ABSTRACT

Title of dissertation: THE INFLUENCE OF INSTRUCTIONAL MODEL ON THE CONCEPTUAL UNDERSTANDING OF PRESERVICE ELEMENTARY TEACHERS

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This study compared the influence of two instructional models—explicit instruction and problem-based instruction—on the procedural and conceptual understanding attained by prospective elementary teachers from a unit on place value in different number bases. In the explicit instruction the instructor included intuitive conceptual explanations. In the problem-based instruction students worked on tasks intended to elicit the students' discovery of these same conceptual understandings. Students worked in groups and then explained their insights and approaches to the whole class. The essential difference in the instructional models was who was responsible for providing explanations: the teacher or the students.

The students answered procedural and conceptual questions on an immediate post-test and on a delayed post-test, and wrote a written reflection on place value. The differences in scores on the post-tests and written reflection were not attributable to the differences in instructional model, even after using math SAT scores as a covariate. A

mild interaction was observed between treatment and math SAT score on the conceptual portion of the delayed post-test, in which students with lower math SAT scores who were in the problem-based group did somewhat better than those with lower math SAT scores in the explicit instruction group.

Seven students were interviewed to learn more about their understanding, attitudes, and persistence in problem solving. The interview analysis suggested that students' differences in understanding, attitudes, and persistence were due to their prior (pre-study) experience rather than to differences in instructional model for the study unit. The students from the problem-based group, however, were more likely to identify working in groups as a helpful learning strategy than those from the explicit instruction group.

THE INFLUENCE OF INSTRUCTIONAL MODEL ON THE CONCEPTUAL UNDERSTANDING OF PRESERVICE ELEMENTARY TEACHERS

by

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Chapter 1

Elementary school teachers have a critical role in the development of mathematical skills, understanding, and attitudes in their young students. Many have argued that to fulfill that role elementary school teachers must know mathematics well. With a good understanding of mathematics, teachers know not just the individual topic of the day's lesson but also the larger context of the mathematics they are teaching their students—how it relates to the concepts learned last year and to those the students will encounter in the future. They are able to evaluate unusual or novel approaches their students may use to solve problems, and they are better prepared to respond to students' misunderstandings. They can help their students to gain the mathematical proficiency that is essential to the students' future success in life. (Ball, 1991; Ball, 1993; CBMS, 2001; Leitzel, 1991; Ma, 1999).

Many elementary teachers take courses in college or university that are designed to prepare them to have this deep understanding of the math that is taught in elementary school. The purpose of my study is to compare two types of conceptual instruction given to preservice elementary teachers in such a course, to determine if the type of instruction influences the preservice teachers' conceptual understanding of the topic.

Rationale

Many preservice elementary teachers do not know math well (Ball, 1990a; Ball, 1990b; McClain, 2003; Simon, 1993). The mathematical understanding needed by teachers is broad and complex. It includes, but is not limited to, the procedural fluency and conceptual understanding needed by their students (Ma, 1999; Shulman, 1987).

Unfortunately, however, when preservice teachers begin their college career they often have not acquired the foundation of procedural and conceptual knowledge that would ideally be gained from their K-12 school experiences (Ball, 1990a; Ball, 1990b; Ma, 1999; Simon, 1993).

The reasons for this lack have been attributed to deep-seated and widespread weaknesses in the mathematics teaching most students receive in grades K-12. In many schools, students are taught an "underachieving curriculum" that treats many topics briefly; students never get an in-depth understanding because inadequate time is spent exploring any one idea (McKnight, et al., 1987). In addition, much of the teaching focuses on procedural practice and skills rather than on the underlying concepts that support these procedures (Stigler & Hiebert, 1997).

Thus, it seems likely that the type of teaching prevalent in the K-12 school systems in the United States contributes to the lack of conceptual understanding these preservice teachers evidence. Unfortunately without some intervention to break this cycle of procedurally-focused instruction leading to rule-bound knowledge they will most likely teach in the same way they were taught, and the next generation of children in the United States will continue to gain little conceptual understanding in mathematics.

Most preservice teachers take both math content and math methods courses in college. Therefore efforts to intervene have focused considerable attention on the way these courses could be improved (CBMS, 2001; Leitzel, 1991). One set of recommendations addresses content needs: these preservice teachers must return to the elementary mathematics they know in a limited way and re-examine it in order to develop meaning for the symbols and algorithms. They also need to make connections between these ideas and ideas in algebra and geometry so they will understand the broader context of the mathematics they teach (CBMS, 2001).

Recommendations have also been made regarding the types of instructional experiences these college classrooms should offer in order to accomplish these learning goals. Some call for the priority of problem-solving experiences, "classroom experiences in which [preservice teachers'] ideas for solving problems are elicited and taken seriously, their sound reasoning affirmed, and their missteps challenged in ways that help them make sense of their errors" (CBMS, 2001, p. 17).

Others envision a traditional teacher-centered presentation focusing on the mathematical arguments and content that a professional mathematician judges to be most relevant to the content of elementary school mathematics (Beckmann, 2005; Wu, 1997). This "top-down" approach seeks to provide a broad framework based on how mathematicians think about their field with the goal of helping students organize their understanding into a cohesive, connected body of knowledge. Unfortunately, however, this type of "professional explicit instruction" can be so far removed from the experience of preservice teachers that they may not grasp the connections nor appreciate the framework provided. Moreover it is unlikely to translate easily into explanations that could be offered to elementary-school-aged students. For example, Beckmann (2005, pp. 71-72) cites the example of forming equivalent fractions. One justification for stating that two fractions are equivalent, such as $3/5 = 12/20$, is to appeal to multiplication by 1: $3/5 \cdot 4/4 = 12/20$. From a professional viewpoint this explanation might be preferred because it fits neatly into the larger framework of viewing the rational numbers as a mathematical field. However, she notes that this requires the student to understand the conceptually complex topic of fraction multiplication.

Others take a more moderate view, envisioning instructor-provided explanations that are thoughtfully crafted to connect to preservice teachers' experiences along with opportunities for them to participate and interact (Lester, 1988). These explanations are often linked to concrete materials or diagrams that ground the reasoning in practical experience. For example, a more intuitive justification that 3/5 = 12/20 would involve subdividing a diagram into more same-size pieces: initially 3 of 5 pieces are shaded, but after each piece is subdivided into fourths there will be 12 of 20 pieces shaded (Beckmann, 2005, p. 71). This "intuitive explicit instruction" is also closer to the experiences and explanations that preservice teachers may someday offer to their young students.

These positions are part of a much larger discussion about mathematical understanding and how it is achieved. The notions of procedural knowledge and conceptual understanding have been analyzed and researched for many years, their interrelationship probed both theoretically and empirically (Hiebert & Lefevre, 1986). In more recent years, the discussion has centered on becoming "mathematically proficient," a goal that encompasses procedural and conceptual understanding along with additional aspects such as productive attitudes and strategies (NRC, 2001). Despite decades of research, it remains unclear *how* to ensure all students achieve mathematical proficiency (Kilpatrick, 1992).

Procedural knowledge and conceptual understanding have a particularly complex relationship. The desired outcome is often described as a rich network of mental connections related to each math concept, some connections involving procedures, and others related concepts (Hiebert & Carpenter, 1992; Ma, 1999). Ideally, of course, students learn both "how" and "why," achieving a practical yet flexible understanding of mathematics that has been termed "relational understanding" (Skemp, 1987). Many studies with elementary school students have found that procedural knowledge and conceptual understanding were indeed highly correlated (Siegler, 2003)—students more commonly had both or neither. Research with preservice teachers, however, has indicated that they often possess only procedural competence without understanding the conceptual basis of the algorithms they can perform (Ball, 1990a; Ball, 1990b; Ma, 1999; McClain, 2003; Simon, 1993).

Students who achieve competence in both procedural and conceptual understanding also seem to differ in the order in which they acquired this knowledge. Cases have been documented of both orders of acquisition: procedural first followed by conceptual (Hiebert & Wearne, 1996; Resnick, 1986b; Ross, 2001) as well as conceptual first followed by procedural (Kamii, Lewis, & Livingston, 1993; Pesek & Kirschner, 2000; Wearne & Hiebert, 1988). However, studies suggest that instruction linking both skills and concepts is more effective than instruction on skills alone (Fuson & Briars, 1990; Hiebert & Wearne, 1996). In fact, some results indicate that students who master a rote skill first may have difficulty later attaching meaning to the procedure (Mack, 1990; Pesek & Kirschner, 2000; Resnick & Omanson, 1987; Wearne & Hiebert, 1988).

Even accepting that conceptual knowledge is best taught before or concurrently with procedural knowledge leaves critical questions unanswered, however. In particular, what type of teaching helps students attain this conceptual knowledge? How is this rich network of mental connections forged by the student? Must it be through a lengthy, possibly meandering, process of conjecturing, testing, discussing, rethinking, and eventually linking ideas? Is it possible the process can be made more efficient by the teacher explicitly explaining the connections to the student? Or will this lead to the student simply memorizing the connections as yet another isolated bit of information? (Hiebert & Carpenter, 1992)

Unfortunately, this comparison—between intuitive conceptually based explanations and problem-based student-centered instruction—has not been adequately researched. At the elementary school level, a number of studies have compared one of these approaches to procedurally-focused instruction (Fuson & Briars, 1990; Kamii, Lewis, & Livingston, 1993; Pesek & Kirshner, 2000). In these cases, the conceptually-focused initiative resulted in better learning by the students. However, in these cases the explicit instruction used in the traditional classes did not attempt to offer students a conceptual basis for their learning.

Similarly, at the postsecondary level, studies have compared small-group learning with traditional, professionally rigorous, lectures ("professional explicit instruction") (Daves, 2002; Seymour, 2002; Springer, Donovan, & Stanne, 1999). In these studies, the small-group learning promoted achievement and improved attitudes. However, small-group learning was not compared with an instructor-focused class offering intuitive explanations ("intuitive explicit instruction"). Since explicit instruction of the needed conceptual links using intuitive explanations generally takes less time than student discovery of these links through problem-solving experiences, it would be useful to know if the students learn the concepts as thoroughly from this more "efficient" process.

Statement of the Problem

The purpose of my study was to examine the results of two types of conceptual instruction in the particular context of a unit on number bases in the curriculum for preservice elementary teachers. One group of students was taught in a teacher-centered manner, but with a focus on providing clear, intuitive explanations. The other group of students was taught using a student-centered, problem-based approach, where the students worked in groups to develop solution methods for the problems posed. A variety of measures were used to attempt to gain insight into the understanding achieved by the students—responses to exam questions (both conceptual and procedural), a short reflection paper on the meaning of place value, and interviews with a selected subset of the students from each treatment.

Research Questions

I investigated the skills and understanding the students gained in a number of areas—not only their skill translating between bases but also their ability to articulate the meaning of place value in general and to identify how systems with place value differ from counting systems that do not use place value. Since coming to understand our place value number system is a major task of children in the early years of elementary school, this content goal is in keeping with the recommendations that have been made for preservice elementary teachers' math learning during their college education (CBMS, 2001).

In particular, I examined the following:

1. Did either instructional approach result in better skill in translating from base ten to other bases, and from other bases to base ten?

2. Did either instructional approach result in better retention of how to translate between base ten and other bases (as evaluated on the final exam, approximately two months later)?

3. Did either instructional approach result in better ability to make conceptual connections, such as evaluating whether a given counting system exhibits place value or extending the meaning of place value to fractional place values?

4. Did either instructional approach result in better retention of the ability to make conceptual connections (as evaluated on the final exam, approximately two months later)?

5. Did either instructional approach result in better ability to articulate what place value means, using examples and non-examples to illustrate the essential differences between place-value systems and non-place-value systems?

6. Did either instructional approach result in greater ability to explain their thinking about how to solve problems involving number bases?

7. Did either instructional approach result in students' having better problem-solving skills in approaching problems involving number bases?

Overview of Research Design

My study examined the results of two types of conceptual instruction in the particular context of a unit on number bases in the curriculum for preservice elementary teachers. I selected this topic for my study because I believed it would be new to nearly all, if not all, of the students. Thus, it would be less likely that a student's achievement on the post-test measures would be influenced by earlier experiences with this topic.

To facilitate the comparison, I used a quasi-experimental approach. Each treatment group consisted of several intact sections of the same course, a math content course for preservice elementary teachers. After the unit, I compared the scores of each group of students on an initial unit test, a delayed final exam (occurring about two months later), and a written paper. In addition I interviewed a subset of students from each group to listen to their thinking processes as they solved problems related to number bases.

A recently published study by Pesek and Kirshner (2000) served as a model for my plan, although it is a contrast between somewhat different constructs. In that study, fifth graders who had not yet learned the area and perimeter formulas for various shapes were separated into two groups for instruction. One group received five days of instruction on the formulas with no attempt to relate the formulas to the diagrams or to the students' intuition. This was followed by three days of conceptual development. A second group received only the conceptual treatment; the teacher gave no explicit instruction on formulas.

On a posttest measure of both skill-based and conceptual items the second (conceptual-only) group scored slightly higher, although the difference was not significant at the 0.05 level. Interviews with several of the children, however, gave evidence that the students in the first group, who had first practiced the formulas, failed to develop robust notions of area and perimeter during the conceptual instruction. For example, five of the six children did not recognize that when painting a room one must determine the area, not the perimeter, of the walls to be painted. Thus, the interviews provided some evidence for the claim that rote procedural learning can hinder a student's later attempt to develop conceptual understanding.

My study contrasted somewhat different instructional methods. One group was taught via a teacher-centered approach using "intuitive conceptual explicit instruction": in addition to explicitly teaching the algorithms, the connections between our number system and other base systems were also emphasized, using diagrams of base ten blocks and base five blocks as visual aids. The other group experienced a student-centered problem-based approach: the instructor introduced a task related to number bases, and then students worked in groups to formulate a solution to the task. Following the group work the instructor led a whole-class discussion that included students sharing insights and summarizing remarks from the instructor. In general, the first group in my study listened to the *instructor* explain the meaning and connections, while the other group engaged in tasks and discussion designed to help the *students* articulate the meaning and connections.

One caveat emerged in my pilot tests: I found the lecture method went more quickly. Thus the group receiving explicit instruction was taught for 3 – 4 class sessions, whereas the group developing their own procedures worked on the unit for 4 – 5 class sessions.

Significance

As described above there is little known about the relative merits of explicit intuitive instruction on conceptual principles versus student-centered problem solving.

Some research studies have compared conceptual explicit instruction to procedural explicit instruction (Fuson & Briars, 1990; Good, Grouws, & Ebmeier, 1983); other studies have contrasted problem-based instruction with traditional procedural instruction (Kamii, Lewis, & Livingston, 1993; Springer, Donovan, & Stanne, 1999).

In postsecondary settings, even fewer comparative studies have been done. Although cooperative groups have been found to result in better than learning than traditional professional lectures (Daves, 2002; Seymour, 2002; Springer, Donovan, & Stanne, 1999), little is known about how students' learning from intuitive explanations compares to "discovered" understanding.

Additionally, in some cases claims are made about the effectiveness or lack of effectiveness of certain curriculum programs without detailed knowledge of how the curriculum was implemented (Hiebert, 1999). There is evidence, however, that the way in which a curriculum is implemented can have an impact on its effectiveness (Huntley, Rasmussen, Villarubi, Sangtong, & Fey, 2000; Schoen, Cebulla, Finn, & Fi, 2003).

This research study has the potential to provide information about these areas where data is scant. The effectiveness of carefully constructed conceptual explanations will be compared with that of student-generated insights, this contrast will be examined in the context of college classes, and the implementation will be monitored closely to ensure consistency and fidelity to the written plans.

Limitations

I did some of the teaching, nearly all of the grading, and all of the interviewing in this study. To help avoid bias I was careful during the grading and interview processes to remove any information that would identify which group the paper or student came from. In addition, the writing assignment was graded independently by three people and the sum used as the student's score.

My personal experiences undoubtedly impacted my perceptions and interpretations in this study. In my prior experience as a teacher, using both of the types of teaching included in the study, students seemed to vary in the extent to which they engaged, questioned, and reflected. Many students become more engaged when given a challenging problem and an opportunity to discuss their thinking with their classmates. However, other students participated more readily in a teacher-led explanation, giving evidence that they were following and even anticipating the development of the mathematical concepts. I had also observed that some students disengaged when given a group task; other students seemed to listen passively when I did all (or most of) the talking.

As a student I typically attempted to actively follow a mathematical lecture. When a question was asked, I was frequently ready to offer a thought. I also enjoyed the challenge of a problem, but I sometimes found it difficult and distracting to "think out loud" with my classmates. I generally preferred to think about the problem on my own at first. Thus, from the perspective of both teaching and learning I had seen the potential for benefit or frustration in both types of instruction. These experiences predisposed me to believe there is a complex relationship between the characteristics of the student, teacher, and task that all may influence what is "best" for learning.

Discussion of Terms Used

The following terms are used frequently throughout this paper, so an expanded explanation of the sense in which I am using them is given here.

• Procedural understanding or knowledge: Procedural proficiency involves the ability to apply known algorithms efficiently and accurately, such as being able to quickly and correctly multiply two multi-digit numbers.

• Conceptual understanding or knowledge: Conceptual understanding involves the ability to relate a mathematical idea to other ideas, to other settings, and to examples. Mental networks of connections linking mathematical ideas are a hallmark of conceptual understanding.

• Relational understanding: A blend of procedural and conceptual understanding. The student not only knows what ideas and calculations are relevant but is also proficient in carrying out the necessary computations. Commonly accepted as the overall goal of mathematics instruction, although the specific content areas in which students should achieve this competence is less-agreed-upon.

• Explicit instruction: The teacher takes on the responsibility of providing a clear explanation to the student. This explanation may take many forms—it may provide only a procedure to follow, a mathematically rigorous proof of a fact or procedure, or an intuitive explanation linking the math procedure or idea to something the teacher believes to be in the experiential knowledge of the student.

• Conceptual explicit instruction: The teacher takes on the responsibility of providing a clear explanation that the teacher believes will be "linkable" to prior knowledge for the student. For example, the teacher may explain that the division problem 12 ÷ 1/2 can be thought of as asking the question, "If I have 12 pounds of M&Ms, how many 1/2-pound bags can I make?"

Note: Explicit instruction may also make use of cooperative groups for more constrained tasks, such as practice problems using a method explained by the teacher.

• Student-centered problem solving: The teacher selects and defines the problem so that the students are clear about the goal. The students then take on the responsibility of determining how to solve the problem, how to explain what they have done, and why they believe their approach is correct. The teacher guides the students' thinking primarily by asking questions, but may also suggest an alternate way of thinking if students are stuck. The teacher also guides whole-class discussion after the groups have worked on their methods in order to have the students explain their thinking to each other and allow the students to see various approaches to the problem. This type of instruction is often called "reform," "Standards-based," "constructivist," or "inquiry-based."

Ideally, students in mathematics classes will not merely memorize algorithms but will link meaning with the various mathematical symbols and processes they encounter. What is less clear is what range of instructional experiences may support this development of meaningful understanding, and with what effectiveness. In this study I investigated whether the type of instruction experienced by these preservice elementary teachers seemed to influence them to probe for understanding rather than simply master the procedures needed to answer the homework questions.

Chapter 2

What constitutes effective mathematics teaching has been a central theme of discussions for decades, and continues to be a source of substantial disagreement. Some of the conflict reflects different goals for mathematics learning; some involves differing experiences of teaching that did or did not achieve those goals.

Learning Goals

Mathematical proficiency or expertise is characterized differently by different observers of and participants in the enterprise of education. To many parents and teachers, students primarily need procedural skill, or what the National Research Council (NRC) terms "procedural fluency" (2001, p. 116)–the ability to carry out computations quickly and accurately. Mathematicians and mathematics educators would quickly add that this alone is insufficient: students also need conceptual understanding linked to these skills that enables them to understand what situations are modeled by the computation, why the computation procedure "works," and how it might be modified to work in a slightly different situation (Hiebert & Lefevre, 1986; Skemp, 1987; Wu, 1999).

To these abilities, researchers in mathematics education have recently added additional desired outcomes. These have been termed (NRC, 2001, p. 116) "strategic competence" (the ability to solve problems), "adaptive reasoning" (the ability to think logically, explain, and justify), and "productive disposition" (the belief that mathematics is useful, logical, and understandable). These additional outcomes are often associated with a view that equates learning mathematics with becoming participants in a discourse community of mathematicians, albeit in a limited way (Ball, 1993; Sfard, 1998). In this model, or "participation metaphor" (Sfard, 1998), students are not seen as acquiring knowledge but as becoming better skilled at *doing* mathematics—conjecturing, explaining, justifying, and critiquing in the context of solving mathematical problems.

This contrasts with the traditional view that learning results in the learner possessing some knowledge or skill he or she did not have before, the "acquisition metaphor" (Sfard, 1998). Prior to the introduction of the participation metaphor, the acquisition metaphor was the only one available. This eventually led to dissatisfaction for many who recognized that thinking of knowledge as a possession is inadequate to describe the process of carrying out mathematical thinking. An analogy might be describing what it means to be a skilled carpenter. Not everyone who owns a set of wood-working tools is equally proficient in their use; the ability to use these tools is an important part of what it means to be a carpenter.

Sometimes the participation metaphor is used to the exclusion of the acquisition metaphor. This can lead to logical difficulties, however. For example, if the learner does not "have" some kind of knowledge or skill, how is it possible that sometimes people do successfully transfer knowledge practiced in one setting to a new setting (Sfard, 1998)? Sfard points out that it is helpful to have both ways of thinking about learning, and recommends they be viewed as complementary rather than competing descriptions.

My project will include several of the aspects of learning described above. My primary focus is on the relative effectiveness of two instructional models in helping students achieve conceptual understanding of number bases. Conceptual understanding is intricately intertwined with procedural ability, however (Resnick & Omanson, 1987; Rittle-Johnson & Alibali, 1999; Sfard, 2003; Siegler, 2003), so my assessments will include questions addressing each of these types of knowledge. In addition, the written reflection assessment will give students an opportunity to explain how place value systems are constructed and contrast them with non-place-value systems, an element of adaptive reasoning. Finally, the interviews will include an opportunity for students to extend their reasoning about place value to a new representation and novel tasks, which are elements of strategic competence.

Conceptual Understanding

A variety of ways of thinking about conceptual understanding have been proposed. Several researchers use the notion of "relationships" as a key idea (Hiebert, 1990; Hiebert & Carpenter, 1992; Ma, 1999; Skemp, 1987). In fact, Hiebert and Lefevre (1986) state that it is impossible for a conceptual insight to exist in isolation—by definition it is conceptual knowledge only if it exists in relationship(s) with other pieces of knowledge.

Hiebert and Lefevre (1986) also note that these relationships can be constructed in two distinct ways. Previously known pieces of information can suddenly be recognized as related, causing a reorganization of one's existing knowledge. Alternately, new information can be linked to previous information, a process that involves assimilating the new knowledge into an existing network or structure.

Some researchers (Hiebert, 1990; Hiebert & Carpenter, 1992) describe understanding in mathematics as seeing connections *between* or *within* representations. Mathematical ideas can be represented in various ways--with words, concrete objects, pictures, an actual situation, or written symbols (Lesh, Post, & Behr, 1987). A firstgrader who recognizes that combining three pennies and two nickels into one set of five coins can be written as $3 + 2 = 5$, is making a connection *between* representations, connecting pictures with written symbols. In a later grade, students may use base ten blocks as a model for decimal fractions. Recognizing that each block is 1/10 of the next-larger block can allow them to predict how the model could be continued into smaller and smaller pieces--a connection *within* the representation (Hiebert, 1990).

Skemp (1987) uses the term "instrumental understanding" to refer to rote procedural skill, or "rules without reasons" (p. 153). He notes that this is the sense in which many students and teachers in school use the term "understand": if a student is able to correctly use the formula $A = lw$ to find the area of a rectangle, both the student and teacher may conclude that the student "understands." "Relational understanding," on the other hand, includes "knowing both what to do and why" (p. 153). A person with a relational understanding of area, for example, knows that finding the area amounts to counting the number of square units inside a figure; the formula $A = lw$ is simply an efficient way to count how many squares fill a rectangle. Were the shape to be varied, a person with relational understanding could adjust their approach to find the area, or a reasonable estimate, of the new shape (Pesek & Kirshner, 2000). Thus, having relational understanding confers the ability to improvise or devise an alternate approach as needed. Skemp compares having instrumental understanding with knowing only one set of directions to get from your present location to a desired location; having relational understanding, however, is like having a "mental map" (p. 162) of the area that enables one to get to his or her destination by a variety of routes.

Ma (1999) describes the conceptual knowledge of the Chinese teachers in her study as composed of structured "knowledge packages" (p. 15) of related concepts. For example, the "knowledge package" she inferred from discussions of how to teach a two-digit subtraction-with-regrouping problem included two key concepts the teachers repeatedly emphasized: "addition and subtraction within 20" and "composing and decomposing a higher value unit"; additional concepts linked to these included "the composition of 10," "addition and subtraction as inverse operations," and "the composition of numbers within 100" (pp. 17 - 19). The Chinese teachers saw a single problem as illustrating a whole complex set of concepts, not simply one concept.

In contrast to the conceptual knowledge described above (Hiebert, 1990; Hiebert & Carpenter, 1992; Skemp, 1987), these knowledge packages were particularly structured to be useful for teaching. The concepts organized by these teachers tended to be very finely differentiated and sequenced (distinguishing, for example, "addition and subtraction within 10" from "additional and subtraction within 20"). The basic characteristic of connectedness and relationships, however, is common to all three of these descriptions of conceptual knowledge.

Procedural Knowledge

Procedural knowledge consists of two parts: the symbol system of mathematics, and the algorithms for carrying out certain tasks (Hiebert & Lefevre, 1986). Knowing the symbol system includes recognizing what forms are "syntactically" permitted, according to the "grammar" rules of mathematics. For example, $3x + 5$ is permissible; $3^{2=}$ is not.

Knowing the algorithms involves being able to follow sequential actions to accomplish tasks.

