ABSTRACT

Title of thesis: INVESTIGATION INTO SOLVABLE QUINTICS

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Solving quintics has fascinated and challenged mathematicians for centuries. David Dummit in Solving Solvable Quintics [3] gives a powerful method that permits one to determine when a quintic is solvable and to solve for its roots. G.N. Watson in 1948 [1], and Kobayashi and Nakagawa [4] have come up with similar methods.

This paper focuses on two families of quintics that pose different challenges for solving them. The first family is a famous group of quintics that are called Emma Lehmer's quintics [5],[6]. These quintics are known to have $\mathbf{Z}_5$ as their Galois group and one might hope that expressing the roots in terms of radicals would give simple expressions from which Emma Lehmer's polynomials could be recovered. However, we show that the expressions of the roots in terms of radicals is much more complicated than expected.

We also consider the simple equation $f(x) = x^5 + ax + p$ and show that, for a fixed nonzero integer p, the polynomial f is solvable by radicals for only finitely many $a \in \mathbf{Z}$.

INVESTIGATION INTO SOLVABLE QUINTICS

by

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Thesis submitted to the Faculty of the Graduate School of the University of Maryland, College Park in partial fulfillment of the requirements for the degree of Master of Science
2004

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Dedication

To Victoria, Rafael and Natalia

ACKNOWLEDGMENTS

All my gratitude goes to Professor Washington, who made the achievement of this thesis possible. He always made himself readily available and has tought me tremendously. I would also like to thank my husband for his crucial support at home.

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Chapter 1

Summary of Dummit's method

In this section, Dummit's method is summarized. The main steps are:

(1) a sextic resolvent is constructed which has a rational root if and only if the general reduced quintic $f(x) = x^5 + px^3 + qx^2 + rx + s \in \mathbf{Q}[x]$ is solvable;

(2) the Lagrange resolvents r_i of the roots of f are defined;

(3) the fifth power of the resolvents are expressed as linear combinations of roots of unity.

Let x_1, x_2, x_3, x_4 and x_5 be the roots of the general quintic polynomial $x^5 - s_1x^4 + s_2x^3 - s_3x^2 + s_4x - s_5$ where the s_i are the elementary symmetric functions in the roots. We assume that $s_1, s_2, s_3, s_4, s_5 \in \mathbf{Q}$. Let

$$\begin{aligned}\theta =& x_1^2x_2x_5 + x_1^2x_3x_4 + x_2^2x_1x_3 + +x_2^2x_4x_5 + x_3^2x_1x_5 + \\ & x_3^2x_2x_4 + x_4^2x_1x_2 + x_4^2x_3x_5 + x_5^2x_1x_4 + x_5^2x_2x_3\end{aligned}$$

The stabilizer of θ in S_5 is precisely F_{20}, the Frobenius group of order 20, with generators (12345) and (2354). Since S_3, generated by (12) and (123), is a complement of F_{20} in S_5, (that is, every element of S_5 can be written uniquely as an element of S_3 times an element of F_{20}), it follows that θ and its conjugates satisfy a polynomial $g(x)$ of degree 6 over $\mathbf{Q}$.

By making a translation, we may assume our quintic is

$$f(x) = x^5 + px^3 + qx^2 + rx + s \tag{1.1}$$

Dummit proves the following theorem:

Theorem 1 *The irreducible quintic* $f(x) = x^5 + px^3 + qx^2 + rx + s \in \mathbf{Q}[x]$ *is solvable by radicals if and only if the polynomial* $g(x)$ *has a rational root. If this is the case, the sextic* $g(x)$ *factors into the product of a linear polynomial and an irreducible quintic.*

Proof: The polynomial $f(x)$ is solvable if and only if the Galois group of $f(x)$, considered as a permutation group on the roots, is contained in a solvable subgroup of S_5. It can be shown that all solvable subgroups of S_5 are contained in the conjugates of F_{20}. It follows that $f(x)$ is solvable by radicals if and only if θ or one of its conjugates is rational. This proves the first assertion. We may assume θ is rational so the Galois group of f is contained in the specific group F_{20} above.

If $g(x)$ has a rational root, then it factors as a linear times a quintic. It can be shown that this quintic is irreducible. See [3] for details.

Henceforth, we assume that the Galois group of $f(x)$ is solvable, hence is a subgroup of F_{20}. Let ζ be a fixed primitive 5th root of unity. Dummit defines the usual Lagrange resolvents of x_1:

$$(x_1, 1) = x_1 + x_2 + x_3 + x_4 + x_5$$

$$r_1 = (x_1, \zeta) = x_1 + x_2\zeta + x_3\zeta^2 + x_4\zeta^3 + x_5\zeta^4$$

$$r_2 = (x_1, \zeta^2) = x_1 + x_2\zeta^2 + x_3\zeta^4 + x_4\zeta + x_5\zeta^3$$

$$r_3 = (x_1, \zeta^3) = x_1 + x_2\zeta^3 + x_3\zeta + x_4\zeta^4 + x_5\zeta^2$$

$$r_4 = (x_1, \zeta^4) = x_1 + x_2\zeta^4 + x_3\zeta^3 + x_4\zeta^2 + x_5\zeta$$

so that:

$$
\begin{aligned}
x_1 &= (r_1 + r_2 + r_3 + r_4)/5 \\
x_2 &= (\zeta^4 r_1 + \zeta^3 r_2 + \zeta^2 r_3 + \zeta r_4)/5 \\
x_3 &= (\zeta^3 r_1 + \zeta r_2 + \zeta^4 r_3 + \zeta^2 r_4)/5 \\
x_4 &= (\zeta^2 r_1 + \zeta^4 r_2 + \zeta r_3 + \zeta^3 r_4)/5 \\
x_5 &= (\zeta r_1 + \zeta^2 r_2 + \zeta^3 r_3 + \zeta^4 r_4)/5
\end{aligned}
\tag{1.2}
$$

Let

$$
R_1 = r_1^5 = l_0 + l_1\zeta + l_2\zeta^2 + l_3\zeta^3 + l_4\zeta^4 \tag{1.3}
$$

where l_0 is the sum of terms involving powers ζ^j of ζ with j divisible by 5, l_1 involves powers of ζ where $j \equiv 1 \pmod 5$, etc. Explicitly,

$$
\begin{aligned}
l_0 =& 30x_2x_4^2x_5^2 + 20x_1x_4x_5^3 + 20x_1^3x_2x_5 + 20x_2x_3x_5^3 + x_2^5 + x_5^5 \\
&+ x_1^5 + x_3^5 + x_4^5 + 20x_1^3x_3x_4 + 30x_1^2x_2^2x_4 + 30x_1^2x_2x_3^2 + 20x_1x_2^3x_3 \\
&+ 30x_1^2x_3x_5^2 + 30x_1^2x_4^2x_5 + 30x_2^2x_3^2x_5 + 30x_2^2x_3x_4^2 \\
&+ 20x_2^3x_4x_5 + 20x_2x_3^3x_4 + 20x_1x_2x_4^3 + 30x_1x_2^2x_5^2 + 30x_1x_3^2x_4^2 \\
&+ 20x_1x_3^3x_5 + 120x_1x_2x_3x_4x_5 + 30x_3^2x_4x_5^2 + 20x_3x_4^3x_5
\end{aligned}
\tag{1.4}
$$

l_1, l_2, l_3, l_4 are defined similarly. See [3] for details. Similarly, let

$$
\begin{aligned}
R_2 = r_2^5 &= l_0 + l_3\zeta + l_1\zeta^2 + l_4\zeta^3 + l_2\zeta^4 \\
R_3 = r_3^5 &= l_0 + l_2\zeta + l_4\zeta^2 + l_1\zeta^3 + l_3\zeta^4 \\
R_4 = r_4^5 &= l_0 + l_4\zeta + l_3\zeta^2 + l_2\zeta^3 + l_1\zeta^4
\end{aligned}
\tag{1.5}
$$

Since $l_0(1+\zeta+\zeta^2+\zeta^3+\zeta^4)=0$, we may write

$$\begin{aligned}
R_1 &= (l_1-l_0)\zeta+(l_2-l_0)\zeta^2+(l_3-l_0)\zeta^3+(l_4-l_0)\zeta^4 \\
R_2 &= (l_1-l_0)\zeta^2+(l_2-l_0)\zeta^4+(l_3-l_0)\zeta+(l_4-l_0)\zeta^3 \\
R_3 &= (l_1-l_0)\zeta^3+(l_2-l_0)\zeta+(l_3-l_0)\zeta^4+(l_4-l_0)\zeta^2 \\
R_4 &= (l_1-l_0)\zeta^4+(l_2-l_0)\zeta^3+(l_3-l_0)\zeta^2+(l_4-l_0)\zeta
\end{aligned} \tag{1.6}$$

and these expressions are unique.

Let $\sigma=(12345)$, $\tau=(2354)$ and $\omega:\zeta\mapsto\zeta^3$

Then σ fixes $l_1-l_0,\quad l_2-l_0,\quad l_3-l_0,\quad l_4-l_0$. Also,

$$\tau(l_1-l_0)=(l_2-l_0),\tau(l_2-l_0)=(l_4-l_0),\tau(l_4-l_0)=(l_3-l_0),\tau(l_3-l_0)=(l_1-l_0)$$

Since the fixed field of σ and τ is $\mathbf{Q}$ (since they generate F_{20}), it follows that

$$l_1-l_0,\qquad l_2-l_0,\qquad l_3-l_0,\qquad l_4-l_0$$

are the roots of a quartic polynomial over $\mathbf{Q}$ and that the field

$L=\mathbf{Q}(l_1-l_0,l_2-l_0,l_3-l_0,l_4-l_0)$ is a cyclic extension of $\mathbf{Q}$ of degree 4 with the unique quadratic subfield $\mathbf{Q}(\Delta)$. Therefore l_1,l_2,l_3,l_4 are the roots of a quartic over $\mathbf{Q}$ which factors over $\mathbf{Q}(\Delta)$ into the product of two conjugate quadratics:

$$[x^2+(T_1+T_2\Delta)x+(T_3+T_4\Delta)][x^2+(T_1-T_2\Delta)x+(T_3-T_4\Delta)] \tag{1.7}$$

with $T_1,T_2,T_3,T_4\in\mathbf{Q}$. The roots of one of these two quadratic factors are either the conjugate pair l_1,l_4, or the other conjugate pair l_2,l_3. Supposing l_1,l_4 are the roots of the first factor, we obtain the following set of equations:

$$l_1 + l_4 = -T_1 - T_2\Delta$$

$$l_2 + l_3 = -T_1 + T_2\Delta$$

$$l_1 l_4 = T_3 + T_4\Delta$$

$$l_2 l_3 = T_3 - T_4\Delta$$

Dummit then provides us with a rule to choose the fifth roots of the R_i to obtain the resolvents r_i. He proves that each of the five possible choices of r_1 uniquely defines the choices for r_2, r_3, r_4, hence uniquely defines the five roots of the quintic. Since $r_1 r_4$ and $r_2 r_3$ are fixed by σ, $\tau\omega^{-1}$ and by τ^2, they are elements of the corresponding fixed field $\mathbf{Q}(\Delta\sqrt{5})$. Dummit then shows that given r_1 there is a unique choice of r_2, r_3, r_4 such that $r_1 r_4, r_2 r_3 \in \mathbf{Q}(\Delta\sqrt{5})$ and such that the 2 following equations are satisfied:

$$r_1 r_2^2 + r_4 r_3^2 = u + v\Delta\sqrt{5} \tag{1.8}$$

$$r_3 r_1^2 + r_2 r_4^2 = u - v\Delta\sqrt{5}$$

with $u = -\frac{25}{2}q$. All of the above is summarized in the following theorem:

Theorem 2 *: Suppose the irreducible polynomial $f(x) = x^5 + px^3 + qx^2 + rx + s \in \mathbf{Q}[x]$ is solvable by radicals and let θ be the unique rational root of the associated resolvent sextic $g(x)$ as in Theorem 1. Fix any square root Δ of the discriminant D of $f(x)$ and fix any primitive fifth root of unity ζ. Then the Galois group of $f(x)$ is:*

(a) the Frobenius group of order 20 if and only if the discriminant D of $f(x)$ is not a square, which occurs if and only if the quadratic factors in (1.7) are irreducible over $\mathbf{Q}(\sqrt{D})$.

