ABSTRACT

Title of dissertation: DEFINITIONS OF CHAOS

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As a relatively new field in mathematics, chaos theory and its fundamentals have not been set. Specifically, there exist many different mathematical definitions of chaos and what it means for a function to be classified as chaotic.

This paper examines eight major definitions that either have been used to classify a function as chaotic or have been considered a major charactersitic of a chaotic function, first illustrating the ideas and definitions with the Tent function. The body of the paper is spent proving implications between definitions or examining functions that are chaotic according to one definition of chaos but not according to another.

DEFINITIONS OF CHAOS

by

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Introductory Concepts and Examples

1.1 Introduction

Since the surprising output from Lorenz's weather machine in 1961, the field of chaos has become increasingly popular among specialists and non-specialists alike. This popularity has been manifested in an explosion of popular science-fiction books and movies, from Jurassic Park to "The Butterfly Effect." Thus, in most cases, even a non-specialist has had some encounter with the relatively new scientific idea of chaos and has some idea of what it means for a system to be chaotic. Many mathematicians, however, have developed their own individual ideas of what constitutes a chaotic system, and the result has been numerous distinct definitions of chaos. As Martelli et al wrote in "Defining Chaos" [5], "We could say, with a bit of exaggeration, that there are as many definitions of chaos as experts in this new area of knowledge ....Moreover, and this is certainly not a desirable situation, the various definitions are not equivalent to each other."

In Section 1.2 we will present a collection of concepts used in defining a function as chaotic. In Chapter 2 we will introduce several potential definitions of chaos and examine relationships between them. Finally, in Chapter 3 we will briefly consider the applications of these definitions in different fields.

1.2 Preliminary Definitions

In the following definitions and throughout this paper, f will be a real-valued continuous function mapping an interval X of real numbers to X. The interval X can be compact or non-compact. If one restricts attention to compact sets, certain results may vary.

We begin with the fundamental notions on which the entire discussion in the paper rests.

Definitions: For $x \in X$, we call $f(x)$ the **iterate** of x. Next, we define the **orbit** of x as the collection of the successive iterates of x; in other words, the orbit of x is the set $\{x_0, x_1, x_2, ...\}$, where $x_0 = x$ and $x_{n+1} = f(x_n)$ for $n = 0, 1, 2, ...$ If we let $f^{[1]}(x) = f(x)$, then for $n = 1, 2, 3, ...,$ we define the $(n+1)$st iterate of x for f to be $f^{[n+1]}(x) = f(f^{[n]}(x))$. If $f(x) = x$ for some $x \in X$, we call x a **fixed point** of f. If $f(x) \neq x$, but there exists a positive integer n such that $f^{[n+1]}(x) = f^{[n]}(x)$, we call x an **eventually fixed point**. If there exist a $p \in X$ and some positive integer n such that $f^{[n]}(p) = p$ and $f^{[k]}(p) \neq p$ for all integers k such that $1 \leq k \leq n-1$, we call p a **periodic point** with period n, or a **period-n point**. Note that under this definition, a fixed point is a periodic point with period $n = 1$, or a period-1 point. Finally, we define x to be an **eventually periodic point** if x is not a periodic point but some iterate of x is periodic.

In some of the proofs to follow we will also use the definition of a dense set in a real interval. A set B is **dense** in a real interval X if every nonempty open subinterval of X contains an element of B. For example, consider the interval $X = [0, 1]$. The set of rational numbers is dense in X, since every open interval in X contains a rational number. Similarly, the set of irrational numbers is dense in X. However, the set $B = \{\frac{1}{n} : n = 1, 2, 3, ...\}$ is not dense in X, since the open interval $(\frac{1}{2}, 1)$ does not contain any elements of B.

1.3 The Tent function and some of its properties

To illustrate the definitions above and the definitions to follow, we will use the Tent function (see Figure 1.1) $T : [0, 1] \to [0, 1]$ defined by

$$T(x) = \begin{cases} 2x & : \quad 0 \leq x \leq \frac{1}{2} \\ 2 - 2x & : \quad \frac{1}{2} < x \leq 1 \end{cases}$$

In order to find the fixed points of T, we let $T(x) = x$, which implies either $2x = x$ or $2 - 2x = x$. Thus, the only two fixed points of T are $x = 0$ and $x = \frac{2}{3}$. To find an eventually fixed point of T, we need an x that is not 0 or $\frac{2}{3}$ such that some iterate of x equals either 0 or $\frac{2}{3}$.

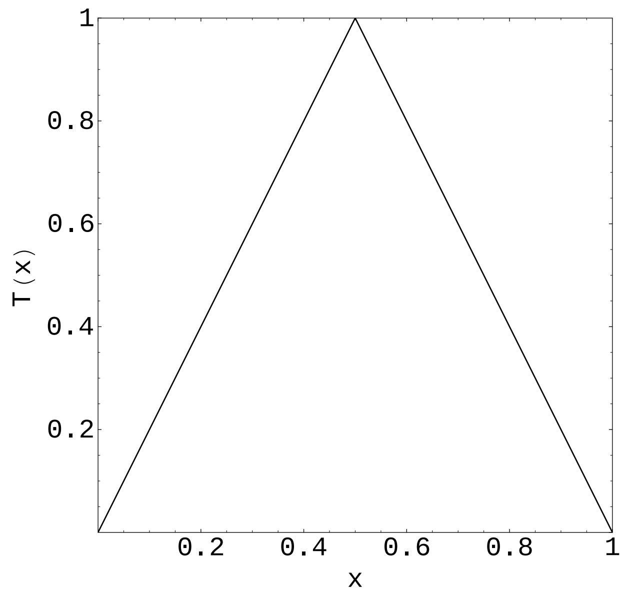

Figure 1.1: The Tent function

Consider $x = \frac{1}{12}$:

$$T\left(\frac{1}{12}\right) = 2 \cdot \frac{1}{12} = \frac{1}{6},$$
$$T\left(\frac{1}{6}\right) = 2 \cdot \frac{1}{6} = \frac{1}{3},$$
$$T\left(\frac{1}{3}\right) = 2 \cdot \frac{1}{3} = \frac{2}{3},$$

and $\frac{2}{3}$ is a fixed point. Thus, $T^{[3]}(\frac{1}{12}) = \frac{2}{3}$, and therefore $T^{[n]}(\frac{1}{12}) = \frac{2}{3}$ for all positive integers $n \geq 3$, so $x = \frac{1}{12}$ is an eventually fixed point.

Next we illustrate periodic points. Let $x = \frac{2}{9}$:

$$T\left(\frac{2}{9}\right) = 2 \cdot \frac{2}{9} = \frac{4}{9},$$
$$T\left(\frac{4}{9}\right) = 2 \cdot \frac{4}{9} = \frac{8}{9},$$
$$T\left(\frac{8}{9}\right) = 2 - 2 \cdot \frac{8}{9} = \frac{2}{9}$$

Thus $T^{[3]}(\frac{2}{9}) = \frac{2}{9}$ and $T^{[n]}(\frac{2}{9}) \neq \frac{2}{9}$ for $n = 1$ and 2, so $x = \frac{2}{9}$ is a periodic point of T with period $n = 3$. Similarly, one can show that $x = \frac{4}{13}$ is a periodic point of T with period $n = 6$.

Now we will illustrate eventually periodic points. Let $x = \frac{1}{18}$:

$$T\left(\frac{1}{18}\right) = 2 \cdot \frac{1}{18} = \frac{1}{9},$$

$$T\left(\frac{1}{9}\right) = 2 \cdot \frac{1}{9} = \frac{2}{9},$$

and $\frac{2}{9}$ is a periodic point, as we have just shown. Thus, $x = \frac{1}{18}$ is an eventually periodic point.

In some of the proofs to follow, we will associate each $x \in [0, 1]$ with a sequence $x_0x_1x_2\cdots$ by means of a function h that is related to T and which we now define. Let h map the interval $[0, 1]$ to the set of all sequences of 0's and 1's by

$$h(x) \quad = \quad \text{the sequence } x_0x_1x_2\cdots \tag{1.1}$$

where

$$x_n = \begin{cases} 0 & : \quad 0 \le T^{[n]}(x) \le \frac{1}{2} \\ 1 & : \quad \frac{1}{2} < T^{[n]}(x) \le 1 \end{cases}$$

and where $T^{[0]}(x) = x_0$ by definition. Notice that by the definition of h, if $h(x) = x_0x_1x_2x_3\cdots$, then $h(T(x)) = x_1x_2x_3x_4\cdots$, and by induction, $h(T^{[n]}(x)) = x_nx_{n+1}x_{n+2}\cdots$. Thus, $h \circ T$ is a "left shift" on the set of all sequences of 0's and 1's, in the sense that $h(T(x)) = x_1x_2\cdots$, so that under h, the sequence $x_0x_1x_2\cdots$ is shifted to the left, with x_0 disappearing. We will associate a number $x \in [0, 1]$ with its image under h by writing $x \sim x_0x_1x_2\cdots$, where $h(x) = x_0x_1x_2\cdots$. For example, $x = \frac{2}{9} \sim 001\overline{001}\cdots$, since

$$x = \frac{2}{9} \in [0, \frac{1}{2}] \text{ implies } x_0 = 0,$$

$$T^{[1]}(x) = \frac{4}{9} \in [0, \frac{1}{2}] \text{ implies } x_1 = 0,$$

$$T^{[2]}(x) = \frac{8}{9} \in (\frac{1}{2}, 1] \text{ implies } x_2 = 1,$$

and then the sequence repeats itself since $x = \frac{2}{9}$ is a periodic point with period $n = 3$.

