ABSTRACT

Title of Thesis: AN INVESTIGATION OF ALEXANDER POLYNOMIALS

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We explore the Alexander polynomial for a knot. We prove that an arbitrary reciprocal polynomial with integral coefficients that sum to one exists as an Alexander polynomial for some knot. In addition, we explore parallels between the Alexander and Weil polynomials.

AN INVESTIGATION OF ALEXANDER POLYNOMIALS

by

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0.1 Introduction

In 1928, J. Alexander introduced the idea of associating to every knot a polynomial. The Alexander polynomial, as it is known, is a powerful knot invariant. It can be used to distinguish a great number of knots although it is not a universal invariant. In Chapter 1, we will explore one definition of the Alexander polynomial given in [CL]. This definition requires an understanding of a Seifert surface for a knot. We work out an example for the knot 7_4, constructing the Seifert surface and calculating the Alexander polynomial. Multiple definitions for the Alexander polynomial exist. In [HS], we are presented with an alternative definition. In Chapter 2, we prove that the two definitions yield equivalent polynomials.

Symmetry conditions arise on the Seifert matrix forcing certain conditions on the Alexander polynomial. In Chapter 3, we show that it is reciprocal and that its coefficients sum to one as described in [HS]. In [HS], we also find that any arbitrary polynomial with integral coefficients that is reciprocal and whose coefficients sum to one comes from a Seifert matrix. In Chapter 4, we prove this assertion. In addition, we explain a process for creating a Seifert surface given an arbitrary matrix satisfying the symmetry conditions.

In Chapter 5, we describe a general result on reciprocal polynomials. Its significance for knots does not seem to be known. Then, in Chapter 6, we run through the list of Alexander polynomials for knots with up to nine crossings in order to see which polynomials have roots with absolute value one. The Alexander polynomial in knot theory parallels the Weil polynomial in number theory. The final chapter in this thesis explains the motivation behind this investigation of Alexander polynomials by pointing out the analogous results for the Weil polynomial.

I wish to thank Lawrence Washington for all of his guidance on this thesis.

Chapter 1

The Alexander polynomial

Attempting to distinguish two knots is an essential problem in the study of knots. If two diagrams portray the same knot, then we can deform one into the other using Reidemeister moves, a series of moves that do not alter the fundamental form of the knot. However, if one knot can be deformed into another knot, they should possess some other identifying characteristics to show that they are, in fact, the same knot. These characteristics are called knot invariants. One powerful knot invariant is a polynomial associated to each knot known as the Alexander polynomial. Although the Alexander polynomial is not a universal invariant since it cannot completely distinguish all knots, it has proven to be an essential tool in distinguishing knots.

To calculate the Alexander polynomial of a knot, we will begin by creating a Seifert surface for the knot as in [CL]. The Seifert surface of a knot is an orientable surface with the knot as its boundary. To create a Seifert surface for a knot, begin by choosing an oriented diagram of the knot and create a collection of circles as follows (Note: An example for knot 7_4 follows the description). Choose an arbitrary point on the diagram and begin tracing around the knot in the direction of the orientation. When we arrive at a crossing, switch the arcs along which we are tracing, continuing in the direction of the orientation. If at some point we retrace a portion of the knot we have already covered, then we choose a new point on an untraced portion of the knot and repeat the process. The resulting figure will be a collection of oriented circles known as Seifert circles.

Consider the circles to be boundaries of disks in the plane. Connect the disks by attaching twisted bands where the original crossings would have been. The twist in the band will correspond to the direction of the original crossing, right-handed or left-handed according to the original diagram of the knot (see figure 1.1). If any disks are nested, lift the inner disk above the outer disk and attach the bands. The resulting surface has the knot as its boundary and is one Seifert surface for the knot. A natural collection of oriented curves arise on the surface. Once we have the surface, we can define a top side by choosing a nonvanishing vector normal to the surface. The positive push of x, x_i^*, is a curve that lies slightly above the surface and runs parallel to x_i. By calculating the linking numbers of the various curves, we create a Seifert matrix for the surface. To find the linking number of the two curves, assign to each crossing a $+1$ or a -1 depending on whether the crossing is right-handed or left-handed, respectively.

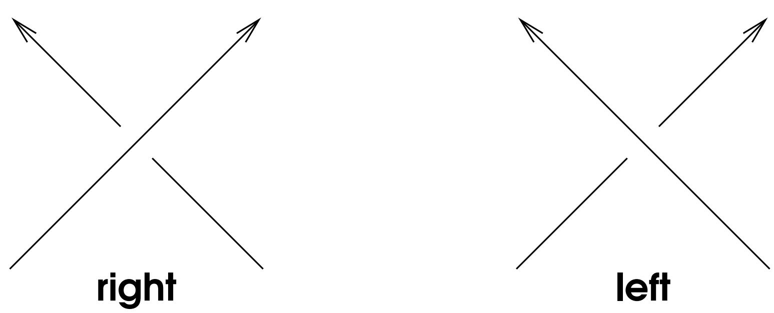

Figure 1.1: Crossing Labels for the Linking Number of Two Curves

The *linking number* of two curves, $\mathrm{lk}(x_i, x_j)$, is defined to be the sum of the signs of the crossings divided by two. The entries of the Seifert matrix $V = (v_{i,j})$ then equal $\mathrm{lk}(x_i, x_j^*)$ for all i and j. Due to the way the curves link on the surface, symmetry conditions for the matrix arise:

Fundamental Symmetry Conditions:

$$a_{i,j} = a_{j,i} \text{ if } (i,j) = (2r+1, 2r+2) \text{ or } (2r+2, 2r-1), r \geq 0;$$

$$a_{2r+1,2r+2} = a_{2r+2,2r+1} + 1, r \geq 0.$$

Finally, the *Alexander polynomial* for the knot K is given by

$$A_K(t) = \det(V - tV^t) \tag{1.1}$$

Example: Consider the following example for the knot 7_4. Begin with an oriented diagram of the knot.

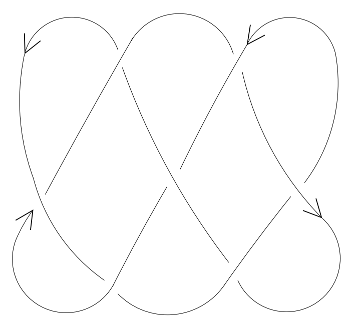

Figure 1.2: An oriented diagram of knot 7_4

Starting at any point, we draw the Seifert circles for the knot, preserving the original orientation, to achieve six unnested circles. Then, we join the disks by twisted bands, the twists agreeing with the original crossings. The result will be a Seifert surface for the knot (Figure 1.3).

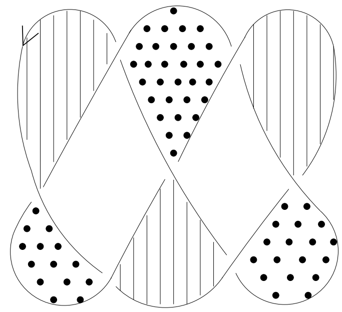

Figure 1.3: A Seifert surface for knot 7_4

From here, we can manipulate the figure into a disk with twisted bands attached. Essentially, we choose one portion of the surface to be the disk. Let us choose the lower middle portion. Then, we slide the other crossings up around the surface to achieve

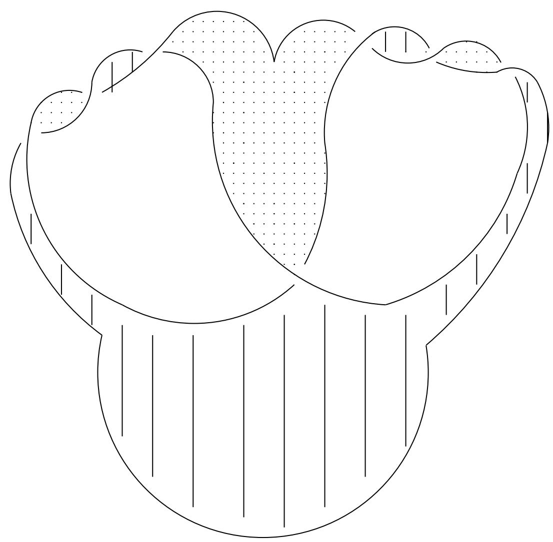

Figure 1.4: A Manipulation of the Seifert Surface for 7_4

Next, we further divide the thick middle band down to the crossing and flip each of the divided portions over. This creates two additional crossings on each side. We now have a disk with two twisted bands attached (See Figure 1.5).