(b) the dihedral group of order 10 if and only if D is a square and the rational quadratics in (1.7) are irreducible over $\mathbf{Q}$.

(c) the cyclic group of order 5 if and only if D is a square and the rational quadratics in (1.7) are reducible over $\mathbf{Q}$.

Let r_1 be any fifth root of R_1 in (1.3), and let r_2, r_3, r_4 be the corresponding fifth roots of R_2, R_3, R_4 as in (1.5) and (1.8). Then the formulas (1.2) give the roots of $f(x)$ in terms of radicals and x_1, x_2, x_3, x_4, x_5 are permuted cyclically by some 5-cycle in the Galois group.

Chapter 2

Application of Dummit's Method to Emma Lehmer's Quintics

Emma Lehmer [5],[6] defined the following family of polynomials, for $n \in \mathbf{Z}$:

$$\begin{aligned} f_n(x) = & x^5 + n^2x^4 - (2n^3 + 6n^2 + 10n + 10)x^3 + (n^4 + 5n^3 + 11n^2 + 15n + 5)x^2 \\ & + (n^3 + 4n^2 + 10n + 10)x + 1. \end{aligned}$$

It is easy to see that $f_n(x)$ is irreducible for any $n \in \mathbf{Z}$ by considering it modulo 2. The $f_n(x)$ have Z_5 as their Galois group. Their discriminants are equal to

$$(n^3 + 5n^2 + 10n + 7)^2(n^4 + 5n^3 + 15n^2 + 25n + 25)^4$$

Z_5 permutes the roots of $f_n(x)$ cyclically according to the following transformation

$$x \to \frac{(n+2) + nx - x^2}{1 + (n+2)x} \tag{2.1}$$

By making a translation, we may change our quintic to

$$f_1(x) = x^5 + a_1x^3 + a_2x^2 + a_3x + a_4 \tag{2.2}$$

with

$$\begin{aligned} a_1 &= -\frac{2}{5}n^4 - 2n^3 - 6n^2 - 10n - 10 \\ a_2 &= \frac{4}{25}n^6 + \frac{6}{5}n^5 + \frac{23}{5}n^4 + 11n^3 + 17n^2 + 15n + 5 \\ a_3 &= -\frac{3}{125}n^8 - \frac{6}{25}n^7 - \frac{28}{25}n^6 - \frac{16}{5}n^5 - \frac{28}{5}n^4 - 5n^3 + 2n^2 + 10n + 10 \\ a_4 &= \frac{4}{3125}n^{10} + \frac{2}{125}n^9 + \frac{11}{125}n^8 + \frac{7}{25}n^7 + \frac{13}{25}n^6 + \frac{2}{5}n^5 - \frac{3}{5}n^4 - 2n^3 - 2n^2 + 1 \end{aligned}$$

In order to provide an application of Dummit's method to this family of quintics, we compute l_0, l_1, l_2, l_3, l_4 in terms of n with the program Mathematica. Then, we show how to prove that the polynomials in terms of n are correct and give an example, where the $l_i(n)$ are used to compute the roots of f.

2.1 Computation of the l_i with Mathematica

In order to come up with a polynomial in terms of n for each l_0, l_1, l_2, l_3, l_4, we computed several values of the l_i. We then applied the method of finite differences to each set of values.

An example of the computations of l_0, l_1, l_2, l_3, l_4 for the case $n = -2$ is provided in Appendix A. The quintic for $n = -2$ is in this case $f(x) = x^5 + 4x^4 + 2x^3 - 5x^2 - 2x + 1$. The 5 roots of f were solved for and produced in a cyclic order according to transformation (2.1). Then $R_1 = (x_1, \zeta)^5$, $R_2 = (x_1, \zeta^2)^5$, $R_3 = (x_1, \zeta^3)^5$ and $R_4 = (x_1, \zeta^4)^5$ were computed. Let $R_0 = (x_1, 1)^5 = l_0 + l_1 + l_2 + l_3 + l_4$. We used R_1 as in (1.3) and R_2, R_3, R_4 as in (1.5), and computed $R_0 + R_1 + R_2 + R_3 + R_4$.

Since $1 + \zeta + \zeta^2 + \zeta^3 + \zeta^4 = 0$, $R_0 + R_1 + R_2 + R_3 + R_4 = 5l_0 + (l_1 + l_2 + l_3 + l_4)(1 + \zeta + \zeta^2 + \zeta^3 + \zeta^4) = 5l_0$. Similarly,

$$
\begin{aligned}
R_0 + R_1\zeta^4 + R_2\zeta^3 + R_3\zeta^2 + R_4\zeta &= 5l_1 \\
R_0 + R_1\zeta^3 + R_2\zeta + R_3\zeta^4 + R_4\zeta^2 &= 5l_2 \\
R_0 + R_1\zeta^2 + R_2\zeta^4 + R_3\zeta + R_4\zeta^3 &= 5l_3 \\
R_0 + R_1\zeta + R_2\zeta^2 + R_3\zeta^3 + R_4\zeta^4 &= 5l_4
\end{aligned}
\tag{2.3}
$$

The same process was repeated to get the values of the l_i for values of n ranging from 11 to -10 for l_0 and from 8 to -5 for l_1, l_2, l_3, l_4.

Appendix B provides a list of the 13 values of l_1, indexed in order by n, ranging from 8 to -5. We used the method of finite differences to determine $l_1(n)$. The difference of each pair of consecutive terms were computed to obtain 12 new consecutive values. The difference of these differences was computed in turn, to get 11 more values. The process stopped once a new row of values became a constant. The number of times the process was repeated provided us with the highest degree d of the polynomial. The constant was divided by $d!$, and the resulting value corresponded to the coefficient g of n^d. The process was carried further when one substracted from each value of l_1 the corresponding gn^d to obtain a new set of consecutive values. The process was repeated until the entire polynomial of l_1 in terms of n was obtained. The same method was applied to compute l_0, l_2, l_3, l_4. The computations yielded the following results:

$$\begin{aligned}
l_0(n) &= -n^{10} - 10n^9 - 55n^8 - 175n^7 - 325n^6 \\
&\quad - 250n^5 + 375n^4 + 1250n^3 + 1250n^2 - 625 \\
l_1(n) &= 5n^9 + 35n^8 + 125n^7 + 225n^6 - 1125n^4 - 3125n^3 - 4375n^2 - 3125n - 625 \\
l_2(n) &= -2n^8 - 10n^7 - 15n^6 + 50n^5 + 325n^4 + 875n^3 + 1375n^2 + 1250n + 500 \\
l_3(n) &= n^9 + 10n^8 + 55n^7 + 195n^6 + 475n^5 + 800n^4 + 875n^3 + 500n^2 - 125 \\
l_4(n) &= -4n^8 - 35n^7 - 160n^6 - 475n^5 - 975n^4 - 1375n^3 - 1250n^2 - 625n - 125
\end{aligned}$$

The $l_i(n)$ were used in order to compute the roots of f in the next section.

2.2 Proof the l_i are the right ones

In this section we show how to prove that the polynomials in terms of n, obtained in the previous section are correct. We do the following:

1. we prove that the degree of $l_0(n)$ is bounded by 60.

2. we use the $l_i(n)$ to solve for the roots of f

Theorem 3 *The degree of $l_0(n)$ is bounded by 60.*

Proof: First, we prove $l_0(n)$ is a polynomial. Now, $f_n(x) \in \mathbf{Q}[n][x]$. Since the roots of $f_n(x)$ are integral over $\mathbf{Q}[n]$, $l_0(n)$ as in (1.4) is integral over $\mathbf{Q}[n]$. But $l_0(n) \in \mathbf{Q}(n)$ since it is fixed by F_{20}. Since $\mathbf{Q}[n]$ is a UFD and is therefore integrally closed, $l_o(n) \in \mathbf{Q}[n]$.

Secondly, we prove that the degree of $l_0(n)$ is bounded by 60. Note that l_0 as in (1.4) is homogeneous of degree 5 in x_1, x_2, x_3, x_4, x_5. Also, l_0 is fixed by F_{20}. Let

$$A = \prod_{\sigma \in \mathbf{S}_3} \sigma(l_0)$$

with S_3, generated by $(12), (123)$. Then A is fixed by S_5. Therefore, A is a polynomial in the elementary symmetric functions s_1, s_2, s_3, s_4, s_5, homogeneous of degree 30 in x_1, x_2, x_3, x_4, x_5. Therefore, A can be written as:

$$A = \sum_{a,b,c,d,e} s_1^a s_2^b s_3^c s_4^d s_5^e$$

with $2b + 3c + 4d + 5e = 30$, since A has to be of degree 30 in the x_i. Note that we use a quintic in its reduced form, so $s_1 = 0$. In the case of Emma Lehmer's quintics in reduced form, as in (2.2), $s_1 = 0$, s_2 has degree 4; s_3 has degree 6, s_4 has degree 8 and s_5 has degree 10. Therefore, s_2^b has degree $4b$; s_3^c has degree $6c$; s_4^d has degree $8d$; s_5^e has degree $10e$. Therefore, $4b + 6c + 8d + 10e = 60$, which must be the total degree in n of A. Since $l_0(n)$ divides A, the degree of $l_0(n)$ is at most 60. We applied the method of finite differences to 22 values of l_0 and came up with the polynomial in terms of n above. One could compute 61 values of l_0 to verify that $l_i(n)$ is of no higher degree than 10. This would, however, be very cumbersome and would surely give the same result.