Another important feature of the association of each $x \in [0, 1]$ with its image under h is the division of the interval $[0, 1]$ into blocks of length $\frac{1}{2^n}$ for every positive integer n (see Figure 1.2). For example, if $0 \le x \le \frac{1}{2}$, then $x_0 = 0$, whereas if $\frac{1}{2} < x \le 1$, then $x_0 = 1$. It follows that by considering x_0, we can determine which *half* of the $[0, 1]$ interval $x \sim x_0x_1x_2\cdots$ lies in.

Similarly,

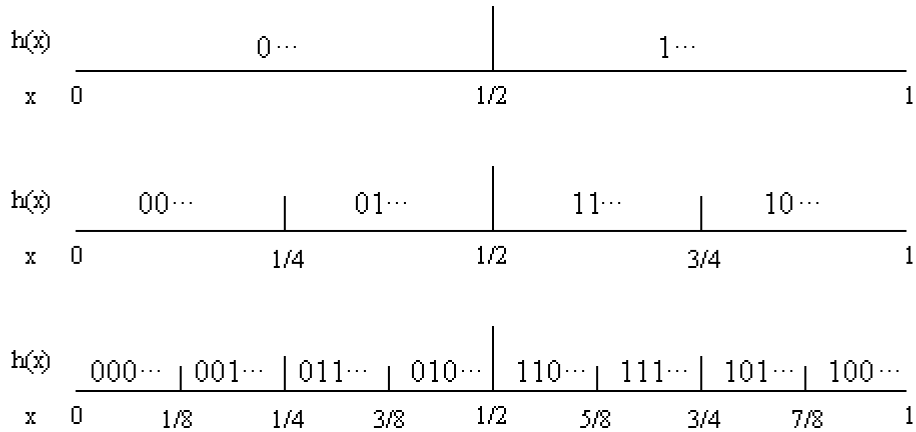

Figure 1.2: The division of the unit interval under the function h.

if $0 \le x \le \frac{1}{4}$, then $0 \le T(x) \le \frac{1}{2}$, so that $x_0 = 0$, $x_1 = 0$.

If $\frac{1}{4} < x \le \frac{1}{2}$, then $\frac{1}{2} < T(x) \le 1$, so that $x_0 = 0$, $x_1 = 1$.

If $\frac{1}{2} < x < \frac{3}{4}$, then $\frac{1}{2} < T(x) \le 1$, so that $x_0 = 1$, $x_1 = 1$.

If $\frac{3}{4} \le x \le 1$, then $0 \le T(x) \le \frac{1}{2}$, so that $x_0 = 1$, $x_1 = 0$.

By considering x_0x_1, we can determine which *quarter* of the $[0,1]$ interval $x \sim x_0x_1x_2\cdots$ lies in. Similarly, by considering $x_0x_1\cdots x_n$, we can determine which subinterval of $[0,1]$ of length $\frac{1}{2^n}$ that x lies in.

It can also be shown that h is one-to-one and onto the set of sequences of 0's and 1's, excluding "finite" sequences of the form $x_0x_1x_2\cdots\bar{0}\cdots$ (that is, sequences in which all terms to the right of a given term are 0). We will denote by A this set of sequences which excludes finite sequences. Thus, in future examples we will construct sequences in A whose associated numbers in $[0,1]$ have specified properties for T.

The Tent function is an important function because it exhibits all of the characteristics that we will present in Chapter 2 that are associated with chaotic functions. Let us briefly explore one of the most important characteristics of the Tent function: long term iterates.

Let $x = \frac{2}{7}$, $y = \frac{9}{32}$, and $z = \frac{\pi}{11}$. Then $|x - y| < 0.005$ and $|x - z| < 0.002$, so x, y, and z are close together. However, if we look at the 29th iterates of these numbers, we see that they have separated over the interval $[0, 1]$:

$$T^{[29]}\left(\frac{2}{7}\right) = \frac{6}{7}, \quad T^{[29]}\left(\frac{9}{32}\right) = 0, \text{ and } T^{[29]}\left(\frac{\pi}{11}\right) \approx 0.169955$$

We see that small differences in the initial starting points lead to large differences in higher iterates.

Chapter 2

Relationships between Definitions

2.1 Definitions of Chaos

Definition 1: PSDIC

A function f has **pointwise sensitive dependence on initial conditions** (**PSDIC**) if for every x in the domain of f, there is an $\epsilon_x > 0$ such that for any $\delta > 0$, there exist a y in the domain and a positive integer n that satisfy $|x - y| < \delta$ and $|f^{[n]}(x) - f^{[n]}(y)| > \epsilon_x$.

Note that y and n depend on x, δ, and ϵ_x, and that ϵ_x depends on x.

Proposition 1: T has **PSDIC**.

Proof: Postponed; after the next definition, we will show that T has **USDIC**, which implies T has **PSDIC** (see Proposition 2).

Definition 2: USDIC

A function f has **uniform sensitive dependence on initial conditions** (**USDIC**) if there is an $\epsilon > 0$ such that for any x in the domain and for any $\delta > 0$, there exist a y in the domain and a positive integer n that satisfy $|x - y| < \delta$ and $|f^{[n]}(x) - f^{[n]}(y)| > \epsilon$.

Note that in this definition, ϵ does not depend on x. This means that if a function has **USDIC**, then it automatically has **PSDIC**.

This definition has been used to say a function is chaotic.

Proposition 2: T has **USDIC**.

Proof: The following proof is due to Gulick in [4]. Choose ϵ in the definition of **USDIC** to be

$\frac{1}{4}$, let $x \in [0,1]$, and let $\delta > 0$. We will show that there exist a dyadic rational v, an irrational w, and a positive integer m such that either

$$|T^{[m]}(x) - T^{[m]}(v)| > \frac{1}{4} \qquad \text{or} \qquad |T^{[m]}(x) - T^{[m]}(w)| > \frac{1}{4}$$

It is shown in [4] that for any dyadic rational v, there exists a positive integer n such that $T^{[n]}(v) = 0$, and, since 0 is a fixed point, it follows that $T^{[k]}(v) = 0$ for all positive integers k such that $k \geq n$. It is also shown in [4] that any irrational w is not fixed or eventually fixed, or even eventually periodic for T. Then, if some iterate of w is in $(0, \frac{1}{2})$, some future iterate of w must be greater than $\frac{1}{2}$, since T doubles each number in $(0, \frac{1}{2})$. So there exists some integer $m > n$ such that $T^{[m]}(w) > \frac{1}{2}$. Also, since dyadic rationals and irrationals are dense in $[0,1]$, for any δ neighborhood U of x, there exist a dyadic rational v and an irrational w in both U and $[0,1]$. The idea behind this proof is that for any x, there is a dyadic rational and an irrational close to x. Eventually iterates of v will be 0, whereas there are infinitely many iterates of w that will be greater than $\frac{1}{2}$. Since an iterate of x cannot be close to both 0 and $\frac{1}{2}$, the associated iterates of x must be a certain distance (in this case, the distance is $\frac{1}{4}$) from either the associated iterates of v or the associated iterates of w.

Formally, let n be a positive integer such that $T^{[n]}(v) = 0$, and let $m > n$ be a positive integer such that $T^{[m]}(w) > \frac{1}{2}$. Since $m > n$, $T^{[m]}(v) = 0$, and thus $|T^{[m]}(v) - T^{[m]}(w)| > \frac{1}{2}$. Using the triangle inequality, we see that

$$\frac{1}{2} < |T^{[m]}(v) - T^{[m]}(w)| \leq |T^{[m]}(v) - T^{[m]}(x)| + |T^{[m]}(x) - T^{[m]}(w)|.$$

Thus either

$$\frac{1}{4} < |T^{[m]}(v) - T^{[m]}(x)| \qquad \text{or} \qquad \frac{1}{4} < |T^{[m]}(x) - T^{[m]}(w)|.$$

Therefore, T has **USDIC**. □

Note that since we could choose the $\epsilon = \frac{1}{4}$ independently of the choice of x, T has **USDIC** and not just **PSDIC**.

