THE MAIN PROBLEM IN SATELLITE THEORY REVISITED

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Abstract. Using the elimination of the parallax followed by the Delaunay normalization, we present a procedure for calculating a normal form of the main problem (J_2 perturbation only) in satellite theory. This procedure is outlined in such a way that an object-oriented automatic symbolic manipulator based on a hierarchy of algebras can perform this computation. The Hamiltonian after the Delaunay normalization is presented to order six explicitly in closed form, that is, in which there is no expansion in the eccentricity. The corresponding generating function and transformation of coordinates, too lengthy to present here to the same order; the generator is given through order four.

Keywords: Lie transformation, normal form, closed form, computer algebra.

1. Introduction

The techniques astronomers use to handle perturbations to Keplerian behavior has changed over time with discoveries of new methods and the availability of computer technology. Before Brouwer, the Hamiltonian was expressed in Cartesian coordinates and velocities. These in turn can be expressed in terms of osculating elements. However, when we do this, the Hamiltonian will no longer be a function of the mean anomaly ℓ. This in turn requires a solution to Kepler's equation, which could not be solved in closed form, but rather must be expressed as a power series in the eccentricity e. Thus, astronomers would resort to the artifice of introducing the coordinates and velocities expressed as a Fourier series in ℓ, with their coefficients being power series in e. First the perturbation was expanded, then a Poincaré-von Zeipel transformation used to eliminate the short-period term ℓ. This was the approach of Delaunay in the lunar theory (1846, 1855), Poincaré (1905), Tisserand (1888), and Smart (1953).

Brouwer (1959) gave up the explicit representation of the Hamiltonian as a Fourier series in ℓ, and settled for an implicit representation. He obtained expressions that were $1/r^2$ times a Fourier series in the true anomaly f, with the coefficients of this Fourier series being rational functions of e and $\eta = \sqrt{1-e^2}$. The success of this method in carrying the Delaunay normalization in closed form for the main problem in satellite theory was shown by Brouwer himself, who presented results to order one, Aksnes (1965a, b), who presented results to order two, Deprit and Coffey (1982) who presented results to order three and four, and the present paper, with results at order five and six.

There are two critical factors that make the normalization to sixth order possible in closed form. First, the preparatory calculation of the elimination of the parallax, introduced by (Deprit, 1981), that simplifies considerably the ensuing Delaunay normalization. This was used by Deprit and Coffey (1982) to successfully calculate the third and fourth order terms. Second, the capability of doing algebra by machine; specifically, an object-oriented hierarchical algebraic processor. Through this hierarchy we are able to localize the levels at which we apply the simplification rules. This has two advantages: it limits the number of operations, and helps in defining and making systematic the simplification rules. Because of the object orientation, we may attach the simplification rules to objects; we conceive of the solution as one of applying simplification rules to particular instances of an algebra, not of programs executed with data representing the expression.

Of course, in the satellite problem it does not make sense to expand the main problem to order six. This is not the point; rather, it is to indicate the power of these new techniques: they open an avenue towards simplifying a number of problems in celestial mechanics. The mechanism described here represents a formulation of rules for simplification and substitution that when combined allow for the algorithmic computation of Lie simplifications (Deprit and Miller, 1989). Furthermore, these rules, like the canonical representation of ellipse parameters given in Section 6, when applied independently, will prove useful in their own right for other celestial mechanics and astrodynamics problems.

2. Structuring the Problem for Automatic Symbolic Manipulation

The process of performing a Delaunay normalization involves copious amounts of algebra. Even with the elimination of the parallax, an example of what Deprit and Ferrer (1989) later called simplification; that is, approaching the normalization through a succession of stages, the calculation quickly exceeds the capacity of even the most patient mathematician as the perturbation order increases, requiring the use of computer-based symbolic algebra codes. However, not all such codes will do; the commonly available general purpose symbolic manipulation codes have the wrong model: while able to do a multitude of operations from all branches of mathematics on a wide range of small to medium size expressions, they do not efficiently handle simple algebra on a restricted class of large expressions, on the order of tens of thousands of terms.

We require a system that allows us to incorporate knowledge of the structure of the expressions in our problem, such that their representation, both internal and external, is efficient. The program PMAO (Portable Mechanized Algebraic Operations), a derivative of MAO (Miller and Deprit, 1986), reflects such a hierarchy of algebras. For a particular problem, one declares the variables and algebras (e.g., polynomial or Fourier) and their relationships as a hierarchy of algebras over a domain of coefficients which are themselves algebras. Thereafter, all expressions are stored in compact form because variables are known in advance, and computations are performed in the context of and with the appropriate organization. There is no need to coerce the computer into showing the desired form, or to frequently pass back and forth between different representations.

For example, consider the hierarchy

$$F[f, g] \triangleright P[e, \eta, \beta] \triangleright P[s^2] \triangleright \mathbf{Q}. \tag{1}$$

In this notation, each algebra has a triangle '$\triangleright$' pointing to its algebra of coefficients, P designates a polynomial algebra in the variable(s) indicated, F designates a Fourier algebra, and **Q** is the field of rationals. We mean by this that the Fourier algebra in f and g has coefficients that are polynomials of e, η and β, which in turn have coefficients that are polynomials in s^2, which finally has rational coefficients. For example,

$$\begin{aligned} &e^2\left(\frac{45}{256}s^6+\frac{21}{128}s^4-\frac{15}{32}s^2+\frac{3}{16}\right)\cos 2f\,+\,\eta^2\left[\left(\frac{135}{128}s^6-\frac{27}{16}s^4+\frac{21}{32}s^2\right)-\right.\\ &\left.\quad-\frac{135}{128}s^6+\frac{27}{16}s^4-\frac{21}{32}s^2\right]\cos(2f+2g) \end{aligned} \tag{2}$$

is in this algebra. The effect of the hierarchy is to show us how the expression is to be factored among any number of equivalent representations of the same expression.

With the algebraic organization reflected in object-oriented code, we may target algebraic simplifications at the appropriate algebra, ignoring the others. Then, as we pass an expression to the function that performs our algebraic simplifications, the simplification takes place at the appropriate algebra, with further simplifications being called recursively on the domain of coefficients. This greatly reduces the amount of computation over a free-form representation or an algorithmic approach to simplification. We employ this simplification strategy in the Delaunay normalization to handle expressions of e, η, and β (Section 6) and $\cos jf$ and $\sin jf$ (Section 7). Given a known algebraic structure, the conversion from one form to another, as occurs in algebraic simplification, is designed so that it happens a minimal number of times and only at strategically chosen places so that expressions are kept in a canonical form of compact size.

The choice of an algebraic structure, derivatives, and algebraic simplification strategy are not at all obvious in many problems. The goals one strive for are first, of course, correctness; next, literate code, in the sense of (Knuth, 1992): "Instead of imagining that our main task is to instruct the computer what to do, let us concentrate rather on explaining to human beings what we want a computer to do;" and finally efficiency, that the computation completes in a tolerable amount of time on a readily accessible machine. Deprit (1982) following Jeffreys (1971) discusses the problem areas of *simplification* and *closure*, the former meaning conversion of an expression from one representation to an equivalent one at each step of the calculation, the latter meaning the finding of the generator in closed form from its image. In the present discussion, we view these as part of the issue of *structure*. Structure means finding the appropriate algebraic hierarchy to solve the problem, including the right *canonical* form for each algebra and the procedures required to put arbitrary expressions into that canonical form. The canonical algebraic form is a unique way of representing an expression in the algebra. Since we are now assuming that there will be no expansion in the eccentricity, the issue of closure is really one of *inversion of the Lie operator*. Inversion means finding the generator whose image under the Lie operator is the term we need to eliminate. This includes finding the right set of variables such that the operator can be inverted, an issue discussed in the next section, and structuring the overall problem such that no terms that can't be inverted ever occur.

In practice, this is a matter of trial-and-error, sometimes requiring recomputation hundreds of times. It happens often that one can find an organization that produces the correct answer in a satisfactory time to a given order, but going to the next order results in the appearance of terms that cannot be handled or that there are unacceptable computation delays. In the present problem, this is mostly the case with the Delaunay normalization. Of course, this is true independent of whether automatic symbolic computation is employed. Here is where the use of computers is a great advantage; after all, if we wish merely to solve only one problem and none related, we could with patience do it by hand to low order. Repeatedly tackling the same problem, each time modifying the strategy to find the best would be unbearably tedious without a computer.

The organization presented here is, the author feels, fairly efficient, being both in a form that is reasonably literate for someone reading the code who knows LISP and PMAO, and efficient (about 8 min on a Silicon Graphics 175 MHz R10000, computing to sixth order). It is also correct in that, through fourth order, it has been checked against previous results, and at all orders, passes internal checks; primarily, that the generator computed does generate the transformation and that there are no unwanted terms left in the new Hamiltonian. Nevertheless, it is entirely possible that the reader may discover an organization that is easier to understand, more efficient to compute, or both. If so, he or she is encouraged to contact the author.

As part of its natural representation of mathematics to the user, PMAO prints out expressions in a customary form; Greek letters are drawn properly, numerical fractions are written with one integer atop another, and exponents are true superscripts. For PMAO expressions of greater significance, the equivalent LaTeX (Lamport, 1994) expression can, through a built-in mechanism, be inserted directly into an editor buffer. Thus expressions put into a paper or a report will be free of transcription error. Expressions from PMAO in this paper have been generated in this fashion. Readers interested in this software should contact the author.

The descriptions below for the elimination of the parallax and the Delaunay normalization are idealized versions of how they are actually coded. Sometimes, it is necessary to introduce intermediate variables for a particular stage of the computation that are not needed elsewhere; they have been deleted for clarity. Moreover, all algebra hierarchies presented here are in actuality merged together in one large one.

3. The Lie Transformation Method

Deprit (1969) gives a method for performing canonical transformations based on a small parameter; this Lie transformation method has been widely used for computing the normal form (or, more generally, Lie simplification) of various dynamical systems, especially perturbed Keplerian systems. This method has the advantage, among other things, that it is suited for automatic symbolic computation. PMAO has a mathematically equivalent but more efficient procedure due to Miller built-in. The user solving a particular problem need only supply a few things; obviously, the algebra hierarchy and variables is of paramount importance before even any expression can be given to the computer. Beyond this, there are basically three other things: the initial Hamiltonian (which may come from a previous calculation as in the case of the Delaunay normalization), separators and integrators for each stage, and any needed algebraic simplifiers used *en route*. Because LISP is incrementally linked and treats functions and their environments as data objects, we merely write the declarations and functions for the specific problem in a separate file, then compile them and call the appropriate function to compute the whole thing.

Each Lie simplification involves a strategy or choice on *what* canonical transformation is performed. Deprit shows how the general mechanism works; with $\mathcal{H}_{n0}$ the terms in the expansion of the original Hamiltonian and $\mathcal{H}_{0n}$ the terms of the transformed, the quantities in the Lie triangle are computed by

$$\mathcal{H}_{q,p+1} = \mathcal{H}_{q+1,p} + \sum_{m=0}^{q} \binom{q}{m} (\mathcal{H}_{q-m,p}; W_{m+1}). \tag{3}$$

This must be solved recursively; the term $\mathcal{H}_{q+1,p}$ in the right-hand side is solved by the same formula, repetitively until the last step $p = 0$ and $q = n - 1$. At this point, we need to solve for $\mathcal{H}_{n-1,1}$,

$$\mathcal{H}_{n-1,1} = \mathcal{H}_{n0} + (\mathcal{H}_{n-1,0}; W_1) + \cdots + (n-1)(\mathcal{H}_{10}; W_{n-1}) + (\mathcal{H}_{00}; W_n). \tag{4}$$

The provisional element $\tilde{\mathcal{H}}_{0n}$ is the entire recursively calculated $\mathcal{H}_{0n}$, but with the last term missing, $\tilde{\mathcal{H}}_{0n} \equiv -(\mathcal{H}_{00}; W_n)$. Then we may write the final Hamiltonian $\mathcal{H}_{0n}$ in terms of the provisional element and an unknown term which is the image of the generator at order n, W_n

$$\mathcal{H}_{0n} = \tilde{\mathcal{H}}_{0n} + (\mathcal{H}_{00}; W_n) = \tilde{\mathcal{H}}_{0n} + \mathcal{L}_0(W_n). \tag{5}$$

At each order n, one obtains a provisional Hamiltonian. From this point, however, we must choose the W_n in order to effect some desirable property in the transformed Hamiltonian. This choice is dependent on the given problem, for example, eliminate the terms periodic in an angle θ. This dictates the image of the generator $\mathcal{L}_0(W_n)$ under the Lie operator $\mathcal{L}_0 \equiv (\,\cdot\,; \mathcal{H}_0)$, where $\mathcal{H}_0$ is the unperturbed Hamiltonian. We next face the inversion problem: finding the pre-image W_n. This could be quite difficult if we are not clever. As we shall see, the only effective solution to this problem is to change the variables so that the Lie operator becomes proportional to a simple partial derivative. Once we have a strategy for choosing this image term and finding the appropriate generator, a general method may proceed to calculate the next order provisional element, and the new Hamiltonian and generator at the current order n. What we wish to leave in the final Hamiltonian is a member of the kernel of the Lie operator in the case of a normalization (such as the Delaunay normalization); this need not be the case in a general simplification (such as the elimination of the parallax).

PMAO controls perturbation series, including the Lie simplification, by *lazy series*. Given the perturbation parameter, say δ, one gives instructions what the lowest order is and how to calculate the next order, perhaps basing it on another lazy series. A particular order is calculated only if required, either by direct request or indirectly, because some other lazy series needs this calculation. For Lie simplifications, the interrelationships are built-in, one needs only to specify the initial Hamiltonian, the separator which identifies terms to be kept in the new Hamiltonian and the terms in the image of the Lie operator that will be eliminated, and the integrator, which finds the generator from its image that the separator produced. The separator and integrator are functions. Thus, for the problem described here, the initial Hamiltonian to the Delaunay normalization is essentially the final Hamiltonian of the parallax elimination, which is passed as the value of a variable; therefore, any order requested of the Delaunay-normalized Hamiltonian results in the computation of the parallax elimination to the appropriate order without the user being aware of it.

In the sections that follow, I describe the two stages of computation of the main problem in satellite theory, the parallax elimination and the Delaunay normalization, in the context of the discussion above. The next section deals with the elimination of the parallax, a relatively simple task, and Sections 5–7 deal with the Delaunay normalization, the latter sections being discussions of algebraic simplifications needed to make the calculations proceed expeditiously.

4. Elimination of the Parallax

The Hamiltonian for the main problem in satellite theory is

$$\mathcal{H} \;=\; \frac{1}{2}\left(R^2 + \frac{\Theta^2}{r^2}\right) - \frac{\mu}{r}\left[1 - \frac{J_2}{2}\left(\frac{\alpha}{r}\right)^2 (3s^2 \sin^2\theta - 1)\right]. \tag{6}$$

where r, θ, R, Θ is the phase space in Whittaker (polar) variables, α is the radius of the earth, μ is the Keplerian constant, and $s = \sin I$, the sine of the inclination. The goal of the elimination of the parallax is to find a near-identity transformation which does not have explicit θ, a technique developed by Deprit (1981).

