

ABSTRACT

Title of Dissertation: LIFTING OF CHARACTERS AND FUNCTIONS
 ON METAPLECTIC GROUPS

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We study the lifting of representations between the N -fold metaplectic covers of $\mathrm{SL}(n, \mathbb{F})$ where $n|N$ and $\mathrm{PGL}(n, \mathbb{F})$, \mathbb{F} a local field, and obtain a formula relating irreducible characters of $\mathrm{PGL}(n, \mathbb{F})$ and $\widetilde{\mathrm{SL}}(n, \mathbb{F})$ (covers of $\mathrm{SL}(n, \mathbb{F})$). This is achieved by generalizing the approach of Adams [1].

In the second part of the thesis we study the lifting of functions between $\widetilde{\mathrm{SL}}(n, \mathbb{F})$ and $\mathrm{PGL}(n, \mathbb{F})$. Using orbital integrals we obtain the formula for the lifting of characters as a dual to the lifting of functions. This is based on the methods of Flicker and Kazhdan [7].

Finally we use our methods of lifting of orbital integrals to provide alternate proof of a well-known fact about p -adic fields (under certain restrictions). We show that for a Galois extension \mathbb{E}/\mathbb{F} , $\mathbb{F}^*/N(\mathbb{E}^*) \simeq \mathrm{Gal}(\mathbb{E}/\mathbb{F})^{ab}$ where $\mathrm{Gal}(\mathbb{E}/\mathbb{F})$ is the Galois group of \mathbb{E} over \mathbb{F} and $\mathrm{Gal}(\mathbb{E}/\mathbb{F})^{ab}$ denotes its abelianization and $N : \mathbb{E}^* \rightarrow \mathbb{F}^*$ is the norm map.

LIFTING OF CHARACTERS AND FUNCTIONS
ON METAPLECTIC GROUPS

by

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DEDICATION

I dedicate this dissertation to my parents. The gift of unwavering love and support has no equal.

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Chapter 1

Introduction

1.1 Overview

The study of non-linear groups has gained much importance since Shimura defined a correspondence between the space of cusp forms of half-integral weight and the space of cusp forms of even integral weight. An important example involving non-linear groups is of the oscillator representation of the two fold cover of $\mathrm{Sp}(2n)$. Shimura's correspondence gives a correspondence between automorphic forms on $\mathrm{PGL}(2)$ and the two-fold cover $\widetilde{\mathrm{SL}}(2)$ of $\mathrm{SL}(2)$.

Flicker, Kazhdan, and Patterson have extensively studied automorphic forms on arbitrary covers $\widetilde{\mathrm{GL}}(n)$ of $\mathrm{GL}(n)$ [6], [7], [9]. A complete description of the correspondence between the automorphic forms between these two groups for $n = 2$ is given in [6]. This was generalized to any n by Flicker and Kazhdan in [7]. In comparison to $\widetilde{\mathrm{GL}}(n)$, we know much less about automorphic forms on $\widetilde{\mathrm{SL}}(n)$. However, there exist results for the case $n = 2$. Waldspurger has done a deep study of automorphic forms on $\widetilde{\mathrm{SL}}(2)$ in his work [23], [24], [25].

Kazhdan and Patterson have shown that certain covers of $\mathrm{GL}(n, \mathbb{F})$ have interesting properties. This work focuses on the genuine representations of N -fold

covers of the general linear group $\mathrm{GL}(n, \mathbb{F})$ and the special linear group $\mathrm{SL}(n, \mathbb{F})$ where \mathbb{F} is a p -adic field containing μ_N , satisfying $|\mu_N(\mathbb{F})| = N$, where $\mu_N(\mathbb{F})$ is the group of the N^{th} roots of unity in the field \mathbb{F} . A representation of $\widetilde{\mathrm{GL}}(n, \mathbb{F})$ is said to be genuine if it does not factor to any proper quotient of $\widetilde{\mathrm{GL}}(n, \mathbb{F})$. We study the lifting (transfer) of characters of representations between $\mathrm{PGL}(n, \mathbb{F})$ and the N -fold cover $\widetilde{\mathrm{SL}}(n, \mathbb{F})$. In lifting theory one tries to obtain a relationship between the characters of representations of the two groups in question. This is important because it provides methods for obtaining information about the representations of one group by knowing about the representations of the other. Lifting or transfer of functions defined on the two groups is dual to the lifting of representations of the given groups.

The problem of lifting of representations has been studied extensively. Howe's theta correspondence and Langlands' functoriality of L-packets conjectures are important examples of the phenomenon of lifting. The Theta correspondence relates representations of the members of a reductive dual pair imbedded in the Metaplectic group. Langlands' functoriality conjecture provides methods to transfer representations between linear groups by using homomorphisms between L-groups.

Adams [3] proved a correspondence between characters of $\mathrm{SO}(p, q)$ (here $p + q = 2n + 1$) and the two-fold cover $\widetilde{\mathrm{Sp}}(2n)$ over \mathbb{R} . Renard [16] obtained an orbital integral correspondence between continuous functions with compact support on the above groups and proved that this correspondence is dual to the one obtained by Adams. Schultz [18] obtained the correspondence between characters for the case $n = 1$ for \mathbb{F} a p -adic field. This is the same as the correspondence between the 2-fold cover of $\mathrm{SL}(2)$ and $\mathrm{PGL}(2)$. Adams [1] generalized this to a correspondence

between the n -fold cover of $\mathrm{SL}(n, \mathbb{F})$ and $\mathrm{PGL}(n, \mathbb{F})$.

We generalize the work of Adams [1] by obtaining a correspondence between the characters of certain covers of $\mathrm{SL}(n, \mathbb{F})$ and $\mathrm{PGL}(n, \mathbb{F})$. As mentioned before, the problem of lifting of functions is dual to the problem of lifting of characters and we exhibit a transfer of functions between $\widetilde{\mathrm{SL}}(n, \mathbb{F})$ and $\mathrm{PGL}(n, \mathbb{F})$ in that direction. We consider N -fold covers of $\mathrm{GL}(n, \mathbb{F})$ where $n|N$. We do this because our methods work only for this case. Only when $n|N$ can we obtain representations of $\widetilde{\mathrm{SL}}(n, \mathbb{F})$ from our analysis. In the second part of the thesis we study the lifting of functions between $\widetilde{\mathrm{SL}}(n, \mathbb{F})$ and $\mathrm{PGL}(n, \mathbb{F})$. Using orbital integrals we obtain the formula for the lifting of characters as a dual to the lifting of functions. This is based on the methods of Flicker and Kazhdan [7]. Finally, using analysis of orbital integrals, we provide alternate proofs of a well-known fact about p -adic fields (under certain restrictions). We show that for a Galois extension \mathbb{E}/\mathbb{F} , $\mathbb{F}^*/N(\mathbb{E}^*) \simeq \mathrm{Gal}(\mathbb{E}/\mathbb{F})^{ab}$ where N is the norm map (section 1.3).

1.2 Main Results

First we introduce some notation. Let \mathbb{F} denote a p -adic field i.e. a finite extension of the p -adic numbers, \mathbb{Q}_p . We will be considering central extensions of $\mathrm{GL}(n, \mathbb{F})$ and $\mathrm{SL}(n, \mathbb{F})$. Let $\mu_N(\mathbb{F})$ be the group of N^{th} roots of unity in \mathbb{F} , and assume that $|\mu_N(\mathbb{F})| = N$ and that $n|N$ and $N_o = N/n$.

Let $\widetilde{\mathrm{SL}}(n, \mathbb{F})$ be a perfect group (a group which is its own commutator subgroup) fitting into the exact sequence

$$1 \rightarrow \mu_N(\mathbb{F}) \rightarrow \widetilde{\mathrm{SL}}(n, \mathbb{F}) \rightarrow \mathrm{SL}(n, \mathbb{F}) \rightarrow 1$$

with $\mu_N(\mathbb{F})$ central in $\widetilde{\mathrm{SL}}(n, \mathbb{F})$. The group $\widetilde{\mathrm{SL}}(n, \mathbb{F})$ is unique up to isomorphism

and is given by a Steinberg cocycle [13]. We let $\widetilde{\text{GL}}(n, \mathbb{F})$ to be that extension of $\text{GL}(n, \mathbb{F})$ which contains $\widetilde{\text{SL}}(n, \mathbb{F})$. This amounts to taking $c = 0$ in the notation of Kazhdan-Patterson [9] (section 2.3).

Thus there is an exact sequence :

$$1 \rightarrow \mu_N(\mathbb{F}) \rightarrow \widetilde{\text{GL}}(n, \mathbb{F}) \rightarrow \text{GL}(n, \mathbb{F}) \rightarrow 1.$$

It should be noted that there are other central extensions of $\text{GL}(n, \mathbb{F})$ and are obtained by twisting the cocycle of $\widetilde{\text{GL}}(n, \mathbb{F})$ obtained above by powers of the N^{th} Hilbert symbol.

The group $\widetilde{\text{GL}}(n, \mathbb{F})$ is a non-linear group, i.e it cannot be imbedded inside any matrix group. Let T denote a Cartan subgroup of $\text{GL}(n, \mathbb{F})$ and p the projection map from $\widetilde{\text{GL}}(n, \mathbb{F})$ to $\text{GL}(n, \mathbb{F})$. A key point is that $p^{-1}(T)$ is not abelian. This makes the analysis $\widetilde{\text{GL}}(n, \mathbb{F})$ very different from that of $\text{GL}(n, \mathbb{F})$.

1.2.1 Lifting of Characters.

We study the representations of $\widetilde{\text{SL}}(n, \mathbb{F})$ by restricting representations of $\widetilde{\text{GL}}(n, \mathbb{F})$.

For any k , let $\widetilde{G}_+^k = \{g \in \widetilde{\text{GL}}(n, \mathbb{F}) \mid \det(g) \in \mathbb{F}^{*k}\}$

We have the following inclusions:

$$\widetilde{\text{SL}}(n, \mathbb{F}) \subseteq \widetilde{G}_+^{N/d} \subseteq \widetilde{G}_+^m \subseteq \widetilde{\text{GL}}(n, \mathbb{F}) \tag{1.1}$$

where $d = (n - 1, N)$.

We have our first theorem below. We will be using this theorem to prove our main result.

Theorem 1.2.1 *Let Π be an irreducible, genuine representation of $\widetilde{\text{GL}}(n, \mathbb{F})$. Let π be an irreducible summand of the restriction of Π to \widetilde{G}_+^m (that $\Pi|_{\widetilde{G}_+^m}$ decomposes*

as a sum of its irreducible constituents follows because \widetilde{G}_+^n is a normal subgroup of finite index). Also let σ be an irreducible component of π restricted to $\widetilde{G}_+^{N/d}$ (it turns out that all components of $\pi|_{\widetilde{G}_+^{N/d}}$ are isomorphic). Then we have:

$$\Theta_\sigma(g) = \frac{d}{Nn} \sum_{z \in \widetilde{Z}^{N_0/d} / \widetilde{Z}^{N/d}} \chi_\pi(z)^{-1} \Theta_\Pi(zg)$$

where $\widetilde{Z}^{N_0/d}$ denotes the pullback of $Z^{N_0/d} (\subset \mathrm{GL}(n, \mathbb{F}))$ to $\widetilde{\mathrm{GL}}(n, \mathbb{F})$ and similarly for $\widetilde{Z}^{N/d}$ and $\Theta_\pi, \Theta_\sigma$ and Θ_Π are characters of representations as functions on regular semi-simple elements of respective groups, i.e they are functions on regular semi-simple elements of the respective group so that integration of a function against them yields the representation as a distribution acting on the given function. Existence of characters as a function on the elements of the group follows from the work of Harish-Chandra ([8]). Also χ_π denotes the central character of the representation π .

Proof. We refer to theorem 3.3.1 for the proof. ■

Since $\widetilde{G}_+^{N/d} = \widetilde{\mathrm{SL}}(n, \mathbb{F})Z(\widetilde{G}_+^{N/d})$ (where $Z(G)$ denotes the center of G), the above theorem gives us a formula for the characters of $\widetilde{\mathrm{SL}}(n, \mathbb{F})$ in terms of those of $\widetilde{\mathrm{GL}}(n, \mathbb{F})$. The main point here is that the restriction of an irreducible representation from $\widetilde{\mathrm{GL}}(n, \mathbb{F})$ to \widetilde{G}_+^n is easily understood by means of Clifford theory. This is where we really need $n|N$.

The corresponding problem is difficult in the case of $\mathrm{GL}(n, \mathbb{F})$ as is illustrated in [22].

Flicker, Kazhdan and Patterson have defined a lifting theory similar to endoscopy for linear groups. They conjecture that under certain conditions, an irreducible, unitary character π of $\mathrm{GL}(n, \mathbb{F})$ lifts to an irreducible, genuine, unitary character $L(\pi)$ of $\widetilde{\mathrm{GL}}(n, \mathbb{F})$ or to zero [6], [7], [9]. These conditions generally

hold for tempered representations [1]. They compute the character of $L(\pi)$ in terms of π . We follow their approach and relate the characters of representations of $\widetilde{\text{SL}}(n, \mathbb{F})$ to those of a linear group, $\text{PGL}(n, \mathbb{F})$.

Theorem 1.2.1 expresses the character of an irreducible constituent of $L(\pi)$ restricted to $\widetilde{\text{SL}}(n, \mathbb{F})$ in terms of characters of $\widetilde{\text{GL}}(n, \mathbb{F})$. Combining the results of theorem 1.2.1 together with results of [7] on lifting between $\text{GL}(n, \mathbb{F})$ and $\widetilde{\text{GL}}(n, \mathbb{F})$ we relate the characters of $\widetilde{\text{SL}}(n, \mathbb{F})$ and $\text{PGL}(n, \mathbb{F})$.

Our main result exhibits a correspondence between representations of $\text{PGL}(n, \mathbb{F})$ and $\widetilde{\text{SL}}(n, \mathbb{F})$. We prove it by taking an appropriate sum of representations of $\widetilde{\text{SL}}(n, \mathbb{F})$ and relating the character of the sum to an irreducible character of $\text{PGL}(n, \mathbb{F})$. We describe this sum now. The constituents of $L(\pi)$ restricted to \widetilde{G}_+^n are parametrized by their central characters. We describe that now. Let μ denote a genuine character (fixed once and for all) of $\widetilde{Z}^{N_o/d}$ such that $\mu(z, \zeta) = \zeta \forall z \in Z^N$. Consider the characters $\{\nu \mid \nu \in \widehat{\mathbb{F}^*} \text{ and } \nu^N = \chi_\pi\}$. We use χ_ν to denote the genuine character of $\widetilde{Z}^{N_o/d}$ parametrized by ν given by $\chi_\nu(z^{N_o/d}, \zeta) = \nu(z^{N_o/d})\mu(z^{N_o/d}, \zeta)$.

Let $L(\pi, \chi_\nu)$ be the irreducible summand of $L(\pi)$, with central character χ_ν , restricted to $\widetilde{\text{SL}}(n, \mathbb{F})$.

This turns out to be an irreducible genuine representation of $\widetilde{\text{SL}}(n, \mathbb{F})$. For any character α of \mathbb{F}^* we have $L(\pi\alpha^N, \chi_\nu\alpha^n) \simeq L(\pi, \chi_\nu)$; we sum over $\widehat{\mathbb{F}^*}/\widehat{\mathbb{F}^*}^n \simeq \widehat{\mu}_n$ and define

$$L_{st}(\pi, \chi_\nu) = \sum_{\alpha \in \widehat{\mu}_n / \widehat{\mu}_n^{n/e}} L(\pi\alpha^{N_o}, \chi_\nu\alpha).$$

where $e = \text{gcd}(n, N_o)$.

The representation $\pi\nu^{-N_o}$ factors to $\text{PGL}(n, \mathbb{F})$, and we get the character of $L_{st}(\pi, \chi_\nu)$ in terms of the character of $\pi\nu^{-N_o}$.

We have a definition:

Definition 1.2.1 *Let ϕ be the orbit correspondence map between $\mathrm{PGL}(n, \mathbb{F})$ and $\mathrm{SL}(n, \mathbb{F})$ given by: $\phi(g) = \det(g^{-N_o})g^N \in \mathrm{SL}(n, \mathbb{F})$.*

We have our main theorem:

Theorem 1.2.2

$$\Theta_{L_{st}(\pi, \chi_\nu)}(g) = \sum_{\phi(h)=p(g)} \Delta_\mu(h, g) \Theta_{\pi_\nu^{-N_o}}(h).$$

In the above theorem, ϕ is the orbit correspondence map and p is the projection map from $\widetilde{\mathrm{SL}}(n, \mathbb{F})$ to $\mathrm{SL}(n, \mathbb{F})$. Here $\Delta_\mu(h, g)$ is a transfer factor having the property that $|\Delta_\mu(h, g)| = |\Delta(h)|/|\Delta(g)|$ where Δ denotes the usual Weyl-denominator ([1]). Also g in any regular element of $\mathrm{GL}(n, \mathbb{F})$.

Proof. We refer to section 4.5 for the proof and for further details regarding the notation in the above theorem. ■

The set $\Pi(\pi, \chi_\nu) = \{L(\pi\alpha^{N_o}, \chi_\nu\alpha) \mid \alpha \in \widehat{\mu}_n\}$ appearing here is analogous to an L-packet for a linear group [1]. But it should be noted that $\Pi(\pi, \chi_\nu)$ is not the set of constituents of the restriction of a representation of $\widetilde{\mathrm{GL}}(n, \mathbb{F})$ and specifically $\Theta_{L_{st}(\pi, \chi_\nu)}$ is in general not $\widetilde{\mathrm{GL}}(n, \mathbb{F})$ conjugation invariant [1].

In [1], the case $n = N$ was considered. We prove that the methods of restriction which we are using work only in the case when $n|N$. Also, in the case $n|N$, we follow a two-step restriction process (cf. (1)). A representation of $\widetilde{\mathrm{GL}}(n, \mathbb{F})$ is restricted to two intermediate subgroups before obtaining a representation $\widetilde{\mathrm{SL}}(n, \mathbb{F})$. For a precise statement refer to Theorem 1.2.1. In [1], these two subgroups coincide and so it is a one-step restriction process. In the formula for $L(\pi)$, we need a supplementary character, $\tilde{\omega}$ of $\tilde{Z}^{N/d}$ (cf. [7], section 26). This is

not needed in [1]. But it is interesting to note that $\tilde{\omega}$ cancels finally and the final character formula does not depend upon $\tilde{\omega}$.

1.2.2 Lifting of functions.

From now on, we restrict to the case $n = N$. We expect all the results to hold without any major changes in the proofs. We are dealing with $n = N$ here because the exposition is much easier in this case. We will use the notation $L(\pi, \nu)$ to denote $L(\pi, \chi_\nu)$ (from the previous section). This will simplify our presentation (see section 5.1).

In the rest of the thesis, we concentrate on the lifting of functions between \tilde{G}_+^n and $\widetilde{\text{GL}}(n, \mathbb{F})$. As we shall see, this will also provide a lifting of representations between $\widetilde{\text{SL}}(n, \mathbb{F})$ and $\text{PGL}(n, \mathbb{F})$. In [7], a lifting of orbital integrals and functions has been obtained between $\widetilde{\text{GL}}(n, \mathbb{F})$ and $\text{GL}(n, \mathbb{F})$ and has been used to define a lifting of representations between $\text{GL}(n, \mathbb{F})$ and $\widetilde{\text{GL}}(n, \mathbb{F})$. We are essentially following the same approach. We also obtain theorem 1.2.1 as a dual to the lifting of functions and orbital integrals between \tilde{G}_+^n and $\widetilde{\text{GL}}(n, \mathbb{F})$. We explain the above results in the rest of this section.

Definition 1.2.2 *Let $\iota : \mu_n \rightarrow \mathbb{C}^*$ be an injective character of μ_n . Fix ι once and for all. A function $\tilde{f} \in C_c^\infty(\widetilde{\text{SL}}(n, \mathbb{F}))$ is said to be a genuine function (with respect to ι .) if $\tilde{f}(g, \zeta) = \iota(\zeta)\tilde{f}(g, 1) \quad \forall g \in \widetilde{\text{SL}}(n, \mathbb{F}), \quad \forall \zeta \in \mu_n$.*

Let $\alpha \in \hat{\mu}_n$. Let $C_c^\infty(\widetilde{\text{SL}}(n, \mathbb{F}))_\alpha$ denote the functions \tilde{f} in $C_c^\infty(\widetilde{\text{SL}}(n, \mathbb{F}))$ satisfying

$$\tilde{f}(z_\zeta g) = \chi_\alpha^{-1}(z_\zeta)\tilde{f}(g)$$

for any $g \in \widetilde{\mathrm{SL}}(n, \mathbb{F})$ and any z_ζ where z_ζ is any pullback of ζI via the map $p : \widetilde{\mathrm{SL}}(n, \mathbb{F}) \rightarrow \mathrm{SL}(n, \mathbb{F})$. Also $\chi_\alpha(z_\zeta) = \mu(z_\zeta)\alpha(\zeta)$.

Let $\tilde{f} \in C_c^\infty(\widetilde{\mathrm{SL}}(n, \mathbb{F}))_\alpha$ be a genuine function. Choose a character ν of \mathbb{F}^* such that $\nu|_{\mu_n} = \alpha$. We define a function $A(\tilde{f}, \nu)$ on \widetilde{G}_+^n by extension via the center. Specifically:

Definition 1.2.3 *Let $\tilde{f} \in C_c^\infty(\widetilde{\mathrm{SL}}(n, \mathbb{F}))_\alpha$ and $\nu \in \widehat{\mathbb{F}^*}$ such that $\nu|_{\mu_n} = \alpha$. Define:*

$$A(\tilde{f}, \nu)(zg) = \chi_\nu^{-1}(z)A(\tilde{f}, \nu)(g)$$

for any $g \in \widetilde{\mathrm{SL}}(n, \mathbb{F})$, $z \in Z(\widetilde{G}_+^n) = p^{-1}\{zI \mid z \in \mathbb{F}^*\}$ (Lemma 3.2.4).

We are using z to denote an element of $Z(\widetilde{G}_+^n) = \widetilde{Z}$. (which is just the pullback of $Z(\mathrm{GL}(n, \mathbb{F}))$ via p .) This defines $A(\tilde{f}, \nu)$ on the whole of $\widetilde{G}_+^n = \widetilde{\mathrm{SL}}(n, \mathbb{F})\widetilde{Z}$. We also note that $A(\tilde{f}, \nu)$ is well-defined because of that transformation properties of \tilde{f} on $\widetilde{\mathrm{SL}}(n, \mathbb{F})$.

Related to $A(\tilde{f}, \nu)$, we make a few more definitions:

Definition 1.2.4 1. *Let $B(\tilde{f}, \nu)$ be defined on $\widetilde{\mathrm{GL}}(n, \mathbb{F})$ by extending $A(\tilde{f}, \nu)$ outside \widetilde{G}_+^n by zero.*

2. *$C(\tilde{f}, \nu)$ is defined on $\mathrm{GL}(n, \mathbb{F})$ to be a Kazhdan-Flicker lift of $B(\tilde{f}, \nu)$ i.e $C(\tilde{f}, \nu)$ satisfies the following:*

$$\Theta_{L(\pi)}(B(\tilde{f}, \nu)) = \Theta_\pi(C(\tilde{f}, \nu))$$

where π is any irreducible admissible representation of $\mathrm{GL}(n, \mathbb{F})$ and $L(\pi)$ is its lift on $\widetilde{\mathrm{GL}}(n, \mathbb{F})$ as defined earlier in this section (see section 4.1 [7] for further details. Note that $L(\pi)$ exists only for certain π . Hypothesis I and II in section 4.1 list some π for which $L(\pi)$ exists).

3. $D(\tilde{f}, \nu)$ is defined on $\mathrm{GL}(n, \mathbb{F})$ by:

$$D(\tilde{f}, \nu)(g) = \nu(\det(g))C(\tilde{f}, \nu)(g).$$

We state an important property of $D(\tilde{f}, \nu)$.

Lemma 1.2.1 $D(\tilde{f}, \nu)$ is trivial on the center of $\mathrm{GL}(n, \mathbb{F})$ and is hence a function on $\mathrm{PGL}(n, \mathbb{F})$. Also $D(\tilde{f}, \nu)$ is independent of the choice of the extension ν of α .

Proof. see Lemma 5.1.3

■

The above Lemma follows by considering the orbital integral of $C(\tilde{f}, \nu)$ over $\mathrm{GL}(n, \mathbb{F})$ and using the Weyl Integration formula. Details are given in section 5.1. Since $D(\tilde{f}, \nu)$ is independent of the choice of the extension ν of α we use $D(\tilde{f})$ instead to denote $D(\tilde{f}, \nu)$.

Let $\tilde{f} \in C_c^\infty(\widetilde{\mathrm{SL}}(n, \mathbb{F}))$ be a genuine function. Then $\tilde{f} = \sum_{\alpha \in \hat{\mu}_n} \tilde{f}^\alpha$ where each $\tilde{f}^\alpha \in C_c^\infty(\widetilde{\mathrm{SL}}(n, \mathbb{F}))_\alpha$ is a genuine function. We make the following definition.

Definition 1.2.5 Let $\tilde{f} \in C_c^\infty(\widetilde{\mathrm{SL}}(n, \mathbb{F}))$ be a genuine function and $\tilde{f} = \sum_{\alpha \in \hat{\mu}_n} \tilde{f}^\alpha$ as above. Define:

$$\Gamma(\tilde{f}) = \sum_{\alpha \in \hat{\mu}_n} D(\tilde{f}^\alpha) \in C_c^\infty(\mathrm{PGL}(n, \mathbb{F})).$$

We state certain important relations that hold between the functions defined above.

Lemma 1.2.2 For any $\tilde{f} \in C_c^\infty(\widetilde{\mathrm{SL}}(n, \mathbb{F}))_\alpha$, choose $\nu \in \widehat{\mathbb{F}^*}$ satisfying $\nu|_{\mu_n} = \alpha$, we have

$$\Theta_{L(\pi, \nu)}(\tilde{f}) = \Theta_{\pi\nu^{-1}}(D(\tilde{f})).$$

For a proof we refer to proposition 5.1.1.

From the above Lemma, it follows that:

Theorem 1.2.3 *Let $\tilde{f} \in C_c^\infty(\widetilde{\text{SL}}(n, \mathbb{F}))$. We have*

$$\Theta_{L_{st}}(\tilde{f}) = \Theta_{\pi\nu^{-1}}(\Gamma(\tilde{f})).$$

The first equality in the above theorem follows because $\Theta_{L(\pi, \nu)}(\tilde{f}^\alpha) = 0$ if $\nu|\mu_n \neq \alpha$. We also see from this fact that stabilization (using L_{st} instead of merely $L(\pi, \nu)$) is indeed necessary. The fact that $D(\tilde{f}, \nu)$ is independent of the extension ν of α is also important. This enables us to define a distribution on $C_c^\infty(\widetilde{\text{SL}}(n, \mathbb{F}))$ using the above theorem. We refer to Theorem 5.1.1 for further details.

