ABSTRACT

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The principal topic of this thesis is the study of Riemannian submersions and Riemannian foliations of Lie groups. We classify Riemannian submersions of tori T with left invariant (hence bi-invariant) metric. In this case, the fiber through the identity F_e of the torus is a subgroup of T, and the submersion itself must be the coset projection $T \to T/F_e$. Then we characterize Riemannian submersions $f : G \to H$ where G and H are Lie groups with left invariant metric. These are homomorphisms if and only if the basic lift of a left invariant vector field in H is a left invariant vector field in G. A similar theorem is proven to characterize antihomomorphisms. We then study a Ricatti equation associated with a Riemannian foliation, giving new proofs of several results of Walschap. We also give a relationship that must hold between the fiber dimension and base dimension of a Riemannian submersion, thus partially answering a question of Wilhelm. We then

turn to the study of homogeneous Riemannian foliations. First we give a sufficient condition for a Riemannian foliation to be homogeneous, and use it to show that any Riemannian submersion of a compact Lie group with totally geodesic connected fibers is homogeneous if the fiber through the identity is a Lie subgroup. This is a result related to work of Ranjan [19]. Finally, we discuss the homogeneity of flat foliations of symmetric spaces of compact type, culminating in a proof that compact simple Lie groups with bi-invariant metric do not admit Riemannian foliations of codimension 1 with closed leaves.

RIEMANNIAN SUBMERSIONS AND LIE GROUPS

by

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Chapter 1

Introduction

The central topic of this thesis is the study of Riemannian submersions. A *Riemannian submersion* is a smooth submersion $f : M \to B$ between two Riemannian manifolds (M, g) and (B, h) with the property that at any point $p \in M$,

$$g_p(v, w) = h_{f(p)}(f_*(v), f_*(w))$$

for any v, w in the tangent space M_p to M at p, that are perpendicular to the kernel of f_*. Here g and h are the metrics for M and B, respectively. A Riemannian foliation of (M, g) is a smooth foliation of M locally given by Riemannian submersions. One can characterize this notion as follows. A foliation $\mathcal{F}$ is Riemannian if and only if any geodesic that starts out perpendicular to a leaf of the foliation, stays perpendicular to any leaf of the foliation it meets. The definition makes the notion of Riemannian submersion a natural counterpart to the notion of isometric immersion. Moreover, every known metric (B, h) of positive curvature is the base space of a Riemannian submersion of (G, g), where G is a compact Lie group with some left-invariant metric g. Actually one construction suffices to cover all known cases. One takes a subgroup H of $G \times G$ and lets H act on G in the following manner:

$$(h_1, h_2) \cdot g = h_1 g h_2^{-1}$$

for $(h_1, h_2) \in H$ and $g \in G$. If this action is free, then the orbit space $G//H$ is a manifold, called a *biquotient*. If one takes the metric on G to be left invariant and invariant by the action of H, one gets an action of H on G by isometries, and then there is a unique metric on $G//H$ such that the quotient map is a Riemannian submersion. If one is lucky, this quotient metric has positive sectional curvature. If one takes the metric on G to be bi-invariant, the quotient in any case will be seen to admit a metric of non-negative sectional curvature. This is because bi-invariant metrics on a compact Lie group have non-negative sectional curvature, and Riemannian submersions are known to be curvature nondecreas ing. This idea was first exploited in this way in [10] to construct a metric of non-negative sectional curvature on a seven dimensional exotic sphere. It was later used in [5], [6] to construct metrics of positive curvature on certain biquotients of $SU(3)$, and in [1] to construct metrics of positive curvature on certain biquotients of $SU(5)$. These metrics, together with the various homogeneous examples, make up almost all of the list of known examples of complete Riemannian metrics of positive sectional curvature. In fact, every known diffeomorphism type of manifold admitting positive sectional curvature is a biquotient, but recently Dearricott has found positively curved metrics on certain Eschenburg spaces which are not biquotient metrics (see [3]). So the question to which this thesis is devoted is this: how far can this idea be pushed? In other words, try to classify all Riemannian submersions of, say, a given homogeneous space. This kind of question was asked in connection with the search for new positively curved manifolds in [13]. This has been done in the past for Riemannian submersions of Euclidean spheres, for instance in [18], [7], [25], [8], [9]. Also a partial study was made of Riemannian submersions of compact Lie groups with totally geodesic fibers, concluding that under certain hypotheses, they were projection maps to coset spaces [19]. So a natural first question in this direction is then the following. Is it the case that any Riemannian foliation of a compact Lie group with bi-invariant metric is homogeneous? Namely, is there a Lie algebra of Killing fields for the metric on G which span the leaf directions? In this thesis we will examine several instances where the answer is yes. Our first case is the instance of tori, with bi-invariant metric. In chapter 3 we will show that any Riemannian submersion of a torus is the quotient of the torus by a subgroup, up to a fi nite cover. In chapter 4, we will examine the relationship between homomorphisms of Lie Groups and Riemannian submersions between Lie Groups. In chapter 5, we will study a Ricatti equation associated to a Riemannian foliation, and use it to derive restrictions on the existence of Riemannian foliations in positive and non-negative curvature. No characterization of homogeneous Riemannian foliations is known. In chapter 6, we will give a sufficient condition for a Riemannian foliation of a Riemannian manifold to be homogeneous, and conclude with a proof that there are no codimension 1 Riemannian foliations of compact simple Lie groups (with bi-invariant metric) with closed leaves.

Chapter 2

Basic Concepts

2.1 Basic Definitions

In this chapter, we will discuss the basic concepts associated to Riemannian submersions and recall results of others. O'Neill [15] defined certain basic tensors associated with them and wrote down formulas relating curvatures of base and total space. We will review this work and other work here. As stated before, a *Riemannian submersion* is a submersion $f : (M, g) \to (B, h)$ of smooth manifolds with the property that at any $p \in M$,

$$g_p(v, w) = h_{f(p)}(f_{*p}(v), f_{*p}(w))$$

for $v, w \in T_pM$ perpendicular to $\ker f_{*p}$. The integrable distribution $\mathcal{V}$ in M defined by the equation $\mathcal{V}_p = \ker f_{*p}$ is called the *vertical distribution* of the submersion. The distribution $\mathcal{H}$ defined by $\mathcal{H}_p = \mathcal{V}_p^\perp$ is called the *horizontal distribution* of the submersion. One also uses $\mathcal{V}$ and $\mathcal{H}$ to denote the orthogonal projections onto the distributions $\mathcal{V}$ and $\mathcal{H}$, respectively. Often one uses E^v or E^h to denote these projections as well. A vector field (or a vector) E on M is said to be *vertical* if $E^v = E$, and *horizontal* if $E^h = E$. One usually uses letters like V, W to denote vertical vectors and vector fields, and letters like X, Y to denote horizontal vector fields.

2.2 Basic Vector Fields

A vector field E is said to be *basic* if E is horizontal, and f-related to a vector field on B. That is, E is the horizontal lift of a vector field on the base manifold B. This is equivalent to saying that E is a horizontal vector field such that $[E, V]$ is vertical for any vertical vector field V. For any vector field $\bar{X}$ in B, there is a unique basic vector field X in M which is f-related to $\bar{X}$. X is called the *basic lift* of $\bar{X}$. Similarly, given $p \in M$ and a horizontal vector x at p, there is a basic vector field X on M such that $X_p = x$. To see this, extend $f_*(x)$ to a vector field $\bar{X}$ on B, then let X be the basic lift of $\bar{X}$.

2.3 The O'Neill tensors

In his paper [15], O'Neill defined two (2,1) tensors associated with the submersion f. They are defined as follows:

$$T_E F = (\nabla_{E^v} F^v)^h + (\nabla_{E^v} F^h)^v \tag{2.1}$$

which gives the second fundamental form on each fiber, and

$$A_E F = (\nabla_{E^h} F^v)^h + (\nabla_{E^h} F^h)^v \tag{2.2}$$

which can be viewed as the "second fundamental form" of the horizontal distribution $\mathcal{H}$. Here ∇ is the Levi-Civita connection of (M, g). We will record some of their basic properties in this section.

Lemma 1. *let $f : M \to B$ be a Riemannian submersion, and let the O'Neill tensors*

A and T be defined by the equations (2.1) and (2.2) above. Then A and T have the following properties:

1. *They exchange horizontal and vertical subspaces. That is, T_E takes vertical vectors to horizontal vectors, and vice versa, and A_E does the same.*

2. *A_E and T_E are skew-symmetric operators. That is,*

$$< A_E F, G > = - < F, A_E G >,$$

$$< T_E F, G > = - < F, T_E G >,$$

   where $<, >$ denotes the metric on M.

3. *$T_V W = T_W V$, and $A_X Y = -A_Y X$.*

4. *on horizontal fields, $A_X Y = (1/2)[X, Y]$.*

Some authors prefer to define the tensors $A : \mathcal{H} \times \mathcal{H} \to \mathcal{V}$ and $S : \mathcal{H} \times \mathcal{V} \to \mathcal{V}$, using the formulas $A_X Y = (\nabla_X Y)^v$ and $S_X V = -(\nabla_V X)^v$, and then work with the adjoints where necessary. This leads to cosmetic changes in some formulas, and in Chapter 5, these cosmetic changes will lead to a derivation of a useful equation.