Jumping a little ahead, we shall anticipate what the best algebraic hierarchy for the final Hamiltonian of the elimination of the parallax, consider the variables

$$C = e\cos g, \quad S = e\sin g, \tag{7}$$

and the semi-latus rectum of the ellipse, $p = \Theta^2/\mu$. We consider a polynomial algebra in these variables (allowing negative powers of p) over coefficients in polynomials of s. On the other side, we make this algebra the coefficient domain of a Fourier algebra in the satellite's polar angle θ, which, in turn, is a coefficient of the polynomials in the conjugate momentum Θ, then each term of the perturbation Hamiltonian is in the algebraic hierarchy

$$\mathcal{A}_P = \frac{1}{r^2}P[\Theta, \alpha, J_2] \triangleright F[\theta] \triangleright P\left[S, C, p, \frac{1}{p}\right] \triangleright P[s^2] \triangleright \mathbf{Q}. \tag{8}$$

The computation of the provisional Hamiltonian requires Poisson brackets which involve the derivatives

$$\begin{aligned}
\frac{\partial S}{\partial r} &= -\frac{P}{r}\sin\theta, & \frac{\partial C}{\partial r} &= -\frac{P}{r}\cos\theta,\\
\frac{\partial S}{\partial R} &= -\frac{rP}{\Theta}\cos\theta, & \frac{\partial C}{\partial R} &= -\frac{rP}{\Theta}\sin\theta,\\
\frac{\partial S}{\partial \Theta} &= \frac{1}{\Theta}[S + (1+P)\sin\theta], & \frac{\partial C}{\partial \Theta} &= \frac{1}{\Theta}[C + (1+P)\cos\theta]\\
\frac{\partial s^2}{\partial \Theta} &= \frac{2}{\Theta}(1-s^2), & \frac{\partial p}{\partial \Theta} &= \frac{2p}{\Theta},
\end{aligned}$$

with $P \equiv p/r = 1 + C\cos\theta + S\sin\theta$. All other derivatives (except $\sin\theta$ and $\cos\theta$, of course) are zero. The actions of the Lie operator on the variables is of special interest. It is possible to show (Deprit, 1981) that the Lie operator acting on these variables are all zero

$$\mathcal{L}_0(p) = \mathcal{L}_0(S) = \mathcal{L}_0(C) = 0 \tag{9}$$

except

$$\mathcal{L}_0(\theta) = \frac{\Theta}{r^2}. \tag{10}$$

The quantities $\mathcal{L}_0(s^2)$ and $\mathcal{L}_0(\Theta)$ are also zero. We have thus solved the inversion problem for the elimination of the parallax, by reducing the Lie operator to a multiple of a simple partial derivative. In the algebra $\mathcal{A}_P$, the Lie operator is

$$\mathcal{L}_0 = \frac{\Theta}{r^2}\frac{\partial}{\partial\theta}, \tag{11}$$

so its inverse is

$$\mathcal{L}_0^{-1} = \frac{r^2}{\Theta}\int \mathrm{d}\theta. \tag{12}$$

Using $p/r = 1 + e\cos f$, where f is the true anomaly $f = \theta - g$, we find that $1/r$ can be expressed in $\mathcal{A}_P$,

$$\frac{1}{r} = \frac{1}{p}(1 + C\cos\theta + S\sin\theta). \tag{13}$$

With $R = (\Theta/p)e\sin f$, the original Hamiltonian (6) is

$$\begin{aligned}\mathcal{H} \; = \; & -\frac{1}{2}\eta^2\frac{1}{p^2}\Theta^2 + \\ & + \delta\frac{\Theta^2}{r^2}\frac{\alpha^2}{p^2}J_2\left[\frac{3}{4}s^2 - \frac{1}{2} + C\left(\frac{3}{8}s^2 - \frac{1}{2}\right)\cos\theta + S\left(\frac{9}{8}s^2 - \frac{1}{2}\right)\sin\theta - \right.\\ & \left. - \frac{3}{4}s^2\cos 2\theta - \frac{3}{8}s^2 C\cos 3\theta - \frac{3}{8}s^2 S\sin 3\theta\right]\end{aligned} \tag{14}$$

expressing the Hamiltonian in $\mathcal{A}_P$.

The strategy of the elimination of the parallax is to remove the θ-dependence. This is not a complete short-period elimination like the Delaunay normalization, because we leave behind the factor of $1/r^2$. We merely remove *explicit* θ dependencies of (14). The coefficient algebra of $F[\theta]$, $P[S, C, p, 1/p]$, is in the kernel of $\mathcal{L}_0$, as shown above. We call this the ‘background algebra’. We view $1/r$ as a Fourier series in θ with the coefficients in the background algebra. Consider terms of the type F/r^2, for $F \in \mathcal{A}_P$; call it $\mathcal{B} \subset \mathcal{A}_P$. The elimination of parallax gives an element of $\mathcal{C} = 1/r^2 \ker \mathcal{L}_0$, $\mathcal{C} \subset \mathcal{B}$. Mathematically, the elimination of the parallax consists in reducing any element of $\mathcal{B}$ to an element of $\mathcal{C}$.

The Hamiltonian and the generator after the elimination of the parallax are shown in the appendix. The generator may be used to calculate the transformation (and its inverse) of any quantity. In particular, propagation under the Hamiltonian is computed by transforming the desired coordinates with the generator, propagating under the normalized Hamiltonian, then transforming back to the original coordinates with the inverse generator.

5. Delaunay Normalization

The goal of the Delaunay normalization is, from the Hamiltonian $\mathcal{H}_P$ produced by the parallax simplification, normalize such that the last of the short-period factors, $1/r^2$, is removed, that is, to put the final perturbation Hamiltonian in the algebra

$$\mathcal{A}_D = P\left[\left(\frac{\alpha}{p}\right)^2 J_2\right] \triangleright F[g] \triangleright P\,[e, \eta, \beta] \triangleright P[s^2] \triangleright \mathbf{Q}. \tag{15}$$

The initial Hamiltonian perturbation, after the minor conversion from the elimination of the parallax to substitute $\mu = \Theta^2/p$, $R = (\Theta/p)e\sin f$, $C = e\cos g$, $S = e\sin g$, is in the algebra

$$\frac{\Theta^2}{r^2} \triangleright \mathcal{A}_D, \tag{16}$$

i.e., Θ^2/r^2 times elements of $\mathcal{A}_D$. This initial Hamiltonian is presented in the appendix to sixth order. The generator will be in the algebra

$$\Theta \triangleright P\left[\left(\frac{\alpha}{p}\right)^2 J_2\right] \triangleright P[\phi] \triangleright F[f, g] \triangleright P\,[e, \eta, \beta] \triangleright P[s^2] \triangleright \mathbf{Q}, \tag{17}$$

with $\phi = f - l$ the equation of the center.

The zero order (unperturbed) Hamiltonian is nL where $n = \mu^2/L^3$ is the mean motion. Therefore, the Lie operator for the Delaunay normalization is $\mathcal{L}_0 = n\partial/\partial\ell$. Except for actual computations of the generator, we avoid introducing n into any of the Poisson brackets.

We first classify all terms that we expect to encounter in the Delaunay normalization, and then show how we intend to treat them. The method presented here was inspired by a treatment of the satellite theory for an earlier incarnation of PMAO by Shannon Coffey and Bruce Miller.

5.1. CLASSIFICATION OF TERMS

From the given Hamiltonian, we must anticipate what sorts of terms will come in the provisional Hamiltonian in the process of doing the normalization. We concern ourselves with short-period variables, that is, those that are dependent on the mean anomaly ℓ. In the Delaunay variables, these are the true anomaly f, the radial distance r, and the equation of center ϕ. The form of the provisional Hamiltonian is

$$\sum_{m=0,2}\sum_{k\geqslant 0}\frac{\phi^k}{r^m}\sum_{j\geqslant 0}(C_{jkm}\cos jf + S_{jkm}\sin jf), \tag{18}$$

where C_{jkm}, S_{jkm} are in the kernel of $\mathcal{L}_0$, that is, they are not dependent on ℓ. The justification of this form is given presently. We shall consider the different types of terms, distinguished by how they are integrated. The essential features for integration are whether each of the coefficient of f, the exponent of ϕ, and the negative exponent of r, are nonzero or zero. A nonzero coefficient or exponent is indicated by the superscript $+$. Thus, for example, $C_{j^+k^+0}$ represents coefficients of terms that have a dependence on ϕ and a cosine of f and are independent of r. The symbol* X in place of C or S will be user to mean either sine or cosine, e.g.

* When $j = 0$, there is no f dependence and thus no sine terms, so C_{0km} is synonymous with X_{0km}.

X_{j+k+0} means C_{j+k+0} and S_{j+k+0} . Sometimes these symbols S_{jkm}, C_{jkm}, X_{jkm} will be used to represent the whole term instead of just the coefficient, when there is no danger of confusion.

5.2. DERIVATIVES

We choose our method of doing Poisson brackets such that the explicit exponent of r is preserved, that is, if A and B are independent of r, then $(r^m A; r^n B) = r^{m+n} C$, where C is independent of r. We compute our generator in such a way that it will always be independent of explicit r terms. The exponent of r may be preserved by careful choice of the form used to express the derivatives of the quantities used, p, r, ϕ, f, e, η, β, s, with respect to the Delaunay variables (ℓ, g, h, L, G, H) in the Poisson bracket.

$$
\begin{aligned}
\frac{\partial r}{\partial \ell} &= \frac{erP}{\eta^3}\sin f, \qquad \frac{\partial r}{\partial L} = \frac{r\eta}{Ge}\left(2 - \frac{P}{e}\cos f\right), \\
\frac{\partial r}{\partial g} &= 0, \qquad \frac{\partial r}{\partial G} = \frac{rP}{Ge}\cos f, \\
\frac{\partial \phi}{\partial \ell} &= \frac{P^2}{\eta^3} - 1, \qquad \frac{\partial \phi}{\partial L} = \eta \sin f\left(\frac{2 + e\cos f}{Ge}\right), \\
\frac{\partial \phi}{\partial g} &= 0, \qquad \frac{\partial \phi}{\partial G} = -\sin f\left(\frac{2 + e\cos f}{Ge}\right), \\
\frac{\partial p}{\partial G} &= \frac{2p}{G}, \\
\frac{\partial f}{\partial \ell} &= \frac{P^2}{\eta^3}, \qquad \frac{\partial f}{\partial L} = \eta \sin f\left(\frac{2 + e\cos f}{Ge}\right), \\
\frac{\partial f}{\partial g} &= 0, \qquad \frac{\partial f}{\partial G} = -\sin f\left(\frac{2 + e\cos f}{Ge}\right), \\
\frac{\partial e}{\partial L} &= \frac{\eta^3}{Ge}, \qquad \frac{\partial e}{\partial G} = -\frac{\eta^2}{Ge}, \\
\frac{\partial \eta}{\partial L} &= -\frac{\eta^2}{G}, \qquad \frac{\partial \eta}{\partial G} = \frac{\eta}{G}, \\
\frac{\partial \beta}{\partial L} &= \frac{\beta^2\eta^2}{G}, \qquad \frac{\partial \beta}{\partial G} = \frac{\beta^2\eta}{G}, \\
\frac{\partial s}{\partial G} &= \frac{1/s - s}{G}.
\end{aligned}
$$

Using

$$
P \equiv \frac{p}{r} = (1 + e\cos f), \tag{19}
$$

we have insured that r derivatives have an explicit factor of r on the right-hand side, and no other derivatives do; thus the exponent of r is maintained in the Poisson bracket.

Noting the method of the computation of the provisional element we see that by our choice of expressing the derivative, each term in each element $\tilde{\mathcal{H}}_{n,p+1}$ is either independent of r or proportional to $1/r^2$. We know each generator W_k is independent of r, each original Hamiltonian $\mathcal{H}_{n0}$ (from the parallax computation, $\mathcal{H}_P$) is proportional to $1/r^2$ for $n \geqslant 1$ and independent of r for $\mathcal{H}_{00}$, and each final Hamiltonian $\mathcal{H}_{0n}$ should be independent of r. Because of the selection of the method for computing the Poisson bracket, the end result is that terms in the provisional Hamiltonian are independent of r or proportional to $1/r^2$; thus we have only terms of the type X_{jk0} and X_{jk2}.

Knowing that the initial Hamiltonian from the parallax elimination has been put in the form (16), that all generators at previous orders are in the algebra (17), the new Hamiltonian at previous orders is in the algebra $\mathcal{A}_D$, and that the Poisson bracket preserves the power of r, we have the r-dependence in the form (18) for the short-period dependence of terms in the provisional Hamiltonian.

5.3. C_{000} AND C_{002}

We proceed now with the treatment of specific terms. We start with C_{000} and C_{002} because it involves images we will need for other terms as well. The constant term C_{000} is clearly in the kernel. C_{002} is split between kernel and image. With the Lie operator $\mathcal{L}_0 = n\partial/\partial\ell$, the equation of the center $\phi = f - \ell$, and $\partial f/\partial\ell = G/(nr^2)$ and the product rule for derivatives,

$$\mathcal{L}_0(\phi) = \frac{G}{r^2} - n. \tag{20}$$

If the generator is proportional to ϕ, terms of the type C_{000} and C_{002} will be in the image in the proportion $-n/G$. Thus, to treat the terms $C_{002}/r^2 + C_{000}$, use the generator $C_{002}\phi/G$. Under the Lie operator,

$$\mathcal{L}_0\left(\frac{1}{G}C_{002}\phi\right) = \frac{1}{r^2}C_{002} - \frac{n}{G}C_{002}. \tag{21}$$

Thus, the constant term is modified to become

$$C'_{000} = C_{000} - \frac{n}{G}C_{002} \tag{22}$$

which is in the kernel.

5.4. ϕ-DEPENDENT TERMS

The most difficult terms are those that are dependent on ϕ. In the algorithm, these are computed first, as we shall see. They are eliminated recursively: the term that generates a particular exponent of ϕ also generates lesser exponents of ϕ which, in turn, may be eliminated by the same method.

The ϕ-dependence of (18) may be understood as follows. Because the image term to be eliminated at first order $\mathcal{H}_{10} - \mathcal{H}_{01}$ has a factor $G/r^2 - n$, the generator at that order, W_1, will have a factor of ϕ from (20). At successive orders $k > 1$, the presence of W_m for $m < k$ in Poisson brackets leads to terms in the image of the form

$$\left(\frac{G}{r^2} - n\right)\phi^{k-1}\cos\left(jf + i\frac{\pi}{2}\right) \tag{23}$$

which will necessitate the appearance of a ϕ^k term in the generator W_k; applying the Lie operator to such a term, we see the desired image plus another term:

$$\begin{aligned}\mathcal{L}_0\left[\phi^k\cos\left(jf + i\frac{\pi}{2}\right)\right] \;=\; & k\left(\frac{G}{r^2} - n\right)\phi^{k-1}\cos\left(jf + i\frac{\pi}{2}\right) - \\ & - j\phi^k\frac{G}{r^2}\sin\left(jf + i\frac{\pi}{2}\right). \end{aligned} \tag{24}$$

As we can see from this Lie derivative, f-dependent terms are quite a bit more complicated than the f-independent terms. Consider a generator made out of sums of the previous generator:

$$W_{jk} = -\frac{1}{G}\sum_{i=0}^{k}\frac{k!}{(k-i)!}j^{-(i+1)}\phi^{k-i}\cos\left[jf + (i+i')\frac{\pi}{2}\right], \tag{25}$$

where i' is an arbitrary integer. Using the rule (24), the image is

$$\begin{aligned}\mathcal{L}_0(W_{jk}) \;=\; & -\frac{1}{G}\sum_{i=0}^{k}\frac{k!}{(k-i)!}j^{-(i+1)}\times \\ & \times\left\{(k-i)\left(\frac{G}{r^2} - n\right)\phi^{k-i-1}\cos\left[jf + (i+i')\frac{\pi}{2}\right] - \right. \\ & \left. - j\frac{G}{r^2}\phi^{k-i}\sin\left[jf + (i+i')\frac{\pi}{2}\right]\right\} - \\ =\; & -\frac{1}{G}\left\{\sum_{i=0}^{k}\frac{k!}{(k-i)!}j^{-(i+1)}(k-i)\left(\frac{G}{r^2} - n\right)\times\right. \\ & \times\phi^{k-i-1}\cos\left[jf + (i+i')\frac{\pi}{2}\right] - \end{aligned}$$

$$
\left. - \sum_{i=-1}^{k-1} \frac{k!}{(k-i-1)!} j^{-(i+1)} \frac{G}{r^2} \phi^{k-i-1} \cos\left[jf + (i+i')\frac{\pi}{2} \right] \right\}
$$

$$
= \underbrace{\frac{\phi^k}{r^2} \sin\left(jf + i'\frac{\pi}{2} \right)}_{\text{desired image term}} +
$$

$$
\underbrace{+\frac{n}{G} \sum_{i=1}^{k} \frac{k!}{(k-i)!} j^{-i} \phi^{k-i} \sin\left[jf + (i+i')\frac{\pi}{2} \right]}_{\text{entourage}}. \tag{26}
$$