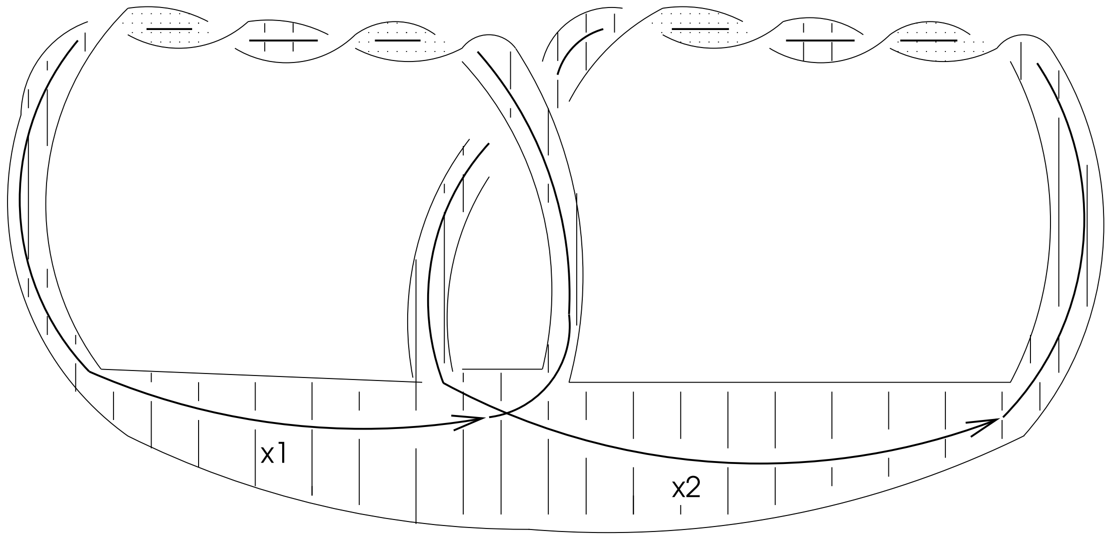

Figure 1.5: The Seifert Surface for 7_4 Drawn as a Disk with Twisted Bands Attached

We can calculate the Alexander polynomial for 7_4 using the directions above. First, we must calculate the linking numbers of each pair of curves. Consider x_1 and x_1^*. Isolating these curves, we have

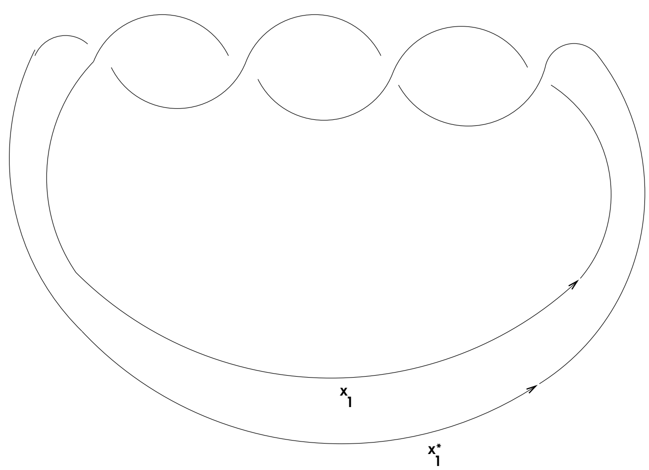

Figure 1.6: The linking of x_1 and x_1^*

We can see that we have four left-handed crossings so $\mathrm{lk}(x_1, x_1^*) = -2$. If we isolate x_2 and x_2^*, we have a figure identical to that above and, thus, $\mathrm{lk}(x_2, x_2^*) = -2$ as well. We will refer to Figure 1.5 to see how curves x_1 and x_2 link together. Then we can see that $\mathrm{lk}(x_1, x_2^*) = 1$ and $\mathrm{lk}(x_2, x_1^*) = 0$. These values constitute the Seifert matrix

$$V = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}.$$

Then, we calculate the Alexander polynomial using (1.1):

$$\begin{aligned} A_{7_4}(t) &= \det\left(\begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix} - t\begin{bmatrix} -2 & 0 \\ 1 & -2 \end{bmatrix}\right) \\ &= \begin{vmatrix} -2+2t & 1 \\ -t & -2+2t \end{vmatrix} \\ &= 4t^2 - 7t + 4. \end{aligned}$$

The process described above assigns to each knot a polynomial. For knots with up to nine crossings, only eight pairs among the eighty-seven knots have the same Alexander polynomial [CL, p.3].

Chapter 2

Alternate Definition of the Alexander Polynomial

In [HS], we find a different formula for the Alexander polynomial. As we will be using both formulas, it is necessary that we prove that they are interchangeable. Begin with an altered form of the matrix in [CL]:

$$\Gamma = \begin{bmatrix} v_{1,2} & -v_{1,1} & \cdots & v_{1,2h} & -v_{1,2h-1} \\ v_{2,2} & -v_{2,1} & \cdots & v_{2,2h} & -v_{2,2h-1} \\ \vdots & \vdots & \ddots & \vdots & \\ v_{2h-1,2} & -v_{2h-1,1} & \cdots & v_{2h-1,2h} & -v_{2h-1,2h-1} \\ v_{2h,2} & -v_{2h,1} & \cdots & v_{2h,2h} & -v_{2h,2h-1} \end{bmatrix}$$

where Γ adheres to the*fundamental symmetry conditions* mentioned earlier. From here, the Alexander polynomial for a knot K is given by

$A_K(x) = \det(\mathrm{I} - \Gamma + x\Gamma) = \det(\mathrm{I} + \Gamma(x-1)) =$

$$\begin{vmatrix} 1+v_{1,2}(x-1) & -v_{1,1}(x-1) & \cdots & v_{1,2h}(x-1) & -v_{1,2h-1}(x-1) \\ v_{2,2}(x-1) & 1-v_{2,1}(x-1) & \cdots & v_{2,2h}(x-1) & -v_{2,2h-1}(x-1) \\ \vdots & \vdots & \ddots & \vdots & \\ v_{2h-1,2}(x-1) & v_{2h-1,1}(x-1) & \cdots & 1+v_{2h-1,2h}(x-1) & v_{2h-1,2h-1}(x-1) \\ v_{2h,2}(x-1) & -v_{2h,1}(x-1) & \cdots & v_{2h,2h}(x-1) & 1-v_{2h,2h-1}(x-1) \end{vmatrix}.$$

Using the symmetry conditions to rewrite $v_{i,j}$ so that $i \leq j$, this becomes

$$\begin{vmatrix} 1+v_{1,2}(x-1) & -v_{1,1}(x-1) & \cdots & v_{1,2h}(x-1) & -v_{1,2h-1}(x-1) \\ v_{2,2}(x-1) & x-v_{1,2}(x-1) & \cdots & v_{2,2h}(x-1) & -v_{2,2h-1}(x-1) \\ \vdots & \vdots & \ddots & \vdots & \\ v_{2,2h-1}(x-1) & v_{1,2h-1}(x-1) & \cdots & 1+v_{2h-1,2h}(x-1) & v_{2h-1,2h-1}(x-1) \\ v_{2,2h}(x-1) & -v_{1,2h}(x-1) & \cdots & v_{2h,2h}(x-1) & x-v_{2h-1,2h}(x-1) \end{vmatrix}.$$

Theorem 1

$$det(V - xV^t) = det(I + \Gamma(x-1))$$

Proof: Let V be defined as follows:

$$V = \begin{bmatrix} v_{1,1} & v_{1,2} & \cdots & v_{1,2h-1} & v_{1,2h} \\ v_{2,1} & v_{2,2} & \cdots & v_{2,2h-1} & v_{2,2h} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ v_{2h-1,1} & v_{2h-1,2} & \cdots & v_{2h-1,2h-1} & v_{2h-1,2h} \\ v_{2h,1} & v_{2h,2} & \cdots & v_{2h,2h-1} & v_{2h,2h} \end{bmatrix}$$

The first step will be to take the transpose of the expression to get $\det(V^t - xV)$. This will not change the determinant. Next, we can write out the matrices involved.

$$det(V^t - xV) = \begin{vmatrix} v_{1,1} & v_{2,1} & \cdots & v_{2h-1,1} & v_{2h,1} \\ v_{1,2} & v_{2,2} & \cdots & v_{2h-1,2} & v_{2h,2} \\ \vdots & \vdots & \ddots & \vdots & \\ v_{1,2h-1} & v_{2,2h-1} & \cdots & v_{2h-1,2h-1} & v_{2h,2h-1} \\ v_{1,2h} & v_{2,2h} & \cdots & v_{2h-1,2h} & v_{2h,2h} \end{vmatrix}$$

$$-x \begin{vmatrix} v_{1,1} & v_{1,2} & \cdots & v_{1,2h-1} & v_{1,2h} \\ v_{2,1} & v_{2,2} & \cdots & v_{2,2h-1} & v_{2,2h} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ v_{2h-1,1} & v_{2h-1,2} & \cdots & v_{2h-1,2h-1} & v_{2h-1,2h} \\ v_{2h,1} & v_{2h,2} & \cdots & v_{2h,2h-1} & v_{2h,2h} \end{vmatrix}$$