The algorithms associated with mathematical ideas are further subdivided. Some algorithms are performed on symbols; others relate to manipulatives or diagrams or other objects. For example, young children may use counters to solve simple addition problems. The symbolic algorithms, however, are the ones most commonly associated with school mathematics—the conventional procedures done with paper-and-pencil that record tasks such as multidigit subtraction or long division.

Procedural knowledge exhibits connections of a sort, but of only very particular types (Hiebert & Lefevre, 1986). One relationship is hierarchical, where one procedure can include subprocedures. For example, multiplying 1.58 x 2.3 is usually performed by using the multiplication procedure for whole numbers followed by a special procedure for determining where to place the decimal point. The second relationship is the sequential nature of the steps within a procedure. Knowing a procedure does not require a global sense of what the procedure accomplishes; the person doing the procedure only needs to know what action to perform at each stage to get to the next state. For example, a student in elementary calculus may successfully complete a homework problem by differentiating a polynomial function using the power rule, adjoining "= 0," and solving for x, yet fail to recognize that he or she has found the point(s) where the function has a horizontal tangent line. Thus, procedural knowledge is qualitatively different from conceptual knowledge: procedural knowledge has a linear, sequential structure, while conceptual knowledge includes relationships of many types (Hiebert & Lefevre, 1986).

Procedural skill has many benefits--speed and efficiency are gained, and the automatic nature of the responses frees the student's memory for other, more creative or reflective thinking. Thus a student will not be distracted or slowed in the process of a more difficult problem by an inefficient or cognitively taxing algorithm (Hiebert, 1990; Skemp, 1987, p. 62). However, if procedural knowledge is learned separately from conceptual knowledge, it remains limited to the particular context(s) in which it has been specifically applied; it does not transfer to other contexts that share deeper common features (Hatano, 1988; Hiebert & Lefevre, 1986). The procedure or fact is stored in isolation by the learner, rather than as part of a network of related information. The student has routine expertise, but not adaptive expertise.

Procedural knowledge can also be known in a meaningful way, however, when it is linked to conceptual knowledge. In this case the two types of knowledge each increase the usefulness and power of the other: procedures "translate the conceptual knowledge into something observable" (Hiebert & Lefevre, 1986, p.9) and provide objects and actions for reflective thought (Sfard, 2003); conceptual understanding gives insight that can be used to select, recall, adapt, or even generate procedures as needed (Hiebert & Carpenter, 1992).

Mathematical Symbols and Symbolic Procedures

Historically, mathematics has gained power as it has distilled ideas and procedures into succinct forms. Symbols enable a concept to be named and discussed, such as dy/dx for "the derivative of y with respect to x" (Hiebert & Lefevre, 1986; Skemp, 1987). Complex ideas compressed into symbols can be written and manipulated with much greater facility (Hiebert & Lefevre). For example, the ancient Greeks stated the Pythagorean theorem as "the sum of the squares on the legs of a right triangle are equal in area to the square on the hypotenuse." Today we write $a^2 + b^2 = c^2$, and solving for an unknown value is a straightforward procedure accessible to many middle school students. Well-designed representations systems can even give rise to new conceptual insights (Hiebert & Lefevre, 1986; Skemp, 1987). For example, place value notation laid a foundation that made possible the invention of logarithms (Katz, 1998, p. 418).

Historically, procedures developed as conceptual knowledge was laboriously applied to solve problems (Hiebert & Lefevre, 1986). Gradually the processes of tasks such as multi-digit multiplication were routinized and made much more efficient. Thus many routine procedures condense a substantial amount of conceptual understanding. Once developed, these symbolic procedures are sometimes divorced from their original settings. These symbolic systems can then be analyzed syntactically, by the logical rules that govern these transformations, rather than semantically, by the external meanings they can represent (Goldin, 2003; Hiebert & Lefevre, 1986; Resnick, 1986a; Resnick & Omanson, 1987).

Unfortunately, in school mathematics the syntactic symbolic manipulations have often become the exclusive focus of instruction--without adequate connection to the semantic meanings the symbols can represent (Stevenson & Stigler, 1992; Stigler & Hiebert, 1997). There is widespread evidence that many students fail to take into account the meaning of the symbols they manipulate, even in the face of unreasonable answers. By the second grade, it is common for children to develop "buggy" algorithms with errors in the procedures for multi-digit subtraction (Brown & Burton, 1978; Resnick &

Omanson, 1987). Similarly, it is common for students learning to add fractions to simply add the numerators and the denominators separately (Rittle-Johnson & Siegler, 1998), even when that results in a "total" that is smaller than the number they started with (e.g., claiming 1/2 + 1/3 = 2/5 even though 2/5 is less than 1/2).

Resnick (1986a) notes that in mathematics there is a unique tendency to neglect "linking symbols to their referents"(p. 145) that does not arise when students learn to read or write. In both mathematics and natural language students are expected to attend to both the semantics (meaning) and the syntax (grammar). In natural language students do not routinely ignore the meaning of the words in the course of reading or writing, but in mathematics it is common for teaching and learning to focus almost exclusively on the formal rules, as though mathematical expressions are "nothing but strings of syntactically well-formed symbols" (Resnick, 1986a, p. 145).

Even when the teacher or text presents conceptual meanings in the course of instruction, students may not attend to these contexts and meanings (Perry, 1991). Skemp (1987) theorizes that students settle for the instrumental understanding of syntactic rules because of the difficulties that accompany two systems being taught simultaneously: a conceptual system and a symbol system. If the student does not firmly grasp the concepts to which the symbols relate he or she begins to think exclusively in terms of the system of symbols.

Students show evidence of a deeply ingrained motivation to seek order and meaning, however (Sfard, 2003). If the student conceives of mathematics as a strictly symbolic system, the student begins to look for and generalize on symbolic patterns that may bear incorrect relationships to the underlying conceptual structure (Skemp, 1987, pp.

187-188). Resnick (1986a) cites the consistency of incorrect rules for multi-digit subtraction such as "subtract the smaller from the larger" (Brown & Burton, 1978) and common algebra fallacies (such as $(a + b)^2 = a^2 + b^2$) as evidence that students are seeking to generalize logically, but on the basis of purely procedural knowledge without linking the symbols to meaningful referents. Similarly, an analysis of students' errors in interpreting decimal fractions revealed that students' errors appeared to be systematic generalizations from more familiar number systems—from whole numbers or fractions (Resnick, et al., 1989).

Paradoxically, mathematics has developed as a discipline by moving away from the initial referents for symbols and focusing on extending rules and domains syntactically, according to explicit axioms suggested by but not dictated by the initial referential meanings. Historically this process has been arduous at times: negative numbers were not accepted as "legitimate" numbers for centuries in Europe. Eventually the rules for multiplying two negative numbers or evaluating negative exponents were constructed in order to make prior rules continue to work, not because of any meaningful external context (Goldin, 2003). Thus, as students pursue the study of mathematics they will need to reenact this struggle to leave behind the referents when the need arises. Nonetheless, research suggests that students experience greater success when they begin their experience of mathematics by linking symbols with representations of meaningful situations.

The Complex Relationship Between Conceptual and Procedural Understanding

Professional mathematicians and math educators agree that a blend of both conceptual and procedural understanding is essential to "understanding" or knowing mathematics (Hiebert & Carpenter, 1992; Hiebert & Lefevre, 1986; Ma, 1999; Skemp, 1987; Wu, 1999). Despite this agreement on the goal, however, much remains to be understood about how the development of conceptual and procedural knowledge are related: Does one type commonly precede the other? Do they develop in tandem? Does the development of one ever occur independently of the other? Does the development of one support the development of the other? Does the development of one interfere with the development of the other? Complicating the answers to these questions is the evidence that different people experience this road to understanding differently, so there may be several valid ways to answer.

Learning concepts before procedures has many advantages. Conceptual understanding in mathematics involves connections and relationships; these can provide referents for the symbols used in mathematics so that symbol manipulation is carried out in meaningful ways to obtain reasonable answers. This understanding might enable students to invent or adapt procedures for new situations (Hiebert & Lefevre, 1986; Hiebert & Wearne, 1996). Understanding may also make it easier for a student to adopt a procedure they see demonstrated, since the student would have a way to connect the new information to prior knowledge (Hiebert & Wearne, 1996). By linking procedures and concepts in a meaningful coherent network, students might retain the procedural knowledge better or be able to regenerate it when needed (Hiebert & Carpenter, 1992; Hiebert & Lefevre, 1986). Understanding the meanings that symbols represent may also prevent students from constructing incorrect procedures (Cauley, 1988; Hiebert & Lefevre, 1986; Rittle-Johnson & Alibali, 1999). Finally, conceptual understanding can enable students to see similarities in problems that are structurally similar even when superficially different, enabling transfer to occur (Hiebert & Lefevre, 1986). Thus, in theory it seems preferable to develop conceptual understanding prior to procedural skill.

Sfard (2003), however, points out that this may not always be possible. She theorizes that the relationship between procedural skill and conceptual understanding is circular, based on an iterative process of actions followed by reflection followed by new actions. Paradoxically, "meaningfulness can arise only from using a concept but at the same time is prerequisite to successful use of that concept" (p. 359). Ideally this inherent "unstable equilibrium" will cause a student to practice yet press for meaning so that he or she can gradually grasp the concept and subsequently apply it to new situations. Sfard therefore advocates that students be taught skills via "reflective practice," viewing practice and understanding as inexorably linked. She states that students must be taught that some practice may need to precede understanding, that "persistent doing in situations of only partial understanding is necessary for progress" (p. 365).

A similar goal of joint construction of meaning and skill influences an instructional project described by Yackel, Cobb, Wood, Wheatley, and Merkel (1990). Young children were not given procedures to follow, but were asked to figure out a procedure to do a challenging problem, such as 39 + 53. Initially the students only recognized these quantities as 39 "ones" and 53 "ones." By using procedures such as counting out objects to represent the quantities, they had actions on which to reflect. Eventually students were guided to recognize the importance of groups of tens in our number system, and students began to make use of these groupings in their counting strategies. As the students repeatedly made representations for quantities in tens and ones they gradually internalized the connection between the addition process and its effect on the groups of tens and ones. By requiring that the students explain their methods for addition using mathematical reasoning, students' knowledge of place value increased concurrently with their construction of increasingly efficient procedures.

Evidence exists that some mathematics students do coordinate procedural actions with a reflective mental attitude that results in conceptual understanding. Resnick (1986b) related anecdotal evidence of successful math learners who "suspended" their need for meaning temporarily when first practicing a new procedure, but continued to reflect and expected to understand eventually. Some students even seem to carry this out without encouragement from the type of instruction or the teacher. Rittle-Johnson and Alibali (1999) carried out a study in which they found that some elementary school students seemed to derive the concept of what the equals symbol represents from a purely procedural explanation. These students were shown a problem like 3 + 5 + 9 = ____ + 9 and taught that they could group the 3 and 5 to get 8, the correct number to put into the blank. Some of the students who received this explanation were able to improve their ability to solve different but conceptually similar problems; for example, they could determine what to put in the blank for problems like 3 + 5 + 9 = ___ + 10. Rittle-Johnson and Alibali (1999) hypothesized that these students reflected about *why* the procedure was correct rather than simply performing it.

Hiebert (1990) agreed that reflecting on procedures can be valuable, and encouraged teachers to lead classroom discussions to focus students' attention on patterns that can be perceived within the symbol system. He cautioned, however, that this reflective practice is distinctly different from practice that focuses on automaticity and speed. Conceptual understanding is not likely to be activated in the context of speed-focused goals, because the slower, more reflective cognitive processes required for conceptual understanding would interfere with efficiency (Hiebert, 1990).

Resnick and Omanson (1987) concurred with this, citing evidence from their attempt to reteach students who had mis-learned the multi-digit subtraction algorithm. Even though they were successful in helping students understand the concept of regrouping in the context of using base ten blocks, the students reverted to their incorrect algorithms when they resumed using paper and pencil. They hypothesized that students who learn a procedural skill at the automatic level have a very difficult time returning to the reflective state where conceptual understanding can inform and constrain the procedure (Resnick & Omanson, 1987).

Thus, ideally conceptual understanding should develop with and inform the development of procedural skill whenever possible. When this is not possible, the procedures should be practiced reflectively, in a spirit of continuing meaning-seeking. Speed and automaticity should not be sought until conceptual understanding is well in hand. What actually happens in practice, however, may not be completely congruent with this ideal.

Rittle-Johnson and Siegler (1998) reviewed a number of studies investigating in what order children and students seem to actually develop mathematical skill and conceptual understanding. They began by identifying four possible order relationships:

1. Procedural knowledge develops before conceptual knowledge.

2. Procedural knowledge develops after conceptual knowledge.

3. Procedural and conceptual knowledge develop concurrently.

4. Procedural and conceptual knowledge develop iteratively, with small increases in one leading to small increases in the other, which trigger new increases in the first (p. 77).

They noted, however, that nearly all prior studies had investigated only the first two of these four possibilities, largely due to the difficulties in assessing the last two.

The earliest mathematical competency that has been studied in these terms is the counting of preschoolers. The conceptual principles believed to underlie correct use of counting have been named by Gelman and Gallistel (1978) as the one-one principle (each object is touched once), the stable order principle (the counting words must be used in the same order every time), the cardinality principle (the last counting word tells "how many"), the abstraction principle (objects of different types can be counted as one set), and the order irrelevance principle (the objects can be touched in any order without affecting the count). Despite Gelman and Gallistel's (1978) conclusions that children as young as 2 1/2 understand these principles, later researchers using somewhat different tasks reported that many 4- and 5- year-olds show limited understanding of the principles despite being able to count skillfully (Baroody, 1984; Baroody, 1992; Briars & Siegler, 1984). Thus Rittle-Johnson and Siegler (1998) concluded that children generally acquire the procedural skill of counting before they understand the conceptual principles identified above.

In the very earliest school mathematics instruction, conceptual understanding seems to naturally precede procedural skill. Resnick (1986b) studied the development of children's mathematical intuition in early elementary school. She found evidence that the additive composition of quantity functions as a "cognitive primitive" for children, an idea they understand without instruction. Resnick's analysis of the strategies used by children to solve problems involving addition and subtraction indicated many children intuitively grasp the notion that larger numbers are composed of smaller ones additively. Baroody and Gannon (1984) and Cowan and Renton (1996) found a similar outcome when studying kindergartners' understanding of the commutative property of addition and their use of it to solve single-digit addition problems (e.g., solving 2 + 5 by counting "5, 6, 7"): most students understand the concept of commutativity before beginning to use this procedure.

Rittle-Johnson & Siegler (1998) noted conceptual understanding also preceded procedural skill in the case of fraction addition (Byrnes & Wasik, 1991) and in a particular instance of proportional reasoning (determining the temperature of water obtained by combining two containers with differing amounts of water at different temperatures) (Dixon & Moore, 1996). In fraction multiplication students performed the procedure correctly even when they did not succeed on conceptual tasks, but this may be due to their "inventing" a procedure by treating the numerators and denominators as whole numbers. This interpretation is supported by the observation that a similar procedure is often used by children for fraction addition (Rittle-Johnson & Siegler, 1998).

The situation is more complicated with studies involving students' understanding of and performance on multidigit addition and subtraction, however. The algorithms for multidigit arithmetic depend on the concept of place value, a topic that typically is difficult for children to learn (Cauley, 1988; Resnick & Omanson, 1987; Rittle-Johnson

& Siegler, 1998). Studies reviewed by Fuson (1990) indicate that many American elementary school children demonstrate neither conceptual understanding nor procedural skill in multidigit arithmetic. In studies of Korean children (Fuson & Kwon, 1992) and Japanese and Chinese children (Stevenson & Stigler, 1992), nearly all the students possessed both types of knowledge. This suggests that these two types of understanding are highly correlated (Rittle-Johnson & Siegler, 1998), but does not yield insight into which type of understanding, if either, occurs first. Some evidence suggests, in fact, that the order may vary for different students (Hiebert & Wearne, 1996).

In a longitudinal study by Hiebert and Wearne (1996), 70 children in two different instructional treatments were followed from first grade to the beginning of fourth grade as they learned about place value and multidigit arithmetic. They were assessed frequently on both conceptual and procedural items to determine the order in which conceptual and procedural knowledge appeared. Two-thirds of the students demonstrated conceptual understanding either before or concurrently with using a correct procedure; over one-third of the children, however, used a correct procedure before demonstrating conceptual understanding.

When subdivided according to the type of classroom instruction experienced, differences emerge: in the conventional, procedure-focused classroom, 54% of the students demonstrated procedural skill before conceptual understanding; in the alternative instructional environment, where conceptual understanding was encouraged, only 28% of the students used the procedure before attaining conceptual understanding. Thus, it seems that alternative instructional strategies can influence more students to develop conceptual understanding first, but that some students may still develop procedural abilities first.

Students who had this pattern of later-understanding did not seem to be at a disadvantage in the long-term: "early understanding was not essential as long as it developed eventually" (Hiebert & Wearne, 1996, p. 276). However, consistent with the theoretical advantages of prior conceptual knowledge stated at the beginning of this section, using invented procedures was highly correlated with initial conceptual understanding; students who did not develop conceptual understanding until later tended to adopt algorithms demonstrated by others (Hiebert & Wearne, 1996).

In summary, Rittle-Johnson and Siegler (1998) suggested the order of attainment of conceptual and procedural understanding is affected by the environment of the child. Concepts will usually develop first if the child has frequent experience with the concept in his or her environment (proportional reasoning with water at different temperatures; alternative instruction on multidigit arithmetic) or does not see demonstrations of the procedure (adding by starting with the larger number); procedures will usually develop first if the child sees the procedure demonstrated frequently before he or she has developed an understanding of the key concepts (counting or conventional instruction in multidigit arithmetic) or if the procedure is analogous to a known procedure and can be derived from it (fraction multiplication). Variations occur for individual students, however.

Recommendations for Practice

Researchers have considered what characteristics seem to be most successful in introducing students to a new mathematical concept. Many have recommended a new idea be introduced first with concrete models of the concept accompanied by verbal explanations (Fuson & Briars, 1990; Hiebert, 1990; Skemp, 1987). Hiebert (1990) asserted that students benefit most from instruction that encourages them to begin by building relationships *between* the symbols and other representations before working within the symbol system. Support for this position is provided by Resnick (1986b), who found evidence that children who are successful mathematics learners routinely link the symbols they are taught in school to "reference situations" that provide meaning for the symbols, while less successful mathematics learners tend to disassociate the symbols from the quantities and operations they represent. As another example, Hiebert (1990) noted that students who form solid connections between the symbols for decimal fractions and the base ten block representation can even invent meaningful procedures for operations based on the blocks: for example, addition must proceed by adding "like" place values.

Once the reference situations, or concrete representations, are well understood then symbolic notation can be introduced. However, it is very important that the notation connect to the students' conceptual structures (Hiebert & Carpenter, 1992). Lengthy and idiosyncratic notation may be preferable if it is meaningful to the students. Gradually, through discussion and the desire for greater efficiency, the students can be led to the use of conventional notation (Skemp, 1987, p. 188). Researchers have cautioned that this groundwork is essential before students are encouraged to routinize procedures (Resnick & Omanson, 1987; Wearne & Hiebert, 1988).

Teaching methods vary, however, in how explicitly the students are instructed on the steps and symbols of conventional procedures. For example, in "mapping instruction" (Resnick, 1983; Resnick & Omanson, 1987) the student was taken step-by-step through the procedure of making "fair trades" with the blocks, and guided to notate the steps as they occur with the standard algorithm. In other studies (Hiebert & Wearne, 1996; Kamii, Lewis, & Livingston, 1993; Wearne & Hiebert, 1988) students were guided to develop meaningful referents for symbols, then asked to invent their own procedures to solve problems they were given.

In addition to carefully planning the development of the mathematics content, the teacher must ensure the values of the classroom culture encourage conceptual understanding. Hatano (1988) pointed out the need to attend to more than cognitive accessibility when considering what type of classroom will encourage students' conceptual understanding: students must be motivated to seek understanding, since it requires substantial effort. He noted that certain conditions seem to encourage the development of only "routine expertise," or procedural skills without accompanying conceptual knowledge: situations in which the same procedure can be used repeatedly to answer the questions posed, where speed is valued, where correct answers are valued by explanations are not needed, and where external rewards are the prime focus. He contrasted this with the characteristics of a setting that can encourage conceptual development: when unusual questions are posed regularly that require nonroutine approaches, when understanding is valued over efficiency, when there is no urgent need for external rewards, and when one must engage in dialogue about the topic being investigated. Under these circumstances one is much more likely to develop the flexible, concept-rich understanding he terms "adaptive expertise" (Hatano, 1988).

Manipulatives. Manipulative materials such as base ten blocks are frequently recommended as an instructional aid for students in elementary school. Having concrete objects to operate on can serve as a source of insights and a center of discussion (Hiebert & Carpenter, 1992). However, the mere presence of the manipulatives is not a guarantee that students will gain the understanding desired (Clements & McMillen, 1996; Ball 1992). The manipulatives must be used in ways that support students' meaningful observations.

First, the students must be guided to recognize the important mathematical relationships embodied in the objects. For example, students need to become familiar with the 10:1 ratio of the sizes of base ten blocks. Second, teachers need to guard against being overly prescriptive in how they use the blocks; students can substitute rote learning of movements with blocks for true insight and understanding (Clements & McMillen, 1996).

Finally, it is important to build relationships *between* the objects and other representations, particularly between the objects and symbolic notation (Hiebert, 1990; Hiebert & Carpenter, 1992). This process may be taken in multiple steps, moving from representations that are conceptually "close" to the contextual situation to representations that are "contextually more distant" (Hiebert & Carpenter, 1992, p. 71). For example, students may represent a quantity first using base ten blocks, in which the physical sizes of the blocks correspond to the quantity pictured, then on an abacus, in which the beads are all the same size but different columns represent different values, and finally to a written place-value numeral.

Thus, introducing manipulatives into instruction can be worthwhile, if the manipulatives are used in ways that encourage students to focus, reflect, and connect significant mathematical ideas both within and between representations.

Cooperative learning. Small groups can be used for a variety of learning tasks. In many reform-style classes small groups are the setting in which students discuss nonroutine problems that will motivate concepts the teacher wishes to emphasize (Hiebert, et al., 1997). This was not considered an essential ingredient of a discovery-focused classroom, however; type of classroom organization was specifically identified as optional by the researchers (Hiebert, et al., 1997, p. 165). On the other hand, cooperative learning may be used for other types of tasks besides problem-solving—for example, discussing concepts, collecting data, or practicing skills (Davidson & Kroll, 1991). Although these two types of classrooms (problem-focused and small-group-organized) overlap, they are not equivalent, nor does the use of one imply the presence of the other.

Early research indicated that the use of small group structures in mathematics classes may or may not improve student achievement; only rarely, however, was traditional instruction significantly better than small group work (Davidson & Kroll, 1991). Some evidence exists that the efficacy of small group work in mathematics is related to the type of task: for practicing skills, small group work might not be better than traditional teaching, but for problem-solving tasks cooperative methods may be significantly better (Dees, 1991). Slavin (1991) found that the reward structure used in conjunction with the cooperative task was also important: in order to be effective in enhancing achievement cooperative learning needed to reward groups on the basis of each individual group member's achievement. In this way it seemed more likely all would take seriously the understanding of each individual in the group (Slavin, 1991).

Some research has focused on renegotiating the classroom norms to make helping behavior a baseline expectation, part of the classroom culture, rather than dependent on external rewards or imposed structures (Yackel, Cobb, & Wood, 1991). Specifically, Yackel, Cobb, and Wood identify and describe how a second-grade teacher in their research guided the students to act in accordance with helpful expectations: students worked together to solve problems and achieved consensus (ideally on both method and answer, but if that was not possible, at least on the answer); students had a responsibility to explain their answer to their partner; students had a responsibility to make sense of their partner's thinking; it was more important to be thinking meaningfully than to just "get" the right answer; persistence in figuring out one challenging problem was more important than completing a lot of problems (Yackel et al., 1991). Rather than just presenting these expectations as a list of rules to follow, the teacher reinforced her expectations by how she interacted with the groups and how she used spontaneous occurrences in group work as examples of problematic or successful learning interactions.

Webb (1991) reviewed a number of studies in order to glean information about more specific characteristics of group participation that might relate to achievement, and about characteristics of group composition that might relate to how the students participated in the group. When a student gave elaborated content-related explanations, he or she tended to have higher achievement (even after controlling for prior achievement). Conversely, receiving explanations did not seem to be related to achievement, and receiving only superficial information (e.g. the correct answer to a problem without any explanation) was negatively correlated with achievement. In terms of group composition and its effect on participation, Webb found that in groups with a wide range of abilities the high-achieving students tended to focus on explaining the content to the low-achieving students, and the medium-achieving students were left out of the conversation. However, when medium-achieving students were grouped in a narrow-range group they were active participants, giving and receiving explanations. Not surprisingly, students in this type of group also showed higher achievement than medium-achieving students in a wide-range group (Webb, 1991).

In homogenous high-achieving groups, students tended not to give explanations to each other, and consequently did not achieve as well as high-achieving students in mixed-ability groups. In homogenous low-achieving groups students also tended not to give explanations; low-achieving students in mixed-ability groups achieved better than low-achieving students in homogenous groups (Webb, 1991). In summary, active participation, particularly as an explanation-giver, seemed to be very important for student learning.

Yackel et al. (1991) described in richer detail some of the ways that students' dialog helped the students to increase their understanding. Using examples drawn from video tapes of the classroom, they illustrated how students could use their partner's comments to think about a problem in a new way, and how a student could extend their own thinking in the process of trying to explain an error to his or her partner. Their examples illustrated in more detail how the process of "giving elaborated explanations" may lead to new conceptualizations of the problem and hence greater understanding for the student giving the explanation.