Definition 3: ESDIC

A function f has **extreme sensitive dependence on initial conditions** (**ESDIC**) if there is an $\epsilon > 0$ such that for any x in the domain and for any $\delta > 0$, there exists a y in the domain such that $|x - y| < \delta$, $\limsup_{n\to\infty} |f^{[n]}(x) - f^{[n]}(y)| \geq \epsilon$, and $\liminf_{n\to\infty} |f^{[n]}(x) - f^{[n]}(y)| = 0$.

This definition is due to Du in [3].

Proposition 3: T has **ESDIC**.

Proof: We will show that T has extreme sensitive dependence on initial conditions with $\epsilon = \frac{1}{4}$ in the definition of **ESDIC**. Let $x \in [0, 1]$, $x \sim x_0x_1x_2\cdots$, a non-finite sequence. Let

$$x'_n = \begin{cases} 1 & : \text{ if } x_n = 0 \\ 0 & : \text{ if } x_n = 1 \end{cases}$$

and choose $y \in [0, 1]$ such that $y \sim s = x'_0x'_1x_2x_3x'_4x'_5x_6x_7\cdots x_{15}x'_{16}x'_{17}x_{18}x_{19}\cdots x_{63}x'_{64}x'_{65}x_{66}\cdots$, where the y_m, y_{m+1} terms have primes on them, for $m = 0, 1, 2, \ldots.$ Assume that this manipulation of the sequence associated with x does not produce a finite sequence. We will consider the case that s is finite after we examine the non-finite sequence case.

Since A is the set of all non-finite sequences and s is not a finite sequence by assumption, s is in A, and thus a $y \in [0.1]$ such that $h(y) = s$, or such that $y \sim s$, can be chosen since h is onto A. Consider the blocks of length $\frac{1}{4}$ of the interval $[0, 1]$ and the corresponding first two terms of the associated sequence (see Figure 1.2).

Let $z \in [0, 1]$, $z \sim z_0z_1z_2\cdots$, and let $w \sim z'_0z'_1z_2\ldots.$ If $z_0z_1 = 00$, then $w \sim z'_0z'_1z_2\cdots = 11\cdots$, so that $|z - w| > \frac{1}{4}$. If $z_0z_1 = 01$, then $w \sim z'_0z'_1z_2\cdots = 10\cdots$, and $|z - w| > \frac{1}{4}$. Similarly, if $z_0z_1 = 11$ or 10, then $|z - w| > \frac{1}{4}$.

Now consider $|x - y|$. By the statements above concerning z and w, we can see that

$$|x - y| > \frac{1}{4}, \text{ since } y \sim x'_0x'_1\cdots.$$

We also have that

$$|T^{[4]}(x) - T^{[4]}(y)| > \frac{1}{4}, \text{ since } T^{[4]}(y) \sim x'_4x'_5\cdots,$$

$$|T^{[16]}(x) - T^{[16]}(y)| > \frac{1}{4}, \text{ since } T^{[16]}(y) \sim x'_{16}x'_{17}\cdots.$$

More generally, for any positive integer n,

$$|T^{[4^n]}(x) - T^{[4^n]}(y)| > \frac{1}{4}, \text{ since } T^{[4^n]}(y) \sim x'_{4^n}x'_{4^n+1}\cdots.$$

Thus, $\limsup_{n\to\infty} |T^{[n]}(x) - T^{[n]}(y)| > \frac{1}{4}$.

Now it remains to show that $\liminf_{n\to\infty} |T^{[n]}(x) - T^{[n]}(y)| = 0$. But

$$|T^{[2]}(x) - T^{[2]}(y)| < \frac{1}{4}, \text{ since } T^{[2]}(y) \sim x_2x_3\cdots,$$

$$|T^{[6]}(x) - T^{[6]}(y)| < \frac{1}{2^{10}}, \text{ since } T^{[6]}(y) \sim x_6x_7\cdots x_{15}\cdots.$$

For any positive integer n, we observe that the number of identical initial terms of $T^{[4^n+2]}(y)$ and $T^{[4^n+2]}(x)$ is $4^{n+1} - 4^n - 2$. Thus

$$|T^{[4^n+2]}(x) - T^{[4^n+2]}(y)| < \frac{1}{2^{(4^{n+1}-4^n-2)}} \to 0 \text{ as } n \to \infty.$$

Thus, $\liminf_{n\to\infty} |T^{[n]}(x) - T^{[n]}(y)| = 0$.

If the sequence s is finite, then we can still use the same ideas as just discussed, but we will need to slightly modify the sequence, because a $y \in [0,1]$ such that $y \sim s$ does not necessarily exist, sine the sequence is not in A. Instead, we consider the sequence

$$t = x'_0x'_1x_2x_3x'_4x'_5 \;\; 1 \;\; x_7x_8\cdots x_{15}x'_{16}x'_{17} \;\; 1 \;\; x_{19}x_{20}\cdots x_{63}x'_{64}x'_{65} \;\; 1 \;\; x_{67}x_{68}\cdots$$

(We have replaced x_{4^n+2} for $n = 1, 2, 3, ...$ with the number one to guarantee that t is non-finite). Now $t \in A$, and therefore there exists a $y \in [0,1]$ such that $h(y) = t$, or $y \sim t$, since h is onto A. With such a sequence, we clearly still have that for any positive integer n,

$$|T^{[4^n]}(x) - T^{[4^n]}(y)| > \frac{1}{4}, \text{ since } T^{[4^n]}(y) \sim x'_{4^n}x'_{4^n+1}\cdots.$$

The argument considering the $\liminf_{n\to\infty} |T^{[n]}(x) - T^{[n]}(y)|$ is slightly altered.

$$|T^{[7]}(x) - T^{[7]}(y)| < \frac{1}{2^9}, \text{ since } T^{[7]}(y) \sim x_7x_8\cdots x_{15}\cdots,$$

$$|T^{[19]}(x) - T^{[19]}(y)| < \frac{1}{2^{45}}, \text{ since } T^{[19]}(y) \sim x_{19}x_{20}\cdots x_{63}\cdots.$$

For any positive integer n, we observe that the number of identical initial terms of $T^{[4^n+3]}(y)$ and $T^{[4^n+3]}(x)$ is $4^{n+1} - 4^n - 3$. Thus

$$|T^{[4^n+3]}(x) - T^{[4^n+3]}(y)| < \frac{1}{2^{(4^{n+1}-4^n-3)}} \to 0 \text{ as } n \to \infty,$$

and $\liminf_{n\to\infty} |T^{[n]}(x) - T^{[n]}(y)| = 0$.

So in both cases, T has **ESDIC**. □

Definition 4: Lyapunov Exponent

The **Lyapunov exponent** $\lambda(x)$ of f at x is defined by $\lambda(x) = \lim_{n\to\infty} \frac{1}{n} \ln |(f^{[n]})'(x)|$, if the limit exists.

To find a formula that is more tractable, notice that

$$\begin{aligned}
\lambda(x) &= \lim_{n\to\infty} \frac{1}{n} \ln |(f^{[n]})'(x)| \\
&= \lim_{n\to\infty} \frac{1}{n} \ln |f'(f^{[n-1]}(x)) \cdot (f^{[n-1]})'(x)|, \text{ by the chain rule} \\
&= \lim_{n\to\infty} \frac{1}{n} (\ln |f'(x_{n-1})| + \ln |(f^{[n-1]})'(x)|) \\
&= \lim_{n\to\infty} \frac{1}{n} (\ln |f'(x_{n-1})| + \ln |f'(f^{[n-2]}(x)) \cdot (f^{[n-2]})'(x)|), \text{ by the chain rule again} \\
&= \lim_{n\to\infty} \frac{1}{n} (\ln |f'(x_{n-1})| + \ln |f'(x_{n-2})| + \ln |(f^{[n-2]})'(x)|) \\
&\quad \vdots \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln |f'(x_k)|, \qquad (2.1)
\end{aligned}$$

if the limit exists. There are examples of continuous functions for which the limit does not necessarily exist for all x (or for any x, for that matter) in the domain of f, so that the Lyapunov exponent does not necessarily exist for such x. In fact, T is such a function, since $T'(\frac{1}{2})$ does not exist. In some definitions, a function f is considered chaotic if the limit exists for a dense set of x in the domain of f, and if for these values of x, the Lyapunov exponent of f is positive. We will say f has a positive Lyapunov exponent (**PLE**) if $\lambda(x)$ exists for a dense set of x and if $\lambda(x) > 0$ for all x in the domain of f such that $\lambda(x)$ exists. Moreover, f is **PLE chaotic** if f has a positive

Lyapunov exponent.

This definition of **PLE chaos** has been used to characterize a function as chaotic.