2.4 Holonomy

The distribution $\mathcal{H}$ is a connection (in the sense of Ehresmann [4]) on the bundle f. Thus, given any smooth path $c : [0, 1] \to B$ in B, and any point p in $f^{-1}(c(0))$, there is a unique curve $\bar{c}$ in M starting at p, such that $f \circ \bar{c} = c$ and $\bar{c}$ is horizontal, ie $\dot{\bar{c}}(t) \in \mathcal{H}_{c(t)}$ for all t. This enables one to compare the fibers of f over different points of the curve c in B. To do this, one defines a diffeomorphism $h_c : F_{c(0)} \to F_{c(1)}$, called the *holonomy diffeomorphism.* (Here F_b denotes the fiber of f over the point $b \in B$). It is defined by sending $p \in F_{c(0)}$ to $\bar{c}(1) \in F_{c(1)}$, the endpoint of the unique lift $\bar{c}$ of c starting at p.

We now show that $h_c : F_{c(0)} \to F_{c(1)}$ is a smooth map with smooth inverse $h_{\tilde{c}}$, where $\tilde{c}$ is the path c traversed in reverse. To see that h_c is smooth, choose a tubular neighborhood U of c in B, and choose a smaller tubular neighborhood V of c such that $\bar{V} \subseteq U$. One can then choose a smooth vector field $\tilde{X}$ in B with the following properties: 1) $\tilde{X}_{c(t)} = \dot{c}(t)$ whenever $0 \leq t \leq 1$, and 2) $\tilde{X} = 0$ outside of U. Let X be the basic lift of $\tilde{X}$. Then X is a smooth vector field on M with the property that the integral curve of X through $p \in F_{c(0)}$ is the horizontal lift of c starting at p. Therefore the local flow of X will carry $F_{c(0)}$ diffeomorphically onto $F_{c(t)}$ for $0 \leq t \leq 1$, since diffeomorphisms take submanifolds to submanifolds.

It follows that a Riemannian submersion f has the structure of a fiber bundle with structure group the group of diffeomorphisms of the fiber F_b, for any given $b \in B$. In fact the structure group is the group of holonomy diffeomorphisms associated to loops in the base, with group structure given by concatenation of loops. Notice that $h_{c_1} \circ h_{c_2} = h_{c_1 * c_2}$.

When c is a geodesic, the differential of h_c can be described in terms of Jacobi fields. In this case, since f is a Riemannian submersion, the horizontal lifts of c are geodesics of M. Choose a point $p \in F_{c(0)}$ and choose a curve $\alpha : (-\epsilon, \epsilon) \to F_{c(0)}$ representing a tangent vector v at p. At each point $\alpha(s)$ take the horizontal lift of c, starting at $\alpha(t)$, denoted by $c_{\alpha(s)}$. This gives you a variation $\theta(t, s) = c_{\alpha(s)}(t)$

of the geodesic $\bar{c}$ starting at $\alpha(0) = p$ through geodesics. Hence the variation field $J(t) = \theta_{*(t,0)}(\partial_s)$ of this variation is a Jacobi field along $\bar{c}$. But since the endpoints of the holonomy lifts generate a smooth curve $\tilde{\alpha}$ in $F_{c(1)}$, the tangent vector to $\tilde{\alpha}$ at $t = 0$ is $(h_c)_*(v)$. But then $J(1) = \tilde{\alpha}'(0) = \theta_{*(1,0)}(\partial_s)$.

It is also possible to derive a formula for the covariant derivative of J along a horizontal geodesic c. It is

$$J' = \nabla_{\dot{c}} J = -S_{\dot{c}}(J) - A^*_{\dot{c}}(J),$$

where $A^*_{\dot{c}}$ is the adjoint of $A_{\dot{c}}$. (We are, of course, using the other notation for the O'Neill tensors here). To see this, we compute the vertical and horizontal components of J'. Since $J'^h = -A^*_{\dot{c}} J$ by definition, we see that the horizontal component is as desired. To compute the vertical component of J', we again look at the variation $\theta(t, s)$ of c through horizontal geodesics. Let the variation fields associated to this variation be denoted by $X(t, s) = \theta_{*(t,s)}(\partial_t)$ for the horizontal direction and $W(t, s) = \theta_{*(t,s)}(\partial_s)$ for the vertical direction. Then $W(t, 0) = J(t)$, and $X(t, 0) = \dot{c}(t)$. It follows that $\nabla_X W = \nabla_W X$, since X and W are variation fields for the variation θ, and hence $J'^v(t) = (\nabla_X W)^v(t, 0) = (\nabla_W X)^v(t, 0) = -S_{\dot{c}}(J)$.

2.5 Parallel Fields

Another type of vector field associated to the bundle f are the *vertically parallel* fields. This comes from the connection $\hat{\nabla}$ on the vertical bundle $\mathcal{V}$, defined by the equation $\hat{\nabla}_Z V = (\nabla_Z V)^v$ for $Z \in \Gamma(TM)$ and $V \in \mathcal{V}$. Often, if we are discussing vector fields $V(t)$ along a curve c, we will write V'^v for $\hat{\nabla}_{\dot{c}}(V)$. Given a horizontal geodesic γ in M and a vertical vector v at $\gamma(0)$, there is a vertical vector field $V(t)$ along γ which is parallel with respect to $\hat{\nabla}$, ie $V'^v = 0$, and such that $V(0) = v$. A vector field $V(t)$ along a curve c such that $\hat{\nabla}_{\dot{c}} V = 0$, and which is also vertical, will be said to be *vertically parallel.* We get an associated notion of parallel transport which preserves the metric on $\mathcal{V}$. That is, if $v, w \in M_{\gamma(0)}$, the vertically parallel fields V, W along γ extending v, w satisfy $< v, w > = < V(t), W(t) >$ for all t. This is because

$$\begin{aligned} < V(t), W(t) >' &= < V(t)', W(t) > + < V(t), W(t)' > \\ &= < V(t)'^v, W(t) > + < V(t), W(t)'^v > \\ &= 0. \end{aligned}$$

Which means that $< V(t), W(t) >$ is constant along γ. In a later chapter, we will use vertically parallel fields along a horizontal geodesic to write down a matrix Ricatti equation satisfied by a Riemannian foliation $\mathcal{F}$.

2.6 Curvature Formulas

There exist a number of formulas using the O'Neill tensors to relate the curvature of the base and the total space of the bundle f. They were first derived by O'Neill and can be found in his paper [15]. We list them below for easy reference.

Lemma 2. *If U, V, W, F are vertical vector fields and X, Y, Z, H are horizontal vector fields, then the following equations hold:*

$$< R(V, U)W, F > = < \hat{R}(V, U)W, F > - < T_U W, T_V F > + < T_V W, T_U F > \quad (\{0\})$$

$$< R(V,U)W,X >=< (\nabla_V T)_U W,X > - < (\nabla_U T)_V W,X > \qquad (\{1\})$$

$$< R(V,X)Y,W >=< (\nabla_X T)_V W,Y > + < (\nabla_V A)_X Y,W > \qquad (\{2\})$$

$$- < T_V X, T_W Y > + < A_X V, A_Y W >$$

$$< R(Y,X)Z,V >=< (\nabla_Z A)_X Y,V > + < A_X Y, T_V Z > \qquad (\{3\})$$

$$- < A_Y Z, T_V X > - < A_Z X, T_V Y >$$

$$< R(Y,X)Z,H >=< R^*(Y,X)Z,H > -2 < A_X Y, A_Z H > \qquad (\{4\})$$

$$+ < A_Y Z, A_X H > + < A_Z X, A_Y H > .$$

Here $\hat{R}$ denotes the curvature tensor of the intrinsic metric on the fibers, and R^* is used to denote both the curvature tensor of the metric on B and its horizontal lift to M.

The convention used to define the curvature tensor is $R_{XY}Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$, the opposite of O'Neill's paper. Like O'Neill, I have numbered these curvature formulas by the number of horizontal vectors. Thus for example $\{0\}$ is the first formula on the above list, with no horizontal vectors. In the rest of this thesis I will refer to them in this way. From the above formulas, we get the following expressions for the sectional curvature of a two-plane in M. Before proceeding, recall that the metric $<,>$ on a Riemannian manifold N is extended to a fiber metric on the bundle $\pi : \wedge(TN)^k \to N$ in a canonical way. Namely $< A_1 \wedge \cdots \wedge A_k, B_1 \wedge \cdots \wedge B_k >= \det(< A_i, B_j >)$. The determinant in this last equation is the determinant of the matrix whose (i,j) entry is $< A_i, B_j >$.

Lemma 3. *Let $f : M \to B$ be a Riemannian submersion. If U, V are vertical vec-*

tors and X, Y are horizontal vectors, then the following formulas give the sectional curvature of a two-plane.

1. $K(U \wedge V) = \hat{K}(U \wedge V) - \frac{<T_U U, T_V V> - <T_U V, T_U V>}{\|U \wedge V\|^2}$.