Now, we will use the symmetry conditions to rewrite all of the $v_{i,j}$ such that $i \leq j$. The above equation becomes

$$= \begin{vmatrix} v_{1,1}(1-x) & v_{2,1} - xv_{1,2} & \cdots & v_{2h-1,1}(1-x) & v_{2h,1}(1-x) \\ v_{1,2} - xv_{2,1} & v_{2,2}(1-x) & \cdots & v_{2h-1,2}(1-x) & v_{2h,2}(1-x) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ v_{1,2h-1}(1-x) & v_{2,2h-1}(1-x) & \cdots & v_{2h-1,2h-1}(1-x) & v_{2h,2h-1} - xv_{2h-1,2h} \\ v_{1,2h}(1-x) & v_{2,2h}(1-x) & \cdots & v_{2h-1,2h} - xv_{2h,2h-1} & v_{2h,2h}(1-x) \end{vmatrix}$$

$$
= \begin{vmatrix}
v_{1,1}(1-x) & v_{1,2}(1-x)-1 & \cdots & v_{1,2h-1}(1-x) & v_{1,2h}(1-x) \\
v_{1,2}(1-x)+x & v_{2,2}(1-x) & \cdots & v_{2,2h-1}(1-x) & v_{2,2h}(1-x) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
v_{1,2h-1}(1-x) & v_{2,2h-1}(1-x) & \cdots & v_{2h-1,2h-1}(1-x) & v_{2h,2h-1}(1-x)-1 \\
v_{1,2h}(1-x) & v_{2,2h}(1-x) & \cdots & v_{2h-1,2h}(1-x)+t & v_{2h,2h}(1-x)
\end{vmatrix}.
$$

Finally, we will multiply the h even columns by -1 and rewrite the remaining columns in terms of $(x-1)$. Then we can interchange column i with column $i+1$ for i odd yielding

$$
= \begin{vmatrix}
1+v_{1,2}(x-1) & -v_{1,1}(x-1) & \cdots & v_{1,2h}(x-1) & -v_{1,2h-1}(x-1) \\
v_{2,2}(x-1) & x-v_{1,2}(x-1) & \cdots & v_{2,2h}(x-1) & -v_{2,2h-1}(x-1) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
v_{2,2h-1}(x-1) & -v_{1,2h-1}(x-1) & \cdots & 1+v_{2h-1,2h}(x-1) & -v_{2h-1,2h-1}(x-1) \\
v_{2,2h}(x-1) & -v_{1,2h}(x-1) & \cdots & v_{2h,2h}(x-1) & x-v_{2h-1,2h}(x-1)
\end{vmatrix}.
$$

The effect of the last two steps is multiplication of the determinant by $(-1)^h$ for the even column change followed by multiplication by another $(-1)^h$ for the h column switches. Thus, the determinant remains unchanged since this equates to multiplication by $(-1)^h \cdot (-1)^h = (-1)^{2h} = 1$. Thus, $\det(V - xV^t) = \det(\mathrm{I} + \Gamma(x-1))$.

Chapter 3

Conditions on the Alexander Polynomial

From either definition of the Alexander polynomial, certain conditions on the polynomial arise.

Theorem 2 *If $A_K(t) = det(V - tV^t) = c_{2h}t^{2h} + c_{2h-1}t^{2h-1} + \cdots + c_1 t + c_0$, then the coefficients of $A_K(t)$ satisfy the following conditions:*

$$c_0 + c_1 + \cdots + c_{2h} = 1 \tag{3.1}$$

$$c_i = c_{2h-i}, 0 \leq i \leq h-1 \tag{3.2}$$

Proof: To prove (3.1), we can simply let $t = 1$ in the above equation. Then,

$$\begin{aligned} A_K(1) = c_o + c_1 + \cdots + c_{2h} &= \det(V - V^t) \\ &= \det \begin{bmatrix} 0 & 1 & & & & & \\ -1 & 0 & & & & & \\ & & 0 & 1 & & & \\ & & -1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & 1 \\ & & & & & -1 & 0 \end{bmatrix} = 1 \end{aligned}$$

Note in the above $2h \times 2h$ matrix, all unspecified entries are zero. Recall that the matrix V adheres to certain symmetry conditions, for example, that $v_{i,j} = v_{j,i}$ for all $(i,j) \neq (2r+1, 2r+2)$ or $(2r+2, 2r+1)$. These conditions yield 1's for $v_{i,j}$ when $(i,j) = (2r+2, 2r+1)$, -1's for $v_{i,j}$ when $(i,j) = (2r+1, 2r+2)$, and 0's everywhere else.

Condition (3.2) arises from properties of the determinant. Essentially, to show $c_i = c_{2h-i}, 0 \leq i \leq h-1$, we want to show that $t^{2h}A_K(t^{-1}) = A_K(t)$:

$$\begin{aligned}
t^{2h}A_K(t^{-1}) &= t^{2h}\det(V - t^{-1}V^t) \\
&= \det(t(V - t^{-1}V^t)) \\
&= \det(tV - V^t) \\
&= \det(tV - V^t)^t \\
&= \det(tV^t - V) \\
&= \det(V - tV^t).
\end{aligned}$$

For the last equality, we use the fact that the dimension of the matrix is even. We can see that the Alexander polynomial satisfies the above conditions. In summary, the value of the Alexander polynomial at 1 is 1 and the polynomial is reciprocal. This fact will be used again in Chapter 4, Section 2.

Chapter 4

Reversing Alexander's Argument

We have seen how to calculate the Alexander polynomial for any knot. Now we will show that given an arbitrary reciprocal polynomial with integral coefficients that sum to 1, we can find a Seifert surface with a knot K as its boundary corresponding to this polynomial. We will begin in Section 1 by showing that given an arbitrary matrix Γ with the specific symmetry conditions, we can find a Seifert surface with boundary K corresponding to this matrix. Then, in section 2, we will show that given an arbitrary polynomial with the conditions above, we can find a matrix V such that $\det(\mathrm{I} - \Gamma + x\Gamma)$ yields the polynomial as in [HS].

4.1 Every Matrix Comes from a Knot

Let A be an $n \times n$ Seifert matrix with n even, that is, a matrix defined as follows:

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix}. \tag{4.1}$$

The Seifert matrix adheres to the *fundamental symmetry conditions* mentioned earlier. To construct the surface corresponding to A, begin with a disk in the plane. We will first add n twisted bands to the disk, where n is the dimension of the Seifert matrix, as follows. Attach band i to the disk with $2|a_{i,i}|$ twists. The twists will be right-handed if $a_{i,i} > 0$ and left-handed if $a_{i,i} < 0$. Once band i is added, a natural curve x_i forms along the band and surface. We then see

$$lk(x_i, x_i^*) = a_{i,i}$$

by construction. Thus, the linking number of a curve x_i and the positive push of the curve x_i^* equals the diagonal entry $a_{i,i}$ as desired.

Next we need to consider how the entries of the matrix will determine how two bands are linked together. Let us first look at bands $2r + 2$ and $2r + 1$ for $r \geq 0$. Remove the ends closest to each other from the disk and twist the $(2r + 1)^{th}$ band over the $(2r + 2)^{th}$ band $|a_{2r+1,2r+2}|$ times if $a_{2r+1,2r+2} \geq 0$ and $a_{2r+2,2r+1} \geq 0$. Otherwise, we twist the $(2r+2)^{th}$ over $(2r+1)^{th}$ band $|a_{2r+2,2r+1}|$ times. Reattach the bands to the disk in such a way that the resulting curves will have to cross one more time on the disk before closing. Then the linking number of these curves will be

$$lk(x_{2r+1}, x_{2r+2}^*) = a_{2r+1,2r+2}$$

by construction.

We need to ensure the linking of these bands coincides with the symmetry conditions, namely that

$$lk(x_{2r+2}, x_{2r+1}^*) = a_{2r+2,2r+1}.$$

Yet, this is easily seen. When we take the positive push of x_{2r+1}, we will change one crossing in the linking of x_{2r+1} and x^*_{2r+2}, thus decreasing the sum of the signs of the crossings by two. Hence, the linking number decreases by one. This satisfies the symmetry conditions for the matrix when $(i, j) = (2r + 1, 2r + 2)$ or $(2r + 2, 2r + 1)$.