Let $p_1 = l_1 - l_0, \quad p_2 = l_2 - l_0, \quad p_3 = l_3 - l_0, \quad p_4 = l_4 - l_0$. Factoring the p_i yields the following

$$p_1 = (n^4 + 5n^3 + 15n^2 + 25n + 25)(n^5 + 10n^4 + 25n^3 - 100n - 125)n$$

$$p_2 = (n^4 + 5n^3 + 15n^2 + 25n + 25)(n^4 + 25)(n^2 + 5n + 5)$$

$$p_3 = (n^4 + 5n^3 + 15n^2 + 25n + 25)(n^2 + 5n + 10)(n^2 + 5n + 5)n^2$$

$$p_4 = (n^4 + 5n^3 + 15n^2 + 25n + 25)(n^3 - 10n - 25)(n^2 + 5n + 5)n$$

Note that $(n^4 + 5n^3 + 15n^2 + 25n + 25)$ is a factor in the discriminants of the $f_n(x)$. The formulas

$$R_1 = p_1\zeta + p_2\zeta^2 + p_3\zeta^3 + p_4\zeta^4 \tag{2.4}$$

$$R_2 = p_1\zeta^2 + p_2\zeta^4 + p_3\zeta + p_4\zeta^3$$

permit us to compute R_1 and R_2. Then, R_3 is R_2's complex conjugate, and R_4 is R_1's complex conjugate.

Example For $n = 12$, the polynomial is:

$$f_1 2(x) = x^5 - \frac{63722}{5}x^3 + \frac{22334561}{25}x^2 - \frac{2898012838}{125}x + \frac{660368960021}{3125}$$

The formulas $x_1 = \frac{1}{5}(r_1 + r_2 + r_3 + r_4 + r_5), r_i = \sqrt[1/5]{R_i}$ and equations (1.6) permit us to write the following expression for x_1 in terms of radicals in the case $n = 12$:

$$\begin{aligned} x_1 = \frac{1}{5}[&(190426952244\zeta + 138246440189\zeta^2 + 205202172384\zeta^3 + 126493395204\zeta^4)^{\frac{1}{5}} + \\ &(190426952244\zeta^2 + 138246440189\zeta^4 + 205202172384\zeta + 126493395204\zeta^3)^{\frac{1}{5}} + \\ &(190426952244\zeta^3 + 138246440189\zeta + 205202172384\zeta^4 + 126493395204\zeta^2)^{\frac{1}{5}} + \\ &(190426952244\zeta^4 + 138246440189\zeta^3 + 205202172384\zeta^2 + 126493395204\zeta)^{\frac{1}{5}}] \end{aligned}$$

Appendix C provides an example of the computation of R_1 and R_2 for $n = 12$. We obtain the following values:

$$R_1 = -1.79922 \times 10^{11} + 2.14488 \times 10^{10} i \qquad R_2 = -1.50262 \times 10^{11} + 1.01258 \times 10^{11} i$$

Choose r_1 to be any fifth root of R_1. We chose $r_1 = -178.44624 + 4.23539i$. Then r_4 is r_1's complex conjugate. Then u is computed, and r_2 and its complex conjugate r_3 are chosen so that

$$r_1 r_2^2 + r_4 r_3^2 - u = v\Delta\sqrt{5}$$

$$r_3 r_1^2 + r_2 r_4^2 - u = -v\Delta\sqrt{5}$$

with $u = -\frac{25}{2}a_2 = -\frac{22334561}{2}$.

We obtained: $r_2 = -177.24271 + 21.11921i$. Using equations (1.2) we computed x_1, x_2, x_3, x_4, x_5 and obtained:

$$x_1 = -142.27558$$

$$x_2 = 41.87644$$

$$x_3 = 28.79959$$

$$x_4 = 42.87644$$

$$x_5 = 28.72311$$

2.3 Computation of θ in terms of n

In this section, we compute θ_n, the rational root of the sextic resolvent, $g_n(x)$ of Emma Lehmer's quintic, in terms of n, by using again the method of finite differences. A formula for $g(x)$ in terms of the coefficients of $f(x)$ in reduced form as in (1.1) is provided by Dummit [3]. We use Dummit's formula and substitute the coefficients of the quintics as in (2.2) in order to obtain a formula for the sextic resolvent in terms of n. Appendix D provides the computation of $g_n(x)$. We then compute $g_n(x)$ for values of n ranging from 4 to -6. We solve for the roots of the 11 sextics obtained. Each 11 sets of six roots includes a rational root, which are the θ_i. We apply the method of finite differences to these 11 values of θ to obtain the following polynomial in terms of n:

$$\theta_n = 4n^8 + 40n^7 + 145n^6 + 200n^5 - 525n^4 - 3125n^3 - 7625n^2 - 10000n - 6875$$

which factored is:

$$\theta_n = (n^4 + 5n^3 + 15n^2 + 25n + 25)(4n^4 + 20n^3 - 15n^2 - 125n - 275)$$

Note that θ_n has the first factor in common with the discriminant of Emma Lehmer's quintics.

As in the case of the coefficients l_0, l_1, l_2, l_3, l_4, the formula for θ_n is much more complicated than what one might expect. The polynomials $f_n(x)$ have polynomials of moderate degree in n as coefficients. It could be hoped that expressing the roots of $f_n(x)$ in terms of radicals would involve simple expressions in n. Instead, our results show the opposite is the case. The expressions are rather complicated.

Chapter 3

$f(x) = x^5 + ax + p$

In this section, we will prove that for each fixed integer $p \neq 0$, the polynomial f is solvable by radicals for only finitely many integer a. When p is odd, we will show that if the polynomial is irreducible and solvable, then its Galois group is F_{20}. We will investigate more closely the case where p is either 1 or a prime number and determine exactly when $f(x)$ is irreducible. We will compare these results with those obtained by classical methods that use the complex roots to obtain cycles in the Galois group.

3.1 When is f irreducible?

First note that $-f(-x) = x^5 + ax - p$ has the same Galois group as $f(x)$. So we may assume p is positive.

Proposition 1 *For each nonzero fixed integer p, there are only finitely many integer values of a for which the polynomial is reducible.*

Proof: For each p, the rational root test allows only finitely many rational roots. Each of these is a factor of p. Each such root allows only one value of a.

Now, assume f factors as $(x^3 + ux^2 + vx + s)(x^2 + wx + t) = x^5 + ax + p$. By Gauss's Lemma, if it factors, the factors have integer coefficients. Then we can deduce the following set of equations:

1. $u + w = 0$
2. $t + uw + v = 0$
3. $ut + vw + s = 0$
4. $sw + tv = a$
5. $st = p$

From (1) we obtain $u = -w$, which we substitute into (2) to get $v = w^2 - t$. Substituting v and u in (3), we get $-wt + (w^2 - t)w + s = 0$, from which we deduce the equation:

$$g(w) = w^3 - 2tw + s = 0 \tag{3.1}$$

This has at most 3 rational roots w. For each w, there is only one u, by 1, and then by 2, there is only one v. By 4, we get one a. Therefore, each (s, t) yields at most 3 values of a. There are only finitely many (s, t), so this proves the result. In the particular case where p is 1 or an odd prime, we prove the following:

Proposition 2 *When $p = 1$, $f(x)$ is irreducible if and only if $a \neq -2, 1$. When $p = 3$, $f(x)$ is irreducible if and only if $a \neq -82, -80, -5, -4, 2$. When $p \geq 5$ is prime, $f(x) = x^5 + ax + p$ is irreducible over $\mathbf{Z}$ iff $a \neq p - 1, -p - 1, 1 - p^4, -p^4 - 1$.*

Proof: By Gauss's Lemma, if $f(x)$ factors over $\mathbf{Q}$, then $f(x)$ factors over $\mathbf{Z}$.

Suppose $f(x)$ has a rational root. Then, by the root test, $f(-1)$, $f(1)$, $f(p)$ or $f(-p) = 0$.

If $f(-1) = 0$, then $a = p - 1$ and $f(x) = x^5 + px - x + p$ factors as

$(x+1)(x^4 - x^3 + x^2 - x + p)$.

If $f(1) = 0$, then $a = -p-1$ and $f(x) = x^5 - px - x + p$ factors as $(x-1)(x^4 + x^3 + x^2 + x - p)$. Note that when $p = 1, a = -2$ and the quintic $f(x) = x^5 - 2x + 1$ factors as $f(x) = (x-1)(x^4 + x^3 + x^2 + x - 1)$.

If $f(-p) = 0$, then $a = 1 - p^4$ and $f(x) = x^5 - p^4x + x + p$ factors as $(x+p)(x^4 - px^3 + p^2x^2 - p^3x + 1)$.

If $f(p) = 0$, then $a = -p^4 - 1$ and $f(x) = x^5 - p^4x - x + p$ factors as $(x-p)(x^4 + px^3 + p^2x^2 + p^3x - 1)$.

In the case where $f(x)$ factors as a cubic times a quadratic, we use equation (3.1) above and consider the following 4 cases: $(s = 1, t = p)$, $(s = p, t = 1)$, $(s = -1, t = -p)$, $(s = -p, t = -1)$.

1. If $(s = 1, t = p)$, then $g(w) = w^3 - 2pw + 1$. By the root test, if g has a rational root, then $g(1) = 0$ or $g(-1) = 0$.

   a) If $g(1) = 0$, then $p = 1$ and we deduce from equations 1 to 5 that $u = -1, v = 0, a = 1$. Then, $f(x) = x^5 + x + 1$ factors as $(x^3 - x^2 + 1)(x^2 + x + 1)$.

   b) If $g(-1) = 0$, then $p = 0$ and $f(x)$ is clearly reducible.

2. If $(s = p, t = 1)$, then $g(w) = w^3 - 2w + p$. By the root test, if g has a rational root, then $g(1)$, $g(-1)$, $g(p)$, or $g(-p) = 0$.

   a) If $g(1) = 0$, then $p = 1$, and we obtain the same result as 1.a).

   b) If $g(-1) = 0$, then $p = -1$ which is a case we do not need to cover.

   c) If $g(p) = 0$, then $p = 0, \pm 1$. The case $p = 1$ is covered above. The other 2 cases don't need to be covered.

d) If $g(-p) = 0$, then $p = 0$.

3. If $(s = -1, t = -p)$, then $g(w) = w^3 + 2pw - 1$. By the root test, if g has a rational root, then $g(1) = 0$ or $g(-1) = 0$.

   a) If $g(1) = 0$, then $p = 0$.

   b) If $g(-1) = 0$, then $p = -1$.

4. If $(s = -p, t = -1)$, then $g(w) = w^3 + 2w - p$. By the root test, if g has a rational root, then $g(1) = 0, g(-1) = 0, g(p) = 0$ or $g(-p) = 0$.

   a) If $g(1) = 0$, then $p = 3$. From equations 1 to 5, we get $u = -1, v = 2, a = -5$. Then $f(x) = x^5 - 5x + 3$ factors as $(x^3 - x^2 + 2x - 3)(x^2 + x - 1)$.

   b) If $g(-1) = 0$, then $p = -3$, which we don't need to cover.

   c) If $g(p) = 0$, then $p = 0$ is the only rational solution.

   d) If $g(-p) = 0$, then $p = 0$ is the only rational solution.

