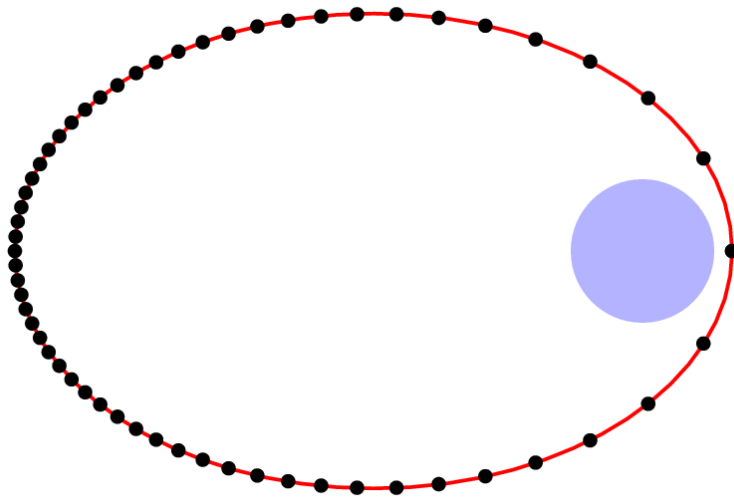
(a) Equal mean anomaly (n=0) with 58 steps.
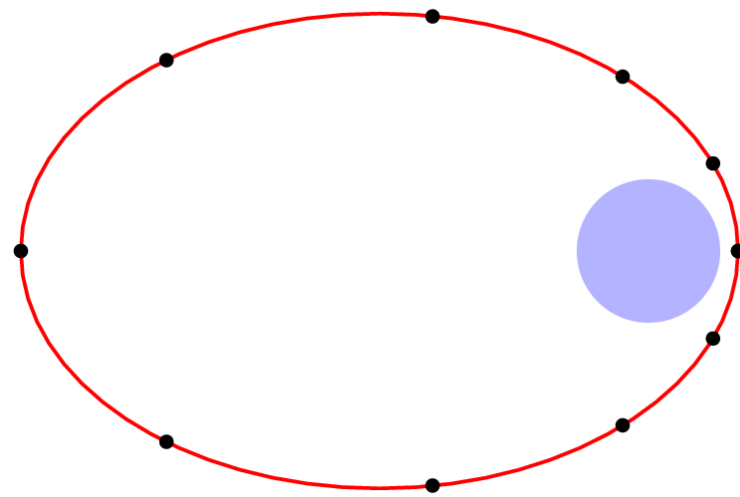
(b) Equal intermediate anomaly (n=3/2) with 10 steps.

$$e = 0.75$$

Implementation

- Generalized Sundman Transformation with $n = 3/2$ and $c = 1/\sqrt{\mu}$ implemented into Gauss-Jackson integrator.

- Known as s-integration.

- Step size set so step at perigee is same as t-integration.

- Accelerations from force model must be converted into s derivative, $\boldsymbol{r}''$,

$$\boldsymbol{r}'' = \frac{1}{\mu} \left(\frac{3}{2} r(\boldsymbol{r} \cdot \dot{\boldsymbol{r}}) \dot{\boldsymbol{r}} + r^3 \ddot{\boldsymbol{r}} \right).$$

- $\boldsymbol{r}' = dr/ds$ must be converted to $\dot{\boldsymbol{r}}$.

- Time must be found by integrating a seventh differential equation,

$$t' = \frac{1}{\sqrt{\mu}} r^{\frac{3}{2}}.$$

Regularization

- Equation of motion contains a singularity at $r = 0$,

$$\ddot{\boldsymbol{r}} + \frac{\mu}{r^3}\boldsymbol{r} = \boldsymbol{P}.$$

- Introducing $\mathcal{E}$ and $\boldsymbol{B}$ can remove the singularity,

$$\boldsymbol{r}'' = \frac{3}{\mu}\mathcal{E}r\boldsymbol{r}' + \frac{1}{2}\boldsymbol{r} - \frac{3}{2\mu}r\boldsymbol{B} + \frac{1}{\mu}r^3\boldsymbol{P}.$$

- Regularization may improve accuracy of s-integration.

- Implemented in SpecialK to test the benefit.

Test Cases

- Consider 6 orbits:

  - Eccentricities of 0.0, 0.25, and 0.75.