Proposition 4: T has **PLE**.

Proof: Let $x \in [0,1]$.

$$T'(x) = \begin{cases} 2 & : \quad 0 < x < \frac{1}{2} \\ \\ -2 & : \quad \frac{1}{2} < x < 1 \end{cases}$$

So $|T'(x_k)| = 2$ if $T'(x_k)$ exists. Thus, if $\lambda(x)$ exists,

$$\begin{aligned} \lambda(x) &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln |T'(x_k)| \\ &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln(2) \\ &= \lim_{n\to\infty} \frac{1}{n} (n)(\ln(2)) \\ &= \lim_{n\to\infty} \ln(2) \\ &= \ln(2), \end{aligned}$$

and $\ln(2) > 0$. Now it only remains to show that the Lyapunov exponent actually exists for a dense set of values in the domain of T. But if x is irrational, then $T^{[n]}(x) \neq 0, \frac{1}{2},$ or 1 (the only possible values where the derivative of T fails to exist) for any positive integer n, so $\lambda(x)$ exists. The set of irrational numbers is dense in $[0,1]$, and thus, T has **PLE**. $\square$

Definition 5: Transitive

A real-valued function $f : X \to X$ is **transitive** if for every pair of non-empty open intervals U and V in X, there exists a positive integer n such that $f^{[n]}(U) \cap V \neq \emptyset$.

Proposition 5: T is transitive.

Proof: In order to show that T is transitive, we will first show that there is an $x \in [0,1]$ such that the orbit of x under T is dense in $[0,1]$.

Consider the sequence s:

$$s = \underbrace{0\,1}_{\text{1 block}}\ \underbrace{00\,01\,10\,11}_{\text{2 block}}\ \underbrace{000\,001\,010\,011\,100\,101\,110\,111}_{\text{3 block}}\cdots,$$

where s is composed of all blocks of singles, doubles, triples, etc. of 0's and 1's. Since s is a non-finite sequence of 0's and 1's, $s \in A$. Then, since h is onto A, there exists some $x^* \in [0, 1]$ such that $h(x^*) = s$, which implies that $x^* \sim x_0x_1x_2x_3x_4 \cdots = 01\ 00011011\ 000001010011100101110111 \cdots$. Therefore,

$$x^* \in [0, \frac{1}{2}] \text{ since } x^* \sim 0x_1x_2 \cdots,$$

$$T(x^*) \in [\frac{1}{2}, 1] \text{ since } T(x^*) \sim 1x_2x_3 \cdots,$$

$$T^{[2]}(x^*) \in [0, \frac{1}{4}] \text{ since } T^{[2]}(x^*) \sim 00x_4x_5 \cdots,$$

$$T^{[4]}(x^*) \in [\frac{1}{4}, \frac{1}{2}] \text{ since } T^{[4]}(x^*) \sim 01x_6x_7 \cdots,$$

$$T^{[6]}(x^*) \in [\frac{3}{4}, 1] \text{ since } T^{[6]}(x^*) \sim 10x_8x_9 \cdots,$$

etc. In general for any interval $L = [\frac{k}{2^n}, \frac{k+1}{2^n}]$, there will exist a positive integer m such that $T^{[m]}(x) \in L$. Thus, the orbit of x under T is dense in $[0, 1]$.

Now we will show T is transitive. Let U and V be non-empty open intervals in $[0, 1]$. Since the orbit of x is dense, there exists a positive integer n such that $y \equiv T^{[n]}(x) \in U$. But then, again since the orbit of x is dense, there exists a positive integer m such that $T^{[n+m]}(x) = T^{[m]}(y) \in V$. Thus, T is transitive. $\square$

Note that the second part of the proof can be generalized to show that if a function has an element in the domain with a dense orbit, then the function is transitive.

Definition 6: Shared Periodic Orbit

A function f has a **shared periodic orbit** if for every pair of non-empty open intervals U and V in X, there is a periodic point $p \in U$ such that $f^{[n]}(p) \in V$ for some positive integer n.

Proposition 6: T has a shared periodic orbit.

Proof: Let U and V be non-empty open intervals in $[0, 1]$. There exist positive integers k and

n such that $[\frac{k}{2^n}, \frac{k+1}{2^n}] \subseteq U$. Similarly, there exist positive integers l and m, with $m > n$, such that $[\frac{l}{2^m}, \frac{l+1}{2^m}] \subseteq V$. We can construct a sequence $s = x_0x_1x_2\cdots$ in a manner similar to the one in Proposition 5 by specifying $x_0, x_1, x_2, ..., x_{a+b}$ such that $h(x^*) = s$ for some $x^* \in [0,1]$ and $T^{[a]}(x^*) \in [\frac{k}{2^n}, \frac{k+1}{2^n}] \subseteq U$, $T^{[a+b]}(x^*) \in [\frac{l}{2^m}, \frac{l+1}{2^m}] \subseteq V$, where h is as defined in (1.1). If we then let $s = \overline{x_0x_1x_2\cdots x_{a+b}}$, x^* will be periodic. Thus, T has a shared periodic orbit. $\square$

Definition 7: Dense Set of Periodic Points

A function $f : X \to X$ has a **dense set of periodic points** if for every non-empty open set $U \subseteq X$, there exists a periodic point p such that $p \in U$.

Proposition 7: T has a dense set of periodic points.

Proof: Note that any number in $[0,1]$ of the form $\frac{\text{even integer}}{\text{odd integer}}$ is a periodic point for T (shown in [4]; in Section 1.3 we looked at the example $x = \frac{2}{9}$). Thus, to show that T has a dense set of periodic points, it suffices to show that the numbers of the form $\frac{\text{even integer}}{\text{odd integer}}$ are dense in $[0,1]$. Let U be a non-empty open set in $[0,1]$. Then U must contain some interval $[a,b]$. Let $d = b - a$ and choose a positive odd integer n such that $n > \frac{2}{d}$. Then for any positive integer l,

$$\frac{l}{n} - \frac{l-1}{n} = \frac{1}{n} < \frac{d}{2}.$$

Since $[a,b] \subseteq U$ has length d, there must exist a positive integer k, $2 \le k \le n-1$ such that $\frac{k}{n}$ and $\frac{k-1}{n}$ are both in $[a,b] \subseteq U$. Then either k or $k-1$ is even, so that either $\frac{k}{n}$ or $\frac{k-1}{n}$ is of the form $\frac{\text{even integer}}{\text{odd integer}}$, which implies there exists a number of the form $\frac{\text{even integer}}{\text{odd integer}} \in [a,b] \subseteq U$. Thus, U contains a periodic point, and therefore T has a dense set of periodic points. $\square$

Definition 8: Strong Chaos

A function f has **strong chaos** if f has **USDIC** as well as a dense set of periodic points, and is also transitive.

The set of conditions here is employed to define chaos in Devaney [2], and has been used extensively by mathematicians as the definition of chaos.

Proposition 8: T has **Strong Chaos**.

Proof: By Propositions 2, 5, and 7, T has **USDIC**, transitivity, and a dense set of periodic points. Thus, T has strong chaos. $\square$

The above definitions and conditions have all been used either solely (where indicated after the definition) or in combined (as in the definition of strong chaos) to characterize a function as chaotic. In the next section we will look at the relationships between these definitions.

2.2 Relationships Between Definitions

In this section, we will examine the relationships between pairs of definitions. In some cases two or more of the conditions defined in Chapter 1 have been used to characterize a function as chaotic. Thus, the comparisons presented in this section will help clarify previous potential definitions of chaos. When a condition implies another condition, a proof is provided or referenced; when a condition fails to imply another condition, a counter-example is provided. Figure 2.1 summarizes the main results of Chapter 2, with the number on an arrow referring to the proposition in which a proof or counterexample of the relationship is provided.

As in Chapter 1, we will consider real-valued functions on intervals of the real line.

2.3 Proofs of Implications

Proposition 9: ESDIC implies USDIC.