Now let us consider how to twist the remaining bands, that is, bands i and j such that $i \neq j$ and $(i, j) \neq (2r + 1, 2r + 2)$ or $(2r + 2, 2r + 1)$. Again, we will twist the bands according to the $(i, j)^{th}$ entry of the matrix. Since $a_{i,j} = a_{j,i}$ for the current i and j, we may assume $i < j$. Now, twist the i^{th} band over the j^{th} band $|a_{i,j}|$ times if $a_{i,j} > 0$. Otherwise, twist the j^{th} band over the i^{th} band $|a_{i,j}|$ times. Unlike the special case above, reattach the bands to the disk such that the corresponding curves will not cross again on the disk. Then,

$$lk(x_i, x_j) = lk(x_j, x_i) = a_{i,j} = a_{j,i}.$$

Since the curves do not cross on the disk, one curve and the positive push of the other curve will cross $a_{i,j}$ times just like the actual x_i and x_j curves on the bands.

Example: Consider the following example. For simplicity, we can allow some nonessential matrix entries to be zero.

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -2 & 2 & 1 & 0 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 4 & -1 \end{bmatrix}$$

Begin with the diagonal entries. Add four bands to a disk since $\dim(A) = 4$. Natural curves arise which we can label x_1 through x_4. Mark each curve with the same orientation. We will twist the bands according to the rules above. Consider, for example, the curve x_1. Since $a_{1,1} = 1$, we will add two right-handed twists to the band containing x_1. We can easily see then that $lk(x_1, x_1^*) = 1$ as desired. We can repeat this process for each of the three remaining curves.

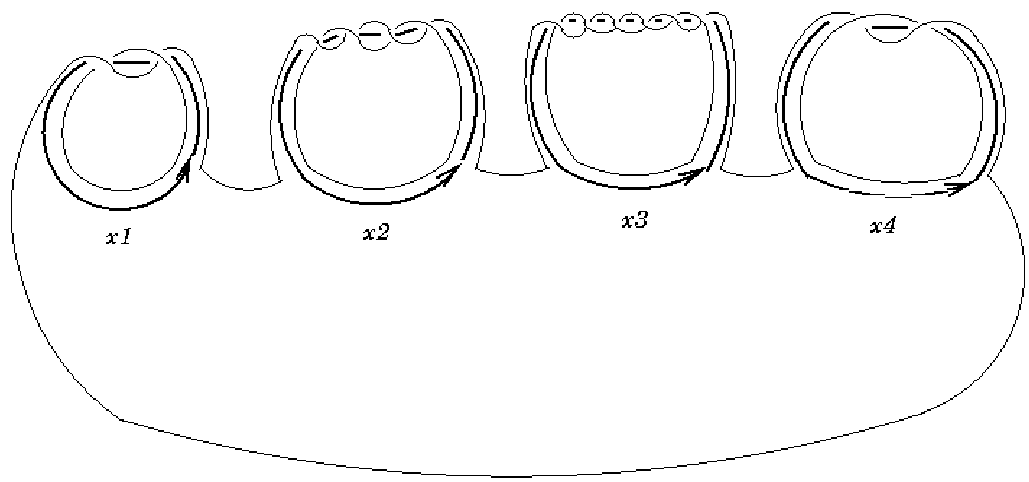

Figure 4.1: The Disk with Four Twisted Bands

Next, we will see how the non-diagonal entries affect the surface. Consider the special entries, $a_{i,j}$ with $(i, j) = (2r + 2, 2r + 1)$ or $(2r + 1, 2r + 2), r \geq 0$. Specifically, consider $a_{1,2}, a_{2,1}, a_{3,4},$ and $a_{4,3}$. First, twist the 2^{nd} band over the 3^{rd} band $|a_{2,1}| = 2$ times. Reattach bands so that they will cross again on the disk. Note $lk(x_1, x_2^*) = -1$ and $lk(x_2, x_1^*) = -2$. Similarly, we will twist the 3^{rd} band over the 4^{th} band $|a_{3,4}| = 5$ times and reattach so that the bands will cross once more on the disk. Then $lk(x_3, x_4^*) = 5$ and $lk(x_4, x_3^*) = 4$. The remaining entries of our example are all zero with the exception that $a_{3,2} = a_{2,3} = 1$. So now we twist the 2^{nd}band over the 3^{rd} band $|a_{2,3}| = 1$ time and reattach the bands so that they do not cross again on the disk. Then, $lk(x_2, x_3^*) = lk(x_3, x_2^*) = 1$ as desired. The resulting Seifert surface for matrix A appears in Figure 4.2.

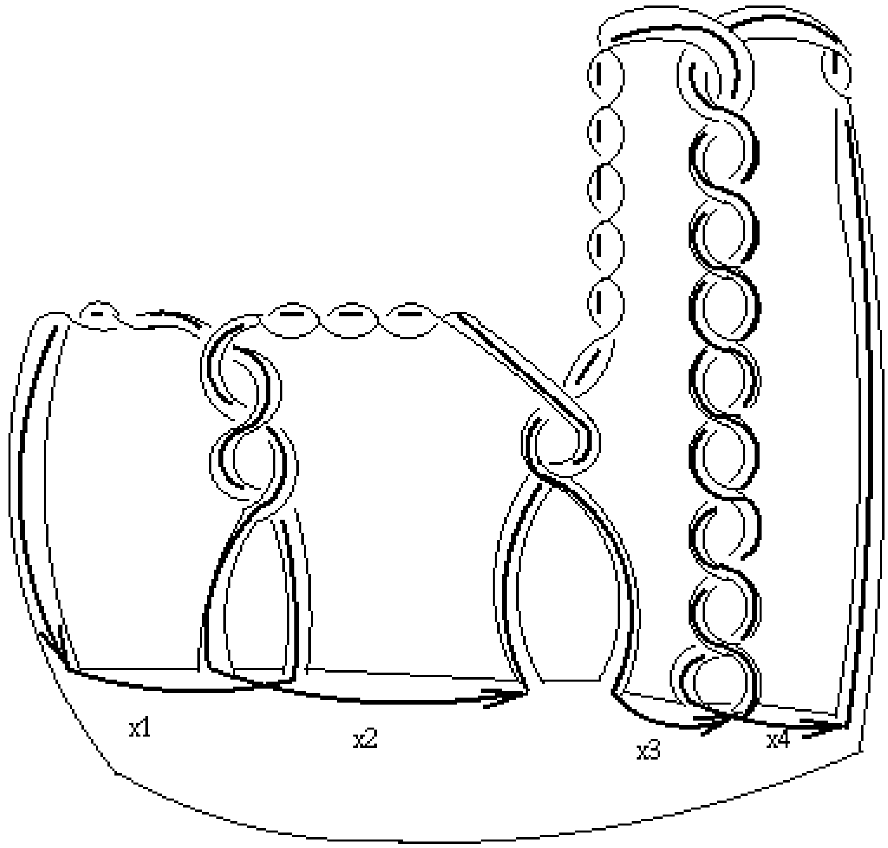

Figure 4.2: A Seifert Surface Created from Matrix A

4.2 Every Polynomial Comes from a Matrix

It remains to show that the coefficients of our polynomial can be chosen arbitrarily. That is, we must show that there exists a Γ such that any polynomial $c_{2h}t^{2h}+\cdots+c_1t+c_0$ with $c_{h-1},\cdots,c_0$ satisfying (3.1) and (3.2) will be of the form $\det(\mathrm{I}-\Gamma+t\Gamma)$. We will refer to Seifert's form of the matrix and the Alexander polynomial for the proof. Begin by allowing $v_{i,j} = 0$ for all i and j with the following exceptions:

$$v_{1,4} = v_{3,6} = v_{5,8} = \cdots = v_{2h-3,2h} = v_{2,2} = 1, \text{ and}$$

$$v_{1,3} = v_{3,5} = v_{5,7} = \cdots = v_{2h-2,2h} = v_{1,1} \text{ are as yet undetermined.}$$

Next, let Λ_h denote $\det(\mathrm{I} - \Gamma + t\Gamma)$ with these value for $v_{i,j}$. Then we have, for example,

$$\Lambda_1 = \det \begin{bmatrix} 1 & -v_{1,1}(x-1) \\ x-1 & x \end{bmatrix}$$

$$\Lambda_2 = \det \begin{bmatrix} 1 & -v_{1,1}(x-1) & x-1 & -v_{1,3}(x-1) \\ x-1 & x & 0 & 0 \\ 0 & -v_{1,3}(x-1) & 1 & 0 \\ 0 & -(x-1) & 0 & x \end{bmatrix}$$