The first term is interpreted as $(\phi^k/r^2)\sin jf$ or $(\phi^k/r^2)\cos jf$ depending on whether $i' = 0$ or 1, and is the term of type X_{j+k2} we aim to eliminate. The other terms, of type X_{j+q0} for $q < k$, are what we may call the 'entourage' – they come along with the desired term in the image. The algorithm for treating ϕ^k terms is recursive (Figure 1), starting with the highest exponent k of ϕ; call these terms P. We know of only one way that such a term can be in the image of the Lie operator, and that is with the generator (25) whose image is (26). These terms are accumulated into the generator we are calculating. Before doing this however, we add the entourage terms to the Hamiltonian as they appear in (26), call them E_i, and subtract them in canonical form, terms of the type X_{pq0} for $p \leqslant j-2$ and C_{0k0}, C_{1k0}, S_{1k0} and S_{2k0} (Section 7) call them E_c, thus effectively adding zero: $P = P + E_i - E_c$. The image under (26) is $P + E_i$, so we have replaced the term P in the provisional Hamiltonian with $-E_c$. These terms are X_{pq2} for $q < k$, and the

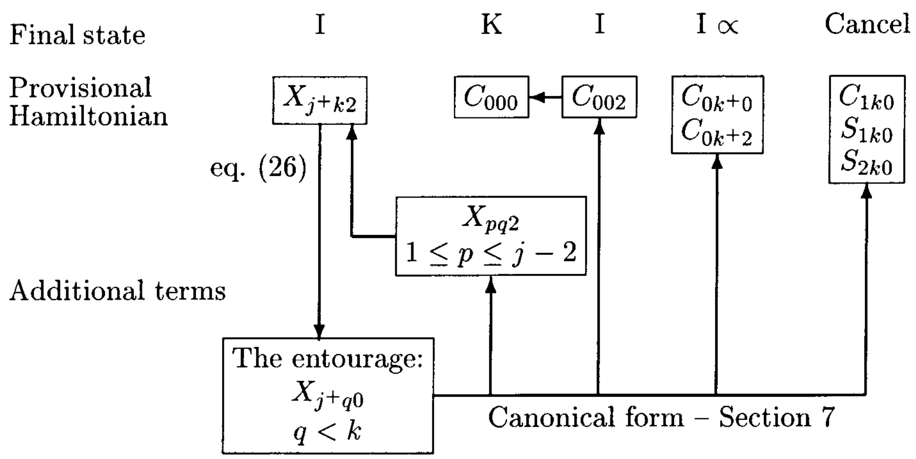

Figure 1. The handling of terms in the provisional Hamiltonian. For final state, 'I' represents an image term, 'I ∝' is an image term if the coefficients have the proper proportionality, 'K' represents a kernel term, 'cancel' means that the terms must cancel out and not be in the final image. Boxes below the top line represent terms converted to other forms.

two terms of the type C_{0k0} and C_{1k0} if X is cosine or S_{1k0} and S_{2k0} if it is sine, and are added to the remaining terms (if any) already present in the Hamiltonian. The X_{pq2}, is the 'trigger' for the recursion; it is treated as the original X_{j+k2} but now the new highest exponent of ϕ is lower than before. At the end of the recursion, there will be no terms with ϕ. The remaining terms are handled as described above and below, once all X_{j+k2} terms are gone.

5.5. OTHER TERMS

After the X_{j+k2} has been iteratively reduced, we are left with a linear combination of terms of the type C_{0k0}, C_{1k0}, S_{1k0}, and S_{2k0}.

The terms C_{000} and C_{002} have been treated in Section 5.3. In contrast, the analogous terms dependent on ϕ, C_{0k+0} and C_{0k+2} *must* be proportional in the ratio $-n/G$ in order to integrate, because C_{0k+0} is not in the kernel as C_{000} is. From (20) we propose a generator for these two terms,

$$\mathcal{L}_0\left(\frac{\phi^{k+1}}{k+1}\right) = \phi^k\left(\frac{G}{r^2} - n\right). \tag{27}$$

As is easy to see, if the two terms have this proportionality, the generator will be

$$W = -\frac{C_{0k0}}{n} = \frac{C_{0k2}}{G}. \tag{28}$$

This proportionality indeed prevails after the iteration, in computations through order six.

Finally, the integrals of C_{1k0}, S_{1k0} and S_{2k0} may be expressed in terms of the eccentric anomaly (Kelly, 1989; Tables I and II for $k = 0$). As it happens, however, terms of these types present in the provisional Hamiltonian, together with those produced by the iteration, cancel off entirely through sixth order, making integration unnecessary.

6. Canonical Form for e, η, β, and the Mean Motion

A frequent problem in algebraic manipulation is the standard representation of terms that have some algebraic relation between them. Here, we find that in the computation of the provisional Hamiltonian of the Delaunay normalization we obtain an abundance of terms involving e and $\eta = \sqrt{1-e^2}$. The problem of reduction to a standard, or Gröbner, basis for polynomials is well-studied in the field of computer algebra, having been initiated by Buchberger (Davenport et al., 1988). Here, however, our expressions are not confined to polynomials; negative powers of e and η are possible.

In order to simplify some of the computations, we introduce the variable $\beta = 1/(1+\eta)$, we can write any term $e^h\eta^i\beta^j$ for h, i, and j integers in an algebraic canonical form. With numbers K and M whose choice will be explained later, this monomial is a linear combination of $e^K \eta^M$ times nonnegative powers of η and β times e with exponent no higher than 1.*

Using the definitions $e^2 + \eta^2 = 1$ and $\beta = 1/(1+\eta)$, we have the five relations

$$\eta\beta = 1 - \beta, \tag{29a}$$

$$\frac{\beta}{\eta} = \frac{1}{\eta} - \beta, \tag{29b}$$

$$\eta = 1 - e^2\beta, \tag{29c}$$

$$\frac{1}{\eta} = \frac{e^2}{\eta} + \eta, \tag{29d}$$

$$\beta = \frac{1}{2}(1 + e^2\beta^2). \tag{29e}$$

We first use substitution $e^2 \to 1 - \eta^2$ to remove all terms whose e exponent is greater than $K + 1$. Then, the rules (29a) and (29b) may be applied repetitively to all terms to insure that η and β never occur together in a product or quotient (a negative power of β should never occur but is easily reduced using $1/\beta = 1 + \eta$ anyway). Finally, we eliminate e^k for $k < K$; to eliminate $e^k\eta^m$ for $m > M$, use (29c); to eliminate it for $m < M$ use (29d), and to eliminate $e^k\beta^n$ use (29e). Note that the actual individual substitutions must be applied repetitively; not only individual rules must be reapplied, but the application of later rules will sometimes produce terms that necessitate the reapplication of earlier rules.

While heuristic and recursive (or iterative) simplifiers are easy to implement in PMAO, as opposed to its predecessor Poisson series processors (Deprit, 1982), it is wise to forgo the compact elegance of recursion in favor of the alacrity of explicit forms of monomials of e, η, and β. These forms are

$$e^{-2\kappa}\eta^{2\mu} = \sum_{i=0}^{\kappa-1}(-1)^i \binom{\mu}{i} e^{2(i-\kappa)} + (-1)^\kappa \sum_{i=0}^{\mu-\kappa} \binom{\mu - i - 1}{\kappa - 1} \eta^{2i}, \tag{30a}$$

$$e^{-2\kappa}\eta^{-2\mu} = \sum_{i=1}^{\kappa} \binom{\mu + \kappa - i - 1}{\kappa - i} e^{-2i} + \sum_{i=1}^{\mu} \binom{\mu + \kappa - i - 1}{\kappa - 1} \eta^{-2i}, \tag{30b}$$

* Without the introduction of β, it is possible to have another canonical form which is a linear combination of terms η^m, $e\eta^m$, e^k, $e^k\eta$ each multiplied by $e^K\eta^M$ for $k < 0$ and $m \geqslant 0$. If we have a term $e^k\eta^m$ with $k > K + 1$, we make the substitution $e^2 \to 1 - \eta^2$ enough times to reduce it. If, on the other hand, $k < K$ and $m > M + 1$, we use the substitution $\eta^2 \to 1 - e^2$. If $k < K$ and $m < M$, use (29d). The formulæ (30) apply, except that (30c) and (30d) have $e^{2(i-\kappa)}\eta$ replacing $e^{2(i-\kappa)}(1 - e^2\beta)$, and $e^{-2i}\eta$ replacing $e^{-2i}(1 - e^2\beta)$, respectively.

$$\begin{aligned}
e^{-2\kappa}\eta^{2\mu+1} &= \sum_{i=0}^{\kappa-1}(-1)^i\binom{\mu}{i}\left(e^{2(i-\kappa)} - e^{2(i-\kappa+1)}\beta\right) + \\
&\quad + (-1)^\kappa \sum_{i=0}^{\mu-\kappa}\binom{\mu-i-1}{\kappa-1}\eta^{2i+1}, \qquad \text{(30c)}\\
e^{-2\kappa}\eta^{-2\mu+1} &= \sum_{i=1}^{\kappa}\binom{\mu+\kappa-i-1}{\kappa-i}\left(e^{-2i} - e^{-2i+2}\beta\right) + \\
&\quad + \sum_{i=1}^{\mu}\binom{\mu+\kappa-i-1}{\kappa-1}\eta^{-2i+1}, \qquad \text{(30d)}\\
\frac{\beta^n}{\eta^m} &= \sum_{i=1}^{m}(-1)^{m-i}\binom{n+m-i-1}{m-i}\frac{1}{\eta^i} + \\
&\quad + (-1)^m \sum_{i=1}^{n}\binom{n+m-i-1}{m-1}\beta^i, \qquad \text{(30e)}\\
\eta^m\beta^n &= (-1)^{m+n}\binom{m-1}{n-1} + \sum_{i=1}^{n}(-1)^{m+n-i}\binom{m}{n-i}\beta^i + \\
&\quad + \sum_{i=1}^{m-n}(-1)^{m+n-i}\binom{m-i-1}{n-1}\eta^i, \qquad \text{(30f)}\\
e^{-2\kappa}\beta^n &= \sum_{i=1}^{\kappa}\frac{1}{2^{n+2\kappa-2i}}\left[\binom{n+2\kappa-2i-1}{\kappa-i} - \binom{n+2\kappa-2i-1}{\kappa-i-1}\right]e^{-2i} + \\
&\quad + \sum_{i=1}^{n+\kappa-1}\frac{1}{2^{n+2\kappa-i-1}}\left[\binom{n+2\kappa-i-2}{\kappa-1} - \right.\\
&\quad \left. - \binom{n+2\kappa-i-2}{\kappa-i-1}\right]\beta^{i+1}, \qquad \text{(30g)}
\end{aligned}$$

where κ, m, μ and n are all positive, a binomial coefficient $\binom{a}{b}$ is zero if $b \leqslant 0$ and a sum $\sum_a^b$ is zero if $b < a$. The factor $e^K\eta^M$ on both sides of the equation has been removed for clarity. Such formulæ are very difficult to discover, but straightforward if tedious to prove by induction once known. These proofs are left to the reader as an exercise.

Generally, the K specified will be even and is governed by a generalization of the D'Alembert characteristic: each term of $\sin(kf+qg)$ or $\cos(kf+qg)$ should have a factor of $e^{|k-q|}$ multiplying it. This is based on the observation that the Lie algebra is really built on the variables $e\cos g$, $e\sin g$, $\cos(f+g)$, and $\sin(f+g)$, so that any difference from equal coefficients should be accompanied by a factor of

e. Of course, in the final Hamiltonian after the Delaunay normalization, there will be no dependence on f, so we merely use the absolute value of the coefficient of g. We should expect that all exponents of e that are not $|k-q|$ will cancel off.

We wish to express the final Hamiltonian in terms of the mean motion n, where

$$n = \frac{\mu^2}{L^3} = \frac{\Theta\eta^3}{p^2}. \tag{31}$$

When we present the final Delaunay Hamiltonian, we wish to identify all factors of the mean motion. Other times, as when PMAO computes the provisional element, it is preferable to keep the mean motion out of the expressions in order to keep computations simple. In the former case, we need to identify factors of η^3 for substitution of n, and thus choose $M = 3$ in the canonical form of e, η and β. In the latter, we substitute for any n using (31).

7. Canonical Form of $r^0\cos(jf)$ and $r^0\sin(jf)$

In this section, we discuss how to express X_{jk0} terms in a canonical form. This canonical form consists of terms of the type X_{ik2} for $i \leqslant j-2$ and terms of the type C_{0k0} and C_{1k0} for C_{jk0} or S_{1k0} and S_{2k0} for S_{jk0}, and holds for $e \neq 0$. The conversion to canonical form uses p/r expressed in terms of the true anomaly (19),

$$\frac{p^2}{r^2} = 1 + \frac{e^2}{2} + 2e\cos f + \frac{e^2}{2}\cos 2f, \tag{32}$$

and the trigonometric product relations

$$\cos nf\,\cos mf = \tfrac{1}{2}\cos(n+m)f + \tfrac{1}{2}\cos(n-m)f \tag{33a}$$

and

$$\sin nf\,\cos mf = \tfrac{1}{2}\sin(n+m)f + \tfrac{1}{2}\sin(n-m)f. \tag{33b}$$

It is possible to write a canonical form for $\cos jf$ and $\sin jf$ which has only $1/r^2$ terms and constants, $\cos f$, $\sin f$, and $\sin 2f$ terms:

$$\cos jf = C_{j0} + C_{j1}\cos f + \frac{p^2}{r^2}\sum_{\Delta=2}^{j}\left(1 - \frac{1}{2}\delta_{j-\Delta}\right) X_\Delta \cos(j-\Delta)f, \tag{34a}$$

$$\sin jf = S_{j1}\sin f + S_{j2}\sin 2f + \frac{p^2}{r^2}\sum_{\Delta=2}^{j} X_\Delta \sin(j-\Delta)f, \tag{34b}$$

where δ_i is 1, if $i = 0$, and 0 otherwise, and the power series in e are

$$\begin{aligned} C_{j0} \;=\; & (-1)^{j/2}(1 - j \bmod 2) + \\ & + \sum_{m=1-j\bmod 2}^{j-1}{}_{2}\; (-1)^{\frac{j+m-1}{2}} 2^m j \binom{\frac{j+m-1}{2}}{m} \frac{n}{n+1} e^{-m-1}, \end{aligned} \tag{35a}$$

$$C_{j1} = \mathop{{\sum}_2}\limits_{m=1-j \bmod 2}^{j-1} (-1)^{\frac{j+m-1}{2}} 2^m j \binom{\frac{j+m-1}{2}}{m} e^{-m}, \tag{35b}$$

$$S_{j1} = \mathop{{\sum}_2}\limits_{m=3-j \bmod 2}^{j-1} (-1)^{\frac{j+m+1}{2}} 2^m (m-1) \binom{\frac{j+m-1}{2}}{m} e^{-m}, \tag{35c}$$

$$S_{j2} = \mathop{{\sum}_2}\limits_{m=j \bmod 2}^{j-2} (-1)^{\frac{j+m}{2}-1} 2^m (m+1) \binom{\frac{j+m}{2}}{m+1} e^{-m}, \tag{35d}$$

$$X_{\Delta} = \mathop{{\sum}_2}\limits_{m=2+\Delta \bmod 2}^{\Delta} (-1)^{\frac{\Delta+m}{2}} 2^m (m-1) \binom{\frac{\Delta+m}{2}-1}{m-1} e^{-m}. \tag{35e}$$

The summation ${\sum}_2{}_{m=a}^{b}$ means sum over every *other* term, i.e., $m = a, a+2, a+4, \ldots, b$.