  - Perigee heights of 300 km and 1000 km.

  - All orbits have a $40°$ inclination, and a $0.01 \ \mathrm{m}^2/\mathrm{kg}$ ballistic coefficient.

- Test integration accuracy over 3 days with and without perturbations.

- Perturbations include 24 $\times$ 24 WGS-84 geopotential, Jacchia 70 drag model, and lunar/solar forces.

- Use a 30 sec time step in t-integration.

- s-integration time step is 30 sec at perigee.

Determining Integration Error

- Use analytic solution as the reference 2 body comparisons.

- For testing with perturbations, integrate forward, and use final value to integrate backwards.

- Define an error ratio:

$$\rho_r = \frac{1}{r_A N_{\text{orbits}}} \sqrt{\frac{1}{N} \sum_{i=1}^{N} (\Delta r_i)^2},$$

where $\Delta r = |r_{\text{computed}} - r_{\text{ref}}|$.

Accuracy Comparison - 2 Body Only

Test Case		Error Ratio		
e	h_p (km)	t	s	s-reg
0	300	8.40×10^{-17}	8.94×10^{-12}	8.94×10^{-12}
0.25	300	8.05×10^{-16}	1.47×10^{-13}	1.40×10^{-13}
0.75	300	6.76×10^{-15}	1.55×10^{-14}	1.95×10^{-14}
0	1000	7.36×10^{-17}	4.33×10^{-11}	4.33×10^{-11}
0.25	1000	8.25×10^{-17}	1.43×10^{-13}	1.45×10^{-13}
0.75	1000	1.04×10^{-15}	1.14×10^{-13}	1.07×10^{-13}

Accuracy Comparison - With Perturbations

Test Case		Error Ratio		
e	h_p (km)	t	s	s-reg
0	300	4.93×10^{-9}	1.69×10^{-8}	1.69×10^{-8}
0.25	300	4.17×10^{-10}	9.38×10^{-9}	9.31×10^{-9}
0.75	300	5.78×10^{-9}	1.61×10^{-8}	1.56×10^{-8}
0	1000	3.38×10^{-12}	1.18×10^{-9}	1.18×10^{-9}
0.25	1000	2.17×10^{-13}	2.72×10^{-10}	2.80×10^{-10}
0.75	1000	3.61×10^{-12}	1.92×10^{-10}	1.18×10^{-10}

Speed Comparison

Test Case		Time for 30 Day Run (sec)		
e	h_p (km)	t	s	s-reg
0	300	*21*	*21*	*22*
0.25	300	29	20	21
0.75	300	28	4.7	4.8
0	1000	31	31	32
0.25	1000	29	20	20
0.75	1000	28	4.6	4.7

Preliminary Conclusions

- With step sizes equal at perigee:

  - t-integration is more accurate.

  - s-integration is faster.

- Question: When is s-integration faster than t-integration if they have the same accuracy?

- Perform a new test:

  - Find the step size that gives an error ratio of 1×10^{-9}.

  - Perform 30 day speed test with this step.

Equal Error Speed Comparison - 1000 km

e	Step Size (sec)		Time for 30 Day Run (sec)		Speed Ratio
	t	s	t	s	
0	50	30	18.3	31.0	0.59
0.05	55	54	16.7	16.4	1.0
0.10	64	58	14.2	14.0	1.0
0.15	73	72	12.2	10.3	1.2
0.20	70	74	12.6	9.17	1.4
0.25	68	80	12.8	7.80	1.6
0.50	61	61	14.0	5.80	2.4
0.75	65	51	13.1	2.94	4.5

Equal Error Speed Comparison - 300 km

	Step Size (sec)		Time for 30 Day Run (sec)		
e	t	s	t	s	Speed Ratio
0	6	4	*99.6*	*153*	0.65
0.15	50	26	18.1	27.9	0.65
0.25	32	15	27.0	38.9	0.69
0.30	17	16	50.0	33.0	1.5
0.35	32	30	26.7	16.2	1.6
0.40	50	36	17.2	12.2	1.4
0.50	50	36	17.1	9.55	1.8
0.75	29	10	28.8	13.0	2.2

Conclusions

- Use s-integration for $e > 0.15$, outside drag regime.

- Use s-integration for $e > 0.30$, inside drag regime.

- Round-off error is a concern for s-integration.

- Regularization does not significantly improve the s-integration results.