1.2.3 Lifting of Orbital Integrals.

Next we study the relationship between various orbital integrals. We use the notation:

$$F_G(\gamma, f) = \Delta(\gamma) \int_{G(\gamma)\backslash G} f(g^{-1}\gamma g) dg$$

where G is any linear or non-linear group, $\gamma \in G$ is any regular semi-simple element, $G(\gamma)$ is the centralizer of γ in G , f is compactly supported mod the center or compactly supported and may transform by some character of the center of G , dg is a right-invariant measure on the homogeneous space $G(\gamma)\backslash G$ whose normalization will be specified later. We multiply by $\Delta(\gamma)$, the Weil denominator, in order to normalize the orbital integrals so that they extend by continuity to the singular elements of G . When we assume G to be a non-linear group (some subgroup of $\widetilde{\text{GL}}(n, \mathbb{F})$), we will further assume f to be a genuine function and

denote it by \tilde{f} . We let an element in the non-linear group to be regular (semi-simple) if its projection in the linear group is regular (semi-simple).

We now state some results regarding relationships between orbital integrals of various functions defined before.

Lemma 1.2.3 *Let $\tilde{\gamma} \in \widetilde{\mathrm{SL}}(n, \mathbb{F})$ be a regular, semi-simple element and T be the Cartan subgroup of $\mathrm{GL}(n, \mathbb{F})$ containing $p(\tilde{\gamma})$ with $p(\tilde{\gamma}) \in T^n$. Let $\tilde{f} \in C_c^\infty(\widetilde{\mathrm{SL}}(n, \mathbb{F}))_\alpha$ and $\nu \in \widehat{\mathbb{F}^*}$ such that $\nu|_{\mu_n} = \alpha$. We use k_1 to denote $|\widetilde{G}_+^n \backslash \widetilde{G}_+^n \tilde{T}|$ and z_y to denote a pullback in \tilde{Z} of $yI \in Z$ (note that this choice does not matter as we are dealing with genuine functions and characters). We then have:*

$$F_{\widetilde{\mathrm{GL}}(n, \mathbb{F})}(z_y \tilde{\gamma}, B(\tilde{f}, \nu)) = \frac{1}{k_1} \chi_\nu^{-1}(z_y) \sum_{h \in \widetilde{G}_+^n \backslash \widetilde{\mathrm{GL}}(n, \mathbb{F})} (y, \det(h)) F_{\widetilde{G}_+^n}(\tilde{\gamma}^h, A(\tilde{f}, \nu))$$

where $\tilde{\gamma}^h = h^{-1} \tilde{\gamma} h$.

Proof. We refer to Lemma 5.3.3 for proof. ■

Upon taking χ_ν to the other side in the above lemma and summing over all $z_y \in \widetilde{Z}^n \backslash \tilde{Z}$ we obtain:

Theorem 1.2.4

$$F_{\widetilde{G}_+^n}(\tilde{\gamma}, A(\tilde{f}, \nu)) = \frac{k_1}{|\mathbb{F}^*/\mathbb{F}^{*n}|} \sum_{z \in \widetilde{Z}^n \backslash \tilde{Z}} \chi_\nu(z) F_{\widetilde{\mathrm{GL}}(n, \mathbb{F})}(z \tilde{\gamma}, B(\tilde{f}, \nu))$$

Proof. We refer to Theorem 5.3.1 for proof. ■

Next we use lemmas 1.2.2 and 1.2.3 and theorems 1.2.3 and 1.2.4 to again obtain theorem 1.2.2. We outline the proof (see section 5.4 for details).

Define a function on $\widetilde{\mathrm{SL}}(n, \mathbb{F})$:

$$l^*(\pi, \nu)(\tilde{\gamma}) = \frac{1}{|\mathbb{F}^*/\mathbb{F}^{*n}|} \sum_{z \in \tilde{Z}^n \setminus \tilde{Z}} \chi_\nu^{-1}(z) \Theta_{L(\pi)}(z\tilde{\gamma})$$

and a distribution $\Theta_{l^*(\pi, \nu)}$ on $C_c^\infty(\widetilde{\mathrm{SL}}(n, \mathbb{F}))$ by integrating against $l^*(\pi, \nu)$. Using Weyl Integration formula we obtain:

$$\Theta_{l^*(\pi, \nu)}(\tilde{f}) = \Theta_{L(\pi, \nu)}(\tilde{f})$$

for every $\tilde{f} \in C_c^\infty(\widetilde{\mathrm{SL}}(n, \mathbb{F}))$ and thus we obtain

$$\Theta_{L(\pi, \nu)} = \Theta_{l^*(\pi, \nu)}$$

as characters considered to be functions of elements of $\widetilde{\mathrm{SL}}(n, \mathbb{F})$.

1.3 Connection to Local Class Field Theory

We have used the Norm residue symbol in all our analysis (construction of the covering group and performing calculations regarding commutators etc). Since this symbol comes out of Local Class Field Theory we will give an alternate explicit definition of a symbol, τ , satisfying the properties of the Norm residue symbol (section 2.1). We can then derive all the results of this thesis using this symbol and its properties (all that has been used regarding the Norm residue symbol are the properties stated in section 2.1) and then we would be having all our results without using the results from Local Class Field Theory.

Let \mathbb{F} be a p -adic field containing μ_n , the n^{th} roots of unity. Assume that n is coprime to p , the residual characteristic of \mathbb{F} . Let \mathbb{E} be an extension field of \mathbb{F} such that $|\mathbb{E}/\mathbb{F}| = n$. We mention here that n is what it has been in previous sections i.e coming from $\mathrm{SL}(n, \mathbb{F})$. Hence we can associate \mathbb{E} to a Cartan subgroup of

$\mathrm{GL}(n, \mathbb{F})$. Let $N : \mathbb{E}^* \rightarrow \mathbb{F}^*$ be the norm map. Let $\tau(,)$ denote the alternate to the n^{th} Norm residue symbol (whose definition we provide explicitly) with respect to \mathbb{F} . We consider orbital integrals of various functions over \tilde{T} , the pullback of T to $\widetilde{\mathrm{GL}}(n, \mathbb{F})$ via the covering map $p : \widetilde{\mathrm{GL}}(n, \mathbb{F}) \rightarrow \mathrm{GL}(n, \mathbb{F})$. We obtain the following result:

Theorem 1.3.1 *Let $x \in \mathbb{F}^*/\mathbb{F}^{*n}$. Then $\tau(x, y) = 1 \quad \forall y \in \mathbb{E}^{*n} \cap \mathbb{F}^*/\mathbb{F}^{*n}$ if and only if $x \in N(\mathbb{E}^*)/\mathbb{F}^{*n}$.*

From the fact that τ is a perfect pairing, we have an immediate corollary:

Corollary 1.3.1

$$\mathbb{E}^{*n} \cap \mathbb{F}^*/\mathbb{F}^{*n} \simeq \mathbb{F}^*/N(\mathbb{E}^*).$$

We use the above corollary to obtain the following well-known results about Local Fields:

Theorem 1.3.2 *Let \mathbb{E}/\mathbb{F} be a finite Galois extension of degree n . Assume that $\mu_n \subset \mathbb{F}$ and that n is coprime to the residual characteristic of \mathbb{F} . Let $\mathrm{Gal}(\mathbb{E}/\mathbb{F})^{ab}$ denote the abelianization of $\mathrm{Gal}(\mathbb{E}/\mathbb{F})$. Then there exists a map $\sigma : \mathbb{F}^* \rightarrow \mathrm{Gal}(\mathbb{E}/\mathbb{F})^{ab}$ such that the sequence*

$$1 \rightarrow N(\mathbb{E}^*) \rightarrow \mathbb{F}^* \xrightarrow{\sigma} \mathrm{Gal}(\mathbb{E}/\mathbb{F})^{ab} \rightarrow 1$$

is exact.

For abelian extensions, we obtain:

Theorem 1.3.3 *Let \mathbb{E}/\mathbb{F} be an abelian field extension of \mathbb{F} and \mathbb{E}_1 and \mathbb{E}_2 be two abelian field extensions of \mathbb{F} inside \mathbb{E} with N^i ($i=1,2$) the corresponding Norm maps. Then $N^1(\mathbb{E}_1) = N^2(\mathbb{E}_2)$ if and only if $\mathbb{E}_1 = \mathbb{E}_2$. We assume that $\mu_n \subseteq \mathbb{F}$ where $n = |\mathbb{E}/\mathbb{F}|$ and that n is coprime to the residual characteristic of \mathbb{F} .*

For general extensions we have:

Theorem 1.3.4 *Let \mathbb{E}/\mathbb{F} be a finite extension of degree n . Assume $\mu_n \subset \mathbb{F}$ and n coprime to the residual characteristic of \mathbb{F} . Let \mathbb{E}^1 be the maximal abelian extension of \mathbb{F} inside \mathbb{E} . Then*

$$N_{\mathbb{F}}^{\mathbb{E}}(\mathbb{E}^*) = N_{\mathbb{F}}^{\mathbb{E}^1}(\mathbb{E}^{1*}).$$

We are using the assumption $(n, p) = 1$ because for the case $p|n$ the explicit formula for the Norm-residue symbol is very complicated and it is not easy from there to obtain the property of non-degeneracy. If we assume the existence of the Norm-residue symbol and its properties, the same proof of theorem 1.3.4 works exactly the same with τ replaced by the Norm-residue symbol and proves theorem 1.3.4 for the general case $p|n$. We are using local class field theory but still providing a completely different proof of theorem 1.3.4. We refer to [20], Pg. 172, for the classical proof.

Above analysis raises some questions. Why are we able to obtain number-theoretic results like these using methods from harmonic analysis? What is the role played by non-linear groups?

Chapter 2

Metaplectic Groups

2.1 The Hilbert symbol

Let \mathbb{F} be a non-archimedean local field. Fix an integer $N \geq 2$. Let $\mu_N(\mathbb{F}) = \{x \in \mathbb{F} : x^N = 1\}$ i.e. the N^{th} roots of unity in \mathbb{F} . Sometimes we will denote the roots of unity in \mathbb{F} by just μ_N .

We will assume that $|\mu_N(\mathbb{F})| = N$, i.e., that \mathbb{F} contains the full group of N^{th} roots of unity. We will denote the N^{th} Hilbert symbol by $(\ , \)_{N, \mathbb{F}}$ over \mathbb{F} . We will abbreviate it by $(\ , \)$ when there is no chance of ambiguity. This is a map

$$(\ , \) : \mathbb{F}^* \times \mathbb{F}^* \rightarrow \mu_N(\mathbb{F})$$

satisfying, for a, a', b in \mathbb{F}^* ,

1. $(a, b)(a', b) = (aa', b)$
2. $(a, b)(b, a) = 1$
3. $(a, 1 - a) = 1$ for $a \neq 1$.
4. $\{a : (a, x) = 1 \forall x \in \mathbb{F}^*\} = \mathbb{F}^{*N}$

5. $(a, Nb)_{\mathbb{F}} = (a, b)_{\mathbb{E}}$ where \mathbb{E} is a finite field extension of \mathbb{F} where $b \in \mathbb{E}^*$ and $N : \mathbb{E}^* \rightarrow \mathbb{F}^*$ denotes the norm map.

where $\mathbb{F}^{*N} = \{x^N | x \in \mathbb{F}^*\}$. We refer to [20] for more information on the Hilbert symbol. In particular, we note that $(\ , \)_N$ is a perfect pairing on $\mathbb{F}^*/\mathbb{F}^{*N}$ and gives an isomorphism of $\mathbb{F}^*/\mathbb{F}^{*N}$ with $\widehat{\mathbb{F}^*/\mathbb{F}^{*N}}$.

Remark 2.1.1 *It must be noted that all the results in this thesis use only the above properties of the Norm Residue symbol. In particular if we can define explicitly another symbol having the above properties of the Norm Residue symbol then we can use that symbol to derive the results of this thesis regarding $\mathrm{GL}(n, \mathbb{F})$. We will be doing this in chapter 6 for the case when the residual characteristic of \mathbb{F} is co-prime to N .*

2.2 Basics

We discuss some basic material regarding properties of characters of \mathbb{F}^* and covering groups. ([1])

We consider covering groups:

$$1 \rightarrow \mu_N \rightarrow \tilde{G} \xrightarrow{p} G \rightarrow 1$$

with μ_n central in \tilde{G} (Section 2.3). Let χ_π be the central character of a representation π . We say a representation π of \tilde{G} is genuine if π has a central character χ_π whose restriction to μ_N is injective. If π is not genuine then π factors to a representation of a cover of G with kernel a proper subgroup of μ_N . If $\iota : \mu_N \hookrightarrow \mathbb{C}^*$ is an embedding we say π is of type ι if $\chi_\pi|_{\mu_N} = \iota$.

We have the following exact sequences. They will play a very important role throughout.

$$1 \rightarrow \mu_N \rightarrow \mathbb{F}^* \xrightarrow{N} \mathbb{F}^{*N} \rightarrow 1 \quad (2.1)$$

$$1 \rightarrow \mathbb{F}^{*N} \rightarrow \mathbb{F}^* \rightarrow \mathbb{F}^*/\mathbb{F}^{*N} \rightarrow 1 \quad (2.2)$$

We also have their Pontriagin duals:

$$1 \rightarrow \widehat{\mathbb{F}^{*N}} \rightarrow \widehat{\mathbb{F}^*} \xrightarrow{\text{res}} \widehat{\mu_N} \rightarrow 1 \quad (2.3)$$

$$1 \rightarrow \widehat{\mathbb{F}^*/\mathbb{F}^{*N}} \rightarrow \widehat{\mathbb{F}^*} \xrightarrow{\text{res}} \widehat{\mathbb{F}^{*N}} \rightarrow 1 \quad (2.4)$$

Suppose μ_N is in the kernel of $\lambda \in \widehat{\mathbb{F}^*}$. Then by (2.3) $\lambda(x) = \nu(x^N)$ for some character ν of $\widehat{\mathbb{F}^{*N}}$, which by (2.4) extends to $\tau \in \widehat{\mathbb{F}^*}$. This gives us the following lemma which will be used repeatedly:

Lemma 2.2.1 *Let $\lambda \in \widehat{\mathbb{F}^*}$. Then $\lambda = \nu^N$ for some $\nu \in \widehat{\mathbb{F}^*}$ if and only if $\lambda(\zeta) = 1$ for all $\zeta \in \mu_N$.*

We identify the center Z of $\text{GL}(n, \mathbb{F})$ with \mathbb{F}^* and the central character χ_π of a representation of $\text{GL}(n, \mathbb{F})$ with an element of $\widehat{\mathbb{F}^*}$.

For $\alpha \in \widehat{\mathbb{F}^*}$ we write α for the character $\alpha \circ \det$ of $\text{GL}(n, \mathbb{F})$, and also for the character $\alpha \circ p$ of $\widetilde{\text{GL}}(n, \mathbb{F})$. Note that for π a representation of $\text{GL}(n, \mathbb{F})$ (with a central character)

$$\chi_{\pi\alpha} = \chi_\pi \alpha^n. \quad (2.5)$$

We write Θ_π for the global character of a representation π , considered as a function on the set of regular semisimple elements.

2.3 Group Structure

We continue with the notation of Section 2.2. Most of this material can be found in [1]. We first define the group \widetilde{G}_1 ([14], [21]): this is a topological group which fits in an exact sequence:

$$1 \rightarrow \mu_N \xrightarrow{\iota} \widetilde{\mathrm{SL}}(n, \mathbb{F}) \xrightarrow{p} \mathrm{SL}(n, \mathbb{F}) \rightarrow 1 \quad (2.6)$$

with ι, p continuous, ι closed and p open. The classes of such extensions are parametrized by the group of (bilinear) Steinberg cocycles with values in μ_N . Let $(\ , \)_N : \mathbb{F}^* \times \mathbb{F}^* \rightarrow \mu_N$ denote the N^{th} norm residue symbol for \mathbb{F} . For properties of $(\ , \)_N$ see section 2.1. Each Steinberg cocycle is given by $c(x, y) = (x, y)_N^k$ for some $k \in \mathbb{Z}$. Write $G[k]$ for the group defined by the cocycle $(x, y)_N^k$. Then $G[k]$ and $G[k']$ are equivalent extensions if and only if $k \equiv k' \pmod{N}$. We let $\widetilde{\mathrm{SL}}(n, \mathbb{F}) = G[1]$. The proof of above facts can be found in [21] and [12].

Once and for all we fix an embedding

$$\iota : \mu_N(\mathbb{F}) \hookrightarrow \mathbb{C}^*$$

and we identify μ_N with its image. Henceforth we assume all genuine representations are of type ι .

The Steinberg cocycle defines a cover $\widetilde{\mathrm{GL}}(n, \mathbb{F})$ of $\mathrm{GL}(n, \mathbb{F})$ by [9], and we let $\widetilde{\mathrm{GL}}(n, \mathbb{F})$ to be that cover which contains $\widetilde{\mathrm{SL}}(n, \mathbb{F})$ as a subgroup (we are taking $c = 0$ in the notation of [9]).

We write $c(\ , \)$ for the cocycle defining $\widetilde{\mathrm{GL}}(n, \mathbb{F})$. Then

$$\widetilde{\mathrm{GL}}(n, \mathbb{F}) = \{(g, \zeta) \mid g \in \mathrm{GL}(n, \mathbb{F}), \zeta \in \mu_n\}$$

with multiplication $(g, \zeta)(g', \zeta') = (gg', \zeta\zeta'c(g, g'))$.

Commutators play an important role. Suppose g and h are commuting elements of $\mathrm{GL}(n, \mathbb{F})$. Let \tilde{g}, \tilde{h} be inverse images of g, h in $\widetilde{\mathrm{GL}}(n, \mathbb{F})$. Then $\eta = \tilde{g}\tilde{h}\tilde{g}^{-1}\tilde{h}^{-1} \in \mu_N$ is independent of the choices of \tilde{g} and \tilde{h} . We also use $\{g, h\}$ to denote $\tilde{g}\tilde{h}\tilde{g}^{-1}\tilde{h}^{-1}$.

Now let $z \in Z$, the center of $\mathrm{GL}(n, \mathbb{F})$ (scalar matrices). We write $z = xI$ for some $x \in \mathbb{F}^*$. For $g \in \mathrm{GL}(n, \mathbb{F})$, we wish to compute $\{z, g\}$. To do this we will first compute $\{h, k\}$ where h and k are diagonal elements in $\mathrm{GL}(n, \mathbb{F})$. We have a lemma:

Lemma 2.3.1 *Let $h = \mathrm{diag}(h_i)$ and $k = \mathrm{diag}(k_j)$. Then*

$$\{h, k\} = \prod_l (h_l, k_l)_N^{-1} \cdot (\det(h), \det(k))_N$$

Proof. This lemma can be proved using the properties of the Hilbert symbol. ([9]) ■

Using the above lemma we have:

Proposition 2.3.1 *Let $z = xI$. Then the map*

$$\zeta_z : \widetilde{\mathrm{GL}}(n, \mathbb{F}) \rightarrow \mu_N(\mathbb{F})$$

given by

$$\zeta_z(g) = \{z, g\}$$

is a homomorphism and

$$\zeta_z(g) = (x, \det(g))_N^{n-1}$$

Proof. The map ζ_z is a homomorphism because $\{z, g\}$ is in the center of $\widetilde{\mathrm{GL}}(n, \mathbb{F})$ for every g . It factors through the determinant because $\widetilde{\mathrm{SL}}(n, \mathbb{F})$ is the commutator subgroup of $\widetilde{\mathrm{GL}}(n, \mathbb{F})$ and $\mu_N(\mathbb{F})$ is an abelian group. Thus $\zeta_z(g) = \zeta_z(d_g)$

where d_g could be any diagonal matrix with one entry $\det(g)$ and the rest equal to 1. From the above lemma we obtain:

$$\zeta_z(g) = \{z, d_g\} = (x, \det(g))^{-1}(x^n, \det(g)) = (x, \det(g))^{n-1}$$

We are as before using the N^{th} Hilbert symbol. ■

We refer to this formula as the commutator formula. We will be using this formula repeatedly in the next chapter.

Chapter 3

Representation Theory

3.1 Restriction of genuine representations to certain subgroups.

We wish to study how a genuine irreducible representation of \tilde{G} behaves upon restricting to some particular subgroups. We start by considering a very general setting. Notation is the same as in chapter 2.

We are considering a central extension \tilde{G} of G by a finite cyclic group A . Here G can be any linear group. Later on we will consider the case when $G = GL(n)$. Consider the exact sequence:

$$1 \rightarrow A \rightarrow \tilde{G} \xrightarrow{p} G \rightarrow 1.$$

Let Z be the centre of G and \tilde{Z} the pullback of Z in \tilde{G} . Define subgroups \tilde{H} and \tilde{K} of \tilde{G} :

1. $\tilde{H} = Cent_{\tilde{G}}(\tilde{Z})$.
2. $\tilde{K} = \tilde{H}\tilde{Z}$.

Assume that \tilde{H} and \tilde{K} are proper normal subgroups of finite index. Also

assume that $Cent_{\tilde{G}}(\tilde{H}) = \tilde{Z}$ and $Cent_{\tilde{G}}(Z(\tilde{K})) = \tilde{K}$. These conditions will be naturally satisfied in our setting, i.e when $G = GL(n, \mathbb{F})$ and $\tilde{G} = \widetilde{GL}(n, \mathbb{F})$.

We study the restriction of a genuine representation of \tilde{G} to \tilde{H} . Since $\tilde{H} \subseteq \tilde{K} \subseteq \tilde{G}$, our approach will be to restrict a representation of \tilde{G} to \tilde{K} and then restrict from \tilde{K} to \tilde{H} .

Lemma 3.1.1 *Let Π be an irreducible genuine representation of \tilde{G} and let σ be an irreducible representation of \tilde{K} occurring in the restriction $\Pi|_{\tilde{K}}$. We use σ^g to denote the conjugate representation of σ by g i.e $\sigma^g(x) = \sigma(gxg^{-1})$. Then $\Pi|_{\tilde{K}} = \bigoplus_{g \in \tilde{G}/\tilde{K}} \sigma^g$ where all σ^g 's in the direct sum are distinct irreducible representations of \tilde{K} .*

Proof. Let χ_σ be the central character of σ . Choose a $g \in \tilde{G}$ such that $g \notin \tilde{K}$. Let $z \in Z(\tilde{K})$.

$$\begin{aligned} \chi_{\sigma^g}(z) &= \chi_\sigma(gzg^{-1}z^{-1}z) \\ &= \chi_\sigma(\{g, z\}z) \end{aligned}$$

Since $\{g, z\} \in A$, we obtain $\chi_{\sigma^g}(z) = \chi_\sigma(z)\{g, z\}$.

Choose a $z \in Z(\tilde{K})$ such that $\{g, z\} \neq 1$. This can be done because $Cent_{\tilde{G}}(Z(\tilde{K})) = \tilde{K}$ and $g \notin \tilde{K}$. Thus we get $\chi_\sigma \neq \chi_{\sigma^g}$ and hence $\sigma^g \not\cong \sigma$. By use of results from Clifford theory (Pg. 345 [4]) we obtain the lemma. ■

If $\Pi \in Irr(\tilde{K})$, what happens on restricting to \tilde{H} ?

Lemma 3.1.2 *Let Π be an irreducible, genuine representation of \tilde{K} . Let σ be an irreducible summand in $\Pi|_{\tilde{H}}$. Then $\sigma^g \simeq \sigma$ for all $g \in \tilde{K}$.*

Proof. : We can assume $g \in \tilde{Z}$ in our claim as $\tilde{K} = \tilde{H}\tilde{Z}$.

$$\Theta_{\sigma^g}(h) = \Theta_\sigma(\{g, h\}h) \quad \forall h \in \tilde{H}.$$

$\{g, h\} = 1$ follows by definition of \tilde{H} . Hence ;

$$\Theta_{\sigma^g} = \Theta_\sigma \quad \forall g \in \tilde{K}.$$

By Clifford theory ([4], Pg. 345) there exists a positive integer m such that:

$$\Pi|_{\tilde{H}} = m\sigma. \quad \underbrace{(\sigma + \dots + \sigma)}_m. \quad (3.1)$$

■

Next we determine m . We have a lemma:

Lemma 3.1.3 *Let Π be an irreducible, genuine representation of \tilde{K} . Let σ be an irreducible summand in $\Pi|_{\tilde{H}}$. Then $\Pi|_{\tilde{H}} = m\sigma$ where $m = \sqrt{|\tilde{K}/\tilde{H}|}$.*

Proof. By Frobenius Reciprocity:

$$\text{Ind}_{\tilde{H}}^{\tilde{K}}(\sigma) = m\Pi \oplus \dots. \quad (3.2)$$

We use the formula for the character of an induced representation (very similar to [19], chapter 7) to obtain:

$$\begin{aligned} \Theta_{\text{Ind}_{\tilde{H}}^{\tilde{K}}}(h) &= |\tilde{K}/\tilde{H}|\Theta_\sigma(h) & h \in \tilde{H} \\ &= 0 & h \notin \tilde{H}. \end{aligned}$$

From the above formula and the next lemma, we conclude:

$$\Theta_{\text{Ind}_{\tilde{H}}^{\tilde{K}}} = \frac{|\tilde{K}/\tilde{H}|\Theta_\Pi}{m}.$$

Using this and equation 3.2, we have:

$$\frac{|\tilde{K}/\tilde{H}|\Theta_\Pi}{m} = m\Theta_\Pi + \dots$$

where $+\dots$ stands for sum of characters of representations other than Π . Since the characters of distinct representations are linearly independent, we can compare coefficients and conclude that $|\tilde{K}/\tilde{H}| = m^2$. This proves the lemma. ■

Lemma 3.1.4 Θ_Π vanishes outside \tilde{H} .

Proof. : Let $g \notin \tilde{H}$. Choose z such that $\{z, g\} \neq 1$. ($z \in \tilde{Z}$). (here we only use the definition of \tilde{H} .)

$$\begin{aligned}\Theta_\Pi(g) &= \Theta_\Pi(zgz^{-1}) \\ &= \Theta_\Pi(\{z, g\}g) \\ &= \Theta_\Pi(g)\{z, g\}.\end{aligned}$$

Here we have used the fact that Π is genuine. Since g can be chosen so that $\{z, g\} \neq 1, \Rightarrow \Theta_\Pi(g) = 0$. ■

3.2 Subgroups of \tilde{G} .

Let us state the notation first. From now on we will use the following notation:

- $G := \text{GL}(n, \mathbb{F})$.
- $\tilde{G} := \widetilde{\text{GL}}(n, \mathbb{F})$.
- For $l \in \mathbb{Z}$, let $G_+^l := \{g \in G \mid \det(g) \in \mathbb{F}^{*l}\}$.
- For $l \in \mathbb{Z}$, let $\tilde{G}_+^l := \{g \in \tilde{G} \mid \det(g) \in \mathbb{F}^{*l}\}$.
- $G_1 := \text{SL}(n, \mathbb{F})$.