By substituting (32) into these formulæ and applying (33), one can check they indeed reduce to the left-hand sides. This is a time-consuming activity and not recommended for those in a hurry. With a symbolic manipulation code, it is easy to try specific values of j and then substitute.

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Appendix A. Expressions

This appendix gives the normalized Hamiltonian after the elimination of the parallax and after the Delaunay transformation through sixth order. The corresponding generating functions, coordinate transformations and inverse coordinate transformations are given, but only to second order for space considerations. The full expressions are available electronically from the author; the full expressions at even fifth order makes this paper several thousand pages long. The expressions are given so that someone attempting to reproduce the results presented in this paper can check agreement as the calculation progresses from the contents of this paper alone.

In these expressions, the symbol δ is used to keep track of the order of the perturbation, but can otherwise be considered to be one. For expressions that are given as a table, the final expression is the sum of all the subexpressions given in the table. The subexpressions are numbered for convenience.

A.1. THE NEW HAMILTONIAN AFTER THE ELIMINATION OF THE PARALLAX

$$
\begin{aligned}
&-\frac{1}{2}\eta^2\frac{1}{p^2}\Theta^2+\delta\frac{\Theta^2}{r^2}\frac{\alpha^2}{p^2}J_2\left(\frac{3}{4}s^2-\frac{1}{2}\right)+\\
&+\delta^2\frac{\Theta^2}{r^2}\frac{\alpha^4}{p^4}J_2^2\left[S^2\left(\frac{105}{128}s^4-\frac{15}{32}s^2-\frac{3}{16}\right)+\right.\\
&\left.+C^2\left(-\frac{75}{128}s^4+\frac{27}{32}s^2-\frac{3}{16}\right)-\frac{21}{32}s^4+\frac{21}{16}s^2-\frac{5}{8}\right]+\\
&+\delta^3\frac{\Theta^2}{r^2}\frac{\alpha^6}{p^6}J_2^3\left[S^2\left(\frac{6285}{1024}s^6-\frac{2427}{256}s^4+\frac{273}{64}s^2-\frac{29}{32}\right)+\right.\\
&+C^2\left(\frac{4575}{1024}s^6-\frac{2115}{256}s^4+\frac{285}{64}s^2-\frac{29}{32}\right)+\\
&\left.+\frac{105}{32}s^6-\frac{987}{128}s^4+\frac{189}{32}s^2-\frac{13}{8}\right]+\\
&+\delta^4\frac{\Theta^2}{r^2}\frac{\alpha^8}{p^8}J_2^4\left[S^4\left(-\frac{156015}{131072}s^8+\frac{111267}{16384}s^6-\frac{68283}{8192}s^4+\right.\right.\\
&\left.+\frac{6147}{2048}s^2-\frac{261}{1024}\right)+\\
&+S^2C^2\left(-\frac{1362555}{65536}s^8+\frac{177795}{4096}s^6-\frac{108387}{4096}s^4+\frac{4635}{1024}s^2-\frac{261}{512}\right)+\\
&+S^2\left(\frac{2438835}{32768}s^8-\frac{1314221}{8192}s^6+\frac{207853}{2048}s^4-\frac{5541}{512}s^2-\frac{1211}{256}\right)+\\
&+C^4\left(-\frac{519255}{131072}s^8+\frac{144873}{16384}s^6-\frac{48195}{8192}s^4+\frac{3123}{2048}s^2-\frac{261}{1024}\right)+\\
&+C^2\left(-\frac{707985}{32768}s^8+\frac{656573}{8192}s^6-\frac{192847}{2048}s^4+\frac{21489}{512}s^2-\frac{1211}{256}\right)-\\
&\left.-\frac{4605}{512}s^8+\frac{16683}{512}s^6-\frac{43719}{1024}s^4+\frac{6303}{256}s^2-\frac{167}{32}\right]+\\
&+\delta^5\frac{\Theta^2}{r^2}\frac{\alpha^{10}}{p^{10}}J_2^5\left[S^4\left(\frac{29137563}{262144}s^{10}-\frac{7858677}{32768}s^8+\frac{6745003}{40960}s^6-\right.\right.\\
&\left.-\frac{1406151}{32768}s^4+\frac{42831}{4096}s^2-\frac{3429}{1024}\right)+\\
&+S^2C^2\left(\frac{13254489}{65536}s^{10}-\frac{19975239}{32768}s^8+\frac{10553465}{16384}s^6-\right.\\
&\left.-\frac{4521261}{16384}s^4+\frac{23175}{512}s^2-\frac{3429}{512}\right)+
\end{aligned}
$$

$$+S^2\left(\frac{48801939}{262144}s^{10}-\frac{81636667}{131072}s^8+\frac{124355759}{163840}s^6-\right.$$
$$\left.-\frac{3530111}{8192}s^4+\frac{274689}{2048}s^2-\frac{12859}{512}\right)+$$
$$+C^4\left(\frac{697113}{262144}s^{10}-\frac{758787}{16384}s^8+\frac{8777699}{81920}s^6-\right.$$
$$\left.-\frac{3165587}{32768}s^4+\frac{142569}{4096}s^2-\frac{3429}{1024}\right)+$$
$$+C^2\left(\frac{133993809}{262144}s^{10}-\frac{183200081}{131072}s^8+\frac{236938061}{163840}s^6-\right.$$
$$\left.-\frac{6038521}{8192}s^4+\frac{410139}{2048}s^2-\frac{12859}{512}\right)+$$
$$+\frac{404259}{8192}s^{10}-\frac{411735}{2048}s^8+\frac{332763}{1024}s^6-\frac{542895}{2048}s^4+$$
$$\left.+\frac{56391}{512}s^2-\frac{599}{32}+\right]$$
$$+\delta^6\frac{\Theta^2}{r^2}\frac{\alpha^{12}}{p^{12}}J_2^6\left[S^6\left(-\frac{7236966195}{67108864}s^{12}+\frac{96799787355}{234881024}s^{10}-\right.\right.$$
$$-\frac{8239746849}{14680064}s^8+\frac{9636567}{28672}s^6-\frac{28720581}{327680}s^4+$$
$$\left.+\frac{398949}{40960}s^2-\frac{11343}{16384}\right)+$$
$$+S^4C^2\left(-\frac{48059257095}{67108864}s^{12}+\frac{538583765931}{234881024}s^{10}-\right.$$
$$-\frac{195133337523}{73400320}s^8+\frac{12407852817}{9175040}s^6-\frac{38108511}{131072}s^4+$$
$$\left.+\frac{1993329}{81920}s^2-\frac{34029}{16384}\right)+$$
$$+S^4\left(\frac{55634100807}{29360128}s^{12}-\frac{214996266099}{36700160}s^{10}+\frac{241083403999}{36700160}s^8-\right.$$
$$\left.-\frac{346585066749}{114688000}s^6+\frac{243122949}{819200}s^4+\frac{62422131}{409600}s^2-\frac{32277}{1024}+\right)+$$
$$+S^2C^4\left(-\frac{10898388405}{67108864}s^{12}+\frac{153118795509}{234881024}s^{10}-\frac{69449015151}{73400320}s^8+\right.$$
$$\left.+\frac{2912367393}{4587520}s^6-\frac{31651641}{163840}s^4+\frac{398241}{20480}s^2-\frac{34029}{16384}\right)+$$
$$+S^2C^2\left(-\frac{58758813387}{14680064}s^{12}+\frac{920787392109}{73400320}s^{10}-\right.$$

$$
\begin{aligned}
&-\frac{1400921947657}{91750400}s^8+\frac{11934593383}{1310720}s^6-\frac{455013213}{163840}s^4+\\
&+\frac{7375341}{16384}s^2-\frac{32277}{512}\Bigg)+\\
&+S^2\left(\frac{767302869}{2097152}s^{12}-\frac{19058965801}{5242880}s^{10}+\frac{630082669}{81920}s^8-\right.\\
&\left.-\frac{25212883601}{4096000}s^6+\frac{647379881}{409600}s^4+\frac{2236305}{8192}s^2-\frac{34147}{256}\right)+\\
&+C^6\left(\frac{2018417535}{67108864}s^{12}-\frac{26491912827}{234881024}s^{10}+\frac{11455322847}{73400320}s^8-\right.\\
&\left.-\frac{796935831}{9175040}s^6+\frac{6494829}{655360}s^4+\frac{397533}{81920}s^2-\frac{11343}{16384}\right)+\\
&+C^4\left(-\frac{1619759139}{4194304}s^{12}+\frac{25957745127}{14680064}s^{10}-\frac{558916431277}{183500800}s^8+\right.\\
&\left.+\frac{604078775023}{229376000}s^6-\frac{251664041}{204800}s^4+\frac{60980697}{204800}s^2-\frac{32277}{1024}\right)+\\
&+C^2\left(\frac{3628920303}{1048576}s^{12}-\frac{40431331979}{5242880}s^{10}+\frac{4359896319}{1310720}s^8+\right.\\
&\left.+\frac{4137242069}{1024000}s^6-\frac{890157553}{204800}s^4+\frac{11314407}{8192}s^2-\frac{34147}{256}\right)-\\
&-\frac{25074573}{131072}s^{12}+\frac{32672199}{32768}s^{10}-\frac{174246807}{81920}s^8+\\
&+\frac{9816903}{4096}s^6-\frac{12390699}{8192}s^4+\frac{1042323}{2048}s^2-\frac{2293}{32}\Bigg]+\mathcal{O}(\delta^7).
\end{aligned}
$$

A.2. THE GENERATOR FOR THE ELIMINATION OF THE PARALLAX

Parallax generator at order 4

No.	Expression
1	$\Theta\frac{\alpha^2}{p^2}J_2C\left(\frac{3}{8}s^2-\frac{1}{2}\right)\sin\theta$
2	$-\frac{3}{8}\Theta\frac{\alpha^2}{p^2}J_2s^2\sin 2\theta$
3	$-\frac{1}{8}\Theta\frac{\alpha^2}{p^2}J_2s^2C\sin 3\theta$

Parallax generator at order 4 (*continued*)

No.	Expression
4	$\Theta\frac{\alpha^2}{p^2}J_2S\left(-\frac{9}{8}s^2+\frac{1}{2}\right)\cos\theta$
5	$\frac{1}{8}\Theta\frac{\alpha^2}{p^2}J_2s^2S\cos 3\theta$
6	$\delta\Theta\frac{\alpha^4}{p^4}J_2^2C\left(-\frac{195}{32}s^4+\frac{131}{16}s^2-2\right)\sin\theta$
7	$\delta\Theta\frac{\alpha^4}{p^4}J_2^2S^2\left(-\frac{39}{256}s^4+\frac{3}{16}\right)\sin 2\theta$
8	$\delta\Theta\frac{\alpha^4}{p^4}J_2^2C^2\left(-\frac{9}{256}s^4+\frac{3}{8}s^2-\frac{3}{16}\right)\sin 2\theta$
9	$\delta\Theta\frac{\alpha^4}{p^4}J_2^2\left(-\frac{21}{16}s^4+\frac{5}{4}s^2\right)\sin 2\theta$
10	$\delta\Theta\frac{\alpha^4}{p^4}J_2^2C\left(\frac{29}{64}s^4-\frac{5}{16}s^2\right)\sin 3\theta$
11	$\delta\Theta\frac{\alpha^4}{p^4}J_2^2S^2\left(-\frac{81}{256}s^4+\frac{15}{64}s^2\right)\sin 4\theta$
12	$\delta\Theta\frac{\alpha^4}{p^4}J_2^2C^2\left(\frac{75}{256}s^4-\frac{15}{64}s^2\right)\sin 4\theta$
13	$\frac{3}{64}\delta\Theta\frac{\alpha^4}{p^4}J_2^2s^4\sin 4\theta$
14	$\frac{3}{64}\delta\Theta\frac{\alpha^4}{p^4}J_2^2s^4C\sin 5\theta$
15	$-\frac{3}{256}\delta\Theta\frac{\alpha^4}{p^4}J_2^2s^4S^2\sin 6\theta$
16	$\frac{3}{256}\delta\Theta\frac{\alpha^4}{p^4}J_2^2s^4C^2\sin 6\theta$
17	$\delta\Theta\frac{\alpha^4}{p^4}J_2^2S\left(-\frac{141}{32}s^4+\frac{23}{16}s^2+2\right)\cos\theta$
18	$\delta\Theta\frac{\alpha^4}{p^4}J_2^2SC\left(-\frac{45}{128}s^4-\frac{3}{8}s^2+\frac{3}{8}\right)\cos 2\theta$
19	$\delta\Theta\frac{\alpha^4}{p^4}J_2^2S\left(-\frac{35}{64}s^4+\frac{5}{16}s^2\right)\cos 3\theta$

Parallax generator at order 4 (*continued*)

No.	Expression
20	$\delta\Theta\frac{\alpha^4}{p^4}J_2^2SC\left(-\frac{39}{64}s^4+\frac{15}{32}s^2\right)\cos 4\theta$
21	$-\frac{3}{64}\delta\Theta\frac{\alpha^4}{p^4}J_2^2s^4S\cos 5\theta$
22	$-\frac{3}{128}\delta\Theta\frac{\alpha^4}{p^4}J_2^2s^4SC\cos 6\theta$
23	$\delta^2\Theta\frac{\alpha^6}{p^6}J_2^3S^2C\left(\frac{6153}{512}s^6-\frac{58593}{2048}s^4+\frac{4275}{256}s^2-\frac{69}{64}\right)\sin\theta$
24	$\delta^2\Theta\frac{\alpha^6}{p^6}J_2^3C^3\left(\frac{18411}{2048}s^6-\frac{35565}{2048}s^4+\frac{2247}{256}s^2-\frac{69}{64}\right)\sin\theta$
25	$\delta^2\Theta\frac{\alpha^6}{p^6}J_2^3C\left(\frac{20067}{2048}s^6-\frac{5683}{256}s^4+\frac{1235}{64}s^2-\frac{69}{8}\right)\sin\theta$
26	$\delta^2\Theta\frac{\alpha^6}{p^6}J_2^3S^2\left(-\frac{56529}{4096}s^6+\frac{58495}{2048}s^4-\frac{3917}{256}s^2+\frac{75}{64}\right)\sin 2\theta$
27	$\delta^2\Theta\frac{\alpha^6}{p^6}J_2^3C^2\left(-\frac{31455}{4096}s^6+\frac{21329}{2048}s^4-\frac{469}{256}s^2-\frac{75}{64}\right)\sin 2\theta$
28	$\delta^2\Theta\frac{\alpha^6}{p^6}J_2^3\left(-\frac{3069}{256}s^6+\frac{311}{16}s^4-\frac{61}{8}s^2\right)\sin 2\theta$
29	$\delta^2\Theta\frac{\alpha^6}{p^6}J_2^3S^2C\left(-\frac{2853}{4096}s^6+\frac{3411}{2048}s^4-\frac{255}{256}s^2+\frac{15}{64}\right)\sin 3\theta$
30	$\delta^2\Theta\frac{\alpha^6}{p^6}J_2^3C^3\left(\frac{1185}{4096}s^6-\frac{661}{2048}s^4+\frac{49}{256}s^2-\frac{5}{64}\right)\sin 3\theta$
31	$\delta^2\Theta\frac{\alpha^6}{p^6}J_2^3C\left(-\frac{43657}{2048}s^6+\frac{257}{8}s^4-\frac{683}{64}s^2\right)\sin 3\theta$
32	$\delta^2\Theta\frac{\alpha^6}{p^6}J_2^3S^2\left(-\frac{2781}{2048}s^6-\frac{2411}{1024}s^4+\frac{401}{128}s^2\right)\sin 4\theta$
33	$\delta^2\Theta\frac{\alpha^6}{p^6}J_2^3C^2\left(-\frac{11121}{2048}s^6+\frac{9115}{1024}s^4-\frac{401}{128}s^2\right)\sin 4\theta$
34	$\delta^2\Theta\frac{\alpha^6}{p^6}J_2^3\left(\frac{69}{128}s^6-\frac{233}{512}s^4\right)\sin 4\theta$
35	$\delta^2\Theta\frac{\alpha^6}{p^6}J_2^3S^2C\left(-\frac{873}{4096}s^6-\frac{2097}{10240}s^4+\frac{189}{640}s^2\right)\sin 5\theta$