Proof: Let $f : X \to X$ have **ESDIC**, let $x \in X$, and let U be any open neighborhood of x. Then by the definition of **ESDIC**, there exists $\delta > 0$ and there is a $y \in U$ such that $\limsup_{n\to\infty} |f^{[n]}(x) - f^{[n]}(y)| \geq \epsilon$. So clearly there there exists a positive integer n such that $|f^{[n]}(x) - f^{[n]}(y)| \geq \frac{\epsilon}{2}$, and thus f has **USDIC**. $\square$

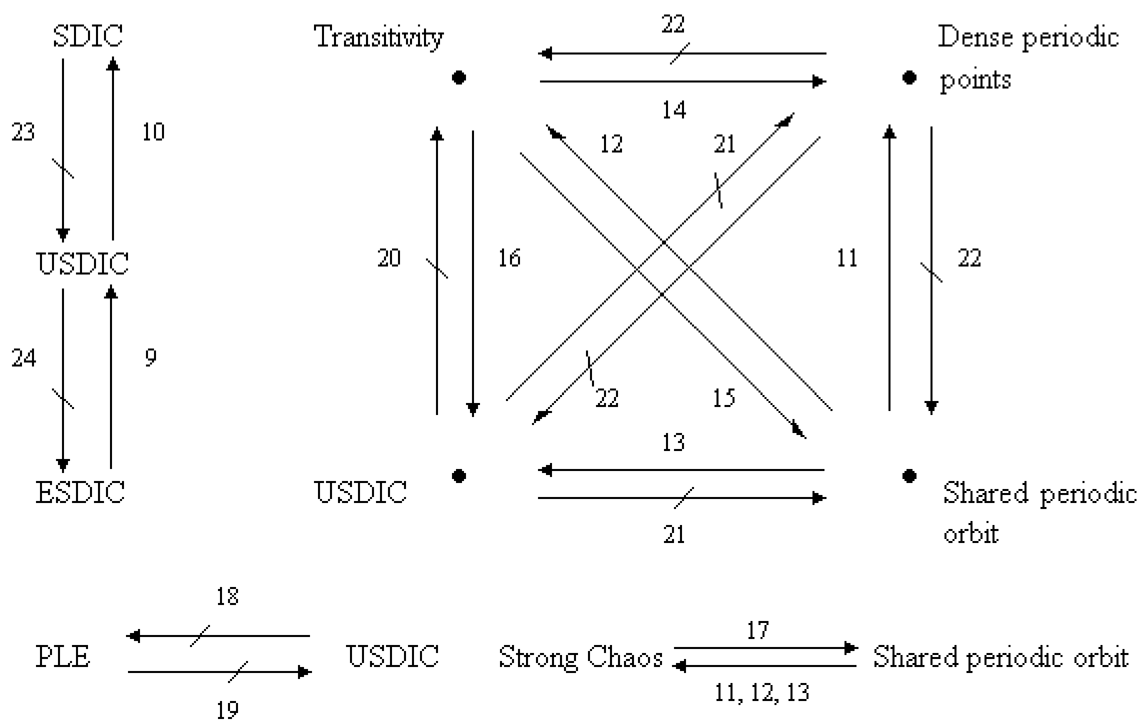

Figure 2.1: Summary of Results. Numbers correspond with Propositions in which the relationship is proven.

Proposition 10: USDIC implies PSDIC.

Proof: The only difference between the definitions of **USDIC** and **PSDIC** relates to the conditions on ϵ. In the definition of **USDIC**, the ϵ is independent of x, whereas in the definition of **PSDIC**, the ϵ is dependent on x. Thus, this implication is trivial. $\square$

Proposition 11: A shared periodic orbit implies a dense set of periodic points.

Proof: Let $f : X \to X$ have a shared periodic orbit, and let U be any non-empty open subset of X. Since every non-empty open set in X contains a non-empty open interval, by the definition of shared periodic orbit, for any non-empty open subset V of X, there exists a positive integer n such that $f^{[n]}(p) \in V$ for some periodic point $p \in U$. Thus, there exists a periodic point $p \in U$, and therefore, f has a dense set of periodic points. $\square$

Proposition 12: A shared periodic orbit implies transitivity.

Proof: Let $f : X \to X$ have a shared periodic orbit, and let U and V be non-empty open intervals in X. Then by the definition of shared periodic orbit, there is a periodic point $p \in U$ and a positive integer n such that $f^{[n]}(p) \in V$. Thus since $f^{[n]}(p) \in f^{[n]}(U)$, we have $f^{[n]}(p) \in (f^{[n]}(U) \cap V)$, which implies $f^{[n]}(U) \cap V \neq \emptyset$, so f is transitive. $\square$

Proposition 13: A shared periodic orbit implies USDIC.

Proof: Let $f : X \to X$ have a shared periodic orbit. Then by Proposition 12, f is transitive. By Proposition 11, f has a dense set of periodic points. Then, as shown in [1], a function that is transitive and has a dense set of periodic points has **USDIC**. $\square$

Proposition 14: Transitivity implies a dense set of periodic points.

Proof: In [7] Vellekoop and Berglund show that if f is transitive, then f has a dense set of periodic points. $\square$

Proposition 15: Transitivity implies a shared periodic orbit.

Proof: Let $f : X \to X$ be transitive, and let U and V be non-empty open intervals in X. By the definition of transitivity, there exists $x \in U$ such that $f^{[n]}(x) \in V$ for some positive integer n. Since V is an open set, there exists an $\epsilon > 0$ such that $(f^{[n]}(x) - \epsilon, f^{[n]}(x) + \epsilon) \subseteq V$. By hypothesis f is continuous, which implies that $f^{[n]}$ is continuous. By the definition of continuity, there exists some $\delta > 0$ such that if $|x - y| < \delta$, then $|f^{[n]}(x) - f^{[n]}(y)| < \epsilon$, which implies that $f^{[n]}(y) \in (f^{[n]}(x) - \epsilon, f^{[n]}(x) + \epsilon) \subseteq V$. Let $W \equiv ((x - \delta, x + \delta) \cap U)$. Then W is open. By Proposition 14, since f is transitive, f has a dense set of periodic points. So there exists a periodic point $p \in W$. But $f^{[n]}(W) \subseteq (f^{[n]}(x) - \epsilon, f^{[n]}(x) + \epsilon) \subseteq V$, so $f^{[n]}(p) \in V$. Since $W \subseteq U$, we know that $p \in U$. Thus, we have found a periodic point p such that $p \in U$ and $f^{[n]}(p) \in V$. Therefore, f has a shared periodic orbit. $\square$

Proposition 16: Transitivity implies USDIC.

Proof: Let f be transitive. Then by Proposition 14, f has a dense set of periodic points. In [1] it is shown that if f is transitive and has a dense set of periodic points, then f has **USDIC**. Thus, f has **USDIC**. $\square$

Proposition 17: Strong Chaos implies a shared periodic orbit.

Proof: Let $f : X \to X$ have strong chaos. So f is transitive and has a dense set of periodic points, by the definition of strong chaos. Let U and V be non-empty open intervals in X. Since f is transitive there exists a positive integer n such that $W \equiv (f^{[n]}(U) \cap V) \neq \emptyset$. Therefore, let $x \in U$ such that $f^{[n]}(x) \in W$. Consider the inverse image of W under $f^{[n]}$:

$$f^{-[n]}(W) = f^{-[n]}(f^{[n]}(U) \cap V) \supseteq (U \cap f^{-[n]}(V)).$$

So $f^{-[n]}(W)$ contains an open set, since U and $f^{-[n]}(V)$ are non-empty and open, and the intersection of two non-empty open sets is open. Also, $f^{-[n]}(W)$ is non-empty, since $x \in f^{-[n]}(W)$. Thus, since f has a dense set of periodic points, there exists a periodic point $p \in f^{-[n]}(W)$, which implies that $p \in U$ and $f^{[n]}(p) \in V$. Therefore, f has a shared periodic point. $\square$

Note that this proof only used the facts that f was transitive and had a dense set of periodic points. We did not use the fact that f had **USDIC**. Thus, this proof actually shows that transitivity and a dense set of periodic points together imply a shared periodic orbit.

2.4 Counterexamples

Proposition 18: USDIC does not imply PLE (and therefore PSDIC does not imply PLE).

Proof: Let $F : [0, \infty) \to [0, \infty)$ be defined by $F(x) = (\sqrt{x} + 1)^2$ (See Figure 2.2). This function and the following proof are from Pennings in [6].

First we will show that F has **USDIC**.

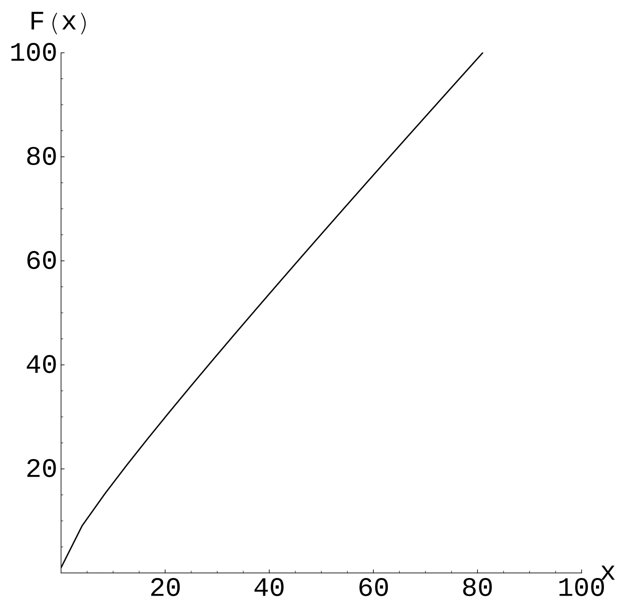

Figure 2.2: $F(x)$: A function with **USDIC** but not **PLE**

By an induction argument we will show that $F^{[n]}(x) = (\sqrt{x} + n)^2$ for all positive integers n.