2. $K(U \wedge X) = \frac{<(\nabla_X T)_V V, X> + <A_X V, A_X V> - <T_V X, T_V X>}{\|X\|^2 \|V\|^2}$.

3. $K(X \wedge Y) = K^*(X \wedge Y) - \frac{3<A_X Y, A_X Y>}{\|X \wedge Y\|^2}$.

Here $K(A \wedge B)$ denotes the sectional curvature of the two-plane spanned by the vectors A and B, $\hat{K}$ denotes the intrinsic curvature of the fibers, and K^* denotes the sectional curvature of the manifold B.

The first formula above is of course simply the Gauss formula giving the intrinsic curvature of a submanifold. The formula {2} will be very useful in the last chapter, and is the origin of the matrix Ricatti equation that we will derive and use there. The third formula in the above lemma implies that Riemannian submersions are curvature nondecreasing, and that the quantity

$$K(X \wedge Y) + \frac{3 < A_X Y, A_X Y >}{\|X \wedge Y\|^2}$$

is constant along the fibers when X and Y are basic.

2.7 Positive definite and Positive semi-definite Matrices

Here we will recall some facts about positive definite and positive semi-definite matrices, and turn them into a form more convenient for our use. Suppose M is a square symmetric matrix of order n. We say that M is *positive semi-definite* if and only if for any $X \in \mathbb{R}^n$, we have $X^t M X \geq 0$. We say that M is *positive definite* if and only if it is positive semi-definite and invertible. Given a set of vectors $\{v_1, \ldots, v_k\}$ in an inner product space $(V, <, >)$, we define their *Gram matrix* to be the matrix $G_{\{v_1,\ldots,v_k\}}$ whose (i,j) entry is $< v_i, v_j >$. Any Gram matrix is symmetric, since inner products are symmetric. Here is a theorem connecting these notions which will be needed in the sequel.

Theorem 4. *Let $\{v_1, \ldots, v_n\}$ be a set of vectors in the inner product space $(V, <, >)$. Then their Gram matrix $G_{\{v_1,\ldots,v_k\}}$ is positive semi-definite. If $\{v_1, \ldots, v_n\}$ is a linearly independent set, then $G_{\{v_1,\ldots,v_k\}}$ is positive definite. Conversely, suppose that M is a positive semi-definite matrix. Then there is an inner product space $(V, <, >)$ and a set of vectors $\{v_1, \ldots v_n\}$ in V such that $M = G_{\{v_1,\ldots,v_k\}}$. Therefore, the induced inner product on $\wedge^k(V)$ defined by $< v_1 \wedge \ldots \wedge v_k, w_1 \wedge \ldots \wedge w_k >= det(< v_i, w_j >)$ is positive definite.*

Proof. Any matrix M such that $M = PP^t$ is positive semidefinite, since in this case $X^t M X = X^t P P^t X = (P^t X)^t (P^t X) \geq 0$ for $X \in \mathbb{R}^n$. To write $G_{\{v_1,\ldots,v_k\}}$ in this way, choose an orthonormal basis $\{e_1, \ldots, e_n\}$ of V, and let $P_{ij} =< v_i, e_j >$. Then $G_{\{v_1,\ldots,v_k\}} = PP^t$. If the vectors form a linearly independent set, then the matrix P is invertible, and hence $G_{\{v_1,\ldots,v_k\}}$ is positive definite. Conversely, if M is a positive semi-definite matrix, then by the spectral theorem there is an orthogonal matrix Q such that $QMQ^t = D$, where D is a diagonal matrix with non-negative entries. Therefore, defining $P = Q^t\sqrt{D}$, where $(\sqrt{D})_{ij} = \sqrt{D_{ij}}$, implies that $M = PP^t$. Now choose an inner product space $(V, <, >)$ of dimension equal to the order of M, and an orthonormal basis $\{e_1, \ldots, e_n\}$ of V. Setting

$$v_i = \sum_k P_{ik} e_k$$

shows that $M_{ij} =< v_i, v_j >$, and hence that $M = G_{\{v_1,\ldots,v_k\}}$. For the last statement, recall that a set of vectors $\{v_1, \ldots, v_k\}$ is linearly independent if and only if $v_1 \wedge \ldots \wedge v_k \neq 0$. So if $v_1 \wedge \ldots \wedge v_k \neq 0$, then

$$< v_1 \wedge \ldots \wedge v_k, v_1 \wedge \ldots \wedge v_k >= \det(G_{\{v_1,\ldots,v_k\}}) > 0,$$

since we know that $G_{\{v_1,\ldots,v_k\}}$ is positive definite in this case. $\square$

Chapter 3

Riemannian Submersions of Tori

3.1 Statement of Theorem

In this chapter we will classify Riemannian submersions $\pi : (T, g) \to (B, h)$ where $T = (S^1)^{n+k}$ is an $(n+k)$-dimensional torus with left-invariant metric g, and (B, h) is a Riemannian manifold of dimension n. Recall that a Riemannian metric g on a Lie group G is said to be *bi-invariant* if and only if the left and right translation maps defined by $L_y(x) = yx$ and $R_y(x) = xy$ are isometries of the metric for all $y \in G$. Any left invariant metric on a torus is automatically right invariant since Tori are abelian Lie Groups. Here is the statement of the theorem we will prove in this chapter.

Theorem 5. *Let $\pi : T^{n+k} \to B^n$ be a Riemannian submersion of a torus with bi-invariant metric.*

1. *Suppose that the fibers of T are connected. Then F_e, the fiber of π containing the identity of T is a closed subgroup of T, B is a torus, diffeomorphic to the coset space T/F_e, and in fact π is the standard projection defined by $\pi(x) = xF_e$ for $x \in T$.*

2. *Suppose the fibers of π are disconnected. Then there is a torus $\bar{B} = (T^n, g)$ with bi-invariant metric g, a local isometric covering map $\varphi : \bar{B} \to B^n$ and*

a Riemannian submersion $\bar{\pi} : T^{n+k} \to \bar{B}$ *with connected fibers, such that* $\pi = \varphi \circ \bar{\pi}$.

3.2 Bi-invariant metrics on Tori

There are many possible bi-invariant metrics on T, so our first step is to describe the bi-invariant metrics on T in a useful way. We do this in the following lemma. In this chapter d_m will denote the standard metric on $\mathbb{R}^m$.

Lemma 6. *Let* (T, g) *be an* m*-dimensional torus with bi-invariant metric* g*. Then* (T, g) *is isometric to* $(\mathbb{R}^m, d_m)/\mathcal{L}$*, where* $\mathcal{L}$ *is a free subgroup of* $(\mathbb{R}^m, +)$ *of rank* m *and* d_m *is the standard metric on* $\mathbb{R}^m$*.*

Proof. Think of T as $\mathbb{R}^m/\mathbb{Z}^m$, where $\mathbb{Z}^m$ is the standard lattice in $\mathbb{R}^m$, generated by the standard basis vectors $\{e_1, \ldots e_m\}$. Pull back the metric g on T to $\mathbb{R}^m$. This gives $\mathbb{R}^m$ a bi-invariant metric since the covering map $\phi : \mathbb{R}^m \to T$ is a Lie group homomorphism. Since $(\mathbb{R}^m, +)$ is abelian, this metric is flat. Since $\mathbb{R}^m$ is simply connected, $(\mathbb{R}^m, (\phi)^*(g))$ is isometric to $\mathbb{R}^m$ with its standard metric d_m. Choose an isometry $\theta : (\mathbb{R}^m, (\phi)^*(g)) \to (\mathbb{R}^m, d_m)$, and let $\mathcal{L}$ be the subgroup of $(\mathbb{R}^m, +)$ generated by the images of the standard basis vectors $\{e_1, \ldots e_m\}$. Then $\mathcal{L}$ is a free subgroup of $\mathbb{R}^m$ of rank m, and θ induces an isometry $\hat{\theta} : (T, g) \to (\mathbb{R}^m, d_m)/\mathcal{L}$. $\square$

Thus from now on, we will think of (T, g) as $(\mathbb{R}^{n+k}, d_{n+k})/\mathcal{L}$, where $\mathcal{L}$ is a free lattice subgroup of rank m in $\mathbb{R}^{n+k}$.

3.3 Proof in the case of connected fibers

Now, let $\pi : T^{n+k} \to B^n$ be a Riemannian submersion, where the superscripts denote the dimensions of the manifolds involved. To analyze π we will pass to a Riemannian submersion $\pi_1 : \mathbb{R}^{n+k} \to \tilde{B}$ between the universal covers of T and B so that we have a commutative diagram:

$$\begin{array}{ccc} \mathbb{R}^{n+k} & \xrightarrow{p} & T^{n+k} \\ \pi_1 \downarrow & & \pi \downarrow \\ \tilde{B} & \xrightarrow{\bar{p}} & B \end{array}$$

In this case π_1 exists by covering space theory, and is smooth because the covering maps $p : \mathbb{R}^{n+k} \to T$ and $\bar{p} : \tilde{B} \to B$ are local diffeomorphisms. We give $\mathbb{R}^{n+k}$ and $\tilde{B}$ the pullback metrics, so that the covering projections p and $\bar{p}$ are local isometries. Notice that the domain of p, $\mathbb{R}^{n+k}$, then gets the standard metric d_{n+k}, by the preceding lemma.