One dilemma seems to arise in the context of using cooperative learning groups to solve more complex mathematical tasks, a difficulty that has not been researched as thoroughly. Sometimes the act of talking to another seems to interfere with the process of figuring out the problem (Kieran, 2001; Sfard, 2003). Sfard (2003) noted that difficult mathematical problems require extreme concentration and intellectual effort. Communication is also effort-ful, and may distract from the problem-solver's ability to focus on understanding the problem. Kieran (2001) hypothesized that this may have been one reason why some pairs of 13-year-old algebra students in a research study had interactions in which only one member achieved insight into the problem, despite the students' talk. At one point she noted that the student who seemed to be making progress understanding the problem "spoke quietly [rather than to his partner], as if he did not want to lose his train of thought" (p. 217). At another point she noted that this student seemed to be "distracted by [his partner's] question, at the very moment that he was trying to grab hold of a newly emerging idea" (p. 217). Unfortunately, this student did not return to answer his partner's confusion after he had figured out how to think clearly about the problem. The norms for using cooperative learning in the situation of problems that require intensive thought and effort may need to be modified to include individual reflection, perhaps in substantial quantities.

Problem-Based Instruction

A number of researchers have analyzed the characteristics of a variety of successful recent projects in order to glean information about the essential common elements that seem to be responsible for students' learning with understanding (Carpenter & Lehrer, 1999; Hiebert, et al., 1997; Stevenson & Stigler, 1992). All of these projects included problem-based instruction as a key element. Critical features were discerned in several aspects of the classroom experience.

The tasks assigned to the students were very important. The tasks were true problems that did not have an immediate solution path, not routine exercises (Carpenter & Lehrer, 1999). The problems were challenging yet accessible and focused the students' attention on important mathematical ideas (Hiebert, et al., 1997). The goal of the tasks was understanding, not completion (Carpenter & Lehrer, 1999). A successful lesson usually contained a small set of substantial problems with ample time allowed for students to think about the problem and work toward a solution (Stevenson & Stigler, 1992).

In all of these successful programs extensive class time was dedicated to students' explaining, justifying, and critiquing original solution methods. The social norms of the classroom included the expectation that alternative strategies and conceptions would be discussed, and that explanations were required (Carpenter & Lehrer, 1999). Mathematical explanations (in the words and symbols of the students), rather than the teacher's approval, was viewed as the warrant for acceptance of a method (Hiebert, et al., 1997).

In addition, tools, such as manipulatives, were used in meaningful and purposeful ways (Hiebert, et al., 1997). The tools were a means to solve problems and explain solutions (Carpenter & Lehrer, 1999). Discussion helped students connect the tools with symbolic procedures or the mathematical ideas they were intended to represent (Carpenter & Lehrer, 1999).

Important aspects of the teacher's role included selecting and sequencing tasks, sharing information (but without prescribing a solution method), focusing students on solution methods, helping students articulate their ideas, and respecting students' ideas and insights (Hiebert, et al., 1997). The teacher used questions to help students value reflection and articulate ideas: he or she asked what process the student was using, why that process works, or how two different solution strategies compare (Carpenter & Lehrer, 1999). In the studies reviewed by Hiebert, et al. (1997) no teacher demonstrated a solution method, but it was acknowledged that this may have been due to these studies' all being situated in elementary school classrooms. It was noted that in higher grades the teacher may wish to demonstrate a procedure as one possible solution approach; in no case, however, was it considered appropriate for the teacher to prescribe a solution method as the only acceptable approach (Hiebert, et al., 1997, p. 164).

Whole Class Discourse

As is evident from the descriptions above, problem-based classrooms that successfully supported meaningful learning characteristically included a substantial amount of time in which students were talking. In addition to the talking that occurred in small groups, whole class discussion was a feature of nearly all reform classrooms.

When used in conjunction with small groups, whole class discussion gives an opportunity for solutions developed in the small group setting to be explained and justified to the whole class, and for classmates to ask questions and critique the method presented. When the entire lesson is taught via a whole-class discussion the topic may be a problem that students are seeing for the first time or one they have thought about individually (e.g. as homework).

Many researchers have spent time analyzing, reflecting on, and elaborating the particular characteristics of classroom discussion that seem to encourage mathematical learning. A rich source of inspiration has been the "zig-zag" of mathematical discourse in the professional community as new ideas are explored, challenged, and clarified (Lampert, 1990), particularly as this process has been articulated by Lakatos (1976) and Polya (1973).

In order for this talk to be mathematically worthwhile, both the teacher and students must learn new and quite different roles (Lampert, 1990). In the traditional classroom, student talk is usually limited to brief responses to teacher questions, often recall questions about the procedure being demonstrated. The teacher is considered the source of new information and the sole authority for determining whether an answer is correct or not (Silver & Smith, 1996). In reform classrooms, however, students and teacher share the responsibility to determine correctness based on mathematical warrants (Yackel, Cobb, Wood, Wheatley, et al., 1990).

The teacher continues to have an important leadership role in the class, however. First, the teacher is the one who establishes an atmosphere that supports meaningful discussions and learning (Yackel, Cobb, Wood, Wheatley, et al., 1990). He or she needs to respect each student's thinking and probe for the meaning the student sees in his or her approach. The teacher encourages this approach in student-to-student interactions, as well. He or she may need to explicitly state that no disrespect or ridicule will be tolerated (Silver & Smith, 1996). He or she may need to direct students' attention to trying to understand each other's reasoning. These norms help clarify that the students need to listen respectfully and carefully and try to understand each other's thinking.

Second, the teacher needs to direct students to evaluate the correctness of their own or another student's approach based on mathematical reasons. Sometimes the source of an error becomes apparent as the student attempts to explain or justify what he or she did (Yackel, Cobb, Wood, Wheatley, et al., 1990). At other times the teacher may need to ask probing questions or present a counter-example to expose a contradiction implied by the student's approach. Thirdly, the teacher guides the direction of the discussion (Clements & Battista, 1990; Yackel, Cobb, Wood, Wheatley, et al.). He or she may help students clarify their explanations, or help them think about the situation in an alternate way. The teacher also draws the students' attention to important mathematical ideas by "subtly highlighting" selected ideas (Yackel, Cobb, Wood, Wheatley, et al.).

One research project enacted these characteristics in an eighth grade classroom by naming three specific strategies and guiding the students to make use of these: "explain," "build," and "go beyond" (Sherin, Mendez, & Louis, 2000, p. 189). The teacher encouraged students' explanations by pressing for more details when they gave answers: "Can you explain why?"; "So, how does it curve?" The directive to "build" reflected a commitment to help students listen carefully to each other's explanations and respond to them, a key characteristic of a learning community. The teacher encouraged this by following a student explanation with a question to the class: "So what do people think about that?" Students quickly adopted a type of response beginning, "I agree because . . . " or "I disagree because . . . " (Sherin et al., 2000, p. 192). The final strategy, "go beyond," was intended to encourage generalizations beyond the specifics of the particular problem. The students occasionally used this strategy; more often the teacher would use this as an opportunity to introduce a relevant mathematical concept into the discussion for students to make use of in their thinking and responding.

Challenges attend the attempt to put this type of teaching into practice. In addition to the need to develop a respectful environment, both teachers and students need to focus on true conceptual understanding; a discussion in which explanations are justified by simply naming or stating the steps in the procedure that was used has drifted from the goal of using mathematical concepts to explain and justify procedures (Silver & Smith, 1996). Choosing good problem tasks that do not have straightforward procedural approaches will help to achieve this goal. This, however, is much easier said than done. In first grade, asking students to add 39 + 53 may count as a problem. In fifth grade, however, the problem Lampert (1990) chose to stimulate students' thinking about exponent relationships was the following: "Figure out the last digit in 5^4, 6^4, and 7^4 without multiplying" (Lampert, 1990, p. 39). Finding or creating rich tasks such as this one is a substantial challenge for the teacher.

Another challenge relates to leading the class discourse. When the discussion includes multiple conflicting opinions, stated and elaborated, confusion can result (Ball, 1993). Ball struggled with the dilemma of allowing her students to persist in their effort to resolve conflicting opinions about the answer to $6 + -6$. However, she felt that simply telling them that the answer is zero would be unhelpful—not only would they not work out for themselves why this is true, but they also might internalize a belief that they *could* not have figured it out for themselves.

Sometimes students arrive at misconceptions when they are encouraged to consider arguments for differing opinions. In a study asking students to analyze a problem involving the relative speeds of two objects moving in a rotating pattern, some students regressed from an initial correct belief (the outer portion of the object is moving faster in a linear sense) to a misbelief (the entire object is moving at the same speed) based on arguments given by their peers (Levin & Druyan, 1993). Clearly, the teacher must at times intervene with challenges to incorrect conclusions and redirect students to a more accurate conceptualization. Sfard (2003) echoes this conviction, noting that it is particularly important at junctures where students must overcome prior beliefs in order to progress to the next level. This commonly occurs in school with the introduction of new types of numbers, such as rationals or negatives. In these cases, prior patterns and expectations no longer hold: for example, multiplying by a fraction less than one results in a smaller product, contradicting a student's experience in the realm of whole numbers.

Explicit Instruction

Although not a substantial element of the present reform recommendations, explicit instruction has historically been the subject of research in an attempt to articulate what effective mathematics teaching looks like. In mathematics, substantial variation has been found in the quality of learning associated with different forms of explicit instruction (Good, Grouws, & Ebmeier, 1983, p. 219); not all "teacher-telling" is created equal. The descriptions of and recommendations for teacher-centered mathematics instruction indicate there are three general categories of explicit instruction to consider.

First, procedural instruction focuses almost exclusively on skills and definitions, with little attention to explaining why the steps make sense. In this model a student learning to divide fractions would simply be taught to memorize the pattern "invert and multiply," and would be given extensive practice to help them fix this routine in their memory.

Second, professional explanations give justifications in the language and symbols that mathematicians have developed. A teacher with this orientation might justify the procedure for dividing fractions for the students by illustrating the following: $2/3 \div 3/4$ can be written as a complex fraction, $(2/3) / (3/4)$. Multiplying the top and bottom of a fraction by the same quantity will preserve its value, so this fraction can be multiplied by $(4/3) / (4/3)$, to obtain $(2/3)(4/3) / (3/4)(4/3) = (2/3)(4/3) / 1 = (2/3)(4/3)$. Thus, $2/3 \div 3/4 = (2/3)(4/3)$. Doing this sequence of steps using variables would count as a mathematical proof of the general procedure. Depending on the student's background, this type of explanation may connect to previous knowledge and be a satisfying explanation; other students may find this sequence of steps fails to connect with other knowledge that lends meaning, and thus the explanation would not fulfill its intended purpose of producing conceptual understanding.

Third, explicit instruction can be based on intuitive explanations that attempt to link mathematical ideas to students' informal experience. Using this approach to justify the algorithm for fraction division the teacher might proceed as follows: $12 \div 3$ can be thought of as representing a problem like "Sally has 12 cookies. If she eats 3 cookies each day, how many days will her cookies last?" Clearly the answer is the quotient, 4. This problem could be varied, however. Suppose Sally begins with 12 cookies, but eats only 1/2 cookie each day. Now how many days will her cookies last? The students can model this situation mentally or physically to agree that the answer is 24. Connecting this to the symbolic representation, the students have just affirmed that $12 \div 1/2 = 24$.

More examples could be generated based on Sally's eating 1/3 of a cookie each day or 2/3 of a cookie each day, as well. Justification for the general procedure can be seen in the pattern of answers, but can also be reinforced by adding words to the equation: 12 cookies ÷ 1/2 cookie per day = 12 cookies x 2/1 days per cookie. By giving students a variety of examples and representations embedded in situations close to their own experience the teacher is using explicit instruction to help the students connect the new information to existing networks, rather than attempting to transmit isolated facts and unrelated information. Thus, explicit instruction can take a variety of forms, and any evaluation of the quality or outcomes must take into account the type of explicit instruction in the specific situation.

More Examples of Each Type of Explicit Instruction

Unfortunately, most school mathematics instruction in the United States is procedural (Hiebert, et al., 2003; Stevenson & Stigler, 1992, pp. 194 – 195; Stigler & Hiebert, 1997). This characteristic is particularly striking when American instruction is contrasted with that in other countries. For example, in eighth-grade mathematics lessons in the US observed in 1999, only 9% of the problems solved included some discussion of concepts or connections; the rest were approached procedurally or dispensed with by simply naming the final answer. In contrast, in Japanese classrooms 70% of the problems solved included discussion of concepts or connections (Hiebert, et al., 2003). Even during the introduction to the lesson, when it might be expected that more development of concepts and ideas might occur, teachers in American classrooms generally just stated the relevant concepts and procedures rather than taking time to motivate the content (Stigler & Hiebert, 1997).

Professional explanations tend to occur in higher level courses in high school and in college math courses (Stage, 2001; Wilson, 1997). The traditional college mathematics classroom is a performance by the expert mathematician, who introduces ideas, notations, and then demonstrates how to solve a series of problems using these (Stage, 2001). In Stage's observations of a typical finite mathematics class taught by an award-winning instructor, he concluded, "shared meaning . . . was virtually nonexistent. All meaning was the instructor's meaning" (Stage, 2001, p. 226).

Confounding this situation, in traditional first year courses, such as finite mathematics and calculus, procedural competence in the symbol system of mathematics is the desired outcome (Stage, 2001; Wilson, 1997). Perry (1991) has noted that when children are presented with both concepts and procedures they may attend more to the procedures and virtually ignore the concepts. College students concerned about performing well on an exam that will test procedural competence are likely to make the same choice of how to invest their attention and effort.

Historically, the "new math" of the 50's and 60's introduced professional explanations into school math at the elementary and middle school levels. The new math was motivated by a desire to offer an alternative to skill-focused teaching, a concern that resonates with those today who advocate reform. One of the developers of the new math summarized the shift in perspective regarding arithmetic as follows: "it is being presented not as a set of procedures but as a body of knowledge" (Moise, et al., 1969). Clearly the intent was to help students develop connections, the hallmark of understanding and conceptual knowledge.

The result, however, was conceptual explanations that failed to help students develop concepts: mathematical logic was followed rigorously, but without adequate attention to the more informal models that could have helped students develop mathematical intuition (Neyland, 1995; Moise, et al., 1969). Kline (Moise, et al., 1969) pointed out that this is not surprising, because presenting mathematics as a logically ordered "body of knowledge" is the opposite of how mathematics has developed historically. Mathematical concepts develop gradually from real-life experience as people notice patterns and problems, name and symbolize these, and develop successful approaches to predicting or solving. Only after this groundwork has been laid (often long after) is the content formalized and made rigorous. Kline therefore advocated that good teaching follow a similar pattern, claiming, "though one can compress history and avoid many of the wasted efforts and pitfalls, one can not eliminate it" (Moise, et al., 1969).

Good, Grouws, and Ebmeier (1983) concurred with this emphasis on the importance of informal, intuitive explanations to help students understand mathematical ideas. They emphasized the importance of "development" in a mathematics lesson, defined as "the process whereby a teacher facilitates the meaningful acquisition of an idea by a learner. Meaningful acquisition means that an idea is related in a logical manner to a learner's previously acquired ideas in ways that are independent of a particular wording or a special symbol system"(Good, Grouws, & Ebmeier, 1983, pp. 206-207). As examples of ways to accomplish this they cited teacher explanations, demonstrations, class discussion, using visual and manipulative materials, and hands-on projects (Good, Grouws, & Ebmeier, 1983, p. 199).

Leinhardt (1986) described in detail the "development" portion of a mathematics lesson taught by Dorothy Conway, a second grade teacher introducing her students to the procedure for subtraction with regrouping. Ms. Conway began with examples of subtraction problems using bundled sticks and trades made with pieces on the felt board. She carefully constructed her lesson so that the early problems could be solved without regrouping: for instance, if a student had 4 10-bundles of sticks and 6 single sticks she asked him to give 5 back, thus modeling 26 – 5. After several examples like this, she then asked a student who had 2 10-bundles and 6 sticks to give 8 back. The class immediately saw the contradiction, and responded with glee. This thorough groundwork prepared her students to recognize the situations when regrouping would be required (Leinhardt, 1986).

She then moved from bundled sticks to felt pieces to deliberately give students another, somewhat more abstract, model for the situation. She did not proceed immediately to the algorithm, however, but instead spent a substantial amount of time illustrating trades, demonstrating to the students that the quantity stayed the same even after a trade had been made (Leinhardt, 1986). Ms. Conway's development focused on the conceptual understandings that would help the students make sense of the written algorithm when she introduced it. She made use of multiple representations to give her students "referents" for the place value quantities they would be manipulating. In addition, she carefully sequenced her use of concrete materials to move from more concrete to more abstract (Leinhardt, 1986).

Additional analysis has revealed more specific qualities of intuitive explanations that differ in their effectiveness. Bromme and Steinbring (1994) analyzed how two different teachers negotiated the explanation of a probability lesson that contained an application problem. Both teachers used the same application problem and derived symbolic representations of probability from the problem, but there were substantial differences in how well they connected the contextual problem and the symbolic statements. Teacher A, the "expert teacher," spent approximately equal amounts of time explaining the contextual problem and symbolic representations and the relationships between these two. Teacher B, the "non-expert teacher," spent a large block of time developing the contextual problem and another large block of time developing the symbolic representation, but much less time exploring the relationship between the two (Bromme & Steinbring, 1994).

In addition, the two teachers differed in how they handled transitions between the context and symbolism: teacher A moved smoothly between the two, via remarks commenting on their relationship; teacher B moved abruptly from one context to the other, seemingly unaware of the confusion this caused for the students. Teacher B made some relational statements only after requests from the students for additional explanation (Bromme & Steinbring, 1994).

Leinhardt (1988) also contrasted the explicit instruction of novice and expert teachers on topics related to fractions. She identified several key elements distinguishing the lessons of experts that made the concepts more transparent to students. The expert teachers tended to re-use contexts and activities in order to help students connect related ideas: for example, one teacher had used paper-folding and coloring to introduce fractions, and then used it again to introduce equivalent fractions. The expert teachers also tended to connect the new concepts to similar concepts students knew from previous contexts. For example, the notion that equivalent fractions are equal despite their different appearances was related to the more general idea that "things that look different . . . can have the same value" (p. 61). The procedure for generating equivalent fractions by multiplying the top and bottom by the same number was connected to the known facts that numbers like 2/2 or 3/3 represent one, and multiplying by one does not change the value of a quantity (Leinhardt, 1988).

In summary, explicit instruction can be carried out in a wide variety of ways. Research has identified some specific characteristics that seem to help students gain conceptual understanding. Since explicit instruction tends to be more "efficient" in terms of covering more content than a problem-based approach, it may be useful to know if using high-quality intuitive explicit instruction is a reasonable alternative to problem-based instruction in terms of how successful students are in achieving conceptual understanding.

Innovations in College Mathematics Instruction

As has been noted above, traditional college mathematics instruction for most students (that is, 100-level courses such as finite math and calculus) tend to consist of professional explanations and procedures, with tests focusing primarily on procedural knowledge. In recent years, however, changes that emphasize a more active and conceptual type of learning have been implemented in a number of schools (Daves, 2002; Seymour, 2002; Simon & Blume, 1996; Wilson, 1997; Yackel, Rasmussen, & King, 2000), and more changes are proposed (Wilson, 2000). In some of these courses, such as reform calculus, students worked in small groups to analyze applied problems and wrote detailed explanations of their thinking. In others, the teacher led whole-class discussions with an emphasis on students' explaining their thinking and justifying their approach mathematically.

The use of these reform elements has been shown to decrease failures and withdrawal rates, increase student interest in mathematics, and improve attitudes toward learning (Springer, Donovan, & Stanne, 1999; Wilson, 1997). Results in terms of achievement are more difficult to determine. Opponents of reform assert that students from reform calculus classes don't know the calculation methods they need in subsequent classes (Wilson, 1997). Reformers point out that calculation is not one of the goals of the new courses, and claim that traditional courses teach students only a superficial facility with skills they do not understand (Wilson, 1997). Different goals aligned with different types of assessment mean that the results are not comparable measures, a difficulty with measuring the results of reform at all levels.

Assessment

Assessment should attend to the different levels of reasoning elicited by different types of tasks. At the "reproduction" level, students are recalling facts and using practiced procedures (Shafer & Romberg, 1999). This procedural knowledge is usually elicited when the problem presented is very similar to one that has been solved many times in the past using a known procedure (Hatano, 1988).

The "connections" level entails connecting prior knowledge with new ideas or new contexts (Shafer & Romberg, 1999). Assessment items that draw on this conceptual understanding are often set in novel contexts and have multiple solution paths, although often just one correct answer. The student may also be asked to explain their reasoning (Shafer & Romberg, 1999). In the studies reviewed by Rittle-Johnson and Siegler (1998) conceptual knowledge was assessed by one of the following four types of measures: evaluating nonroutine procedures developed by someone else; adapting a known procedure to a new problem; giving a verbal explanation of why a routine works; giving a representation of a concept using manipulatives or alternate symbols (Rittle-Johnson & Siegler, 1998).

Shafer and Romberg (1999) point out that procedural knowledge can be assessed within the context of conceptual problems. For the purposes of research, however, this poses a difficulty: if the student cannot successfully carry out at least part of the conceptual task they will be unable to demonstrate the procedural knowledge that they have. If procedural knowledge and conceptual knowledge are assessed independently it is possible to identify when one type is present without the other, so separate tasks are desirable (Rittle-Johnson & Siegler, 1998).

Shafer and Romberg (1999) include a level beyond "connections," as well. At this "analysis" level mathematics is used to interpret, explore, and make decisions in a complex situation, often one with more than one correct answer. The students may develop innovative solutions, and need to include all of their assumptions as well as their reasoning in order to justify their conclusions.

For the purposes of this study, the "analysis" level was not be used. The tasks posed included "reproduction" items that tested procedural knowledge as well as "connections" items probing students' conceptual understanding.

Interviews

Interviews offer an opportunity to gather more information from students regarding their conceptual understanding. Paper-and-pencil tasks may leave students' reasoning opaque: they may have had a good idea but failed to carry it out correctly, or they may have gotten the right answer for the wrong reason (Huinker, 1993). With an interview, students have the opportunity to explain the reasoning behind their written work, or to explain their thinking while they use manipulatives and diagrams.

Ginsburg, Kossan, Schwartz, and Swanson (1983) identified three common types of protocols used in mathematical interviews: talk-aloud, clinical, and mixed cases. In the talk-aloud style of interview the student is given a task and asked to report out loud on his or her thoughts while solving it. The interviewer rarely intervenes, allowing the student to select the level and type of report. In the clinical interview, the interview proceeds somewhat differently. The interviewer may ask a series of questions, or pose a problem for the student to work out. After the student has solved the problem, the interviewer may ask questions about how the student figured it out. The interviewer is more active in seeking to elicit the types of information desired, asking follow-up questions based on the student's responses or actions. Huinker (1993) cautioned, however, not to use the interview to teach or ask leading questions; this would interfere with the goal of determining the student's depth of understanding. This variability in how the interview may proceed is the reason this type is sometimes called a "semi-structured clinical interview" (Zazkis & Hazzan, 1999).

Since the specifics of each interview will vary, questions may arise about the comparability of the outcomes obtained. Ginsburg, et al. (1983) noted that if the interviews are being conducted for exploratory, hypothesis generation purposes this poses no difficulty. The interviews in that circumstance are not meant to be compared, but rather to be considered as a corpus of information from which inferences will be drawn. In the case of a hypothetically well-researched domain, on the other hand, comparability can be achieved by use of standardized contingencies based on the different types of responses known to occur (for example in studies of whether children conserve number). When using interviews in an intermediate type of situation, "hypothesis testing in the discovery stage," (Ginsburg, et al., 1983, p. 44), the interview should not be considered a "measurement" activity but at most a "classification" activity (Ginsburg, et al., 1983, p. 43). Additionally, interviews can be used in conjunction with other measurements, such as performance on written instruments, as a source of validation information. Zazkis and Hazzan (1999) pointed out that interviews can specifically give information on a student's "strength of belief" about their answer to a task, differentiating between a student who was working somewhat blindly from one who worked out the answer with confidence.

Zazkis and Hazzan (1999) identified the types of questions that are often included in a clinical interview focusing on student understanding. "Performance" questions are standard types of computations, but followed up with a request for the student to explain how they solved the problem and why. The interviewer is not as interested in whether the student can carry out the procedure, but in their strategies and choices. A second type of question is the "unexpected why" that prompts a student to reflect on something he or she probably takes for granted—for example, why is it not "legal" to have a zero in the denominator of a fraction? A "twist" question asks students a question that looks somewhat familiar but contains an unusual element. For example a student may be asked to interpret 0.3 in base six. "Construction" tasks and "give an example" tasks both give students some set of properties and ask them to describe a particular instance of these properties. For example, Ball (1990a) asked students for a word problem to illustrate 1 3/4 ÷ 1/2. Finally, "reflection" tasks give a report of what some other (possibly fictitious) student said or did, and asks for the student to respond. For example, they may be told "Ben believes 4 ÷ 0 = 0 because 4 x 0 = 0. Is his reasoning correct? Explain" (Zazkis & Hazzan).

Huinker (1993) advocated asking questions that explore the students' ability to make connections between various representations. Three common types of representations are symbolic, concrete/pictorial, and real-world contexts. Six types of questions could be drawn from exploring these connections, (from symbolic to concrete/pictorial, from symbolic to real-world contexts, from concrete/pictorial to symbolic, and so on) (Huinker, 1993, pp. 83-34).

Ginsburg, et al. (1983) noted that verbal reports from interviews have been questioned in terms of validity due to a variety of concerns. It is possible that the person being interviewed is not aware of all of his or her thinking, so cannot describe, for instance, how he knew to divide rather than multiply. In addition, the request to talk aloud during the problem solving process may affect or interfere with the thinking process. Zazkis and Hazzan (1999) noted that the interview process can also be faulted for being inadequate to its purported goal—it is not possible to truly know what is going on inside another person' mind.

However, even with these limitations, verbal reports offer information that is generally not available from other types of assessment, and can be used in conjunction with other assessments to collectively give a fuller picture of thought processes and understanding. It is wise to be tentative in the conclusions drawn. As Skemp (1987) stated, understanding can never be directly assessed; it must be inferred (Skemp, 1987, p. 166). With this in mind, interviews offer an additional opportunity to gather words and actions from which inferences might be drawn (Zazkis & Hazzan, 1999).