   This completes the proof of Proposition 2.

3.2 Is f solvable? Dummit's method

In 1885, Runge proved the following result:

Theorem 4 *Assume that $f(x) = x^5 + ax + b \in \mathbf{Q}[x]$ is irreducible and $a \neq 0$. Then f is solvable by radicals if and only if there are $s, t \in \mathbf{Q}$ such that*

$$a = \frac{3125st^4}{(s-1)^4(s^2 - 6s + 25)}, \quad b = \frac{3125st^5}{(s-1)^4(s^2 - 6s + 25)}$$

For a proof, see Cox [2]. We have not been able to use the result to obtain information more precise than what we give below.

Spearman and Williams [7] gave the following characterization of solvable irreducible quintics:

Theorem 5 *Let a and b be rational numbers such that the quintic trinomial $x^5 + ax + b$ is irreducible. Then the equation $x^5 + ax + b = 0$ is solvable by radicals if and only if there exist rational numbers $\epsilon = \pm 1, c \geq 0$ and $e \neq 0$ such that*

$$a = \frac{5e^4(3 - 4\epsilon c)}{c^2 + 1}, \qquad b = \frac{-4e^5(11\epsilon + 2c)}{c^2 + 1}$$

in which case the roots of $x^5 + ax + b = 0$ are

$$x_j = e(\omega^j u_1 + \omega^{2j} u_2 + \omega^{3j} u_3 + \omega^{4j} u_4) \qquad (j = 0, 1, 2, 3, 4)$$

where $\omega = \exp(2\pi i/5)$ and

$$u_1 = (\frac{\mu_1^2 \mu_3}{D^2})^{1/5}, \qquad u_2 = (\frac{\mu_3^2 \mu_4}{D^2})^{1/5}, \qquad u_3 = (\frac{\mu_2^2 \mu_1}{D^2})^{1/5}, \qquad u_4 = (\frac{\mu_4^2 \mu_2}{D^2})^{1/5}$$

with

$$\mu_1 = \sqrt{D} + \sqrt{D - \epsilon\sqrt{D}}$$
$$\mu_2 = -\sqrt{D} - \sqrt{D + \epsilon\sqrt{D}}$$
$$\mu_3 = -\sqrt{D} + \sqrt{D + \epsilon\sqrt{D}}$$
$$\mu_1 = \sqrt{D} - \sqrt{D - \epsilon\sqrt{D}}$$

$$D = c^2 + 1$$

It is a rather complicated characterization that is difficult to apply and we were not able to use that result either.

According to Dummit's method, $f(x)$ is solvable if and only if its resolvent sextic

$$g(x) = x^6 + 8ax^5 + 40a^2x^4 + 160a^3x^3 + 400a^4x^2 + (512a^5 - 3125p^4)x + 256a^6 - 9375ap^4$$

has a rational solution. By the root test, if $g(x)$ has a rational root, then $g(q) = 0$ for a given divisor of $256a^6 - 9375ap^4$. There is no systematic way to find such a q and one has to deal with it on a case by case basis. We can prove, however, the following:

Proposition 3 *For each fixed nonzero integer p, $g(x, a) = 0$ has a rational root for only finitely many a, so $f(x)$ is solvable for only finitely many a.*

By proposition 1 of the previous section, we may exclude finitely many values of a and assume f is irreducible. In order to prove the above proposition, we need Faltings famous theorem:

Theorem 6 *Faltings Theorem: Let C be an algebraic curve over the rationals of genus g. Then the number of rational points on C is finite when g is greater than or equal to 2.*

In order to apply Faltings theorem, we need to prove that $g(x, a) = 0$ is a curve of genus greater than or equal to 2. For ease of computation, let $c = 3125p^4$.

Let

$$x = \frac{4}{5}t - \frac{y+c}{20}t^{-4}$$
$$a = \frac{1}{10}t + \frac{y+c}{40}t^{-4}$$

This an invertible change of variables:

$$t = x + 2a$$

$$y = 16(x + 2a)^5 - 20x(x + 2a)^4 - c$$

The equation

$$g(x, a) = x^6 + 8ax^5 + 40a^2x^4 + 160a^3x^3 + 400a^4x^2 + (512a^5 - c)x + (256a^6 - 3ca) = 0$$

changes to

$$\frac{1}{80t^4}(y^2 + 64t^{10} - 88t^5c - c^2) = 0$$

The curve

$$y^2 = -64t^{10} + 88ct^5 + c^2 = 0$$

has genus 4 since it is a hyperelliptic curve and the discriminant of the right hand side is non zero when $c \neq 0$. Therefore the genus is $\frac{10}{2} - 1 = 4$.

Since $g(x, a) = 0$ has genus at least 2, by Faltings Theorem $g(x, a) = 0$ has only finitely many rational solutions.

Proposition 4 *Assume p is an odd integer. When $f(x) \in \mathbf{Z}[x]$ is irreducible and solvable, it can only have F_{20} as its Galois group.*

Proof: Suppose $g(x)$ has a rational solution and therefore $f(x)$ is solvable. Let $D = 2^8a^5 + 5^5p^4$ be the discriminant of f . By Dummit's Theorem 2, if D is a square, then the Galois group of f is either D_5 or Z_5. Otherwise, it is F_{20}, the Frobenius group of order 20.

Now, suppose $D = n^2$, $n \in \mathbf{Z}$. Since n^2 is odd, n is odd. So $n^2 \equiv 1 \pmod 8$. But $D \equiv 5 \pmod 8$. Therefore, D cannot be a square. So, the Galois group of $f(x)$ must be F_{20}.

3.3 Method of looking at the number of complex roots of f to determine whether f is solvable

In this section, we will be investigating the solvability of f with a more classical method. We will study the derivative of f and its zeros in order to come up with the number of complex roots of f. This will give us information about some cycle types of the Galois group of f. We assume f is irreducible. Let$f'(x) = 5x^4 + a$ be the derivative of f.

Case 1: $a > 0$, then $f'(x) > 0$ for all x. Therefore $f(x)$ has one real root and 4 complex ones.

Case 2: $a < 0$, then $f'(x)$ has two real roots $x = \pm\sqrt[4]{-\frac{a}{5}}$. Note that $-\frac{a}{5} > 0$ since $a < 0$.

We evaluate f at $(\pm\sqrt[4]{-\frac{a}{5}})$:

$f(-\sqrt[4]{-\frac{a}{5}}) = -(\frac{4}{5})a\sqrt[4]{-\frac{a}{5}}$ is > 0 for all x.

$f(\sqrt[4]{-\frac{a}{5}}) = (\frac{4}{5})a\sqrt[4]{-\frac{a}{5}} + p$. Let $b = -\frac{a}{5}$. Then $f(\sqrt[4]{-\frac{a}{5}}) = -4b\sqrt[4]{b} + p = -4\sqrt[4]{b^5} + p$.

When $b < \sqrt[5]{(p/4)^4}$, i.e. when $a > -5\sqrt[5]{(\frac{p}{4})^4}$, then $f(\sqrt[4]{-\frac{a}{5}}) > 0$ and f has one real root and 4 complex ones.

When $a < -5\sqrt[5]{(\frac{p}{4})^4}$, then $f(\sqrt[4]{-\frac{a}{5}}) < 0$ and f has 3 real roots and two complex ones.

We conclude that when $a < -5\sqrt[5]{(\frac{p}{4})^4}$, f has two complex roots that are conjugate roots. The Galois group of f therefore contains a transposition. Moreover, f is an irreducible quintic, so its Galois group also contains a 5-cycle. Since S_5 is generated by any 5-cycle and any transposition, we conclude that the Galois group of f must be S_5 and therefore f is not solvable.

When $a > 0$, or when $a > -5\sqrt[5]{(\frac{p}{4})^4}$, f has 1 real root and four complex ones. Its Galois group G therefore contains a (2,2)-cycle. Since D, the discriminant of f, is not a square, G is either F_{20} or S_5, but it cannot be A_5. But both S_5 and F_{20} contain 4-cycles. In F_{20} all the 2-Sylow subgroups are isomorphic to Z_4. Therefore, any (2,2)-cycle is the square of a 4-cycle, and in S_5 any (2,2)-cycle is the square of a 4-cycle. Therefore, complex conjugation is a (2,2)-cycle which is the square of a 4-cycle. Since f is an irreducible quintic, its Galois group contains a 5-cycle. F_{20} is generated by a 5-cycle and a 4-cycle. The question is whether a 5-cycle and a 4-cycle can generate all of S_5. Let the 5-cycle be (12345) and the 4-cycle be (1234). The 5-Sylow subgroup is normal in F_{20}. But, (1234)(12345)(4321) = (23415) which is not in the subgroup generated by (12345). Therefore a 5-cycle and a 4-cycle could generate all of S_5. Looking at the number of complex roots of f does not allow us to determine its Galois group, so more complicated methods such as Dummit's are needed.

Chapter 4

Appendix A

Computation of l_0, l_1, l_2, l_3, l_4 in the case $n = -2$

The attached appendix is an example of the computations achieved with the program Mathematica to obtain l_0, l_1, l_2, l_3, l_4 for values of n ranging from 11 to -10. For each case the Emma Lehmer quintic is computed and equations (2.3) are used to compute the l_i. Appendix A gives computations for the case $n = 2$

Line 1 and 2: computation of E. Lehmer's quintic for $n = -2$

Line 3: the 5 roots of f are solved for

Line 9 to 14: the 5 roots are produced in a cyclic order, according to transformation (2.1)

Line 15: gives the equation $x_1+x_2z+x_3z^2+x_4z^3+x_5z^4$ for computing the resolvents r_i

Line 16: gives a numerical value for a fixed fifth root of unity

Line 17: gives the equation $R_0 + R_1 + R_2 + R_3 + R_4 = 5l_0$

Line 18: gives l_0

Line 19: gives the equation $R_0 + R_1\zeta^4 + R_2\zeta^3 + R_3\zeta^2 + R_4\zeta = 5l_1$

Line 20: gives l_1

Line 21: gives the equation $R_0 + R_1\zeta^3 + R_2\zeta + R_3\zeta^4 + R_4\zeta^2 = 5l_2$

Line 22: gives l_2

Line 23: gives the equation $R_0 + R_1\zeta^2 + R_2\zeta^4 + R_3\zeta + R_4\zeta^3 = 5l_3$

Line 24: gives l_3

Line 25: gives the equation $R_0 + R_1\zeta + R_2\zeta^2 + R_3\zeta^3 + R_4\zeta^4 = 5l_4$

Line 26: gives l_4

Computations with Mathematica:

$In[1] : f = x^5 + n^2x^4 - (2n^3 + 6n^2 + 10n + 10)x^3 + (n^4 + 5n^3 + 11n^2 + 15n + 5)x^2 + (n^3 + 4n^2 + 10n + 10)x + 1$