We can therefore analyze π_1 using theorems of Gromoll and Walschap [11], [12]. They proved that any Riemannian submersion of Euclidean space $(\mathbb{R}^m, d_m)$ with connected fibers was the orbit map of an isometric action on $\mathbb{R}^m$ by generalized glide rotations. Since we will use this theorem in a key way, we quote the main result of their two papers here.

Lemma 7. *(Gromoll-Walschap) Suppose that $f : (\mathbb{R}^{n+k}, d_{n+k}) \to M^n$ is a Riemannian submersion of Euclidean space with connected fibers. Then*

1. *The fiber F over a soul of M is an affine subspace of Euclidean space, which, up to congruence, may be taken to be $F = \mathbb{R} \times 0$.*

2. *Furthermore, f is the orbit fibration of the free isometric group action ψ of $\mathbb{R}^k$ on $\mathbb{R}^{n+k} = \mathbb{R}^k \times \mathbb{R}^n$ given by*

$$\psi(v)(u, x) = (u + v, \phi(v)x), \tag{3.1}$$

where $u, v \in \mathbb{R}^k$, $x \in \mathbb{R}^n$, and $\phi : \mathbb{R}^k \to SO(n)$ is a homomorphism.

Recall that a *soul* of a non-negatively curved manifold M is a compact, totally convex submanifold without boundary of M. In the case of the theorem each soul of M is a point ([2]).

Therefore we will know that π_1 is the orbit map of an isometric group action of $\mathbb{R}^k$ on $\mathbb{R}^{n+k}$, once we establish that π_1 is a Riemannian submersion of $(\mathbb{R}^{n+k}, d_{n+k})$ with connected fibers.

We've established that π_1 is a smooth submersion above, so we first need to check that π_1 is a Riemannian submersion. To see this, pick $b \in \mathbb{R}^{n+k}$, and pass to the differentials of each map at p. Then we have that p_{*b} and $\bar{p}_{*\pi_1(b)}$ are isometries on their respective tangent spaces. Since we now have a commutative diagram of tangent maps, we see that p_{*b} takes ker $(\pi_1)_{*b}$ isomorphically to ker π_*, and hence for v, w orthogonal to ker $(\pi_1)_{*b}$, we have

$$\begin{aligned} < v, w > &=< p_{*b}(v), p_{*b}(w) > \\ &=< (\pi \circ p)_{*b}(v), (\pi \circ p)_{*b}(w) > \\ &=< (p_1 \circ \pi_1)_{*b}(v), (p_1 \circ \pi_1)_{*b}(w) > \\ &=< (\pi_1)_{*b}(v), (\pi_1)_{*b}(w) > . \end{aligned}$$

Therefore π_1 is a Riemannian submersion. The map π_1 is therefore a fibration (see

Chapter 2), and therefore the fact that the fibers are connected follows by looking at the long exact sequence of the fibration π_1.

Let ψ be the action guaranteed by the theorem of Gromoll and Walschap, and let $\phi : \mathbb{R}^k \to SO(n)$ be the Lie group homomorphism generating the action. The rest of the proof consists of analyzing the action (3.1) under the assumption that π satisfies the relation

$$p_1 \circ \pi_1 = \pi \circ p.$$

The objective is to show that ϕ is the trivial homomorphism. Then we will know that the action ψ is given by

$$\psi(v)(u, x) = (u + v, x).$$

This will mean that the foliation generated by π_1 is the one whose leaves are the cosets of F. We begin by studying the fiber F guaranteed by the theorem of Gromoll and Walschap. Observe that F is an orbit of the action ψ generating π_1. Our first aim is to show that the cosets $F + v$ are also orbits of ψ, for $v \in \mathcal{L}$. By construction, F maps to a single point in $\tilde{B}$ and therefore to a single point in B, which we call q. Fix $v \in \mathcal{L}$. By definition, $p(F + v) = p(F)$, and therefore $\pi \circ p(F) = \pi \circ p(F + v) = q$. Choose $f_1, f_2 \in F$, and observe that by construction, $\pi_1(f_1 + v)$ and $\pi_1(f_2 + v)$ are points in $\bar{p}^{-1}(q)$. Since $\bar{p}^{-1}(q)$ is discrete (as $\bar{p}$ is a covering map), and $F + v$ is connected, $\pi_1(f_1 + v) = \pi_1(f_2 + v)$, so that π_1 is constant on $F + v$. Since $F + v$ is diffeomorphic to F, this means that $F + v$ is also an orbit of ψ for $v \in \mathcal{L}$.

Now that we know that the cosets $F + v$, for $v \in \mathcal{L}$ are also orbits of ψ and fibers of π_1, we can choose $x \in F = \mathcal{R}^{n+k}$, and calculate $\phi(x) \in SO(n)$. To do this, let $F^\perp$ be the orthogonal complement of F in $\mathbb{R}^{n+k}$, that is $F^\perp = \{(0, y) \mid y \in \mathbb{R}^{n+k}\}$. Fix $v \in \mathcal{L}$ again, and let $(0, w)$ be the unique point of interesection of $F + v$ and $F^\perp$. Observe that $(0, w)$ is simply the orthogonal projection of v on $F^\perp$. Since we have seen that $F + v$ is also an orbit, then the points $(x, \phi(x)w)$ must also lie in $F + v$. But the orthogonal projection of any point (a, b), where $a \in \mathbb{R}^k$ and $b \in \mathbb{R}^n$ on $F^\perp$ is $(0, b)$. Since the orthogonal projection of $F + v$ on $F^\perp$ is $(0, w)$, we see that $\phi(x)w = w$.

The orthogonal projection onto $F^\perp$ is a linear surjection; therefore since $\mathcal{L}$ is a basis of $\mathbb{R}^{n+k}$, we see that $\mathcal{L}$ must project onto a sp anning set of $F^\perp$. On the other hand, we can apply the argument in the previous paragraph to all the vectors in $\mathcal{L}$. Therefore we conclude that $\phi(x)w = w$, where w ranges through a spanning set in $F^\perp = \mathbb{R}^n$. Therefore $\phi(x) = I \in SO(n)$ for all $x \in \mathbb{R}^k$, and therefore ϕ is the trivial homomorphism.

Now assume the fibers of π are connected. We know that any fiber of π_1 must map onto a fiber of π. Also the fibers of π_1 are affine subspaces, which are mapped by p onto closed submanifolds of T (namely, the fibers of π). Therefore the fibers of π are totally geodesic since p is a local isometry. Since p is a homomorphism of Lie groups, the fiber F_e of π containing the identity is a closed subgroup of T, being the image under p of the fiber $\tilde{F}_e$of π_1 containing the identity. Moreoever, the other fibers of π are cosets of F_e, since they are images of the cosets of $\tilde{F}_e$ under p. $\square$

3.4 Proof of Part 2

Here the fibers are disconnected. First an example to show that this can actually happen.

Example The standard 2 torus T^2 is a double cover of the Klein bottle K. So precomposing the covering projection p with the standard projection $\pi : T^3 \to T^2$, we get a Riemannian submersion $p \circ \pi$ with disconnected fibers if we give T^3, T^2, and K their standard metrics. Observe that this submersion is a composition of a covering map together with a Riemannian submersion with connected fibers, which is what we want to prove in general. We now proceed to the proof of this statement.

Proof of Part 2. Suppose that $\pi : T^{n+k} \to B$ is a Riemannian submersion with disconnected fibers. We again study the lift of π to the universal covers of T and B, getting a Riemannian submersion $\pi_1 : \mathbb{R}^{n+k} \to \tilde{B}$. The fibers of π_1 are again connected, and as before the fibers of π are t he cosets of some affine subspace F of $\mathbb{R}^{n+k}$. Therefore, as before, the fiber of π_1 containing the identity, namely F, is mapped under p to some closed subgroup of T, $p(F) = F_e$ Moreover, the connected components of the fibers of π are cosets of this subgroup.

This lets us define the torus in the theorem and the Riemannian submersion: $\bar{B} = T^{n+k}/F_e$, and $\bar{\pi}$ is the orbit map. Then $\varphi : T^n \to B$ is defined as $\varphi(xF_e) = \pi(x)$. This is well defined, since if x and y belong to the same coset of F_e, then they must be in the same fiber of π.

Since π_1 is a Riemannian submersion, and $\bar{p}$ is a covering map, we conclude that there is a group of isometries of T acting transitively on the connected components of π. This action descends to a covering space action of $\bar{B}$. Therefore φ is a covering space projection. $\square$

Chapter 4

Riemannian Submersions with Domain and Image a Lie Group

In this chapter we will consider the set of Riemannian submersions $f : G \to H$, where G and H have left-invariant metrics, and characterize the ones which are homomorphisms and the ones which are antihomomorphisms. We will also prove a similar result for Riemannian foliations. In this chapter $L_g : K \to K$ will denote the map $L_g(x) = gx$ and $R_g : K \to K$ will denote the map $R_g(x) = xg$, for K a Lie group. The letter e_G will always be used to denote the identity element of the group G, and similarly e_H, e_K and so on.

Here is the main theorem of this chapter.