Base case: $n = 1$

Then $F^{[n]}(x) = F^{[1]}(x) = (\sqrt{x}+1)^2$.

Now assume that $F^{[n]}(x) = (\sqrt{x}+n)^2$ for all n such that $1 \leq n \leq k$, where k is an arbitrary positive integer. Consider the case $n = k+1$.

$$\begin{aligned} F^{[n]}(x) = F^{[k+1]}(x) &= F(F^{[k]}(x)) \\ &= F((\sqrt{x}+k)^2), \text{ by the inductive hypothesis} \\ &= (\sqrt{(\sqrt{x}+k)^2}+1)^2 \\ &= (\sqrt{x}+k+1)^2. \end{aligned}$$

Thus, by induction, for all positive integers n,

$$F^{[n]}(x) \quad = \quad (\sqrt{x}+n)^2. \tag{2.2}$$

Let $x \in [0,\infty)$, and let $\delta > 0$ be arbitrary. Then by (2.2),

$$\begin{aligned} |F^{[n]}(x) - F^{[n]}(x+\delta)| &= |(\sqrt{x}+n)^2 - (\sqrt{x+\delta}+n)^2| \\ &= |2n\sqrt{x} - \delta - 2n\sqrt{x+\delta}| \\ &= \delta + 2n(\sqrt{x+\delta} - \sqrt{x}) \end{aligned}$$

As n increases to infinity, so will $\delta + 2n(\sqrt{x+\delta} - \sqrt{x}) = |F^{[n]}(x) - F^{[n]}(x+\delta)|$, and thus $|F^{[n]}(x) - F^{[n]}(x+\delta)| > \epsilon$ for a large enough n, regardless of the choice of x and ϵ. Thus, F has **USDIC**.

Now we will show that F does not have **PLE**.

For all $x \in [0,\infty)$, $F'(x) = 2(\sqrt{x}+1)\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{x}} = \frac{\sqrt{x}+1}{\sqrt{x}}$. By (2.2), $x_k = F^{[k]}(x_0) = (\sqrt{x_0}+k)^2$, so we have $\sqrt{x_k} = \sqrt{x_0}+k$. Therefore, $F'(x_k) = \frac{\sqrt{x_k}+1}{\sqrt{x_k}} = \frac{\sqrt{x_0}+k+1}{\sqrt{x_0}+k}$. Thus,

$$\begin{aligned} \sum_{k=0}^{n-1} \ln|F'(x_k)| &= \sum_{k=0}^{n-1} \ln|\frac{\sqrt{x_0}+k+1}{\sqrt{x_0}+k}| \\ &= \ln(\frac{\sqrt{x_0}+1}{\sqrt{x_0}}) + \ln(\frac{\sqrt{x_0}+2}{\sqrt{x_0}+1}) + ... + \ln(\frac{\sqrt{x_0}+n}{\sqrt{x_0}+n-1}) \\ &= [\ln(\sqrt{x_0}+1) - \ln(\sqrt{x_0})] + [\ln(\sqrt{x_0}+2) - \ln(\sqrt{x_0}+1)] + \dots \\ &\quad +[\ln(\sqrt{x_0}+n) - \ln(\sqrt{x_0}+n-1)] \end{aligned}$$

$$= \ln(\sqrt{x_0}+n) - \ln(\sqrt{x_0}) \tag{2.3}$$

Therefore,

$$\begin{aligned}
\lambda(x_0) &= \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \ln|F'(x_k)|, \text{ by (2.1)} \\
&= \lim_{n\to\infty} \frac{(\ln(\sqrt{x_0}+n) - \ln(\sqrt{x_0}))}{n}, \text{ by (2.3)} \\
&= \lim_{n\to\infty} \frac{\frac{1}{\sqrt{x_0}+n}}{1}, \text{ by l'Hôpital's rule} \\
&= \lim_{n\to\infty} \frac{1}{\sqrt{x_0}+n} \\
&= 0
\end{aligned}$$

So the Lyapunov exponent at an arbitrary x_0 is 0, and thus F does not have **PLE**. □

Proposition 19: PLE does not imply PSDIC (and thus PLE does not imply USDIC or ESDIC).

Proof: Let $G : [0,1] \to [0,1]$ as follows:

$$G(x) = \begin{cases} 0 & : \quad x = 0 \\ 4x - 3(\frac{2^n-1}{2^n}) & : \quad x \in (\frac{2^n-1}{2^n}, \frac{2^{n+2}-3}{2^{n+2}}], n = 0,1,2,... \\ -2x + 3(\frac{2^{n+1}-1}{2^{n+1}}) & : \quad x \in (\frac{2^{n+2}-3}{2^{n+2}}, \frac{2^{n+1}-1}{2^{n+1}}], n = 0,1,2,... \\ 1 & : \quad x = 1 \end{cases}$$

Note that G is defined so that $G'(x) = 4$ for $x \in (0, \frac{1}{4}), (\frac{1}{2}, \frac{5}{8}), (\frac{3}{4}, \frac{13}{16}), (\frac{7}{8}, \frac{29}{32}), ...$ and $G'(x) = -2$ for $x \in (\frac{1}{4}, \frac{1}{2}), (\frac{5}{8}, \frac{3}{4}), (\frac{13}{16}, \frac{7}{8}), ...$ (G is an infinite number of lopsided, shrinking tents along the line $y = x$. See Figure 2.3.)

Now consider $\lambda(x)$ for $x \in [0,1]$ such that x is not a dyadic rational. Then

$$\begin{aligned}
\lambda(x) &= \lim_{n\to\infty} \frac{1}{n} \ln|(G^{[n]})'(x)| \\
&= \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \ln|G'(x_k)|, \text{ by (2.1)}
\end{aligned}$$

But if $G'(x_k)$ exists, then $G'(x_k) = 4$ or -2, so $\ln|G'(x_k)| = \ln(2)$ or $\ln(4)$. Thus for all x where the Lyapunov exponent exists,

$$\lambda(x) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \ln|G'(x_k)|$$

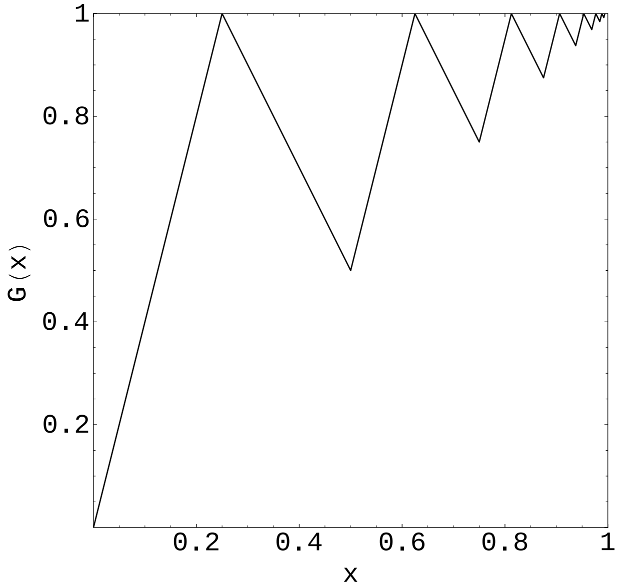

Figure 2.3: $G(x)$: A function with **PLE** but not **PSDIC**

$$\begin{aligned}
&\geq \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \ln(2) \\
&= \lim_{n\to\infty} \frac{1}{n} n \cdot \ln(2) \\
&= \lim_{n\to\infty} \ln(2) \\
&= \ln(2) \\
&> 0
\end{aligned}$$

So clearly $\lambda(x) > 0$ wherever $\lambda(x)$ exists. Now we must show that $\lambda(x)$ does exist for a dense set of $x \in [0,1]$. The only points where G' fails to exist are the points $x = \frac{2^n-1}{2^n}$ or $x = \frac{2^{n+2}-3}{2^{n+2}}$ for any positive integer n. Thus, clearly when x is not a dyadic rational, $G^{[n]}(x)$ will never be of the form $\frac{2^n-1}{2^n}$ or $\frac{2^{n+2}-3}{2^{n+2}}$ for any n, so $G'(x_k)$ will always exist, which implies that $\lambda(x)$ exists and that G has **PLE**.