Place Value

History of place value. Place value is a concept we take for granted in our number system, but it is conceptually quite sophisticated. Most of the early counting systems make use of a "symbol value" system, where a particular shape stands for the size of the group. For instance, the Romans used V to mean 5 and X to mean 10. The symbols are repeated to indicate how many, so 2 groups of ten and 1 group of five would be written XXV. (Of course, the Roman system incorporates subtractive elements to shorten some representations, but the basic idea is still symbol value, not place value.

Interestingly, a couple of ancient cultures developed a combination system that has place value components. For example, the Babylonians used corners (<) and wedges (V) to represent 1 and 10, respectively, a form of symbol value. When they got to the quantity 60, however, they started to enumerate groups of sixty separately from the rest of the number. For instance, the value 75 would be thought of as 1 (60) + 15 and notated < V <<<<<. (The leftmost < indicates one group of sixty, and the V<<<<< represents 15 additional units, for a total of 75). They repeated this grouping when they reached 60 sixties (3600) and higher quantities; they also used it to represent fractional quantities as "sexagecimals," or base sixty "decimals." Unfortunately, the Babylonian system had no zero, so the place values were ambiguous; apparently context was required to determine what values were represented (Bunt, Jones, & Bedient, 1988).

The Mayan culture, similarly, used a combination of symbol value and place value, with the added sophistication of a zero. Their symbols included a horizontal line for five and a dot for one, grouped into place values of 1, 20, 360, 360(20), $360(20^2)$, and so on.   Note that the pattern of using powers of 20 was interrupted for the 360, apparently because this was nearly equal to the number of days in a year (Bunt, et al., 1988).

The ancient Greeks, despite their sophistication in many areas of mathematics, did not use place value. Their system was a "ciphered" system, with different symbols for each of the values 1, 2, 3, . . . , 9 , 10, 20, 30, . . ., 90, 100, 200, 300, . . . 900. Thus, their representation of a quantity like 123 looked superficially like ours, with a single symbol indicating how many hundreds, a symbol for the number of tens, and a symbol for the number of ones. In base ten, however, we can reuse a single symbol, say 3, in various locations in the number to indicate 3 hundreds, 3 tens, and 3 ones: 333. The Greeks, in contrast, would have written 333 using three different symbols (Bunt, et al., 1988).

The base ten system that we use traces its origins to the Far East, where the ancient Chinese represented numbers on a counting board with columns whose values were powers of ten. The same arrangements of sticks were re-used in the columns, with the column indicating whether the quantity was ones, tens, hundreds, and so on (with the caveat that the stick arrangements were rotated 90° in adjacent columns, presumably to help distinguish columns more easily). The stick arrangements gradually evolved into written symbols. At first, each written symbol was accompanied by a "value" indicator to identify whether it meant 4 "hundreds" or 4 "ones." Gradually, however, the value indicators were dropped and the positions retained the values. These symbols and the underlying logic were adapted and adopted by Middle Eastern peoples, and eventually transmitted to Europe with the advent of trade between Europe and the Arab world (Katz, 1998).

Place value instruction. When children encounter place value in elementary school they are expected to traverse this substantial conceptual territory--ideas that required humankind millennia to develop--in the space of a few short years. Children first learn to identify each digit with its "ones" quantity: 3 means three objects. Even in a multi-digit number, students often identify the symbol with its "ones" meaning: for example, in one research study most 6 – 8 year-olds claimed that the 1 in 16 represented 1 object (Kamii & Joseph, 1988).

Even when children are taught to associate pictures with ten-groups and singles with conventional notation, they may not form a robust understanding of place value. For example, a child may state that a picture showing 5 ten-groups and 3 singles is showing the quantity fifty-three, yet be uncertain how to answer when asked how many counters there would be if he counted them (Van de Walle, 2001, p. 151). For this reason, Van de Walle (2001) recommended that teachers incorporate "base-ten language" when naming quantities: for example, "five tens and 3, fifty-three." He cautioned, however, that students need to construct this relationship for themselves by their own reflection.

Van de Walle (2001) also recommended that students have extensive experience with "proportional" materials, materials that incorporate the ten-to-one relationship in their sizes. Interlocking cubes can be fastened into groups of ten, base ten blocks are pregrouped as tens and hundreds, sticks can be formed into bundles of ten. Although these representations are valuable for teaching students to count groups of objects by counting tens and hundreds, Varelas and Becker (1997) pointed out that these types of manipulatives do not model the idea of re-using the same symbol with a different value. They hypothesized that this makes the link between proportional concrete materials and the standard written numbers more difficult. Varelas and Becker (1997) invented an intermediate representation of numbers using chips that showed the "face value" (such as 3) on one side and the "complete value" (such as 30, if the digit is in the tens place) on the opposite side. They found that students who worked with this system were able to differentiate between the face value and complete value of each digit in a multi-digit number, and to understand that the complete values of the digits add up to the total value of the number.

Place value knowledge of elementary teachers. Studies suggest limited understanding of place value persists into adulthood for many elementary education majors and elementary school teachers (Ma, 1999; McClain, 2003). In Ma's research, teachers were asked to diagnose the cause of a student's error in multi-digit multiplication (the student failed to "move over" when writing the partial products). Seventy percent of the American teachers discussed the student's misunderstanding in procedural terms. Although these teachers used the term "place value," they used it to mean the labels on the columns, not the values of the digits. Thus, they explained that they would advise the student that when multiplying by the digit in the tens column you need to align the partial product so it ends in the tens column (Ma, p. 29).

In contrast, most of the Chinese teachers used the values of the digit in their explanation. Two types of explanation were suggested. One group said they would remind the student that when multiplying by the 4 in the tens column you are really multiplying by 40. Thus 40 x 123 = 4920, not 492. Another group of teachers did not introduce the zeros, but rather emphasized that it is possible to name numbers as "tens." When multiplying by the 4 in the tens column your product is the quantity of tens—492 tens, rather than 492 ones (Ma, 1999, pp. 42 – 43).

It is not surprising, then, that the preservice elementary teachers I have taught have consistently had difficulty constructing a counting system that exhibits place value. Their inventions are nearly always variations of a Roman-numeral type of system, indicating a preference for symbolic representations of value rather than positional representations. Most likely the instruction they received in their school experience did not encourage them to reflect on the different values a digit takes on as it moves from one place to another, but focused on procedural explanations such as those offered by the majority of the American teachers in Ma's (1999) study. Thus, this concept may not be easily accessible to them. When couched in an unusual setting, however, such as base five, students must take note of these difficult issues. When trying to count in base 5 students must grapple with the transition from 4_5 to 10_5. When interpreting 23_5, students cannot rely on procedural habits but must explicitly work out how many groups of five and how many singles are indicated. When counting in base six, students must reflect to understand why the "canonical" form now allows a maximum of five singles rather than nine. As they re-enact the construction of meaning for each position in a multi-digit number in another base, hopefully they gain a richer understanding of place value and its role as a foundational concept in the mathematics they will one day teach (CBMS, 2001, p. 63).

Chapter 3

It has been noted that what teachers say and do is associated with student achievement, and needs further study (Grouws, 1991; Good, Grouws, & Ebmeier, 1983; NRC, 2001, p. 359; Sowder, 1989, p. 27). In particular, there is a need for research at the college level to address questions of methodology (Becker & Pence, 1994). My study was an attempt to compare the relative effectiveness of two types of closely monitored instruction at the college level: teacher-provided explanations (explicit instruction) and student-centered problem-based learning (problem-based instruction). The study took place during a 1 $^{1}/_{2}$-week unit on place value in a mathematics content course for elementary education majors.

It is important to note that both instructional models included conceptual understanding as a core goal. The explicit instruction included conceptual explanations developed with sensitivity to the particular background of this population of students. For example, rather than introducing the topic symbolically (base b consists of the digits, 1, 2, . . . , b-1 and place values of b^0, b^1, etc.) students began by examining counting systems based on tens, then counting systems based on fives (see Appendix A). Base ten blocks and base five blocks were used to illustrate the groupings concretely. In the problem-based group, students were asked to invent a counting system using base five blocks and the symbols 0, A, B, C, and D. After the initial inventions were discussed and identified as non-place value systems, students were challenged to revise their systems to use place value based on groups of five.

The essential difference between the two models was *who* in the classroom had responsibility to provide explanations. In the explicit instruction class, the teacher took this responsibility; in the problem-based class, the responsibility was shifted to the students. The teacher guided the students' thinking via questioning and highlighting points made by the students, but the students were expected to provide the core insights and explanations.

Of course, in practical implementation it is nearly impossible to separate these two instructional approaches completely. The Teacher Observation Protocol (see Appendix B) used to evaluate the fidelity of the classroom implementation in this study contains a "continuum" with five categories: "highly expository," "expository with limited interaction," "expository with moderate/extended interaction," "guided discovery," and "highly pure discovery." The goal for the actual implementation was for explicit instruction classes to stay primarily in the first two expository categories and for problem-based instruction classes to remain primarily in the two discovery categories.

In order to carry out this comparison my study used a quasi-experimental approach. Data was collected from eight sections of approximately 26 students each during Fall 2003 and Spring 2004 in a math content course for elementary education majors. Of these, four were taught the place value unit via intuitive conceptual explicit instruction, and four were taught via problem-based instruction followed by whole class discussion. Of the eight sections, I taught three (one of each treatment type in the Fall, and a problem-based section in the Spring). The other five sections were taught by four different graduate teaching assistants (TAs). All four of these TAs were experienced teachers, three of them in this specific course for preservice elementary teachers. As experienced TAs in our department, all were familiar with the explicit instruction approach and were adept at developing explanations for non-math majors. Those who had taught this course before had also used the problem-based approach in a number of lessons.

Since the students had selected their section prior to the start of the study based on the time slot that best fit into their schedule, it was not possible to have random assignment to groups; however, sections were randomly assigned to treatments. A variety of measures were used to compare the characteristics of the groups before the study began—an attitudes and beliefs scale (Kloosterman & Stage, 1992) (see Appendix C), a pretest on the course content (see Appendix D), and math SAT scores.

Pre-Study Implementation

The TAs were introduced to the study during our pre-semester orientation meeting. I explained the goals, discussed the Teacher Observation Protocol, and gave detailed instructions for the first three classes. In these first three classes, as well as the remaining pre-study classes, instruction for all students alternated between problem-based and explicit instruction. In this way all of the instructors and all of the students experienced both types of instruction prior to the study unit; no one experienced a radical shift in instruction.

The lesson plans I provided for explicit instruction lessons were virtually scripts, with specific examples and explanations provided. The lesson plans for problem-based lessons included instructions for introducing the problem(s) the students would be working on, possible roadblocks the students would encounter with suggested responses (usually in question form), and key questions or points to highlight in the whole-class discussion at the end of class. Thus the instructors had taught four or five specific examples of classes using each instructional approach prior to the study unit.

During the first three weeks of the semester, prior to the start of my actual study, I met weekly with the TAs. To verify the classes were being taught as intended, my research assistant and I observed each of the sections (including mine) twice during this time period: once during a lesson taught via explicit instruction and once during a lesson using problem-based instruction (summaries of the pre-study lessons are included in Appendix E).

In addition, all students completed an initial written reflection (see Appendix F) which was graded and returned prior to the written reflection assigned as part of the study, thus ensuring all had at least one experience with one of the types of assessments planned for the place value unit. In these ways I attempted to minimize confounding influences that might impact the study outcomes. Finally, all students were provided with information regarding the study and given the opportunity to decline to participate (see Appendix G for sample forms).

Study Implementation

For the unit involved in this study I randomly assigned the TAs to the two treatment types. We met twice during this time, but as separate "teams" depending on which instructional type each was implementing. At each meeting, we reviewed the next lessons in detail (see Appendix A for study lesson plans). In addition, my research assistant and I observed each section twice during the unit and took notes on what was occurring during the lesson, again using an observation checklist (Appendix B). In these ways we attempted to ensure that the treatments were implemented as closely as possible to the stated expectations and that the treatments were consistent within each treatment type.

Student attendance was also monitored during the pre-study and study weeks. Scores of students who attended fewer than 70% of the pre-study sessions or fewer than 75% of the study lessons were not considered in the analysis. After attendance omits were accounted for, data was available for 69 students in the explicit-instruction group and 60 students in the problem-based instruction group.

Treatment Types

In this study I wanted to compare high-quality, authentic implementations of explicit instruction and problem-based instruction. Research has shown that students in a teacher-centered class learn more when the teacher provides a high-quality conceptual development of the lesson's topic rather than just a rote demonstration of a procedure (Good, Grouws, & Ebmeier, 1983; Leinhardt, 1986). Therefore, in the classes receiving explicit instruction, the TAs did not simply teach the algorithms for changing between bases; they also explained the connections between our number system and other base systems. The instructors did not simply demonstrate how to do the various tasks that were expected of the students on the test but also described the underlying concepts.

Researchers developing problem-based mathematics classes have noted the importance of selecting good tasks, maintaining high expectations for students to engage in sense-making, and guiding whole-class discussion where students are expected to explain their thinking and question their peers (Clements & Battista, 1990; Yackel, Cobb, Wood, Merkel, et al., 1990). In keeping with this vision, the problem-based treatment group in my study had a series of tasks to work on in small groups followed by whole-class discussion. For example, the first task in the problem-based treatment involved inventing a counting system using the symbols 0, A, B, C, and D and relating the system to base five blocks. Consistent with my past experience, most groups invented a Roman-numeral type system where A = 1, B = 5, C = 25, and so on. In follow-up class discussion the students were asked to re-examine our counting system and attempt to create a system with the symbols 0, A, B, C, and D that uses place value.

In my pilot tests I found the lecture method finished the unit more quickly. Thus, the group receiving explicit instruction was scheduled to be taught for 3-4 50-minute class sessions; the group developing their own procedures was scheduled to work on the unit for 4-5 50-minute class sessions. During the additional class time available for the explicit instruction group students worked on additional lessons of an unrelated topic.

Table 1 summarizes the distinctions between the two types of instruction used in this study:

Table 1: Comparison of Two Treatments

Element	Explicit Instruction	Problem-based Instruction
Topic	number bases	number bases
Goal	conceptual understanding of place value	conceptual understanding of place value
Class Organization	whole class for lecture small groups for practice	small groups for problem-solving whole class for discussion
Student Tasks	listen to explanations practice procedures	solve problems explain solutions
Classroom Discourse	teacher explanations limited student input	student explanations limited teacher input (probing and summarizing)
Length of Unit	3 – 4 class periods	4 – 5 class periods

Thus, one treatment group in this study listened to the *teacher* explain the meaning, connections, and procedures while the other group engaged in tasks and discussion designed to help the *students* articulate the meaning and connections and invent appropriate procedures. This distinction highlights essential differences in the recommendations of opposing voices in the "math wars." On one hand the reformers caution, "When a teacher demands that students use set mathematical methods, the sense-making activity of students is seriously curtailed" (Clements & Battista, 1990, p. 35); the opponents of reform respond, "Why not consider the alternative approach of teaching these algorithms *properly* before advocating their banishment from classrooms?" (Wu, 1999, p. 4) [emphasis in original]. The reformers posit, "Knowledge is actively created or invented by the child, not passively received from the environment" (Clements & Battista, 1990, p. 34); the opponents reply, "With the guidance of good teachers, . . . a student can grasp and integrate in twelve years a body of mathematics that it has taken hundreds of geniuses thousands of years to devise" (Ross, 2001).

Content Assessment

Following these two developments of the unit students took a unit exam with several questions related to number bases, some procedural and some conceptual (see Appendix H). With the help of my research assistant, I removed all identifying information (name and class) from these questions and then graded all of these questions using a standard protocol to ensure consistency in assessment and assignment of partial credit.

Students were also asked to complete a 1-2 page written reflection in which they explained what place value is, giving examples of counting systems with and without place value (see Appendix I). These written reflections were scored using a rubric that was provided to the students. Each student's reflection was graded independently by three different raters who had been trained using a set of anchor papers.

Approximately two to three weeks after the end of the unit I began the interviews. I used a "purposeful sampling" strategy to select students for these interviews (Gall, Borg, & Gall, 1996, p. 217). Using the coded data, I separated the students into high-scoring, middle-scoring, and low-scoring based on their scores on the number base questions portion of the unit exam. We invited students to volunteer to participate in the interviews. From those that volunteered my assistant contacted two high-scoring students, two middle-scoring students, and one low-scoring student from each treatment to schedule a time to be interviewed. In this way I prevented my knowing beforehand which instructional approach each student received, and thus tried to safeguard against my possible bias in the interview process (Gall, Borg, & Gall, 1996, p. 323).

The interviews were semi-structured (the actual questions asked are in Appendix J). Questions probed a variety of connections between representations, concepts and procedures, in keeping with the definition of "understanding" as "connections" informing this study (Hiebert & Carpenter, 1992; Shannon, 1999). I attempted to follow recommendations for mathematical assessment interviews, such as refraining from teaching but asking questions that seek to clarify and probe the student's understanding (Bush & Greer, 1999; Huinker, 1993). In addition, I audio-taped each interview and later transcribed portions of each for more in-depth examination.

Finally, procedural and conceptual questions relating to the study unit were asked on the final exam approximately two months later, as well (see Appendix K). I graded this portion of the exam after identifying information had been removed.

Analysis of Results

The pre-study assessments were used in the analysis in two ways—to assess the similarity of the individual sections comprising each treatment group, and, in the case of the math SAT scores, as a covariate when comparing the students' scores on the post-study measures. The decision was made a priori to include this covariate both as a control for possible group differences and to increase power.

For each of the post-study assessments, the four sections comprising each treatment group were combined and considered to be a single group that received (essentially) the same treatment. The students' scores on the two exams were considered to be four separate scores, as follows: immediate post-test procedural questions, immediate post-test conceptual questions, delayed post-test procedural questions, and delayed post-test conceptual questions. These scores were entered into the statistical software SPSS and compared to investigate the following questions:

1. Does either instructional approach result in better skill in translating from base ten to other bases, and from other bases to base ten?

2. Does either instructional approach result in better retention of how to translate between base ten and other bases (as evaluated on the final exam, approximately two months later)?

3. Does either instructional approach result in better ability to make conceptual connections, such as evaluating whether a given counting system exhibits place value or extending the meaning of place value to fractional place values?

4. Does either instructional approach result in better retention of the ability to make conceptual connections (as evaluated on the final exam, approximately two months later)?

The scores on the written reflection were also compared to see if there is a significant difference between the mean scores of the two treatment groups. In this case, the research question of interest was

5. Does either instructional approach result in better ability to articulate what place value means, using examples and non-examples to illustrate the essential differences between place-value systems and non-place-value systems?

The interview analysis considered the student's reported thinking processes, responses, and reasoning or justification. Analysis included considerations such as students' approaches to problems, responses to being "stuck," and students' conceptual understanding of number bases. The interviews gave insight about the first and third research questions above as well as the following:

6. Does either instructional approach result in students' greater ability to explain their thinking about how to solve problems involving number bases?

7. Does either instructional approach result in students' having better problem-solving skills in approaching problems involving number bases?

Table 2 summarizes this information and shows how it links to the initial research questions of this proposal.

Table 2: Summary of Research Questions and Analysis Procedures

Research Question	Data Collected	Analysis
1. Does either instructional approach result in better skill in translating from base ten to other bases, and from other bases to base ten?	immediate post-test: procedural questions; interviews	t-test for equality of means; ANCOVA written descriptions, categories of responses that emerge in analysis
2. Does either instructional approach result in better retention of how to translate between base ten and other bases?	delayed post-test: procedural questions	t-test for equality of means; ANCOVA
3. Does either instructional approach result in better ability to make conceptual connections, such as evaluating whether a given counting system exhibits place value or extending the meaning of place value to fractional place values?	immedate post-test: conceptual questions; interviews	t-test for equality of means; ANCOVA see above
4. Does either instructional approach result in better retention of the ability to make conceptual connections, as described above?	delayed post-test: conceptual questions	t-test for equality of means; ANCOVA
5. Does either instructional approach result in better ability to articulate what place value means?	written reflection	t-test for equality of means; ANCOVA
6. Does either instructional approach result in students' greater ability to explain their thinking about how to solve problems involving number bases?	interviews	written descriptions, categories of responses that emerge in analysis
7. Does either instructional approach result in students' having better problem-solving skills in approaching problems involving number bases?	interviews	see above

The methodology of this study, utilizing larger sample sizes and traditional statistical measures was chosen to seek insight into possible differences in what students come to understand from different instructional experiences. On the other hand, interpretation must take into consideration the specifics of this situation—the particular content, the types of assessment used and the ways responses are evaluated, the specific group of students, and the locale of the study (a major southeastern university) among other things all serve to limit and define the appropriate generalizations that might be drawn.

Chapter 4

This study was undertaken to examine differences in student understanding that might be explained by the type of instruction received. Sections of students taking a mathematics course for preservice elementary teachers in Fall 2003 and Spring 2004 were assigned to two types of instruction during a unit introducing number bases: one group was taught via intuitive conceptual explicit instruction and the other via a problem-based approach. The research questions investigated were the following:

1. Did either instructional approach result in better skill in translating from base ten to other bases, and from other bases to base ten?

2. Did either instructional approach result in better retention of how to translate between base ten and other bases (as evaluated on the final exam, approximately two months later)?

3. Did either instructional approach result in better ability to make conceptual connections, such as evaluating whether a given counting system exhibits place value or extending the meaning of place value to fractional place values?

4. Did either instructional approach result in better retention of the ability to make conceptual connections (as evaluated on the final exam, approximately two months later)?

5. Did either instructional approach result in better ability to articulate what place value means, using examples and non-examples to illustrate the essential differences between place-value systems and non-place-value systems?

6. Did either instructional approach result in greater ability to explain their thinking about how to solve problems involving number bases?

7. Did either instructional approach result in students' having better problem-solving skills in approaching problems involving number bases?

Since it was not possible to randomly assign students to treatments, the eight intact classes (five from the Fall and three from the Spring) were randomly assigned to treatments and various pretest measures used to determine whether the two treatment groups differed significantly on variables such as prior achievement or attitudes towards math. In addition, it was decided a priori to use students' math SAT scores as a covariate in the analysis to control for possible group differences and to increase the power of the statistical test to find differences due to treatment.

Results of Pre-Study Measures

Pre-study assessments were recorded for the following areas:

• prior experience in the topic of number bases ("Pretest" below)

• attitudes and beliefs about mathematical problem-solving (adapted from Kloosterman & Stage, 1992), specifically

  •• the student's belief that some word problems *cannot* be solved with memorized formulas or procedures ("Memorization Belief" below)

  •• the student's belief that understanding concepts in mathematics is important ("Concepts Belief" below)

  •• the student's confidence that he or she can solve a time-consuming math problem ("Confidence Belief" below)

• prior mathematics achievement, as indicated by the student's math SAT score

The Pretest consisted of a number base question that was scored as 0, 1, or 2, according to whether the student had no correct work, partially correct work, or completely correct work, respectively. The math SAT scores are based on a nationally normed scale ranging from 200 to 800, with a national mean of approximately 500.

Each of the survey questions offered students the opportunity to respond via a Likert-type scale (strongly agree, agree, undecided, disagree, strongly disagree) to six statements pertaining to each belief. For each belief, half of the statements were written in a positive format (e.g., "Math problems that take a long time don't bother me") and half in a negative format (e.g., "If I can't solve a math problem quickly, I quit trying"). Each statement was then scored from 1 (least productive attitude) to 5 (most productive attitude), resulting in a possible range of 6 to 30 points for the belief score, with a higher score indicating a more productive attitude toward mathematical problem-solving.

The outcomes of these assessments were compared for those in each treatment group. The results of those analyses confirmed that no significant differences in prior experience and attitudes existed between the two treatment groups prior to the implementation of the study. In addition, the results of the Pretest confirmed the researcher's belief that students would have had little or no prior experience with the study topic of number bases: no student scored a 2, and only four students (out of 128) scored a 1. Table 3 shows the means and standard deviations for each pre-study measure separated into treatment groups, and the relevant significance value from the t-test for equality of means. Since the t-test significance values are all greater than 0.05, the differences in sample means are considered to be statistically insignificant.

Table 3: Pre-Study Measures by Treatment

Assessment	Explicit Instruction		Problem-Based Inst.		t-test sig
Pretest	Mean	0.03	Mean	0.01	0.295*
	N	69	N	59	
	SD	0.146	SD	0.065	
Memorization Belief	Mean	16.87	Mean	17.40	0.311**
	N	69	N	60	
	SD	2.812	SD	3.104	
Concepts Belief	Mean	23.64	Mean	23.28	0.519**
	N	69	N	60	
	SD	2.965	SD	3.263	
Confidence Belief	Mean	22.06	Mean	21.77	0.654**
	N	69	N	60	
	SD	3.338	SD	4.018	
Math SAT	Mean	604.91	Mean	578.13	0.098**
	N	53	N	48	
	SD	70.401	SD	90.075	

*Equal variances not assumed, per Levene's Test for Equality of Variances
**Equal variances assumed

In addition, an ANOVA test was used to ensure that there were no significant differences in the means of the treatment groups between the Fall and Spring semesters. For this test the students were separated into four groups, Fall Explicit Instruction, Fall Problem-Based Instruction, Spring Explicit Instruction, and Spring Problem-Based Instruction. Table 4 gives the means and outcome of the ANOVA for these five measures across the four groups. Since the F-test significance values are all greater than 0.05, the differences in group scores for the pre-study measures are also considered to be statistically insignificant.