$In[2] : \%1/.n- > -2$

$Out[2] : 1 - 2x - 5x^2 + 2x^3 + 4x^4 + x^5$

$In[3] : NSolve[\%2 == 0, x, 100]$

$Out[3] : \{\{x- > -2.68250706566236233772362329783873543502658499684 1$
$07579728530023536330221380328454648617304533957092\},$

$\{x- > -1.830830026003772851058548298459246407048009820929073624852446 66$
$4756879561889745979442742283831928 45\},$

$\{x- > -0.715370323453429719112414662767260662417897277748031343163834 42$
$6881989020798252592196435025752654762\},$

$\{x- > 0.3097214678905701281138501449325871063675823986738568552125912 0$
$9011790179898709784184341793576939660\},$

$\{x- > 0.9189859472289947797807361141326553981249096968443239100889901 1$
$7990380616592573333941008561347214472\}\}$

$In[4] : \%3[[1, 1, 2]]$

$Out[4] : -2.68250706566236233772362329783873543502658499684107579728$
$530023536330221380328454648617304533957092$

$In[5] : \%3[[2, 1, 2]]$

$Out[5] : -1.830830026003772851058548298459246407048009820929073622485$

24466647568795618894597944274228383192845

$In[6]: \%3[[3,1,2]]$

$Out6: -0.715370323453429719112414662767260662417897277748031343163$

834426881989020798252592196435025752654762

$In[7]: \%3[[4,1,2]]$

$Out[7]: 0.309721467890570128113850144932587106367582398673856855212$

591209011790179898709784184341793576939660

$In[8]: \%3[[5,1,2]]$

$Out[8]: 0.918985947228994779780736114132655398124909696844323910088$

990117990380616592573333941008561347214472

$In[9]: c = (-2x - x^2)$

$In[10]: x_2 = \%9/.x- > \%4$

$Out[10]: -1.830830026003772851058548298459246407048009820929073624$

8524466647568795618897459794427422838319284

$In[11]: x_3 = \%9/.x- > \%5$

$Out[11]: 0.30972146789057012811385014493258710636758239867385685521$

25912090117901798987097841843417935769397

$In[12]: x_4 = \%9/.x- > \%7$

$Out[12]: -0.7153703234534297191124146627672606624178972777480313431$

63834426881989020798252592196435025752654762

$In[13] : x_5 = \%9/.x- > \%6$

$Out[13] : 0.918985947228994779780736114132655398124909696844323910088990117990380616592573333941008561347214477$

$In[14] : x_1 = \%9/.x- > \%8$

$Out[14] : -2.682507065662362337723623297838735435026584996841075797285300235363302213803284546486173045339570923$

$In[15] : x_1 + x_2 z + x_3 z^2 + x_4 z^3 + x_5 z^4$

$Out[15] : -2.682507065662362337723623297838735435026584996841075$

$7972853002353633022138032845464861730453395709233$

$- 1.830830026003772851058548298459246407048009820929073624852446664$

$4466647568795618897459794427422838319284z$

$+ 0.309721467890570128113850144932587106367582398673856855212591209$

$0117901798987097841843417935769397z^2$

$- 0.715370323453429719112414662767260662417897277748031343163834426 8$

$81989020798252592196435025752654762z^3$

$+ 0.918985947228994779780736114132655398124909696844323910088990117 9$

$903806165925733339410085613472144 7z^4$

$In[16] : rt = N[Exp[2\Pi i/5], 100]$

$Out[16] : 0.309016994374947424102293417182819058860154589902881$

$4310677243113526302314094512248536036020946955687$

$+ 0.951056516295153572116439333379382143405698634125750222447305 6$

$4443015317008519350171879281097081138 17i$

$In[17] : (\%15/.z- > 1)^5 + (\%15/.z- > rt)^5 + (\%15/.z- > rt^2)^5 + (\%15/.z- > rt^3)^5 + (\%15/.z- > rt^4)^5$

$Out[17] : -45$

$In[18] : L_0 = \%17/5$

$Out[18] : -9$

$In[19] : (\%15/.z- > 1)^5 + (\%15/.z- > rt)^5 rt^4 + (\%15/.z- > rt^2)^5 rt^3 + (\%15/.z- > rt^3)^5 rt^2 + (\%15/.z- > rt^4)^5 rt$

$Out[19] : -375$

$In[20] : L_1 = \%19/5$

$Out[20] : -75$

$In[21] : (\%15/.z- > 1)^5 + (\%15/.z- > rt)^5 rt^3 + (\%15/.z- > rt^2)^5 rt + (\%15/.z- > rt^3)^5 rt^4 + (\%15/.z- > rt^4)^5 rt^2$

$Out[21] : -2300$

$In[22] : L2 = \%21/5$

$Out[22] : -460$

$In[23] : (\%15/.z- > 1)^5 + (\%15/.z- > rt)^5 rt^2 + (\%15/.z- > rt^2)^5 rt^4 + (\%15/.z- > rt^3)^5 rt + (\%15/.z- > rt^4)^5 rt^3$

$Out[23] : -925$

$In[24] : L3 = \%23/5$

$Out[24] : -185$

$In[25] : (\%15/.z- > 1)^5 + (\%15/.z- > rt)^5 rt + (\%15/.z- > rt^2)^5 rt^2 + (\%15/.z- > rt^3)^5 rt^3 + (\%15/.z- > rt^4)^5 rt^4$

$Out[25] : -1475$

$In[26] : L4 = \%25/5$

$Out[26] : -295$

Chapter 5

Appendix B

List of l_1 for $n = 8$ to $n = -5$ and computation of highest coefficient of $l_1(n)$

In appendix B, the method of finite differences is used to deduce the highest term of a polynomial in terms of n for l_1. The 14 first lines of appendix 2 list in order the 14 values of l_1, indexed by n and computed with Mathematica as in Appendix 1 in the case $n = -2$.

Line 1 to 14: list of 14 values of l_1, indexed by n in decreasing order.

Step 1: Line 15 to 27: differences of $l_{i+1} - l_i$. We obtain 13 new values.

Step 2: Line 28 to 39: differences of the 13 values previously computed. We obtain 12 new values.

Step 3: Line 40 to 50: differences of the 12 values previously computed. We obtain 11 new values.

Step 4: Line 51 to 60: differences of the 11 values previously computed. We obtain 10 new values.

Step 5: Line 61 to 69: differences of the 10 values previously computed. We obtain 9 new values.

Step 6: Line 70 to 77: differences of the 9 values previously computed. We obtain 8 new values.

Step 7: Line 78 to 84: differences of the 8 values previously computed. We obtain

7 new values.

Step 8: Line 85 to 90: differences of the 7 values previously computed. We obtain 6 new values.

Step 9: Line 91 to 95: differences of the 6 values previously computed. We obtain 5 new values which are the constant 1814400. The process stops here. It took 9 steps to obtain a constant, so the highest term of $l_1(n)$ is of degree 9. We divide 1814400 by 9! and obtain 5, which is the coefficient of n^9.

To obtain the other terms of $l_1(n)$, we substract $5n^9$ to each value of l_1. The method of finite differences is repeated with these new values to obtain the second highest term, etc. until we get the entire polynomial.

Computations with Mathematica:

$In[1] : L_8 = 1572903975$

$In[2] : L_7 = 528940095$

$In[3] : L_6 = 152354765$

$In[4] : L_5 = 35499375$

$In[5] : L_4 = 6002955$

$In[6] : L_3 = 540575$

$In[7] : L_2 = -25455$

$In[8] : L_1 = -11985$

$In[9] : L_0 = -625$

$In[10] : L_{-1} = 255$

$In[11] : L_{-2} = -75$

$In[12] : L_{-3} = -15505$

$In[13] : L_{-4} = -289485$

$In[14] : L_{-5} = -2750625$

$In[15] : L_8 - L_7$

$Out[15] : 1043963880$

$In[16] : L_7 - L_6$

$Out[16] : 376585330$

$In[17] : L_6 - L_5$

$Out[17] : 116855390$

$In[18] : L_5 - L_4$

$Out[18] : 29496420$

$In[19] : L_4 - L_3$

$Out[19] : 5462380$

$In[20] : L_3 - L_2$

$Out[20] : 566030$

$In[21] : L_2 - L_1$

$Out[21] : -13470$

$In[22] : L_1 - L_0$

$Out[22] : -11360$

$In[23] : L_0 - L_{-1}$

$Out[23] : -880$

$In[24] : L_{-1} - L_{-2}$

$Out[24] : 330$

$In[25] : L_{-2} - L_{-3}$

$Out[25] : 15430$

$In[26] : L_{-3} - L_{-4}$

$Out[26] : 273980$

$In[27] : L_{-4} - L_{-5}$

$Out[27] : 2461140$

$In[28] : \%15 - \%16$

$Out[28] : 667378550$

$In[29] : \%16 - \%17$

$Out[29] : 259729940$

$In[30] : \%17 - \%18$

$Out[30] : 87358970$

$In[31] : \%18 - \%19$

$Out[31] : 24034040$

$In[32] : \%19 - \%20$

$Out[32] : 4896350$

$In[33] : \%20 - \%21$

$Out[33] : 579500$

$In[34] : \%21 - \%22$

$Out[34] : -2110$

$In[35] : \%22 - \%23$

$Out[35] : -10480$

$In[36] : \%23 - \%24$

$Out[36] : -1210$

$In[37] : \%24 - \%25$

$Out[37] : -15100$

$In[38] : \%25 - \%26$

$Out[38] : -258550$

$In[39] : \%26 - \%27$

$Out[39] : -2187160$

$In[40] : \%28 - \%29$

$Out[40] : 407648610$

$In[41] : \%29 - \%30$

$Out[41] : 172370970$

$In[42] : \%30 - \%31$

$Out[42] : 63324930$

$In[43] : \%31 - \%32$

$Out[43] : 19137690$

$In[44] : \%32 - \%33$

$Out[44] : 4316850$

$In[45] : \%33 - \%34$

$Out[45] : 581610$

$In[46] : \%34 - \%35$

$Out[46] : 8370$

$In[47] : \%35 - \%36$

$Out[47] : -9270$

$In[48] : \%36 - \%37$

$Out[48] : 13890$

$In[49] : \%37 - \%38$

$Out[49] : 243450$

$In[50] : \%38 - \%39$

$Out[50] : 1928610$

$In[51] : \%40 - \%41$

$Out[51] : 235277640$

$In[52] : \%41 - \%42$

$Out[52] : 109046040$

$In[53] : \%42 - \%43$

$Out[53] : 44187240$

$In[54] : \%43 - \%44$

$Out[54] : 14820840$

$In[55] : \%44 - \%45$

$Out[55] : 3735240$

$In[56] : \%45 - \%46$

$Out[56] : 573240$

$In[57] : \%46 - \%47$

$Out[57] : 17640$

$In[58] : \%47 - \%48$

$Out[58] : -23160$

$In[59] : \%48 - \%49$

$Out[59] : -229560$

$In[60] := \%49 - \%50$

$Out[60] = -1685160$

$In[61] := \%51 - \%52$

$Out[61] = 126231600$

$In[62] := \%52 - \%53$

$Out[62] = 64858800$

$In[63] := \%53 - \%54$

$Out[63] = 29366400$

$In[64] := \%54 - \%55$

$Out[64] = 11085600$

$In[65] := \%55 - \%56$

$Out[65] = 3162000$

$In[66] := \%56 - \%57$

$Out[66] = 555600$

$In[67] := \%57 - \%58$

$Out[67] = 40800$

$In[68] := \%58 - \%59$

$Out[68] = 206400$

$In[69] := \%59 - \%60$

$Out[69] = 1455600$

$In[70] := \%61 - \%62$

$Out[70] = 61372800$

$In[71] := \%62 - \%63$

$Out[71] = 35492400$

$In[72] := \%63 - \%64$

$Out[72] = 18280800$

$In[73] := \%64 - \%65$

$Out[73] = 7923600$

$In[74] := \%65 - \%66$

$Out[74] = 2606400$

$In[75] := \%66 - \%67$

$Out[75] = 514800$

$In[76] := \%67 - \%68$

$Out[76] = -165600$

$In[77] := \%68 - \%69$

$Out[77] = -1249200$

$In[78] := \%70 - \%71$

$Out[78] = 25880400$

$In[79] := \%71 - \%72$

$Out[79] = 17211600$

$In[80] := \%72 - \%73$

$Out[80] = 10357200$

$In[81] := \%73 - \%74$

$Out[81] = 5317200$

$In[82] := \%74 - \%75$

$Out[82] = 2091600$

$In[83] := \%75 - \%76$

$Out[83] = 680400$

$In[84] := \%76 - \%77$

$Out[84] = 1083600$

$In[85] := \%78 - \%79$

$Out[85] = 8668800$

$In[86] := \%79 - \%80$

$Out[86] = 6854400$

$In[87] := \%80 - \%81$

$Out[87] = 5040000$

$In[88] := \%81 - \%82$

$Out[88] = 3225600$

$In[89] := \%82 - \%83$

$Out[89] = 1411200$

$In[90] := \%83 - \%84$

$Out[90] = -403200$

$In[91] := \%85 - \%86$

$Out[91] = 1814400$

$In[92] := \%86 - \%87$

$Out[92] = 1814400$

$In[93] := \%87 - \%88$

$Out[93] = 1814400$

$In[94] := \%88 - \%89$

$Out[94] = 1814400$

$In[95] := \%89 - \%90$

$Out[95] = 1814400$

Chapter 6

Appendix C

Use of l_i to compute the roots of f

In appendix C, we use the $l_0(n), l_1(n), l_2(n), l_3(n), l_4(n)$ in the case $n = 12$ in order to compute with Mathematica values for R_1, R_2, R_3, R_4. Then the fifth roots of the R_i are chosen according to Dummit's rule. Using equations (1.2), we compute the roots of f. We also solve for the roots directly with Mathematica and compare the result.

Line 1 to 10: $p_1 = l_1 - l_0, p_2 = l_2 - l_0, p_3 = l_3 - l_0, p_4 = l_4 - l_0$ are computed in terms of n and factored

Line 6: gives a value for a fifth root of unity

Line 11 to 12: R_1 is computed with the equation $R_1 = p_1\zeta + p_2\zeta^2 + p_3\zeta^3 + p_4\zeta^4$

Line 13: solves for the fifth roots of R_1

Line 14: r_1 is picked to be any fifth root of R_1

Line 15: r_4 is the conjugate of r_1

Line 17 to 18: R_2 is computed similarly to R_1

Line 19: solves for the fifth roots of R_2

Line 20 to 24: the fifth roots of R_2 are labeled $r_{21}, r_{22}, r_{23}, r_{24}, r_{25}$

Line 25 to 29: the fifth roots of R_3 are the conjugates of $r_{21}, r_{22}, r_{23}, r_{24}, r_{25}$ and are labeled $r_{31}, r_{32}, r_{33}, r_{34}, r_{35}$

Line 30 and 31: the reduced quintic f is computed for $n = 12$

Line 32: u is computed with the coefficient d of $f_3(x)$

Line 33 to 42: computations to choose r_2 and therefore r_3

Line 43 to 47: the roots of f are computed using equations (1.2)

Line 48: the roots of f are solved for. They are equal to the ones computed with the $l_i(n)$.

Computations with Mathematica:

$In[1] := L_0 = -n^{10} - 10n^9 - 55n^8 - 175n^7 - 325n^6 - 250n^5 + 375n^4 + 1250n^3 + 1250n^2 - 625$

$Out[1] = -625 + 1250n^2 + 1250n^3 + 375n^4 - 250n^5 - 325n^6 - 175n^7 - 55n^8 - 10n^9 - n^{10}$

$In[2] := L_1 = -625 - 3125n - 4375n^2 - 3125n^3 - 1125n^4 + 225n^6 + 125n^7 + 35n^8 + 5n^9$

$Out[2] = -625 - 3125n - 4375n^2 - 3125n^3 - 1125n^4 + 225n^6 + 125n^7 + 35n^8 + 5n^9$

$In[3] := L_2 = 2500 + 6250n + 6875n^2 + 4375n^3 + 1625n^4 + 250n^5 - 75n^6 - 50n^7 - 10n^8$

$Out[3] = 2500 + 6250n + 6875n^2 + 4375n^3 + 1625n^4 + 250n^5 - 75n^6 - 50n^7 - 10n^8$

$In[4] := L_3 = -625 + 2500n^2 + 4375n^3 + 4000n^4 + 2375n^5 + 975n^6 + 275n^7 + 50n^8 + 5n^9$

$Out[4] = -625 + 2500n^2 + 4375n^3 + 4000n^4 + 2375n^5 + 975n^6 + 275n^7 + 50n^8 + 5n^9$

$In[5] := L_4 = -20n^8 - 175n^7 - 800n^6 - 2375n^5 - 4875n^4 - 6875n^3 - 6250n^2 - 3125n - 625$

$Out[5] = -625 - 3125n - 6250n^2 - 6875n^3 - 4875n^4 - 2375n^5 - 800n^6 - 175n^7 - 20n^8$

$In[6] := rt = N[Exp[2\Pi i/5], 100]$

$Out[6] = 0.30901699437494742410229341718281905886015458990288143106772431135263023140945122485360360209469 55687$

$+\,0.951056516295153572116439333379382143405698634125750222447305644430153170085193501718792810970811381 7i$

$In[7] := Factor[p_1 = L_1 - L_0]$

$Out[7] = n(25 + 25n + 15n^2 + 5n^3 + n^4)(-125 - 100n + 25n^3 + 10n^4 + n^5)$

$In[8] := p_1/.n- > 12$

$Out[8] = 190426952244$

$In[9] := Factor[p_2 = L_2 - L_0]$

$Out[9] = (5 + 5n + n^2)(25 + n^4)(25 + 25n + 15n^2 + 5n^3 + n^4)$

$In[10] := p_2/.n- > 12$

$Out[10] = 138246440189$

$In[11] := Factor[p_3 = L_3 - L_0]$

$Out[11] = n^2(5 + 5n + n^2)(10 + 5n + n^2)(25 + 25n + 15n^2 + 5n^3 + n^4)$

$In[12] := p_3/.n- > 12$

$Out[12] = 205202172384$

$In[13] := Factor[p_4 = L_4 - L_0]$

$Out[13] = n(5 + 5n + n^2)(-25 - 10n + n^3)(25 + 25n + 15n^2 + 5n^3 + n^4)$

$In[14] := p_4/.n- > 12$

$Out[14] = 126493395204$

$In[15] := Expand[R_1 = \%8rt + \%10rt^2 + \%12rt^3 + \%14rt^4]$

$Out[15] = -1.7992199104140923892452145489156805458834516125942928660 4$

$804002361970622144920024523672385668023089 \times 10^{11}$

$+ 2.14488340921545497276126087687947300480440711654053224241045667008$

$365276825258757833023616224616 72 \times 10^{10} i$

$In[16] := Solve[x^5 == R_1, x]$

$Out[16] = \{\{x- > -178.446242477119445729088736961610775420771507 51$

$16348678472282631685538588726625905121505512588496 30$

$+ 4.23539204762577101849723135531897731400731208355804611929896821963$

$6154108100808486632034995380509i\},$

$\{x- > -59.17101871374168680451438596840678720104979858948620712571870$

$73758568715300177953581669425919680 48$

$- 168.40365359569260573344735450943884204328263163976345907667499871 5$

$19870191410542961590817782612222 05i\},$

$\{x- > -51.11482430182336090684227582755670387350601118263964762383236$

$04992324886807662899439640713462000 54$

$+ 171.021269836806345604225261941045215585942614480936297226460750845$

$1703967313565395850340240950398 65i\},$

$\{x- > 141.87654176307099853499564545581582163780672673417487259566937$

$6332489614133340900387884704062223563$

$- 108.31457379942724401229109526679494209214536398012943370295765143 3$

$982767920624659719298346903297685i\},$

$\{x- > 146.8555437296134949054497533017584448575205905495858500011099 54$

$71115360495010577542639686113479 4170$

$+ 101.4615655106877331230159564798695912354780690553985494338729310843749189952727412635404656389995 16i\}\}$

$In[17] := r_1 = \%16[[1, 1, 2]]$

$Out[17] = -178.44624247711944572908873696161077542077150751163486784722826316855385887266259051215055125884963 0$

$+ 4.23539204762577101849723135531897731400731208355804611929896821963615410810080848663203499538050 9i$

$In[18] := r_4 = Conjugate[r_1]$

$Out[18] = -178.44624247711944572908873696161077542077150751163486784722826316855385887266259051215055125884963 0$

$- 4.23539204762577101849723135531897731400731208355804611929896821963615410810080848663203499538050 9i$

$In[19] := R_2 = \%12rt + \%8rt^2 + \%14rt^3 + \%10rt^4$

$Out[19] = (-1.502624889690907610754785451084319454116548387405707133951959976380293778550799754763276143319769 11 \times 10^{11}$

$+ 1.012578873620795777174006812966190465922857759155541372329099376386949947620773965317633636070234 82 \times 10^{11}i$

$In[20] := Solve[x^5 == R_2, x]$

$Out[20] = \{\{x- > -177.24271152959076133147063873687315574862805171225885407996629608815445397642873152724909838309556 9$

$+ 21.119214228713013404199885437818831932270831898623159699846562859 78$

$4434716954521804388206225233351i\}$,

$\{x->-74.856576302990814457705033657785001390645742029927826243794160858906461501653227947853170596603722$

$-162.04163966152192095229030353943550138310942225009261519928323054494522578370833818693157197020339 8i\}$,

$\{x->-34.685443680489140033791791842760412558652895889266426096983538565766614798587660561126439319962755$