Theorem 8. *Suppose that $f : G \to H$ is a Riemannian submersion of Lie groups with connected fibers, where G and H both have left-invariant metrics. Suppose also that $f(e_G) = e_H$. Then:*

1. *f is a homomorphism if and only if the basic lift of any left invariant vector field in H is a left invariant vector field in G.*

2. *f is an antihomomorphism if and only if the basic lift of any right invariant vector field in H is a left invariant vector field in G.*

Proof.

1. Suppose that f is a Lie group homomorphism. Let X be a left invariant vector

field on H, and let $\bar{X}$ be its horizontal lift to G. We want to show that $\bar{X}$ is left invariant. Since f is a homomorphism, we have $(f \circ L_g)(x) = (L_{f(g)} \circ f)(x))$, for any $g \in G$. Since $\bar{X}$ is the basic lift of X, we also have $(df) \circ \bar{X} = X \circ f$, where df is the differential of f and we think of X and $\bar{X}$ as sections of the tangent bundles of H and G, respectively. Therefore any left invariant vector field V on G which is vertical at $e \in G$, is a vertical vector field:

$$(df)_p(V_p) = ((df)_p \circ (dL)_e)(V_e) = (dL \circ df)_e(V_e) \quad = (dL)_{f(e)}((df)_e(V_e)) = 0.$$

Now let X' be the unique left invariant vector field X' such that $X'_e = \bar{X}_e$. Then I claim that $X' = \bar{X}$. First, X' is horizontal, since X' is left invariant, and therefore for any vertical V (which we can take to be left-invariant by above)

$$< V_p, X'_p >=< V_e, X'_e >= 0$$

for any $p \in G$, since the metric is assumed to be left-invariant, and X'_e is horizontal. On the other hand, since X' is left invariant, we have

$$(df)_p(X'_p) = (df \circ (dL_p)_e(X'_e) = (dL_{f(p)} \circ df)_e(X'_e) \quad = (dL_{f(p)})_p(X_f(e)) = X_{f(p)}$$

since X is left invariant.

Conversely, suppose that $f : G \to H$ is a Riemannian submersion with the property that basic lifts of left-invariant vector fields are left-invariant. First we show that F_e, the fiber of f containing e_G, is a normal subgroup of G. To see this, take $v \in T_eG$ with the property that v is orthogonal to $\mathcal{H}_e$, the horizontal space of f at e, and extend v to a left-invariant vector field V. Then since $\mathcal{H}$ can be taken to be spanned by left invariant vector fields, we see that V is a vertical vector field, since for all p, and all basic X we must have that $< V_p, X_p >=< V_e, X_e >= 0$. On the other hand $\mathcal{V}$ is an integrable distribution, since any two vector fields belonging to $\mathcal{V}$ are f-related to the zero section of H. This shows that $\mathcal{V}$ is a Lie subalgebra of $\mathfrak{g}$, the Lie Algebra of G. In fact, $\mathcal{V}$ is spanned by left-invariant vector fields, and is also an ideal, since if X and Y are basic fields, and V is vertical and left invariant, we have $< [V, X], Y >= 0$, since the bracket of a basic field and a vertical field is vertical. Therefore, if E is left invariant, we can write $E = E^h + E^v$ where E^h and E^v are left invariant. Therefore $[E, V]$ is vertical for any vertical, left-invariant V.

Therefore the fibers of f are the cosets $pF_e = (F_e)p$ of the normal subgroup F_e of G. Moreover, since the fibers of f coincide with the fibers of the group homomorphism $G \to G/F_e$, f must be a homomorphism.

2. First suppose that f is an antihomomorphism, that is $f(xy) = f(y)f(x)$ for all $x, y \in G$. In other words $f \circ L_x(y) = R_{f(x)}(y)$. Setting $y = exp(tv)$ and taking the derivative, we get the equation

$$(R_{f(x)})_{*e}(f_{*e}(v)) = (f_*)_e((L_x)_{*e}(v)$$

for $v \in G_e$. Thus by a similar argument as above, we conclude that the basic lift of a right invariant vector field in H is a left-invariant vector field in G.

Conversely if the basic lift of any right invariant vector field is left invariant, we reason in a similar manner as above, to conclude that the fibers of f

must coincide with the antihomomorphism $\bar{f} : G \to G/F_e$ defined by $\bar{f}(x) = x^{-1}F_e$. □

Chapter 5

Restrictions on Existence of Riemannian Foliations

In this chapter, we will derive restrictions on the existence of Riemannian foliations. We will show that under certain curvature assumptions, Riemannian foliations of certain types cannot exist. The main technique will be an analysis of a matrix Ricatti equation which we will derive in the first section below.

5.1 The Matrix Ricatti Equation of a Riemannian Foliation

In this section we will derive a matrix Ricatti equation satisfied by the leaves of a Riemannian foliation. To do this, let γ be a horizontal geodesic, and let S denote the second fundamental form of the leaves of the folation $\mathcal{F}$, so that

$$S_{\dot\gamma}V = -(\nabla_V \dot\gamma)^v,$$

for V a vertical vector at $\gamma(t)$. Now extend V to a vertically parallel field along γ as in Chapter 1. We will work with the connection

$$\hat\nabla_{\dot\gamma}V = V'^v = (\nabla_{\dot\gamma}V)^v$$

along the curve γ. Therefore

$$S'^v_{\dot\gamma}W \equiv (S_{\dot\gamma}W)'^v - S_{\dot\gamma}(W'^v),$$

for W a vector field along γ, since γ is a geodesic. In the case that W is vertically parallel, this means that the second term on the right hand side is zero, and so we find that

$$S_{\dot{\gamma}}^{\prime v} W \equiv (S_{\dot{\gamma}} W)^{\prime v}$$

Now we can state the equation from which we will derive our results in this chapter. Before doing so, here is some notation. We can choose an orthonormal basis $\{V_1, \ldots, V_k\}$ of vertically parallel fields along γ. With respect to this basis for $\mathcal{V}$ along γ, we can then define the $k \times k$ matrices $S(t)$, $R(t)$ and $B(t)$ by the following formulas:

(a) $S_{ij} =< S_{\dot{\gamma}} V_i, V_j >$,

(b) $R_{ij} =< R(V_i, \dot{\gamma})\dot{\gamma}, V_j >$,

(c) $B_{ij} =< A_{\dot{\gamma}} A_{\dot{\gamma}}^* V_i, V_j >$.

Theorem 9. *Suppose that $\mathcal{F}$ is a Riemannian foliation of (M, g), of leaf dimension k, and let γ be a horizontal geodesic. Then the following matrix Ricatti equation holds along γ:*

$$S' = S^2 + R - B$$

where $\{V_1, \ldots, V_k\}$ is an orthonormal basis of vertically parallel fields along γ, and the entries of the $k \times k$ matrices S, R and B are given by the formulas above.

In the above lemma, the matrices are therefore functions of the parameter t, and differentiation is done componentwise, so that the matrix S' is simply the matrix whose (i,j) entry is the derivative with respect to t of the (i,j) entry of S. We begin the proof by deriving an invariant form of the equation from O'Neill's curvature formula $\{2\}$ stated in Chapter 1. This seems to first have been found by Kim and Tondeur [14]. Walschap [22] gave a proof using holonomy Jacobi fields. We will give a different proof here, which mainly involves translating curvature equation $\{2\}$, which uses the O'Neill tensors T and A, into an equation using the tensors S and A, mentioned in the introduction.

Lemma 10. *Let $\mathcal{F}$ be a Riemannian foliation of (M,g), let S denote the second fundamental form of the fibers, and let A denote the A-tensor of the folation $\mathcal{F}$. Then the following equation holds along a horizontal geodesic γ.*

$$S'^{v} = S_{\dot\gamma}^2 + R_{\dot\gamma}^v - A_{\dot\gamma}A_{\dot\gamma}^*,$$

where $R_{\dot\gamma}^v(W) \equiv (R(W,\dot\gamma)\dot\gamma)^v$.