Parallax generator at order 4 (*continued*)

No.	Expression
36	$\delta^2\Theta\frac{\alpha^6}{p^6}J_2^3C^3\left(\frac{693}{4096}s^6+\frac{207}{10240}s^4-\frac{63}{640}s^2\right)\sin 5\theta$
37	$\delta^2\Theta\frac{\alpha^6}{p^6}J_2^3C\left(\frac{3699}{2048}s^6-\frac{1967}{1280}s^4\right)\sin 5\theta$
38	$\delta^2\Theta\frac{\alpha^6}{p^6}J_2^3S^2\left(-\frac{3609}{4096}s^6+\frac{1699}{2048}s^4\right)\sin 6\theta$
39	$\delta^2\Theta\frac{\alpha^6}{p^6}J_2^3C^2\left(\frac{4017}{4096}s^6-\frac{1699}{2048}s^4\right)\sin 6\theta$
40	$\frac{15}{256}\delta^2\Theta\frac{\alpha^6}{p^6}J_2^3s^6\sin 6\theta$
41	$\delta^2\Theta\frac{\alpha^6}{p^6}J_2^3S^2C\left(-\frac{21951}{57344}s^6+\frac{5049}{14336}s^4\right)\sin 7\theta$
42	$\delta^2\Theta\frac{\alpha^6}{p^6}J_2^3C^3\left(\frac{8073}{57344}s^6-\frac{1683}{14336}s^4\right)\sin 7\theta$
43	$\frac{135}{2048}\delta^2\Theta\frac{\alpha^6}{p^6}J_2^3s^6C\sin 7\theta$
44	$-\frac{207}{8192}\delta^2\Theta\frac{\alpha^6}{p^6}J_2^3s^6S^2\sin 8\theta$
45	$\frac{207}{8192}\delta^2\Theta\frac{\alpha^6}{p^6}J_2^3s^6C^2\sin 8\theta$
46	$-\frac{81}{8192}\delta^2\Theta\frac{\alpha^6}{p^6}J_2^3s^6S^2C\sin 9\theta$
47	$\frac{27}{8192}\delta^2\Theta\frac{\alpha^6}{p^6}J_2^3s^6C^3\sin 9\theta$
48	$\delta^2\Theta\frac{\alpha^6}{p^6}J_2^3S^3\left(-\frac{2307}{512}s^6+\frac{8829}{2048}s^4-\frac{249}{256}s^2+\frac{69}{64}\right)\cos\theta$
49	$\delta^2\Theta\frac{\alpha^6}{p^6}J_2^3SC^2\left(\frac{26097}{2048}s^6-\frac{42447}{2048}s^4+\frac{1779}{256}s^2+\frac{69}{64}\right)\cos\theta$
50	$\delta^2\Theta\frac{\alpha^6}{p^6}J_2^3S\left(-\frac{168117}{2048}s^6+\frac{33491}{256}s^4-\frac{3797}{64}s^2+\frac{69}{8}\right)\cos\theta$
51	$\delta^2\Theta\frac{\alpha^6}{p^6}J_2^3SC\left(-\frac{80103}{2048}s^6+\frac{49809}{1024}s^4-\frac{431}{32}s^2+\frac{75}{32}\right)\cos 2\theta$
52	$\delta^2\Theta\frac{\alpha^6}{p^6}J_2^3S^3\left(\frac{549}{4096}s^6-\frac{997}{2048}s^4+\frac{103}{256}s^2-\frac{5}{64}\right)\cos 3\theta$

Parallax generator at order 4 (*continued*)

No.	Expression
53	$\delta^2\Theta\frac{\alpha^6}{p^6}J_2^3 SC^2\left(-\frac{4101}{4096}s^6+\frac{3075}{2048}s^4-\frac{201}{256}s^2+\frac{15}{64}\right)\cos 3\theta$
54	$\delta^2\Theta\frac{\alpha^6}{p^6}J_2^3 S\left(-\frac{1079}{2048}s^6-\frac{47}{4}s^4+\frac{683}{64}s^2\right)\cos 3\theta$
55	$\delta^2\Theta\frac{\alpha^6}{p^6}J_2^3 SC\left(\frac{2067}{512}s^6-\frac{5763}{512}s^4+\frac{401}{64}s^2\right)\cos 4\theta$
56	$\delta^2\Theta\frac{\alpha^6}{p^6}J_2^3 S^3\left(\frac{63}{4096}s^6+\frac{189}{2048}s^4-\frac{63}{640}s^2\right)\cos 5\theta$
57	$\delta^2\Theta\frac{\alpha^6}{p^6}J_2^3 SC^2\left(-\frac{1503}{4096}s^6-\frac{1359}{10240}s^4+\frac{189}{640}s^2\right)\cos 5\theta$
58	$\delta^2\Theta\frac{\alpha^6}{p^6}J_2^3 S\left(-\frac{3429}{2048}s^6+\frac{1967}{1280}s^4\right)\cos 5\theta$
59	$\delta^2\Theta\frac{\alpha^6}{p^6}J_2^3 SC\left(-\frac{3813}{2048}s^6+\frac{1699}{1024}s^4\right)\cos 6\theta$
60	$\delta^2\Theta\frac{\alpha^6}{p^6}J_2^3 S^3\left(\frac{6939}{57344}s^6-\frac{1683}{14336}s^4\right)\cos 7\theta$
61	$\delta^2\Theta\frac{\alpha^6}{p^6}J_2^3 SC^2\left(-\frac{23085}{57344}s^6+\frac{5049}{14336}s^4\right)\cos 7\theta$
62	$-\frac{135}{2048}\delta^2\Theta\frac{\alpha^6}{p^6}J_2^3 s^6 S\cos 7\theta$
63	$-\frac{207}{4096}\delta^2\Theta\frac{\alpha^6}{p^6}J_2^3 s^6 SC\cos 8\theta$
64	$\frac{27}{8192}\delta^2\Theta\frac{\alpha^6}{p^6}J_2^3 s^6 S^3\cos 9\theta$
65	$-\frac{81}{8192}\delta^2\Theta\frac{\alpha^6}{p^6}J_2^3 s^6 SC^2\cos 9\theta$

A.3. THE NEW HAMILTONIAN AFTER THE DELAUNAY SIMPLIFICATION

$$-\frac{1}{2}\frac{1}{\eta}\Theta n+\delta\Theta n\frac{\alpha^2}{p^2}J_2\left(\frac{3}{4}s^2-\frac{1}{2}\right)+$$

$$+\delta^2\Theta n\frac{\alpha^4}{p^4}J_2^2\left[\eta^2\left(-\frac{15}{128}s^4-\frac{3}{16}s^2+\frac{3}{16}\right)+\right.$$

$$+\eta\left(-\frac{27}{32}s^4+\frac{9}{8}s^2-\frac{3}{8}\right)-\frac{105}{128}s^4+\frac{15}{8}s^2-\frac{15}{16}+$$
$$+e^2\left(-\frac{45}{64}s^4+\frac{21}{32}s^2\right)\cos 2g\Bigg]+$$
$$+\delta^3\Theta n\frac{\alpha^6}{p^6}J_2^3\left\{\eta^3\left(\frac{225}{512}s^6+\frac{105}{256}s^4-\frac{75}{64}s^2+\frac{15}{32}\right)+\right.$$
$$+\eta^2\left(-\frac{2193}{512}s^6+\frac{1881}{256}s^4-\frac{237}{64}s^2+\frac{27}{32}\right)+$$
$$+\eta\left(\frac{945}{512}s^6-\frac{1395}{256}s^4+\frac{315}{64}s^2-\frac{45}{32}\right)+$$
$$+\frac{4575}{512}s^6-\frac{4773}{256}s^4+\frac{807}{64}s^2-\frac{105}{32}+$$
$$+\left[e^2\eta\left(\frac{675}{256}s^6-\frac{135}{32}s^4+\frac{105}{64}s^2\right)+e^2\beta\left(\frac{135}{128}s^6-\frac{27}{16}s^4+\frac{21}{32}s^2\right)+\right.$$
$$\left.\left.+e^2\left(\frac{225}{1024}s^6-\frac{69}{64}s^4+\frac{3}{4}s^2\right)\right]\cos 2g\right\}+$$
$$+\delta^4\Theta n\frac{\alpha^8}{p^8}J_2^4\left\{\eta^5\left(-\frac{29925}{32768}s^8+\frac{5985}{4096}s^6-\frac{2961}{4096}s^4+\frac{63}{256}s^2-\frac{63}{512}\right)+\right.$$
$$+\eta^4\left(-\frac{774765}{131072}s^8+\frac{95445}{8192}s^6-\frac{23337}{4096}s^4-\frac{909}{2048}s^2+\frac{405}{1024}\right)+$$
$$+\eta^3\left(\frac{314415}{16384}s^8-\frac{47685}{1024}s^6+\frac{85275}{2048}s^4-\frac{4575}{256}s^2+\frac{885}{256}\right)+$$
$$+\eta^2\left(-\frac{741165}{65536}s^8+\frac{28131}{4096}s^6+\frac{19245}{1024}s^4-\frac{21207}{1024}s^2+\frac{2745}{512}\right)+$$
$$+\eta\left(-\frac{704025}{32768}s^8+\frac{248463}{4096}s^6-\frac{263133}{4096}s^4+\frac{4041}{128}s^2-\frac{3195}{512}\right)-$$
$$-\frac{782265}{131072}s^8+\frac{436125}{8192}s^6-\frac{394479}{4096}s^4+\frac{133347}{2048}s^2-\frac{15015}{1024}+$$
$$+\left[e^2\eta^3\left(-\frac{4725}{8192}s^8-\frac{1575}{4096}s^6+\frac{1827}{1024}s^4-\frac{441}{512}s^2\right)+\right.$$
$$+e^2\eta^2\left(-\frac{157095}{32768}s^8+\frac{200817}{16384}s^6-\frac{21465}{2048}s^4+\frac{1449}{512}s^2\right)+$$
$$+e^2\eta\left(-\frac{31725}{8192}s^8+\frac{56385}{4096}s^6-\frac{14793}{1024}s^4+\frac{2409}{512}s^2\right)+$$
$$+e^2\beta\left(\frac{1485}{2048}s^8+\frac{63}{1024}s^6-\frac{375}{256}s^4+\frac{3}{4}s^2\right)+$$

$$
+e^2\left(-\frac{1760835}{32768}s^8+\frac{2204209}{16384}s^6-\frac{223987}{2048}s^4+\frac{15159}{512}s^2\right)\Bigg]\cos 2g
$$

$$
+\Bigg[e^4\eta\left(-\frac{14175}{16384}s^8+\frac{6615}{4096}s^6-\frac{3087}{4096}s^4\right)+
$$

$$
+e^4\beta\left(-\frac{2025}{4096}s^8+\frac{945}{1024}s^6-\frac{441}{1024}s^4\right)+
$$

$$
+e^4\left(\frac{103815}{65536}s^8-\frac{11385}{4096}s^6+\frac{9891}{8192}s^4\right)\Bigg]\cos 4g\Bigg\}+
$$

$$
+\delta^5\Theta n\frac{\alpha^{10}}{p^{10}}J_2^5\Bigg\{\eta^6\left(\frac{89775}{16384}s^{10}-\frac{101745}{8192}s^8+\frac{20853}{2048}s^6-\right.
$$

$$
\left.-\frac{4473}{1024}s^4+\frac{441}{256}s^2-\frac{63}{128}\right)+
$$

$$
+\eta^5\left(\frac{12534165}{524288}s^{10}-\frac{21744135}{262144}s^8+\right.
$$

$$
\left.+\frac{26019}{256}s^6-\frac{221445}{4096}s^4+\frac{6741}{512}s^2-\frac{3465}{2048}\right)+
$$

$$
+\eta^4\left(\frac{16370829}{524288}s^{10}-\frac{19761687}{262144}s^8+\right.
$$

$$
\left.+\frac{102221}{2048}s^6+\frac{93711}{32768}s^4-\frac{23463}{2048}s^2+\frac{4599}{2048}\right)+
$$

$$
+\eta^3\left(\frac{4474575}{262144}s^{10}+\frac{8836755}{131072}s^8+\right.
$$

$$
\left.-\frac{2123295}{8192}s^6+\frac{1174035}{4096}s^4-\frac{138135}{1024}s^2+\frac{24525}{1024}\right)+
$$

$$
+\eta^2\left(-\frac{121807785}{262144}s^{10}+\frac{178174443}{131072}s^8+\right.
$$

$$
\left.-\frac{24530515}{16384}s^6+\frac{13177569}{16384}s^4-\frac{60339}{256}s^2+\frac{37305}{1024}\right)+
$$

$$
+\eta\left(\frac{19274085}{524288}s^{10}-\frac{60045255}{262144}s^8+\right.
$$

$$
\left.+\frac{3783015}{8192}s^6-\frac{867915}{2048}s^4+\frac{189009}{1024}s^2-\frac{63945}{2048}\right)+
$$

$$
+\frac{253613325}{524288}s^{10}-\frac{409922679}{262144}s^8+
$$

$$
+\frac{32588595}{16384}s^6-\frac{42514209}{32768}s^4+\frac{942555}{2048}s^2-\frac{153153}{2048}+
$$

$$
\begin{aligned}
&+\Bigg[e^2\eta^4\left(\frac{14175}{4096}s^{10}-\frac{12537}{1024}s^6+\frac{1575}{128}s^4-\frac{441}{128}s^2\right)+\\
&+e^2\eta^3\left(-\frac{169155}{8192}s^{10}+\frac{1546839}{32768}s^8-\frac{1142883}{32768}s^6+\right.\\
&\left.+\frac{42651}{4096}s^4-\frac{945}{512}s^2\right)+e^2\eta^2\left(\frac{14986485}{262144}s^{10}-\frac{8222841}{65536}s^8+\right.\\
&\left.+\frac{6906261}{81920}s^6-\frac{189385}{16384}s^4-\frac{14499}{4096}s^2\right)+\\
&+e^2\eta\left(\frac{14289435}{65536}s^{10}-\frac{22920081}{32768}s^8+\frac{27152705}{32768}s^6-\frac{1778897}{4096}s^4+\right.\\
&\left.+\frac{10983}{128}s^2\right)+e^2\beta\left(\frac{2722455}{32768}s^{10}-\frac{2175333}{8192}s^8+\frac{2568485}{8192}s^6-\right.\\
&\left.-\frac{83683}{512}s^4+\frac{4089}{128}s^2\right)+e^2\left(\frac{29253495}{131072}s^{10}-\frac{85122473}{131072}s^8+\right.\\
&\left.+\frac{119439379}{163840}s^6-\frac{6449157}{16384}s^4+\frac{359493}{4096}s^2\right)\cos 2g+\\
&+\Bigg[e^4\eta^2\left(\frac{42525}{8192}s^{10}-\frac{53865}{4096}s^8+\frac{22491}{2048}s^6-\frac{3087}{1024}s^4\right)+\\
&+e^4\eta\left(-\frac{2038365}{262144}s^{10}+\frac{2225475}{131072}s^8-\frac{384867}{32768}s^6+\frac{41013}{16384}s^4\right)+\\
&+e^4\beta^2\left(\frac{6075}{8192}s^{10}-\frac{7695}{4096}s^8+\frac{3213}{2048}s^6-\frac{441}{1024}s^4\right)+\\
&+e^4\beta\left(-\frac{534195}{65536}s^{10}+\frac{625725}{32768}s^8-\frac{119241}{8192}s^6+\frac{14679}{4096}s^4\right)+\\
&+e^4\left(-\frac{2629395}{262144}s^{10}+\frac{4806609}{131072}s^8-\frac{693951}{16384}s^6+\frac{255547}{16384}s^4\right)\Bigg]\cos 4g\Bigg\}+\\
&+\delta^6\Theta n\frac{\alpha^{12}}{p^{12}}J_2^6\Bigg\{\eta^8\left(-\frac{2784375}{2097152}s^{12}+\frac{30375}{131072}s^{10}+\frac{625725}{131072}s^8-\right.\\
&\left.-\frac{22275}{4096}s^6+\frac{15795}{8192}s^4-\frac{1215}{4096}s^2+\frac{405}{4096}\right)+\eta^7\left(-\frac{336925575}{16777216}s^{12}+\right.\\
&+\frac{107747415}{2097152}s^{10}-\frac{86572719}{2097152}s^8+\frac{1958985}{262144}s^6+\frac{267705}{131072}s^4+\frac{33453}{32768}s^2-\\
&\left.-\frac{9639}{16384}\right)+\eta^6\left(\frac{269944785}{16777216}s^{12}+\frac{557103975}{14680064}s^{10}-\frac{4027787889}{18350080}s^8+\right.\\
&\left.+\frac{1352789301}{4587520}s^6-\frac{114281511}{655360}s^4+\frac{8422869}{163840}s^2-\frac{116025}{16384}\right)+
\end{aligned}
$$