Table 4: Pre-Study Measures by Treatment and Semester

Assessment	Explict Fall		Explicit Spring		Problem-B Fall		Problem-B Spring		F-test sig
Pretest	Mean	0.04	Mean	0.00	Mean	0.02	Mean	0.00	0.475
	N	54	N	15	N	29	N	30	
	SD	0.164	SD	0.000	SD	0.093	SD	0.000	
Memorization Belief	Mean	16.85	Mean	16.93	Mean	18.17	Mean	16.63	0.165
	N	54	N	15	N	30	N	30	
	SD	2.949	SD	2.344	SD	2.601	SD	3.409	
Concepts Belief	Mean	23.67	Mean	23.53	Mean	24.07	Mean	22.50	0.235
	N	54	N	15	N	30	N	30	
	SD	3.120	SD	2.416	SD	2.638	SD	3.665	
Confidence Belief	Mean	21.91	Mean	22.60	Mean	20.97	Mean	22.57	0.324
	N	54	N	15	N	30	N	30	
	SD	3.641	SD	1.882	SD	4.222	SD	3.702	
Math SAT	Mean	605.24	Mean	603.64	Mean	570.00	Mean	585.60	0.367
	N	42	N	11	N	23	N	25	
	SD	69.114	SD	78.648	SD	85.334	SD	95.354	

Treatment Fidelity

Using periodic observations of each section, my research assistant and I confirmed that each course seemed to be following the content provided and teaching style assigned. The observation instrument (see Appendix B), adapted from the "Discovery-Expository Instrument" (Gordon, 1979), consisted of six categories of classroom teaching activities: highly expository, expository with limited interaction, expository with moderate/extended interaction, guided discovery, highly pure discovery, and neutral/miscellaneous. The observer evaluated the classroom at three-minute intervals, identifying the category and possibly annotating with remarks about the specific activity or content under discussion.

A classroom experiencing explicit instruction would be expected to remain primarily in the categories of "highly expository" and "expository with limited interaction" when evaluated by this instrument; a classroom utilizing problem-based instruction would be expected to spend most of its time in the "highly pure discovery" and "guided discovery" columns, with some time in the "expository with moderate/extended interaction" category (particularly during whole-class portions of the lesson, usually at the beginning or end of the period).

In nearly every lesson observed at least 75% of the time was spent in the designated style. Two notable exceptions occurred, however. In one problem-based lesson, almost half of the lesson became expository with limited interaction when the students encountered a significant conceptual roadblock. This was a particularly challenging lesson for the students--making the transition from thinking about a Roman-numeral type system using the symbols 0, A, B, C, D, to inventing a place-value system with these symbols. In contrast to a strictly explicit-instruction lesson, however, the instructor included opportunities for students to explain, as well.

The second exception occurred with an instructor assigned to explicit instruction who, due to personal style and familiarity with the content, tended to include thought-provoking questions within the lecture. For instance, after giving examples of counting in base four and in base six, this instructor asked the class, "What kind of patterns do you see?" and spent several minutes allowing students to reflect and offer insights. After about ten minutes the instructor reverted to a less interactive style, inviting only limited remarks. However, she continued to use pauses and questions effectively to establish a goal of sense-making in the class.

Observing her class posed a unique dilemma for me: I admired and generally advocate the interactive style she used. In the context of this study, however, the intention was to clearly contrast which classroom participant(s) were responsible to provide explanations—the teacher or the students. Her class was identified as part of the explicit instruction group, in which the teacher is responsible for those explanations and the students are expected to be recipients. Her style blurred the edges of this distinction somewhat.

In summary, for the most part the classes remained true to the plan, with occasional variations due to the specific nature of the content or the teacher's individual style.

Results of Study Measures Without Covariate

The dependent variables were based on three post-assessments that were given to all of the students to measure their understanding of the number base content after the instruction: a unit test which came immediately after the instruction (Immediate Post-Test below), a written reflection completed about two weeks after the instruction (Written Reflection below), and the final exam which took place approximately two months after the instruction (Delayed Post-Test below). Each of the two post-tests were separated into conceptual and procedural portions that were scored separately, resulting in a total of five dependent variables.

The Immediate Post-Test consisted of four procedural and four conceptual items (see Appendix H). The four procedural questions were worth 3, 4, or 5 points (depending on difficulty), with a total possible score of 16 points for the procedural portion. The first conceptual question was a multi-part question worth 8 points; the remaining conceptual questions were worth 2 or 3 points (depending on difficulty) with a total possible score of 16 points for the conceptual portion.

The Delayed Post-Test (see Appendix K) consisted of two procedural questions worth 3 points each, for a total of 6 points possible; the conceptual portion consisted of four questions worth 3 points each, giving a total of 12 possible points on this portion.

Students were also asked to complete a 1-2 page Written Reflection in which they explained what place value is, giving examples of counting systems with and without place value (see Appendix I). These written reflections were scored via a rubric that was provided to the students. Each student's reflection was scored independently by three different raters who had been trained using a set of anchor papers. The sum of the three ratings was used as the student's score, giving a total possible of 30 points. The inter-rater reliability was evaluated using Cronbach's Alpha; a reliability score of 0.851 was calculated.

Table 5 summarizes the outcomes of these measures based on the raw scores, without taking into account the students' prior achievement as indicated by math SAT scores. As visible in the chart, the means are very close for the two groups, well within one standard deviation in every case.

Table 5: Means of Study Measures (without covariate)

Measure	Treatment Group			
	Explicit Instruction		Problem-Based Instruction	
Immediate Post-Test Procedural (16 points possible)	Mean	14.16	Mean	13.52
	N	69	N	60
	SD	3.04	SD	3.94
Immediate Post-Test Conceptual (16 points possible)	Mean	10.72	Mean	9.48
	N	69	N	60
	SD	4.26	SD	4.43
Written Reflection (30 points possible)	Mean	21.36	Mean	20.58
	N	66	N	59
	SD	3.96	SD	4.33
Delayed Post-Test Procedural (6 points possible)	Mean	5.09	Mean	5.25
	N	64	N	59
	SD	1.72	SD	1.53
Delayed Post-Test Conceptual (12 points possible)	Mean	8.59	Mean	7.97
	N	64	N	59
	SD	3.44	SD	3.25

Analysis of Potential Covariates

Math SAT scores had been chosen as a covariate prior to the study, based on the expectation that prior success in math learning would influence a student's success in the material taught during the study unit. In addition, the other pre-measures were checked for possible correlation with each of the dependent variables. A correlation of 0.4 or greater would indicate that the measure should also be used as a covariate in the analysis.

Since this study specifically identified conceptual understanding as potentially distinct from procedural understanding, the procedural and conceptual portions of the two tests were considered separately when calculating the student's performance and its possible correlation with the pre-measures. Table 6 summarizes the results of this analysis. As expected, there was a significant correlation greater than 0.4 when students' math SAT scores were compared with their scores on the immediate post-test (both procedural and conceptual portions), and with their scores on the conceptual portion of the delayed post-test. Somewhat unexpectedly, the correlation was below 0.4 when SAT scores were compared with scores on the written reflection and on the procedural portion of the delayed post-test. Examining the correlation values between other pre-measures and the dependent variables reveals that no other pre-measures need to be used as covariates in the analysis.

Table 6: Pearson Correlation Coefficients: Pre-Measures with Dependent Variables

Pre-Measure	Immediate Post-Test Procedural	Immediate Post-Test Conceptual	Written Reflection	Delayed Post-Test Procedural	Delayed Post-Test Conceptual
Pre-Test	0.103	0.017	-0.041	0.028	-0.004
Memorization Belief	-0.018	0.071	0.075	0.030	0.014
Concepts Belief	0.098	0.093	0.056	0.081	0.099
Confidence Belief	0.131	0.277**	0.095	0.135	0.224*
Math SAT	0.444**	0.512**	0.245*	0.241*	0.542**

* Correlation is significant at the 0.05 level (2-tailed)
** Correlation is significant at the 0.01 level (2-tailed)

Results of Study Measures With Covariate

In order to analyze the results using SAT scores as a covariate, it was first necessary to check for homogeneity of regression; if this test fails it indicates there is interaction between the treatment and SAT scores that needs to be accounted for in the analysis. To test for interaction, a product variable was created from the treatment and SAT variables. Each dependent variable was then regressed on three independent variables--SAT, treatment, and SAT x treatment--and the coefficients of each variable checked for significance. In this case, a more lenient alpha value of .25 was used in order to minimize the likelihood of overlooking a significant interaction (Pedhazur, 1997, p. 563). According to Pedhazur (1997, p. 563), this is appropriate to minimize type II error (failure to reject the null hypothesis when it should have been rejected) when one is testing whether two regression lines are parallel (have the same slope). Using this test, the delayed post-test conceptual items were found to be affected by the interaction of SAT and treatment. Table 7 summarizes the results for all five dependent variables.

Table 7: Significance of SAT, Treatment, and Interaction for Each Dependent Variable

Dependent Variable	Significance of Each Coefficient (p values)		
	Math SAT	treatment	SAT x treatment
Immediate Post-Test Procedural	0.006	0.730	0.779
Immediate Post-Test Conceptual	0.001	0.633	0.752
Written Reflection	0.137	0.962	0.932
Delayed Post-Test Procedural	0.424	0.414	0.351
Delayed Post-Test Conceptual	0.000	0.175	0.173*

*considered significant at the 0.25 level

Based on these outcomes, each of the first four dependent variables was analyzed using a univariate analysis of variance with treatment as a fixed factor and SAT as a covariate. In each case, being in a different treatment group did not have a significant impact on how a student scored. The results are summarized in Table 8. In each case, a p value is reported; values above 0.05 are considered to indicate likely random variation, whereas values below 0.05 are considered to indicate a significant relationship between the predictor variable (SAT or treatment) and the dependent variable (the study measure).

Table 8: Significance of SAT and Treatment for Four Dependent Variables (excluding Delayed Post-Test Conceptual)

Dependent Variable	Significance (p values)	
	Math SAT	Treatment
Immediate Post-Test Procedural	0.000*	0.608
Immediate Post-Test Conceptual	0.000*	0.213
Written Reflection	0.014*	0.783
Delayed Post-Test Procedural	0.014*	0.407

*significant at the 0.05 level

Because the Delayed Post-Test Conceptual scores showed evidence of interaction, the students were divided into two groups based on treatment before the analysis was carried out. The students' Delayed Post-Test Conceptual scores were then regressed on SAT scores. For the explicit instruction group, SAT scores were a somewhat better predictor (R square of 0.330 versus 0.271 for the problem-based group). This suggests that the scores of students in the problem-based group were less influenced by their prior achievement (as measured by SAT score). Examining the scatterplots for each group seems to confirm this impression.

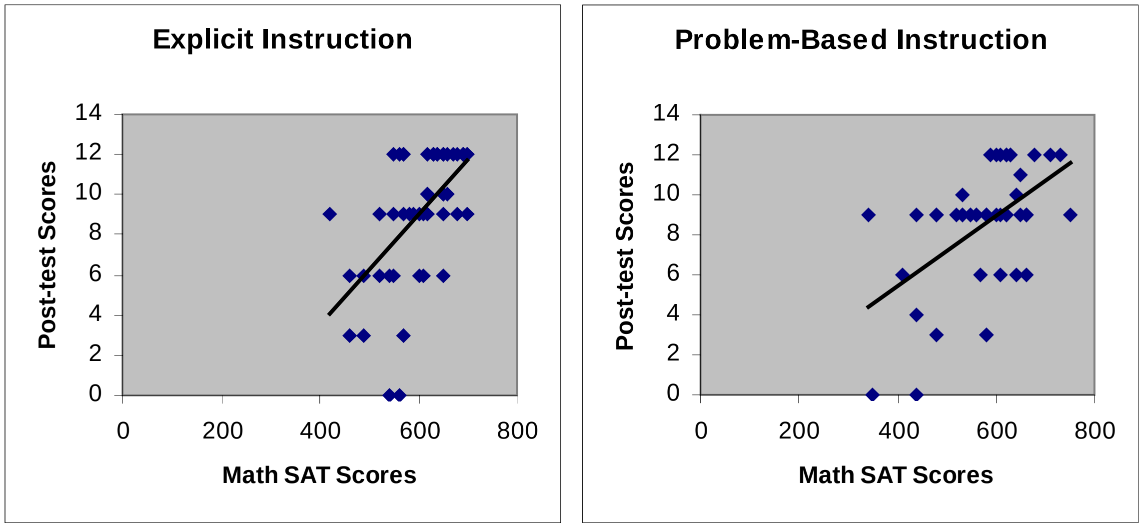

Figure: Delayed Conceptual Post-test Scores vs. Math SAT Scores

The different slopes on these lines suggest that students with lower SAT s

found the problem-based instruction more helpful than students with higher SAT scores. In fact, the point of intersection of the two regression lines occurs at 577.5 on the SAT scale. For students with SAT scores below this intersection point, the predicted score on the posttest was higher for those receiving problem-based instruction; for students with SAT scores above this intersection, the predicted score on the posttest was higher for those receiving explicit instruction. This is consistent with other research results comparing reform-based instruction with traditional instruction (for example, Huntley, Rasmussen, Villarubi, Sangtong, & Fey (2000)). In addition, it is logical that students who have been successful in the past with traditional instruction will continue to be successful with that style of presentation, whereas students who were less successful in the past might find a new, different approach more helpful.

Summary of Statistical Analysis

Overall, it was found that the differences in instructional treatment did not significantly explain the students' scores on any of the dependent measures—procedural or conceptual, immediate or delayed, test questions or a written reflection. Students' scores were significantly predicted by their prior achievement as indicated by SAT scores. Mild interaction between treatment and SAT score was observed for one dependent measure—the delayed conceptual post-test; in this case the problem-based treatment seemed to be more effective for students with lower math SAT scores.

Interview Analysis

Several students were interviewed from each treatment group in order to gather richer data about the students' understanding of number bases, their response to being "stuck" in a problem situation, and their beliefs about the classroom experiences that promote learning. These understandings and beliefs were compared to the type of instruction the student experienced, to see if any connection was apparent.

The interview data was difficult to interpret. Some data seemed to indicate that prior knowledge --rather than instructional model--was more influential when analyzing the students' conceptual understanding of number bases. This would be consistent with the statistical results described earlier, in which math SAT scores correlated with students' scores. Among the students interviewed, the three students with the strongest high school math background, coursework that included either calculus or a college-level statistics course, did the best on the mathematical interview questions. One of these (Esther) was from the problem-based group; the other two (Debbi and Mandy) were from the explicit instruction group.

There were a couple of anomalies, however. One student from the explicit instruction group (Frank) had only a moderately strong high school background yet scored almost as well as the students with stronger prior achievement. In addition, one student from the problem-based group (Steph) had only a moderately strong background, a negative attitude towards math, and low confidence scores on the beliefs pretest yet she scored far better than another student with a similar background, attitude, and confidence score (Cathy). The only apparent difference is revealed by the transcripts of their interviews: Steph was very persistent when faced with a problem, so she eventually made headway, where other students in the same situation dropped the problem quickly. It would be gratifying to think that she learned this from her experiences in the problem-based class, but why then did the other problem-based students, Cathy and Brenda, not have more persistence? The more likely answer is that Steph simply has a more persistent nature or personality.

There did seem to be a relationship, however, between instructional type and beliefs about classroom experiences that fostered their learning. The students who had experienced the problem-based number base unit were much more positive about the helpfulness of working in groups.

Students were invited to participate based on their expressed willingness, which group they were enrolled in, and how well they scored on the first assessment. Originally there were to be ten interviews; however due to non-response and technical problems seven interviews were eventually available for analysis. These seven included the following students:

Student (pseudonym)	Instruction Group	Post-test score (out of 32)
Esther	problem-based	32
Steph	problem-based	23
Brenda	problem-based	21
Cathy	problem-based	18
Frank	explicit	30
Debbi	explicit	25
Mandy	explicit	24

These students varied quite a bit in their prior experience and attitudes towards math. Esther came directly to this university from high school, which she completed in 2003, and this course was her first college math course. She had scored 710 on the math portion of the SAT, a very strong score. Her high school math background was strong, as well, including precalculus and AP statistics. She considered math classes generally to be her favorite classes, particularly when they involved problem-solving, numbers, and algebra. On the beliefs pretest taken at the beginning of the semester, Esther scored within one standard deviation of the (coursewide) mean on all three beliefs ("some problems cannot be solved with memorized formulas," "understanding concepts in math is important," and "I can do a time-consuming math problem"). Interestingly, Esther was the only one of the students interviewed who had submitted an SAT score when applying to our university; the others transferred in college credit so did not need this qualification.

Steph had attended three colleges before arriving at our university. During that time she had taken intermediate algebra and college algebra. In high school, which she completed in 1997, she had taken Algebra 1, Algebra 2, geometry, and math analysis. She didn't like math at all, claiming, "I don't understand math and have never had anybody explain it to me properly! I learned everything by rote and don't have an idea why answers are the way they are." However, she did like algebra "because I liked solving for x and plugging in numbers to get an answer." On the beliefs pretest, Steph's confidence score was about 1.8 standard deviations below the mean; her other beliefs scores were within one standard deviation of the mean.

Brenda had spent one semester at another university, where she took college algebra. Prior to that, she had taken Algebra 1, Geometry, Algebra 2, and precalculus in high school, graduating in 2002. She felt math classes were "ok, because even though math is not my favorite subject the classes help me understand math concepts." Brenda's scores on the beliefs pretest were all within one standard deviation of the coursewide mean.

Cathy had attended another four-year college for two years before transferring to our campus. During that time she took a calculus class. In high school she had taken Algebra 1, Algebra 2, geometry and trig, finishing in 2001. When asked her feelings about math classes, she stated, "I'm not very good at math. It takes me awhile to understand the concepts." Consistent with this was her score on the beliefs pretest, which was about 1.5 standard deviations below the coursewide mean. On the other hand, her belief that understanding math concepts was important was 1.1 standard deviations above the mean.

Frank had attended a community college for about two years before transferring in. He took a couple of lower-level math courses during that time. In high school he had taken the traditional sequence of Algebra 1, Algebra 2, geometry, and precalculus, and graduated in 1997. His attitude toward math classes seemed to depend on the quality of the teaching. When the teacher was inexperienced or "talked over everybody's head" he disliked it. Unfortunately, he felt that although he "used to like math (somewhat)," his experiences with college math classes had "turned [him] off of it completely." On the beliefs pretest Frank was unusual in his belief that math concepts are important: his score affirming this belief was more than two standard deviations above the coursewide mean.

Debbi, like Esther, came directly from high school, although she had gone to a community college during her senior year in high school where she had taken a statistics course. This was a special "parallel enrollment program" where she received college credit at the same time she completed her high school diploma requirements, graduating with her class in 2002. In addition, she had taken geometry, trig, and college algebra during her earlier years of high school. She liked math until she encountered high school geometry. Trig was a struggle because she didn't like her teacher and felt she didn't explain things. She continued to like algebra, however, which she found came easily for her. On the beliefs pretest, she was a little more definite than her classmates in her affirmation that some word problems cannot be solved with memorized formulas—her score was about 1.1 standard deviations above the mean.

Mandy had attended a community college for one year before coming to this university; she took precalculus and calculus there. In high school, which she completed in 2001, she had taken honors geometry as a freshman, honors algebra, precalculus, and calculus. She considered math and science her favorite subjects, and was considering teaching math some day. All of her scores on the beliefs pretest were within one standard deviation of the coursewide means.

The interview (see Appendix J) included mathematics questions that were similar to those asked on the post-test as well as novel items. In addition, students' awareness of connections was probed by asking for similarities and differences between base ten and other number bases. Finally, students were asked about the instruction elements they found most helpful to their learning and about their view of how mathematical understanding is achieved.

Mathematical understanding. Success rates for solving the mathematics items are summarized in Table 9. A student received a "2" for a completely correct response without cueing from the interviewer, a "1" for a partially correct response or a response that required cueing, and a "0" for a completely incorrect response or no response.

Table 9: Interviewees' Scores on Mathematics Items

Group	Student	Change 2012_3 to base ten	Change 2002 to base thirteen	What number before 4550_6 ?	Represent 22 sticks in base six groups	Represent 50 sticks in base six groups	Property if base six # ends in zero?	Total score (12 possible)
Problem-	Esther	2	2	2	2	2	1	11
Based	Steph	2	0	1	2	2	1	8
	Brenda	0	0	0	0	0	0	0
	Cathy	0	0	1	2	0	0	3
Explicit	Frank	2	0	2	2	2	2	10
	Debbi	2	1	2	2	2	2	11
	Mandy	2	2	2	2	2	2	12

The first question was a familiar task, one that had been practiced in class and appeared on the first post-test. All of the successful students immediately launched an efficient routine of making marks over or under each digit to indicate something about its value. Some wrote the whole-number value of each column (1, 3, 9, etc.) and some wrote the exponents associated with the base (0, 1, 2, etc.). They than used these to organize their work of multiplying and adding to arrive at the final answer. All but Steph worked efficiently through the procedure to the conclusion; Steph seemed to recall bits and pieces of the process, and was ultimately successful, but in a haphazard, inefficient way.

Cathy, however, started an efficient-looking routine then lost her way. She began in a similar manner as several successful students, writing powers of three under each digit. She even called these the "place values," but when she described what she thought she should do next she suggested multiplying each digit by 3, regardless of its position. The routine and the term appeared correct, but she didn't understand what they represented. On the other hand, Brenda did not even know how to begin a helpful routine: she only wrote down powers of ten, got stuck, and abandoned the problem.

The second question, likewise, was a familiar task, seen in class and on the post-test. Again, most students responded with an attempt to retrieve a procedure, although fewer were successful this time. Three students (Esther, Mandy, and Debbi) began the procedure correctly, with what seemed to be a well-practiced routine of making a grid with powers of thirteen heading each column. Debbi, however, got derailed when she could not recall how to put a value of eleven into a single column. Of the students who did not do this problem correctly, one (Brenda) used the opposite procedure (changing base 13 to base 10); the other three (Steph, Cathy, and Frank) quickly abandoned their effort, stating they didn't remember how to do this type of problem. They made no attempt to reason about the problem.

The third question was less familiar, although it had been part of the unit. All except one of the students who made progress began with a procedural approach: changing 4550_6 to base ten. At this point, however, two of the students (Esther and Debbi) recognized what the base six value would be without using a procedure to convert back. Mandy carried out the entire solution procedurally, but was able to compare her final answer conceptually when prompted at the end. The remaining successful student, Frank, thought through the problem conceptually, using an analogy with base ten.

The fourth question was the first of two that involved representing base six numbers with popsicle sticks. Since this was not a task done in class the students needed to apply their understanding of number bases to this task rather than rely on a memorized procedure. The students were shown a prompt page before beginning illustrating how bundles of ten sticks are sometimes used to introduce first graders to place value. With this example, all except Brenda quickly made bundles of six from the twenty-two sticks they were given and reported the value as 34_6. Brenda, however, was confused about the task, counting the twenty-two sticks and then writing $22_6 = 14$. Even when the interviewer clarified, "here's a quantity of sticks; figure out how many bundles of six there are and how many left over" she was unable to complete it. The interviewer eventually demonstrated the task.

The fifth question required the students to extend what had been illustrated in the prompt page, since the 50 sticks would be represented in base six as a three-digit number, using place values of 1, 6, and 36. Most students (five out of the seven) completed this task successfully. The successful students varied in how they thought about the "large" group of thirty-six, however: one (Debbi) counted out thirty-six ones and bundled them; others made eight groups of six and then bundled six groups of six.

A particularly interesting misconception appeared in both Brenda's and Cathy's thinking on this problem. In both cases, the students made eight bundles of six, but never regrouped six of the bundles into a thirty-six. Instead, they arranged the eight bundles across two columns. Brenda did this by putting three of the bundles into the third column, so her arrangement looked like BBB BBBBB 11 (B = bundle and 1 = single stick). She then reported the written form as 352_6. When asked to explain her thinking, she quoted a rule that you "can't go over 5 in base six." Cathy's answer was a variation of this. She arranged her bundles as BB BBBBBB 11, or 262_6. When prompted for her reasoning, she explained that in base six you "can't go higher than 6." In both cases, the students seemed to be recalling a rule from class, but misapplying it by failing to see the need to regroup to a new size of group. Their concept of place value did not seem to include multiple sizes of groups. When Cathy's work on the first question was compared with this problem, her misunderstanding seems consistent: although Cathy began working that question by writing powers of three under each digit, and called these the "place values," she speculated that the base ten value would be determined if each digit was multiplied by 3. She only considered groups of 3, the base, rather than a sequence of successively larger and larger groups. This misconception prevailed in her thinking despite writing down the correct values for the larger groups. Cathy seemed to use the term "place value" as a label only, failing to connect the value with actual group size.

Finally, the last question asked students to think more generally than any of the previous tasks in class or in the interview. Ideally, the student would have related a conceptual understanding of why numbers ending in zero in base ten are divisible by ten to a new setting where numbers are grouped in sixes. Unfortunately, this proved to be a difficult question to ask in a way that clearly indicated the type of result desired without partially giving away the answer. Thus, several of the students initially gave superficial responses such as "there are no ones." However, when prompted to think in terms of divisibility, these students all gave the correct divisibility property and were able to explain clearly why it would be true. (In the scoring, these students that required this "cue" scored a "1" for the task.)

Hierarchy of understanding of place value. The analysis of these students' responses indicated a variety of levels of understanding related to place value. At the lowest level, Brenda and Cathy only related the concept of place value to forming groups of size "n" in base n. Although Cathy made some progress on the question "What comes before 4550_6?" this was only after the interviewer made the suggestion that she look at just the two rightmost digits, restricting her attention to the ones and sixes columns.

Steph seemed to understand the number base problems at a somewhat higher level. She moved easily from groups of size six to clustering six sixes into a "large group" of thirty-six. Similarly, in her discussion of the first problem she described the different sizes of groups involved: "27 of them times the 2 gives you the 54…". She apparently understood the concept of multiple sizes of groups, formed either as powers of the base or as groups-of-groups. However, she struggled to make sense of the question "What comes before 4550_6?" Her description of her thinking seemed to reveal a struggle to mentally "unbundle" a group of six and think about the result. She was confident when forming groups and counting them, but struggled with the challenge of changing to an equivalent form by ungrouping.

Both Steph and Frank additionally demonstrated an ability to change from a base ten quantity to a base n representation in the concrete, but without connecting this to the symbolic paper-and-pencil routine that would yield the same outcome. Both successfully represented 50 in base six when they were able to use popsicle sticks to form groups of six and thirty-six, but seemed unable to relate that process to the task "change 2002 to base thirteen."