Now let us define Λ'_h as the $(2h-1) \times (2h-1)$ subdeterminant obtained by removing the last column and the next to last row. We can expand this determinant using $-(x-1)$, the only nonzero entry in the last row. For example, $\Lambda'_1 = x - 1$ and

$$\Lambda_2' = \begin{vmatrix} 1 & -v_{1,1}(x-1) & x-1 \\ x-1 & x & 0 \\ 0 & -(x-1) & 0 \end{vmatrix}$$

$$= (x-1) \cdot \begin{vmatrix} 1 & (x-1) \\ (x-1) & 0 \end{vmatrix}$$

$$= (x-1) \cdot -(x-1)^2$$

$$= -(x-1)^2 \cdot \Lambda_1'$$

$$= -(x-1)^3$$

Next we show $\Lambda_h' = -(x-1)^2\Lambda_{h-1}' = (-1)^{h-1}(x-1)^{2h-1}$. We see that it is true for $h = 2$. Assume it is true for $h-1$, that is,

$$\Lambda_{h-1}' = -(x-1)^2\Lambda_{h-2}' = (-1)^{h-2}(x-1)^{2h-3}.$$

We can then show it is true for h. For logistical purposes, let

$$M = \begin{bmatrix} 1 & -v_{1,1}(x-1) & (x-1) & -v_{1,3}(x-1) \\ x-1 & x & 0 & 0 \\ 0 & -v_{1,3}(x-1) & 1 & 0 \\ 0 & -(x-1) & 0 & x \end{bmatrix}$$

since M will appear in most of the matrices to follow. Then $\Lambda_h' =$

$$\begin{vmatrix} M & & & & & & \\ & \ddots & & & & & \\ & & 0 & (x-1) & -v_{2h-7,2h-5}(x-1) & 0 & 0 & 0 \\ & & x & 0 & 0 & 0 & 0 & 0 \\ & & -v_{2h-7,2h-5}(x-1) & 1 & 0 & (x-1) & -v_{2h-5,2h-3}(x-1) & 0 \\ & & -(x-1) & 0 & x & 0 & 0 & 0 \\ & & 0 & 0 & -v_{2h-5,2h-3}(x-1) & 1 & 0 & (x-1) \\ & & 0 & 0 & -(x-1) & 0 & x & 0 \\ & & 0 & 0 & 0 & 0 & -(x-1) & 0 \end{vmatrix}$$

Expanding along the last row, we have $\Lambda_h' =$

$$(x-1) \cdot \begin{vmatrix} M & & & & & \\ & \ddots & & & & \\ & & 0 & (x-1) & -v_{2h-7,2h-5}(x-1) & 0 & 0 \\ & & x & 0 & 0 & 0 & 0 \\ & & -v_{2h-5,2h-7}(x-1) & 1 & 0 & (x-1) & 0 \\ & & -(x-1) & 0 & x & 0 & 0 \\ & & 0 & 0 & -v_{2h-5,2h-3}(x-1) & 1 & (x-1) \\ & & 0 & 0 & -(x-1) & 0 & 0 \end{vmatrix}$$

Then we can expand along the last column to obtain

$$= -(x-1)^2 \cdot \begin{vmatrix} M & & & & & \\ & \ddots & & & & \\ & & 0 & (x-1) & -v_{2h-7,2h-5}(x-1) & 0 \\ & & x & 0 & 0 & 0 \\ & & -v_{2h-7,2h-5}(x-1) & 1 & 0 & (x-1) \\ & & -(x-1) & 0 & x & 0 \\ & & 0 & 0 & -(x-1) & 0 \end{vmatrix}$$

$$= -(x-1)^2 \Lambda'_{h-1}$$

$$= -(x-1)^2 \cdot (-1)^{h-2} \cdot (x-1)^{2h-3}$$

$$= (-1)^{h-1} \cdot (x-1)^{2h-1}$$

Using the derived formula for Λ'_h, we can now calculate Λ_h. First, let us consider an example. Begin with a 6×6 matrix as described in [HS].

$$\Gamma = \begin{bmatrix} v_{1,2} & -v_{1,1} & v_{1,4} & -v_{1,3} & v_{1,6} & -v_{1,5} \\ v_{2,2} & -v_{2,1} & v_{2,4} & -v_{2,3} & v_{2,6} & -v_{2,5} \\ v_{3,2} & -v_{3,1} & v_{3,4} & -v_{3,3} & v_{3,6} & -v_{3,5} \\ v_{4,2} & -v_{4,1} & v_{4,4} & -v_{4,3} & v_{4,6} & -v_{4,5} \\ v_{5,2} & -v_{5,1} & v_{5,4} & -v_{5,3} & v_{5,6} & -v_{5,5} \\ v_{6,2} & -v_{6,1} & v_{6,4} & -v_{6,3} & v_{6,6} & -v_{6,5} \end{bmatrix}$$

Then, $\det(\mathrm{I} - \Gamma + x\Gamma) =$

$$\begin{vmatrix} 1+v_{1,2}(x-1) & -v_{1,1}(x-1) & v_{1,4}(x-1) & -v_{1,3}(x-1) & v_{1,6}(x-1) & -v_{1,5}(x-1) \\ v_{2,2}(x-1) & x-v_{1,2}(x-1) & v_{2,4}(x-1) & -v_{2,3}(x-1) & v_{2,6}(x-1) & -v_{2,5}(x-1) \\ v_{2,3}(x-1) & -v_{1,3}(x-1) & 1+v_{3,4}(x-1) & -v_{3,3}(x-1) & v_{3,6}(x-1) & -v_{3,5}(x-1) \\ v_{2,4}(x-1) & -v_{1,4}(x-1) & v_{4,4}(x-1) & x-v_{3,4(x-1)} & v_{4,6}(x-1) & -v_{4,5}(x-1) \\ v_{2,5}(x-1) & -v_{1,5}(x-1) & v_{4,5}(x-1) & -v_{3,5}(x-1) & 1+v_{5,6}(x-1) & -v_{5,5}(x-1) \\ v_{2,6}(x-1) & -v_{1,6}(x-1) & v_{4,6}(x-1) & -v_{3,6}(x-1) & v_{6,6}(x-1) & x-v_{5,6}(x-1) \end{vmatrix}$$

Now we have

$$\Lambda_3 = \begin{vmatrix} 1 & -v_{1,1}(x-1) & (x-1) & -v_{1,3}(x-1) & 0 & 0 \\ (x-1) & x & 0 & 0 & 0 & 0 \\ 0 & -v_{1,3}(x-1) & 1 & 0 & (x-1) & -v_{3,5}(x-1) \\ 0 & -(x-1) & 0 & x & 0 & 0 \\ 0 & 0 & 0 & -v_{3,5}(x-1) & 1 & 0 \\ 0 & 0 & 0 & -(x-1) & 0 & x \end{vmatrix}$$

We can calculate Λ_3 by expanding along the last column.

$$\Lambda_3 = -v_{3,5}(x-1) \begin{vmatrix} 1 & -v_{1,1}(x-1) & (x-1) & -v_{1,3}(x-1) & 0 \\ (x-1) & x & 0 & 0 & 0 \\ 0 & -(x-1) & 0 & x & 0 \\ 0 & 0 & 0 & -v_{3,5}(x-1) & 1 \\ 0 & 0 & 0 & -(x-1) & 0 \end{vmatrix}$$

$$+x \begin{vmatrix} 1 & -v_{1,1}(x-1) & (x-1) & -v_{1,3}(x-1) & 0 \\ (x-1) & x & 0 & 0 & 0 \\ 0 & -v_{1,3}(x-1) & 1 & 0 & (x-1) \\ 0 & -(x-1) & 0 & x & 0 \\ 0 & 0 & 0 & -v_{3,5}(x-1) & 1 \end{vmatrix}$$

We can expand the first determinant along the last column and expand the resulting determinant along the last row. For the second determinant, we will expand along the last column.