$+175.094031870556958682226310291420372801006491106484012187667744121925466573420096931215390692231070i\}$,

$\{x->130.978803092892491020960770727242412329546588547369131419581931016519586177853104908624392016092113$

$-121.266455132296567136563957922980624092811122627936666935536679905637995793858881336562397581171664i\}$,

$\{x->155.805928420178224802006693510176157368380101084083975001162064496307944098816515127604316283569933$

$+87.094848694548516002428065733176920742643221872922110247305603468873320287192600787890372633910642i\}\}$

$In[21]:=r_{21}=\%20[[1,1,2]]$

$Out[21]=-177.242711529590761331470638736873155748628051712258854079966296088154453976428731527249098383095569$

$+21.1192142287130134041998854378188319322708318986231596998465628597844347169545218043882062252333 51i$

$In[22] := r_{22} = \%20[[2,1,2]]]$

$Out[22] = -74.856576302990814457705033657785001390645742029927826$

$2437941608589064615016532279478531705966 03722$

$- 162.041639661521920952290303539435501383109422250092615199283230544$

$9452257837083381869315719702033 98i$

$In[23] := r_{23} = \%20[[3,1,2]]$

$Out[23] = -34.685443680489140033791791842760412558652895889266426$

$09698353856576661479858766056112643931996 2755$

$+ 175.094031870556958682226310291420372801006491106484012187667744 12$

$19254665734200969312153906922310 70i$

$In[24] := r_{24} = \%20[[4,1,2]]$

$Out[24] = 130.978803092892491020960770727242412329546588547369131$

$41958193101651958617785310490862439201 6092113$

$- 121.266455132296567136563957922980624092811122627936666935536679 90$

$56379957938588813365623975811716 64i$

$In[25] := r_{25} = \%20[[5,1,2]]$

$Out[25] = 155.805928420178224802006693510176157368380101084083 97$

$50011620644963079440988165151276043162835 69933$

$+ 87.094848694548516002428065733176920742643221872922110247305603468$

$87332028719260078789037263391064 2i$

$In[26] := R_3 = Conjugate[R_2]$

$Out[26] = -1.50262488969090761075478545108431945411654838740570 71$

$339519599763802937785507997547632761433197691 1 \times 10^{11}$

$- 1.01257887362079577717400681296619046592285775915554137232909937638$

$694994762077396531763363607023482 \times 10^{11} i$

$In[27] := r_{31} = Conjugate[r_{21}]$

$Out[27] = -177.2427115295907613314706387368731557486280517122588 5$

$4079966296088154453976428731527249098383095569$

$- 21.1192142287130134041998854378188319322708318986231596998465628597$

$844347169545218043882062252333 51 i$

$In[28] := r_{32} = Conjugate[r_{22}]$

$Out[28] = -74.85657630299081445770503365778500139064574202992782624379 4$

$1608589064615016532279478531705966037 22$

$+ 162.041639661521920952290303539435501383109422250092615199283230544$

$945225783708338186931571970203398 i$

$In[29] := r_{33} = Conjugate[r_{23}]$

$Out[29] = -34.685443680489140033791791842760412558652895889266426$

$0969835385657666147985876605611264393199627 55$

$- 175.094031870556958682226310291420372801006491106484012187667744121$

$9254665734200969312153906922310 70 i$

$In[30] := r_{34} = Conjugate[r_{24}]$

$Out[30] = 130.978803092892491020960770727242412329546588547369131$

$4195819310165195861778531049086243920160921 13$

$+ 121.266455132296567136563957922980624092811122627936666935536679905$

$637995793858881336562397581171664i$

$In[31] := r_{35} = Conjugate[r_{25}]$

$Out[31] = 155.805928420178224802006693510176157368380101084083975 0$

$0116206449630794409881651512760431628356993 3$

$- 87.094848694548516002428065733176920742643221872922110247305603468 87$

$332028719260078789037263391064 2i$

$In[32] := f = 1 - 2n^2 - 2n^3 - 3/5n^4 + 2/5n^5 + 13/25n^6 + 7/25n^7 + 11/125n^8 + 2/125n^9 + 4/3125n^10 + (10 + 10n + 2n^2 - 5n^3 - 28/5n^4 - 16/5n^5 - 28/25n^6 - 6/25n^7 - 3/125n^8)y + (5 + 15n + 17n^2 + 11n^3 + 23/5n^4 + 6/5n^5 + 4/25n^6)y^2 + (-10 - 10n - 6n^2 - 2n^3 - 2/5n^4)y^3 + y^5$

$In[33] := h = f/.n- > 12$

$Out[33] = 660368960021/3125 - 2898012838/125y + 22334561/25y^2 - 63722/5y^3 + y^5$

$In[34] := u = -25/2(22334561/25)$

$Out[34] = -22334561/2$

$In[35] := r_1 r_{21}^2 + r_4 r_{31}^2 - u$

$Out[35] = 178108.404577801998793231705648617980343433853888957052752$

$4876428400615180293652547506066436633 91$

$In[36] := r_{31}r_1^2 + r_{21}r_4^2 - u$

$Out[36] = -178108.404577801998793231705648617980343433853888957052752487642840061518029365254750606643663391$

$In[37] := x_1 = (r_1 + r_4 + r_{21} + r_{31})/5$

$Out[37] = -142.275581602684082824223750279393572467759823689557488770877823702683325139636528815759859856778079$

$In[38] := x_2 = (rt^4 r_1 + rt^3 r_{21} + rt^2 r_{31} + rtr_4)/5$

$Out[38] = 41.8764416473419455580657670730477813979496359605777309509318815988301821672200900734560979749 47952$

$In[39] := x_3 = (rt^3 r_1 + rtr_{21} + rt^4 r_{31} + rt^2 r_4)/5$

$Out[39] = 28.7995869706490721790978878575893773867499394078632095029263175408988573793810189914174762152 76179$

$In[40] := x_4 = (rt^2 r_1 + rt^4 r_{21} + rtr_{31} + rt^3 r_4)/5$

$Out[40] = 42.8764392330327434004815414452221636316615323379633785994743351066891997339012959307033058969 04323$

$In[41] := x_5 = (rtr_1 + rt^2 r_{21} + rt^3 r_{31} + rt^4 r_4)/5$

$Out[41] = 28.7231137516603216865785539035342500513987159831531697175452894562650858591341238201829797696 49626$

$In[42] := NSolve[h == 0, y, 10]$

$Out[42] = \{\{y- > -142.2755816\}, \{y- > 28.723\}, \{y- > 28.800\}, \{y- > 41.8764\}, \{y- > 42.8764\}\}$

Chapter 7

Appendix D

Computation of several values of θ with Mathematica.

In appendix D, we compute the sextic resolvent, $g_n(x)$, of Emma Lehmer's quintics in reduced form ($s_1 = 0$). Then we compute $g_n(x)$ for values of n ranging from 4 to -6. Then we solve for each sextic and pick up the rational root in each case.

Line 1 to 5: computation of $g_n(x)$ with the coefficients of Emma Lehmer's quintics after a translation.

Line 6: $g_4(x)$ is computed

Line 7: $g_4(x)$ is solved for. We obtain the rational root $\theta_4 = \frac{1212949}{125}$

Line 8: $g_3(x)$ is computed

Line 9: $g_3(x)$ is solved for. We obtain the rational root $\theta_3 = \frac{35629}{125}$

Line 10 through 27: the same process is repeated as in line 6 to 9 to get values of θ_i for n ranging from 2 to -6.

Computations with Mathematica:

$In[1] := p = Expand[-2/5n^4 - 2n^3 - 6n^2 - 10n - 10]$

$In[2] := q = Expand[4/25n^6 + 6/5n^5 + 23/5n^4 + 11n^3 + 17n^2 + 15n + 5]$

$In[3] := r = Expand[-3/125n^8 - 6/25n^7 - 28/25n^6 - 16/5n^5 - 28/5n^4 - 5n^3 + 2n^2 + 10n + 10]$

$In[4] := s = Expand[4/5^5n^{10} + 2/125n^9 + 11/125n^8 + 7/25n^7 + 13/25n^6 + 2/5n^5 - 3/5n^4 - 2n^3 - 2n^2 + 1]$

$In[5] := g = Expand[x^6 + 8rx^5 + (2pq^2 - 6p^2r + 40r^2 - 50qs)x^4 + (-2q^4 + 21pq^2r - 40p^2r^2 + 160r^3 - 15p^2qs - 400qrs + 125ps^2)x^3 + (p^2q^4 - 6p^3q^2r - 8q^4r + 9p^4r^2 + 76pq^2r^2 - 136p^2r^3 + 400r^4 - 50pq^3s + 90p^2qrs - 1400qr^2s + 625q^2s^2 + 500prs^2)x^2 + (-2pq^6 + 19p^2q^4r - 51p^3q^2r^2 + 3q^4r^2 + 32p^4r^3 + 76pq^2r^3 - 256p^2r^4 + 512r^5 - 31p^3q^3s - 58q^5s + 117p^4qrs + 105pq^3rs + 260p^2qr^2s - 2400qr^3s - 108p^5s^2 - 325p^2q^2s^2 + 525p^3rs^2 + 2750q^2rs^2 - 500pr^2s^2 + 625pqs^3 - 3125s^4)x + (q^8 - 13pq^6r + p^5q^2r^2 + 65p^2q^4r^2 - 4p^6r^3 - 128p^3q^2r^3 + 17q^4r^3 + 48p^4r^4 - 16pq^2r^4 - 192p^2r^5 + 256r^6 - 4p^5q^3s - 12p^2q^5s + 18p^6qrs + 12p^3q^3rs - 124q^5rs + 196p^4qr^2s + 590pq^3r^2s - 160p^2qr^3s - 1600qr^4s - 27p^7s^2 - 150p^4q^2s^2 - 125pq^4s^2 - 99p^5rs^2 - 725p^2q^2rs^2 + 1200p^3r^2s^2 + 3250q^2r^2s^2 - 2000pr^3s^2 - 1250pqrs^3 + 3125p^2s^4 - 9375rs^4)]$

$Out[5] = 4460328125 + 50779937500n + 283835987500n^2 + 1039738796875n^3$
$+ 2808338385000n^4 + 5961402726250n^5 + 10344843715750n^6 + 15062406627750n^7$
$+ 18731920776150n^8 + 20137470395875n^9 + 18853913272280n^{10}$
$+ 15423445699180n^{11} + 11007263515836n^{12} + 6795992211705n^{13}$
$+ 17771985630761/5n^{14} + 7474112232432/5n^{15} + 10659969228771/25n^{16}$

$+ 10529731057/25n^{17} - 12534893201859/125n^{18} - 2040523945496/25n^{19}$

$- 26879530096884/625n^{20} - 10315204110729/625n^{21} - 12951484375479/3125n^{22}$

$- 57612659691/625n^{23} + 9263317878221/15625n^{24} + 6189125909307/15625n^{25}$

$+12858492149581/78125n^{26}+3723109664222/78125n^{27}+3019360776551/390625n^{28}$