- $\widetilde{G}_1 := \widetilde{\text{SL}}(n, \mathbb{F})$.

Now we consider the case when $G = \text{GL}(n, \mathbb{F})$ and $A = \mu_N$. We are assuming that \mathbb{F} contains the group of N^{th} roots of unity. Let p be the residual characteristic of \mathbb{F} . We assume p is coprime to N .

We have an exact sequence:

$$1 \rightarrow \mu_N \rightarrow \widetilde{G} \rightarrow G \rightarrow 1.$$

Here \widetilde{G} is the extension defined in chapter 2. Let $d=(n-1,N)$ and $k=(n,N)=(n,N/d)$. Let $g, h \in \widetilde{G}$ such that $p(g), p(h)$ commute in G .

Lemma 3.2.1 *With above definitions and the notation from previous section, $\widetilde{H} = \widetilde{G}_+^{N/d}$.*

Proof. $\widetilde{H} = \text{Cent}_{\widetilde{G}}(\widetilde{Z})$ by definition.

Let $g \in \widetilde{H}$. Then

$$\begin{aligned} \{g, z_x\} &= 1 \quad \forall x \in \mathbb{F}^* \\ (\det(g)^{n-1}, x)_N &= 1 \quad \forall x \in \mathbb{F}^* \\ &\Leftrightarrow \det(g)^{n-1} \in \mathbb{F}^{*N}. \end{aligned}$$

From the next lemma, we conclude that above statements are necessary and sufficient for $\det(g) \in \mathbb{F}^{*N/d}$. ■

Lemma 3.2.2 *Let $x \in \mathbb{F}^*$. Then $x^a \in \mathbb{F}^{*b} \Leftrightarrow x \in \mathbb{F}^{*b/(a,b)}$ where (a, b) denotes the greatest common divisor of a and b . We assume all b roots of unity are in \mathbb{F}^* .*

Proof. Denote $d = (a, b)$. If $x \in \mathbb{F}^{*b/d}$ then $x^a \in \mathbb{F}^{*b}$. For the other side write $d = ar + bs$ where r, s are integers.

Then $x^a = y^b \Rightarrow x^{ar} = y^{br} \Rightarrow x^{d-bs} = y^{br} \Rightarrow x^d = y^{b(r+s)}$.

Now the fact that all b^{th} roots of unity are in \mathbb{F}^* completes the proof. ■

Above arguments also prove that \tilde{H} and \tilde{Z} form a dual pair (i.e they are centralizers of each other in \tilde{G}) in \tilde{G} (It is obvious that \tilde{Z} commutes with the whole of \tilde{H} . Nothing more can commute because $G_1 \subseteq p(\tilde{H})$.) We state this as a lemma:

Lemma 3.2.3 *The subgroups \tilde{H} and \tilde{Z} form a dual pair in \tilde{G} .*

We have $\tilde{K} = \tilde{H}\tilde{Z} = \tilde{G}_+^{N/d}\tilde{Z}$. Since $\tilde{G}_1 \subseteq \tilde{K}$, we have $\tilde{K} = \tilde{H}\tilde{Z} = \tilde{G}_+^{N/d}\tilde{G}_1\tilde{Z} = \tilde{G}_+^{N/d}\tilde{G}_+^n$. Since $k = (n, N/d)$, this gives us $\tilde{K} = \tilde{G}_+^k$. That $\tilde{G}_+^{N/d}\tilde{G}_+^n \subseteq \tilde{G}_+^k$ is obvious. The other implication follows from the fact if $d = (a, b)$ and r and s are integers such that $d = ra + sb$ then $r(a/d) + s(b/d) = 1$. Therefore $g = g^{r(a/d)}g^{s(b/d)}$ for any positive integers a and b . This gives us that $\tilde{G}_+^k \subseteq \tilde{G}_+^{N/d}\tilde{G}_+^n$.

From the definition of \tilde{K} we get that $Z(\tilde{K}) = Z(\tilde{H})$ (If something commutes with \tilde{K} then it must commute with \tilde{Z} and therefore must be in \tilde{H}). Also $Z(\tilde{H}) = \tilde{H} \cap \tilde{Z}$. This follows from lemma 3.2.3 and the fact that anything in the center of \tilde{H} must be in \tilde{Z} . An application of lemma 3.2.2 gives us:

Lemma 3.2.4 $Z(\tilde{K}) = Z(\tilde{H}) = \tilde{H} \cap \tilde{Z} = \{z_x | x \in \mathbb{F}^{*N/dk}\}$.

Next we prove that $\text{Cent}_{\tilde{G}}Z(\tilde{K}) = \tilde{K}$.

Lemma 3.2.5 $\text{Cent}_{\tilde{G}}Z(\tilde{K}) = \tilde{K}$.

Proof. Let $z_x \in Z(\tilde{K})$. By lemma 3.2.4 $x = a^{N/dk}$ for some $a \in \mathbb{F}^*$.

$$\begin{aligned} \{g, z_x\} &= (\det(g)^{n-1}, x)_N \\ &= (\det(g)^{n-1}, a^{N/dk})_N. \end{aligned}$$

$$\begin{aligned} \text{Thus } g \in \text{Cent}_{\tilde{G}} Z(\tilde{K}) &\Leftrightarrow (\det(g)^{(n-1)N/dk}, a)_N = 1 \quad \forall a \in \mathbb{F}^* \\ &\Leftrightarrow \det(g)^{(n-1)N/dk} \in \mathbb{F}^{*N} \\ &\Leftrightarrow \det(g) \in \mathbb{F}^{*k} \quad (\text{lemma 3.2.2}) \end{aligned}$$

$$\begin{aligned} \text{We get } \text{Cent}_{\tilde{G}}(Z(\tilde{K})) &= \tilde{G}_+^k \\ &= \tilde{K}. \end{aligned}$$

■

Thus all conditions imposed on \tilde{K} and \tilde{H} in section 3.1 are satisfied when we specialize to the case $G = GL(n)$.

Now we have $\tilde{H} = \tilde{G}_+^{N/d}$ and $\tilde{K} = \tilde{G}_+^k$.

Also $\tilde{G}_1 Z(\tilde{H}) = \tilde{G}_+^{nN/dk}$.

We are ultimately interested in understanding the restriction of representations to \tilde{G}_1 . Restriction from \tilde{H} to \tilde{G}_1 is easily understood if the extension from \tilde{G}_1 to \tilde{H} is central (i.e $\tilde{G}_1 Z(\tilde{H}) = \tilde{H}$). That is the case precisely when $\tilde{G}_+^{nN/dk} = \tilde{G}_+^{N/d}$ i.e iff $n/k = 1$. Since $k = (n, N)$, we are in the above case only when $n|N$. We summarize this in a lemma:

Lemma 3.2.6 $\tilde{H} = \tilde{G}_1 Z(\tilde{H})$ if and only if $n|N$.

Therefore, from now on, we further assume that $n|N$.

We examine our situation again:

$$1 \rightarrow \mu_N \rightarrow \tilde{G} \rightarrow G \rightarrow 1$$

where n divides N and we assume $N = nN_0$.

3.3 A character formula.

Given a genuine, admissible, irreducible representation Π of \tilde{G} , we will write the character of $\Pi|_{\tilde{G}_1}$ in terms of Θ_Π . We will apply the results of section 3.1 to our situation.

We have:

$\tilde{G}_1 \subset \tilde{H} \subset \tilde{K} \subset \tilde{G}$. From the calculations in section 3.2, we have:

1. $\tilde{H} = \tilde{G}_+^{N/d}$ and
2. $\tilde{K} = \tilde{G}_+^n$.

The lemma 3.1.1 gives us:

Lemma 3.3.1

$$\Pi|_{\tilde{G}_+^n} = \sum_{x \in \mathbb{F}^*/\mathbb{F}^{*n}} \pi^x$$

where π is an irreducible representation of \tilde{G}_+^n .

A word on notation. Here x is a representative of a coset of \tilde{G}/\tilde{G}_+^n . We are writing $x \in \mathbb{F}^*/\mathbb{F}^{*n}$ because $\tilde{G}/\tilde{G}_+^n \simeq \mathbb{F}^*/\mathbb{F}^{*n}$ via the determinant map. Therefore $x \in \mathbb{F}^*/\mathbb{F}^{*n}$ represents an element of $g \in \tilde{G}$ with determinant x .

We have shown earlier that (proof of lemma 3.1.1) $\chi_{\pi^x}(z_y) = \chi_\pi(z_y)(x, y)_N^{n-1}$ where z_x denotes any element of \tilde{G} of the form (xI, ζ) . Here ζ is a N^{th} root of unity.

We also have:

1. $Z(\tilde{G}) = \{z_x | x \in \mathbb{F}^{*N/d}\}$
2. $Z(\tilde{G}_+^n) = \{z_x | x \in \mathbb{F}^{*N_0/d}\}$. (lemma 3.2.4.)

We perform a calculation: $\chi_\pi(zz')^{-1}\Theta_\Pi(zz'g) = \chi_\pi(z)^{-1}\Theta_\Pi(zg)$ for $z \in \widetilde{Z^{N_0/d}}, z' \in \widetilde{Z^{N/d}}$.

Therefore $\chi_\pi(z)^{-1}\Theta_\Pi(zg)$ is well-defined for $z \in \widetilde{Z^{N_0/d}}/\widetilde{Z^{N/d}}$.

$$\begin{aligned} \sum_{z \in \widetilde{Z^{N_0/d}}/\widetilde{Z^{N/d}}} \chi_\pi(z)^{-1}\Theta_\Pi(zg) &= \sum_{z \in \widetilde{Z^{N_0/d}}/\widetilde{Z^{N/d}}} \chi_\pi(z)^{-1} \sum_{x \in \mathbb{F}^*/\mathbb{F}^{*n}} \Theta_{\pi^x}(zg) \\ &= \sum_{x \in \mathbb{F}^*/\mathbb{F}^{*n}} \sum_{z \in \widetilde{Z^{N_0/d}}/\widetilde{Z^{N/d}}} \chi_\pi(z)^{-1} \chi_{\pi^x}(z) \Theta_{\pi^x}(g). \end{aligned}$$

We evaluate

$$\sum_{z \in \widetilde{Z^{N_0/d}}/\widetilde{Z^{N/d}}} \chi_\pi(z)^{-1} \chi_{\pi^x}(z).$$

Let us use the notation $z = z_y$ where $y \in \mathbb{F}^{*N_0/d}$ and use the fact $\mathbb{F}^{*N_0/d}/\mathbb{F}^{*N/d} \simeq \mathbb{F}^*/\mathbb{F}^{*n}$ (this is true because $\mu_N \subset \mathbb{F}^*$), then by lemma 3.1.1:

$$\begin{aligned} \sum_{z \in \widetilde{Z^{N_0/d}}/\widetilde{Z^{N/d}}} \chi_\pi(z)^{-1} \chi_{\pi^x}(z) &= \sum_{y \in \mathbb{F}^*/\mathbb{F}^{*n}} (x, y^{N_0/d})_N^{n-1} \\ &= \sum_{y \in \mathbb{F}^*/\mathbb{F}^{*n}} (x, y)_N^{(N_0)n-1/d} \\ &= \sum_{y \in \mathbb{F}^*/\mathbb{F}^{*n}} (x^{N_0(n-1/d)}, y)_N. \end{aligned}$$

If $(x^{N_o(n-1/d)}, y)_N = 1$ for every $y \in \mathbb{F}^*$ then we have $x^{N_o(n-1/d)} \in \mathbb{F}^{*N}$ by the non-degeneracy of the Norm-residue symbol. This forces $x \in \mathbb{F}^{*n}$ by lemma 3.2.2.

We obtain:

$$\sum_{z \in \widetilde{Z^{N_o/d}} / \widetilde{Z^{N/d}}} \chi_\pi(z)^{-1} \chi_{\pi^x}(z) \neq 0$$

if and only if $x = 1$.

Therefore,

$$\sum_{z \in \widetilde{Z^{N_o/d}} / \widetilde{Z^{N/d}}} \chi_\pi(z)^{-1} \Theta_\Pi(zg) = |Z^{N_o/d} / Z^{N/d}| \Theta_\pi(g).$$

We obtain,

$$\Theta_\pi(g) = \left| \frac{1}{\mathbb{F}^{*N_o/d} / \mathbb{F}^{*N/d}} \right| \sum_{z \in \widetilde{Z^{N_o/d}} / \widetilde{Z^{N/d}}} \chi_\pi(z)^{-1} \Theta_\Pi(zg).$$

Now we use the fact that $\pi|_{\widetilde{G}_+^{N/d}} = |N_o/d|\sigma$ (lemma 3.1.3 applied to our situation).

We have:

Theorem 3.3.1 *Let Π be an irreducible, genuine representation of \widetilde{G} . Let π be a summand in the restriction of Π to \widetilde{G}_+^n . Also let σ be the irreducible component of π restricted to $\widetilde{G}_+^{N/d}$. Then:*

$$\Theta_\sigma(g) = \frac{d}{N_o} \frac{1}{|\mathbb{F}^* / \mathbb{F}^{*n}|} \sum_{z \in \widetilde{Z^{N_o/d}} / \widetilde{Z^{N/d}}} \chi_\pi(z)^{-1} \Theta_\Pi(zg).$$

This gives us the character of any summand of Π restricted to $\widetilde{G}_+^{N/d}$ in terms of its central character and the character of Π . Because upon restriction from $\widetilde{G}_+^{N/d}$ to \widetilde{G}_1 , an irreducible representation remains irreducible, the above theorem also gives the character of σ as a representation of \widetilde{G}_1 .

3.4 Centers of Cartan subgroups.

We define a Cartan subgroup of \tilde{G} or \tilde{G}_1 to be the inverse image of a Cartan subgroup of the corresponding linear group. These are non-abelian in general, and their centers play an important role. We will be able to see some aspects of this in the last chapter of the thesis when we study orbital integrals. An element of a covering group is said to be regular (semi-simple) if its projection in the linear group is regular (semi-simple).

We state a lemma.

Lemma 3.4.1 *Let T be a Cartan subgroup of G with inverse image \tilde{T} in \tilde{G} . Then*

1. *The center of \tilde{T} is $p^{-1}(Z^{N/d}T^N)$.*
2. *The center of $\tilde{T} \cap \tilde{G}_1$ is $p^{-1}(Z^{N_0/d}T^N \cap G_1)$.*

Proof.

For the first part we refer to [9] for proof. For the second part we note that if $g \notin p^{-1}(Z^{N_0/d}T^N \cap G_1)$ then $zg \notin p^{-1}(Z^{N/d}T^N)$ for any $z \in \widetilde{Z^{N_0/d}}$. By the first part, $\Theta_\Pi(zg) = 0$ for every genuine representation Π of \tilde{G} (Θ_Π vanishes on any element of \tilde{T} not in the center of \tilde{T} because T is commutative and Θ_Π is a conjugation invariant genuine function). By theorem 3.3.1 $\Theta_\sigma(g) = 0$ for every genuine irreducible representation σ of \tilde{G}_1 . Given any σ we can extend via center to $\tilde{G}_+^{N/d}$ and then induce to \tilde{G} (via \tilde{G}_+^n) to obtain an irreducible, genuine representation Π of \tilde{G} and then apply theorem 3.3.1. By the fact that genuine representation separate non-conjugate points, we obtain that $g \notin Z(\tilde{T} \cap \tilde{G}_1)$. The proves for regular elements. For a general element, we use a continuity argument.

The other inclusion is obvious since $\widetilde{Z^{N_0/d}}$ commutes with every element of \widetilde{G}_1 and $p^{-1}(T^n)$ commutes with \widetilde{T} .

■

Definition 3.4.1 *We say that a regular element $g \in \widetilde{T}$ is relevant if it is contained in the center of \widetilde{T} [1].*

We conclude this chapter by proving the existence of a special kind of genuine character.

3.5 Existence of a special kind of character on $\widetilde{Z^{N_0/d}}$.

We want to define a genuine character μ of $\widetilde{Z^{N_0/d}}$ such that $\mu(z_{x^N}, \zeta) = \zeta \quad \forall x \in \mathbb{F}^*$.

Lemma 3.5.1 *The cocycle c used for the definition of \widetilde{G} splits over $\widetilde{Z^N}$.*

Proof.

Let $x = a^N, y = b^N$ with $a, b \in \mathbb{F}^*$. By the formula for calculating cocycles ([9]):

$$\begin{aligned}
 c(z_x, z_y) &= \prod_{i < j} (x, y)_N \\
 &= (x, y)_N^{n(n-1)/2} \\
 &= (a, b)_N^{N^2 n(n-1)/2} \\
 &= 1.
 \end{aligned}$$

■ Since $\widetilde{Z^N} = Z^N \times \mu_N$ (the cocycle is trivial on Z^N), we can define a character χ of $Z^N \times \mu_N$ by $\chi(x^N I, \zeta) = \zeta$.

Let $\bar{\mu} = \text{Ind}_{\widetilde{Z^N}}^{\widetilde{Z^{N_0/d}}}(\chi) = \oplus \mu_i$ where i runs over some finite set. Since $\widetilde{Z^{N_0/d}}$ is abelian (can be verified by the commutator formula, 2.3.1) all μ_i 's are characters. Let μ be any of the μ_i 's. Then, by Frobenius reciprocity:

$$\langle \bar{\mu}, \mu \rangle_{\widetilde{Z^{N_0/d}}} = \langle \chi, \mu|_{\widetilde{Z^N}} \rangle_{\widetilde{Z^N}} .$$

Hence we have a character μ of $\widetilde{Z^{N_0/d}}$ such that $\mu(z_{x^N}, \zeta) = \zeta$. This also gives us that μ is genuine.

Chapter 4

Lifting of Characters

4.1 Lifting from G to \tilde{G} .

In this section we summarize results on lifting of characters from G to \tilde{G} . This is very similar what has been done in [1]. There are some changes because in our case n may not be equal to N .

We first define transfer factors in this setting. Recall the Weyl denominator for G is given by $\Delta(g) = \prod_{i < j} |x_i - x_j|_{\overline{\mathbb{F}}} / |x_i x_j|_{\overline{\mathbb{F}}}^{\frac{1}{2}}$ if g is a regular semisimple element with (distinct) eigenvalues x_i (in an algebraic closure $\overline{\mathbb{F}}$ of \mathbb{F}).

Definition 4.1.1 *Suppose $h \in G$, $g \in \tilde{G}$ are regular semisimple elements satisfying $h^N x^{N/d} = p(g)$ for some $x \in Z(G)$.*

We denote $s(h)^N u(h)$ by h^ . Here $u(h) = \pm 1 \in \mu_n$ is defined by [10] (we take $u(h) = 1$ if N is odd), and $s : G \rightarrow \tilde{G}$ is any section. We define $\tau(h, g)$ by*

$$(h^*)^{-1}g = (p(h^*)^{-1}g, \tau(h, g)). \quad (4.1)$$

Also let

$$\Delta^{\tilde{\omega}}(h, g) = b \frac{\Delta(h)}{\Delta(g)} \tilde{\omega}((h^*)^{-1}g). \quad (4.2)$$

Here $\tilde{\omega}$ is any genuine character of $\widetilde{Z^{N/d}}$ and $b = N/d |N^n/d|^{\frac{1}{2}}$ ([7].)

Let π be a representation of $GL(n)$ with central character χ_π satisfying $\chi_\pi(\zeta I) = 1$ for all $\zeta \in \mu_N$. Suppose g is a regular semisimple element of \tilde{G} , so $p(g)$ is contained in a Cartan subgroup T of G . Let $\tilde{\omega}$ be a character of $\widetilde{Z^{N/d}}$ satisfying $\tilde{\omega}(x^N, 1) = \chi_\pi(x)$ for all $x \in \mathbb{F}^*$. Let

$$t_*^{\tilde{\omega}}(\Theta_\pi)(g) = \sum_{\{h \in T/Z | (h^*)^{-1}g \in \widetilde{Z^{N/d}}\}} \Delta^{\tilde{\omega}}(h, g) \Theta_\pi(h). \quad (4.3)$$

We fix $\tilde{\omega}$ once and for all.

This is a conjugation invariant function on the regular semisimple elements of \tilde{G} . If there exists an irreducible, genuine, admissible representation of \tilde{G} , say $\tilde{\pi}$, such that $\Theta_{\tilde{\pi}} = t_*^{\tilde{\omega}}(\Theta_\pi)$ we say that π and $\tilde{\pi}$ correspond via $\tilde{\omega}$ and denote $\tilde{\pi}$ by $t_*^{\tilde{\omega}}(\pi)$. Note that in the above case $\tilde{\omega} = \chi_{\tilde{\pi}}$.

We state the conjecture of Kazhdan-Flicker in two hypotheses [1].

Hypotheses I Let π be an irreducible representation of G such that $\chi_\pi(\zeta I) = 1$ for all $\zeta \in \mu_N$. We say *Hypotheses I holds for π* if $t_*^{\tilde{\omega}}(\pi)$ is 0 or \pm the character of an irreducible representation of \tilde{G} . If this holds we define the virtual representation $t_*^{\tilde{\omega}}(\pi)$ by $t_*^{\tilde{\omega}}(\Theta_\pi) = \Theta_{t_*^{\tilde{\omega}}(\pi)}$. Furthermore if $t_*^{\tilde{\omega}}(\pi) \neq 0$ define $\epsilon(\pi) = \pm 1$ so that $\epsilon(\pi)t_*^{\tilde{\omega}}(\pi)$ is a representation.

Hypotheses II Every genuine irreducible unitary representation of \tilde{G} is isomorphic to $\epsilon(\pi)t_*^{\tilde{\omega}}(\pi)$ for some irreducible unitary representation π satisfying Hypothesis I.

We state some conditions when the above hypotheses are true [1]. Hypotheses I and II are true for $n = 2$ [6]. Hypotheses I is true if π is a discrete series representation. In this case $t_*^{\tilde{\omega}}$ turns out to be a bijection between a subset of discrete series of G and the genuine discrete series of \tilde{G} . This means that Hypotheses II

holds for the discrete series representations. If $t_*^{\tilde{\omega}}(\pi)$ is supercuspidal then π is supercuspidal but the converse is not true. If π is tempered then Hypotheses I holds under certain conditions ([7], Proposition 26.2). If π is tempered and satisfies Hypotheses I then $t_*^{\tilde{\omega}}(\pi)$ is tempered and $\epsilon(\pi) = 1$. Subject to certain conditions ([7]) $t_*^{\tilde{\omega}}$ bijectively maps a subset of irreducible tempered representations of G to genuine, irreducible, tempered representations of \tilde{G} .

We also have for any $\alpha \in \widehat{\mathbb{F}^*}$

$$t_*^{\tilde{\omega}}(\pi\alpha^N) = t_*^{\tilde{\omega}}(\pi)\alpha. \quad (4.4)$$

This follows immediately from (4.3).

4.2 Parameters for \tilde{G}_1 .

Let π be an irreducible, admissible representation of G satisfying $\chi_\pi(\zeta I) = 1 \quad \forall \zeta \in \mu_N$. Let $t_*(\pi) = \Pi$. (We assume π satisfies hypothesis I.)

Definition 4.2.1 *Let X be the set of pairs (π, χ) where:*

1. π is an irreducible representation of G , with central character χ_π satisfying $\chi_\pi(\zeta I) = 1 \quad \forall \zeta \in \mu_N$.
2. χ is the central character of some summand of $t_*^{\tilde{\omega}}(\pi)$ restricted to \tilde{G}_+^n . Call the above summand $L_o(\pi, \chi)$. All χ 's are distinct (lemma 3.3.1 and the remarks after it).
3. Let $L(\pi, \chi)$ be the irreducible component obtained upon restriction of $L_o(\pi, \chi)$ to $\tilde{G}_+^{N/d}$. We are considering $L(\pi, \chi)$ as a representation of \tilde{G}_1 (lemma 3.1.3).

Let $\alpha \in \widehat{\mathbb{F}^*}$. We want to define an action of α on the symbols $L_o(\pi, \chi)$. We use the same letter α as an element of $\widehat{\mathbb{F}^*}$ and as a character on G (which is defined naturally using the determinant).

Let $\alpha * L_o(\pi, \chi) = L_o(\pi\alpha, \chi\beta)$ for some $\beta \in \widehat{\mathbb{F}^*}$. We will determine what α and β should be to make the action meaningful.

Since $\chi_{\pi\alpha}(\mu_N) = 1 \Leftrightarrow \alpha = \alpha_o^{N_o}$ for some $\alpha_o \in \widehat{\mathbb{F}^*}$.

To determine β , we note that in $L(\pi, \chi)$, χ satisfies the following equation:

$$\chi(x^N I, 1) = \chi_\pi(x) \quad \forall x \in \mathbb{F}^* \quad (4.5)$$

This is true because $\chi_\pi(x) = \chi_\Pi(x^N I, 1) \quad \forall x \in \mathbb{F}^*$ and the central characters of all summands in the restriction to \widetilde{G}_+^n are extension of χ_Π . Actually the above criterion characterizes the summands in the restriction.

Using the above fact we get:

$$\begin{aligned} \beta\chi(x^N I, 1) &= \chi_{\pi\alpha}(x) \\ \Rightarrow \beta(x^N)\chi(x^N I, 1) &= \chi_\pi(x)\alpha(x^n) \\ \text{Because } \chi(x^N I, 1) &= \chi_\pi(x) \\ \Rightarrow \beta(x^N) &= \alpha_o(x^N) \\ \Rightarrow \beta &= \alpha_o\gamma \quad \text{where } \gamma \in \widehat{\mathbb{F}^*/\mathbb{F}^{*N}}. \end{aligned}$$

We will take $\gamma = 1$. We have chosen $\beta = \alpha_o$. Since $\alpha\tilde{\omega}(x^N, 1) = \chi_{\pi\alpha^{N_o}}(x)$, it is a valid choice for the central character of the lift of $\pi\alpha^{N_o}$. Also $\alpha\chi = \alpha\tilde{\omega}$ on $\widetilde{Z^{N/d}}$. Hence the above defined action is well-defined.

Definition 4.2.2 For any $\alpha \in \widehat{\mathbb{F}^*}$ define $\alpha * L_o(\pi, \chi) = L_o(\pi\alpha^{N_o}, \alpha\chi)$.

We observe that $\alpha^{N_o} = 1 \Rightarrow \alpha * L_o(\pi, \chi) = L_o(\pi, \chi)$.

$\alpha^n * L_o(\pi, \chi) = L_o(\pi \alpha^N, \alpha^n \chi) = L_o(\pi, \chi) \otimes \alpha$. The last equality is true because $t_*(\pi \alpha^N) = t_*(\pi) \alpha$.

Now χ is a genuine character of $\widetilde{Z^{N_o/d}}$. All such χ 's satisfying $\chi(x^N I, 1) = \chi_\pi(x) \quad \forall x \in \mathbb{F}^*$ can be parametrized by elements ν of $\widehat{\mathbb{F}^{*N_o/d}}$ satisfying $\nu(x^N) = \chi_\pi(x)$ for x in \mathbb{F}^* .