$$
\begin{aligned}
&+\eta^5\Big(-\frac{9673330077}{16777216}s^{12}+\frac{2071925325}{1048576}s^{10}-\frac{5592426501}{2097152}s^8+\\
&+\frac{479346353}{262144}s^6-\frac{91782579}{131072}s^4+\frac{5257161}{32768}s^2-\frac{346353}{16384}\Big)+\\
&+\eta^4\Big(-\frac{51208294995}{117440512}s^{12}+\frac{41893358289}{29360128}s^{10}-\frac{12691702939}{7340032}s^8+\\
&+\frac{8638684127}{9175040}s^6-\frac{60869169}{327680}s^4-\frac{1572981}{81920}s^2+\frac{79335}{16384}\Big)+\\
&+\eta^3\Big(\frac{36588668175}{16777216}s^{12}-\frac{16710042705}{2097152}s^{10}+\frac{24859748631}{2097152}s^8-\\
&-\frac{2470794973}{262144}s^6+\frac{579800283}{131072}s^4-\frac{40397793}{32768}s^2+\\
&+\frac{2719575}{16384}\Big)+\eta^2\Big(-\frac{54607489635}{117440512}s^{12}+\frac{1105303905}{7340032}s^{10}+\\
&+\frac{55427623831}{18350080}s^8-\frac{6804687627}{1146880}s^6+\frac{3068410629}{655360}s^4-\frac{281270643}{163840}s^2+\\
&+\frac{4160085}{16384}\Big)+\eta\Big(-\frac{21147194475}{16777216}s^{12}+\frac{1321752339}{262144}s^{10}-\\
&-\frac{17631518211}{2097152}s^8+\frac{2004074883}{262144}s^6-\frac{536802993}{131072}s^4+\frac{40875435}{32768}s^2-\\
&-\frac{2778111}{16384}\Big)-\frac{2240757369}{117440512}s^{12}+\frac{65543049645}{29360128}s^{10}-\frac{58308931017}{7340032}s^8+\\
&+\frac{104441906927}{9175040}s^6-\frac{265657671}{32768}s^4+\frac{29443221}{10240}s^2-\frac{6789783}{16384}+\\
&+\Big[e^2\eta^6\Big(-\frac{2278125}{524288}s^{12}+\frac{1306125}{131072}s^{10}-\frac{2265975}{262144}s^8+\frac{674325}{131072}s^6-\\
&-\frac{52245}{16384}s^4+\frac{8505}{8192}s^2\Big)+e^2\eta^5\Big(-\frac{2399625}{65536}s^{12}+\frac{104279805}{1048576}s^{10}-\\
&-\frac{81864027}{1048576}s^8-\frac{1664469}{524288}s^6+\frac{101493}{4096}s^4-\frac{214893}{32768}s^2\Big)+\\
&+e^2\eta^4\Big(\frac{7492881375}{33554432}s^{12}-\frac{12623427423}{16777216}s^{10}+\frac{5102318259}{5242880}s^8-\\
&-\frac{3197569581}{5242880}s^6+\frac{252001611}{1310720}s^4-\frac{436401}{16384}s^2\Big)+\\
&+e^2\eta^3\Big(-\frac{355108635}{1048576}s^{12}+\frac{89491353}{131072}s^{10}-\frac{160447299}{1310720}s^8-
\end{aligned}
$$

$$-\frac{47726637}{81920}s^6+\frac{15233741}{32768}s^4-\frac{2149623}{20480}s^2\Bigg)+$$

$$+e^2\eta^2\Bigg(\frac{42113478375}{117440512}s^{12}-\frac{44216148123}{41943040}s^{10}+\frac{13504228239}{13107200}s^8-$$

$$-\frac{74050396073}{229376000}s^6-\frac{143122349}{3276800}s^4+\frac{24161397}{819200}s^2\Bigg)+$$

$$+e^2\eta\Bigg(-\frac{623585115}{524288}s^{12}+\frac{4709840055}{1048576}s^{10}-\frac{36207501401}{5242880}s^8+$$

$$+\frac{14327968801}{2621440}s^6-\frac{366358159}{163840}s^4+\frac{61794177}{163840}s^2\Bigg)+$$

$$+e^2\beta^2\Bigg(-\frac{10935}{2048}s^{12}+\frac{5103}{256}s^{10}-\frac{9477}{320}s^8+\frac{14013}{640}s^6-$$

$$-\frac{5157}{640}s^4+\frac{189}{160}s^2\Bigg)+e^2\beta\Bigg(-\frac{179977545}{524288}s^{12}+\frac{154349211}{131072}s^{10}-$$

$$-\frac{1085827807}{655360}s^8+\frac{40262417}{32768}s^6-\frac{20022493}{40960}s^4+\frac{170679}{2048}s^2\Bigg)+$$

$$+e^2\Bigg(-\frac{8797408143}{234881024}s^{12}+\frac{334363458337}{83886080}s^{10}-\frac{294048503303}{26214400}s^8+$$

$$+\frac{10889097472391}{917504000}s^6-\frac{36235796613}{6553600}s^4+\frac{778394433}{819200}s^2\Bigg)+$$

$$+\frac{e^2}{\eta}\Bigg(-\frac{156735}{65536}s^{12}+\frac{73143}{8192}s^{10}-\frac{135837}{10240}s^8+\frac{200853}{20480}s^6-$$

$$-\frac{73917}{20480}s^4+\frac{2709}{5120}s^2\Bigg)\Bigg]\cos 2g+\Bigg[e^4\eta^4\Bigg(-\frac{1366875}{1048576}s^{12}+$$

$$+\frac{91125}{262144}s^{10}+\frac{1269675}{262144}s^8-\frac{93555}{16384}s^6+\frac{59535}{32768}s^4\Bigg)+$$

$$+e^4\eta^3\Bigg(-\frac{102734325}{8388608}s^{12}+\frac{51040935}{1048576}s^{10}-\frac{74121561}{1048576}s^8+$$

$$+\frac{2892915}{65536}s^6-\frac{1298997}{131072}s^4\Bigg)+e^4\eta^2\Bigg(-\frac{765924525}{33554432}s^{12}+$$

$$+\frac{97017129}{1048576}s^{10}-\frac{280228743}{2097152}s^8+\frac{220214061}{2621440}s^6-\frac{5227137}{262144}s^4\Bigg)+$$

$$+e^4\eta\Bigg(-\frac{453166515}{8388608}s^{12}+\frac{75168171}{524288}s^{10}-\frac{133137195}{1048576}s^8+$$

$$+\frac{26892729}{655360}s^6-\frac{2012521}{655360}s^4\Big)+e^4\beta^2\Big(\frac{1080135}{262144}s^{12}-\frac{709155}{65536}s^{10}+$$
$$+\frac{647001}{65536}s^8-\frac{7155}{2048}s^6+\frac{2709}{8192}s^4\Big)+e^4\beta\Big(-\frac{52439535}{1048576}s^{12}+$$
$$+\frac{35028441}{262144}s^{10}-\frac{16005747}{131072}s^8+\frac{7003449}{163840}s^6-\frac{167719}{40960}s^4\Big)+$$
$$+e^4\Big(\frac{24663212709}{33554432}s^{12}-\frac{11713291761}{5242880}s^{10}+\frac{133396066419}{52428800}s^8-$$
$$-\frac{849756711}{655360}s^6+\frac{332976431}{1310720}s^4\Big)\Big]\cos 4g+\Big[e^6\eta^2\Big(-\frac{1366875}{1048576}s^{12}+$$
$$+\frac{1913625}{524288}s^{10}-\frac{893025}{262144}s^8+\frac{138915}{131072}s^6\Big)+e^6\eta\Big(\frac{42045075}{8388608}s^{12}-$$
$$-\frac{56508435}{4194304}s^{10}+\frac{12612915}{1048576}s^8-\frac{1869399}{524288}s^6\Big)+e^6\beta^2\Big(-\frac{91125}{262144}s^{12}+$$
$$+\frac{127575}{131072}s^{10}-\frac{59535}{65536}s^8+\frac{9261}{32768}s^6\Big)+e^6\beta\Big(\frac{19118025}{4194304}s^{12}-$$
$$-\frac{25980345}{2097152}s^{10}+\frac{5871285}{524288}s^8-\frac{882441}{262144}s^6\Big)+e^6\Big(-\frac{811490805}{67108864}s^{12}+$$
$$+\frac{1094010705}{33554432}s^{10}-\frac{7657173}{262144}s^8+\frac{9119871}{1048576}s^6\Big)\Big]\cos 6g\Big\}+\mathcal{O}(\delta^7).$$

A.4. THE GENERATOR FOR THE DELAUNAY SIMPLIFICATION

Delaunay generator at order 4

No.	Expression
1	$\Theta\frac{\alpha^2}{p^2}J_2\phi\left(\frac{3}{4}s^2-\frac{1}{2}\right)$
2	$\delta\Theta\frac{\alpha^4}{p^4}J_2^2\phi\left[\eta^2\left(-\frac{15}{64}s^4-\frac{3}{8}s^2+\frac{3}{8}\right)-\frac{105}{64}s^4+\frac{15}{4}s^2-\frac{15}{8}\right]$
3	$\delta\Theta\frac{\alpha^4}{p^4}J_2^2\phi e^2\left(-\frac{45}{32}s^4+\frac{21}{16}s^2\right)\cos 2g$
4	$\delta\Theta\frac{\alpha^4}{p^4}J_2^2 e\beta\left(-\frac{9}{8}s^4+\frac{3}{2}s^2-\frac{1}{2}\right)\sin f$
5	$\delta\Theta\frac{\alpha^4}{p^4}J_2^2 e^2\beta\left(-\frac{9}{32}s^4+\frac{3}{8}s^2-\frac{1}{8}\right)\sin 2f$

Delaunay generator at order 4 (*continued*)

No.	Expression
6	$\delta^2\Theta\frac{\alpha^6}{p^6}J_2^3\phi^2e^2\left(\frac{675}{256}s^6-\frac{585}{128}s^4+\frac{63}{32}s^2\right)\sin 2g$
7	$\delta^2\Theta\frac{\alpha^6}{p^6}J_2^3\phi\Big[\eta^3\left(\frac{45}{256}s^6+\frac{21}{128}s^4-\frac{15}{32}s^2+\frac{3}{16}\right)+ +\eta^2\left(-\frac{7965}{512}s^6+\frac{6897}{256}s^4-\frac{897}{64}s^2+\frac{99}{32}\right)+\frac{13995}{512}s^6-\frac{14193}{256}s^4+ +\frac{2331}{64}s^2-\frac{297}{32}\Big]$
8	$\delta^2\Theta\frac{\alpha^6}{p^6}J_2^3\phi\Big[e^2\eta\left(\frac{135}{128}s^6-\frac{27}{16}s^4+\frac{21}{32}s^2\right)+e^2\beta\left(\frac{135}{128}s^6-\frac{27}{16}s^4+\frac{21}{32}s^2\right)+ +e^2\left(-\frac{405}{1024}s^6-\frac{99}{64}s^4+\frac{51}{32}s^2\right)\Big]\cos 2g$
9	$\delta^2\Theta\frac{\alpha^6}{p^6}J_2^3\phi e\left(\frac{45}{64}s^6+\frac{21}{32}s^4-\frac{15}{8}s^2+\frac{3}{4}\right)\cos f$
10	$\delta^2\Theta\frac{\alpha^6}{p^6}J_2^3\phi e\left(-\frac{135}{32}s^6+\frac{27}{4}s^4-\frac{21}{8}s^2\right)\cos(f+2g)$
11	$\delta^2\Theta\frac{\alpha^6}{p^6}J_2^3\phi e^2\left(\frac{45}{256}s^6+\frac{21}{128}s^4-\frac{15}{32}s^2+\frac{3}{16}\right)\cos 2f$
12	$\delta^2\Theta\frac{\alpha^6}{p^6}J_2^3\phi\Big[\eta^2\left(\frac{135}{128}s^6-\frac{27}{16}s^4+\frac{21}{32}s^2\right)-\frac{135}{128}s^6+ +\frac{27}{16}s^4-\frac{21}{32}s^2\Big]\cos(2f+2g)$
13	$\delta^2\Theta\frac{\alpha^6}{p^6}J_2^3e^3\beta\left(\frac{405}{256}s^6-\frac{81}{32}s^4+\frac{63}{64}s^2\right)\sin(f-2g)$
14	$\delta^2\Theta\frac{\alpha^6}{p^6}J_2^3\Big[e\eta\left(\frac{135}{256}s^6+\frac{63}{128}s^4-\frac{45}{32}s^2+\frac{9}{16}\right)+e\beta\left(\frac{81}{16}s^6-\frac{387}{32}s^4+ +\frac{153}{16}s^2-\frac{5}{2}\right)+e\left(-\frac{405}{256}s^6-\frac{189}{128}s^4+\frac{135}{32}s^2-\frac{27}{16}\right)+ +\frac{e}{\eta}\left(-\frac{81}{128}s^6+\frac{81}{64}s^4-\frac{27}{32}s^2+\frac{3}{16}\right)\Big]\sin f$

Delaunay generator at order 4 (*continued*)

No.	Expression
15	$\delta^2\Theta\frac{\alpha^6}{p^6}J_2^3\Big[e\eta\Big(-\frac{405}{256}s^6+\frac{81}{32}s^4-\frac{63}{64}s^2\Big)+$
	$+e\Big(\frac{2025}{256}s^6-\frac{405}{32}s^4+\frac{315}{64}s^2\Big)\Big]\sin(f+2g)$
16	$\delta^2\Theta\frac{\alpha^6}{p^6}J_2^3e^4\beta\Big(\frac{405}{1024}s^6-\frac{81}{128}s^4+\frac{63}{256}s^2\Big)\sin(2f-2g)$
17	$\delta^2\Theta\frac{\alpha^6}{p^6}J_2^3\Big[e^2\eta\Big(\frac{135}{1024}s^6+\frac{63}{512}s^4-\frac{45}{128}s^2+\frac{9}{64}\Big)+$
	$+e^2\beta^2\Big(\frac{27}{64}s^6-\frac{27}{32}s^4+\frac{9}{16}s^2-\frac{1}{8}\Big)+$
	$+e^2\beta\Big(\frac{405}{256}s^6-\frac{117}{32}s^4+\frac{45}{16}s^2-\frac{23}{32}\Big)+$
	$+e^2\Big(-\frac{135}{512}s^6-\frac{63}{256}s^4+\frac{45}{64}s^2-\frac{9}{32}\Big)+$
	$+\frac{e^2}{\eta}\Big(-\frac{297}{512}s^6+\frac{297}{256}s^4-\frac{99}{128}s^2+\frac{11}{64}\Big)\Big]\sin 2f$
18	$\delta^2\Theta\frac{\alpha^6}{p^6}J_2^3\Big[\eta^3\Big(\frac{405}{1024}s^6-\frac{81}{128}s^4+\frac{63}{256}s^2\Big)+$
	$+\eta^2\Big(-\frac{1215}{1024}s^6+\frac{243}{128}s^4-\frac{189}{256}s^2\Big)+$
	$+\eta\Big(-\frac{405}{1024}s^6+\frac{81}{128}s^4-\frac{63}{256}s^2\Big)+$
	$+\frac{1215}{1024}s^6-\frac{243}{128}s^4+\frac{189}{256}s^2\Big]\sin(2f+2g)$
19	$\delta^2\Theta\frac{\alpha^6}{p^6}J_2^3\Big[e^3\beta^2\Big(\frac{27}{128}s^6-\frac{27}{64}s^4+\frac{9}{32}s^2-\frac{1}{16}\Big)+$
	$+e^3\beta\Big(\frac{27}{128}s^6-\frac{27}{64}s^4+\frac{9}{32}s^2-\frac{1}{16}\Big)+$
	$+\frac{e^3}{\eta}\Big(-\frac{27}{128}s^6+\frac{27}{64}s^4-\frac{9}{32}s^2+\frac{1}{16}\Big)\Big]\sin 3f$
20	$\delta^2\Theta\frac{\alpha^6}{p^6}J_2^3\Big[e^4\beta^2\Big(\frac{27}{1024}s^6-\frac{27}{512}s^4+\frac{9}{256}s^2-\frac{1}{128}\Big)+$