$-83764495134/78125n^{29}-2753877198319/1953125n^{30}-1249509837714/1953125n^{31}$

$-1927365867239/9765625n^{32}-83772650148/1953125n^{3}3-236708773764/48828125n^{34}$

$+45010690392/48828125n^{35}+183456219996/244140625n^{36}+66550980672/244140625n^{37}$

$+89004646416/1220703125n^{38}+3862833088/244140625n^{39}+17542411536/6103515625n^{40}$

$+2692386816/6103515625n^{41}+1741455616/30517578125n^{42}+7507968/1220703125n^{43}$

$+82479616/152587890625n^{44}+5701632/152587890625n^{45}+1458176/762939453125n^{46}$

$+ 49152/762939453125n^{47} + 4096/3814697265625n^{48} + 303846875x + 2908218750nx$

$+ 13784862500n^2x + 43047668750n^3x + 99408598750n^4x + 180607763750n^5x$

$+ 268110470875n^6x + 333296680000n^7x + 352629413600n^8x + 320738372200n^9x$

$+ 252004142701n^{10}x + 170880520610n^{11}x + 99111966372n^{12}x$

$+ 48007715188n^{13}x + 91196290361/5n^{14}x + 21588691974/5n^{15}x$

$- 12171016016/25n^{16}x - 6365452864/5n^{17}x - 108182766217/125n^{18}x$

$- 49761792514/125n^{19}x - 414899771239/3125n^{20}x - 17114898322/625n^{21}x$

$+ 3979624613/3125n^{22}x + 13539645012/3125n^{23}x + 37224668254/15625n^{24}x$

$+ 13434900186/15625n^{25}x + 17318259406/78125n^{26}x + 538755934/15625n^{27}x$

$- 743501098/390625n^{28}x - 1432280336/390625n^{29}x - 15398306924/9765625n^{30}x$

$- 917064176/1953125n^{31}x - 1078771016/9765625n^{32}x - 208360704/9765625n^{33}x$

$- 166781248/48828125n^{34}x - 22008832/48828125n^{35}x - 11756032/244140625n^{36}x$

$- 196608/48828125n^{37}x - 303104/1220703125n^{38}x - 12288/1220703125n^{39}x$

$- 6144/30517578125n^{40}x - 1209375x^2 - 56250nx^2 + 32266875n^2x^2$

$+ 158960000n^3x^2 + 436411875n^4x^2 + 841097000n^5x^2 + 1244167375n^6x^2$

$+ 1479576400n^7x^2 + 1452952100n^8x^2 + 1196186980n^9x^2 + 830789831n^{10}x^2$

$+ 485251880n^{11}x^2 + 1171136664/5n^{12}x^2 + 444382628/5n^{13}x^2$

$+ 111685897/5n^{14}x^2 - 2666852/25n^{15}x^2 - 100461278/25n^{16}x^2$

$- 342971412/125n^{17}x^2 - 150380783/125n^{18}x^2 - 238879192/625n^{19}x^2$

$- 252927651/3125n^{20}x^2 - 12319506/3125n^{21}x^2 + 90908566/15625n^{22}x^2$

$+ 51820536/15625n^{23}x^2 + 18083369/15625n^{24}x^2 + 23924576/78125n^{25}x^2$

$+ 5056464/78125n^{26}x^2 + 4313856/390625n^{27}x^2 + 588224/390625n^{28}x^2$

$+ 311296/1953125n^{29}x^2 + 120832/9765625n^{30}x^2 + 6144/9765625n^{31}x^2$

$+ 768/48828125n^{32}x^2 - 322500x^3 - 1673750nx^3 - 4262250n^2x^3$

$- 6964500n^3x^3 - 8013650n^4x^3 - 6631500n^5x^3 - 3697300n^6x^3$

$- 843450n^7x^3 + 840370n^8x^3 + 277350n^9x^3 + 5001282/5n^{10}x^3$

$+ 2795036/5n^{11}x^3 + 5865376/25n^{12}x^3 + 1763402/25n^{13}x^3$

$+ 1374552/125n^{14}x^3 - 67038/25n^{15}x^3 - 359358/125n^{16}x^3$

$- 162752/125n^{17}x^3 - 258924/625n^{18}x^3 - 12608/125n^{19}x^3$

$- 298016/15625n^{20}x^3 - 43008/15625n^{21}x^3 - 22528/78125n^{22}x^3$

$- 1536/78125n^{23}x^3 - 256/390625n^{24}x^3 - 2750x^4 - 14250nx^4$

$- 32350n^2x^4 - 45350n^3x^4 - 44180n^4x^4 - 31310n^5x^4 - 16028n^6x^4$

$- 5330n^7x^4 - 456n^8x^4 + 742n^9x^4 + 14542/25n^{10}x^4$

$+ 6464/25n^{11}x^4 + 10114/125n^{12}x^4 + 2304/125n^{13}x^4$

$+ 1856/625n^{14}x^4 + 192/625n^{15}x^4 + 48/3125n^{16}x^4$

$+ 80x^5 + 80nx^5 + 16n^2x^5 - 40n^3x^5 - 224/5n^4x^5 - 128/5n^5x^5$

$-224/25n^6x^5 - 48/25n^7x^5 - 24/125n^8x^5 + x^6$

$In[6] := f_4 = g/.n > 4$

$Out[6] = 645203411828985938984928175486987153013814697265625$

$- 229291902046340266709219163695369/30517578125x$

$+ 6490609440779795713570685348828125x^2 - 46148541739624014396390625x^3$

$+ 170206608821783125x^4 - 15040944/125x^5 + x^6$

$In[7] := Solve[\%6 == 0, x]$

$Out[7] = \{x- > 1212949/125\}$

$In[8] := f_3 = g/.n > 3$

$Out[8] = 18026464964880793140690410512721/3814697265625$

$- 1404561879100479700090044269/30517578125x$

$+ 7867310479384695649893/48828125x^2 - 94934780440349256/390625x^3$

$+ 460418096798/3125x^4 - 2807024/125x^5 + x^6$

$In[9] := Solve[f_3 == 0, x]$

$Out[9] = \{x- > 35629/125\}$

$In[10] := f_2 = g/.n- > 2$

$Out[10] = 671425679733351879074266203701/3814697265625 +$

$44639483939610731771309131/30517578125x + 197243865528651483653/48828125x^2$

$+ 1312579834724904/390625x^3 - 6280641922/3125x^4 - 302544/125x^5 + x^6$

$In[11] := Solve[f_2 == 0, x]$

$Out[11] = \{x- > -68951/125\}$

$In[12] := f_1 = g/.n- > 1$

$Out[12] = 50253833442257301138592831/3814697265625$

$+ 67267538291055772334431/30517578125x + 414712274214831173/48828125x^2$

$- 11102798648676/390625x^3 - 594747262/3125x^4 + 6816/125x^5 + x^6$

$In[13] := Solve[f_1 == 0, x]$

$Out[13] = \{x- > -27761/125\}$

$In[14] := f_0 = g/.n- > 0$

$Out[14] = 4460328125{+}303846875x{-}1209375x^2{-}322500x^3{-}2750x^4{+}80x^5{+}x^6$

$In[15] := Solve[f_0 == 0, x]$

$Out[15] = \{x- > -55\}$

$In[16] := f_{-1} = g/.n- > -1$

$Out[16] = -230566838812460 59/3814697265625$

$+ 953714287461331/30517578125x - 574928454067/48828125x^2$

$- 928420416/390625x^3 + 382118/3125x^4 + 3696/125x^5 + x^6$

$In[17] := Solve[f_{-1} == 0, x]$

$Out[17] = \{x- > -1991/125\}$

$In[18] := f_{-2} = g/.n- > -2$

$Out[18] = -23056683881246059/3814697265625$

$+ 953714287461331/30517578125x - 574928454067/48828125x^2$

$- 928420416/390625x^3 + 382118/3125x^4 + 3696/125x^5 + x^6$

$In[19] := Solve[f_{-2} == 0, x]$

$Out[19] = \{x- > -1991/125\}$

$In[20] := f_{-3} = g/.n- > -3$

$Out[20] = 4955491471940005417393 1/3814697265625$

$+ 2736270081564019803 1/30517578125x + 343313624274813/48828125x^2$

$- 188785447836/390625x^3 - 20778742/3125x^4 + 7936/125x^5 + x^6$

$In[21] := Solve[f_{-3} == 0, x]$

$Out[21] = \{x- > -7781/125\}$

$In[22] := f_{-4} = g/.n- > -4$

$Out[22] = 858015751790833826061969221/3814697265625$

$+ 1594039413178884765882 31/30517578125x + 1829799203901867893/48828125x^2$

$+ 22113395696244/390625x^3 - 880366702/3125x^4 - 63024/125x^5 + x^6$

$In[23] := Solve[f_{-4} == 0, x]$

$Out[23] = \{x- > -27371/125\}$

$In[24] := f_{-5} = g/.n- > -5$

$Out[24] = 1526992474390625 + 45821342909375x + 13405665625x^2$

$- 4771635000x^3 + 16909750x^4 - 7920x^5 + x^6$

$In[25] := Solve[f_{-5} == 0, x]$

$Out[25] = \{x- > -55\}$

$In[26] := f_{-6} = g/.n- > -6$

$Out[26] = 433356203850823091382298089369788981/3814697265625$

$- 35925247664619331683670018904 69/30517578125x$

$+ 2365083396007548024151413/48828125x^2 - 3894236547421781736/390625x^3$

$+ 3304767356558/3125x^4 - 6653264/125x^5 + x^6$

$In[27] := Solve[f_{-6} == 0, x]$

$Out[27] = \{x- > 504169/125\}$

BIBLIOGRAPHY

[1] Bruce C. Berndt, Blair K. Spearman, Kenneth S. Williams *Commentary on an Unpublished Lecture by G.N. Watson on Solving the Quintic* (The Mathematical Intelligencer, Volume 24, (2002), Number 4, 15-33)

[2] David A. Cox *Galois Theory* (Wiley Interscience, 2004)

[3] David S. Dummit *Solving Solvable Quintics* (Mathematics of Computation, Volume 57, (1991), Number 195, 387-401)

[4] Sigeru Kobayashi, Hiroshi Nakagawa *Resolution of Solvable Quintic Equation* (Mathematica Japonica, Volume 37, (1992), Number 5, 883-886)

[5] Lehmer, Emma *Connection Between Gaussian Periods and Cyclic Units* (Mathematics of Computation, Volume 50, (1988), Number 182, 535-541)

[6] Rene Schoof , Lawrence C. Washington *Quintic Polynomials and Real Cyclotomic Fields with Large Class Numbers* (Mathematics of Computation, Volume 50, Number 182, (1988), 543-556)

[7] Blair K. Spearman, Kenneth S. Williams *Characterization of Solvable Quintics* x^5+ax+b (The American Mathematical Monthly, Volume 101, (1994), Number 10, 986-992)