Proof of Lemma. We start with O'Neill's notation for A and T. Recall that for two vertical vectors and two horizontal vectors, we have

$$\begin{aligned} < R(V,X)Y, W > &= < (\nabla_X T)_V W, Y > + < (\nabla_V A)_X Y, W > \\ &\quad - < T_V X, T_W Y > + < A_X V, A_Y W > \end{aligned}$$

where R is the curvature tensor of (M,g). T hus, setting $X = Y = \dot\gamma$ and recalling that $\nabla_E A$ is skew symmetric gives us

$$< R(V,\dot\gamma)\dot\gamma, W > = < (\nabla_{\dot\gamma} T)_V W, \dot\gamma > - < T_V\dot\gamma, T_W\dot\gamma > + < A_{\dot\gamma}V, A_{\dot\gamma}W > .$$

Since $(\nabla_E T)_F$ is skew-symmetric, we have

$$< R(V,\dot\gamma)\dot\gamma, W >= - < (\nabla_{\dot\gamma} T)_V \dot\gamma, W > - < T_V\dot\gamma, T_W\dot\gamma > + < A_{\dot\gamma}V, A_{\dot\gamma}W > .$$

Now we switch notation. Recall that $S_{\dot\gamma}V = -(\nabla_V\dot\gamma)^v = -T_V\dot\gamma$. Therefore $(\nabla_{\dot\gamma}S)_{\dot\gamma}V = -(\nabla_{\dot\gamma}T)_V\dot\gamma$, and also $S'^v_{\dot\gamma}(V) = (\hat\nabla_{\dot\gamma}S)_{\dot\gamma}(V) = -(\hat\nabla_{\dot\gamma}T)_V\dot\gamma$. Hence since $< (\hat\nabla_{\dot\gamma}T)_V\dot\gamma, W >=< (\nabla_{\dot\gamma}T)_V\dot\gamma, W >$, we get the equation

$$< R(V,\dot\gamma)\dot\gamma, W >=< S'^v_{\dot\gamma}V, W > - < S^2_{\dot\gamma}V, W > + < A^*_{\dot\gamma}V, A^*_{\dot\gamma}W >,$$

since $S_{\dot\gamma}$ is a self-adjoint operator. Taking adjoints for A in the last right hand term and making other cosmetic changes gives us the equation

$$< S'^v_{\dot\gamma}V, W >=< R(V,\dot\gamma)\dot\gamma, W > + < S^2_{\dot\gamma}V, W > - < A_{\dot\gamma}A^*_{\dot\gamma}V, W >$$

which yields the result since W is arbitrary. □

Now we are in a position to derive the matrix Ricatti equation.

Proof of Theorem. We work with the distribution $\mathcal{V}$ with fiber metric $g|_V$ and the connection $\hat\nabla_E V = (\nabla_E V)^v$, for $E \in \Gamma(TM)$ and $V \in \Gamma(V)$. Let $\{V_1, \ldots V_k\}$ be an orthonormal basis of vertically parallel fields along the horizontal geodesic γ. (Recall that we showed this was possible in Chapter 2.) Now with the notations of the theorem, we have that

$$< S'^v_{\dot\gamma}V_i, V_j >=< S_{\dot\gamma}V_i, V_j >'$$

as noted above. This gives the result of the theorem, since S is self-adjoint. □

Now for some remarks. First, a Riemannian foliation is said to be *substantial* at a point $p \in M$ if and only if there is a horizontal vector X_p such that $A_{X_p} : \mathcal{H} \to \mathcal{V}$ is onto. (see [9]). This is of course equivalent to saying that $A^*_{X_p} : \mathcal{V} \to \mathcal{H}$ is injective. Second, let us examine the terms in the matrix equation of the theorem:

$$B(t) = -S'(t) + S^2(t) + R(t) \tag{5.1}$$

Now suppose (M, g) is positively curved. Then the bilinear form

$$(V, W) \to < (R(V, \dot{\gamma})\dot{\gamma}, W >$$

is positive definite, and hence its matrix $R(t)$ is positive definite for all t. Similary if M is negatively curved, then $R(t)$ is negative definite. Third, the matrix B is the Gram matrix of the k-vector $\{A^*_{\dot{\gamma}}V_1, \dots, A^*_{\dot{\gamma}}V_k\}$. Hence B is positive semi-definite. B is then positive definite at some point $\gamma(t)$ if and only if the k-vector $A^*_{\dot{\gamma}}V_1 \wedge \cdots \wedge A^*_{\dot{\gamma}}V_k$ has non-zero length, which is in turn true if and only if $A^*_{\dot{\gamma}}$ is injective on $\mathcal{V}$, ie if and only if $\mathcal{F}$ is substantial at the point $\gamma(t)$.

These observations enable us to prove several versions of a theorem first discovered by Walschap [22]. To state the theorems we need a definition. A Riemannian foliation $\mathcal{F}$ is said to be *umbilic along* γ if and only if $S_{\dot{\gamma}} = c(t)I$ for all t, where $c(t)$ is a smooth function of t. Before proceeding, we state a well-known and useful lemma.

Lemma 11. *(a) Suppose that $c : \mathbb{R} \to \mathbb{R}$ is a differentiable function with the property that $c' \geq c^2$ for all $t \in \mathbb{R}$. Then c must be zero.*

(b) *Similarly, if* c *is a differentiable function such that* $c' + c^2 \leq 0$ *for all* t*, then* c *is constant.*

Proof. To see the first statement Let $F(t)$ be an antiderivative of c. Then by differentiating twice, we see that the function $G(t) = exp(-F(t))$ is concave, and bounded below by zero. Hence G is constant, and hence F is constant, so that $F' = c = 0$. To see the second statement, notice that the function $G(t) = exp(F(t))$ where F is an antiderivative of c, is concave and bounded below. □

Proposition 12. *Suppose that* $\mathcal{F}$ *is a Riemannian foliation of* (M, g)*, umbilic along the horizontal geodesic* γ*. If* (M, g) *has positive sectional curvature, then* $\mathcal{F}$ *is substantial at some point* $\gamma(t)$*.*

Proof Applying the hypothesis to the matrix Ricatti equation (5.1) gives us the matrix equation

$$B(t) = (-c'(t) + c^2(t))I + R(t).$$

Now by the remarks above, it suffices to show that B is positive definite at some point t. Since by hypothesis, R is positive definite, we need only show that at some point, $(-c' + c^2)I$ is positive semi-definite, which is true if and only if $-c' + c^2 \geq 0$ at some point t. So if B is not positive definite, then $-c' + c^2 < 0$ for all t. Hence $c = 0$ by the lemma, so that $B(t) = R(t)$ for all t, and hence B is positive definite. □

Observe that the above proof shows that if $\mathcal{F}$ is umbilic along γ, then in fact $\mathcal{F}$ is totally geodesic along γ.

Theorem 13. *Suppose that $\mathcal{F}$ is a Riemannian foliation of (M, g). If (M, g) has negative sectional curvature, then $\mathcal{F}$ cannot be totally geodesic along any horizontal geodesic γ.*

Proof. If $\mathcal{F}$ is totally geodesic, then in (5.1) we have that $S = S' = 0$. Therefore, B, a positive semi-definite matrix, is equal to R, a negative definite matrix. Contradiction. □

Of course, one can prove this by simply looking at the curvature equation {2}. The following is a theorem originally found by Walschap ([22]). We give a new proof below.

Theorem 14. *Suppose that $\mathcal{F}^k$ is a Riemannian foliation of (M^n, g), which is umbilic along the horizontal geodesic γ. If (M, g) has nonnegative sectional curvature, and the leaf dimension k is greater than the codimension $n - k$, then $\mathcal{F}$ is totally geodesic along γ.*

Proof. Choose an orthonormal basis of vertically parallel fields $\{V_1, \ldots V_k\}$. The hypothesis means that the k-vector $A^*_{\dot\gamma}V_1(t) \wedge \cdots \wedge A^*_{\dot\gamma}V_k(t)$ must be zero for all t, since the vectors in that wedge product cannot be linearly independent. Hence, in the matrix Ricatti equation

$$B(t) = (-c'(t) + c^2(t))I + R(t),$$

$B(t)$ is positive semi-definite (but not positive definite!), $R(t)$ is positive semi-definite by the curvature assumption, and $B(t)$ must have determinant 0, or else the squared length of the wedge product $A^*_\gamma V_1(t) \wedge \cdots \wedge A^*_\gamma V_k(t)$ would be bigger than zero. Therefore $(-c'(t) + c^2(t))I$ can never be positive definite for any t, or else $B(t)$ would be positive definite. But this means that $(-c'(t) + c^2(t)) \leq 0$ for all t. By the lemma, this means that $c = 0$, which means that $\mathcal{F}$ is totally geodesic along γ. □

Corallary 15. *An umbilic riemannian foliation of the positively curved manifold (M^n, g) must have leaf dimension smaller than its codimension. In fact, it must be substantial at some point of any horizontal geodesic. Therefore the dual foliation of an umbilic foliation in positive curvature must have an n-dimensional leaf.*

Wilking (unpublished) has defined the notion of *dual* to a Riemannian foliation. Among the results he claims that are related to this notion is a proof that the Sharafutdinov retraction to the soul of a non-negatively curved manifold is smooth. The dual to a Riemannian foliation is defined as follows: Given a (possibly singular) Riemannian foliation of (M, g) and a point p in M, the leaf of the dual foliation through p is defined to be the set of points which can be connected to p via a broken horizontal path.

Proof. We know that along a horizontal geodesic c, there is a t such that $\mathcal{F}$ is substantial at $p = c(t)$. Therefore, for $X_p = \dot{c}(t)$, $A^*_{X_p}$ is injective , and therefore $dim(\mathcal{V}) \leq dim(\mathcal{H}) - 1$, since $A^*_{X_p}$ maps $\mathcal{V}$ to $\mathcal{H} \cap \dot{c(t)}^{\perp}$, since A is skew symmetric. To see the last statement, observe that the tangent space to a leaf of the dual foliation must contain all brackets of horizontal vectors. Since A_{X_p} is onto, this means that the set of brackets $[X, Y]$ for $X, Y \in \mathcal{H}$ spans $T_p(M)$, because $[X, Y] = [X, Y]^h + [X, Y]^v = [X, Y]^h + 2A_X Y$. Hence the dual leaf at p has tangent space equal to all of $T_p M$. □

In fact, Wilking (unpublished) has shown that the dual to any (possibly singular) Riemannian foliation of a positively curved manifold consists of one leaf. Wilhelm has asked if the dimension bound of the above corollary applies to any Riemannian submersion of a positively curved manifold. This question was the impetus for this work. In the next section we will give another partial answer to it.