Next we will show that $x \leq G(x)$, and that then G does not have **PSDIC**. The function G is a piecewise linear function, so the only possible local extrema are at the endpoints of the intervals on which G is piecewise defined. Since $G(x)$ is increasing for $x \in (\frac{2^n-1}{2^n}, \frac{2^{n+2}-3}{2^{n+2}})$, $n = 0, 1, 2, ...$ (the slope on these intervals is 4) and decreasing for $x \in (\frac{2^{n+2}-3}{2^{n+2}}, \frac{2^{n+1}-1}{2^{n+1}})$, $n = 0, 1, 2, ...$ (the slope on these intervals is -2), clearly the local minima occur at numbers of the form $x = \frac{2^{n+1}-1}{2^{n+1}}$, $n = -1, 0, 1, 2, ...$

Let $x = \frac{2^{n+1}-1}{2^{n+1}}$ for an arbitrary $n = -1, 0, 1, 2, ...$ If $n = -1$, then $x = 0$ and $G(0) = 0 = x$. For all other values of n, $G(x) = G(\frac{2^{n+1}-1}{2^{n+1}}) = -2(\frac{2^{n+1}-1}{2^{n+1}}) + 3(\frac{2^{n+1}-1}{2^{n+1}}) = \frac{2^{n+1}-1}{2^{n+1}} = x$. Thus, at the local minima, $G(x) = x$, and therefore $G(x) \geq x$ for all other values of $x \in [0,1]$. So we have shown that for any $x \in [0,1]$, $x \leq G(x)$. Note that this implies that $x \leq G^{[n]}(x)$ for all positive integers n.

Now we will show that G does not have **PSDIC** by showing that G does not have the requirements for **PSDIC** at $x = 1$. Choose any ϵ, with $0 < \epsilon < 1$. Let $0 < \delta < \epsilon$. Then if $|1-y| < \delta$ and $y \in [0,1]$, we must have $y \in (1-\delta, 1]$. Also, for any positive integer n, we have $y \leq G^{[n]}(y)$ by the above argument and $G^{[n]}(1) = 1$ by the definition of G. Thus,

$$\begin{aligned} |G^{[n]}(1) - G^{[n]}(y)| &= |1 - G^{[n]}(y)| \\ &\leq |1-y| \\ &< \delta \\ &< \epsilon \end{aligned}$$

So there does not exist an y and a positive integer n such that $|1-y| < \delta$ and $|G^{[n]}(1) - G^{[n]}(y)| > \epsilon$. Thus, G does not have **PSDIC** at 1, and so G does not have **PSDIC**. □

In this example, G has a positive Lyapunov exponent at a dense set of x in the domain but fails to have **PSDIC** at only one element in the domain, namely $x = 1$.

Proposition 20: USDIC does not imply transitivity.

Proof: Let $K : [0, \frac{3}{2}] \to [0, \frac{3}{2}]$ as follows:

$$K(x) = \begin{cases} 2x & : \quad x \in [0, \frac{1}{2}] \\ -2x+2 & : \quad x \in (\frac{1}{2}, 1] \\ 2x-2 & : \quad x \in (1, \frac{3}{2}] \end{cases}$$

The function K is one and a half tents (see Figure 2.4).

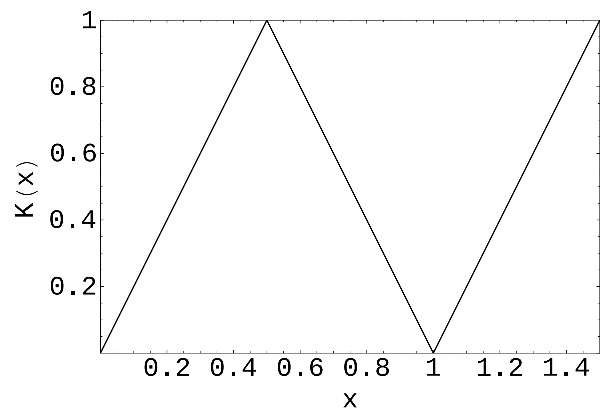

Figure 2.4: $K(x)$: A function with **USDIC** but not transitivity

First we will show that K has **USDIC**. If $x \in [0,1]$, then we are considering K on $[0,1]$, and on $[0,1]$, K is identical to the Tent function defined in Chapter 1. In Proposition 2, we showed that the Tent function has **USDIC** by considering iterates of dyadic rationals and irrational numbers. Using the same argument, we can see that for any $x \in [0,1]$ and for any $\delta > 0$ there exists a dyadic rational v, an irrational w, and a positive integer n such that $|x-v| < \delta$, $|x-w| < \delta$, and either $|K^{[n]}(x) - K^{[n]}(v)| > \frac{1}{4}$ or $|K^{[n]}(x) - K^{[n]}(w)| > \frac{1}{4}$.

Now consider $x \in (1, \frac{3}{2}]$ and let $\delta > 0$. Since the dyadic rational and irrationals are dense, there exists a dyadic rational $v \in (1, \frac{3}{2}]$ and an irrational $w \in (1, \frac{3}{2}]$ such that $|x-v| < \delta$ and $|x-w| < \delta$. Clearly $K(v) \in [0,1]$ and $K(w) \in [0,1]$. Also, by the definition of K, $K(v)$ must be a dyadic rational and $K(w)$ must be an irrational. So as in Proposition 4, there exists a positive integer n such that $K^{[n]}(K(v)) = K^{[n+1]}(v) = 0$ and $K^{[n]}(K(w)) = K^{[n+1]}(w) > \frac{1}{2}$. So either $|K^{[n+1]}(x) - K^{[n+1]}(v)| > \frac{1}{4}$ or $|K^{[n+1]}(x) - K^{[n+1]}(w)| > \frac{1}{4}$.

Thus, for any $x \in [0, \frac{3}{2}]$ and any $\delta > 0$, there exists some $y \in [0, \frac{3}{2}]$, where y is either a dyadic rational or an irrational, and some positive integer n such that $|x-y| < \delta$ and $|K^{[n]}(x) - K^{[n]}(y)| > \frac{1}{4}$. Therefore, K has **USDIC**.

Now we will show that K is not transitive. Let $U = (0,1)$ and $V = (1, \frac{3}{2})$. Then $K(U) \subseteq [0,1]$, which by the definition of K implies that $K^{[n]}(U) \subseteq [0,1]$ for all positive integers n. But $[0,1] \cap V = \emptyset$, which implies that $K^{[n]}(U) \cap V = \emptyset$ for all positive integers n. So K is not transitive. $\square$

Proposition 21: USDIC does not imply a dense set of periodic points and does not imply a shared periodic point.

Proof: Consider F as in Proposition 18. Then F has **USDIC**, and $F^{[n]}(x) > x$ for all $x \in [0, \infty)$ and for any positive integer n. Therefore, $\{F^{[n]}(x)\}_{n=1}^{\infty}$ is an increasing sequence, which implies F does not have any periodic points. Thus, clearly F does not have a dense set of periodic points nor a shared periodic orbit, since F has no periodic points. $\square$

Proposition 22: A dense set of periodic points does not imply transitivity, does not imply a shared periodic orbit, and does not imply USDIC.

Proof: Consider the identity function $f(x) = x$ on the interval $[0, 2]$. Clearly the identity function has a dense set of periodic points, since every point is a fixed point. However, $f^{[n]}(U) = U$ for any non-empty open interval U and for any positive integer n. Thus, if we let $U = (0, 1)$ and $V = (1, 2)$, then $f^{[n]}(U) \cap V = U \cap V = \emptyset$ for all positive integers n, which implies f is not transitive. Then, since f is not transitive, f cannot have a shared periodic orbit by Proposition 12. Finally, we show that f does not have **USDIC**. Let $\epsilon > 0$, and choose any $x \in [0, 2]$. Let $\delta < \epsilon$, and let y be any number in the interval $(x - \delta, x + \delta) \cap [0, 2]$. Then $|f^{[n]}(x) - f^{[n]}(y)| = |x - y| < \delta < \epsilon$ for every positive integer n. Thus, the identity function does not have **USDIC**. $\square$

Proposition 23: PSDIC does not imply USDIC.

Proof: Define a function $h_n : [0, \frac{1}{2^n}] \to [0, \frac{1}{2^n}]$ for $n = 1, 2, 3, ...$ such that

$$h_n(x) = \begin{cases} 2x & : \quad x \in [0, \frac{1}{2^{n+2}}] \\ -2x + \frac{1}{2^n} & : \quad x \in (\frac{1}{2^{n+2}}, \frac{1}{2^{n+1}}] \\ 2x - \frac{1}{2^n} & : \quad x \in (\frac{1}{2^{n+1}}, \frac{1}{2^n}] \end{cases}$$

Now define a function $H : [0, 1) \to [0, 1)$ by letting

$$H(x) = h_1(x) \text{ for } x \in [0, \frac{1}{2}),$$

$$H(x) = h_2(x - \frac{1}{2}) + \frac{1}{2} \text{ for } x \in [\frac{1}{2}, \frac{3}{4}),$$

$$H(x) = h_3(x - \frac{3}{4}) + \frac{3}{4} \text{ for } x \in [\frac{3}{4}, \frac{7}{8}),$$

etc, so that

$$H(x) = h_n(x - \frac{2^{n-1}-1}{2^{n-1}}) + \frac{2^{n-1}-1}{2^{n-1}} \text{ for } x \in [\frac{2^{n-1}-1}{2^{n-1}}, \frac{2^n-1}{2^n})$$

(see Figure 2.5).