In fact any such χ is given by:

$$\chi(x^{N_o/d} I, \zeta) = \mu(x^{N_o/d} I, \zeta) \nu(x^{N_o/d}).$$

where μ is defined in section 3.5.

We note that $L_o(\pi, \mu \nu)$ is the same as $L_o(\pi, \chi)$ because $\chi(x^{N_o} I, \zeta) = \mu(x^{N_o} I, \zeta) \nu(x^{N_o})$. We will often denote $L_o(\pi, \mu \nu)$ by $L_o(\pi, \chi_\nu)$ or even by $L_o(\pi, \chi)$ if there is no chance of ambiguity. Since μ is fixed, we will suppress it in the notation.

Instead of looking at ν we can consider ν as a character of \mathbb{F}^* and then look at $\nu^{N_o/d}$. This we do by extending ν to a character of \mathbb{F}^* and calling it ν_1 and then observing that $\nu(x^{N_o}/d) = \nu_1(x)^{N_o/d}$ and then labelling ν_1 also by ν . We will be doing this from now on. This will not matter as we will always be considering $\nu^{N_o/d}$ in any calculations. We will also refer to such a χ by χ_ν .

From $L(\pi, \chi)$, we want to define a representation of $\text{PGL}(n, \mathbb{F})$.

Definition 4.2.3 Define $M(\pi, \chi_\nu) = \pi \nu^{-N_o}$.

Since $\chi_{\pi \nu^{-N_o}}(x) = \chi_\pi(x) \nu^{-N}(x) = 1$, $M(\pi, \chi_\nu)$ defines a representation of $\text{PGL}(n, \mathbb{F})$.

Also $\pi \alpha^{N_o} (\nu \alpha)^{-N_o} = \pi \nu^{-N_o}$. This gives us $M(\pi, \chi_\nu) = M(\pi \alpha^{N_o}, \alpha \chi_\nu)$.

From the relation $\alpha^n * L_o(\pi, \chi) = L_o(\pi, \chi)\alpha$, we get that $L(\pi\alpha^N, \chi\alpha^n) = L(\pi, \chi)$.

Let $x = (\pi, \chi_\nu)$, $x' = (\pi', \chi'_{\nu'})$.

$$\begin{aligned} M(x) &= M(x') \\ \Rightarrow \pi\nu^{-N_o} &= \pi'\nu'^{-N_o} \\ \Rightarrow \pi &= \pi'(\nu\nu'^{-1})^{N_o}. \end{aligned}$$

Hence we obtain:

$$(\nu\nu'^{-1}) * L_o(\pi', \chi').$$

$M(x) = M(x') \Rightarrow x' = \alpha * x$ for some $\alpha \in \widehat{\mathbb{F}^*}$.

Definition 4.2.4 *Define*

$$L_{st}^n(\pi, \chi) = \sum_{\alpha \in \widehat{\mathbb{F}^*} / \mathbb{F}^{*n} \simeq \hat{\mu}_n} L(\pi\alpha^{N_o}, \chi\alpha).$$

We also define:

$$L_{st}(\pi, \chi) = \sum_{\alpha \in \hat{\mu}_n / \hat{\mu}_n^{\frac{n}{e}}} L(\pi\alpha^{N_o}, \chi\alpha)$$

where $e=(n, N_o)$.

We are factoring by $\hat{\mu}_n^{\frac{n}{e}}$ because we want to take care of those characters α of $\hat{\mu}_n$ which satisfy $\alpha^{N_o} = 1$.

Let π be an irreducible representation of $\text{PGL}(n, \mathbb{F})$, and let π' denote π pulled back to G . Assume $\pi'\alpha^{N_o}$ satisfy Hypothesis 1 for all $\alpha \in \widehat{\mathbb{F}^*}$. Define $L_{st}(\pi) = L_{st}(\pi', 1)$.

We have $L_{st}(\pi, \chi) = L_{st}(\pi\alpha^{N_o}, \chi\alpha) \quad \forall \alpha \in \widehat{\mathbb{F}^*}$.

4.3 Orbit Correspondence.

For $g \in G$, let \bar{g} be the image of g in $\mathrm{PGL}(n, \mathbb{F})$.

Definition 4.3.1 For h in G , define a map $\phi : G \rightarrow G_1$ by $\phi(h) = \det(h^{-N_0})h^N$.

Since $\phi(zh) = \phi(h) \quad \forall z \in Z$, ϕ actually gives a map from $\mathrm{PGL}(n, \mathbb{F})$ to G_1 .

Lemma 4.3.1 1. For every $h \in \mathrm{PGL}(n, \mathbb{F})$ and $g \in G$, $\phi(\bar{g}h\bar{g}^{-1}) = g\phi(h)g^{-1}$.

2. If h is a regular semi-simple element, then any pullback of $\phi(h)$ is relevant.

Proof. 1.) follows from the definitions while 2.) follows from the fact that $Z(\tilde{T} \cap \tilde{G}_1) = \widetilde{Z^{N_0/d} \tilde{T}^N} \cap \tilde{G}_1$. (see lemma 3.4.1.) ■

Suppose $h \in G, g \in G_1$ satisfy:

$$h^N x^{N/d} = zg \quad \text{for some } z, x \in Z.$$

Multiplying by $\det(h^{-N_0})$, we get:

$$\begin{aligned} \phi(h) &= g \det(h^{-N_0})z \\ &= g^{-1}\phi(h) = \det(h^{-N_0})zx^{-N/d}. \end{aligned}$$

$g^{-1}\phi(h)$ has determinant 1 as $g \in G_1$ Hence the Right-hand-side is a scalar matrix with determinant 1.

Thus $g = \zeta\phi(h)$ where $\zeta \in \mu_n$. Note that ζ does not depend on the choice of z, x .

Conversely, If $g = \zeta\phi(h)$ with $\zeta \in \mu_n$ then $h^N = zg$.

Note that if $h^N x^{N/d} = zg$ with $z, x \in Z$, $g \in G_1$ then $z \in \mu_n Z^{N_o/d}$. But we have assumed that the field \mathbb{F} has all the N^{th} roots of unity. This gives us that $\mu_n = \mu_N^{N_o}$. Hence z actually comes from $Z^{N_o/d}$. This observation will be very important when we define the transfer factors in the next section.

Definition 4.3.2 We say $h \in \text{PGL}(n, \mathbb{F}), g \in G_1$ weakly correspond, written as, $h \leftrightarrow g$, if for any (equivalently all) $h' \in G$ with $\bar{h}' = h$, we have

$$h'^N x'^{N/d} = zg \quad \text{for some } z, x \in Z.$$

Equivalently, $g = \zeta \phi(h)$ for $\zeta \in \mu_n$.

If $h \overset{\text{weak}}{\leftrightarrow} g$ define $\zeta(h, g) \in \mu_n$ by $g = \zeta(h, g) \phi(h)$.

4.4 Transfer Factors.

Definition 4.4.1 Suppose $h \in G, g \in \widetilde{G}_1$ satisfy

$$h^N x^{N/d} = p(zg). \quad (\text{this implies } z \in \widetilde{Z^{N_o/d}}.)$$

Define

$$c_n = \frac{nde}{|\mathbb{F}^*/\mathbb{F}^{*n}|N_o}$$

where $e = (n, N_o)$. and

$$\Delta_{\mu}^{\tilde{\omega}}(h, g) = c_n \mu(z)^{-1} \Delta^{\tilde{\omega}}(h, zg)$$

where μ is as defined in section 3.5 and because $z \in \widetilde{Z^{N_o/d}}$, everything is well-defined.

When we finally derive the stable character formula, it will be clear why we defined c_n in the above manner.

Lemma 4.4.1 $\Delta_{\mu}^{\tilde{\omega}}(h, g)\tilde{\omega}(x^{-N/d})$ is independent of z, x and hence of $\tilde{\omega}$. (By $\tilde{\omega}(x^{N/d})$ we mean $\tilde{\omega}(x^{N/d}, 1)$).

Proof. We have $h^N x^{N/d} = p(zg), h^N u^{N/d} = p(yg) \Rightarrow y = z\zeta(\frac{u}{x})^{-N/d}$.

Also

$$\begin{aligned} \mu(y)^{-1}\Delta^{\tilde{\omega}}(h, yg)\tilde{\omega}(u^{-N/d}) &= \mu(z\zeta(\frac{u}{x})^{N/d})^{-1}\Delta^{\tilde{\omega}}(h, z\zeta g(\frac{u}{x})^{N/d})\tilde{\omega}(u^{-N/d}) \\ &= b\mu(z)^{-1}\zeta^{-1}\frac{\Delta(h)}{\Delta(g)}\tilde{\omega}((h^*)^{-1}zg)\tilde{\omega}(x^{-N/d})\zeta. \\ &= \mu(z^{-1})\Delta^{\tilde{\omega}}(h, zg)\tilde{\omega}(x^{-N/d}). \end{aligned}$$

Above steps follow because of properties of μ and the fact that $\tilde{\omega}$ is genuine. That proves $\Delta_{\mu}^{\tilde{\omega}}(h, g)\tilde{\omega}$ is independent of z, x . ■

From now on we will denote $\Delta_{\mu}^{\tilde{\omega}}(h, g)\tilde{\omega} (h \overset{\text{weak}}{\leftrightarrow} g)$ by Δ_{μ} .

Lemma 4.4.2 $\Delta_{\mu}(h, g) = \Delta_{\mu}(\lambda h, g) \quad \forall \lambda \in \mathbb{F}^*$.

Proof. Note that if $p(zg) = h^N x^{N/d}$ then $p(z\lambda^N, 1)g = (\lambda h)^N x^{N/d}$.

Also $\Delta_{\mu}(\lambda h, g) = c_n \mu((\lambda^N, 1)z)^{-1}\Delta^{\tilde{\omega}}(\lambda h, z(\lambda^N, 1)g)\tilde{\omega}(x^{-N/d})$. Since $\mu|_{\widetilde{\mathbb{Z}^N}} = \iota$ we get $\mu((\lambda^N, 1)z) = \mu(z)$. Since $\Delta(h) = \Delta(\lambda h)$, we only need to check the $\tilde{\omega}$ part.

$$\begin{aligned} \tilde{\omega}(((\lambda h)^*)^{-1}z(\lambda^N, 1)g)\tilde{\omega}(x^{-N/d}) &= \tilde{\omega}(z(\lambda^N, 1)g\lambda^{-N}(h^*)^{-1})\tilde{\omega}(x^{-N/d}) \\ &= \tilde{\omega}((h^*)^{-1}zg)\tilde{\omega}(x^{-N/d}) \end{aligned}$$

Therefore $\Delta_{\mu}(\lambda h, g) = \Delta_{\mu}(h, g)$. ■

We note that $\tilde{\omega}((h^*)^{-1}zg)\tilde{\omega}(x^{-N/d}) = \tau(h, zg)$. (as defined in section 2)

Definition 4.4.2 Suppose $h \in \text{PGL}(n, \mathbb{F}), g \in \widetilde{G}_1$ such that $h \overset{\text{weak}}{\leftrightarrow} p(g)$, choose $h' \in G$ such that $\bar{h}' = h$, define $\Delta_{\mu}(h, g) = \Delta_{\mu}(h', g)$.

By lemma 4.4.2, this is independent of the choice of h' .

4.5 Stable Character Formula.

We derive the character formula relating the character of an irreducible representation of $\mathrm{PGL}(n, \mathbb{F})$ to the character of a virtual representation L_{st} of \widetilde{G}_1 .

Fix a μ as defined in section 3.5. We recall:

$$L_{st}(\pi) = \sum_{\alpha \in \widehat{\mu}_n / \widehat{\mu}_n^{n/e}} L(\pi \alpha^{N_o}, \chi \alpha).$$

where $e = (n, N_o)$.

Theorem 4.5.1 *Let π be an irreducible representation of $\mathrm{PGL}(n, \mathbb{F})$, for which $L_{st}(\pi)$ is defined. Then for g , a regular semi-simple element of \widetilde{G}_1 , we have*

$$\Theta_{L_{st}(\pi)}(g) = \sum_{\phi(h)=p(g)} \Delta_\mu(h, g) \Theta_\pi(h).$$

Proof. We first calculate $\Theta_{L(\pi, \chi_\nu)}(g)$ for arbitrary $(\pi, \chi_\nu) \in X$.

$$\begin{aligned} \Theta_{L(\pi, \chi_\nu)} &= \frac{d}{N_o |\mathbb{F}^* / \mathbb{F}^{*n}|} \sum_{z \in \widetilde{Z^{N_o/d}} / \widetilde{Z^{N/d}}} \chi_\nu(z)^{-1} \Theta_{t_*(\pi)}(zg) \\ &= \frac{d}{N_o |\mathbb{F}^* / \mathbb{F}^{*n}|} \sum_{z \in \widetilde{Z^{N_o/d}} / \widetilde{Z^{N/d}}} \sum_{\{h \in T/Z | h^N x^{N/d} = p(zg)\}} \chi_\nu(z)^{-1} \Delta^{\widetilde{\omega}}(h, zg) \Theta_\pi(h) \end{aligned}$$

Next we evaluate $\chi_\nu(z)^{-1} \Delta^{\widetilde{\omega}}(h, zg) \Theta_\pi(h)$.

$$\begin{aligned}
\chi_\nu(z)^{-1}\Delta^{\tilde{\omega}}(h, zg)\Theta_\pi(h) &= \mu(z)^{-1}\nu(p(z))^{-1}\Delta^{\tilde{\omega}}(h, g)\Theta_\pi(h) \\
&= \frac{1}{c_n}\nu(p(z))^{-1}\Delta^{\tilde{\omega}}_\mu(h, g)\Theta_\pi(h) \\
&= \frac{1}{c_n}\nu(p(z))^{-1}\nu^{N_o}(h)\Delta^{\tilde{\omega}}_\mu(h, g)\Theta_{\pi\nu^{-N_o}}(h).
\end{aligned}$$

Now

$$\nu(p(z))^{-1}\nu^{N_o}(h) = \nu(p(z)^{-1})\nu(\det(h)^{N_o}).$$

$$\text{Also } h^N x^{N/d} = p(zg) \Rightarrow \phi(h) = \det(h)^{-N_o}p(g)p(z)x^{N/d}.$$

$$\text{We get } \phi(h)^{-1}p(g)x^{-N/d} = \det(h)^{N_o}p(z^{-1}).$$

$$\text{Therefore, } \chi_\nu(z)^{-1}\Delta^{\tilde{\omega}}(h, zg)\Theta_\pi(h) = \frac{1}{c_n}\nu(\zeta(\bar{h}, g))\Delta_\mu(\bar{h}, g)\Theta_{\pi\nu^{-N_o}}(\bar{h}).$$

where \bar{h} denotes the image of h in $\text{PGL}(n, \mathbb{F})$. We have used the fact that $\chi_\nu(x^{-N/d}, 1) = \tilde{\omega}(x^{-N/d}) \quad \forall x \in \mathbb{F}^*$ and that $\mu(x^{N/d}, 1) = 1 \quad \forall x \in \mathbb{F}^*$.

We finally have:

$$\begin{aligned}
\Theta_{L(\pi, \chi_\nu)}(g) &= \frac{1}{N_o|\mathbb{F}^*/\mathbb{F}^{*n}|} \sum_{z \in \widetilde{Z^{N_o/d}}/\widetilde{Z^{N/d}}} \sum_{h^N x^{N/d} = p(zg)} \frac{d}{c_n} \nu(\zeta(\bar{h}, g))\Delta_\mu(\bar{h}, g)\Theta_{\pi\nu^{-N_o}}(\bar{h}) \\
&= \frac{d}{N_o|\mathbb{F}^*/\mathbb{F}^{*n}|} \frac{1}{c_n} \sum_{h \xleftrightarrow{\text{weak}} g} \nu(\zeta(\bar{h}, g))\Delta_\mu(\bar{h}, g)\Theta_{\pi\nu^{-N_o}}(\bar{h}) \\
&= \frac{d}{c_n N_o|\mathbb{F}^*/\mathbb{F}^{*n}|} \sum \nu(\zeta) \sum_{\phi(h)=\zeta p(g)} \Delta_\mu(h, g)\Theta_{\pi\nu^{-N_o}}(h)
\end{aligned}$$

We have used that $\mu_n = \mu_N^{N_o}$. This makes sure that $\nu(\zeta)$ is well-defined.

Replacing (π, χ_ν) by $(\pi\alpha^{N_o}, \alpha\chi_\nu)$ we get:

$$\frac{d}{c_n N_o|\mathbb{F}^*/\mathbb{F}^{*n}|} \sum (\alpha\nu)(\zeta) \sum_{\phi(h)=\zeta p(g)} \Delta_\mu(h, g)\Theta_{\pi\nu^{-N_o}}(h) \quad (h \in \text{PGL}(n, \mathbb{F})).$$

Summing over α , we get:

$$|(\widehat{\mu}_n)^{\frac{n}{e}}| \Theta_{L_{st}(\pi, \chi_\nu)}(g) = \frac{d}{c_n N_o |\mathbb{F}^*/\mathbb{F}^{*n}|} \sum_{\zeta \in \mu_n} \sum_{\alpha \in \widehat{\mu}_n} \nu(\zeta) \alpha(\zeta) \sum_{\phi(h)=\zeta g} \Delta_\mu(h, g) \Theta_{\pi\nu^{-N_o}}(h).$$

By the orthogonality of characters of μ_n , this equals

$$\frac{nd}{c_n N_o |\mathbb{F}^*/\mathbb{F}^{*n}|} \sum_{\phi(h)=g} \Delta_\mu(h, g) \Theta_{\pi\nu^{-N_o}}(h).$$

By our choice of c_n , we finally have:

$$\Theta_{L_{st}(\pi, \chi_\nu)}(g) = \sum_{\phi(h)=p(g)} \Delta_\mu(h, g) \Theta_{\pi\nu^{-N_o}}(h).$$

This proves the theorem ■

4.6 Inversion

Notation: Let

$$L_{st}^{\widehat{\mu}_n} = \sum_{\alpha \in \widehat{\mu}_n} L(\pi \alpha^{N_o}, \alpha \chi_\nu).$$

For $\zeta \in \mu_n$, define:

$$L_\zeta(\pi, \chi_\nu) = \sum_{\alpha \in \widehat{\mu}_n} \alpha(\zeta) L(\pi \alpha^{N_o}, \alpha \chi).$$

By summing over $\zeta \in \mu_n$ and using Fourier inversion on μ_n , we get $L(\pi, \chi_\nu) = \frac{1}{n} \sum_{\zeta \in \mu_n} L_\zeta(\pi, \chi_\nu)$.

We also note that:

$$\alpha(\zeta) \Theta_{L(\pi \alpha^{N_o}, \alpha \chi_\nu)}(g) = \chi_\nu(z_\zeta)^{-1} \Theta_{L(\pi \alpha^{N_o}, \alpha \chi_\nu)}(z_\zeta g).$$

Inserting this into the definition gives:

Lemma 4.6.1 For all $\zeta \in \mu_n$ we have:

$$\Theta_{L_\zeta(\pi, \chi_\nu)}(g) = \chi_\nu(z_\zeta)^{-1} \Theta_{L_{st}^{\mu_n}(\pi, \chi_\nu)}(z_\zeta g).$$

Using Section 4.5 we obtain:

$$\begin{aligned} \Theta_{L_\zeta(\pi, \chi_\nu)}(g) &= |\widehat{\mu_n}^{\frac{n}{e}}| \chi_\nu(z_\zeta)^{-1} \sum_{h \in \text{PGL}(n, \mathbb{F}), \phi(h) = \zeta p(g)} \Delta_\mu(h, gz_\zeta) \Theta_{\pi\nu - N_o}(h) \\ &= e\nu(\zeta)^{-1} \sum_{h \in \text{PGL}(n, \mathbb{F}), \zeta(h, g) = \zeta^{-1}} \Delta_\mu(h, g) \Theta_{\pi\nu - N_o}(h) \end{aligned}$$

where $e = (n, N_o)$.

Chapter 5

Lifting of Functions and Orbital Integrals

In this chapter we specialize to the case when $n = N$. Recalling the notation from chapter 4 , we observe that $d = N_o = 1$. This means that the two subgroups \widetilde{G}_+^n and $\widetilde{G}_+^{N/d}$ coincide. All the results in this chapter for the case $n = N$ are expected to be true for the general case where $n|N$ without any major changes in the proofs. We present this case because this involves less technicalities and simpler notation making the concepts more conspicuous. The following statements are true:

1. The groups \widetilde{Z} and \widetilde{G}_+^n form a dual pair in \widetilde{G} and $Z(\widetilde{G}_+^n) = \widetilde{Z}$.
2. The group \widetilde{G}_+^n is an extension of \widetilde{G}_1 via its center i.e $\widetilde{G}_+^n = \widetilde{G}_1 \widetilde{Z}$.

We therefore need only a one-step restriction of a genuine, irreducible representation Π from \widetilde{G} to \widetilde{G}_+^n . (section 3.1: \widetilde{H} and \widetilde{K} coincide.)

5.1 Lifting of functions

Notation: We will identify $\widehat{\mu}_n$ with $\widehat{\mathbb{F}^*}/\widehat{\mathbb{F}^*}^n$. A general element of $\widehat{\mu}_n$ will be denoted by α . ν will in general denote an element of $\widehat{\mathbb{F}^*}$ such that $\nu|\mu_n = \alpha$. We will also use ν_α to denote the same when ν is being used to denote a more general

element of $\widehat{\mathbb{F}^*}$. We fix a genuine character μ of \widetilde{Z} (as in previous chapter) and define $\chi_\nu(z_x) = \mu(z_x)\nu(x)$ for any $\nu \in \widehat{\mathbb{F}^*}$. Similarly for any $\alpha \in \widehat{\mu_n}$, we define $\chi_\alpha(z_\zeta) = \mu(z_\zeta)\alpha(\zeta)$. Thus we have defined χ_α on the center of \widetilde{G}_1 and χ_ν is its extension to the center of \widetilde{G}_+^n .

Let \tilde{f} be a genuine function on \widetilde{G}_1 such that $\tilde{f}(\tilde{z}g) = \chi_\alpha(z)^{-1}\tilde{f}(g)$ for every \tilde{z} in $\widetilde{Z}(\widetilde{G}_1)$. Here $\alpha \in \widehat{\mu_n}$ and χ_ν is a character of $Z(\widetilde{G}_1) = p^{-1}\{\zeta I | \zeta \in \mu_n\}$.

From now on the class of such functions will be denoted by $C^\infty(\widetilde{G}_1)_\alpha$.

Let π be an irreducible, admissible representation of G satisfying $\chi_\pi(\zeta) = 1$ where ζ is any n^{th} root of unity. The representation $L(\pi)$ is the lift of π to \widetilde{G} and it breaks up as sum of representations having distinct central characters on \widetilde{G}_+^n . From section 4.2, we see that the above central characters are characterized by elements of $\nu \in \widehat{\mathbb{F}^*}$ such that $\nu(x^n) = \chi_\pi(x) \quad \forall x \in \mathbb{F}^*$. We label each summand as $L_o(\pi, \nu)$ and denote the restriction of $L_o(\pi, \nu)$ to \widetilde{G}_1 by $L(\pi, \nu)$. Here ν is any character of $Z(G)$ such that $\nu^n = \chi_\pi$. Such ν exists because of the assumptions made on π (exact sequence 2.1). Similarly, we will use $L_o(\pi\nu_\alpha, \nu\nu_\alpha)$ and $L(\pi\nu_\alpha, \nu\nu_\alpha)$ to denote similar representations by taking $\pi\nu_\alpha$ and lifting it to $L(\pi\nu_\alpha)$ and then restricting it to \widetilde{G}_+^n and \widetilde{G}_1 .

This simplifies the notation from chapter 4. There we had used $L_o(\pi, \chi_\nu)$ since the parametrization was by χ_ν 's where ν was really a $(N_o/d)^{\text{th}}$ power of a character of \mathbb{F}^* . Here the center of \widetilde{G}_+^n is \widetilde{Z} and hence $\nu \in \widehat{\mathbb{F}^*}$. Therefore parametrization by χ_ν 's is the same as parametrization by ν 's. Similarly we denote the restriction of $L_o(\pi\nu_\alpha, \nu\nu_\alpha)$ to \widetilde{G}_1 by $L(\pi\nu_\alpha, \nu\nu_\alpha)$. The representation $L(\pi\nu_\alpha, \nu\nu_\alpha)$ does not depend on the extension ν_α of α and depends only on α . This was shown in the previous chapter while we were defining $L_{st}(\pi, \nu)$ (section 4.1). We will use this notation to avoid confusion since we are denoting a character of

μ_n by α and that of \mathbb{F}^* by ν .

Assume that for some ν in the above list we have $\nu|_{\mu_n} = \alpha$ (For any other ν , we will see later on that $\Theta_{L(\pi,\nu)}(\tilde{f}) = 0$). We extend \tilde{f} from \widetilde{G}_1 to \widetilde{G}_+^n by extending to \widetilde{Z} :

Definition 5.1.1 *Let $\tilde{f} \in C^\infty(\widetilde{G}_1)_\alpha$. Define the extension by $A(\tilde{f}, \nu)$ satisfying $A(\tilde{f}, \nu)(\tilde{z}g) = \chi_\nu(\tilde{z})^{-1}A(\tilde{f}, \nu)(g)$ where now \tilde{z} could be any element of \widetilde{Z} . Thus $A(\tilde{f}, \nu)$ is defined on the whole of \widetilde{G}_+^n because $\widetilde{G}_+^n = \widetilde{G}_1\widetilde{Z}$.*

Note that $A(\tilde{f}, \nu)$ is well-defined because of the transformation property \tilde{f} satisfies over the center of \widetilde{G}_1 . Center of \widetilde{G}_1 is just the pullback of the n^{th} roots of unity from G_1 . We now compute $\Theta_{L(\pi,\nu)}(\tilde{f})$.

$$\Theta_{L(\pi,\nu)}(\tilde{f}) = \int_{\widetilde{G}_1/\widetilde{Z}(\widetilde{G}_1)} \tilde{f}(g)\Theta_{L(\pi,\nu)}(g)dg$$

Since $\widetilde{G}_1/\widetilde{Z}(\widetilde{G}_1) = \widetilde{G}_+^n/\widetilde{Z}$ we have

$$\Theta_{L(\pi,\nu)}(\tilde{f}) = \int_{\widetilde{G}_+^n/\widetilde{Z}} A(\tilde{f}, \nu)(g)\Theta_{L_o(\pi,\nu)}(g)dg$$

Next we observe that $(\widetilde{G}_+^n/\widetilde{Z}^n)/(\widetilde{Z}/\widetilde{Z}^n) = \widetilde{G}_+^n/\widetilde{Z}$. The constant $|\widetilde{Z}/\widetilde{Z}^n| = n^2$ (in the case when n is coprime to the residual characteristic of \mathbb{F} . In other cases it differs by some power of p). We get:

$$\Theta_{L(\pi,\nu)}(\tilde{f}) = \frac{1}{|\widetilde{Z}/\widetilde{Z}^n|} \int_{\widetilde{G}_+^n/\widetilde{Z}^n} A(\tilde{f}, \nu)(g)\Theta_{L_o(\pi,\nu)}(g)dg$$

We now evaluate the integral

$$\int_{\widetilde{G}_+^n/\widetilde{Z}^n} A(\tilde{f}, \nu)(g)\Theta_{L_o(\pi,\nu')}(g)dg$$

for any ν' satisfying $\nu'^n = \chi_\pi$.