5.2 Restrictions Based on Curvature Bounds

In this section we will derive certain restrictions on the existence of Riemannian submersions based on the curvature properties of the total space and the base space. Here is the main theorem of this section.

Theorem 16. *Suppose that $f : M \to B$ is a Riemannian submersion. Suppose that M has sectional curvature greater than or equal to 1, and suppose that the sectional curvature of any two-plane in B is less than or equal to Q. Then* *the fiber dimension p must satisfy the inequality*

$$p \leq \frac{1}{3}(dim(B) - 1)(Q - 1)$$

In particular, a quarter-pinched manifold cannot be the base space of a Riemannian submersion of a manifold with fiber dimension greater than its codimension.

Proof. We will begin by recalling a result of Kim and Tondeur ([14]). They proved that a Riemannian foliation of a nonnegatively curved (M, g) of dimension p and codimension q is totally geodesic provided that $|A| \leq (\frac{pK}{q})^{1/2}$, where K is the minimum of the sectional curvature of M, and $|A|$ is defined to be the supremum of $|A^*_x y|$, where x and y range through pairs of orthonormal horizontal vectors.

Let $f : M \to B$ be a Riemannian submersion, where the minimum sectional curvature of M is 1. (This can, of course, be assured by scaling the metric on M). By the O'Neill equation, we have

$$\frac{1}{3}(Q - 1) \geq A_X Y$$

where $X \wedge Y$ is an y horizontal two-plane in M. In other words,

$$|A| \leq (\frac{1}{3}(Q - 1))^{1/2}$$

Hence by the theorem of Kim and Tondeur, we conclude that if

$$\frac{p}{q - 1} \geq \frac{1}{3}(Q - 1)$$

the fibers of f must all be totally geodesic. This gives the bound of the theorem. If the metric on B is also quarter pinched, then $(Q-1) \leq 3$, which shows that the fiber dimension must not be bigger than the codimension. $\square$

Chapter 6

Homogeneity Results

6.1 Definition of Homogeneous Riemannian Foliation

One of the aims of the research described in this chapter is to give a characterization of the notion of homogeneous Riemannian foliation. But one must be somewhat careful in defining the notion of a homogeneous Riemannian foliation in the first place. There are several concepts appearing under the word "homogeneous" in the literature which are definitely not equivalent. We'll begin by recording several of these, together with several others that can plausibly be called homogeneous.

Definition 17. *A Riemannian foliation $\mathcal{F}$ of (M, g) is said to be strongly homogeneous if and only if for any $p, q \in M$, there is an isometry of M which preserves $\mathcal{F}$ and takes p to q.*

This is far too strong. It of course implies that M is homogeneous, and if $\pi : M \to B$ is a strongly homogeneous Riemannian submersion, then B is homogeneous as well. This means that, for instance, the Riemannian submersions defining the Eschenburg spaces in [5] are not strongly homogeneous, even though there is a group of isometries acting transitively on the fibers of these submersions. Since foliations are defined by their leaves, one's definition of homogeneous foliation should perhaps also respect the leaves. That is, it makes sense to restrict p and q in the definition of strong homogeneity above to lie in the same leaf. This is basically what Gromoll and Grove do in [9]:

Definition 18. *(see [9]) A Riemannian foliation $\mathcal{F}$ of (M, g) to be homogeneous if and only if there is a group G of isometries of (M, g) preserving the leaves of the foliation such that the isotropy group of each leaf L of $\mathcal{F}$ acts transitively on L.*

The Riemannian submersions defining the Eschenburg spaces in [5] are homogeneous in this sense. In constant curvature (which is all that Gromoll and Grove are concerned with in [9]), a foliation satisfying this condition is automatically Riemannian. However, this is not always the case. In [23] we see examples of subgroups of Lie groups G with left-invariant metric, such that the cosets do not form a Riemannian foliation of G with the given left-invariant metric. With this in mind, we have the following definition:

Definition 19. *(see [23]) A Riemannian foliation $\mathcal{F}$ of (M, g) is said to be orbit homogeneous if and only if the leaves are locally the orbits of a group of isometries of (M, g).*

Since the local action fields associated to an orbit homogeneous foliation are Killing fields for (M, g) where they are defined, we see that the second fundamental form of the horizontal distribution in the sense of Reinhart [20] is identically zero, and so orbit homogeneous foliations are Riemannian.

We will call an orbit foliation $\mathcal{F}$ *almost-free* if and only if the groups in the definition act almost freely. That is, the isotropy groups of the actions in the definition are finite.

Since we will be working with Killing fields rather than groups of isometries in the theorems we prove in this chapter, we will end this section by translating some of these notions into equivalent notions involving Killing fields.

Lemma 20. *A Riemannian foliation $\mathcal{F}$ of (M, g) is orbit homogeneous if and only if for every point $p \in M$ there is a neighborhood U of p and a family of Killing fields $\{K_1, \ldots, K_l\}$ which span the leaf directions in U.*

Proof. Notice that we have not specified the number l in the above lemma. We can take the Lie algebra generated by the K_i to be the Lie algebra of a group of isometries of M, such that the leaves of $\mathcal{F}$ are locally the orbits of this action. Conversely, if $\mathcal{F}$ is orbit homogeneous, then the action fields will serve as the Killing fields in the statement. □

Lemma 21. *An orbit homogeneous Riemannian foliation $\mathcal{F}$ of (M, g) is said to be almost-free if and only if the leaves have dimension k, and for every point $p \in M$, there is a neighborhood U of p and a family of k Killing fields for (M, g) which span the leaf directions.*

Proof. In this case, the group generated by these Killing fields must act almost freely, because of the dimension condition. □

6.2 Known Homogeneity Obstructions

At present the only known obstructions to orbit homogeneity are the following: Given a Riemannian foliation one defines the mean curvature form by the equation $\omega(E) = \mathrm{tr} S_{E^h}$. It is a horizontal form with values in the endomorphism bundle of $\mathcal{V}$. Now if a Riemannian foliation is homogeneous, then the mean curvature form is a basic vector field. If the Riemannian foliation is also one-dimensional, and the mean curvature form is closed, then the foliation is homogeneous. To see this, note that locally we have $\omega = d\phi$ for some smooth function. Then if T is a vector field locally spanning the foliation, one checks that $\exp(-\phi)T$ is a Killing field on (M, g). (for more details, see [8]). However this is certainly not a sufficient condition. Observe that if $\mathcal{F}$ is totally geodesic and one-dimensional, then any unit length vector field T spanning $\mathcal{F}$ is a Killing field. However, this is not the case for the foliation generated by the Hopf map $S^{15} \to S^8$ with fiber S^7. In this case the only symmetries of S^{15} taking each fiber of this map to itself are the identity and the antipodal map (essentially because the octonions are non-associative). In this chapter we will present a sufficient condition for a Riemannian Foliation to be homogeneous. First we will discuss the holonomy map of a homogeneous Riemannian foliation.

6.3 The Holonomy of a Riemannian Foliation

In this section we will assume that the foliation $\mathcal{F}$ is spanned by a family $\{K_1, \ldots, K_k\}$ of holonomy fields, and we will compute the holonomy of $\mathcal{F}$ along the horizontal geodesic γ.

Proposition 22. *Suppose that the Riemannian foliation $\mathcal{F}$ of dimension l is spanned by the family $\{K_1, \ldots, K_l\}$ of Killing fields for (M, g). Let $\gamma : [0, 1] \to M$ be a horizontal geodesic of M, and let v_p be a vertical vector at $p = \gamma(0)$. Extend v to a vert ical Killing vector field K. Then the holonomy Jacobi field $J(t)$ extending v_p along γ is given by $J(t) = K_{\gamma(t)}$.*

Proof. One can extend v to a Killing field by writing $v_p = c_1K_1 + \cdots + c_lK_l$ for $c_i \in \mathbb{R}$, and then setting $K = c_1K_1 + \cdots + c_lK_l$. Now any Killing field K restricted to any geodesic γ is a Jacobi field. This is because the local flow of K is given by isometries, and hence generates a variation of γ through isometries. $K|_\gamma$, the variation field of this variation, is then a Jacobi field along γ. But since our K is vertical, the geodesics of the variation generated by K all project to the same geodesic in the base. That is, the geodesics of the variation are all basic lifts of the same geodesic. Therefore the differential of the holonomy map is given by the variation field of this variation. Observe that this calculation is independent of the choice of Killing field extending v. $\square$

6.4 An Obstruction To Homogeneity

The work in the previous section suggests finding Killing fields via the following procedure: Suppose K is a vector field along the leaf L of $\mathcal{F}$. Further suppose that K is killing along the leaf L; that is, $< \nabla_V K, V >= 0$ for any vertical vector. Then we can extend K to a vector field in a neighborhood of $p \in L \subset M$ by holonomy. Under certain circumstances (by no means all) this process will yield a vertical Killing field for (M, g) in a neighborhood of $p \in M$. Here is one case where it works.