Delaunay generator at order 4 (*continued*)

No.	Expression
	$+e^4\beta\left(\frac{27}{1024}s^6-\frac{27}{512}s^4+\frac{9}{256}s^2-\frac{1}{128}\right)+$
	$+\frac{e^4}{\eta}\left(-\frac{27}{1024}s^6+\frac{27}{512}s^4-\frac{9}{256}s^2+\frac{1}{128}\right)\Big]\sin 4f$
21	$\delta^3\Theta\frac{\alpha^8}{p^8}J_2^4\phi^3e^2\left(\frac{3375}{512}s^8-\frac{4275}{256}s^6+\frac{225}{16}s^4-\frac{63}{16}s^2\right)\cos 2g$
22	$\delta^3\Theta\frac{\alpha^8}{p^8}J_2^4\phi^2\Big[e^2\eta\left(-\frac{2025}{512}s^8+\frac{1215}{128}s^6-\frac{963}{128}s^4+\frac{63}{32}s^2\right)+$
	$+e^2\beta\left(-\frac{2025}{512}s^8+\frac{1215}{128}s^6-\frac{963}{128}s^4+\frac{63}{32}s^2\right)+$
	$+e^2\left(\frac{675}{2048}s^8+\frac{2385}{512}s^6-\frac{2133}{256}s^4+\frac{225}{64}s^2\right)\Big]\sin 2g$
23	$\delta^3\Theta\frac{\alpha^8}{p^8}J_2^4\phi^2e\left(\frac{2025}{256}s^8-\frac{1215}{64}s^6+\frac{963}{64}s^4-\frac{63}{16}s^2\right)\sin(f+2g)$
24	$\delta^3\Theta\frac{\alpha^8}{p^8}J_2^4\phi^2\Big[\eta^2\left(-\frac{2025}{1024}s^8+\frac{1215}{256}s^6-\frac{963}{256}s^4+\frac{63}{64}s^2\right)+$
	$+\frac{2025}{1024}s^8-\frac{1215}{256}s^6+\frac{963}{256}s^4-\frac{63}{64}s^2\Big]\sin(2f+2g)$
25	$\delta^3\Theta\frac{\alpha^8}{p^8}J_2^4\phi\Big[\eta^5\left(-\frac{4275}{4096}s^8+\frac{855}{512}s^6-\frac{423}{512}s^4+\frac{9}{32}s^2-\frac{9}{64}\right)+$
	$+\eta^4\left(-\frac{628065}{32768}s^8+\frac{95895}{2048}s^6-\frac{18117}{512}s^4+\frac{4779}{512}s^2-\frac{297}{256}\right)+$
	$+\eta^3\left(\frac{34785}{4096}s^8-\frac{10347}{512}s^6+\frac{2313}{128}s^4-\frac{513}{64}s^2+\frac{53}{32}\right)+$
	$+\eta^2\left(-\frac{780405}{16384}s^8+\frac{4785}{256}s^6+\frac{3195}{32}s^4-\frac{25593}{256}s^2+\frac{3189}{128}\right)+$
	$+\eta\left(-\frac{1215}{2048}s^8+-\frac{81}{512}s^6+\frac{999}{512}s^4-\frac{27}{16}s^2+\frac{27}{64}\right)+\frac{387915}{32768}s^8+$
	$+\frac{259221}{2048}s^6-\frac{158193}{512}s^4+\frac{117111}{512}s^2-\frac{13521}{256}\Big]$
26	$\delta^3\Theta\frac{\alpha^8}{p^8}J_2^4\phi\Big[e^2\eta^3\left(-\frac{675}{1024}s^8-\frac{225}{512}s^6+\frac{261}{128}s^4-\frac{63}{64}s^2\right)+$

Delaunay generator at order 4 (*continued*)

No.	Expression
	$+e^2\eta^2\left(\frac{40005}{8192}s^8-\frac{18603}{4096}s^6-\frac{1467}{512}s^4+\frac{63}{32}s^2\right)+$
	$+e^2\eta\left(-\frac{3645}{2048}s^8+\frac{6507}{1024}s^6-\frac{1725}{256}s^4+\frac{285}{128}s^2\right)+$
	$+e^2\beta\left(-\frac{10935}{2048}s^8+\frac{14769}{1024}s^6-\frac{51}{4}s^4+\frac{237}{64}s^2\right)+$
	$+e^2\left(-\frac{1639065}{8192}s^8+\frac{2045107}{4096}s^6-\frac{103457}{256}s^4+\frac{27909}{256}s^2\right)\Big]\cos 2g$
27	$\delta^3\Theta\frac{\alpha^8}{p^8}J_2^4\phi\Big[e^4\eta\left(-\frac{2025}{2048}s^8+\frac{945}{512}s^6-\frac{441}{512}s^4\right)+$
	$+e^4\beta\left(-\frac{2025}{2048}s^8+\frac{945}{512}s^6-\frac{441}{512}s^4\right)+$
	$+e^4\left(\frac{111915}{16384}s^8-\frac{6165}{512}s^6+\frac{10773}{2048}s^4\right)\Big]\cos 4g$
28	$\delta^3\Theta\frac{\alpha^8}{p^8}J_2^4\phi\Big[e^3\beta^2\left(-\frac{405}{512}s^8+\frac{459}{256}s^6-\frac{171}{128}s^4+\frac{21}{64}s^2\right)+$
	$+e^3\beta\left(-\frac{1215}{256}s^8+\frac{2889}{256}s^6-\frac{567}{64}s^4+\frac{147}{64}s^2\right)+$
	$+e^3\left(-\frac{675}{2048}s^8-\frac{225}{1024}s^6+\frac{261}{256}s^4-\frac{63}{128}s^2\right)\Big]\cos(f-2g)$
29	$\delta^3\Theta\frac{\alpha^8}{p^8}J_2^4\phi\Big[e\eta^2\left(-\frac{4275}{2048}s^8+\frac{855}{256}s^6-\frac{423}{256}s^4+\frac{9}{16}s^2-\frac{9}{32}\right)+$
	$+e\eta\left(-\frac{405}{512}s^8-\frac{27}{128}s^6+\frac{333}{128}s^4-\frac{9}{4}s^2+\frac{9}{16}\right)+$
	$+e\left(\frac{97515}{2048}s^8-\frac{7371}{64}s^6+\frac{26031}{256}s^4-\frac{1353}{32}s^2+\frac{249}{32}\right)\Big]\cos f$
30	$\delta^3\Theta\frac{\alpha^8}{p^8}J_2^4\phi\Big[e\eta^2\left(\frac{2025}{2048}s^8+\frac{675}{1024}s^6-\frac{783}{256}s^4+\frac{189}{128}s^2\right)+$
	$+e\eta\left(\frac{135}{256}s^6-\frac{27}{32}s^4+\frac{21}{64}s^2\right)+e\beta\left(\frac{405}{256}s^8-\frac{459}{128}s^6+\frac{171}{64}s^4-\frac{21}{32}s^2\right)+$
	$+e\left(-\frac{11475}{2048}s^8+\frac{2439}{1024}s^6+\frac{1737}{256}s^4-\frac{513}{128}s^2\right)\Big]\cos(f+2g)$

Delaunay generator at order 4 (*continued*)

No.	Expression
31	$\delta^3\Theta\frac{\alpha^8}{p^8}J_2^4\phi e^3\left(\frac{2025}{1024}s^8-\frac{945}{256}s^6+\frac{441}{256}s^4\right)\cos(f+4g)$
32	$\delta^3\Theta\frac{\alpha^8}{p^8}J_2^4\phi\Big[e^4\beta^2\left(-\frac{405}{2048}s^8+\frac{459}{1024}s^6-\frac{171}{512}s^4+\frac{21}{256}s^2\right)+ +e^4\beta\left(-\frac{1215}{1024}s^8+\frac{2889}{1024}s^6-\frac{567}{256}s^4+\frac{147}{256}s^2\right)+ +e^4\left(-\frac{675}{8192}s^8-\frac{225}{4096}s^6+\frac{261}{1024}s^4-\frac{63}{512}s^2\right)\Big]\cos(2f-2g)$
33	$\delta^3\Theta\frac{\alpha^8}{p^8}J_2^4\phi\Big[e^2\eta^2\left(-\frac{4275}{8192}s^8+\frac{855}{1024}s^6-\frac{423}{1024}s^4+\frac{9}{64}s^2-\frac{9}{128}\right)+ +e^2\eta\left(-\frac{405}{2048}s^8-\frac{27}{512}s^6+\frac{333}{512}s^4-\frac{9}{16}s^2+\frac{9}{64}\right)+ +e^2\left(\frac{97515}{8192}s^8-\frac{7371}{256}s^6+\frac{26031}{1024}s^4-\frac{1353}{128}s^2+\frac{249}{128}\right)\Big]\cos 2f$
34	$\delta^3\Theta\frac{\alpha^8}{p^8}J_2^4\phi\Big[\eta^4\left(-\frac{2025}{8192}s^8-\frac{675}{4096}s^6+\frac{783}{1024}s^4-\frac{189}{512}s^2\right)+ +\eta^3\left(-\frac{135}{1024}s^6+\frac{27}{128}s^4-\frac{21}{256}s^2\right)+ +\eta^2\left(\frac{135}{2048}s^8+\frac{3501}{1024}s^6-\frac{711}{128}s^4+\frac{561}{256}s^2\right)+ +\eta\left(-\frac{405}{1024}s^8+\frac{1053}{1024}s^6-\frac{225}{256}s^4+\frac{63}{256}s^2\right)-\frac{21195}{8192}s^8+ +\frac{12375}{4096}s^6+\frac{117}{1024}s^4-\frac{345}{512}s^2\Big)\cos(2f+2g)$
35	$\delta^3\Theta\frac{\alpha^8}{p^8}J_2^4\phi\Big[e^2\eta^2\left(-\frac{2025}{4096}s^8+\frac{945}{1024}s^6-\frac{441}{1024}s^4\right)+ +e^2\left(\frac{2025}{4096}s^8-\frac{945}{1024}s^6+\frac{441}{1024}s^4\right)\Big]\cos(2f+4g)$
36	$\delta^3\Theta\frac{\alpha^8}{p^8}J_2^4\phi e\left(-\frac{405}{256}s^8+\frac{459}{128}s^6-\frac{171}{64}s^4+\frac{21}{32}s^2\right)\cos(3f+2g)$
37	$\delta^3\Theta\frac{\alpha^8}{p^8}J_2^4\phi e^2\left(-\frac{405}{2048}s^8+\frac{459}{1024}s^6-\frac{171}{512}s^4+\frac{21}{256}s^2\right)\cos(4f+2g)$

Delaunay generator at order 4 (*continued*)

No.	Expression
38	$\delta^3\Theta\frac{\alpha^8}{p^8}J_2^4e^5\beta\left(-\frac{2025}{4096}s^8+\frac{945}{1024}s^6-\frac{441}{1024}s^4\right)\sin(f-4g)$
39	$\delta^3\Theta\frac{\alpha^8}{p^8}J_2^4\Big[e^3\eta\left(-\frac{675}{2048}s^8-\frac{225}{1024}s^6+\frac{261}{256}s^4-\frac{63}{128}s^2\right)+$
	$+e^3\beta^2\left(-\frac{405}{512}s^8+\frac{459}{256}s^6-\frac{171}{128}s^4+\frac{21}{64}s^2\right)+$
	$+e^3\beta\left(-\frac{135}{64}s^8+\frac{2097}{256}s^6-\frac{2289}{256}s^4+\frac{381}{128}s^2\right)+$
	$+e^3\left(\frac{405}{2048}s^8+\frac{2511}{1024}s^6-\frac{1125}{256}s^4+\frac{231}{128}s^2\right)+$
	$+\frac{e^3}{\eta}\left(\frac{1215}{1024}s^8-\frac{1377}{512}s^6+\frac{513}{256}s^4-\frac{63}{128}s^2\right)\Big]\sin(f-2g)$
40	$\delta^3\Theta\frac{\alpha^8}{p^8}J_2^4\Big[e\eta^3\left(-\frac{4275}{4096}s^8+\frac{855}{512}s^6-\frac{423}{512}s^4+\frac{9}{32}s^2-\frac{9}{64}\right)+$
	$+e\eta^2\left(\frac{20295}{4096}s^8-\frac{4311}{512}s^6+\frac{2559}{512}s^4-\frac{69}{32}s^2+\frac{57}{64}\right)+$
	$+e\eta\left(\frac{129915}{4096}s^8-\frac{38913}{512}s^6+\frac{34083}{512}s^4-\frac{441}{16}s^2+\frac{325}{64}\right)+$
	$+e\beta\left(-\frac{16119}{512}s^8+\frac{12849}{128}s^6-\frac{15147}{128}s^4+\frac{249}{4}s^2-\frac{199}{16}\right)+$
	$+e\left(-\frac{384255}{4096}s^8+\frac{118917}{512}s^6-\frac{108867}{512}s^4+93s^2-\frac{1149}{64}\right)+$
	$+\frac{e}{\eta}\left(\frac{6723}{2048}s^8-\frac{729}{64}s^6+\frac{909}{64}s^4-\frac{489}{64}s^2+\frac{193}{128}\right)+$
	$+\frac{e}{\eta^3}\left(-\frac{729}{2048}s^8+\frac{243}{256}s^6-\frac{243}{256}s^4+\frac{27}{64}s^2-\frac{9}{128}\right)\Big]\sin f$
41	$\delta^3\Theta\frac{\alpha^8}{p^8}J_2^4\Big[e\eta^3\left(\frac{675}{2048}s^8+\frac{225}{1024}s^6-\frac{261}{256}s^4+\frac{63}{128}s^2\right)+$
	$+e\eta^2\left(-\frac{3105}{2048}s^8-\frac{3411}{1024}s^6+\frac{2169}{256}s^4-\frac{483}{128}s^2\right)+$
	$+e\eta\left(\frac{1215}{2048}s^8-\frac{5859}{1024}s^6+\frac{2037}{256}s^4-\frac{381}{128}s^2\right)+$

Delaunay generator at order 4 (*continued*)