$$= v_{3,5}(x-1)^2 \begin{vmatrix} 1 & -v_{1,1}(x-1) & (x-1) \\ (x-1) & x & 0 \\ 0 & -(x-1) & 0 \end{vmatrix}$$

$$+ x \begin{vmatrix} 1 & -v_{1,1}(x-1) & (x-1) & -v_{1,3}(x-1) \\ (x-1) & x & 0 & 0 \\ 0 & -v_{1,3}(x-1) & 1 & 0 \\ 0 & -(x-1) & 0 & x \end{vmatrix}$$

$$+ x(x-1) \begin{vmatrix} 1 & -v_{1,1}(x-1) & (x-1) & -v_{1,3}(x-1) \\ (x-1) & x & 0 & 0 \\ 0 & -(x-1) & 0 & x \\ 0 & 0 & 0 & -v_{3,5}(x-1) \end{vmatrix}$$

$$= v_{3,5}(x-1)^2\Lambda_2' + x\Lambda_2 - x \cdot v_{3,5}(x-1)^2 \begin{vmatrix} 1 & -v_{1,1}(x-1) & (x-1) \\ (x-1) & x & 0 \\ 0 & -(x-1) & 0 \end{vmatrix}$$

$$= v_{3,5}(x-1)^2\Lambda_2' + x\Lambda_2 + xv_{3,5}(x-1)^3\Lambda_2'$$

$$= -v_{3,5}(x-1)^5 + x\Lambda_2 + xv_{3,5}(x-1)^5$$

$$= x\Lambda_2 + v_{3,5}(x-1)^6$$

We will generalize this process beginning with Seifert's $2h \times 2h$ matrix. Making the necessary substitutions to get Λ_h, we can now calculate this determinant. We can easily see that Λ_h contains the previous Λ_i for $1 \leq i \leq h-1$ as submatrices as evidenced in Λ_1, Λ_2, and Λ_3 above. Thus, let us focus on the entries of the bottom right submatrix in Λ_h. Note that in the rows and columns we will use below to calculate the determinant, all of the nonzero entries are listed. Then, $\Lambda_h =$

$$\begin{vmatrix} \ddots & & & & & & \\ & (x-1) & -v_{2h-7,2h-5}(x-1) & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 \\ & 1 & 0 & (x-1) & -v_{2h-5,2h-3}(x-1) & 0 & 0 \\ & 0 & x & 0 & 0 & 0 & 0 \\ & 0 & -v_{2h-5,2h-3}(x-1) & 1 & 0 & (x-1) & -v_{2h-3,2h-1}(x-1) \\ & 0 & -(x-1) & 0 & x & 0 & 0 \\ & 0 & 0 & 0 & -v_{2h-3,2h-1}(x-1) & 1 & 0 \\ & 0 & 0 & 0 & -(x-1) & 0 & x \end{vmatrix}$$

. We will expand the determinant using the last column. Then, $\Lambda_h =$

$$v_{2h-3,2h-1}(x-1)\begin{vmatrix} \ddots & & & & & \\ & (x-1) & -v_{2h-7,2h-5}(x-1) & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ & 1 & 0 & (x-1) & -v_{2h-5,2h-3}(x-1) & 0 \\ & 0 & x & 0 & 0 & 0 \\ & 0 & -(x-1) & 0 & x & 0 \\ & 0 & 0 & 0 & -v_{2h-3,2h-1}(x-1) & 1 \\ & 0 & 0 & 0 & -(x-1) & 0 \end{vmatrix}$$

$$+\,x\begin{vmatrix} \ddots & & & & & \\ & (x-1) & -v_{2h-7,2h-5}(x-1) & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ & 1 & 0 & (x-1) & -v_{2h-5,2h-3}(x-1) & 0 \\ & 0 & x & 0 & 0 & 0 \\ & 0 & -v_{2h-5,2h-3}(x-1) & 1 & 0 & (x-1) \\ & 0 & -(x-1) & 0 & x & 0 \\ & 0 & 0 & 0 & -v_{2h-3,2h-1}(x-1) & 1 \end{vmatrix}$$

We can expand the first determinant along the last column and then expand the resulting determinant along the last row as in the example. For the second determinant, we will expand along the last column. We obtain

$$v_{2h-3,2h-1}(x-1)^2 \begin{vmatrix} \ddots & & & \\ & (x-1) & -v_{2h-7,2h-5}(x-1) & 0 \\ & 0 & 0 & 0 \\ & 1 & 0 & (x-1) \\ & 0 & x & 0 \\ & 0 & -(x-1) & 0 \end{vmatrix}$$

$$+x \begin{vmatrix} \ddots & & & & \\ & (x-1) & -v_{2h-7,2h-5}(x-1) & 0 & 0 \\ & 0 & 0 & 0 & 0 \\ & 1 & 0 & (x-1) & -v_{2h-5,2h-3}(x-1) \\ & 0 & x & 0 & 0 \\ & 0 & -v_{2h-5,2h-3}(x-1) & 1 & 0 \\ & 0 & -(x-1) & 0 & x \end{vmatrix}$$

$$+x(x-1) \begin{vmatrix} \ddots & & & & \\ & (x-1) & -v_{2h-7,2h-5}(x-1) & 0 & 0 \\ & 0 & 0 & 0 & 0 \\ & 1 & 0 & (x-1) & -v_{2h-5,2h-3}(x-1) \\ & 0 & x & 0 & 0 \\ & 0 & -(x-1) & 0 & x \\ & 0 & 0 & 0 & -v_{2h-3,2h-1}(x-1) \end{vmatrix}$$

Expand the third determinant above along the last row. This yields

$$v_{2h-3,2h-1}(x-1)^2 \begin{vmatrix} \ddots & & & \\ & (x-1) & -v_{2h-7,2h-5}(x-1) & 0 \\ & 0 & 0 & 0 \\ & 1 & 0 & (x-1) \\ & 0 & x & 0 \\ & 0 & -(x-1) & 0 \end{vmatrix}$$

$$+ x \begin{vmatrix} \ddots & & & & \\ & (x-1) & -v_{2h-7,2h-5}(x-1) & 0 & 0 \\ & 0 & 0 & 0 & 0 \\ & 1 & 0 & (x-1) & -v_{2h-5,2h-3}(x-1) \\ & 0 & x & 0 & 0 \\ & 0 & -v_{2h-5,2h-3}(x-1) & 1 & 0) \\ & 0 & -(x-1) & 0 & x \end{vmatrix}$$

$$- xv_{2h-3,2h-1}(x-1)^2 \begin{vmatrix} \ddots & & & \\ & (x-1) & -v_{2h-7,2h-5}(x-1) & 0 \\ & 0 & 0 & 0 \\ & 1 & 0 & (x-1) \\ & 0 & x & 0 \\ & 0 & -(x-1) & 0 \end{vmatrix}$$

$$\begin{aligned} &= v_{2h-3,2h-1}(x-1)^2\Lambda'_{h-1} + x\Lambda_{h-1} - x \cdot v_{2h-3,2h-1}(x-1)^2\Lambda'_{h-1} \\ &= x\Lambda_{h-1} + v_{2h-3,2h-1}(x-1)^2\Lambda'_{h-1}(1-x) \\ &= x\Lambda_{h-1} + (-1)^{h-1}v_{2h-3,2h-1}(x-1)^{2h} \qquad (4.2) \end{aligned}$$

Now that we have written Λ_h in terms of Λ'_{h-1}, we will use induction on h to finish the proof. Consider first $h = 1$. Let $f(x) = c_0 + c_1x + c_2x^2$ be an arbitrary polynomial with $c_0 + c_1 + c_2 = 1$ and $c_0 = c_2$ to satisfy conditions (3.1) and (3.2). Substituting $v_{1,1} = c_0$, we find that $f(x)$ can be written as a determinant in the form of Λ_1.

$$\begin{aligned}
\Lambda_1 &= \begin{vmatrix} 1 & -v_{1,1}(x-1) \\ (x-1) & x \end{vmatrix} \\
&= x + v_{1,1}(x-1)^2 \\
&= v_{1,1} + (1 - 2v_{1,1})x + v_{1,1}x^2 \\
&= c_0 + (1 - 2c_0)x + c_0 x^2 \\
&= c_0 + c_1 x + c_2 x^2
\end{aligned}$$

We now assume it to be true for $h - 1$. We claim that we can find values for $v_{1,1}, v_{1,3}, v_{3,5}, \cdots, v_{2h-5,2h-3}$ so that

$$\Lambda_{h-1} = \frac{c_0 + c_1 x + \cdots + c_{2h}x^{2h} - c_0(x-1)^{2h}}{x}$$

We need to show that the right hand side of this equation satisfies (3.1) and (3.2) above. We have $c_0 + c_1 + \cdots + c_{2h} = 1$ since we can again let $x = 1$ to achieve this result. The numerator is the difference of two reciprocal polynomials of the same degree, hence reciprocal. Note also that Λ_{h-1} has degree $\leq 2h - 2$ since $c_0 = c_{2h}$, canceling the highest term. Then, by the induction hypothesis, we can let $v_{2h-3,2h-1} = (-1)^{h-1}c_0$ and (4.2) yields

$$\Lambda_h = c_0 + c_1 x + \cdots + c_{2h}x^{2h}.$$

Thus, given an arbitrary polynomial satisfying (3.1) and (3.2), there exists a $2h \times 2h$ matrix Γ such that $\det(\mathrm{I} - \Gamma + x\Gamma) = \Lambda_h$ equals the polynomial.