This equals

$$\int_{\tilde{G}_+^n/\tilde{Z}} \int_{\tilde{Z}/\tilde{Z}^n} A(\tilde{f}, \nu)(g) \Theta_{L_o(\pi, \nu')}(g) dg$$

The transformation properties of $A(\tilde{f}, \nu)$ and the fact that $\tilde{G}_+^n = \tilde{G}_1 \tilde{Z}$ and $\tilde{G}_+^n/\tilde{Z} = \tilde{G}_1/\tilde{G}_1 \cap \tilde{Z}$ ensure that

$$\begin{aligned} \Theta_{L(\pi, \nu)}(\tilde{f}) &= \int_{\tilde{G}_+^n/\tilde{Z}} \int_{\tilde{Z}/\tilde{Z}^n} A(\tilde{f}, \nu)(g) \Theta_{L_o(\pi, \nu')}(g) dg \\ &= \int_{\tilde{G}_1/\tilde{G}_1 \cap \tilde{Z}} \int_{\tilde{Z}/\tilde{Z}^n} A(\tilde{f}, \nu)(g) \Theta_{L_o(\pi, \nu')}(g) dg \\ &= \int_{\tilde{G}_1/\tilde{G}_1 \cap \tilde{Z}} A(\tilde{f}, \nu)(g) \Theta_{L_o(\pi, \nu')}(g) dg \sum_{\tilde{Z}/\tilde{Z}^n} \nu^{-1} \nu'(\tilde{z}) \\ &= 0 \\ &\text{if } \nu \neq \nu' \end{aligned}$$

The above calculation and the fact that

$$L(\pi) = \sum_{\nu^n = \chi_\pi} L(\pi, \nu)$$

gives us:

$$\Theta_{L(\pi, \nu)}(\tilde{f}) = \frac{1}{|\tilde{Z}/\tilde{Z}^n|} \int_{\tilde{G}_+^n/\tilde{Z}^n} A(\tilde{f}, \nu)(g) \Theta_{L(\pi)}(g) dg$$

Definition 5.1.2 Let $B(\tilde{f}, \nu)$ be the function obtained by extending $A(\tilde{f}, \nu)$ to the whole of \tilde{G} by defining it to be zero outside \tilde{G}_+^n .

We observe that $B(\tilde{f}, \nu)$ satisfies the transformation property by the central character of $L(\pi)$ on the center \tilde{G} .

We have:

Lemma 5.1.1 Let $\tilde{f} \in C_c^\infty(\tilde{G}_1)_\alpha$ and $\nu|_{\mu_n} = \alpha$. Then

$$\Theta_{L(\pi, \nu)}(\tilde{f}) = \Theta_{L_o(\pi, \nu)}(A(\tilde{f}, \nu)) = \frac{1}{|\tilde{Z}^n \setminus \tilde{Z}|} \Theta_{L(\pi)}(B(\tilde{f}, \nu))$$

This equals $\frac{1}{|\mathbb{F}^*/\mathbb{F}^{*n}|} \Theta_{L(\pi)}(B(\tilde{f}, \nu))$ by definition.

Definition 5.1.3 Let $C(\tilde{f}, \nu)$ be a function on G obtained by using Kazhdan-Flicker lifting [7] such that $C(\tilde{f}, \nu)$ satisfies $C(\tilde{f}, \nu)(zg) = \chi_\pi(z)^{-1}C(\tilde{f}, \nu)(g) \quad \forall z \in Z(G)$ and

$$\Theta_{L(\pi)}(B(\tilde{f}, \nu)) = \Theta_\pi(C(\tilde{f}, \nu))$$

Thus we have obtained a relation just like Kazhdan-Flicker [7]. There a similar relation was obtained between \tilde{G} and G . We have a relation between \tilde{G}_1 and G . In Kazhdan-Flicker, they started with a function on \tilde{G} transforming by the central character $\chi_{L(\pi)}$ of $L(\pi)$. We have started with a function on \tilde{G}_1 transforming by the central character of $L(\pi, \nu)$. We have obtained a function on G transforming with respect to the central character χ_π . We have used the results of [7] in doing so.

Here $\Theta_\pi(C(\tilde{f}, \nu))$ is given by the integral:

$$\int_{G/Z} \Theta_\pi(g)C(\tilde{f}, \nu)(g)dg$$

Thus, using $\chi_\pi = \nu^n$, and modifying $C(\tilde{f}, \nu)$ by ν , we obtain:

$$\Theta_\pi(C(\tilde{f}, \nu)) = \int_{G/Z} \Theta_{\pi\nu^{-1}}(g)C(\tilde{f}, \nu)\nu(g)dg$$

where $C(\tilde{f}, \nu)\nu(g) = C(\tilde{f}, \nu)(g)\nu(\det(g))$. Observe that $C(\tilde{f}, \nu)\nu$ is a C_c^∞ function on $G/Z = \mathrm{PGL}(n, \mathbb{F})$. We have $\Theta_{L(\pi, \nu)}(\tilde{f}) = \Theta_{\pi\nu^{-1}}(\nu C(\tilde{f}, \nu))$.

Definition 5.1.4 Let $D(\tilde{f}, \nu)$ be the function $\nu C(\tilde{f}, \nu)$.

It will be shown later in this section that $D(\tilde{f}, \nu)$ is independent of the choice of the extension ν of α from μ_n to the whole of \mathbb{F}^* .

We have proved:

Proposition 5.1.1 *Let $\tilde{f} \in C_c^\infty(\widetilde{G}_1)_\alpha$. Let π be any irreducible, admissible representation of G satisfying $\chi_\pi(\zeta I) = 1 \quad \forall \zeta \in \mu_n$. Also we assume that $\nu \in \widehat{\mathbb{F}}^*$ such that $\nu^n = \chi_\pi$ and $\nu|_{\mu_n} = \alpha$. We have:*

$$\Theta_{L(\pi, \nu)}(\tilde{f}) = \Theta_{\pi\nu^{-1}}(D(\tilde{f}, \nu))$$

Now we make a general observation:

Let $\tilde{f} \in C_c^\infty(\widetilde{G}_1)$. Then \tilde{f} can be written as:

$$\tilde{f} = \sum_{\alpha \in \widehat{\mu}_n} \tilde{f}^\alpha$$

where

$$\tilde{f}^\alpha \in C_c^\infty(\widetilde{G}_1)_\alpha$$

This is true because $C_c^\infty(\widetilde{G}_1) = \bigoplus_{\alpha \in \widehat{\mu}_n} C_c^\infty(\widetilde{G}_1)_\alpha$. We can actually see that

$$\tilde{f}^\alpha(g) = \frac{1}{n} \sum_{z_\zeta} \tilde{f}(gz_\zeta) \chi_\alpha(z_\zeta).$$

Lemma 5.1.2 *Let $\tilde{f} \in C_c^\infty(\widetilde{G}_1)_\alpha$. We assume π, ν as in proposition 5.1.1 and that $\nu^n = \chi_\pi$. Then:*

$$\Theta_{L(\pi, \nu)}(\tilde{f}^\alpha) = 0 \text{ unless } \nu|_{\mu_n} = \alpha.$$

Proof. Let us assume that $\nu|_{\mu_n} = \alpha_o$ and $\alpha_o \neq \alpha$. This allows us to choose a $\zeta \in \mu_n$ such that $\alpha_o(\zeta) \neq \alpha(\zeta)$.

$$\begin{aligned}
\Theta_{L(\pi,\nu)}(\tilde{f}) &= \int_{\widetilde{G}_1} \tilde{f}(g) \Theta_{L(\pi,\nu)}(g) dg \\
&= \int_{\widetilde{G}_1} \tilde{f}(gz_\zeta) \Theta_{L(\pi,\nu)}(gz_\zeta) dg \\
&= \alpha^{-1}(\zeta) \alpha_o(\zeta) \int_{\widetilde{G}_1} \tilde{f}(g) \Theta_{L(\pi,\nu)}(g) dg \\
&= \alpha^{-1}(\zeta) \alpha_o(\zeta) \Theta_{L(\pi,\nu)}(\tilde{f}).
\end{aligned}$$

This forces $\Theta_{L(\pi,\nu)}(\tilde{f}) = 0$. ■

Let $\tilde{f} \in C_c^\infty(\widetilde{G}_1)$. We write $\tilde{f} = \sum_{\beta \in \widehat{\mu}_n} \tilde{f}^\beta$. We recall that

$$\Theta_{L_{st}(\pi,\nu)} = \sum_{\nu_\alpha \in \widehat{\mathbb{F}^*}/\widehat{\mathbb{F}^{*n}}} \Theta_{L(\pi\nu_\alpha, \nu\nu_\alpha)}$$

where we use the identification $\widehat{\mathbb{F}^*}/\widehat{\mathbb{F}^{*n}} \simeq \widehat{\mu}_n$. In the definition of $\Theta_{L_{st}(\pi,\nu)}$, ν_α is any extension of α from μ_n to the whole of \mathbb{F}^* (in other words just the pullback of α via the restriction map from $\widehat{\mathbb{F}^*} \rightarrow \widehat{\mu}_n$). Any two such extensions of α will differ by an element of $\widehat{\mathbb{F}^{*n}}$ and so $L(\pi\nu_\alpha, \nu\nu_\alpha)$ will be independent of the choice of ν_α .

Then we have, using above lemma:

$$\begin{aligned}
\Theta_{L_{st}(\pi,\nu)}(\tilde{f}) &= \sum_{\alpha \in \widehat{\mu}_n} \Theta_{L(\pi\nu_\alpha, \nu\nu_\alpha)}(\tilde{f}) \\
&= \sum_{\alpha \in \widehat{\mu}_n} \Theta_{L(\pi\nu_\alpha, \nu\nu_\alpha)}\left(\sum_{\beta \in \widehat{\mu}_n} \tilde{f}^\beta\right) \\
&= \sum_{\alpha \in \widehat{\mu}_n} \Theta_{L(\pi\nu_\alpha, \nu\nu_\alpha)}(\tilde{f}^{\nu\nu_\alpha|_{\mu_n}}) \\
&= \Theta_{\pi\nu^{-1}}\left(\sum_{\alpha \in \widehat{\mu}_n} D(\tilde{f}^{\nu\nu_\alpha|_{\mu_n}}, \nu\nu_\alpha)\right)
\end{aligned}$$

We summarize the above calculation:

Proposition 5.1.2

$$\Theta_{Lst(\pi, \nu)}(\tilde{f}) = \Theta_{\pi\nu^{-1}}\left(\sum_{\alpha \in \widehat{\mu}_n} D(\tilde{f}^{\nu\nu_\alpha|\mu_n}, \nu\nu_\alpha)\right)$$

For $\tilde{f} \in C_c^\infty(\widetilde{G}_1)_\alpha$, we will show that $D(\tilde{f}, \nu)$ does not depend upon the choice of the extension ν of α .

Let γ be any semi-simple, regular element of G . Let $\tilde{\gamma}$ denote the element $(\gamma, 1)^n u(\gamma)$ of \widetilde{G} . We are considering only those γ for which $\tilde{\gamma}$ is regular, semi-simple (see [10] for definition of u).

Define

$$F(\gamma, B(\tilde{f}^\alpha, \nu)) = \Delta(\tilde{\gamma}) \int_{\widetilde{T}^n \backslash \widetilde{G}} B(\tilde{f}^\alpha, \nu)(g^{-1}\tilde{\gamma}g) dg$$

where $\Delta(\tilde{\gamma})$ is the transfer factor and \widetilde{T} is the pullback of the cartan T which is the centralizer of γ in G . We also note that \widetilde{T} is the centralizer of $\tilde{\gamma}$ inside \widetilde{G} because $\tilde{\gamma} \in \widetilde{T}^n$.

We now have:

$$\Delta(\tilde{\gamma}) \int_{\widetilde{T}^n \backslash \widetilde{G}} B(\tilde{f}^\alpha, \nu)(g^{-1}\tilde{\gamma}g) dg = \Delta(\gamma) \int_{T \backslash G} C(\tilde{f}^\alpha, \nu)(g^{-1}\gamma g) dg. \quad (5.1)$$

(By results of [7].)

Let ν and ρ be characters of \mathbb{F}^* . Suppose they agree on μ_n . Let $\tilde{f} \in C_c^\infty(\widetilde{G}_1)$ and $\tilde{f}^\alpha \in C_c^\infty(\widetilde{G}_1)_\alpha$ be as before. Then \tilde{f}^α transforms with respect to χ_ν as well as χ_ρ . We extend \tilde{f}^α to a function $B(\tilde{f}^\alpha, \nu)$ on \widetilde{G} and then apply Kazhdan-Flicker lift to obtain a function $C(\tilde{f}^\alpha, \nu)$ on G (cf. [7]). Since ν and ρ agree on μ_n , χ_ν also agrees with χ_ρ on the center of \widetilde{G}_1 and so we can also extend \tilde{f}^α to a function $B(\tilde{f}^\alpha, \rho)$ on \widetilde{G} and apply Kazhdan-Flicker lift to obtain $C(\tilde{f}^\alpha, \rho)$ on G . We investigate the relationship between $C(\tilde{f}^\alpha, \rho)$ and $C(\tilde{f}^\alpha, \nu)$.

In the above case, there exists a $\delta \in \widehat{\mathbb{F}^*}$ such that $\nu = \rho\tau$ where $\tau = \delta^n$. (refer chapter 1)

Using the fact that $\widetilde{G}_+^n = \widetilde{G}_1 \widetilde{Z}$, we write $\tilde{\gamma} = z_y \tilde{\gamma}_1$ where $\tilde{\gamma}_1$ is an element of \widetilde{G}_1 .

Let $g^{-1}z_y = z_y g^{-1} \zeta_g$ where $\zeta_g = (1, \zeta_g)$ is an element of μ_n . (recall that \widetilde{G} is the central extension of G via μ_n .)

Each term has the integrand $B(\tilde{f}^\alpha, \nu)(g^{-1}z_y \zeta_g \tilde{\gamma}_1 g)$. Note that ζ_g is central in \widetilde{G} .

Because of the transformation properties satisfied by $B(\tilde{f}^\alpha, \nu)$ we have:

$$B(\tilde{f}^\alpha, \nu)(g^{-1} \tilde{\gamma} g) = \chi_\nu^{-1}(y) B(\tilde{f}^\alpha, \nu)(g^{-1} \zeta_g \tilde{\gamma}_1 g).$$

Since $\tilde{\gamma}_1 \in \widetilde{G}_1$, we have $g^{-1} \zeta_g \tilde{\gamma}_1 g \in \widetilde{G}_1$.

Replacing ν by ρ and using the fact that $B(\tilde{f}^\alpha, \nu)$ agrees with $B(\tilde{f}^\alpha, \rho)$ on \widetilde{G}_1 , we obtain

$$B(\tilde{f}^\alpha, \rho)(g^{-1} \tilde{\gamma} g) = \chi_\rho^{-1}(y) B(\tilde{f}^\alpha, \nu)(g^{-1} \zeta_g \tilde{\gamma}_1 g).$$

From $\nu = \rho\tau$, it follows that $\chi_\rho^{-1}(y) = \chi_\nu^{-1}(y)\tau(y)$.

We have:

$$B(\tilde{f}^\alpha, \rho)(g^{-1} \tilde{\gamma} g) = \chi_\nu^{-1}(y)\tau(y) B(\tilde{f}^\alpha, \nu)(g^{-1} \zeta_g \tilde{\gamma}_1 g).$$

Now we push $\chi_\nu^{-1}(y)$ inside to obtain:

$$B(\tilde{f}^\alpha, \rho)(g^{-1} \tilde{\gamma} g) = \tau(y) B(\tilde{f}^\alpha, \nu)(g^{-1} \tilde{\gamma} g).$$

Here $y = \det(\gamma)\zeta$ where ζ can be any element of μ_n . But then $\tau(\zeta) = 1$ because $\tau = \delta^n$. We have:

$$B(\tilde{f}^\alpha, \rho)(g^{-1} \tilde{\gamma} g) = \tau(\det(\gamma)) B(\tilde{f}^\alpha, \nu)(g^{-1} \tilde{\gamma} g) = \delta(\det(\tilde{\gamma})) B(\tilde{f}^\alpha, \nu)(g^{-1} \tilde{\gamma} g).$$

Now we use equation 5.1 applied to $B(\tilde{f}^\alpha, \nu)$ and $B(\tilde{f}^\alpha, \rho)$ and obtain:

$$\int_{T \backslash G} C(\tilde{f}^\alpha, \rho)(g^{-1}\gamma g) dg = \int_{T \backslash G} \tau(\det(\gamma)) C(\tilde{f}^\alpha, \nu)(g^{-1}\gamma g) dg$$

for every semi-simple $\gamma \in G$.

Therefore by Weyl Integration formula [7] we have

$$\Theta_\pi(C(\tilde{f}^\alpha, \rho)) = \Theta_\pi(\tau C(\tilde{f}^\alpha, \nu))$$

for all π such that π is irreducible and satisfies $\chi_\pi(\zeta) = 1 \quad \forall \zeta \in \mu_n$

Thus we can choose $C(\tilde{f}^\alpha, \rho) = \tau C(\tilde{f}^\alpha, \nu)$

i.e $C(\tilde{f}^\alpha, \rho)(g) = \tau(\det(g)) C(\tilde{f}^\alpha, \nu)(g)$ Note that many other choices of $C(\tilde{f}^\alpha, \rho)$ will satisfy the above equality but we make this choice in order to produce a candidate for $C(\tilde{f}^\alpha, \rho)$ satisfying certain conditions (which form the contents of the next lemma). These conditions ensure that the final function, $\Gamma(\tilde{f})$, we have on $\mathrm{PGL}(n, \mathbb{F})$ is independent of various parameters. That gives us the Stable character formula (theorem 5.1.1). We call this the stable formula because we are summing over all the characters of μ_n and ensuring that it does not depend upon any particular character.

Multiplying both sides of the above equality by $\rho(\det(g))$ gives us

$$\begin{aligned} \rho(\det(g)) C(\tilde{f}^\alpha, \rho)(g) &= \rho(\det(g)) \tau(\det(g)) C(\tilde{f}^\alpha, \nu)(g) \\ &= \nu(\det(g)) C(\tilde{f}^\alpha, \nu)(g). \end{aligned}$$

We have the relation:

Lemma 5.1.3 *With all the above conventions we have:*

$$\rho C(\tilde{f}^\alpha, \rho) = \nu C(\tilde{f}^\alpha, \nu)$$

where $\rho(g)$ means $\rho(\det(g))$ and similarly for ν .

$D(\tilde{f}^\alpha, \rho)(g) = \nu(\det(g))C(\tilde{f}^\alpha, \rho)(g)$ by definition (recall notation from the previous section). Lemma 5.1.3 implies that $D(\tilde{f}^\alpha, \nu)$ is independent of the choice of ν and hence can be denoted by $D(\tilde{f}^\alpha)$. This means that while constructing $D(\tilde{f}^\alpha)$ we could have chosen ν to be any element of $\widehat{\mathbb{F}^*}$ satisfying $\nu|_{\mu_n} = \alpha$.

Definition 5.1.5 *Let $\Gamma(\tilde{f}) = \sum_{\alpha \in \widehat{\mu_n}} D(\tilde{f}^\alpha)$*

Note that for any $\nu \in \widehat{\mathbb{F}^*}$, $\Gamma(\tilde{f})$ also equals $\sum_{\alpha \in \widehat{\mu_n}} D(\tilde{f}^{\nu\nu_\alpha|_{\mu_n}})$ where $\nu_\alpha|_{\mu_n} = \alpha$. With the above calculations in hand and proposition 5.1.2 we have:

Theorem 5.1.1 *Let $\tilde{f} \in C_c^\infty(\widehat{G}_1)$. Then there exists a function $\Gamma(\tilde{f}) \in C_c^\infty(\mathrm{PGL}(n, \mathbb{F}))$ such that we have the following relation:*

$$\Theta_{Lst(\pi, \nu)}(\tilde{f}) = \frac{1}{|\mathbb{F}^*/\mathbb{F}^{*n}|} \Theta_{\pi\nu^{-1}}(\Gamma(\tilde{f})).$$

where π is any admissible, irreducible representation of G satisfying $\chi_\pi(\zeta) = 1$ for every $\zeta \in \mu_n$ and $\nu \in \widehat{\mathbb{F}^*}$ satisfies $\nu^n = \chi_\pi$.

5.2 Cartan subgroups of G and G_+^n

The aim of this section is to obtain a relation between Cartan subgroups of G and those of G_+^n . We also relate the conjugacy classes of Cartan subgroups of the two groups. We are doing all this because later on we will apply the Weyl

Integration Formula to the results of the previous section and then we will need to understand the conjugacy classes of Cartan subgroups of the two groups in question.

First we define a Cartan subgroup.

Definition 5.2.1 *Let G be an algebraic group defined over the field \mathbb{F} . A Cartan subgroup of G is defined to be the centralizer of a regular, semi-simple element of G .*

Let T be a Cartan subgroup of G . Let T_+ denote the subgroup $T \cap G_+^n$. Then T_+ is its own centralizer in G_+^n . This is true because a regular, semi-simple element of G_+^n will also be regular in G . That follows from the fact that G_1 is the derived subgroup of G and $G_+^n = G_1 Z(G)$. By a similar argument one can see that any Cartan subgroup of G_+^n can be obtained by intersecting a Cartan subgroup of G with G_+^n .

We summarize the above:

Lemma 5.2.1 *Let T be a Cartan subgroup of G . Then $T \cap G_+^n$ is a Cartan subgroup of G_+^n and every Cartan subgroup of G_+^n is of the above form.*

Next we investigate the relation between conjugacy classes of Cartan subgroups in G and G_+^n . We wish to understand the distinct conjugacy classes of G_+^n in terms of conjugacy classes of G .

Let $N(T)$ denote the normalizer of T in G and $N_+(T_+)$ the normalizer of T_+ in G_+^n . Also let $W(T)$ denote the Weyl group of T and $W_+(T_+)$ the Weyl group of T_+ in G_+^n . We use $\Phi(G)$ to denote the set of distinct conjugacy classes

of Cartan subgroups in G and denote its general element by $[T]$ (the conjugacy class obtained from T). Similarly, $\Phi(G_+^n)$ denotes the set of distinct conjugacy classes of Cartan subgroups of G_+^n and R denotes its general element.

Because of the above lemma, any conjugacy class of Cartan subgroups in G_+^n comes from a conjugacy class in G . We need to understand how many distinct conjugacy classes can occur in G_+^n from a single conjugacy class in G .

Lemma 5.2.2 *Let T be a Cartan subgroup of G . Let S consist of elements of G which are distinct coset representatives of $G_+^n \backslash G (\simeq \mathbb{F}^* / \mathbb{F}^{*n})$. For any $g \in G$, let $T^g = g^{-1}Tg$. Then the set*

$$\{T_+^g | g \in S\}$$

contains all representatives of all conjugacy classes of Cartan subgroups of G_+^n corresponding to the conjugacy class of T . Also the number of distinct conjugacy classes in the above set is $|G_+^n N(T) \backslash G|$.

Proof. It is obvious that two cartan subgroups T and T' can be conjugate in G but T_+ and T'_+ may not be conjugate inside G_+^n . If they are not then they must be conjugate by some element of $G_+^n \backslash G$ since the element conjugating T and T' must be in some non-trivial coset G_+^n in G . The number of repetitions will be $|G_+^n \backslash N(T)G_+^n|$ because conjugating by any element of T will give back T_+ while conjugating by G_+^n will keep us in the same conjugacy class in G_+^n . This proves the lemma. ■

Definition 5.2.2 *Let Y_T be the coset representatives, h_w , of $G_+^n \backslash G$ such that all $[T_+^{h_w}]$ are distinct conjugacy classes i.e distinct elements of $\Phi(G_+^n)$.*

From the above lemma $|Y_T| = |G_+^n N(T) \backslash G|$.

We state a preliminary result which will be needed in section 5.4.

Lemma 5.2.3 *Let $\Phi(G), \Phi(G_+^n), Y_T$ be as above. Then: $\{[R] \in \Phi(G_+^n)\} = \cup_{[T] \in \Phi(G)} \{[T_+^{h_w}] | h_w \in Y_T\}$.*

The proof of the above lemma follows from lemmas 5.2.1 and 5.2.2 and the definition of Y_T .

The lemma below relates $N_+(T_+)$ to $N(T)$.

Lemma 5.2.4 $N_+(T_+) = N(T) \cap G_+^n$.

Proof. Let $\gamma \in T_+$ be a regular element. Then γ is regular in G . Then $T = Cent_G(\gamma)$. Let $h_w \in N_+(T_+)$ and let $t \in T$. Let $\gamma' = h_w^{-1}\gamma h_w$.

We evaluate $(h_w t h_w^{-1})(\gamma)(h_w t^{-1} h_w^{-1})$. This equals $h_w t \gamma' t^{-1} h_w^{-1}$. Since $h_w \in N_+(T_+), \gamma' \in T_+ \subseteq T$, t commutes with γ' and we obtain that the above expression equals $h_w \gamma' h_w^{-1} = \gamma$. This proves that $h_w t h_w^{-1} \in T$ and hence $h_w \in N(T)$. This proves $N_+(T_+) \subseteq N(T) \cap G_+^n$. The other implication is obvious. ■

Let ι be the injective map:

$$\iota : N_+(T_+) \rightarrow N(T).$$

It is easy to see that ι induces an injective map which also we denote by ι :

$$\iota : W_+(T_+) \rightarrow W(T)$$

Thus $W_+(T_+)$ can be realized as a subgroup of $W(T)$. It is a normal subgroup because both T and $N_+(T_+)$ are normal subgroups of $N(T)$.

We now find an expression regarding the order of $W_+(T_+) \backslash W(T)$. This will be very useful in the next section.

Lemma 5.2.5 $|W_+(T_+) \backslash W(T)| = |TG_+^n \backslash N(T)G_+^n|$

Proof. We have:

$$\begin{aligned}
W_+(T_+) \setminus W(T) &\simeq \\
T(G_+^n \cap N(T)) \setminus N(T) &= \\
TG_+^n \cap N(T) \setminus N(T) &\simeq \\
TG_+^n \setminus N(T)G_+^n &
\end{aligned}$$

because $T(G_+^n \cap N(T)) = TG_+^n \cap N(T)$. This is true because $T \subseteq N(T)$. The last step follows by an isomorphism theorem of groups and because G_+^n is a normal subgroup of G . This proves the lemma. ■

We note from the last line of the proof above that

$$W_+(T_+) \setminus W(T) \simeq TG_1 \setminus N(T)G_1$$

since $Z(G) \subset T$ and $G_+^n = G_1Z(G)$. All results in this section are valid for G_1 in place of G_+^n because of the fact that $G_+^n = G_1Z(G)$.