Debbi demonstrated a clear understanding of the dynamics of forming and unbundling groups, and was able to work symbolically with these processes, but she had not thought through all the implications for the symbol system. She was stymied when her attempt to rewrite 2002 in base thirteen required that she put the quantity "eleven" in just one column. She couldn't recall how this was handled, and did not seem to be able to reason about the need for a digit for eleven when the group size is thirteen. Frank illustrated a similar lack of attentiveness to the relationship between the base and the digits used. In the process of solving the first problem he miscopied the base three numeral as 2013. After he had completed his solution, the interviewer asked, "Would this be a legitimate question in base three for me to ask you, 2013_3? Is there any difficulty with that, or is that just a variation of the problem?" He did not perceive a problem with the question, apparently not connecting the base size of three with the restraint that unique representations require the only legal digits be 0, 1, and 2.

Esther and Mandy, on the other hand, demonstrated not only an ability to relate symbolic manipulations to the conceptual processes but also stated reflective generalizations about these processes. As Esther worked through the third question, "What comes before 4550_6," she took awhile to understand and solve the specific problem. When she was done, she explained her thinking for that specific task, but then summarized her final answer in the general case: "In order to convert to what comes right before you make the first place that has a [nonzero] number into one less number, and then the next place you have to make it the highest it can be…like you're changing a 10 to a 9."

Mandy, on the other hand, tended to generalize by relating her approach to an analogous procedure in base ten. She compared her work on the first problem to base ten by explaining, "With base ten you have your place values of the ones place, tens place, and 10 to the zero is your ones place, 10 to the first is the tens place, and so on." She went on to compare that to using 3^0, 3^1, 3^2 and so on as the place values for base three. Similarly, when prompted to explain an alternate way to solve the third question, "What comes before 4550_6," she described a process of subtracting with regrouping that she explicitly compared to base ten: "Like in our tens system you would carry from the place value before or to the left of it…"

In summary, then, the students work on the number base problems suggested the levels of expertise in their understanding shown in Table 10.

Table 10: Interviewees' Hierarchy of Place Value Understanding

Level of Understanding	Students	
	Explicit	Problem-based
1: Groups of size n only		Brenda, Cathy
2: Static groups of sizes n, n^2…		Steph
3. Concrete groups without symbolic process	Frank	Steph
4: Dynamic groups (form and un-form); lack of attention to digit and base relationships	Frank, Debbi	
5: Dynamic groups, symbolic processes, attention to digit and base relationships, generalizations and connections over different bases	Mandy	Esther

Connections. In addition to indirect observation during the problem tasks, as noted above, the interviewer probed students' recognition of connections in place value by requesting they tell how other number bases are similar to and different from base ten. The number indicated in Table 11 tells how many statements each was able to give.

Table 11: Interviewees' Number of Connections Cited

Group	Problem-based				Explicit		
Student	Esther	Steph	Brenda	Cathy	Frank	Debbi	Mandy
# statements	4	2	1	2	4	2	4

The higher scores for Esther and Mandy in this task are consistent with the observations made during their problem-solving tasks about their tendency to offer generalizations and connections spontaneously in the context of explaining their thinking process. Of the seven students, only Esther and Mandy noted explicitly the consistent pattern of powers of the base for place values. Debbi alluded to a pattern, stating, "instead of 1's, 10's, 100's, 1000's it's 5, 25, 125," but she did not elaborate on what is similar about these two sequences. Moreover, even though Steph, Cathy, Frank and Debbi had correctly written 3^0, 3^1, 3^2, and so on for place values in the process of solving the first question they did not identify this pattern explicitly when asked for similarities. This is even more striking because, as mentioned earlier, these students were clearly enacting a familiar routine when they began this problem; however, their repeated use of the correct procedure did not seem to lead to a generalization in their thinking.

In contrast, some students seemed to be able to state generalizations that did not have robust connections in their mind to tasks and examples. For example, in this context Frank stated that digits go to "wherever the base is," even though he had not attended to that restraint during his solution of the first question discussed above (or perhaps he misunderstood the restraint to allow a digit of n in base n). In a similar vein, Cathy received "credit" on this task for stating the need to "go to a new place value when you reach the base," even though her work on the popsicle stick problems indicated she clearly misunderstood this statement. The source of these misunderstood generalizations is somewhat mysterious. Perhaps the students had heard others in the class or the teacher make these observations, and memorized them as rules without understanding what logic or examples gave rise to the rules.

Responses to impasse situations. The students varied in their response to encountering an impasse during the solution process. Steph was paradoxically the most likely to make negative comments about her likelihood of succeeding, yet also the most persistent. When she encountered a block, she would continue to pose theories about what might be true, although she might not have pursued some of her good ideas if the interviewer had not prompted her to explain her thinking. Debbi, on the other hand, posed a theory for resolving her difficulty with the second question, but when asked if she thought it would help her solve the problem (it would have), she responded "probably not," and abandoned her effort.

Brenda and Cathy, however, were even quicker to abandon their efforts when they didn't remember how or know how to do a problem. Even when the interviewer attempted to cue Brenda to think about the rightmost two digits on question three, Brenda's response was simply to guess ("Forty-four, maybe? Not sure"), and then shut down.

Frank exhibited two quite different responses the two times he got stuck. On the second question he quickly abandoned his efforts, stating, "I don't remember how to do that." He made no effort to reason it out, despite having a clear understanding of the first question. On the number property question, however, a more difficult conceptual problem, he engaged in a mental search process that eventually yielded a good solution: "If it was ten, if it was zero, it could be divided evenly by the base number, I guess. Would that work? Yeah, that would work."

The students' responses to impasse situations seem to bear little relationship to the type of instruction received. The most persistent student (Steph) and the two least persistent students (Brenda and Cathy) were all from the problem-based group; Debbi and Frank were from the explicit instruction group. (The remaining students, Esther and Mandy, from the problem-based and explicit groups respectively, did not encounter any significant impasse situations during the interview.) Thus, while the problem-based instruction was intended to foster the type of theory-posing and persistence that Steph exhibited, it did not seem to have had the desired impact on Brenda and Cathy.

Student beliefs about instruction. After the problem portion of the interview, students were asked to reflect on their experiences during the number base unit. In particular, they were asked what types of classroom experiences they considered helpful in their learning. The experiences mentioned were working by onesself (S), working in groups (G), teacher explanations (TE), teacher's use of visual diagrams or materials (TV), and being able to handle manipulatives (M). Table 12 summarizes the students' responses. The preferred experiences are arranged from left to right in the order above, considered to reflect most traditional to least traditional in style.

Table 12: Student-identified Helpful Class Experiences

Group	Student	Helpful traditional → reform	Sometimes helpful	Not helpful
Problem-based	Esther	S TE		G
	Steph	TV G M		
	Brenda	S TV G		
	Cathy	G M		
Explicit	Frank	TE TV		G
	Debbi	S TE M	G	
	Mandy	TE		

A couple of interesting observations arise from this analysis. First, the students who had scored the highest on the initial post-test, Esther and Frank, were the ones who identified working in groups as definitely not helpful. Frank elaborated with complaints that other students relied on him to explain the material to them when they had missed class.

A second observation is the lack of mention of group work among any of the students from the explicit instruction group, in contrast with the support it enjoyed from the problem-based group (except for Esther). This suggests that the students in the problem-based instruction may have come to value its potential benefit through their experiences.

Summary. The interviews offered more in-depth insight into the students' individual mathematical understanding as well as an opportunity to observe their response to impasse situations and request their reflections on their classroom experiences. The students' mathematical understanding and their persistence in the face of a daunting mathematical problem seemed to be related to the attitudes and prior knowledge they brought into the course, rather than to the type of instruction they received during this study. This is logical given that those attitudes and experiences developed over the course of 12-16 years, and this intervention lasted only $1\,^{1}/_{2}$ weeks.

Students from both treatment groups demonstrated a variety of degrees of understanding and a range of determination in problem-solving. The most successful students were Esther, Debbi, Mandy, and Frank. Esther, Debbi, and Mandy had the strongest high school math backgrounds of those interviewed (included calculus or college-level statistics in their high school math courses). In addition, Esther and Mandy were very highly motivated, considering math their best subject. Debbi was less motivated, but a very strong student overall, having been permitted to attend community college as a senior in high school. Frank seemed to be a strong intuitive thinker who sometimes saw connections in a "flash of insight."

The least successful students were Brenda and Cathy, while Steph was moderately successful. These students are not as easy to categorize. All had similar math courses in high school (or courses with similar titles). Steph and Cathy expressed the least confidence in their ability to do math, yet Steph persisted and achieved some success. Most puzzling was Brenda. She thought math was "ok," and her scores on the beliefs pretest indicate somewhat above-average confidence; yet in the interview she got confused and easily discouraged. It was not clear why she did so poorly.

Students' responses to the classroom experiences they believed helpful in learning math did seem to show a pattern related to which instructional model they had experienced: students who had been in the problem-based group were more likely to identify working in groups as a helpful way to learn math. This may indicate a shift in the students' willingness to accept a new type of mathematics instruction through their experiences in the treatment group.

Thus, the interview data raises further questions. How do specific students, bringing their own unique backgrounds and past experiences, respond to each type of instruction in the classroom? Those with strong backgrounds seem to do well regardless of which type of instruction they receive. However, many highly individual factors could be at work, including the students' anxiety about new types of situations or sensitivity to the social requirements of working in groups. In addition, the interview setting is yet another, possibly intimidating situation. Factors such as these may impact the outcomes described in as-yet-unknown ways.

Chapter 5

This study was undertaken to examine whether differences in student understanding might be explained by the type of instruction received. Sections of students taking a mathematics course for preservice elementary teachers in Fall 2003 and Spring 2004 were assigned to two types of instruction during a unit introducing number bases: one group was taught via intuitive conceptual explicit instruction and the other via a problem-based approach. The research questions investigated were the following:

1. Did either instructional approach result in better skill in translating from base ten to other bases, and from other bases to base ten?

2. Did either instructional approach result in better retention of how to translate between base ten and other bases (as evaluated on the final exam, approximately two months later)?

3. Did either instructional approach result in better ability to make conceptual connections, such as evaluating whether a given counting system exhibits place value or extending the meaning of place value to fractional place values?

4. Did either instructional approach result in better retention of the ability to make conceptual connections (as evaluated on the final exam, approximately two months later)?

5. Did either instructional approach result in better ability to articulate what place value means, using examples and non-examples to illustrate the essential differences between place-value systems and non-place-value systems?

6. Did either instructional approach result in greater ability to explain their thinking about how to solve problems involving number bases?

7. Did either instructional approach result in students' having better problem-solving skills in approaching problems involving number bases?

Overall, it was found that the differences in the scores of the students in the two treatment groups on any of the dependent measures—procedural or conceptual, immediate or delayed, test questions or a written reflection did not seem to be attributable to the type of instruction they received. This suggests that, taken as groups, students from the two treatment groups had comparable outcomes in their learning and in their ability to articulate what they'd learned. Only prior achievement, as indicated by SAT scores, was significantly related to the study measures. However, mild interaction between treatment and SAT score was observed for one dependent measure—the delayed conceptual post-test; in this case the problem-based treatment seemed to be more effective for students with lower math SAT scores.

Clinical interviews with a small sample of students offered more in-depth insight into the students' mathematical understanding as well as an opportunity to observe their response to impasse situations and request their reflections on their classroom experiences. Similar to the results above, the students' mathematical understanding and their persistence in the face of a daunting mathematical problem seemed to be related more to their prior experience and attitudes than to the type of instruction they received during this study. Students from both treatment groups demonstrated a variety of degrees of understanding and a range of determination in problem-solving. Students' responses to the classroom experiences they believed helpful in learning math did seem to show a pattern: students who had been in the problem-based group were more likely to identify working in groups as a helpful way to learn math. This may indicate a shift in the students' willingness to accept a new type of mathematics instruction based on their experiences in the treatment group.

Discussion

One obvious possible reason for the results obtained is that there truly is no difference in the students' learning between the two types of instruction. Perhaps students ultimately gain the same understanding whether they actively engage in a problem-solving process with their peers or listen to a carefully explained lecture on the same content. Informal observation of the differing levels of engagement among students in the same classroom, however, leads to a somewhat revised possibility: perhaps individual students may differ in the quality of understanding they gain from each type of instruction, but on the average each benefits approximately the same number of students. This could also lead to the pattern of "no difference" observed in this study.

Limitations of the study and its implementation may also contribute. This intervention lasted about 1 1/2 weeks. This is a very short period of time in which to effect a change. In addition, the instructors were sometimes unable to overcome their own long-practiced biases, as described in Chapter 4. Thus, the study might not have reached some "critical mass" of time and degree of change in order to have a noticeable impact on the students' learning.

Related to this is the students' own internal beliefs about how to learn—particularly, how to learn math. Within this short period of time, it is unlikely that twelve years of practice in productive or unproductive approaches to learning will undergo substantial change. A striking illustration of this was visible in the interview process.

A recurring theme in the analysis of the interview results was the prevalence of rule-based and procedure-based responses. Even though the instructors in the explicit instruction had explained the concepts behind the rules, and even though the problem-based instruction had not provided rules and procedures, the students all seemed to have attempted to memorize procedures that would enable them to solve the questions they encountered. This was visible in the students' universal enactment of an efficient routine when presented with the first problem. On the second problem, when they could not retrieve the procedure, or got stuck in the middle of the procedure, most of the students aborted their efforts. For example, Frank refused to try the second problem despite his clear ability to reason conceptually on other tasks.

The students' dependence on rule-based approaches is reminiscent of the outcome in the study by Pesek and Kirschner (2000). In their results, fifth-grade students who had first memorized formulas for area and perimeter were less successful thinking conceptually about the need to measure area before painting a room. In the case of the preservice teachers in the present study, they had not learned procedures for number bases prior to the study but they seemed to have internalized an approach to learning mathematics that focused on rules and procedures. When faced with a question that could be solved either procedurally or conceptually they only considered the procedural approach.

An additional confounding student characteristic may be the particular selectivity of the university where this study was undertaken. In recent years the average SAT scores of the incoming freshman class have increased considerably. For 2002-2003, the year in which most of these students would have taken the SAT, the national average on the math portion of the SAT was 519 (CNN.com, August 27, 2003); the average math SAT score for the students in this study was 592. Only 18% of the students in the study scored at or below the national average. Thus, the students in this study have been very successful in classrooms using a traditional form of instruction. They either learn well in that type of environment or have learned to adapt to it. It stands to reason that most are likely to continue to be successful in that more familiar classroom (in the explicit instruction group) and possibly somewhat resistant to change (in the problem-based instruction group).

According to research on American pre-college classrooms, most mathematics instruction is very rote and procedural (Stigler & Hiebert, 1997). Students have learned over the course of many years that the primary expectation in math is that a student can "do" the problem and "get the right answer." Despite this focus, research also shows that some students have learned to see "through" the procedures and are able to infer what general principles or concepts are at work (Rittle-Johnson & Alibali, 1999). It is these students who tend to remain successful through more years of mathematics as they progress through high school (Resnick, 1986b). Given the selectivity of the university where this study was done, it is very likely that a disproportionate number of the students in this study were these latter ones, who grasp concepts quickly but, based on prior experience, focus on memorizing "how to do" the problems for the test. In short, they are successful at "doing" school: studying for exams and getting high grades.

Given this background, perhaps it is more significant than it first appears that some interaction was observed between treatment and SAT score on the conceptual portion of the delayed post-test. Perhaps with a more representative sample of students this interaction would have been more pronounced. Perhaps the students who are less successful with a traditional approach to school math would discover they *can* understand math concepts when they are engaged in a problem or discussing their insights with a classmate.

The possibility of resistance to change in the problem-based group raises an additional potential confounding influence: the rate of absenteeism in the problem-based group during the study lessons was higher than for the explicit instruction groups (18% versus 8%, respectively). If the characteristics of the students who were omitted due to absenteeism is systematically different in the two groups, this could have impacted the study outcome. Did some of the students stop attending the problem-based class because they believed they already understood the main ideas and didn't feel they needed to spend an additional class day in "unproductive" group work?

Thus, there are many possible influences on the students' scores besides the treatment itself. Characteristics of the study, of the instructors, and of the students themselves may have had substantial impact on the outcomes observed.

Recommendations for Future Study

A number of improvements could be made to the study design to increase its potential impact. Most obviously, lengthening the time of the intervention would give the students more time to experience and (hopefully) learn to maximize the potential benefit of an unfamiliar teaching style. In the problem-based classes, students could be exposed to more opportunities to experience active engagement and student-generated solutions; in the intuitive explicit instruction sections, students could become more aware of the conceptual focus and potentially internalize this value in their own learning.

In addition, the instructors could benefit from more time discussing ways to enact challenging lessons in a way that is true to the intended type of instruction. In the problem-based class, learning to ask scaffolding questions that can support students' investigations without "giving away" the answer is a difficult skill. It requires a deep understanding of the content and of the conceptual hurdles faced by the student.

In the conceptually-focused lecture, as well, improvements are possible. Instructors can improve the clarity of their explanations and tighten the connection between the symbolic and conceptual aspects. Students could be given more tasks requesting they "explain why" a procedure works, refocusing them on the conceptual content of the course.

The variations between individual instructors were noted but not analyzed in depth in this study. It is possible that these variations are significant. Does the inclusion of more thought-provoking questions within an explicit-instruction lesson fundamentally change the level of engagement of the students, leading to a difference in their quality of learning? In the problem-based classrooms, what types of verbal interchange did instructors have with students as they worked in groups? Different comments--"That looks good" vs. "Can you explain why?"--are likely to result in different student actions. These analyses were beyond the scope of this study but bear further investigation.

Finally, the dynamics of the group learning in the problem-based classroom seems to have been significant, as well. Both Brenda and Cathy seem to have been passive participants in an investigation where they heard a rule—"you can't go over five in base six"—but did not pursue the questioning or testing needed to understand the rule. Encouraging groups to take responsibility for each person's understanding might help prevent this from occurring.

Concluding Remarks

Both of the instructional types used in this study—intuitive conceptual explicit instruction and problem-based instruction—seemed to result in student learning. The quality of understanding was comparable for students in the two groups, although this may be influenced by confounding factors such as the students' greater familiarity with explicit instruction and the limited length of time in a novel, problem-based, setting. A hint of interaction between students' prior achievement and treatment suggests that further investigation is merited, particularly incorporating a longer intervention and more attentiveness to the specifics of the teaching process.

This study was situated in a very particular, selective university environment. As described earlier, the students in this study were, for the most part, very successful with traditional mathematics instruction. Thus, they are not representative, and certainly do not allow any inferences about a more general population of students. However, this study does suggest that in most cases these successful students continue to succeed in a problem-based classroom.

The larger context of this study includes the students' extensive prior experiences with procedural math learning, primarily memorizing routines to reproduce on tests and quizzes. Given this background, it is not unexpected that students were reluctant to change their focus to the conceptual underpinnings of a new topic. Nevertheless, most of the students demonstrated some degree of conceptual understanding.

As college students, these preservice teachers are in a unique stage of life where most are open to new ideas and challenges. With persistent challenge and encouragement from the instructor, I believe many can be influenced to begin to value and even enjoy thinking conceptually about mathematics, probing "why" as well as "how." Ideally, these same individuals will someday convey this enthusiasm to their young students, as well.

Appendix A: Study Lesson Plans

Explicit Instruction

Summary:

Day 1: Counting Systems Based on Tens

Counting Systems Based on Fives

Counting in Other Bases

Homework:

Read Text 2.3 (Bassarear, 2001a). You do not need to memorize all the symbols for the different symbols discussed, but do understand the new characteristics introduced. Know all the terms in blue used to describe the various systems.

Text 2.3 (Bassarear, 2001a) #11abcd, 12acdf

Day 2: Changing From Other Bases to Base Ten

Changing From Base Ten to Other Bases

Homework:

Text 2.3 (Bassarear, 2001a) #13abg, 14ace

Day 3: Finding Base x

Homework:

Text 2.3 (Bassarear, 2001a) #16, 17

Day 4: Homework Questions and Review

Counting Systems Based on Tens

Base Ten Blocks	Egyptian Hieroglyphics	Hindu-Arabic Notation
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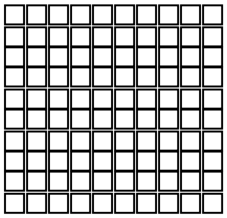

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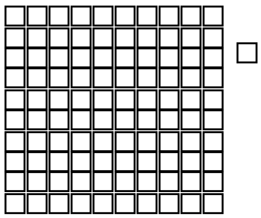

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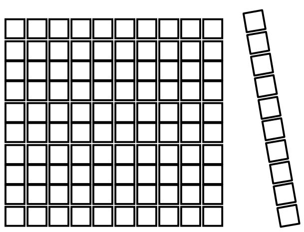

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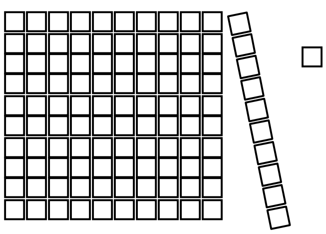

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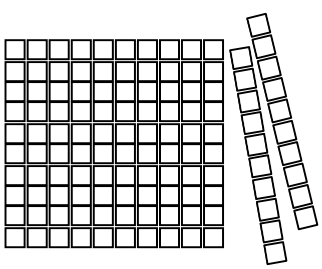

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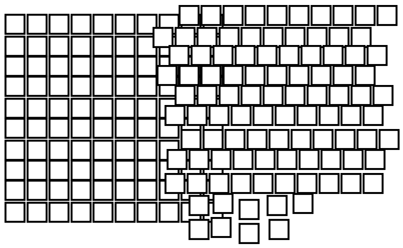

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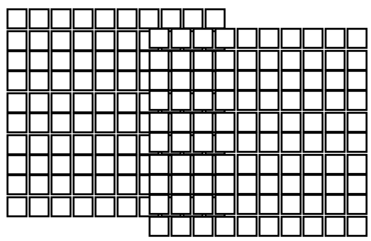

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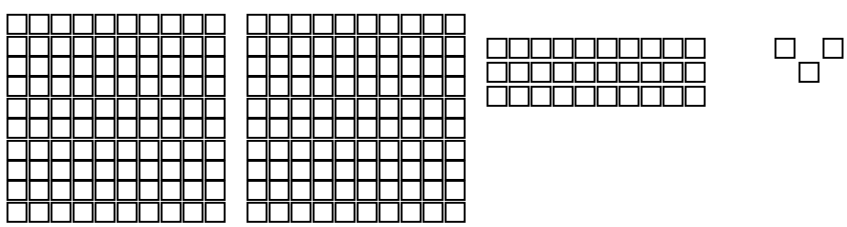
Questions lecturer will pose and answer:

1. Consider the following quantity:



a. How would you represent this quantity in Egyptian hieroglyphics?
b. How would you represent this quantity in Hindu-Arabic notation?

2. In which representation, Egyptian or Hindu-Arabic, does the *symbol* represent the size or *value* of the group (i.e., ones, tens, hundreds) ?

3. In which representation, Egyptian or Hindu-Arabic, does the *position* of the symbol indicate the size or *value* of the group (ones, tens, hundreds)?

4. What do you think "place value" means? Which representation, Egyptian or Hindu-Arabic, seems better-described by the phrase "place value"? Explain.

Counting Systems Based on Fives

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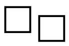
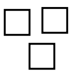
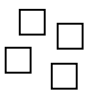
	AA	B
	AAA	C
	AAAA	D
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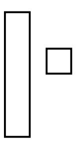
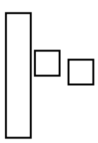
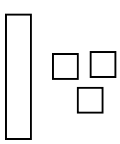
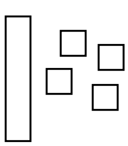
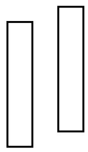
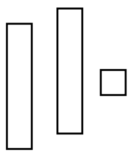
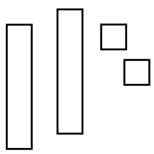
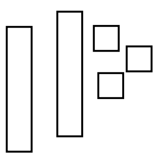
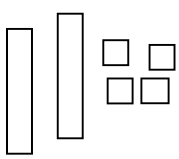
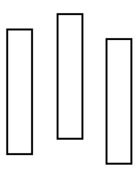
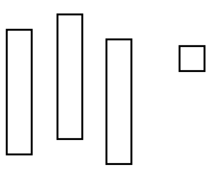
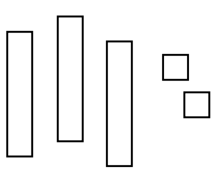
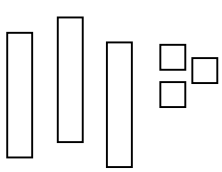
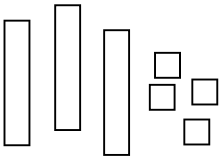
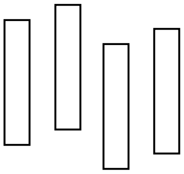
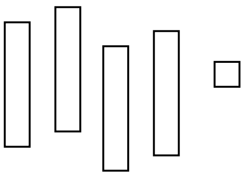
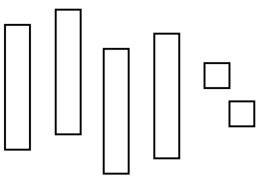
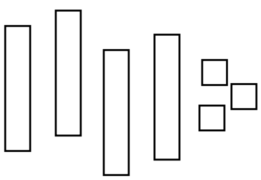
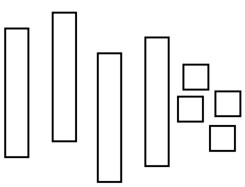
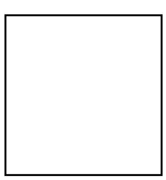
	BA	AA
	BAA	AB
	BAAA	AC
	BAAAA	AD

	BB	B0
	BBA	BA
	BBAA	BB
	BBAAA	BC
	BBAAAA	BD
	BBB	C0
	BBBA	CA
	BBBAA	CB
	BBBAAA	CC

	BBBAAAA	CD
	BBBB	D0
	BBBBA	DA
	BBBBAA	DB
	BBBBAAA	DC
	BBBBAAAA	DD
	C	A00

Questions Lecturer will pose and answer:

1. Continue counting in each of the systems above for another ten numbers.

2. a. Suppose only the symbols 0, A, B, C, and D are available. How would you express one million in each system? b. If you could invent more symbols, how might your answer change?