No.	Expression
	$+e\beta\left(-\frac{405}{128}s^8+\frac{459}{64}s^6-\frac{171}{32}s^4+\frac{21}{16}s^2\right)+$
	$+e\left(\frac{94095}{2048}s^8-\frac{90891}{1024}s^6+\frac{13611}{256}s^4-\frac{1203}{128}s^2\right)+$
	$+\frac{e}{\eta}\left(\frac{1215}{1024}s^8-\frac{1377}{512}s^6+\frac{513}{256}s^4-\frac{63}{128}s^2\right)\Big]\sin(f+2g)$
42	$\delta^3\Theta\frac{\alpha^8}{p^8}J_2^4\Big[e^3\eta\left(\frac{2025}{4096}s^8-\frac{945}{1024}s^6+\frac{441}{1024}s^4\right)+$
	$+e^3\left(-\frac{18225}{4096}s^8+\frac{8505}{1024}s^6-\frac{3969}{1024}s^4\right)\Big]\sin(f+4g)$
43	$\delta^3\Theta\frac{\alpha^8}{p^8}J_2^4e^6\beta\left(-\frac{2025}{16384}s^8+\frac{945}{4096}s^6-\frac{441}{4096}s^4\right)\sin(2f-4g)$
44	$\delta^3\Theta\frac{\alpha^8}{p^8}J_2^4\Big[e^4\eta\left(-\frac{675}{8192}s^8-\frac{225}{4096}s^6+\frac{261}{1024}s^4-\frac{63}{512}s^2\right)+$
	$+e^4\beta^2\left(-\frac{2025}{2048}s^8+\frac{2295}{1024}s^6-\frac{855}{512}s^4+\frac{105}{256}s^2\right)+$
	$+e^4\beta\left(-\frac{675}{512}s^8+\frac{3933}{1024}s^6-\frac{3657}{1024}s^4++\frac{549}{512}s^2\right)+$
	$+e^4\left(-\frac{135}{4096}s^8+\frac{1143}{2048}s^6-\frac{27}{32}s^4+\frac{21}{64}s^2\right)+$
	$+\frac{e^4}{\eta}\left(\frac{4455}{4096}s^8-\frac{5049}{2048}s^6+\frac{1881}{1024}s^4-\frac{231}{512}s^2\right)\Big]\sin(2f-2g)$
45	$\delta^3\Theta\frac{\alpha^8}{p^8}J_2^4\Big[e^2\eta^3\left(-\frac{4275}{16384}s^8+\frac{855}{2048}s^6-\frac{423}{2048}s^4+\frac{9}{128}s^2-\frac{9}{256}\right)+$
	$+e^2\eta^2\left(\frac{11745}{16384}s^8-\frac{2601}{2048}s^6+\frac{1713}{2048}s^4-\frac{51}{128}s^2+\frac{39}{256}\right)+$
	$+e^2\eta\left(\frac{130995}{16384}s^8-\frac{38877}{2048}s^6+\frac{33639}{2048}s^4-\frac{429}{64}s^2+\frac{313}{256}\right)+$
	$+e^2\beta^2\left(-\frac{351}{128}s^8+\frac{531}{64}s^6-\frac{75}{8}s^4+\frac{75}{16}s^2-\frac{7}{8}\right)+$
	$+e^2\beta\left(-\frac{5103}{512}s^8+\frac{16089}{512}s^6-\frac{18801}{512}s^4+\frac{153}{8}s^2-\frac{483}{128}\right)+$

Delaunay generator at order 4 (*continued*)

No.	Expression
	$+e^2\left(-\frac{259785}{16384}s^8+\frac{78825}{2048}s^6-\frac{70365}{2048}s^4+\frac{1869}{128}s^2-\frac{707}{256}\right)+$
	$+\frac{e^2}{\eta}\left(\frac{52839}{16384}s^8-\frac{22563}{2048}s^6+\frac{27843}{2048}s^4-\frac{3717}{512}s^2+\frac{1459}{1024}\right)+$
	$+\frac{e^2}{\eta^3}\left(-\frac{7047}{16384}s^8+\frac{2349}{2048}s^6-\frac{2349}{2048}s^4+\frac{261}{512}s^2-\frac{87}{1024}\right)\Big]\sin 2f$
46	$\delta^3\Theta\frac{\alpha^8}{p^8}J_2^4\Big[\eta^5\left(-\frac{675}{8192}s^8-\frac{225}{4096}s^6+\frac{261}{1024}s^4-\frac{63}{512}s^2\right)+$
	$+\eta^4\left(\frac{135}{1024}s^8+\frac{171}{256}s^6-\frac{693}{512}s^4+\frac{147}{256}s^2\right)+$
	$+\eta^3\left(\frac{675}{2048}s^8+\frac{603}{1024}s^6-\frac{807}{512}s^4+\frac{45}{64}s^2\right)+$
	$+\eta^2\left(-\frac{4995}{2048}s^8+\frac{801}{512}s^6+\frac{2277}{1024}s^4-\frac{765}{512}s^2\right)+$
	$+\eta\left(-\frac{17415}{8192}s^8+\frac{15255}{4096}s^6-\frac{237}{128}s^4+\frac{51}{256}s^2\right)+$
	$+\frac{19305}{2048}s^8-\frac{9405}{512}s^6+\frac{11421}{1024}s^4-\frac{1041}{512}s^2+$
	$+\frac{1}{\eta}\left(\frac{4455}{4096}s^8-\frac{5049}{2048}s^6+\frac{1881}{1024}s^4-\frac{231}{512}s^2\right)\Big]\sin(2f+2g)$
47	$\delta^3\Theta\frac{\alpha^8}{p^8}J_2^4\Big[e^2\eta^3\left(-\frac{2025}{16384}s^8+\frac{945}{4096}s^6-\frac{441}{4096}s^4\right)+$
	$+e^2\eta^2\left(\frac{10125}{16384}s^8-\frac{4725}{4096}s^6+\frac{2205}{4096}s^4\right)+$
	$+e^2\eta\left(\frac{2025}{16384}s^8-\frac{945}{4096}s^6+\frac{441}{4096}s^4\right)+$
	$+e^2\left(-\frac{10125}{16384}s^8+\frac{4725}{4096}s^6-\frac{2205}{4096}s^4\right)\Big]\sin(2f+4g)$
48	$\delta^3\Theta\frac{\alpha^8}{p^8}J_2^4\Big[e^5\beta^2\left(-\frac{405}{1024}s^8+\frac{459}{512}s^6-\frac{171}{256}s^4+\frac{21}{128}s^2\right)+$
	$+e^5\beta\left(-\frac{405}{1024}s^8+\frac{459}{512}s^6-\frac{171}{256}s^4+\frac{21}{128}s^2\right)+$

Delaunay generator at order 4 (*continued*)

No.	Expression
	$+\frac{e^5}{\eta}\Big(\frac{405}{1024}s^8-\frac{459}{512}s^6+\frac{171}{256}s^4-\frac{21}{128}s^2\Big)\Big]\sin(3f-2g)$
49	$\delta^3\Theta\frac{\alpha^8}{p^8}J_2^4\Big[e^3\beta^3\Big(-\frac{27}{128}s^8+\frac{9}{16}s^6-\frac{9}{16}s^4+\frac{1}{4}s^2-\frac{1}{24}\Big)+$
	$+e^3\beta^2\Big(-\frac{1647}{1024}s^8+\frac{153}{32}s^6-\frac{681}{128}s^4+\frac{21}{8}s^2-\frac{31}{64}\Big)+$
	$+e^3\beta\Big(-\frac{1647}{1024}s^8+\frac{153}{32}s^6-\frac{681}{128}s^4+\frac{21}{8}s^2-\frac{31}{64}\Big)+$
	$+\frac{e^3}{\eta}\Big(\frac{189}{128}s^8-\frac{1233}{256}s^6+\frac{1473}{256}s^4-3s^2+\frac{37}{64}\Big)+$
	$+\frac{e^3}{\eta^3}\Big(-\frac{1107}{4096}s^8+\frac{369}{512}s^6-\frac{369}{512}s^4+\frac{41}{128}s^2-\frac{41}{768}\Big)\Big]\sin 3f$
50	$\delta^3\Theta\frac{\alpha^8}{p^8}J_2^4\Big[e\eta\Big(-\frac{1215}{1024}s^8+\frac{1377}{512}s^6-\frac{513}{256}s^4+\frac{63}{128}s^2\Big)+$
	$+e\Big(\frac{3375}{1024}s^8-\frac{3825}{512}s^6+\frac{1425}{256}s^4-\frac{175}{128}s^2\Big)+$
	$+\frac{e}{\eta}\Big(\frac{405}{1024}s^8-\frac{459}{512}s^6+\frac{171}{256}s^4-\frac{21}{128}s^2\Big)\Big]\sin(3f+2g)$
51	$\delta^3\Theta\frac{\alpha^8}{p^8}J_2^4\Big[e^6\beta^2\Big(-\frac{405}{8192}s^8+\frac{459}{4096}s^6-\frac{171}{2048}s^4+\frac{21}{1024}s^2\Big)+$
	$+e^6\beta\Big(-\frac{405}{8192}s^8+\frac{459}{4096}s^6-\frac{171}{2048}s^4+\frac{21}{1024}s^2\Big)+$
	$+\frac{e^6}{\eta}\Big(\frac{405}{8192}s^8-\frac{459}{4096}s^6+\frac{171}{2048}s^4-\frac{21}{1024}s^2\Big)\Big]\sin(4f-2g)$
52	$\delta^3\Theta\frac{\alpha^8}{p^8}J_2^4\Big[e^4\beta^3\Big(-\frac{81}{512}s^8+\frac{27}{64}s^6-\frac{27}{64}s^4+\frac{3}{16}s^2-\frac{1}{32}\Big)+$
	$+e^4\beta^2\Big(-\frac{1593}{4096}s^8+\frac{1125}{1024}s^6-\frac{597}{512}s^4+\frac{141}{256}s^2-\frac{25}{256}\Big)+$
	$+e^4\beta\Big(-\frac{1593}{4096}s^8+\frac{1125}{1024}s^6-\frac{597}{512}s^4+\frac{141}{256}s^2-\frac{25}{256}\Big)+$

Delaunay generator at order 4 (*continued*)

No.	Expression
	$+\frac{e^4}{\eta}\Big(\frac{3051}{8192}s^8-\frac{2259}{2048}s^6+\frac{2499}{2048}s^4-\frac{153}{256}s^2+\frac{7}{64}\Big)+$
	$+\frac{e^4}{\eta^3}\Big(-\frac{405}{4096}s^8+\frac{135}{512}s^6-\frac{135}{512}s^4+\frac{15}{128}s^2-\frac{5}{256}\Big)\Big]\sin 4f$
53	$\delta^3\Theta\frac{\alpha^8}{p^8}J_2^4\Big[e^2\eta\Big(-\frac{1215}{8192}s^8+\frac{1377}{4096}s^6-\frac{513}{2048}s^4+\frac{63}{1024}s^2\Big)+$
	$+e^2\Big(\frac{1215}{4096}s^8-\frac{1377}{2048}s^6+\frac{513}{1024}s^4-\frac{63}{512}s^2\Big)+$
	$+\frac{e^2}{\eta}\Big(\frac{405}{8192}s^8-\frac{459}{4096}s^6+\frac{171}{2048}s^4-\frac{21}{1024}s^2\Big)\Big]\sin(4f+2g)$
54	$\delta^3\Theta\frac{\alpha^8}{p^8}J_2^4\Big[e^5\beta^3\Big(-\frac{81}{2048}s^8+\frac{27}{256}s^6-\frac{27}{256}s^4+\frac{3}{64}s^2-\frac{1}{128}\Big)+$
	$+e^5\beta^2\Big(-\frac{243}{4096}s^8+\frac{81}{512}s^6-\frac{81}{512}s^4+\frac{9}{128}s^2-\frac{3}{256}\Big)+$
	$+e^5\beta\Big(-\frac{243}{4096}s^8+\frac{81}{512}s^6-\frac{81}{512}s^4+\frac{9}{128}s^2-\frac{3}{256}\Big)+$
	$+\frac{e^5}{\eta}\Big(\frac{243}{4096}s^8-\frac{81}{512}s^6+\frac{81}{512}s^4-\frac{9}{128}s^2+\frac{3}{256}\Big)+$
	$+\frac{e^5}{\eta^3}\Big(-\frac{81}{4096}s^8+\frac{27}{512}s^6-\frac{27}{512}s^4+\frac{3}{128}s^2-\frac{1}{256}\Big)\Big]\sin 5f$
55	$\delta^3\Theta\frac{\alpha^8}{p^8}J_2^4\Big[e^6\beta^3\Big(-\frac{27}{8192}s^8+\frac{9}{1024}s^6-\frac{9}{1024}s^4+\frac{1}{256}s^2-\frac{1}{1536}\Big)+$
	$+e^6\beta^2\Big(-\frac{81}{16384}s^8+\frac{27}{2048}s^6-\frac{27}{2048}s^4+\frac{3}{512}s^2-\frac{1}{1024}\Big)+$
	$+e^6\beta\Big(-\frac{81}{16384}s^8+\frac{27}{2048}s^6-\frac{27}{2048}s^4+\frac{3}{512}s^2-\frac{1}{1024}\Big)+$
	$+\frac{e^6}{\eta}\Big(\frac{81}{16384}s^8-\frac{27}{2048}s^6+\frac{27}{2048}s^4-\frac{3}{512}s^2+\frac{1}{1024}\Big)+$
	$+\frac{e^6}{\eta^3}\Big(-\frac{27}{16384}s^8+\frac{9}{2048}s^6-\frac{9}{2048}s^4+\frac{1}{512}s^2-\frac{1}{3072}\Big)\Big]\sin 6f$

References

Aksnes, K.: 1965a, ‘On the dynamical theory of a near-earth satellite, I’, *Astrophys. Norv.* **10**, 69–77.

Aksnes, K.: 1965b, ‘On the dynamical theory of a near-earth satellite, II’, *Astrophys. Norv.* **10**, 149–169.

Brouwer, D.: 1959, ‘Solution of the problem of artificial satellite theory without drag’, *Astron. J.* **64** (1274), 378–397.

Davenport, J. H., Siret, Y. and Tournier, E.: 1988, *Computer Algebra: Systems and Algorithms for Algebraic Computation*, Academic Press, London.

Delauanay, C.: 1846, ‘Mémoire sur une nouvelle méthode pour la détermination du mouvement de la Lune’, *C.R. Acad. Sc. Paris* **22**, 32–36.

Delaunay, C.: 1855, ‘Sur une méthode d’intégratin applicable au calcul de perturbations de planètes et de laurs satellites’, *C.R. Acad. Sc. Paris* **40**, 335–338.

Deprit, A.: 1969, ‘Canonical transformations depending on a small parameter’, *Celest. Mech. & Dyn. Astr.* **1**, 12–30.

Deprit, A.: 1981, ‘The elimination of the parallax in satellite theory’, *Celest. Mech. & Dyn. Astr.* **24**, 111–153.

Deprit, A.: 1982, ‘Delaunay normalisations’, *Celest. Mech. & Dyn. Astr.* **26**, 9–21.

Deprit, A. and Coffey, S. L.: 1982, ‘Third-order solution to the main problem in satellite theory’, *J. Guidance, Control and Dynamics* **5**, 366–371.

Deprit, A. and Ferrer, S.: 1989, ‘Simplifications in the theory of artificial satellites’, *J. Astro. Sci.* **37**(4), 451–463.

Deprit, A. and Miller, B.: 1989, ‘Simplify or perish’, *Celest. Mech. & Dyn. Astr.* **45**, 189–200.

Jeffreys, W. H.: 1971, ‘Automated, closed form integration of formulas in elliptic motion’, *Celest. Mech. & Dyn. Astr.* **3**, 390–394.

Kelly, T. J.: 1989, ‘A note on first-order normalizations of perturbed Keplerian systems’, *Celest. Mech. & Dyn. Astr.* **46**(1), 19–25.

Knuth, D. E.: 1992, Literate programming, No. 27 in CSLI Lecture Notes, Center for the Study of Language and Information, QA 76.6.K644.

Lamport, L.: 1994, *LaTeX : A Document Preparation System*, 2nd edn, Addison-Wesley.

Miller, B. R. and Deprit, A.: 1986, ‘Experiments with LISP mechnies in non-linear dynamics’, Talk presented at the conference on Computers and Mathematics.

Poincaré, H.: 1905, *Leçon de Méchanique Céleste*, Gauthier-Villars, Paris.

Smart, W. M.: 1953, *Celestial Mechanics*, Longmans, Green and Co., London.

Tisserand, F.: 1888, *Traité de Mécanique Céleste*, Vol. 1. Gauthier-Villars, Paris.