Now we consider the non-linear groups and state similar results.

Let $p : \tilde{G} \rightarrow G$ be the map defined by $p(g, \zeta) = g$. We define the Cartan subgroups of G to be those obtained by pulling back the Cartan subgroups of G by means of p . Let $\tilde{T} = p^{-1}(T)$. We define $\tilde{T}_+, N(\tilde{T}), N_+(\tilde{T}_+), W(\tilde{T}), W_+(\tilde{T}_+)$ in exactly the same way and it turns out that each of the above groups is a pullback of the corresponding linear group via the map p . We have $|W(\tilde{T})| = |W(T)|$ and $|W_+(\tilde{T}_+)| = |W_+(T_+)|$ and lemma 5.2.5 holds in the non-linear setting. This is what we will be using in section 5.4. The definition of $\Phi(\tilde{G})$ and $\Phi(\tilde{G}_+^n)$ is exactly analogous to that of $\Phi(G)$ and $\Phi(G_+^n)$. We define, $Y_{\tilde{T}}$ to be those coset representatives of $\tilde{G}_+^n \setminus \tilde{G}$ such that $[\tilde{T}_+^w]$ are all distinct elements of $\Phi(\tilde{G}_+^n)$.

5.3 Matching of orbital integrals

Suppose $\tilde{f} \in C_c^\infty(\widetilde{G}_1)_\alpha$ and $\nu|_{\mu_n} = \alpha$. We relate the orbital integrals of $A(\tilde{f}, \nu)$ over \widetilde{G}_+^n and that of $B(\tilde{f}, \nu)$ over \widetilde{G} . Earlier we had obtained a relationship between $\Theta_{L_o(\pi, \nu)}(A(\tilde{f}, \nu))$ and $\Theta_{L(\pi)}(B(\tilde{f}, \nu))$. The orbital integral relation is dual to this relation. We use this in the next section and apply the Weyl Integration Formula to obtain a relation between the characters of $L(\pi, \nu)$ and $L(\pi)$. Let γ be any regular, semi-simple element of G_1 and let $\tilde{\gamma} = (\gamma, 1)^n u(\gamma)$. This is very similar to the work of Kazhdan and Kazhdan-Flicker where exactly the same approach was followed to obtain a relation between the characters of π and $L(\pi)$.

Denote the orbital integral of a function f over the group G at a regular, semi-simple element $\gamma \in G$ by $F_G(\gamma, f)$ i.e

$$F_G(\gamma, f) = \Delta(\gamma) \int_{G(\gamma) \backslash G} f(g^{-1} \gamma g) dg$$

where $G(\gamma)$ is the centralizer of γ in G and dg denotes the right-invariant quotient measure on the homogeneous space $G(\gamma) \backslash G$. How this is chosen will be specified when it is being used for certain calculations.

We will consider $F_{\widetilde{G}}(\tilde{\gamma}, B(\tilde{f}, \nu))$. Let \tilde{T} be a Cartan subgroup of \widetilde{G} (which means it is the pullback of a Cartan subgroup T of G). We assume that $\tilde{\gamma} \in \tilde{T}$ be a regular element (i.e $(\gamma, 1)^n u(\gamma)$ is regular). We also assume that $\tilde{\gamma} \in \widetilde{G}_1$ because we will be considering orbital integrals of $\tilde{\gamma}$ with respect to three different groups: $\widetilde{G}, \widetilde{G}_+^n, \widetilde{G}_1$.

As before, $\nu \in \widehat{\mathbb{F}^*}$ such that $\nu|_{\mu_n} = \alpha$. By these conventions, we have:

$$F_{\widetilde{G}}(\tilde{\gamma}, B(\tilde{f}, \nu)) = \Delta(\tilde{\gamma}) \int_{\widetilde{G}(\tilde{\gamma}) \backslash \widetilde{G}} B(\tilde{f}, \nu)(g^{-1} \tilde{\gamma} g) dg.$$

where $\widetilde{G}(\tilde{\gamma})$ is the centralizer of $\tilde{\gamma}$ in \widetilde{G} and dg is the right-invariant quotient

measure on the homogeneous space $\tilde{G}(\tilde{\gamma})\backslash\tilde{G}$ with normalizations to be specified later.

We state one immediate property of $F_{\tilde{G}}(\tilde{\gamma}, B(\tilde{f}, \nu))$:

Lemma 5.3.1 *Let $\delta \in \tilde{T}$ such that $\delta \notin Z(\tilde{T})$. Then $F_{\tilde{G}}(\delta, B(\tilde{f}, \nu))$ vanishes for every $\tilde{f} \in C_c^\infty(\tilde{G}_1)_\alpha$ for any α .*

Proof. The proof follows because the orbital integrals are conjugation invariant functions. Since T is commutative and \tilde{T} is not (in our case $Z(\tilde{T}) = \tilde{T}^n$, section 3.4.1) we can find $\delta_o \in \tilde{T}$ such that $\delta_o^{-1}\delta\delta_o = \zeta\delta$ where $\zeta \neq 1$. The lemma follows because \tilde{f} is a genuine function. ■

Let us denote the space of coset representatives of $\tilde{G}(\tilde{\gamma})\tilde{G}_+^n\backslash\tilde{G}$ by $W_{\tilde{\gamma}}$. We use the isomorphism of the homogeneous spaces

$$(\tilde{G}_+^n \cap \tilde{G}(\tilde{\gamma})\backslash\tilde{G}_+^n)\backslash(\tilde{G}(\tilde{\gamma})\backslash\tilde{G}) \simeq \tilde{G}(\tilde{\gamma})\tilde{G}_+^n\backslash\tilde{G}$$

to obtain

$$F_{\tilde{G}}(\tilde{\gamma}, B(\tilde{f}, \nu)) = \Delta(\tilde{\gamma}) \sum_{h \in W_{\tilde{\gamma}}} \int_{\tilde{G}_+^n \cap \tilde{G}(\tilde{\gamma})\backslash\tilde{G}_+^n} A(\tilde{f}, \nu)(h^{-1}g^{-1}\tilde{\gamma}gh) dg.$$

We recall here that $A(\tilde{f}, \nu)$ is just the restriction of $B(\tilde{f}, \nu)$ from \tilde{G} to \tilde{G}_+^n . A word about notation. The elements h are really coset representatives. They are actually elements of \tilde{G} but we make a choice for the calculations. Because of right-invariance of dg the choice of h (as coset representatives) does not matter and the calculations following this do not depend on any particular choice.

We would like to make the summation run over some set independent of $\tilde{\gamma}$. Since dg is right-invariant, this can be achieved by letting h run over $\tilde{G}_+^n\backslash\tilde{G}(\simeq \mathbb{F}^*/\mathbb{F}^{*n})$ and comparing the two sides.

Let $S(g) = \tilde{G}_+^n \backslash \tilde{G}_+^n \tilde{G}(g)$ for any $g \in \tilde{G}$. We have:

$$F_{\tilde{G}}(\tilde{\gamma}, B(\tilde{f}, \nu)) = \frac{1}{|S(\tilde{\gamma})|} \Delta(\tilde{\gamma}) \sum_{h \in \tilde{G}_+^n \backslash \tilde{G}} \int_{\tilde{G}_+^n \cap \tilde{G}(\tilde{\gamma}) \backslash \tilde{G}_+^n} A(\tilde{f}, \nu)(h^{-1}g^{-1}\tilde{\gamma}gh)dg$$

Using the above equation, we deduce that for any $z_y \in \tilde{Z}$ (i.e $p(z_y) = yI$ for some $y \in \mathbb{F}^*$):

$$F_{\tilde{G}}(z_y \tilde{\gamma}, B(\tilde{f}, \nu)) = \frac{1}{|S(z_y \tilde{\gamma})|} \Delta(\tilde{\gamma}) \sum_{h \in \tilde{G}_+^n \backslash \tilde{G}} \int_{\tilde{G}_+^n \cap \tilde{G}(z_y \tilde{\gamma}) \backslash \tilde{G}_+^n} A(\tilde{f}, \nu)(h^{-1}g^{-1}z_y \tilde{\gamma}gh)dg.$$

Since \tilde{G}_+^n commutes with \tilde{Z} , we have $g^{-1}z_y = z_y g^{-1}$.

Also $h^{-1}z_y = z_y h^{-1}(y, \det(h))$. We use these facts along with the transformation properties of $A(\tilde{f}, \nu)$ to obtain:

Lemma 5.3.2 *Let $z_y \in \tilde{Z}$. Then*

$$\begin{aligned} & F_{\tilde{G}}(z_y \tilde{\gamma}, B(\tilde{f}, \nu)) \\ &= \frac{1}{|S(z_y \tilde{\gamma})|} \Delta(\tilde{\gamma}) \sum_{h \in \tilde{G}_+^n \backslash \tilde{G}} (y, \det(h)) \chi_\nu^{-1}(z_y) \\ & \quad \int_{\tilde{G}_+^n \cap \tilde{G}(z_y \tilde{\gamma}) \backslash \tilde{G}_+^n} A(\tilde{f}, \nu)(h^{-1}g^{-1}\tilde{\gamma}gh)dg \end{aligned}$$

Using the fact $\tilde{G}_+^n \cap \tilde{G}(z_y \tilde{\gamma}) = \tilde{G}_+^n \cap \tilde{G}(\tilde{\gamma})$ and taking χ_ν to the other side, we obtain:

$$\begin{aligned}
& \chi_\nu(z_y) F_{\tilde{G}}(z_y \tilde{\gamma}, B(\tilde{f}, \nu)) = \\
& \frac{1}{|S(z_y \tilde{\gamma})|} \Delta(\tilde{\gamma}) \sum_{h \in \tilde{G}_+^n \setminus \tilde{G}} (y, \det(h)) \int_{\tilde{G}_+^n \cap \tilde{G}(z_y \tilde{\gamma}) \setminus \tilde{G}_+^n} A(\tilde{f}, \nu)(h^{-1} g^{-1} \tilde{\gamma} g h) dg = \\
& \frac{1}{|S(z_y \tilde{\gamma})|} \sum_{h \in \tilde{G}_+^n \setminus \tilde{G}} (y, \det(h)) F_{\tilde{G}_+^n}^h(\tilde{\gamma}, A(\tilde{f}, \nu))
\end{aligned}$$

where we use the notation:

$$\begin{aligned}
& F_{\tilde{G}_+^n}^h(\tilde{\gamma}, A(\tilde{f}, \nu)) = \\
& \Delta(\tilde{\gamma}) \int_{\tilde{G}_+^n \cap \tilde{G}(\tilde{\gamma}) \setminus \tilde{G}_+^n} A(\tilde{f}, \nu)(h^{-1} g^{-1} \tilde{\gamma} g h) dg
\end{aligned}$$

The above relation and the fact that $F_{\tilde{G}_+^n}(\tilde{\gamma}, A(\tilde{f}, \nu)) = F_{\tilde{G}_1}(\tilde{\gamma}, \tilde{f})$ (see comments towards the end of this section) imply that $\chi_\nu(z_y) F_{\tilde{G}}(z_y \tilde{\gamma}, B(\tilde{f}, \nu))$ is independent of the choice of the extension ν of α . Furthermore z_y can be assumed to be coming from $\tilde{Z}^n \setminus \tilde{Z} (\simeq \mathbb{F}^* / \mathbb{F}^{*n})$. This can be seen by observing either side of the above lemma. We will sometimes say $y \in \mathbb{F}^* / \mathbb{F}^{*n}$ or just y also instead of $z_y \in \tilde{Z}^n \setminus \tilde{Z}$ when there is no chance of confusion.

Since $h \in \tilde{G}_+^n \setminus \tilde{G}$ and $\tilde{G}_+^n \setminus \tilde{G} \simeq \mathbb{F}^* / \mathbb{F}^{*n}$ via the \det (the determinant) map, we will use h_x instead of h from now on where the subscript x implies that h_x has determinant $x \in \mathbb{F}^* / \mathbb{F}^{*n}$.

Now $F_{\tilde{G}}(\tilde{\gamma}, B(\tilde{f}, \nu))$ vanishes outside $Z(\tilde{T}) = \tilde{T}^n$. If we assume $\tilde{\gamma} \in \tilde{T}^n$, then we have $\tilde{G}(\tilde{\gamma}) = \tilde{T}$ (\tilde{T} commutes with $\tilde{\gamma}$ because \tilde{T}^n is the center of \tilde{T} and the centralizer of $p(\tilde{\gamma})$ is T in the linear group) and $F_{\tilde{G}}(z_y \tilde{\gamma}, B(\tilde{f}, \nu))$ vanishes for $z_y \notin \tilde{Z}^n \setminus \tilde{T}^n \cap \tilde{Z}$ (see lemma 5.3.1) Let $k_1 = |\tilde{G}_+^n \setminus \tilde{G}_+^n \tilde{T}|$, we have:

$$\begin{aligned} & \chi_\nu(z_y)F_{\tilde{G}}(z_y\tilde{\gamma}, B(\tilde{f}, \nu)) = \\ & \frac{1}{k_1} \sum_{h_x \in \tilde{G}_+^n \setminus \tilde{G}} (y, x)F_{\tilde{G}_+^n}^{h_x}(\tilde{\gamma}, A(\tilde{f}, \nu)) \end{aligned}$$

for $z_y \in \tilde{Z}^n \setminus \tilde{T}^n \cap \tilde{Z}$. For remaining $z_y \in \tilde{Z}^n \setminus \tilde{Z} (\simeq \mathbb{F}^*/\mathbb{F}^{*n})$, we have

$$0 = \frac{1}{|S(z_y\tilde{\gamma})|} \sum_{h_x \in \tilde{G}_+^n \setminus \tilde{G}} (y, x)F_{\tilde{G}_+^n}^{h_x}(\tilde{\gamma}, A(\tilde{f}, \nu))$$

Since the LHS is 0, we can replace $|S(z_y\tilde{\gamma})|$ by k_1 . Thus for every $y \in \mathbb{F}^*/\mathbb{F}^{*n}$ (really we mean $z_y \in \tilde{Z}^n \setminus \tilde{Z}$), we have:

Lemma 5.3.3

$$\begin{aligned} & \chi_\nu(z_y)F_{\tilde{G}}(z_y\tilde{\gamma}, B(\tilde{f}, \nu)) = \\ & \frac{1}{k_1} \sum_{h_x \in \tilde{G}_+^n \setminus \tilde{G}} (y, x)F_{\tilde{G}_+^n}^{h_x}(\tilde{\gamma}, A(\tilde{f}, \nu)) \end{aligned}$$

Taking sum over $y \in \mathbb{F}^*/\mathbb{F}^{*n}$:

$$\begin{aligned} & \sum_y \chi_\nu(z_y)F_{\tilde{G}}(z_y\tilde{\gamma}, B(\tilde{f}, \nu)) = \\ & \frac{1}{k_1} \sum_{h_x} \sum_y (y, x)F_{\tilde{G}_+^n}^{h_x}(\tilde{\gamma}, A(\tilde{f}, \nu)) \end{aligned}$$

By orthogonality of characters, we have the theorem:

Theorem 5.3.1 *With above notations:*

$$F_{\tilde{G}_+^n}(\tilde{\gamma}, A(\tilde{f}, \nu)) = \frac{k_1}{|\mathbb{F}^*/\mathbb{F}^{*n}|} \sum_{y \in \mathbb{F}^*/\mathbb{F}^{*n}} \chi_\nu(z_y)F_{\tilde{G}}(z_y\tilde{\gamma}, B(\tilde{f}, \nu))$$

We state another version of lemma 5.3.3 which we are going to use in the next section. Let us choose representatives of $\tilde{G}_+^n \backslash \tilde{G}$ and fix them. As before, we still denote them by h_x . We might sometimes use the notation $h_x \in \mathbb{F}^* / \mathbb{F}^{*n}$. (Since $\tilde{G}_+^n \backslash \tilde{G} \simeq \mathbb{F}^* / \mathbb{F}^{*n}$ via the determinant map. This has been discussed in section 5.2.) By conjugation by h_x , we can define an isomorphism:

$$\theta_{h_x} : \tilde{G}(\tilde{\gamma}) \rightarrow \tilde{G}(\tilde{\gamma})^{h_x}$$

Therefore θ_{h_x} can also be considered as an isomorphism between the groups $\tilde{G}_+^n \tilde{G}(\tilde{\gamma}) \rightarrow \tilde{G}_+^n \tilde{G}(\tilde{\gamma})^{h_x}$ and hence also between the homogeneous spaces $\tilde{G}(\tilde{\gamma}) \backslash \tilde{G}_+^n \tilde{G}(\tilde{\gamma}) \rightarrow \tilde{G}(\tilde{\gamma})^{h_x} \backslash \tilde{G}_+^n \tilde{G}(\tilde{\gamma})^{h_x}$. We also note that as spaces $\tilde{G}(\tilde{\gamma}) \backslash \tilde{G}_+^n \tilde{G}(\tilde{\gamma}) \simeq \tilde{G}(\tilde{\gamma}) \cap \tilde{G}_+^n \backslash \tilde{G}_+^n$ and similarly for $\tilde{G}(\tilde{\gamma})^{h_x}$. Let dg^{h_x} be the measure on $\tilde{G}(\tilde{\gamma})^{h_x} \cap \tilde{G}_+^n \backslash \tilde{G}_+^n$ compatible with dg (we had mentioned in the beginning of this section regarding the how we would choose the measure) in the sense that:

$$\begin{aligned} \int_{\tilde{G}_+^n \cap \tilde{G}(\tilde{\gamma}) \backslash \tilde{G}_+^n} A(\tilde{f}, \nu)(h_x^{-1} g^{-1} \tilde{\gamma} g h_x) dg &= \\ \int_{\tilde{G}_+^n \cap \tilde{G}(\tilde{\gamma})^{h_x} \backslash \tilde{G}_+^n} A(\tilde{f}, \nu)(g^{-1} \tilde{\gamma}^{h_x} g) dg^{h_x} & \end{aligned}$$

which is the same as saying:

$$F_{\tilde{G}_+^n}^{h_x}(\tilde{\gamma}, A(\tilde{f}, \nu)) = F_{\tilde{G}_+^n}(\tilde{\gamma}^{h_x}, A(\tilde{f}, \nu))$$

Thus we have the following version of lemma 5.3.3

Lemma 5.3.4

$$\begin{aligned} \chi_\nu(z_y) F_{\tilde{G}}(z_y \tilde{\gamma}, B(\tilde{f}, \nu)) &= \\ \frac{1}{k_1} \sum_{h_x \in \tilde{G}_+^n \backslash \tilde{G}} (y, x) F_{\tilde{G}_+^n}(\tilde{\gamma}^{h_x}, A(\tilde{f}, \nu)) & \end{aligned}$$

Finally we say a few words as to how the orbital integral of a function $\tilde{f} \in C_c^\infty(\tilde{G}_1)_\alpha$ over \tilde{G}_1 is related to the orbital integral of $A(\tilde{f}, \nu)$ over \tilde{G}_+^n . In order to get the orbital integral over \tilde{G}_1 , we use the bijection of sets:

$$(\tilde{G}(\tilde{\gamma}) \cap \tilde{G}_+^n) \backslash \tilde{G}_+^n \simeq (\tilde{G}(\tilde{\gamma}) \cap \tilde{G}_1) \backslash \tilde{G}_1.$$

to obtain :

$$F_{\tilde{G}_1}(\tilde{\gamma}, \tilde{f}) = F_{\tilde{G}_+^n}(\tilde{\gamma}, A(\tilde{f}, \nu))$$

In the above equality, the measure on the quotient space on the left hand side is normalized according to the isomorphism between the two quotient spaces.

We will prove in the the next section that lemma 5.3.4 and the Weyl Integration formula can be used to obtain a relation between $\Theta_{L(\pi, \nu)}$ and $\Theta_{L(\pi)}$ as functions of elements of respective groups. We had obtained this relation by studying the restriction properties of $L(\pi)$ in the previous chapter.

5.4 The Character formula: Another proof.

Let π, ν be as before and let $\alpha = \nu|_{\mu_n} =$. Let $\tilde{f} \in C_c^\infty(\tilde{G}_1)_\alpha$. Given a subgroup H of G , \tilde{H} denote the pullback of H in \tilde{G} through the map $p : \tilde{G} \rightarrow G$. We have the following relations from the previous sections in this chapter (lemmas 5.1.1, 5.3.4):

$$\Theta_{L(\pi, \nu)}(A(\tilde{f}, \nu)) = \frac{1}{|\mathbb{F}^*/\mathbb{F}^{*n}|} \Theta_{L(\pi)}(B(\tilde{f}, \nu)) \quad (5.2)$$

$$F_{\tilde{G}}(z_y \tilde{\gamma}, B(\tilde{f}, \nu)) = \frac{1}{k_1} \chi_\nu^{-1}(z_y) \sum_{h_x \in \tilde{G}_+^n \backslash \tilde{G}} (y, x) F_{\tilde{G}_+^n}(\tilde{\gamma}^{h_x}, A(\tilde{f}, \nu)) \quad (5.3)$$

We wish to use the above relations to obtain a relation between $\Theta_{L(\pi, \nu)}$ and $\Theta_{L(\pi)}$ as characters evaluated on elements of respective groups. We can use this relation and the one between $\Theta_{L(\pi)}$ and Θ_π (from the work of K-F) to obtain a relation between $\Theta_{L(\pi, \nu)}$ and $\Theta_{\pi\nu^{-1}}$.

Using the Weyl Integration formula, the first relation gives:

$$\begin{aligned} & \sum_{[T] \in \Phi(G_+^n)} |W(\tilde{T})|^{-1} \int_{\tilde{Z} \backslash \tilde{T}} \Delta_{\Theta_{L_o(\pi, \nu)}(\tilde{\gamma})} F_{\tilde{G}_+^n}(\tilde{\gamma}, A(\tilde{f}, \nu)) d\tilde{\gamma} = \\ & \frac{1}{|\mathbb{F}^*/\mathbb{F}^{*n}|} \sum_{[T] \in \Phi(G)} |W(\tilde{T})|^{-1} \int_{\tilde{Z}^n \backslash \tilde{T}} \Delta_{\Theta_{L(\pi)}(t)} F_{\tilde{G}}(t, B(\tilde{f}, \nu)) dt \end{aligned} \quad (5.4)$$

where we are assuming that if \tilde{T} is a maximal torus in \tilde{G} . We choose measures $d\tilde{\gamma}$ on $\tilde{Z} \backslash \tilde{T}$ and dt on $\tilde{Z}^n \backslash \tilde{T}$ so that above equation is valid.

Let us define a function on the regular elements of \tilde{G}_+^n :

$$l^*(\pi, \nu)(\tilde{\gamma}) = \frac{1}{|\mathbb{F}^*/\mathbb{F}^{*n}|} \sum_{y \in \mathbb{F}^*/\mathbb{F}^{*n}} \chi_\nu^{-1}(z_y) \Theta_{L(\pi)}(z_y \tilde{\gamma})$$

We denote by $\Theta_{l^*(\pi, \nu)}$ to denote the distribution obtained on $C_c^\infty(G_+^n)$ by integrating against $l^*(\pi, \nu)$. We consider the sum

$$\sum_{h_w} \int_{\tilde{Z} \backslash \tilde{T}^{h_w} \cap \tilde{G}_+^n} \Delta(\tilde{\gamma}) l^*(\pi, \nu)(\tilde{\gamma}) F_{\tilde{G}_+^n}(\tilde{\gamma}, A(\tilde{f}, \nu)) d\tilde{\gamma} \quad (5.5)$$

where h_w are coset representatives of $\tilde{G}_+^n \backslash \tilde{G} \simeq \mathbb{F}^*/\mathbb{F}^{*n}$ via the determinant map. In h_w , w denotes the determinant of h_w . We will sometimes also use the notation $h_w \in \mathbb{F}^*/\mathbb{F}^{*n}$ though we will mean the same as above. We denote $h_w^{-1} \tilde{T} h_w$ by \tilde{T}^{h_w} . In the above summation, we would like the sum to run only over those h_w 's such that the all the tori, $\tilde{T}^{h_w} \cap \tilde{G}_+^n$, are not conjugate in \tilde{G}_+^n i.e run over Y_T . Later on we will divide by a constant (depending upon \tilde{T}) to provide a remedy to this situation. We consider the above sum because in chapter 3 we proved that

$$\Theta_{L_o(\pi, \nu)}(\tilde{\gamma}) = \frac{1}{|\mathbb{F}^*/\mathbb{F}^{*n}|} \sum_{y \in \mathbb{F}^*/\mathbb{F}^{*n}} \chi_\nu^{-1}(z_y) \Theta_{L(\pi)}(z_y \tilde{\gamma})$$

and now we are interested in obtaining the same result by a different method.

We want to prove:

Lemma 5.4.1 *Let $[T] \in \Phi(G)$. Then*

$$\begin{aligned} \sum_{h_w \in Y_T} |W_+(T_+)|^{-1} \int_{\tilde{Z} \backslash \tilde{T}^{h_w} \cap \tilde{G}_+^n} \Delta(\tilde{\gamma}) \\ l^*(\pi, \nu)(\tilde{\gamma}) F_{\tilde{G}_+^n}(\tilde{\gamma}, A(\tilde{f}, \nu)) d\tilde{\gamma} = \\ |W(T)|^{-1} \frac{1}{|\mathbb{F}^*/\mathbb{F}^{*n}|} \\ \int_{\tilde{Z}^n \backslash \tilde{T}^n} \Delta(\tilde{\gamma}) \Theta_{L(\pi)}(\tilde{\gamma}) F_{\tilde{G}}(\tilde{\gamma}, B(\tilde{f}, \nu)) d\tilde{\gamma} \end{aligned}$$

We will simplify the sum 5.5 and use the results of the previous sections in proving this lemma. Then we sum over all $[T] \in \Phi(G)$ and apply the Weyl integration formula to obtain (this will be explained in detail later on):

Lemma 5.4.2

$$\Theta_{l^*(\pi, \nu)}(A(\tilde{f}, \nu)) = \frac{1}{|\mathbb{F}^*/\mathbb{F}^{*n}|} \Theta_{L(\pi)}(B(\tilde{f}, \nu))$$

for every $\tilde{f} \in C_c^\infty(\tilde{G}_1)_\alpha$.

After some more calculations, this will enable us to establish $l^*(\pi, \nu) = L_0(\pi, \nu)$. Now we proceed to consider the sum 5.5 and simplify it using the results of previous sections.

We note that $F_{\tilde{G}_+^n}(\tilde{\gamma}, A(\tilde{f}, \nu))$ vanishes outside $\tilde{Z} \tilde{T}^{h_w}$. Also we observe that $\tilde{Z} \backslash \tilde{Z} \tilde{T}^{h_w} = \tilde{Z} \cap \tilde{T}^{h_w} \backslash \tilde{T}^{h_w}$. The last group is a quotient of $\tilde{Z}^n \backslash \tilde{T}^{h_w}$ by the subgroup $\tilde{Z}^n \backslash \tilde{Z} \cap \tilde{T}^{h_w}$. Hence we replace $\tilde{Z}^n \backslash \tilde{T}^{h_w}$ for $\tilde{Z} \backslash \tilde{T}^{h_w} \cap \tilde{G}_+^n$ as the set on which we are performing the integration.