3. In which system does the symbol indicate the value of the group?

4. In which system does the position of the symbol indicate the value of the group?

5. Which system has "place value"?

6. Rewrite the place value system using our traditional symbols for each digit and a subscript of "5" to indicate that the places are based on groups of fives.

Counting In Other Bases

Systems using the pattern of our system and the one on the right above are termed "place value" systems because the place or position of the symbol indicates its value . Any base can be used for the groups. For example, a "base 4" system, is generally written:

1, 2, 3, 10, 11, 12, 13, 20, 21, 22, 23, 30, 31, 32, 33, 100, 101, 102, 103, 110

(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20)

Here are some more examples of counting in different place value systems.

Counting in Base Five:

1, 2, 3, 4, 10, 11, 12, 13, 14, 20, 21, 22, 23, 24, 30, 31, 32, 33, 34, 40, 41, 42, 43, 44, 100, 101, 102, etc.

Counting in Base Six:

1, 2, 3, 4, 5, 10, 11, 12, 13, 14, 15, 20, 21, 22, 23, 24, 25, 30, 31, 32, 33, 34, 35, 40, 41, 42, 43, 44, 45, etc.

Counting in bases larger than base 10 (our number system's base) requires additional symbols. For example, in base twelve symbols for ten and eleven are needed. In the example below, T = ten and E = eleven:

1, 2, 3, 4, 5, 6, 7, 8, 9, T, E, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19 1T, 1E, 20, 21, 22, 23, 24, 25, 26, 27, etc.

Caution Regarding the Third (and subsequent) Columns: Make sure that once a third column is added you go through the entire two-column pattern before changing the third column to the next digit.

For example, in base five, the counting would continue as follows:

103, 104, 110 (NOT 201), 111, 112, 113, 114, 120 (NOT 210), etc.

Exercises:

1. Continue counting in base 5 to 1000_5. (Note: The subscript indicates the numeral is in base five).

2. Invent symbols as needed to count in base fifteen. Count to 30 in base fifteen.

3. How many symbols would be needed to count in

    a. base eight? List them.

    b. base twenty-six?

Changing from Other Bases to Base Ten

Our numbers are written with place value in base ten. This means that a number like

3402 means $3 \times 10^3 + 4 \times 10^2 + 0 \times 10^1 + 2 \times 10^0$. (Remember: $10^0 = 1$.)

Numbers can be written in other bases, as well:

In base five, 3402_5 means $3 \times 5^3 + 4 \times 5^2 + 0 \times 5^1 + 2 \times 5^0 = 477$ in base ten.

In base eight, 127_8 means $1 \times 8^2 + 2 \times 8^1 + 7 \times 8^0 = 87$ in base ten.

Use this procedure to change each of the following to base ten from the indicated base.

1. 37_8

2. 146_9

3. 4002_5

4. 10011_2

5. 666_7

6. 21012_3

*7. $3TE_{12}$

8. 21012_4

*Note: Base 12 uses two additional digits: T = ten and E = eleven

Changing from Base Ten to Other Bases

To change a number from base ten to another base, the following procedure may be used:

Example 1: Change 136 to base 7.

Make a place-value grid showing the base seven place values:

$7^3=343$	$7^2=49$	$7^1=7$	$7^0=1$

Include enough columns so you are sure you will be able to represent the quantity correctly; a good rule-of-thumb is to include the first column that is too large for your quantity in the grid.

Now, think as follows:

How many groups of 49 can be made from 136?

$136 \div 49 = 2.7755\ldots$, so **2**. We place a 2 in the 49's column:

$7^3=343$	$7^2=49$	$7^1=7$	$7^0=1$
	2		

How much remains to be represented? $136 - 2\,(49) = 38$

How many groups of 7 can be made from 38?

$38 \div 7 = 5.428\ldots.$, so **5**. We place a 5 in the 7's column:

$7^3=343$	$7^2=49$	$7^1=7$	$7^0=1$
	2	**5**	

How much remains to be represented? $38 - 5\,(7) = 3$.

Only the 1's column remains, so we place a 3 in the 1's column:

7^3=343	7^2=49	7^1=7	7^0=1
	2	**5**	**3**

So our answer is 253_7.

We can check by multiplying out: $253_7 = 2 \cdot 7^2 + 5 \cdot 7^1 + 3 \cdot 7^0 = 2(49) + 5(7) + 3(1) = 136$.

Example 2: Change 136 to base 3.

Make a place-value grid showing the base three place values. Include one column that is too large for the quantity given (136):

3^5=243	3^4=81	3^3=27	3^2=9	3^1=3	3^0=1

How many groups of 81 can be made from 136?

$136 \div 81 = 1.679\ldots$, so **1**. We place a 1 in the 81's column:

3^5=243	3^4=81	3^3=27	3^2=9	3^1=3	3^0=1
	1				

How much remains to be represented? $136 - 1(81) = 55$.

How many groups of 27 can be made from 55?

$55 \div 27 = 2.037\ldots$, so **2**. We place a 2 in the 27's column:

3^5=243	3^4=81	3^3=27	3^2=9	3^1=3	3^0=1
	1	**2**			

How much remains to be represented? $55 - 2(27) = 1$

How many groups of 9 can be made from 1?

0, so we place a 0 in the 9's column.

Similarly, no groups of 3 can be made from 1,

so we place a 0 in the 3's column as well.

The remaining 1 is represented by a 1 in the 1's column.

$3^5=243$	$3^4=81$	$3^3=27$	$3^2=9$	$3^1=3$	$3^0=1$
	1	**2**	**0**	**0**	**1**

So our answer is 12001_3.

We can check by multiplying out:

$$12001_3 = 1 \cdot 3^4 + 2 \cdot 3^3 + 0 \cdot 3^2 + 0 \cdot 3^1 + 1 \cdot 3^0 = 1(81) + 2(27) + 0 + 0 + 1(1) = 136.$$

Changing from Base Ten to Other Bases (alternate procedure)

To change a number from base ten to another base, the following division-like procedure may also be used:

Example 1: Change 136 to base 7.

7	|	136	(Divide 7 into 136. The quotient is 19, remainder 3)
7	|	19 r 3	(Divide 7 into 19. The quotient is 2, remainder 5)
7	|	2 r 5	(Divide 7 into 2. The quotient is 0, remainder 2)
7	|	0 r 2	

The base seven representation of 136 is 253_7 (Read the remainders up from the bottom to get the digits in the base seven representation).

Note: This division procedure is carried out until a quotient of 0 is obtained, as above.

Check: $253_7 = 2 \times 7^2 + 5 \times 7^1 + 3 \times 7^0 = 136.$

Example 2: Change 136 to base 3.

3	|	136	(Divide 3 into 136. The quotient is 45, remainder 1)
3	|	45 r 1	(Divide 3 into 45. The quotient is 15, remainder 0)
3	|	15 r 0	(Divide 3 into 15. The quotient is 5, remainder 0)
3	|	5 r 0	(Divide 3 into 5. The quotient is 1, remainder 2)
3	|	1 r 2	(Divide 3 into 1. The quotient is 0, remainder 1)
3	|	0 r 1	

The base three representation of 136 is 12001_3.

Check: $12001_3 = 1 \times 3^4 + 2 \times 3^3 + 0 \times 3^2 + 0 \times 3^1 + 1 \times 3^0 = 136$

Use either procedure given to change each number below to the indicated base. Be sure to check all answers as illustrated above.

1. 451 to base seven.

2. 136 to base eight.

3. 451 to base two.

4. 128 to base twelve.

5. 67 to base three.

6. 1253 to base eleven.

Finding Base x

Sometimes a problem asks you to find the "mystery" base. For example, in what base x does $32_x = 26$?

Recall what base "x" means: $32_x = 3x^1 + 2x^0$. So this type of problem can be solved by setting up and solving an equation, as follows:

$$3x + 2 = 26$$

$$3x = 24$$

$$x = 8$$

The mystery base is 8.

This can be checked: $3(8) + 2(1) = 24 + 2 = 26$.

Example 2: In what base x does $51_x = 66$?

Solution:

$$5x + 1 = 66$$

$$5x = 65$$

$$x = 13$$

So the mystery base is 13.

If the number in the mystery base has three digits, the equation is a little more complex:

Example 3: In what base x does $201_x = 51$?

Recall again what being written in base x means: $201_x = 2x^2 + 0x^1 + 1x^0$.

Solution:

$$2x^2 + 0x + 1 = 51$$

$$2x^2 = 50$$

$$x^2 = 25$$

$$x = 5$$

Example 4: In what base x does $130_x = 40$?

Solution: $1x^2 + 3x + 0 = 40$

$x^2 + 3x - 40 = 0$

$(x + 8)(x - 5) = 0$

$x = -8$ or $x = 5$

Since the base must be positive, it must be base 5.

Exercises: Find the mystery base.

1. $27_x = 31$

2. $66_x = 48$

3. $103_x = 84$

4. $150_x = 84$

5. $111_x = 13$

Problem-Based Lesson Plans

Summary

Day 1: Inventing a New Counting System

Homework:

Read Text 2.3 (Bassarear, 2001a). You do not need to memorize all the symbols for the different symbols discussed, but do understand the new characteristics introduced. Know all the terms in blue used to describe the various systems.

Day 2: Analyzing Counting Systems

Counting Systems Based on Tens

Counting Systems Reconsidered

Homework:

Text 2.3 (Bassarear, 2001a) #11abcd, 12acdf

Day 3: Exploration 2.8, Parts 1, 3 (Bassarear, 2001b, pp. 43 – 47)

Homework:

Text 2.3 (Bassarear, 2001a) #13abg.

Day 4: Packaging Widgets

Exploration 2.8, Part 4 (Bassarear, 2001b, pp. 47-48, omit "Looking Back…").

Homework:

Text 2.3 (Bassarear, 2001a) #14ace.

Day 5: Finding the Mystery Base

Homework:

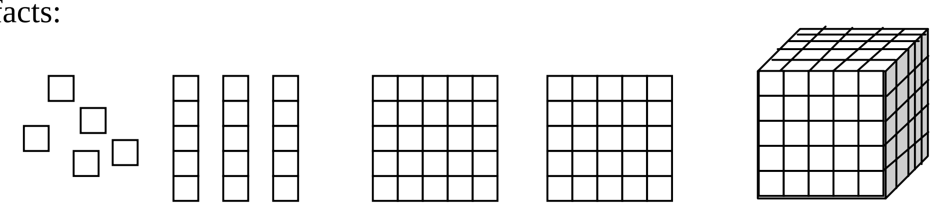
Text 2.3 (Bassarear, 2001a) #16, 17

Inventing a New Counting System

(adapted from Bassarear, 2001b, pp. 37 – 38)

Many ancient tribes had only a few number words and number symbols. Any quantity larger than their largest word or symbol would simply be called "many." Imagine that you are a member of such a tribe. Your present counting system consists of the following: A = 1, B = 2, C = 3, . . . Z = 26. Any quantity larger than 26 you call "many." For the past thousand years your group has been hunter-gatherers, so no one needed more number words. Now, however, you all have settled down and begun to plant crops and keep herds of sheep and cows. You have begun to recognize a need for a more extensive counting system.

The elders of your tribe have decided they want a counting system based on groups of five, using only the symbols 0, A, B, C, D, and related to the following artifacts:



You and a group of other bright young tribe members have been selected to invent a counting system with these characteristics that will serve the needs of the tribe for many years to come.

After deciding how you will use the symbols and artifacts given above, you need to prepare a poster illustrating how the symbols and quantities relate. Show how to count from 1 to 30 in your system, how to notate one hundred, and one thousand. Also indicate what rules summarize the ways that numbers are formed, so that your fellow tribe members will learn quickly how to use your system.

Analyzing Counting Systems

Analyze each group's counting system in terms of the following characteristics:

1. Is it additive?

2. Does it use powers of ten?

3. Powers of any other base?

4. Does it have place value?

5. Does it have a symbol for zero?

Cite examples to justify each of your answers to the above questions.

Counting Systems Based on Tens

Base Ten Blocks	Egyptian Hieroglyphics	Hindu-Arabic Notation
□	|	1
□□	||	2
□□□	|||	3
□□□□	||||	4
□□□□□	|||||	5
□□□□□ □	||||||	6
□□□□□ □□	|||||||	7
□□□□□ □□□	||||||||	8
□□□□□ □□□□	|||||||||	9
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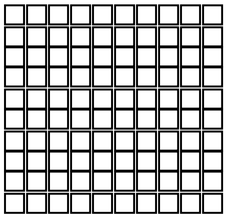

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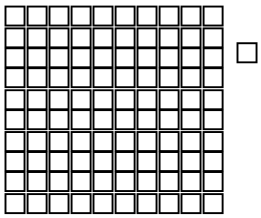
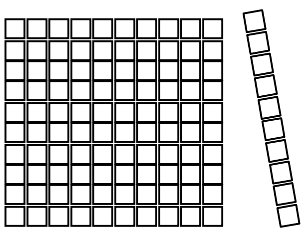
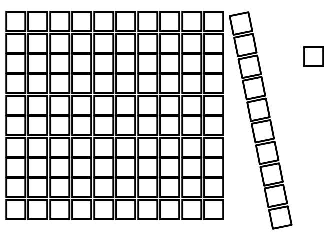


𓍢 | 101

. . . .



𓍢 ∩ 110



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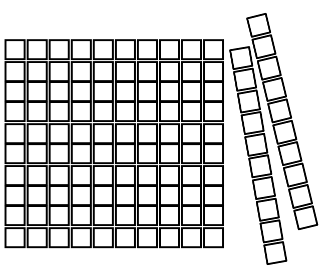

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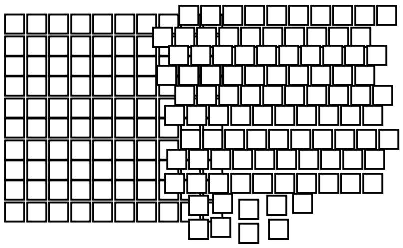

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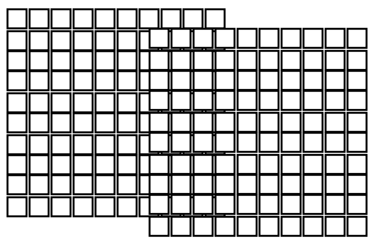

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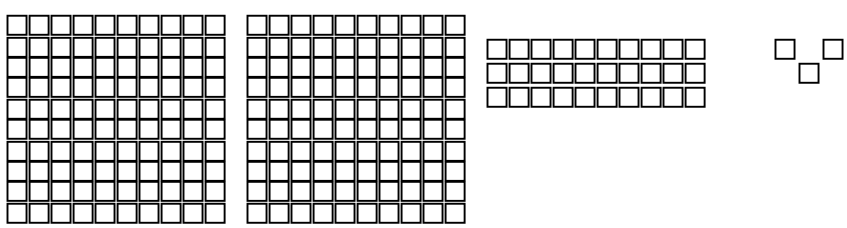
Discussion Questions:

1. Consider the following quantity:



a. How would you represent this quantity in Egyptian hieroglyphics?
b. How would you represent this quantity in Hindu-Arabic notation?

2. In which representation, Egyptian or Hindu-Arabic, does the *symbol* represent the size or *value* of the group (i.e., ones, tens, hundreds) ?

3. In which representation, Egyptian or Hindu-Arabic, does the *position* of the symbol indicate the size or *value* of the group (ones, tens, hundreds)?

4. What do you think "place value" means? Which representation, Egyptian or Hindu-Arabic, seems better-described by the phrase "place value"? Explain.

Counting Systems Reconsidered

Materials needed: Base ten blocks, base five blocks.

1. a. Which, if any, of the counting systems developed by your class remind you of Egyptian hieroglyphics? Why?

   b. Which, if any, have place value? Do the place value systems include a symbol for zero? Are these systems more like the Babylonian system or the Hindu-Arabic system?

2. Reconsider the flats, longs, and units used in your invented counting system. These are often called "base five" blocks. Why?

   Try to construct a counting system using these blocks and 0, A, B, C, and D which is as analogous as possible to base ten. Show how to count from 1 to 26 in this system. You may want to move back and forth between base ten blocks and base five blocks to make as many connections as possible. Some important considerations are the following:

   a. When does the counting system add a new column?

   b. When does the counting system use a new symbol?

   c. How many symbols are required for this counting system?

   d. How does this counting system show "how many" of "what size" blocks make up a given quantity?

3. a. Suppose only the symbols 0, A, B, C, and D are available. How would you express one million in a hieroglyphics-type system?

   b. How would you express one million in a place value system?

   c. If you could invent more symbols, how might your answer(s) change?

4. Using the place value system constructed in #3 above, answer the following:

What number comes after each of the following? Explain.

a. AA b. BD c. BAD d. ABD

What number comes before each of the following? Explain.

a. D0 b. B00 c. D0D0

5. Traditionally, the symbols 0, 1, 2, 3, and 4 are used in base 5 instead of 0, A, B, C, and D. Count from 1 to 26 in base 5 using these symbols. Then translate each of the values in #4 to these symbols and answer #4 again.

EXPLORATION 2.8 Different Bases

In this exploration, we will explore several different bases and their relationships to one another. The primary purpose of these explorations is to deepen your understanding of base 10—that is, your understanding of place value and the function of zero. I have found that the structures of a system are often best seen by putting the familiar in an unfamiliar context; this was the reason for the Alphabitian exploration. Over the past ten years, I have observed that an increasing number of elementary teachers have their students explore different bases. These teachers have told me that the explorations with different bases help their students come to a better understanding of place value and how base 10 works, and that this knowledge results in better problem-solving in base 10.

We will begin by exploring three different bases after noting an important caution: Working in different bases is often difficult at first. If you find yourself groping about, consider the following tools to help you:

Connections

Can I connect what I know about base 10 to this new base?

Problem-Solving Strategies

- What if I made a table? Would that help?
- As I make my tables, what patterns do I see?

Transition numbers are often difficult; in base 10, for example, 99 is a transition number. When you encounter transition numbers, consider the following questions:

- Might it help to use manipulatives?
- Might it help to stop and say what the symbols mean?

Patterns

- Are there patterns that are true for all bases?
- Are there patterns that are true for some bases but not others?

You may have already surmised that the new Alphabitian system is actually a base 5 system that uses A, B, C, and D instead of 1, 2, 3, and 4. Before proceeding with the following explorations, you might want to make a base 5 counting chart (1, 2, 3, 4, 10, 11, 12, 13, 14, 20, 21 . . .) and become comfortable with this system.

PART 1: Learning about different bases

Base 6

The following activities are designed to help you understand base 6.

1. a. If your instructor has base 6 manipulatives, get them if you wish. If your instructor does not have them, you might want to make your own, using graph paper.
   b. Take some time, individually or in a group, to learn how to count in base 6.
   c. As you learn about this new base, remember to have a group check-in frequently to be sure you are not leaving anyone behind. One way to do this is to ask a question and see whether everyone can answer it and explain it.

For example, pick a number and ask what the next number is or what the previous number is. Record your work.

2. Once you have determined that all members of the group understand base 6, respond to the following questions.
   a. What helped you to accomplish this task?
   b. What patterns do you see in this base?
3. Now that you are able to count in base 6, try the following exercises, both to assess your understanding and to stretch your understanding. Answers are on page 48.
   a. What comes after each of the following?[3] 25_6 555_6 1235_6
   b. What comes before each of the following? 40_6 300_6 12340_6
4. Describe one area in which you had initial difficulty—for example, deciding what comes after 555_6. Describe the difficulty; for example, did you have no idea, or were you debating between two possible answers? Describe what it was that helped you get unstuck.

Base 2

The following activities are designed to help you understand base 2.

5. a. If your instructor has base 2 manipulatives, get them if you wish. If your instructor does not have them, you might want to make your own, using graph paper.
   b. Take some time, individually or in a group, to learn how to count in base 2.
   c. As you learn about base 2, remember to have a group check-in frequently, as in Step 1(c).
6. Once you have determined that all members of the group understand base 2, respond to the following questions:
   a. What helped you to accomplish this task?
   b. What patterns do you see in this base?
7. Now that you are able to count in base 2, try the following exercises, both to assess your understanding and to stretch your understanding. Answers are on page 48.
   a. What comes after each of the following? 110_2 1011_2 101111_2
   b. What comes before each of the following? 100_2 111_2 1010_2
8. Describe one area in which you had initial difficulty—for example, deciding what comes after 1011_2. Describe the difficulty; for example, did you have no idea, or were you debating between two possible answers? Describe what it was that helped you get unstuck.
9. Some students find base 2 the hardest to learn, whereas others find it the easiest. Why do you think this is so?

Base 12

The following activities are designed to help you learn about base 12.

[3] A subscript at the end of a number indicates the base. Thus, 25_6 means the amount 25 in base 6. When there is no subscript, the number is in base 10.

10. a. If your instructor has base 12 manipulatives, get them if you wish. If your instructor does not have them, you might want to make your own, using graph paper.
   b. Take some time, individually or in a group, to learn how to count in base 12. Then read part (c).
   c. Base 12 presents a problem that did not come up in the other bases: How do you represent the amount that in base 10 is called "10" in base 12? What do you think? Write down your first thoughts.
   d. Discuss the problem in part (c) in your group. Discuss why this is a problem, and discuss possible ways to resolve it. Write down your thoughts.
11. a. Once you have resolved the problem of new digits by consulting your instructor or from a class discussion, take some time, individually or as a group, to learn how to count in base 12.
   b. As you learn about base 12, remember to have a group check-in frequently, as in Step 1(c).
12. Once you have determined that all members of the group understand base 12, respond to the following questions:
   a. What helped you to accomplish this task?
   b. What patterns do you see in this base?
13. Now that you are able to count in base 12, try the following exercises, both to assess your understanding and to stretch your understanding. (*Note*: There is no answer key for this question because it depends on the symbols that your class invents for the two numbers after 9 in base 12.)
   a. What comes after each of the following? 39_{12} 79_{12} 909_{12}
   b. What comes before each of the following? 100_{12} 450_{12} 1010_{12}
14. Describe one area in which you had initial difficulty—for example, deciding what comes after 19_{12}. Describe the difficulty; for example, did you have no idea, or were you debating between two possible answers? Describe what it was that helped you get unstuck.

PART 3: Translating from another base into base 10

A traveler going from the United States to Great Britain has to convert dollars to pounds. Similarly, we will now investigate the process of converting amounts from one base to another. Before we proceed, I need to emphasize strongly that the point of the following explorations is not to master the procedure for translating from one base to another. Rather, these explorations have two aims: (1) to give you practice in applying the problem-solving tools you are developing, and (2) to deepen your understanding of base 10, which is the base that you will (at least implicitly) teach your future students, whether you are a kindergarten teacher helping your students learn to count, a second-grade teacher helping your students learn how to subtract, or a sixth-grade teacher helping your students learn decimals.

To help keep you focused on this goal and to make the process a bit more fun, let us go into the future and imagine that there are intelligent beings on every planet of the solar system. The beings on Mercury use what we call base 2; the beings on Venus use what we call base 5; the beings on Mars use what we call base 6; and the beings on Jupiter use what we call base 12.

1. Each planet would say that *it* has base "10." Do you see why? Discuss this question in your group, and then write down why each planet would say it has base "10." Imagine explaining this to a classmate who was not present for this discussion and whose initial reaction to the statement is a puzzled expression.
2. The United Nations of the Solar System is having its monthly interplanetary meeting. You are the hotel clerk who is handling reservations. Unfortunately, the faxes that come from each planet contain numbers in that planet's own base. Thus, you have to determine how many rooms to reserve for each delegation.
   a. The Mercury delegation says that it needs 1101 rooms. How many rooms will you reserve?
   b. The Venus delegation says that it needs 34 rooms. How many rooms will you reserve?
   c. The Mars delegation says that it needs 23 rooms. How many rooms will you reserve?
   d. The Jupiter delegation says that it needs 18 rooms. How many rooms will you reserve?
3. Think about how you translated from each of these bases into base 10. Develop an algorithm that can be used for each of the bases, using the following guidelines. Imagine that you are going home soon and that the night clerk is not very confident working with different bases. Write directions for this clerk so that she can make the appropriate number of reservations, regardless of which base is used. At the same time, imagine that the night clerk is curious and wants to

understand. Draw a line down the middle of the page. On the left, describe what to do. On the right, explain why this works; in other words, justify each step in the algorithm.

4. Now imagine that the United Nations of the Solar System is having its biannual interplanetary conference. The number of participants at this conference is much greater than at the monthly meeting. As before, imagine that you are the clerk and need to reserve the appropriate number of rooms.

   a. The Mercury delegation says that it needs 100110 rooms. How many rooms will you reserve?
   b. The Venus delegation says that it needs 403 rooms. How many rooms will you reserve?
   c. The Mars delegation says that it needs 203 rooms. How many rooms will you reserve?
   d. The Jupiter delegation says that it needs 109 rooms. How many rooms will you reserve?

5. Once your group is confident that you have reserved the correct number of rooms, take a moment to look back on what you have learned. Did your algorithm from Step 3 work for larger numbers? That is, did it stretch well or did it break down? If it did not stretch well, explain the new algorithm that you developed.

Packaging Widgets

Materials Needed: (optional) 46 beads, cups

1. The Wacky Widget Company packages widgets as follows:

   5 widgets make a box

   5 boxes make a case

   5 cases make a skid

Package the 46 "widgets" into boxes, cases, etc. Write your result as a base five number.

2. The F. B. Night Widget Company packages widgets differently:

   12 widgets make a box

   12 boxes make a case

   12 cases make a skid

How would the F. B. Night Widget Company package the 46 widgets? Write your result as a base twelve number.

3. The Acme Widget Company packages

   3 widgets to a box

   3 boxes to a case

   3 cases to a skid

How would the Acme Widget Company package the 46 widgets? Write your result as a base three number.