Theorem 23. *Suppose that $\mathcal{F}$ is a totally geodesic Riemannian foliation of (M, g) of dimension k. Fix a leaf L of $\mathcal{F}$ and choose $p \in L$. Suppose that there exist vector fields $X_1, \ldots, X_k$ of L with the following properties:*

(a) The X_i are tangent to the leaf L,

(b) The X_i are Killing along L; that is, $< \nabla_V K, V >= 0$ for any vertical vector,

(c) There exist constants c_{ij}^l such that $< [X_i, X_j], X_l >= c_{ij}^l X_l$ along L

Then the X_i can be extended to Killing fields for (M, g) in a neighborhood of p.

Proof. Observe that the condition above automatically implies that L must have the structure of a Lie group, provided that the vector fields X_i are defined everywhere on L (see [21]). Since the foliation is totally geodesic, any holonomy Jacobi field J along a horizontal geodesic γ is vertically parallel. This is because $J' = -S_{\dot\gamma}J - A^*_{\dot\gamma}J$, so that $J'^v = -S_{\dot\gamma} = 0$. Therefore the holonomy transformations are isometries, since holonomy transport coincides with the parallel transport coming from the vertical connection $\hat\nabla$. Now we extend each X_i to a vector field K_i near p using the holonomy maps. K_i is well defined, if the neighborhood in which the extension takes place is small enough. The K_i are smooth, since along each horizontal geodesic, K_i is the solution to the differential equation $K_i'' = R(K_i, \dot\gamma)\dot\gamma$ with initial conditions $K_i(0) = X_i(\gamma(0))$, $K_i'(0) = -A^*_{\dot\gamma}X_i(\gamma(0))$. Therefore since the initial conditions vary smoothly, K_i is smooth. Moreover, since we have seen that the holonomy maps are isometries (as the leaves are totally geodesic), we must have the bracket relations $c^l_{ij} =< [K_i, K_j], K_l >$ among the extended vector fields. Now we need only check that the K_i are Killing fields. This is equivalent to saying that $< \nabla_B K_i, B >= 0$ for any vector B and any i. We will check this condition by writing $B = B^h + B^v$, expanding out $< \nabla_B K_i, B >$ and checking that each of the four terms in the sum are zero:

(a) $< \nabla_{B^v} K_i, B^v >= 0$ along L by hypothesis. Since the holonomy maps are isometries, they take Killing fields along L to Killing fields along L, and therefore this is true wherever the extension is defined.

(b) $< \nabla_{B^v} K_i, B^h >= 0$ because $\mathcal{F}$ is totally geodesic.

(c) $< \nabla_{B^h} K_i, B^h >= - < A^*_{B^h} K_i, B^h >= - < K_i, A_{B^h} B^h >= 0$.

To see that $< \nabla_{B^h} K_i, B^v >= 0$ we note that the quantity in question is tensorial in B^h and B^v, and so we extend B^h to a vector field Q such that $[Q, K_i] = 0$. Then we have $< \nabla_{B^h} K_i, B^v >=< \nabla_Q K_i, B^v >=< T_Q K_i, B^v >= 0$. □

Corallary 24. *Suppose that π is a Riemannian submersion of a compact Lie group G with bi-invariant metric with totally geodesic fibers, with the property that the fiber through the identity is a Lie subgroup of G. Then π is orbit homogeneous and almost-free.*

Proof. We begin by passing to the universal cover $\tilde{G}$ of G, and lifting the foliation defined by π there. Since $\tilde{G}$ is locally isometric to G, the lifted foliation is still Riemannian and totally geodesic. By hypothesis the left-invariant vector fields tangent to the fiber through the identity must form a Lie subalgebra of $\mathfrak{g}$. Therefore those fields satisfy the requirements of the above theorem, and can therefore be extended by holonomy to Killing fields in a neighborhood of the identity. But then we have a Lie algebra of local Killing fields of G, which can be extended to a global Lie group of isometries of G, by the theorem of Cartan (see [17]). Therefore the submersion π must be orbit homogeneous and almost free, and similarly for the original submersion. □

Ranjan ([19]) has shown that a Riemannian submersion of a compact simple Lie group G with bi-invariant metric with connected totally geodesic fibers must be the coset projection onto one of the coset spaces G/F_e or $F_e\backslash G$, provided that the fiber F_e through the identity e of G contains a maximal torus of G. (As part of his proof, he shows that F_e is a subgroup of G.)

6.5 Flat foliations of symmetric spaces

In this section we will discuss flat foliations of symmetric spaces of compact type. A *flat* Riemannian foliation is one whose A tensor vanishes identically. Here is the theorem that we will prove:

Theorem 25. *Suppose that (M, g) is a simply connected symmetric space of compact type and that $\mathcal{F}$ is a Riemannian foliation of M. If $\mathcal{F}$ is flat, then M splits isometrically, and in fact $\mathcal{F}$ is homogeneous. In particular, a codimension 1 Riemannian foliation of a compact Lie group with bi-invariant metric must be homogeneous.*

Proof. Recall that g must have nonnegative sectional curvature. Therefore in the Riccati equation (5.1)

$$S' = S^2 + R - B$$

we can take traces, getting the equation

$$s' = s^2 + tr((S - sI)^2) + r - b$$

where S, R, and B are all $k \times k$ matrices, and $s = \frac{2}{k+1} tr(S)$, $r = tr(R)$ and $b = tr(B)$. Since $\mathcal{F}$ is flat, we have B = 0, and hence $b = 0$. Therefore s has the property that $s' \geq s^2$. Therefore $s = 0$, which implies that $tr(S^2) + r = 0$, and hence $S = 0$ since S is symmetric. Therefore the leaves of a flat Riemannian foliation in non-negative curvature are totally geodesic, and moreover the vertizontal 2-planes (ie, those spanned by a vertical and a horizontal vector) must be flat, s ince $r = 0$. Therefore the metric g must split locally along the leaves of the Riemannian foliations $\mathcal{H}$ and $\mathcal{V}$. (See also [24] for more on flat Riemannian foliations. The method I used to show that $S = 0$ is well known. I included it here for completeness.) Recalling the criterion of Escobales [7], we conclude that at any point, the geodesic symmetry ϕ_p of M preserves $\mathcal{F}$, since $A = 0$. Therefore, writing $M = G/K$ where G is a compact Lie group and K is the fixed point set of an involution of G, we see that G is a group of isometries of M preserving the foliation $\mathcal{F}$. Therefore near any point $p \in M$, we can find a set of Killing fields (the action fields of the action by G) which lie tangent to the leaves of $\mathcal{F}$. Therefore $\mathcal{F}$ is homogenous. The last sentence in the statement follows because compact Lie groups with bi-invariant metric are symmetric spaces of compact type, and codimension 1 Riemannia n foliations must be flat. So if we lift the given foliation $\mathcal{F}$ to the universal cover $\tilde{G}$ of G, we get a new foliation $\tilde{\mathcal{F}}$ of $\tilde{G}$ which must also be of codimension 1. Therefore the isometries of $\tilde{G}$ preserving $\tilde{\mathcal{F}}$ induce local isometries of G, such that the corresponding action fields lie tangent to the leaves of $\mathcal{F}$. Therefore $\mathcal{F}$ is homogeneous. □

We also have the following theorem.

Theorem 26. *Let (G, g) be a compact simple Lie group with bi-invariant* *metric G. There are no codimension-one Riemannian foliations of G with closed leaves.*

Proof. Suppose that $\mathcal{F}$ is such a foliation. The argument above shows that $\mathcal{F}$ must have totally geodesic leaves, since (G, g) has non-negative curvature. Observe that if $f : G \to G$ is an isometry with the property that at some point $p \in G$, $f(\mathcal{H}_p) = \mathcal{H}_{f(p)}$, then for all $q \in G$, $f(\mathcal{H}_q) = \mathcal{H}_{f(q)}$. In particular, f preserves $\mathcal{F}$ if $f(\mathcal{H}_p) = \mathcal{H}_{f(p)}$ for any point p. Therefore every geodesic symmetry of G preserves $\mathcal{F}$, and therefore every isometry in the identity component of Isom(G) preserves $\mathcal{F}$. Therefore, since G has a bi-invariant metric, we see that for every $g \in G$, the map $c_g(x) = gxg^{-1}$ is an isometry of G preserving $\mathcal{F}$. Since c_g fixes the identity of G, c_g must therefore take the leaf of $\mathcal{F}$ through the identity to itself for every g. I claim this leaf is therefore a subgroup of G. To see this, pick $v, w \in \mathcal{V}_e$, where $\mathcal{V}_e$ is the tangent space at the identity to the leaf of $\mathcal{F}$ through e, set $g_t = exp(tw)$, and observe that the smooth path $v(t) = Ad_{g_t}(v)$ must lie in $\mathcal{V}_e$. Therefore $v'(0) = ad(w)(v)$ must lie in $\mathcal{V}_e$. Therefore the leaf of $\mathcal{F}$ through the identity i s not only a subgroup of G but also a normal subgroup, since it is preserved by every c_g. But then G cannot be simple. Contradiction. ☐

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