Chapter 5

A General Result on Reciprocal Polynomials

We have already seen that Alexander polynomials are reciprocal and of even degree. Let us consider a general property of reciprocal polynomials. Let $f(x)$ be a reciprocal polynomial of degree $2n$. Then there exists a polynomial $h(x)$ such that

$$f(x) = x^n h(x + \frac{1}{x}).$$

The map $x \to \frac{1}{x}$ has two fixed points, namely 1 and -1. The following theorem will give us a formula for $f(1)$ and $f(-1)$.

Theorem 3

$$disc(f) = (-1)^n disc(h)^2 f(1) f(-1).$$

Proof: If we write f in the form

$$f(x) = c \prod_{i=1}^{n} (x - \alpha_i)(x - \alpha_i^{-1}),$$

then the roots of h are of the form $\alpha_i + \alpha_i^{-1}$, and h can be written as

$$h(x) = c \prod_{i=1}^{n} (x - \alpha_i - \alpha_i^{-1}).$$

The discriminant of h is then

$$c^{2n-2}\prod_{i<j}(\alpha_j+\alpha_j^{-1}-\alpha_i-\alpha_i^{-1})^2.$$

Consider the value of $f(1)f(-1)$:

$$\begin{aligned}
f(1)f(-1) &= c^2\prod_{i=1}^{n}(1-\alpha_i)(1-\alpha_i^{-1})(1+\alpha_i)(1+\alpha_i^{-1})\\
&= c^2\prod_{i=1}^{n}(1-\alpha_i^2)(1-\alpha_i^{-2})\\
&= c^2\prod_{i=1}^{n}(\alpha_i^{-1}-\alpha_i)(\alpha_i-\alpha_i^{-1})\\
&= c^2(-1)^n\prod_{i=1}^{n}(\alpha_i-\alpha_i^{-1})^2.
\end{aligned}$$

Now let us calculate the discriminant of f. Order the roots $r_1,\cdots,r_{2n}$ so that $r_i=\alpha_i^{-1}$ for $1\le i\le n$ and $r_{i+n}=\alpha_i$ for $1\le i\le n$. Then,

$$\begin{aligned}
\operatorname{disc}(f) &= c^{4n-2}\prod_{j<k}(r_k-r_j)^2\\
&= c^{4n-2}\prod_{i<j}(\alpha_j^{-1}-\alpha_i^{-1})^2\prod_{i,j}(\alpha_j-\alpha_i^{-1})^2\prod_{i<j}(\alpha_j-\alpha_i)^2. \qquad (5.1)
\end{aligned}$$

Consider the middle product $\prod_{i,j}(\alpha_j-\alpha_i^{-1})^2$. Let us break it into factors with $i<j, i=j$, and $i>j$. We will combine the first and third of these with the first and third products in equation (5.1) above. The first yields

$$\begin{aligned}
&\prod_{i<j}((\alpha_j^{-1}-\alpha_i^{-1})(\alpha_j-\alpha_i^{-1}))^2\\
=&\prod_{i<j}(\alpha_i^{-1}(\alpha_i+\alpha_i^{-1}-\alpha_j-\alpha_j^{-1}))^2. \qquad (5.2)
\end{aligned}$$

The middle is

$$\prod_{j=1}^{n}(\alpha_j - \alpha_j^{-1})^2. \tag{5.3}$$

The third is

$$\prod_{i>j}(\alpha_j - \alpha_i^{-1})^2 \prod_{i<j}(\alpha_j - \alpha_i)^2.$$

However,

$$\prod_{i>j}(\alpha_j - \alpha_i^{-1})^2 = \prod_{i<j}(\alpha_i - \alpha_j^{-1})^2,$$

So the third then becomes

$$\prod_{i<j}(\alpha_i - \alpha_j^{-1})^2(\alpha_j - \alpha_i)^2$$

which equals

$$\prod_{i<j}(\alpha_i(\alpha_j + \alpha_j^{-1} - \alpha_i - \alpha_i^{-1}))^2. \tag{5.4}$$

Then we can combine (5.2), (5.3), and (5.4) so that (5.1) becomes

$$\operatorname{disc}(f) = c^{4n-2}\prod_{i<j}(\alpha_i(\alpha_j + \alpha_j^{-1} - \alpha_i - \alpha_i^{-1})^4 \prod_{j=1}^{n}(\alpha_j - \alpha_j^{-1})^2,$$

which equals

$$(-1)^n \operatorname{disc}(h)^2 f(1) f(-1).$$

So we see $f(1)f(-1)$ is equal to the the discriminant of f up to multiplication by a square. In particular, if f is an Alexander polynomial, $f(1) = 1$ and so we obtain a formula for $f(-1)$.

Chapter 6

Roots of Alexander Polynomials for Knots with up to Nine Crossings

Here we will consider the Alexander polynomial $A_K(t)$ for any knot K with up to nine crossings. In testing to see which polynomials have roots of absolute value 1, we achieve the following results. We use the polynomials listed in [CL].

Polynomials with Roots of Absolute Value 1
$A_{3_1}(t) = t^2 - t + 1$
$A_{5_1}(t) = t^4 - t^3 + t^2 - t + 1$
$A_{5_2}(t) = 2t^2 - 3t + 2$
$A_{7_1}(t) = t^6 - t^5 + t^4 - t^3 + t^2 - t + 1$
$A_{7_2}(t) = 3t^2 - 5t + 3$
$A_{7_3}(t) = 2t^4 - 3t^3 + 3t^2 - 3t + 2$
$A_{7_4}(t) = 4t^2 - 7t + 4$
$A_{7_5}(t) = 2t^4 - 4t^3 + 5t^2 - 4t + 2$
$A_{8_{10}}(t) = t^6 - 3t^5 + 6t^4 - 7t^3 + 6t^2 - 3t + 1$
$A_{8_{15}}(t) = 3t^4 - 8t^3 + 11t^2 - 8t + 3$
$A_{8_{19}}(t) = t^6 - t^5 + t^3 - t + 1$
$A_{8_{20}}(t) = t^4 - 2t^3 + 3t^2 - 2t + 1$
$A_{9_1}(t) = t^8 - t^7 + t^6 - t^5 + t^4 - t^3 + t^2 - t + 1$

$A_{9_2}(t) = 4t^2 - 7t + 4$
$A_{9_3}(t) = 2t^6 - 3t^5 + 3t^4 - 3t^3 + 3t^2 - 3t + 2$
$A_{9_4}(t) = 3t^4 - 5t^3 + 5t^2 - 5t + 3$
$A_{9_5}(t) = 6t^2 - 11t + 6$
$A_{9_6}(t) = 2t^6 - 4t^5 + 5t^4 - 5t^3 + 5t^2 - 4t + 2$
$A_{9_7}(t) = 3t^4 - 7t^3 + 9t^2 - 7t + 3$
$A_{9_9}(t) = 2t^6 - 4t^5 + 6t^4 - 7t^3 + 6t^2 - 4t + 2$
$A_{9_{10}}(t) = 4t^4 - 8t^3 + 9t^2 - 8t + 4$
$A_{9_{13}}(t) = 4t^4 - 9t^3 + 11t^2 - 9t + 4$
$A_{9_{16}}(t) = 2t^6 - 5t^5 + 8t^4 - 9t^3 + 8t^2 - 5t + 2$
$A_{9_{18}}(t) = 4t^4 - 10t^3 + 13t^2 - 10t + 4$
$A_{9_{23}}(t) = 4t^4 - 11t^3 + 15t^2 - 11t + 4$
$A_{9_{35}}(t) = 7t^2 - 13t + 7$
$A_{9_{38}}(t) = 5t^4 - 14t^3 + 19t^2 - 14t + 5$
$A_{9_{49}}(t) = 3t^4 - 6t^3 + 7t^2 - 6t + 3$