We evaluate:

$$\int_{\tilde{Z}^n \backslash \tilde{T}^{h_w}} \Delta(\tilde{\gamma}) l^*(\pi, \nu)(\tilde{\gamma}) F_{\tilde{G}_+^n}(\tilde{\gamma}, A(\tilde{f}, \nu)) d\tilde{\gamma}$$

for any $h_w \in \mathbb{F}^*/\mathbb{F}^{*n}$. We start with $h_w = 1$ and then evaluate for general h_w by comparing it to the case when $h_w = 1$.

We can pull the summation outside and use theorem 5.3.1 and obtain:

$$\int_{\widetilde{Z}^n \setminus \widetilde{T}^n} \Delta(\widetilde{\gamma}) l^*(\pi, \nu)(\widetilde{\gamma}) F_{\widetilde{G}_+^n}(\widetilde{\gamma}, A(\widetilde{f}, \nu)) d\widetilde{\gamma} = \frac{k_1}{|\mathbb{F}^*/\mathbb{F}^{*n}|^2} \sum_{z_v, z_u \in \mathbb{F}^*/\mathbb{F}^{*n}} \int_{\widetilde{Z}^n \setminus \widetilde{T}^n} \chi_\nu^{-1}(z_u z_v^{-1}) \Delta(\widetilde{\gamma}) \Theta_{L(\pi)}(z_u \widetilde{\gamma}) F_{\widetilde{G}}(z_v \widetilde{\gamma}, B(\widetilde{f}, \nu)) d\widetilde{\gamma}$$

We note that both $\Theta_{L(\pi)}$ as well as $F_{\widetilde{G}}$ vanish outside \widetilde{T}^n and hence we can let z_u, z_v^{-1} run over $\widetilde{Z}^n \setminus \widetilde{Z} \cap \widetilde{T}^n$ which we denote by $U_{\widetilde{T}}$. We make a change of variables $z_v \widetilde{\gamma} \rightarrow \widetilde{\gamma}$ and then label $z_y = z_u z_v^{-1}$. Let us denote $|U_{\widetilde{T}}|$ by k_2 . We also use the fact that $d\widetilde{\gamma}$ is left-invariant Haar-measure and therefore:

We have a lemma

Lemma 5.4.3

$$\int_{\widetilde{Z}^n \setminus \widetilde{T}^n} \Delta(\widetilde{\gamma}) l^*(\pi, \nu)(\widetilde{\gamma}) F_{\widetilde{G}_+^n}(\widetilde{\gamma}, A(\widetilde{f}, \nu)) d\widetilde{\gamma} = \frac{k_1 k_2}{|\mathbb{F}^*/\mathbb{F}^{*n}|^2} \sum_{y \in U_{\widetilde{T}}} \chi_\nu^{-1}(z_y) \int_{\widetilde{Z}^n \setminus \widetilde{T}^n} \Delta(\widetilde{\gamma}) \Theta_{L(\pi)}(\widetilde{\gamma}) F_{\widetilde{G}}(z_y \widetilde{\gamma}, B(\widetilde{f}, \nu)) d\widetilde{\gamma}$$

We add here that we can also let the sum run over $y \in U_{\widetilde{T}}$ whenever convenient. We can do this because $F_{\widetilde{G}}$ vanishes outside \widetilde{T}^n .

We first evaluate

$$\int_{\widetilde{Z}^n \setminus \widetilde{T}^n} \Delta(\widetilde{\gamma}) \Theta_{L(\pi)}(\widetilde{\gamma}) F_{\widetilde{G}}(z_y \widetilde{\gamma}, B(\widetilde{f}, \nu)) d\widetilde{\gamma}$$

for any $z_y \in U_{\widetilde{T}}$.

We use equation 5.3 to obtain:

$$\int_{\widetilde{Z}^n \setminus \widetilde{T}^n} \Delta(\widetilde{\gamma}) \Theta_{L(\pi)}(\widetilde{\gamma}) F_{\widetilde{G}}(z_y \widetilde{\gamma}, B(\widetilde{f}, \nu)) d\widetilde{\gamma} = \frac{1}{k_1} \chi_\nu^{-1}(z_y) \sum_{h_x \in \widetilde{G}_+^n \setminus \widetilde{G}} (y, x) \int_{\widetilde{Z}^n \setminus \widetilde{T}^n} \Theta_{L(\pi)}(\widetilde{\gamma}) F_{\widetilde{G}_+^n}(\widetilde{\gamma}^{h_x}, A(\widetilde{f}, \nu)) d\widetilde{\gamma}$$

where $\tilde{\gamma}^{h_x} = h_x^{-1}\tilde{\gamma}h_x$.

The previous lemma holds for \tilde{T} replaced by \tilde{T}^{h_w} where h_w is any coset representative of $\tilde{G}_+^n \backslash \tilde{G}$. If we sum lemma 5.4.3 over all such \tilde{T}^{h_w} , we obtain:

$$\begin{aligned} & \sum_{h_w} \int_{\tilde{Z}^n \backslash \tilde{T}^{h_w}} \Delta(\tilde{\gamma}) l^*(\pi, \nu)(\tilde{\gamma}) F_{\tilde{G}_+^n}(\tilde{\gamma}, A(\tilde{f}, \nu)) d\tilde{\gamma} = \\ & \frac{k_1 k_2}{|\mathbb{F}^* / \mathbb{F}^{*n}|^2} \sum_{y \in U_{\tilde{T}}} \chi_\nu^{-1}(z_y) \sum_{h_w} \int_{\tilde{Z}^n \backslash \tilde{T}^{h_w}} \Delta(\tilde{\gamma}) \Theta_{L(\pi)}(\tilde{\gamma}) F_{\tilde{G}}(z_y \tilde{\gamma}, B(\tilde{f}, \nu)) d\tilde{\gamma} \end{aligned}$$

We are hence eventually interested in simplifying and evaluating:

$$\sum_{h_w \in \mathbb{F}^* / \mathbb{F}^{*n}} \int_{\tilde{Z}^n \backslash \tilde{T}^{h_w}} \sum_{y \in U_{\tilde{T}}} \Delta(\tilde{\gamma}) \Theta_{L(\pi)}(\tilde{\gamma}) F_{\tilde{G}}(z_y \tilde{\gamma}, B(\tilde{f}, \nu)) d\tilde{\gamma}$$

We have another lemma:

Lemma 5.4.4

$$\sum_{h_w \in \mathbb{F}^* / \mathbb{F}^{*n}} \int_{\tilde{Z}^n \backslash \tilde{T}^{h_w}} \Delta(\tilde{\gamma}) \Theta_{L(\pi)}(\tilde{\gamma}) F_{\tilde{G}}(z_y \tilde{\gamma}, B(\tilde{f}, \nu)) d\tilde{\gamma} = 0$$

for $y \neq 1$.

Proof. We have using equation 5.3

$$\begin{aligned} & \int_{\tilde{Z}^n \backslash \tilde{T}^n} \Delta(\tilde{\gamma}) \Theta_{L(\pi)}(\tilde{\gamma}) F_{\tilde{G}}(z_y \tilde{\gamma}, B(\tilde{f}, \nu)) d\tilde{\gamma} = \\ & \frac{1}{k_1} \chi_\nu^{-1}(z_y) \sum_{h_x \in \tilde{G}_+^n \backslash \tilde{G}} (y, x) \int_{\tilde{Z}^n \backslash \tilde{T}^n} \Theta_{L(\pi)}(\tilde{\gamma}) F_{\tilde{G}_+^n}(\tilde{\gamma}^{h_x}, A(\tilde{f}, \nu)) d\tilde{\gamma} \end{aligned}$$

where $\tilde{\gamma}^{h_x} = h_x^{-1}\tilde{\gamma}h_x$.

If we make a substitution $\tilde{g} = \tilde{\gamma}^{h_x}$, and use the fact that $\Theta_{L(\pi)}$ is conjugation invariant, we obtain

$$\frac{1}{k_1} \chi_\nu^{-1}(z_y) \sum_{h_x \in \tilde{G}_+^n \backslash \tilde{G}} (y, x) \int_{\tilde{Z}^n \backslash \tilde{T}^{h_x}} \Theta_{L(\pi)}(\tilde{g}) F_{\tilde{G}_+^n}(\tilde{g}, A(\tilde{f}, \nu)) d\tilde{g}^{h_x}$$

where $d\tilde{g}^{h_x}$ denotes the measure on $\widetilde{Z}^n \setminus \widetilde{T}^n{}^{h_x}$ which is compatible with $d\tilde{\gamma}$ in the sense that it is defined in a manner that the two integrals are equal.

Now we evaluate

$$\int_{\widetilde{Z}^n \setminus \widetilde{T}^n{}^{h_w}} \Delta(\tilde{\gamma}) \Theta_{L(\pi)}(\tilde{\gamma}) F_{\tilde{G}}(z_y \tilde{\gamma}, B(\tilde{f}, \nu)) d\tilde{\gamma}$$

Exactly the same analysis gives us that:

$$\begin{aligned} & \int_{\widetilde{Z}^n \setminus \widetilde{T}^n{}^{h_w}} \Delta(\tilde{\gamma})(z_y) \Theta_{L(\pi)}(\tilde{\gamma}) F_{\tilde{G}}(z_y \tilde{\gamma}, B(\tilde{f}, \nu)) d\tilde{\gamma} = \\ & \frac{1}{k_1} \chi_\nu^{-1}(z_y) \sum_{h_x \in \tilde{G}_+^n \setminus \tilde{G}} (y, x) \int_{\widetilde{Z}^n \setminus \widetilde{T}^n{}^{h_w h_x}} \Theta_{L(\pi)}(\tilde{g}) F_{\tilde{G}_+^n}(\tilde{g}, A(\tilde{f}, \nu)). \end{aligned}$$

Hence, by substituting $h_u = h_w h_x$, we see that

$$\begin{aligned} & \int_{\widetilde{Z}^n \setminus \widetilde{T}^n{}^{h_w}} \Delta(\tilde{\gamma}) \chi_\nu^{-1}(z_y) \Theta_{L(\pi)}(\tilde{\gamma}) F_{\tilde{G}}(z_y \tilde{\gamma}, B(\tilde{f}, \nu)) d\tilde{\gamma} = \\ & \frac{1}{k_1} \chi_\nu^{-1}(z_y) \sum_{h_u \in \mathbb{F}^* / \mathbb{F}^{*n}} (y, u w^{-1}) \int_{\widetilde{Z}^n \setminus \widetilde{T}^n{}^{h_u}} \Theta_{L(\pi)}(\tilde{g}) F_{\tilde{G}_+^n}(\tilde{g}, A(\tilde{f}, \nu)). \end{aligned}$$

We pull out (y, w^{-1}) from the above equation and comparing it to the expression obtained for $w = 1$, we have

$$\begin{aligned} & \int_{\widetilde{Z}^n \setminus \widetilde{T}^n{}^{h_w}} \Delta(\tilde{\gamma}) \Theta_{L(\pi)}(\tilde{\gamma}) F_{\tilde{G}}(z_y \tilde{\gamma}, B(\tilde{f}, \nu)) d\tilde{\gamma} = \\ & (y, w^{-1}) \int_{\widetilde{Z}^n \setminus \widetilde{T}^n} \Delta(\tilde{\gamma}) \Theta_{L(\pi)}(\tilde{\gamma}) F_{\tilde{G}}(z_y \tilde{\gamma}, B(\tilde{f}, \nu)) d\tilde{\gamma} \end{aligned}$$

If we now sum over $h_w \in \mathbb{F}^* / \mathbb{F}^{*n}$, we obtain

$$\begin{aligned} & \sum_{h_w} \int_{\widetilde{Z}^n \setminus \widetilde{T}^n{}^{h_w}} \Delta(\tilde{\gamma}) \Theta_{L(\pi)}(\tilde{\gamma}) F_{\tilde{G}}(z_y \tilde{\gamma}, A(\tilde{f}, \nu)) d\tilde{\gamma} = \\ & \sum_{h_w} (w^{-1}, y) \int_{\widetilde{Z}^n \setminus \widetilde{T}^n} \Delta(\tilde{\gamma}) \Theta_{L(\pi)}(\tilde{\gamma}) F_{\tilde{G}}(z_y \tilde{\gamma}, A(\tilde{f}, \nu)) d\tilde{\gamma} \end{aligned}$$

The RHS equals zero if $y \neq 1$.

This proves the lemma. ■

Thus, using lemma 5.4.3, we have:

$$\begin{aligned} \sum_{h_w} \int_{\widetilde{Z}^n \setminus \widetilde{T}^n{}^{h_w}} \Delta(\tilde{\gamma}) \frac{1}{|\mathbb{F}^*/\mathbb{F}^{*n}|} \left(\sum_{y \in \mathbb{F}^*/\mathbb{F}^{*n}} \chi_\nu^{-1}(z_y) \Theta_{L(\pi)}(z_y \tilde{\gamma}) \right) F_{\widetilde{G}_+^n}(\tilde{\gamma}, A(\tilde{f}, \nu)) d\tilde{\gamma} = \\ \frac{k_1 k_2}{|\mathbb{F}^*/\mathbb{F}^{*n}|} \int_{\widetilde{Z}^n \setminus \widetilde{T}^n} \Delta(\tilde{\gamma}) \Theta_{L(\pi)}(\tilde{\gamma}) F_{\widetilde{G}}(\tilde{\gamma}, B(\tilde{f}, \nu)) d\tilde{\gamma} \end{aligned}$$

Hence, we also get:

$$\begin{aligned} \sum_{h_w} \int_{\widetilde{Z} \setminus \widetilde{Z} \widetilde{T}^n{}^{h_w}} \Delta(\tilde{\gamma}) \frac{1}{|\mathbb{F}^*/\mathbb{F}^{*n}|} \left(\sum_{y \in \mathbb{F}^*/\mathbb{F}^{*n}} \chi_\nu^{-1}(z_y) \Theta_{L(\pi)}(z_y \tilde{\gamma}) \right) F_{\widetilde{G}_+^n}(\tilde{\gamma}, A(\tilde{f}, \nu)) d\tilde{\gamma} = \\ \frac{k_1}{|\mathbb{F}^*/\mathbb{F}^{*n}|} \int_{\widetilde{Z}^n \setminus \widetilde{T}^n} \Delta(\tilde{\gamma}) \Theta_{L(\pi)}(\tilde{\gamma}) F_{\widetilde{G}}(\tilde{\gamma}, A(\tilde{f}, \nu)) d\tilde{\gamma} \end{aligned}$$

because

$$\begin{aligned} \left(\sum_{y \in \mathbb{F}^*/\mathbb{F}^{*n}} \chi_\nu^{-1}(z_y) \Theta_{L(\pi)}(z_y \tilde{z} \tilde{\gamma}) \right) F_{\widetilde{G}_+^n}(\tilde{z} \tilde{\gamma}, A(\tilde{f}, \nu)) = \\ \left(\sum_{y \in \mathbb{F}^*/\mathbb{F}^{*n}} \chi_\nu^{-1}(z_y) \Theta_{L(\pi)}(z_y \tilde{\gamma}) \right) F_{\widetilde{G}_+^n}(\tilde{\gamma}, A(\tilde{f}, \nu)) \end{aligned}$$

for every $\tilde{z} \in \widetilde{Z}$ and

$$(\widetilde{Z}^n \setminus \widetilde{T}^n{}^{h_w} \cap \widetilde{Z}) \setminus (\widetilde{Z}^n \setminus \widetilde{T}^n{}^{h_w}) \simeq \widetilde{Z} \cap \widetilde{T}^n{}^{h_w} \setminus \widetilde{T}^n{}^{h_w} \simeq \widetilde{Z} \setminus \widetilde{Z} \widetilde{T}^n{}^{h_w}.$$

Also $|\widetilde{Z}^n \setminus \widetilde{Z} \cap \widetilde{T}^n{}^{h_w}| = |\widetilde{Z}^n \setminus \widetilde{Z} \cap \widetilde{T}^n| = k_2$ for every $h_w \in \mathbb{F}^*/\mathbb{F}^{*n}$.

In the Weyl Integration formula the sum ranges over distinct Cartan subgroups. We therefore write the above equation such that we include only those $h_w \in \mathbb{F}^*/\mathbb{F}^{*n}$ for which all $[T_+^{h_w}] \neq [T_+]$, i.e over $h_w \in Y_T$. (Recall notation from section 5.2. Also note that it is the same whether we talk of Cartan subgroups in G or in \widetilde{G} .) We are interested in non-conjugate Cartan sub-groups of \widetilde{G}_+^n corresponding to \widetilde{T} . Thus we obtain:

$$\begin{aligned}
& |\widetilde{G}_+^n \setminus N(\widetilde{T}) \widetilde{G}_+^n| \sum_{h_w} \int_{\widetilde{Z} \setminus \widetilde{Z} \widetilde{T}^{h_w}} \\
& \Delta(\widetilde{\gamma}) l^*(\pi, \nu)(\widetilde{\gamma}) F_{\widetilde{G}_+^n}(\widetilde{\gamma}, A(\widetilde{f}, \nu)) d\widetilde{\gamma} = \\
& \frac{k_1}{|\mathbb{F}^* / \mathbb{F}^{*n}|} \int_{\widetilde{Z}^n \setminus \widetilde{T}^n} \Delta(\widetilde{\gamma}) \Theta_{L(\pi)}(\widetilde{\gamma}) F_{\widetilde{G}}(\widetilde{\gamma}, B(\widetilde{f}, \nu)) d\widetilde{\gamma}
\end{aligned}$$

where now h_w runs over the set Y_T .

Now we multiply both sides with $|W_+(T_+)|^{-1}$ and take the constant $|\widetilde{G}_+^n \setminus W(\widetilde{T}) \widetilde{G}_+^n|^{-1}$ to the other side to obtain:

$$\begin{aligned}
& |W_+(T_+)|^{-1} \sum_{h_w \in Y_T} \int_{\widetilde{Z} \setminus \widetilde{Z} \widetilde{T}^{h_w}} \Delta(\widetilde{\gamma}) \\
& l^*(\pi, \nu)(\widetilde{\gamma}) F_{\widetilde{G}_+^n}(\widetilde{\gamma}, A(\widetilde{f}, \nu)) d\widetilde{\gamma} = \\
& |W_+(T_+)|^{-1} |\widetilde{G}_+^n \setminus N(\widetilde{T}) \widetilde{G}_+^n|^{-1} \frac{k_1}{|\mathbb{F}^* / \mathbb{F}^{*n}|} \\
& \int_{\widetilde{Z}^n \setminus \widetilde{T}^n} \Delta(\widetilde{\gamma}) \Theta_{L(\pi)}(\widetilde{\gamma}) F_{\widetilde{G}}(\widetilde{\gamma}, B(\widetilde{f}, \nu)) d\widetilde{\gamma}
\end{aligned}$$

This gives us, from a previous lemma (5.2.5):

$$\begin{aligned}
& \sum_{h_w \in Y_T} |W_+(T_+^{h_w})|^{-1} \int \Delta(\widetilde{\gamma}) \\
& l^*(\pi, \nu)(\widetilde{\gamma}) F_{\widetilde{G}_+^n}(\widetilde{\gamma}, A(\widetilde{f}, \nu)) d\widetilde{\gamma} = \\
& |W(T)|^{-1} \frac{1}{|\mathbb{F}^* / \mathbb{F}^{*n}|} \\
& \int_{\widetilde{Z}^n \setminus \widetilde{T}^n} \Delta(\widetilde{\gamma}) \Theta_{L(\pi)}(\widetilde{\gamma}) F_{\widetilde{G}}(\widetilde{\gamma}, A(\widetilde{f}, \nu)) d\widetilde{\gamma}
\end{aligned} \tag{5.6}$$

This proves the lemma 5.4.1. We recall the definition of the function $l^*(\pi, \nu)$ on \widetilde{G}_+^n . (defined earlier in this section)

$$l^*(\pi, \nu) = \frac{1}{|\mathbb{F}^* / \mathbb{F}^{*n}|} \sum_{y \in \mathbb{F}^* / \mathbb{F}^{*n}} \chi_\nu^{-1}(z_y) \Theta_{L(\pi)}(z_y \widetilde{\gamma}).$$

Note that $l^*(\pi, \nu)$ has the property that $l^*(\pi, \nu)(z_y \widetilde{\gamma}) = \chi_\nu(z_y) l^*(\pi, \nu)(\widetilde{\gamma})$ for every $z_y \in Z(\widetilde{G}_+^n) = \widetilde{Z}$. We will use $l_1^*(\pi, \nu)$ to denote the restriction on \widetilde{G}_1 and

$\Theta_{l_1^*(\pi, \nu)}$ to denote a distribution on $C_c^\infty(\widetilde{G}_1)$ by integrating a function against $l_1^*(\pi, \nu)$. We do not know as of now whether $l_1^*(\pi, \nu)$ is the character of some representation.

Now we consider equation 5.6 and sum it over all non-conjugate maximal tori of \widetilde{G} , i.e over all elements of $\Phi(G)$ (see notation in section 5.2.) and obtain

$$\begin{aligned} & \sum_{[T] \in \Phi(G)} \sum_{h_w \in Y_T} |W_+(T_+^{h_w})|^{-1} \int_{\widetilde{Z} \backslash \widetilde{Z} \widetilde{T}^{h_w}} \Delta(\widetilde{\gamma}) \\ & \quad l^*(\pi, \nu)(\widetilde{\gamma}) F_{\widetilde{G}_+^n}(\widetilde{\gamma}, A(\tilde{f}, \nu)) d\widetilde{\gamma} = \\ & \quad \frac{1}{|\mathbb{F}^*/\mathbb{F}^{*n}|} \sum_{[T] \in \Phi(G)} |W(T)|^{-1} \\ & \quad \int_{\widetilde{Z}^n \backslash \widetilde{T}^n} \Delta(\widetilde{\gamma}) \Theta_{L(\pi)}(\widetilde{\gamma}) F_{\widetilde{G}}(\widetilde{\gamma}, A(\tilde{f}, \nu)) d\widetilde{\gamma}. \end{aligned} \tag{5.7}$$

Now we use lemma 5.2.3 to obtain:

$$\begin{aligned} & \sum_{[T] \in \Phi(G_+^n)} |W_+(T_+^{h_w})|^{-1} \int_{\widetilde{Z} \backslash \widetilde{T}_+^{h_w}} \Delta(\widetilde{\gamma}) \\ & \quad l^*(\pi, \nu)(\widetilde{\gamma}) F_{\widetilde{G}_+^n}(\widetilde{\gamma}, A(\tilde{f}, \nu)) d\widetilde{\gamma} = \\ & \quad \frac{1}{|\mathbb{F}^*/\mathbb{F}^{*n}|} \sum_{[T] \in \Phi(G)} |W(T)|^{-1} \\ & \quad \int_{\widetilde{Z}^n \backslash \widetilde{T}^n} \Delta(\widetilde{\gamma}) \Theta_{L(\pi)}(\widetilde{\gamma}) F_{\widetilde{G}}(\widetilde{\gamma}, A(\tilde{f}, \nu)) d\widetilde{\gamma}. \end{aligned} \tag{5.8}$$

An application of the Weyl Integration Formula gives us:

$$\Theta_{l^*(\pi, \nu)}(A(\tilde{f}, \nu)) = \frac{1}{|\mathbb{F}^*/\mathbb{F}^{*n}|} \Theta_{L(\pi)}(B(\tilde{f})).$$

Using equation 5.2, we get

$$\Theta_{l^*(\pi, \nu)}(A(\tilde{f}, \nu)) = \Theta_{L_o(\pi, \nu)}(A(\tilde{f}, \nu)).$$

Because $A(\tilde{f}, \nu)$ is just the extension of \tilde{f} by χ_ν and $\widetilde{G}_+^n = \widetilde{G}_1 \widetilde{Z}$, we have

Lemma 5.4.5

$$\Theta_{l_1^*(\pi, \nu)}(\tilde{f}) = \Theta_{L(\pi, \nu)}(\tilde{f})$$

for every $\tilde{f} \in C_c^\infty(\widetilde{G}_1)_\alpha$. Recall that $\alpha \in \widehat{\mu}_n$ and $\nu|_{\mu_n} = \alpha$.

We recall that given any $\tilde{f} \in C_c^\infty(\widetilde{G}_1)$ we can write \tilde{f} as

$$\tilde{f} = \sum_{\delta \in \widehat{\mu}_n} \tilde{f}^\delta.$$

We also recall that $\Theta_{L(\pi, \nu)}(\tilde{f}^\delta) = 0$ unless $\delta = \alpha$. This was lemma 5.1.2.

Based on this motivation, we have a similar lemma:

Lemma 5.4.6 $\Theta_{l_1^*(\pi, \nu)}(\tilde{f}^\delta) = 0$ unless $\delta = \nu|_{\mu_n}$.