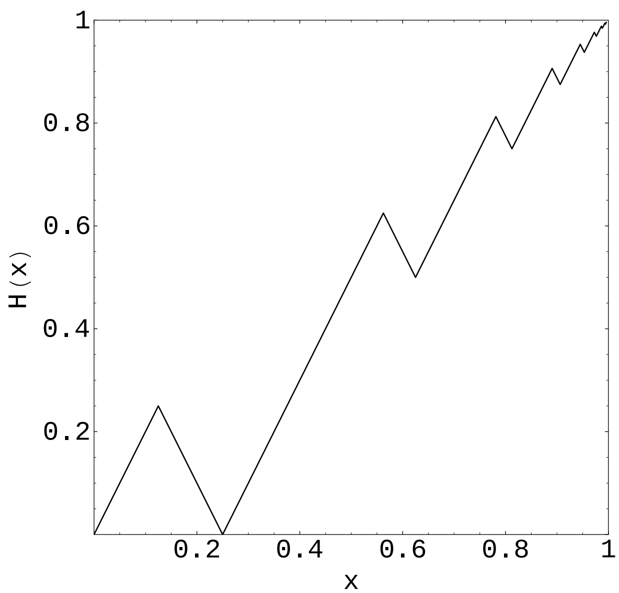

Figure 2.5: $H(x)$: A function with **PSDIC** but not **USDIC**

We will prove that H has **PSDIC** but not **USDIC** by considering each interval of H where H is equal to h_n for some positive integer n, possibly shifted up and to the right. We will see that on $[0, \frac{1}{2^n})$, h_n has **PSDIC** for each $n = 0, 1, 2, \ldots$ by first showing that if $x \in (\frac{1}{2^{n+1}}, \frac{1}{2^n})$, then eventually for some positive integer m, $h_n^{[m]}(x)$ will be in the interval $[0, \frac{1}{2^{n+2}})$. On the interval $[0, \frac{1}{2^{n+2}})$, h_n is a modified Tent function and therefore has **PSDIC**.

First we will show that if $x \in (\frac{1}{2^{n+1}}, \frac{1}{2^n})$, then $h_n(x) < x$. Consider $x \in (\frac{1}{2^{n+1}}, \frac{1}{2^n})$. Then

$$\begin{aligned} h_n(x) - x &= 2x - \frac{1}{2^n} - x \\ &= x - \frac{1}{2^n} \\ &< 0, \text{ since } x < \frac{1}{2^n}. \end{aligned}$$

Thus, since $h_n(x) < x$ and $h_n(x) \in [0, \frac{1}{2^n})$ for $x \in (\frac{1}{2^{n+1}}, \frac{1}{2^n})$, $\{h_n^{[m]}(x)\}_{m=0}^{\infty}$ is a decreasing sequence for $x \in (\frac{1}{2^{n+1}}, \frac{1}{2^n})$, with no fixed points in the interval $(\frac{1}{2^{n+1}}, \frac{1}{2^n})$. So eventually for some positive integer m, $h_n^{[m]}(x) \in [0, \frac{1}{2^{n+1}}]$. Therefore, to prove that h_n has **PSDIC**, it is sufficient to consider h_n on the interval $[0, \frac{1}{2^{n+1}}]$. But once we are considering h_n on the interval $[0, \frac{1}{2^{n+1}}]$, the argument showing that h_n has **PSDIC** is analogous to the argument showing that the Tent function has **USDIC** in Proposition 2, letting ϵ_x in the definition of **PSDIC** be $\frac{1}{2^{n+3}}$. In any δ-neighborhood of any $x \in [0, \frac{1}{2^{n+1}}]$, we can find an irrational number v and a dyadic rational number w such that for some positive integer m,

$$|h_n^{[m]}(x) - h_n^{[m]}(v)| > \epsilon_x \text{ or } |h_n^{[m]}(x) - h_n^{[m]}(w)| > \epsilon_x.$$

So on $[0, \frac{1}{2^n})$, h_n has **PSDIC** for each $n = 0, 1, 2, \ldots$. Shifting h_n up and to the left will not alter the conditions required for **PSDIC**, and thus H has **PSDIC** on every interval of the form $[\frac{2^{n-1}-1}{2^{n-1}}, \frac{2^n-1}{2^n})$ in $[0, 1]$. Therefore, H has **PSDIC**. Note that although we can show that for each positive integer n, h_n has **USDIC**, the ϵ_x in the proof above depends on n, and therefore H does not necessarily have **USDIC**.

In fact, we will now show that H does not have **USDIC**. Let $\epsilon > 0$ be arbitrary. Choose a positive integer n such that $\frac{1}{2^n} < \epsilon$. Now consider h_n. For any $z \in [0, \frac{1}{2^{n+1}}]$, $h_n(z)$ remains in the interval $[0, \frac{1}{2^n}]$. Therefore, let $x = 0$, and let $\delta < \frac{1}{2^n}$. Then for any $y \in [0, \frac{1}{2^n}]$ such that $|x - y| < \delta$, we have $h_n^{[m]}(y) \in [0, \frac{1}{2^n})$ for all positive integers m, which implies $|h_n^{[m]}(x) - h_n^{[m]}(y)| = |0 - h_n^{[m]}(y)| < \frac{1}{2^n} < \epsilon$ for every positive integer m. Now since H is a transposition of h_n on some subinterval of the form $[\frac{2^{n-1}-1}{2^{n-1}}, \frac{2^n-1}{2^n})$, H does not have **USDIC**. □

Proposition 24: USDIC does not imply ESDIC.

Proof: In [3] Du shows that a function with **USDIC** does not necessarily have **ESDIC** by letting $S = \{x | x = x_0 x_1 x_2 \cdots,$ where $x_n = 0$ or $1\}$ and defining the metric d on S by $d(x, y) = \sum_{n=0}^{\infty} \frac{x_n - y_n}{2^{n+1}}$. Finally, define the shift map σ by $\sigma(x) = \sigma(x_0 x_1 x_2 \cdots) = x_1 x_2 \cdots$. Du shows that σ has **USDIC** but not **ESDIC**. □

Proposition 25: PLE does not imply a shared periodic orbit, and PLE does not imply a dense set of periodic points.

Proof: Consider the function $f(x) = 2x$. The Lyapunov exponent $\lambda(x) = \ln 2 > 0$ since $f'(x) = 2$ for all x, so f has a **PLE**. But f has no periodic points other than $x = 0$, and thus f cannot have a shared periodic orbit or a dense set of periodic points. □

Chapter 3

Conclusion

In Chapter 2 we have given a number of definitions that can be put into three groups: geometric (relating to sensitive dependence on initial conditions), analytic (utilizing derivatives, such as PLE), and topological (relating to mixing qualities of the iterates, such as transitive). Purely mathematically, all of these definitions are very interesting, but we are also concerned with which definitions are most suitable in different situations.

On the one hand, for physical applications one can usually assemble only a small number of observational data. It therefore follows that the most general form of sensitive dependence on initial conditions (PSDIC) might be most natural to use. Indeed, if one is actually getting data for weather prediction, and similarly, if one obtains data from a double pendulum, then USDIC and ESDIC and also analytic and topological conditions may well not be applicable. When people outside of mathematics speak of chaos, they are likely thinking of PSDIC; USDIC and ESDIC are important alternate definitions of chaos, but are more valuable as mathematical constructs. However, mathematically speaking, PSDIC has problems. For example, $f(x) = 2x$ has PSDIC, though one would not think of $f(x) = 2x$ as a chaotic function. Thus when considering functions isolated from applications, PSDIC is not a satisfactory condition for considering a function to be chaotic.

On the other hand, if one has a function with a given formula, or a system of differential equations, then derivatives (or their higher dimension analogues) become accessible and can yield much information, not only about separation of iterates but also about the rate at which separation occurs. Thus, the Lyapunov exponent is an important definition. Again, however, using a PLE as the sole characteristic to define a function as chaotic is not satisfactory; the function $G(x)$ introduced in Proposition 19 has a PLE but does not even have PSDIC.

Finally, relationships between dense periodic points, transitivity, and sensitive dependence on initial conditions are very interesting and important, and therefore have been used in combination in some definitions of chaos. One must be careful, however, when combining these characteristics, for, as we have shown, often one or two conditions associated with chaos will imply another.

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