Polynomials with Only Some Roots of Absolute Value 1 (*The number of roots is listed after the polynomial.*)
$A_{6_2}(t) = t^4 - 3t^3 + 3t^2 - 3t + 1; 2$
$A_{7_6}(t) = t^4 - 5t^3 + 7t^2 - 5t + 1; 2$
$A_{8_2}(t) = t^6 - 3t^5 + 3t^4 - 3t^3 + 3t^2 - 3t + 1; 4$
$A_{8_4}(t) = 2t^4 - 5t^3 + 5t^2 - 5t + 2; 2$
$A_{8_5}(t) = t^6 - 3t^5 + 4t^4 - 5t^3 + 4t^2 - 3t + 1; 4$
$A_{8_7}(t) = t^6 - 3t^5 + 5t^4 - 5t^3 + 5t^2 - 3t + 1; 2$
$A_{8_{11}}(t) = 2t^4 - 7t^3 + 9t^2 - 7t + 2; 2$
$A_{8_{14}}(t) = 2t^4 - 8t^3 + 11t^2 - 8t + 2$
$A_{8_{16}}(t) = t^6 - 4t^5 + 8t^4 - 9t^3 + 8t^2 - 4t + 1; 2$

$A_{8_{18}}(t) = t^6 - 5t^5 + 10t^4 - 13t^3 + 10t^2 - 5t + 1; 4$

$A_{8_{21}}(t) = t^4 - 4t^3 + 5t^2 - 4t + 1; 2$

$A_{9_8}(t) = 2t^4 - 8t^3 + 11t^2 - 8t + 2; 2$

$A_{9_{11}}(t) = t^6 - 5t^5 + 7t^4 - 7t^3 + 7t^2 - 5t + 1; 4$

$A_{9_{12}}(t) = 2t^4 - 9t^3 + 13t^2 - 9t + 2; 2$

$A_{9_{15}}(t) = 2t^4 - 10t^3 + 15t^2 - 10t + 2; 2$

$A_{9_{17}}(t) = t^6 - 5t^5 + 9t^4 - 9t^3 + 9t^2 - 5t + 1; 2$

$A_{9_{20}}(t) = t^6 - 5t^5 + 9t^4 - 11t^3 + 9t^2 - 5t + 1; 4$

$A_{9_{21}}(t) = 2t^4 - 11t^3 + 17t^2 - 11t + 2; 2$

$A_{9_{22}}(t) = t^6 - 5t^5 + 10t^4 - 11t^3 + 10t^2 - 5t + 1; 2$

$A_{9_{24}}(t) = t^6 - 5t^5 + 10t^4 - 13t^3 + 10t^2 - 5t + 1; 4$

$A_{9_{25}}(t) = 3t^4 - 12t^3 + 17t^2 - 12t + 3; 2$

$A_{9_{26}}(t) = t^6 - 5t^5 + 11t^4 - 13t^3 + 11t^2 - 5t + 1; 2$

$A_{9_{28}}(t) = t^6 - 5t^5 + 12t^4 - 15t^3 + 12t^2 - 5t + 1; 2$

$A_{9_{29}}(t) = t^6 - 5t^5 + 12t^4 - 15t^3 + 12t^2 - 5t + 1; 2$

$A_{9_{31}}(t) = t^6 - 5t^5 + 13t^4 - 17t^3 + 13t^2 - 5t + 1; 2$

$A_{9_{32}}(t) = t^6 - 6t^5 + 14t^4 - 17t^3 + 14t^2 - 6t + 1; 2$

$A_{9_{36}}(t) = t^6 - 5t^5 + 8t^4 - 9t^3 + 8t^2 - 5t + 1; 4$

$A_{9_{39}}(t) = 3t^4 - 14t^3 + 21t^2 - 14t + 3; 2$

$A_{9_{40}}(t) = t^6 - 7t^5 + 18t^4 - 23t^3 + 18t^2 - 7t + 1; 2$

$A_{9_{42}}(t) = t^4 - 2t^3 + t^2 - 2t + 1; 2$

$A_{9_{43}}(t) = t^6 - 3t^5 + 2t^4 - t^3 + 2t^2 - 3t + 1; 4$

$A_{9_{45}}(t) = t^4 - 6t^3 + 9t^2 - 6t + 1; 2$

$A_{9_{47}}(t) = t^6 - 4t^5 + 6t^4 - 5t^3 + 6t^2 - 4t + 1; 2$

$A_{9_{48}}(t) = t^4 - 7t^3 + 11t^2 - 7t + 1; 2$

Polynomials with No Roots of Absolute Value 1
$A_{4_1}(t) = t^2 - 3t + 1$
$A_{6_1}(t) = 2t^2 - 5t + 2$
$A_{6_3}(t) = t^4 - 3t^3 + 5t^2 - 3t + 1$
$A_{7_7}(t) = t^4 - 5t^3 + 9t^2 - 5t + 1$
$A_{8_1}(t) = 3t^2 - 7t + 3$
$A_{8_3}(t) = 4t^2 - 9t + 4$
$A_{8_4}(t) = 2t^4 - 6t^3 + 9t^2 - 6t + 2$
$A_{8_9}(t) = t^6 - 3t^5 + 5t^4 - 7t^3 + 5t^2 - 3t + 1$
$A_{8_{12}}(t) = t^4 - 7t^3 + 13t^2 - 7t + 1$
$A_{8_{13}}(t) = 2t^4 - 7t^3 + 11t^2 - 7t + 2$
$A_{8_{17}}(t) = t^6 - 4t^5 + 8t^4 - 11t^3 + 8t^2 - 4t + 1$
$A_{9_{14}}(t) = 2t^4 - 9t^3 + 15t^2 - 9t + 2$
$A_{9_{19}}(t) = 2t^4 - 10t^3 + 17t^2 - 10t + 2$
$A_{9_{27}}(t) = t^6 - 5t^5 + 11t^4 - 15t^3 + 11t^2 - 5t + 1$
$A_{9_{30}}(t) = t^6 - 5t^5 + 12t^4 - 17t^3 + 12t^2 - 5t + 1$
$A_{9_{33}}(t) = t^6 - 6t^5 + 16t^4 - 23t^3 + 16t^2 - 6t + 1$
$A_{9_{37}}(t) = 2t^4 - 11t^3 + 19t^2 - 12t + 3$
$A_{9_{41}}(t) = 3t^4 - 12t^3 + 19t^2 - 12t + 3$
$A_{9_{44}}(t) = t^4 - 4t^3 + 7t^2 - 4t + 1$
$A_{9_{46}}(t) = 2t^2 - 5t + 2$

Chapter 7

Curves over Finite Fields

It was pointed out in [JM] that the Weil polynomial in number theory possesses characteristics that parallel the Alexander polynomial in knot theory. As in [DG], let C be a curve of genus g defined over the finite field $\mathbf{F}_q$. The zeta function of C has the form

$$Z_c(T) = \frac{P(T)}{(1-T)(1-qT)},$$

where the degree of $P(T) = 2g$ and $P(0) = 1$. Then $P(T)$ is the characteristic polynomial of the Frobenius acting on some cohomology group. This corresponds to a result presented in [JM] that the Alexander polynomial appears as a characteristic polynomial for an action on the homology group of a space.

We have already seen that the Alexander polynomial is reciprocal. The polynomial $P(T)$ exhibits reciprocal behavior as well. We have the "functional equation"

$$P(T) = q^g T^{2g} P(\frac{1}{qT})$$

If we write the Weil polynomial as

$$P(T) = \prod_{i=1}^{2g} (\alpha_i T - 1),$$

then, by the Riemann hypothesis for C, we have $|\alpha_i| = q^{\frac{1}{2}}$ Note that if we formally let $q = 1$, the functional equation becomes the equation for a reciprocal polynomial and the roots all have absolute value 1. In the last chapter, we saw that some or all roots of many of the knots with up to nine crossings have absolute value 1. From [JT], we know that essentially every $P(T) \in \mathbb{Z}(T)$ satisfying the functional equation and Riemann hypothesis comes from a curve. This is analogous to the fact that any reciprocal polynomial of even degree with integer coefficients comes from a knot. It would be interesting to find out if there is some characterization of the knots whose Alexander polynomials satisfy the "Riemann hypothesis," namely that all roots have absolute value 1, since these should be the knots that are most closely analogous to curves over finite fields.

Finally, solving $T = \frac{1}{qT}$ gives the two fixed points $T = \pm q^{-\frac{1}{2}}$ for the symmetry in the functional equation. The paper [NR] essentially gives a formula for $P(q^{-\frac{1}{2}})P(-q^{-\frac{1}{2}})$. The analogue for this result is the formula derived for $f(1)f(-1)$ where f is a reciprocal polynomial with integer coefficients of degree $2n$, which allows us to write $f(-1)$ in terms of the discriminant of f up to a square.

Given these striking similarities between the Alexander and Weil polynomials, certainly more research in this area would prove fruitful.

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