4. Reflect on the process you used above. Summarize it so that the steps work for any base.

5. Do Exploration 2.8 pp. 47 - 48 (Part 4, but can omit "Looking Back...")

Exploration 2.8, Part 4 (Bassarear, 2001b, pp. 47–48)

PART 4: Translating from base 10 into other bases

1. The United Nations of Earth has held new elections, and it is time to send the new delegations to each of the planets. You are in charge of sending a fax to each planet telling them how many people are in the new delegation. However, you realize that the people who receive the messages may not translate accurately from our base into their base. You have decided that you will tell them the number of people in the delegation in their system. Below are the numbers of people in the delegation to each planet in base 10.
   a. The delegation to Mercury will consist of 34 people. Translate this amount into the base used on Mercury.
   b. The delegation to Venus will consist of 34 people. Translate this amount into the base used on Venus.
   c. The delegation to Mars will consist of 34 people. Translate this amount into the base used on Mars.
   d. The delegation to Jupiter will consist of 34 people. Translate this amount into the base used on Jupiter.
2. Think about how you translated from base 10 into each of these bases. Develop an algorithm that can be used for each of the bases, using the following guidelines. Imagine that you need to give directions to the person who will send faxes from the United Nations of Earth to each of the planets after the next elections. Draw a line down the middle of the page. On the left, describe what to do. On the right, explain why that works; in other words, justify each step in the algorithm.
3. Now imagine that you are a travel agent on Earth handling reservations for tours to each of these planets. You need to reserve the appropriate number of rooms.
   a. The tour to Mercury has 208 people. Translate this amount into the base used on Mercury.
   b. The tour to Venus has 208 people. Translate this amount into the base used on Venus.
   c. The tour to Mars has 208 people. Translate this amount into the base used on Mars.

d. The tour to Jupiter has 208 people. Translate this amount into the base used on Jupiter.

4. Once your group is confident that you have reserved the correct number of rooms, take a moment to look back on what you have learned. Did your algorithm from Step 2 work for larger numbers? That is, did it stretch well, or did it break down?

Looking Back on Exploration 2.8

1. Describe the most important things that you learned while translating to and from base 10. In each case, first describe what it was that you learned and then describe how you learned it.
2. Describe something from this exploration that you are still not clear about.
3. Examine the five process standards: Problem Solving, Reasoning, Communication, Connections, and Representation. Select one of the subheadings from this set that you feel you "own" better as a result of this exploration. Describe what you learned.

Answers to Exploration questions

Exploration 2.8: Base 6, p. 44

30_6 comes after 25_6.	1000_6 comes after 555_6.	1240_6 comes after 1235_6.
35_6 comes before 40_6.	255_6 comes before 300_6.	12335_6 comes before 12340_6.

Exploration 2.8: Base 2, p. 44

111_2 comes after 110_2.	1100_2 comes after 1011_2.	110000_2 comes after 101111_2.
11_2 comes before 100_2.	110_2 comes before 111_2.	1001_2 comes before 1010_2.

Finding the Mystery Base

When the XYZ Widget Company packages 46 widgets they make 5 boxes and have 6 widgets left over. How many widgets do they package per box?

Here is another way of asking the same question: $56_x = 46$. What is x?

(Make sure it makes sense to you why this question is the same as the story problem above.)

Solve each of the following. If it is helpful, imagine or sketch a story problem like the one above, or sketch base blocks (units, longs, flats, etc.).

1. $27_x = 31$

2. $66_x = 48$

3. $103_x = 84$

4. $150_x = 84$

5. $111_x = 13$

6. Summarize the approach you used to help you solve the problems above. State your methods so that it will apply to any similar problem.

Homework

(from Bassarear, 2001a, p. 108)

11. Tell what comes after:

a. 34_5 b. 1011_2 c. 99_{12} d. 7099_{12}

e. 101_2 f. 111_{12} g. 124_5 h. 405_6

12. Tell what comes before:

a. 1010_5 b. 340_5 c. 100_{12} d. 1110_2

e. 1010_2 f. 110_{12} g. 120_4 h. 60_7

13. Convert the following numbers into base 10.

a. 41_5 b. 55_6 c. 210_5 d. 2104_5

e. 101_5 f. 1111_6 g. 303_4 h. 606_7

i. 1101_2 j. 10001_2 k. 99_{12} l. 909_{12}

14. Convert the following numbers from base 10 into the designated base.

a. $44_{10} = ?_5$ b. $152_{10} = ?_5$ c. $92_{10} = ?_2$

d. $206_{10} = ?_2$ e. $72_{10} = ?_{12}$ f. $402_{10} = ?_{12}$

g. $44_{10} = ?_6$ h. $1252_{10} = ?_6$ i. $144_{10} = ?_{12}$

j. $100_{10} = ?_5$ k. $99_{10} = ?_{12}$ l. $1052_{10} = ?_5$

m. $2{,}500{,}000_{10} = ?_5$ n. $2{,}500{,}000_{10} = ?_2$

16. Tell what base makes the following statement true:

$23_{10} = 25_x$

17. For what base x is this statement true? $598_{10} = 734_x$

Text problem answers:

11a) 40_5 b) 1100_2 c) $9T_{12}$ d) $109T_{12}$

12a) 1004_5 c) EE_{12} d) 1101_2 f) $10E_{12}$

13a) $(4 \times 5) + (1 \times 1) = 21$ b) $(5 \times 6) + (5 \times 1) = 35$

g) $(3 \times 16) + (0 \times 4) + (3 \times 1) = 51$

14a) 134_5 c) 1011100_2 e) 60_{12}

16) $x = 9$

17) $x = 9$

Appendix B: Teacher Observation Protocol

adapted from Gordon (1979)

	neutral/misc	highly expository	expository w/ limited interaction	expository w/ moderate/extended interaction	guided discovery	highly pure discovery
1						
2						
3						
4						
5						
6						
7						
8						
9						
10						
11						
12						
13						

	neutral/misc	highly expository	expository w/ limited interaction	expository w/ moderate/extended interaction	guided discovery	highly pure discovery
14						
15						
16						
17						
18						
19						
20						
21						
22						
23						

Instructions:

Using a timer divide the lesson into 3-minute segments. At the end of each segment, place a check mark in the box of the category that best describes the type of instruction that was occurring during that segment.

Definitions:

Neutral/Misc: giving directions, collecting or passing out papers; giving an assignment or announcements

Highly Expository: "The teacher lectures and has no interaction with the students during the major portion of the presentation. The teacher does all of the talking and demonstrating. The teacher neither asks nor answers questions. . . "The teacher completely presents the subject matter for the student to learn, without the student needing to independently create the concepts to be learned or the relationships involved in solving problems related to the presentation."

Sample types of statements:
Teacher states underlying principle, rule, or formula
T describes and executes application of a principle
T supplies conclusions or summarizes the work he or she has just done
T states predictions made from a principle or consequences of a principle
T gives examples of a subtopic he or she is discussing
T solves problems for class and explains procedures
T asks and answers own question w/o waiting for class to respond
T tells student to memorize something
T gives a definition in words or with a formula
T explains why a formula or principle works

Sample types of questions:
T asks a rhetorical question

Expository with Limited Interaction: "In addition to expository presentation characteristics outlined above, the teacher asks or answers several questions. A limited or minimal verbal interchange between student and teacher occurs for the major part of the interval."

Sample types of statements:
[any of those listed above]

Sample types of questions:
T asks student what step to do next
T asks a math-fact recall question

Expository with Moderate/Extended Interaction: The teacher continues to be the central figure and organize the flow of the lesson, but he or she asks more questions that require thought, allows more wait-time, and builds on students' thoughts and ideas when these fit with the flow of the pre-planned lesson. Teacher still talks substantially more than the students.

Sample types of statements:
[can include those listed above]
*T says or suggests that an underlying principle, rule, or formula exists for solving a problem
*T sets up and/or clarifies a problem
*T supplies conclusions of students' input
*T summarizes students' input
*T clarifies a previous statement of his or her own or of a student

Sample types of questions:
[may include those listed above]
T asks students to give overall procedure or plan for a solution
T asks students to predict or apply principles
T asks for generalizations, rules, or formulas about several preceding examples or problems

Guided Discovery: The teacher continues to be the central figure but relinquishes some control over the flow of the lesson, primarily directing students' attention via questioning. Teacher may incorporate short peer interaction segments such as "think-pair-share" into the lesson. Teacher may respond to students' ideas and clarify, build on, extend, summarize these, but in a limited way. Teachers and students talk approximately equal amount of time.

Sample types of statements:
*T says or suggests that an underlying principle, rule, or formula exists for solving a problem
*T sets up and/or clarifies a problem
*T supplies limited conclusions of students' input
*T occasionally summarizes students' input
*T clarifies a previous statement of his or her own or of a student

Sample types of questions:
T asks students to give overall procedure or plan for a solution
T asks students to predict or apply principles
T asks for generalizations, rules, or formulas about several preceding examples or problems
T asks students for conclusions and summaries of preceding student input

Pure Discovery: After the teacher sets up and clarifies the problem to be solved, he/she then expects students to direct the flow of the lesson. Students may work individually or in small groups. Students may present their ideas, justify their methods, question each other. The teacher may ask questions of individual students, groups, or the class. If a student asks a question of the teacher, the teacher turns the question back to the pupil or class, perhaps in a revised form. Students talk more than the teacher.

Appendix C: Attitudes and Beliefs Instrument

Indiana Mathematics Belief Scales

For each statement below, please circle the response that corresponds to your opinion:

SA = strongly agree, A = agree, U = uncertain, D = disagree, SD = strongly disagree.

1. There are word problems that just can't be solved by following a predetermined sequence of steps.	SA	A	U	D	SD
2. Time used to investigate why a solution to a math problem works is time well spent.	SA	A	U	D	SD
3. Math problems that take a long time don't bother me.	SA	A	U	D	SD
4. Learning to do word problems is mostly a matter of memorizing the right steps to follow.	SA	A	U	D	SD
5. A person who doesn't understand why an answer to a math problem is correct hasn't really solved the problem.	SA	A	U	D	SD
6. If I can't do a math problem in a few minutes, I probably can't do it at all.	SA	A	U	D	SD
7. Word problems can be solved without remembering formulas.	SA	A	U	D	SD
8. It's not important to understand why a mathematical procedure works as long as it gives a correct answer.	SA	A	U	D	SD

9. I feel I can do math problems that take a long time to complete.	SA	A	U	D	SD
10. Most word problems can be solved by using the correct step-by-step procedure.	SA	A	U	D	SD
11. Getting a right answer in math is more important than understanding why the answer works.	SA	A	U	D	SD
12. I find I can do hard math problems if I just hang in there.	SA	A	U	D	SD
13. Any word problem can be solved if you know the right steps to follow.	SA	A	U	D	SD
14. In addition to getting a right answer in mathematics, it is important to understand why the answer is correct.	SA	A	U	D	SD
15. If I can't solve a math problem quickly, I quit trying.	SA	A	U	D	SD
16. Memorizing steps is not that useful for learning to solve word problems.	SA	A	U	D	SD
17. It doesn't really matter if you understand a math problem if you can get the right answer.	SA	A	U	D	SD
18. I'm not very good at solving math problems that take a while to figure out.	SA	A	U	D	SD

Appendix D: Pretest on Course Content

Instructions: For each question, first check the appropriate box indicating your familiarity with the problem. Then, if you know how to do any part of the problem, show your work in the space provided. Remember, this is not part of your grade in this course; it is just designed to give the instructor information about your prior experience with these types of math problems.

1. A bottle of medicine contains 3 ounces. Each dose is 1 1/4 ounces. How many doses are in the bottle?

____I have done this type of problem before and remember how to solve it (work show below).
____I have done this type of problem before, but I don't remember how to solve it.
____I have not done this type of problem before, but I can figure out how to solve it (work shown below)
____I have not done this type of problem before, and I do not know how to solve it.

2. Express the quantity one thousand in base five.

____I have done this type of problem before and remember how to solve it (work show below).
____I have done this type of problem before, but I don't remember how to solve it.
____I have not done this type of problem before, but I can figure out how to solve it (work shown below)
____I have not done this type of problem before, and I do not know how to solve it.

3. The greatest common factor of two numbers is 5 and the least common multiple of the same two numbers is 200. The numbers are between 10 and 100. Give an example of two numbers that fit this description.

____I have done this type of problem before and remember how to solve it (work show below).
____I have done this type of problem before, but I don't remember how to solve it.
____I have not done this type of problem before, but I can figure out how to solve it (work shown below)
____I have not done this type of problem before, and I do not know how to solve it.

4. Marie takes a job with a starting salary of $30,000. If she gets a 5% raise each year for the first three years, what will her salary be after the three raises?

____I have done this type of problem before and remember how to solve it (work show below).
____I have done this type of problem before, but I don't remember how to solve it.
____I have not done this type of problem before, but I can figure out how to solve it (work shown below)
____I have not done this type of problem before, and I do not know how to solve it.

Appendix E: Summary of Lessons Before Study

Day 1: Introduction to Problem Solving Strategies (problem-based instruction)

Day 2: Using guess-and-test to motivate algebraic equations (explicit instruction)

Day 3: Identifying similarities in problems; using multiple approaches (problem-based)

Written reflection #1 (see below)

Day 4: Ways to identify common patterns; writing formulas for patterns (explicit)

Day 5: Checkerboard problem (problem-based)

Day 6: Introduction to set notation (explicit)

Day 7: Finding patterns in subsets (problem-based)

Day 8: Introduction to functions and representations (explicit)

Day 9: Sequences with cubes (problem-based)

Day 10: Patio problem (problem-based)

Appendix F:

Written Reflection #1

Consider the following problem:

Maria has a crafts business. She makes two types of ornaments—stars, that take 3 hours to complete, and angels, that take 4 1/2 hours to complete. Last week she worked 45 hours and completed 13 ornaments. How many of each type did she make?

Choose one of the problems done so far in class or for homework that is similar to this problem in structure. Explain the similarities you see (at least two similarities in *structure*, not surface features). Then explain how you could solve this problem using two different methods. Give the solutions for both methods. Be complete.

Grading Rubric (Total possible: 10 points)

Element	0 points	1 point	2 points	2.5 points
Choose an appropriate problem	not chosen or not appropriate	appropriate problem identified	n/a	n/a
Similarity #1	not identified or only a surface feature	identified but not explained	identified and explained	n/a
Similarity #2	not identified or only a surface feature	identified but not explained	identified and explained	n/a
Solution Method #1	missing	less than half done or less than half correct	more than half correct but not completely correct or not clear	clear and correct
Solution Method #2	missing	less than half done or less than half correct	more than half correct but not completely correct or not clear	clear and correct

Appendix G: **INFORMED CONSENT FORM (Students: Use of scores)**

Title of Project:	The Influence of Instructional Model on the Conceptual Understanding of Preservice Elementary Teachers
Statement of age of Participant:	I state that I am over 18 years of age, in good physical health, and wish to participate in a program of research being conducted by Dr. James T. Fey in the Department of Curriculum and Instruction at the University of Maryland, College Park.
Purpose:	The purpose of this research is to gain insight into the possible influence of instructional model on conceptual understanding attained.
Procedures:	The procedures involve: a) my participation in normal class activities b) my completion of a survey on beliefs and attitudes regarding mathematics c) my completion of a survey regarding prior familiarity with course content d) the inclusion of my mathematics SAT score in the data e) the inclusion of my scores on two quizzes and one written reflection completed after the study unit (these are part of the normal course requirements for Math 210)
Confidentiality:	All information collected in this study is confidential to the extent permitted by law. I understand that the data I provide will be grouped with the data of others in reporting and presentation and that my name will not be used.
Risks:	There are no risks involved in my participation in this project. I understand that my participation or nonparticipation will not influence my grade in Math 210; I will still be expected to complete the two surveys, two quizzes and one written reflection as part of the normal course requirements for Math 210, but if I choose not to participate my scores and/or responses will not be included in the study.
Benefits, Freedom to Withdraw, & Ability to Ask Questions:	This experiment is not designed to benefit me personally, but to help the investigator learn more about the influence of instructional model on conceptual understanding. I am free to ask questions or withdraw from participation at any time and without penalty.
Contact Information:	For further information, contact Dr. James T. Fey, 2226 Benjamin Building, 301-405-3151 Karen McLaren, 3319 Math Building, 301-405-5057

Student Name: ____

I agree____ do not agree____ to participate in this study.

[signed]____ Date: ____________

INFORMED CONSENT FORM (Students: Interview)

Title of Project:	The Influence of Instructional Model on the Conceptual Understanding of Preservice Elementary Teachers
Statement of age of Participant:	I state that I am over 18 years of age, in good physical health, and wish to participate in a program of research being conducted by Dr. James T. Fey in the Department of Curriculum and Instruction at the University of Maryland, College Park.
Purpose:	The purpose of this research is to gain insight into the possible influence of instructional model on conceptual understanding attained.
Procedures:	The procedures involve my participation in one interview of approximately 45 minutes. The questions will be about the mathematics content in the study unit and my experiences during the unit.
Confidentiality:	All information collected in this study is confidential to the extent permitted by law. I understand that the data I provide will be grouped with the data of others in reporting and presentation and that my real name will not be used.
Risks:	There are no risks involved in my participation in this project. I understand that my participation or nonparticipation will not influence my grade in Math 210.
Benefits, Freedom to Withdraw, & Ability to Ask Questions:	This experiment is not designed to benefit me personally, but to help the investigator learn more about the possible influence of instructional model on conceptual understanding. I am free to ask questions or withdraw from participation at any time and without penalty.
Contact Information:	For further information, contact Dr. James T. Fey, 2226 Benjamin Building, 301-405-3151 Karen McLaren, 3319 Math Building, 301-405-5057

Student Name: ____

I am willing to be interviewed. I understand that it is possible not everyone who volunteers will be called.

[signed]________________________________ **Date:** ______________________

e-mail:________________________________ **phone:**______________________

preferred way to contact: e-mail___ phone___

I do not wish to be interviewed.

[signed]________________________________**Date:** ______________________

Appendix H: Immediate Post-Test (Unit Exam) Questions

7(8). It has been discovered that Plutonians have 12 fingers, and use a base 12 counting system.

a. A Plutonian says, "I have 25_{12} people in my immediate family." In base 10, how many people are in the Plutonian's immediate family? *Show work.*

b. You, the Earthling, boast of having a *very* large immediate family of 166_{10} people, and brag to the Plutonian "I have $\_\_\_\__{12}$ people in my immediate family." How many people are in your immediate family using the Plutonian base 12 counting system? (That is, write 166_{10} in base 12.)

8(8). You take a trip to Saturn where they have the following counting system,

English	Saturnian	English	Saturnian
1	%		
2	#		
3	?		
4	% <		
5	% %		
6	% #		
7	% ?		
8	# <		
9	# %		
10	# #		
11	# ?		

a. In the space above, write down the next five Saturnian numbers corresponding to 12 through 16.

b. Does the Saturnian system have place value? Explain how you know.

c. What base or group size is the Saturnian counting system based on?

9(8). a. Change 221011_3 to base ten. *Show work.*

b. Change 3000 to base seven. *Show work.*

10(2). How would you write $b^4 + 1$ in base b?

11(6). When you are counting,

a. what comes just before 1000_4 ? *(Write it in base four.)*

b. what comes after 99_{14} ? *(Write it in base fourteen.)*

12(3). Which is the larger quantity: 51_x or 51_{2x} ? Explain.

Grading Rubric for Unit Exam

Procedural Questions: 7, 9 (16 points possible)

Conceptual Questions: 8, 10, 11, 12 (16 points possible)

7. a. +2 for correct set-up but incorrect answer

+3 for set-up with correct answer

b. +3 for correct work (setting up place values and dividing by place values), but incorrect answer (two or more incorrect digits)

+4 for correct work and one incorrect digit in answer

+5 for correct work and correct answer

8. a. +0 if two or fewer numerals from 12-15 are correct *and* the numeral for 16 has fewer than two symbols correct

+2 if at least three of the four numerals from 12-15 are correct *and* at least two of the symbols for 16 are correct

+3 if at least three of the four numerals from 12-15 are correct *and* all three of the three symbols for 16 are correct

b. +1 for correct answer ("yes") but missing or incorrect reasoning

+3 for correct answer with correct reasoning

c. +2 for correct answer ("base 4")

9. a. +2 if place values are correct but digits are not correctly placed in grid

+3 if place values are correct and digits are placed correctly in grid but answer is incorrect

+4 if place values and digit placement and answer are all correct

b. +1 if place values are correct

+2 if place values are correct and three digits are correct

+3 if place values are correct and four digits are correct

+4 if place values are correct and all five digits are correct

10. +2 if correct (no part credit)

11. a. +3 if correct (no part credit)

b. *omitted, because many students seemed to have misread the question*

12. +1 for correct answer but missing or incorrect reasoning

+3 for correct answer and reasoning

Appendix I: Written Reflection #2

Place Value

Instructions: Your response to this is to be typed, double-spaced. Please answer completely, in well-written, well-developed paragraphs.

Due date: ________________________

What is place value? Define the concept in your own words. If you know more than one way to think about place value, include these alternate conceptions. Use mathematical examples in more than one base to illustrate your conceptions. Contrast with a non-place-value system, pointing out the essential differences.

(Rubric on following page.)

Rubric

Base Points (maximum possible = 4)

Element	0 points	1/2 point	1 point
Definition	missing or completely incorrect	partly correct or vague, difficult to interpret	clear and correct
Example in one base	missing or completely incorrect	partly correct or vague, difficult to interpret	clear and correct
Example in a second base	missing or completely incorrect	partly correct or vague, difficult to interpret	clear and correct
Non-place-value example with one difference from place value noted	missing or completely incorrect	partly correct or vague, difficult to interpret	clear and correct

Bonus Points (maximum possible = 1)

Element	0 points	1/2 points	1 point
Additional way to think about place value	missing or completely incorrect	partly correct or vague, difficult to interpret	clear and correct
Additional differences between place value and non-place value	missing or completely incorrect	1 additional correct difference noted	2 or more additional correct differences noted

Appendix J: Interview Questions

Part I

These first problems are similar to ones done in class and on the first exam. For each, work out the problem and then explain what you did and why.

1. Change 2012_3 to base ten.
2. Change 2002 to base thirteen.
3. What number comes right before 4550_6 ?

Part II

We study different number bases in order to compare and contrast those systems with our familiar base ten system.

What similarities have you seen between other number bases and base ten--what do all of these number systems have in common?

> Anything else?
>
> Is that all?

What properties differ from base to base?

Part III

Imagine you have taken a 1-year teaching assignment with the Intergalactic Peace Corps on the planet Plutonium, where base six is used. You will need to teach first and second graders how to represent numbers in base six.

1. [show illustration below of base ten sticks and bundles of several 2-digit quantities from a Scott Foresman 1st grade text (Bolster, et al., 1988)]. In base ten we would

illustrate the quantity 21 for young students as 2 bundles of ten and 1 single stick.

Here is a quantity of unbundled sticks [give student 22 sticks and several rubber bands].

a. How would you bundle them to show Plutonium students what the base six number means?

b. What is the base six number that corresponds with this quantity?

2. Here is another quantity of unbundled sticks [give student 50 sticks].

a. How would you bundle these to show Plutonium students what the base six number means?

b. What is the base six number that corresponds with this quantity?

Part IV

The following question is not one you are expected to know the answer to immediately. Feel free to take time to reflect and try some examples before answering. If you are unable to figure one out, that's fine; just say so and we'll move on.

Numbers in base six that end in zero (for example, 40 or 1120) all share a common property. Try to determine what it is.

Try to explain why this property holds.

Try to make a connection between this property and a similar property in base ten.

Part V

1. What new insights about place value did you gain from your class unit on this (2.3) material? What helped you gain this insight? [teacher explanations? group discussion? etc.]

2. For this unit, different sections of this math course were taught in different ways. In some sections the instructor used a lecture style to explain how to change between other bases and base ten; in other sections a cooperative learning style was used: students worked in groups to develop their own procedures to change between bases.

a. Which teaching style did your section have?

b. How well do you feel it worked for you? Why?

c. Do you think the teaching style in your section worked better for you than the alternative style would have, or do you think you would have preferred the other? Why?

3. Here is a quote about teaching and learning math: "Understanding is generated by individual students, not provided by the teacher."

a. What do you think--do you agree, disagree? Why?

b. Do you think certain styles of teaching are better than others to help students understand math? Explain.

c. What teaching elements do you feel best support *your* learning math with understanding?

d. How do you envision helping your students learn to understand math when you become a teacher?

Following page is the illustration for Part III from *Invitation to Mathematics: Grade 1* (Bolster, et al., 1988) Reprinted by permission of Pearson Education, Inc.

Name ______________________________

Listen to your teacher. Show each number with bundles of ten and single sticks. For each row, write A for the number that is the smallest. Write B for the number that is the next largest. Write C for the number that is the next larger. Finally, write D for the number that is the largest.

19 30 9 21

tens	ones		tens	ones		tens	ones		tens	ones
1	9		3	0			9		2	1

B ____ D ____ A ____ C ____

tens	ones		tens	ones		tens	ones		tens	ones
5	1		2	3		3	2		1	5

____ ____ ____ ____

tens	ones		tens	ones		tens	ones		tens	ones
1	9		3	0		2	9		2	0

____ ____ ____ ____

Listen to your teacher. Think about 23 and 32. Tell how they are alike. Tell why they are very different.

Appendix K: Delayed Post-Test (Final Exam) Questions

4 (9). Show all work for the following:

a) Convert 145 to base 3.

b) Convert $11T_{12}$ to base 10. (Assume T = ten)

c) What base makes the following statement true? $25_x = 19$

5 (9).

a) What comes **after** 199_{11} ?

b) What comes **before** 1000_2 ?

c) Give a reasonable interpretation of 0.2_7

Grading Rubric for Final Exam Questions

Procedural Questions: 4a, 4b (total possible = 6 points)

Conceptual Questions: 4c, 5a, 5b, 5c (total possible = 12 points)

4. a. +1 if place values correct and at least three of five digits are correct

+2 if place values correct and at least four of five digits are correct

+3 if place values correct and all five digits are correct

b. +1 if place values correct and method correct *but* two arithmetic errors

+2 if place values correct and method correct *but* one arithmetic error

+3 if place values correct and method and answer correct

c. +3 if approach and answer correct; +0 otherwise

5. a. +3 if answer is correct and in the given base

b. +3 if answer is correct and in the given base

c. +1 if student indicates "sevenths," but not 2/7

+3 if student correctly states 2/7

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