Proof. Assume $\delta \neq \alpha$. Thus there exists a $\zeta \in \mu_n$ such that $\delta(\zeta) \neq \alpha(\zeta)$. We have

$$\Theta_{l_1^*(\pi, \nu)}(\tilde{f}^\delta) = \int_{\widetilde{G}_1} \Theta_{l_1^*(\pi, \nu)}(\tilde{\gamma}) \tilde{f}^\delta(\tilde{\gamma}) d\tilde{\gamma}$$

We can substitute $\tilde{g} = \zeta \tilde{\gamma}$. The transformation properties of $l_1^*(\pi, \nu)$ and \tilde{f}^δ imply that

$$\Theta_{l_1^*(\pi, \nu)}(\tilde{f}^\delta) = \alpha \delta^{-1}(\zeta) \Theta_{l_1^*(\pi, \nu)}(\tilde{f}^\delta)$$

Since $\alpha \delta^{-1}(\zeta) \neq 0$, we have proved the lemma. ■

We have the theorem:

Theorem 5.4.1

$$\Theta_{L(\pi, \nu)} = l_1^*(\pi, \nu) = \frac{1}{|\mathbb{F}^*/\mathbb{F}^{*n}|} \sum_{y \in \mathbb{F}^*/\mathbb{F}^{*n}} \chi_\nu^{-1}(z_y) \Theta_{L(\pi)}(z_y \tilde{\gamma})$$

Proof. It follows from 5.4.5 and 5.4.6 that

$$\Theta_{l_1^*(\pi, \nu)}(\tilde{f}) = \Theta_{L(\pi, \nu)}(\tilde{f})$$

for every $\tilde{f} \in C_c^\infty(\widehat{G}_1)$. This implies that $\Theta_{L(\pi,\nu)}(\tilde{f})$ can be obtained by integration against the function $l^*(\pi, \nu)$. This proves the theorem. ■

We had derived this in chapter 3 by alternate means. We substitute the relation from section 4.5 (we have done this calculation in 4.5)

$$\Theta_{L(\pi)}(\tilde{\gamma}) = \sum_{h^n=p(\tilde{\gamma})} \Delta(h, \tilde{\gamma})\Theta_\pi(h)$$

into theorem 5.4.1 to obtain:

$$\Theta_{L(\pi,\nu)}(\tilde{\gamma}) = \alpha(\zeta) \sum_{h^n=\zeta p(\tilde{\gamma})} \Delta_\mu(h, \tilde{\gamma})\Theta_{\pi\nu^{-1}}(h).$$

Replacing ν by $\nu\nu_\delta$ and π by $\pi\nu_\delta$ for any $\delta \in \widehat{\mu}_n$, we obtain:

$$\Theta_{L(\pi\nu_\delta,\nu\nu_\delta)}(\tilde{\gamma}) = \alpha\delta(\zeta) \sum_{h^n=\zeta p(\tilde{\gamma})} \Delta_\mu(h, \tilde{\gamma})\Theta_{\pi\nu^{-1}}(h).$$

Taking a sum over $\delta \in \widehat{\mu}_n$, we finally have:

Theorem 5.4.2

$$\Theta_{L_{st}(\pi,\nu)}(\tilde{\gamma}) = \Delta(\tilde{\gamma}) \sum_{h^n=p(\tilde{\gamma})} \Delta_\mu(h, \tilde{\gamma})\Theta_{\pi\nu^{-1}}(h)$$

where, we recall from chapter 3 that:

$$\Theta_{L_{st}(\pi,\nu)} = \sum_{\delta \in \widehat{\mu}_n} L(\pi\nu_\delta, \nu\nu_\delta).$$

Chapter 6

Applications to Local Fields

Let \mathbb{F} be a p -adic field (i.e a finite extension of \mathbb{Q}_p). Let \mathbb{E}/\mathbb{F} be a finite extension of order n with the norm map $N : \mathbb{E}^* \rightarrow \mathbb{F}^*$. We also assume that $\mu_n \subseteq \mathbb{F}^*$ and that n is coprime to p . We will make the above assumptions throughout this chapter except where stated otherwise (at the end of the proof of theorem 6.0.5) We will determine the structure of the group $\mathbb{F}^*/N(\mathbb{E}^*)$ by analyzing orbital integrals. Classical proofs of this are algebraic in nature.

We have used the Norm residue symbol in all our analysis. Since this symbol comes out of Local Class Field Theory we will give an alternate explicit definition of a symbol satisfying the properties of the Norm residue symbol (section 2.1). We can then derive all the results of this thesis using this symbol and its properties (all that has been used regarding the Norm residue symbol are the properties stated in section 2.1) and then we would be having all our results without using the results from Local Class Field Theory.

In this chapter we use the results on orbital integrals from the previous chapter and prove the following theorem about Local Fields ([20], Pages 196, 172):

Theorem 6.0.3 *Let \mathbb{E}/\mathbb{F} be a finite Galois extension of degree n with $\text{Gal}(\mathbb{E}/\mathbb{F})$ denoting the Galois group. Let $\text{Gal}(\mathbb{E}/\mathbb{F})^{ab}$ be the abelianization of $\text{Gal}(\mathbb{E}/\mathbb{F})$.*

Assume $\mu_n \subset \mathbb{F}$. Then there exists a map $\sigma : \mathbb{F}^* \rightarrow \text{Gal}(\mathbb{E}/\mathbb{F})^{ab}$ such that the sequence

$$1 \rightarrow N(\mathbb{E}^*) \rightarrow \mathbb{F}^* \xrightarrow{\sigma} \text{Gal}(\mathbb{E}/\mathbb{F})^{ab} \rightarrow 1$$

is exact.

In order to prove theorem 6.0.3 we first prove the following two theorems:

Theorem 6.0.4 *Let \mathbb{E}/\mathbb{F} be a finite abelian extension of degree n . Assume $\mu_n \subset \mathbb{F}$. Then there exists a map $\sigma : \mathbb{F}^* \rightarrow \text{Gal}(\mathbb{E}/\mathbb{F})$ such that the sequence*

$$1 \rightarrow N(\mathbb{E}^*) \rightarrow \mathbb{F}^* \xrightarrow{\sigma} \text{Gal}(\mathbb{E}/\mathbb{F}) \rightarrow 1$$

is exact.

Theorem 6.0.5 *Let \mathbb{E}/\mathbb{F} be a finite extension of degree n . Assume $\mu_n \subset \mathbb{F}$. Let \mathbb{E}^1 be the maximal abelian extension of \mathbb{F} inside \mathbb{E} . Then*

$$N_{\mathbb{F}}^{\mathbb{E}}(\mathbb{E}^*) = N_{\mathbb{F}}^{\mathbb{E}^1}(\mathbb{E}^{1*}).$$

Theorem 6.0.3 follows from the above mentioned theorems because $\text{Gal}(\mathbb{E}/\mathbb{F})^{ab}$ is the Galois group of the maximal abelian sub-extension of \mathbb{F} inside \mathbb{E} .

The proof of both the above theorems follow from results of last chapter. We proceed to describe them. Assumptions and notation will be the same as in the last chapter. In particular we assume that we are considering the n -fold cover of $G = GL(n)$.

6.1 Explicit definition of a symbol on $\mathbb{F}^* \times \mathbb{F}^*$

Now we will give an alternate explicit definition of a symbol satisfying the properties of the Norm residue symbol (section 2.1). We can then derive all the results of this thesis using this symbol and its properties (all that has been used regarding the Norm residue symbol are the properties stated in section 2.1) and then we would be having all our results without using the results from Local Class Field Theory.

Now we define the symbol τ following [20] (Pg. 210). First we fix some notation. Let $R_{\mathbb{F}}$ be the ring of integers of \mathbb{F} and $R_{\mathbb{F}}^*$ the group of units in $R_{\mathbb{F}}$. We denote the residue field of \mathbb{F} by $k_{\mathbb{F}}$ and let $|k_{\mathbb{F}}| = q$. For any element $x \in R_{\mathbb{F}}$, let \bar{x} be its image in $k_{\mathbb{F}}$.

Let $a, b \in \mathbb{F}^*$. Let α be the valuation of a and β of b . We define a map $c : \mathbb{F}^* \times \mathbb{F}^* \rightarrow R_{\mathbb{F}}^*$ by

$$c(a, b) = (-1)^{\alpha\beta} \frac{a^\beta}{b^\alpha}.$$

Next we define $\tau : \mathbb{F}^* \times \mathbb{F}^* \rightarrow \mu_n$ by

$$\tau(a, b) = \bar{c}(a, b)^{\frac{q-1}{n}}$$

Proposition 6.1.1 *For $a, a', b \in \mathbb{F}^*$, τ satisfies the following properties:*

1. $\tau(a, b)\tau(a', b) = \tau(aa', b)$ and similar for the argument b .
2. $\tau(a, b)\tau(b, a) = 1$
3. $\tau(a, 1 - a) = 1$
4. $\{a : \tau(a, x) = 1 \forall x \in \mathbb{F}^*\} = \mathbb{F}^{*n}$

5. $\tau(a, Nb)_{\mathbb{F}} = \tau(a, b)_{\mathbb{E}}$ where \mathbb{E} is a finite field extension of \mathbb{F} where $b \in \mathbb{E}^*$ and $N : \mathbb{E}^* \rightarrow \mathbb{F}^*$ denotes the norm map.

We note that 4 gives us that $\tau : \mathbb{F}^*/\mathbb{F}^{*n} \times \mathbb{F}^*/\mathbb{F}^{*n} \rightarrow \mu_n$ is a perfect pairing.

Proof. Properties 1 and 2 can be easily verified using the definition of τ . We refer to [13], Pg. 98 for a proof of 3. To prove 4, we first assume that x is a unit and obtain that the valuation of a is a multiple of n . Next we assume x to be the uniformizer and obtain $a \in \mathbb{F}^{*n}$. We use Hensel's lemma ([20], II.2) for this. To prove 5, we prove it first for the cases when \mathbb{E}/\mathbb{F} is unramified, tamely totally ramified, and totally ramified with the degree of ramification a power of p . For the tamely totally ramified case we use the fact that for such extensions the uniformizer in \mathbb{E} satisfies an irreducible polynomial of the form $X^e - \pi = 0$ where e is the degree of ramification and π is a uniformizer in \mathbb{F} . The other two cases follow by definitions. Since we have $\mathbb{F} \subset \mathbb{L} \subset \mathbb{K} \subset \mathbb{E}$ with \mathbb{L}/\mathbb{F} unramified, \mathbb{K}/\mathbb{L} tamely totally ramified and \mathbb{E}/\mathbb{K} totally ramified with order a power of p ([11], chapter 2), this proves 5 for any extension \mathbb{E} . ■

We construct the covering group of G using τ instead of the standard Norm residue symbol. For the construction we refer to chapter 2. All the properties and calculations regarding the covering group can be formulated in terms of τ . For example we can quote the commutator formula (proposition 2.3.1) in terms of τ . Once we have that we can proceed to perform the analysis in chapters 3, 4, and 5 using τ .

6.2 Orbital Integrals

In this section we will assume that the construction of non-linear group in chapter 2 has been done using τ and all the results will be stated in terms of τ . We will mostly quote results from chapter 5 in that regard and also perform some analysis using orbital integrals in from chapter 5. We will perform all our analysis in terms of τ .

Let $\alpha \in \widehat{\mu}_n$. We recall that $Z(\widetilde{G}_1) = p^{-1}\{\zeta I | \zeta \in \mu_n\}$. Let $\tilde{f} \in C_c^\infty(\widetilde{G}_1)_\alpha$. Let $\nu \in \widehat{\mathbb{F}^*}$ such that $\nu|_{\mu_n} = \alpha$. From theorem 5.3.1 and the discussion at the end of section 5.3, we have:

Theorem 6.2.1

$$F_{\widetilde{G}_1}(\tilde{\gamma}, \tilde{f}) = \frac{k_1}{|\mathbb{F}^*/\mathbb{F}^{*n}|} \sum_{y \in \mathbb{F}^*/\mathbb{F}^{*n}} \chi_\nu(z_y) F_{\widetilde{G}}(z_y \tilde{\gamma}, B(\tilde{f}, \nu))$$

where $k_1 = |\widetilde{G}_+^n \backslash \widetilde{G}_+^n \tilde{T}|$ and $\tilde{\gamma}$ is a regular, semi-simple element of \tilde{T} .

Let $\tilde{\gamma} \in \widetilde{T}^n$. Then $\widetilde{G}(\tilde{\gamma}) = \tilde{T}$. Recalling the definition of $W_{\tilde{\gamma}}$ (section 5.3), we note that $W_{\tilde{\gamma}} = \widetilde{G}_+^n \tilde{T} \backslash \widetilde{G}$. If we consider lemma 5.3.2 and take a sum over $z_y \in \widetilde{Z}^n \backslash \widetilde{T}^n \cap \tilde{Z}$, we obtain:

$$\begin{aligned} & \sum_{z_y \in \widetilde{Z}^n \backslash \widetilde{T}^n \cap \tilde{Z}} F_{\widetilde{G}}(z_y \tilde{\gamma}, B(\tilde{f}, \nu)) \\ &= \Delta(\tilde{\gamma}) \sum_{z_y \in \widetilde{Z}^n \backslash \widetilde{T}^n \cap \tilde{Z}} \sum_{h_x \in W_{\tilde{\gamma}}} \tau(y, \det(h_x)) \chi_\nu^{-1}(z_y) \\ & \quad \int_{\widetilde{G}_+^n \cap \tilde{T} \backslash \widetilde{G}_+^n} A(\tilde{f}, \nu)(h_x^{-1} g^{-1} \tilde{\gamma} g h_x) dg. \end{aligned}$$

We change the order of summation on the right hand side, get $\chi_\nu(z_y)$ on the left side, and use lemma 5.3.3 to obtain:

$$\begin{aligned}
& \sum_{z_y \in \widetilde{Z}^n \setminus \widetilde{T}^n \cap \widetilde{Z}} \chi_\nu(z_y) F_{\widetilde{G}}(z_y \widetilde{\gamma}, B(\tilde{f}, \nu)) \\
&= \sum_{h_x \in W_{\widetilde{\gamma}}} \sum_{z_y \in \widetilde{Z}^n \setminus \widetilde{T}^n \cap \widetilde{Z}} \tau(y, \det(h_x)) F_{\widetilde{G}_+^n}^{h_x}(\widetilde{\gamma}, A(\tilde{f}, \nu)).
\end{aligned}$$

Since $\widetilde{\gamma} \in Z(\widetilde{T}) = \widetilde{T}^n$ (lemma 3.4.1), we have $F_{\widetilde{G}}(z_y \widetilde{\gamma}, B(\tilde{f}, \nu)) = 0$ if $z_y \notin \widetilde{Z}^n \setminus \widetilde{Z} \cap \widetilde{T}^n$. Therefore we can extend the summation in the LHS to run over $\mathbb{F}^*/\mathbb{F}^{*n}$. Then we use theorem 5.3.1 to obtain:

$$\begin{aligned}
& \sum_{h_x \in W_{\widetilde{\gamma}}} \sum_{z_y \in \widetilde{Z}^n \setminus \widetilde{T}^n \cap \widetilde{Z}} \tau(y, \det(h_x)) F_{\widetilde{G}_+^n}^{h_x}(\widetilde{\gamma}, A(\tilde{f}, \nu)) \\
&= \frac{|\mathbb{F}^*/\mathbb{F}^{*n}|}{k_1} F_{\widetilde{G}_+^n}(\widetilde{\gamma}, A(\tilde{f}, \nu)).
\end{aligned}$$

Now we use the fact that $F_{\widetilde{G}_+^n}(\widetilde{\gamma}, A(\tilde{f}, \nu)) = F_{\widetilde{G}_1}(\widetilde{\gamma}, \tilde{f})$ (this follows from discussion towards the end of section 5.3) and $W_{\widetilde{\gamma}} = \widetilde{T} \widetilde{G}_+^n \setminus \widetilde{G}$. We use the notation $c_x = \sum_{z_y \in \widetilde{Z}^n \setminus \widetilde{T}^n \cap \widetilde{Z}} \tau(y, \det(h_x))$ to obtain

$$\begin{aligned}
& \sum_{h_x \in \widetilde{T} \widetilde{G}_+^n \setminus \widetilde{G}} c_x F_{\widetilde{G}_1}^{h_x}(\widetilde{\gamma}, \tilde{f}) \\
&= \frac{|\mathbb{F}^*/\mathbb{F}^{*n}|}{k_1} F_{\widetilde{G}_1}(\widetilde{\gamma}, \tilde{f})
\end{aligned} \tag{6.1}$$

In general if $\tilde{f} \in C_c^\infty(\widetilde{G}_1)$ then

$$\tilde{f} = \sum_{\alpha \in \mu_n} \tilde{f}^\alpha.$$

where $\tilde{f}^\alpha \in C_c^\infty(\widetilde{G}_1)_\alpha$. (see discussion after proposition 5.1.1) By equation 6.1, we have for each α ,

$$\begin{aligned}
& \sum_{h_x \in \tilde{T}\tilde{G}_+^n \setminus \tilde{G}} c_x F_{\tilde{G}_1}^{h_x}(\tilde{\gamma}, \tilde{f}^\alpha) \\
&= \frac{|\mathbb{F}^*/\mathbb{F}^{*n}|}{k_1} F_{\tilde{G}_1}(\tilde{\gamma}, \tilde{f}^\alpha)
\end{aligned}$$

Upon summing over $\alpha \in \widehat{\mu}_n$, we obtain

$$\begin{aligned}
& \sum_{h_x \in \tilde{T}\tilde{G}_+^n \setminus \tilde{G}} c_x F_{\tilde{G}_1}^{h_x}(\tilde{\gamma}, \tilde{f}) \\
&= \frac{|\mathbb{F}^*/\mathbb{F}^{*n}|}{k_1} F_{\tilde{G}_1}(\tilde{\gamma}, \tilde{f})
\end{aligned} \tag{6.2}$$

for any $\tilde{f} \in C_c^\infty(\tilde{G}_1)$.

Let us recall here that an element in \tilde{G} is said to be regular (semi-simple) if its projection in G is regular (semi-simple). Since $\tilde{\gamma}$ is a regular, semi-simple element of \tilde{T} , $h_x^{-1}g^{-1}\tilde{\gamma}gh_x$ is conjugate to $\tilde{\gamma}$ in \tilde{G}_1 (here $g \in \tilde{G}_1$) if and only if h_x is the trivial element of $\tilde{T}\tilde{G}_+^n \setminus \tilde{G}$. This is true because $\tilde{G}(\tilde{\gamma}) = \tilde{T}$. We note here that the projections of $\tilde{\gamma}$ and $\tilde{\gamma}^{h_x}$ are also not conjugate inside G_1 . Since $\tilde{\gamma}$ and $\tilde{\gamma}^{h_x}$ (for all $h_x \in \tilde{T}\tilde{G}_+^n \setminus \tilde{G}$) are semi-simple elements their conjugacy classes in the linear group G_1 are closed and we can have a compact open neighborhood, U , of $p(\tilde{\gamma})$ not intersecting some compact open neighborhood of any $p(\tilde{\gamma}^{h_x})$ for any $h_x \in \tilde{T}\tilde{G}_+^n \setminus \tilde{G}$ (we remind here that by h_x we mean coset representatives). We can pull back this compact open subset to \tilde{G}_1 . Define a function \tilde{f} on $p^{-1}(U)$ such that for $(g, \zeta) \in p^{-1}(U)$, $\tilde{f}(g, \zeta) = \zeta$ and zero outside $p^{-1}(U)$.

Therefore we have a genuine function $\tilde{f} \in C_c^\infty(\tilde{G}_1)$ such that $F_{\tilde{G}_1}(\tilde{\gamma}, \tilde{f}) = 1$ and $F_{\tilde{G}_1}^{h_x}(\tilde{\gamma}, \tilde{f}) = 0$ for any non-trivial h_x . Similarly we can find $\tilde{f}_{x_o} \in C_c^\infty(\tilde{G}_1)$ such that $F_{\tilde{G}_1}^{h_{x_o}}(\tilde{\gamma}, \tilde{f}_{x_o}) = 1$ and $F_{\tilde{G}_1}^{h_x}(\tilde{\gamma}, \tilde{f}_{x_o}) = 0$ for $x \neq x_o$ in $\tilde{T}\tilde{G}_+^n \setminus \tilde{G}$. Upon

comparing coefficients on both sides of $F_{\widetilde{G}_1}^{h_x}$ when h_x is non-trivial, which is to say that $h_x \notin \widetilde{T}\widetilde{G}_+^n$, we obtain:

$$c_x = \sum_{z_y \in \widetilde{Z}^n \setminus \widetilde{Z} \cap \widetilde{T}^n} \tau(y, \det(h_x)) = 0$$

Comparing coefficients when h_x is trivial gives:

$$|\widetilde{Z}^n \setminus \widetilde{T} \cap \widetilde{G}_+^n| = \frac{|\mathbb{F}^*/\mathbb{F}^{*n}|}{k_1}$$

where we recall that $k_1 = |\widetilde{G}_+^n \setminus \widetilde{G}_+^n \widetilde{T}|$.

We summarize the above calculation in a lemma:

Lemma 6.2.1 *Let h_x denote any coset representative of $\widetilde{T}\widetilde{G}_+^n \setminus \widetilde{G}$. Also let $c_x = \sum_{z_y \in \widetilde{Z}^n \setminus \widetilde{Z} \cap \widetilde{T}^n} \tau(y, \det(h_x))$. Then $c_x \neq 0$ if and only if $h_x \in \widetilde{T}\widetilde{G}_+^n$. If $h_x \in \widetilde{T}\widetilde{G}_+^n$ then $c_x = |\widetilde{Z}^n \setminus \widetilde{T} \cap \widetilde{G}_+^n|$.*

6.3 Results on Local Fields

We now apply the results of the previous section to Local Class Field Theory. In the case when T is given by a field extension, \mathbb{E} , of \mathbb{F} we note the the image of \widetilde{T} under the det map is $N(\mathbb{E}^*) \subset \mathbb{F}^*$. We get $k_1 = |N(\mathbb{E}^*)/\mathbb{F}^{*n}|$. We have have proved the following:

Proposition 6.3.1 *Let \mathbb{E}/\mathbb{F} be a field extension of order n . Then*

$$\sum_{y \in (\mathbb{F}^* \cap \mathbb{E}^{*n})/\mathbb{F}^{*n}} \tau(y, x) = 0$$

if and only if $x \notin N(\mathbb{E}^)$.*

Remark 6.3.1 *It should be noted here that this proposition gives us a sufficient condition for an element of \mathbb{F}^* to be in $N(\mathbb{E}^*)$.*

Proof. It is easy to see that if $x \in N(\mathbb{E}^*)$, then $\tau(y, x) = 1$ for every $y \in \mathbb{F}^* \cap \mathbb{E}^{*n}/\mathbb{F}^*$ where $N(\mathbb{E}^*)$ denotes the norm map. Let $x = N(a)$ and $y = b^n$ for some $a, b \in \mathbb{E}^*$ with b satisfying $b^n \in \mathbb{F}^*$. The desired identity follows from the fact that $\tau(N(a), b)_{\mathbb{F}} = \tau(a, b)_{\mathbb{E}}$ where $a \in \mathbb{E}^*, b \in \mathbb{F}^*$.

For the other side if $\tau(y, x) = 1$ for every $y \in (\mathbb{F}^* \cap \mathbb{E}^{*n})/\mathbb{F}^{*n}$ then $\sum_{y \in (\mathbb{F}^* \cap \mathbb{E}^{*n})/\mathbb{F}^{*n}} \tau(y, x) \neq 0$. Therefore $c_x \neq 0$ and hence $h_x \in \tilde{T}\tilde{G}_+^n$ from lemma 6.2.1. Since $\det(h_x) = x$, this ensures that $x \in N(\mathbb{E}^*)$. ■

This tells us that the character $\tau(\cdot, x)$ is a non-trivial character of $(\mathbb{F}^* \cap \mathbb{E}^{*n})/\mathbb{F}^{*n}$ if and only if $x \notin N(\mathbb{E}^*)$. Since $\tau : \mathbb{F}^*/\mathbb{F}^{*n} \times \mathbb{F}^*/\mathbb{F}^{*n} \rightarrow \mu_n$ is an exact pairing, we deduce that:

Proposition 6.3.2

$$\mathbb{F}^*/N(\mathbb{E}^*) \simeq ((\mathbb{F}^* \cap \mathbb{E}^{*n})/\mathbb{F}^{*n})^\wedge$$

where $((\mathbb{F}^* \cap \mathbb{E}^{*n})/\mathbb{F}^{*n})^\wedge$ represents the group of characters of $(\mathbb{F}^* \cap \mathbb{E}^{*n})/\mathbb{F}^{*n}$.

If we identify the group of characters, we obtain:

Proposition 6.3.3 *We have a non-canonical isomorphism between the following groups:*

$$\mathbb{F}^*/N(\mathbb{E}^*) \simeq (\mathbb{F}^* \cap \mathbb{E}^{*n})/\mathbb{F}^{*n}$$

Now we will use proposition 6.3.1 to prove theorems 6.0.4 and 6.0.5.

Let \mathbb{E}/\mathbb{F} be a finite abelian extension of order n . By Kummer theory, $\mathbb{E} = \mathbb{F}[a_1, a_2, \dots, a_r]$ where $a_i \in \mathbb{E}$ form a minimal set of generators satisfying $a_i^{n/d_i} =$

$y_i \in \mathbb{F}$ for some set of positive integers $d_i|n$. We choose $\{d_i\}$ such that they are highest integers satisfying $a_i^{n/d_i} \in \mathbb{F}$. Such a set of generators exist because μ_n , the n^{th} roots of unity, are in \mathbb{F} . Then $\text{Gal}(\mathbb{E}/\mathbb{F}) \simeq \bigoplus_{i=1}^r \mathbb{Z}/(n/d_i)\mathbb{Z}$.

We define a map $\sigma : \mathbb{F}^* \rightarrow \text{Gal}(\mathbb{E}/\mathbb{F})$ by

$$\sigma(x)(a_i) = a_i \tau(a_i^n, x).$$

for every generator a_i and $\sigma(x)(a) = a$ for $a \in \mathbb{F}$. Then σ maps each a_i to some root of the equation $x^{n/d_i} - y_i = 0$. Therefore we see that $\sigma(x)$ is a well-defined element of $\text{Gal}(\mathbb{E}/\mathbb{F})$. Also $\sigma(x) \equiv 1$ if and only if $x \in N(\mathbb{E}^*)$ (this follows from proposition 6.3.1).

Therefore

Lemma 6.3.1

$$\sigma : \mathbb{F}^*/N(\mathbb{E}^*) \rightarrow \text{Gal}(\mathbb{E}/\mathbb{F})$$

is an injective map.

Now we compare the order of both groups. By proposition 6.3.3, we have $|(\mathbb{F}^* \cap \mathbb{E}^{*n})/\mathbb{F}^{*n}| = |\mathbb{F}^*/N(\mathbb{E}^*)|$. So by lemma 6.3.1, we obtain $|(\mathbb{F}^* \cap \mathbb{E}^{*n})/\mathbb{F}^{*n}| \leq \prod_{i=1}^r n/d_i$.

But $\mathbb{F}^{*n}\langle a_1^{d_1}, a_2^{d_2}, \dots, a_r^{d_r} \rangle/\mathbb{F}^{*n} \subseteq (\mathbb{F}^* \cap \mathbb{E}^{*n})/\mathbb{F}^{*n}$ as multiplicative subgroups of $\mathbb{F}^*/\mathbb{F}^{*n}$. Since d_i are maximal integers satisfying $a_i^{n/d_i} \in \mathbb{F}^*$, we obtain

$$|\mathbb{F}^{*n}\langle a_1^{d_1}, a_2^{d_2}, \dots, a_r^{d_r} \rangle/\mathbb{F}^{*n}| = \prod_{i=1}^r n/d_i.$$

This proves that

$$\prod_{i=1}^r n/d_i \leq |(\mathbb{F}^* \cap \mathbb{E}^{*n})/\mathbb{F}^{*n}| \leq \prod_{i=1}^r n/d_i$$

giving

Lemma 6.3.2

$$|(\mathbb{F}^* \cap \mathbb{E}^{*n})/\mathbb{F}^{*n}| = \prod_{i=1}^r n/d_i$$

and hence the surjectivity of σ . Theorem 6.0.4 follows from lemmas 6.3.1 and 6.3.2. We state it again.

Theorem 6.3.1 *Let \mathbb{E}/\mathbb{F} be a finite abelian extension of degree n . Assume $\mu_n \subset \mathbb{F}^*$. Then there exists a map $\sigma : \mathbb{F}^* \rightarrow \text{Gal}(\mathbb{E}/\mathbb{F})$ such that the sequence*

$$1 \rightarrow N(\mathbb{E}^*) \rightarrow \mathbb{F}^* \xrightarrow{\sigma} \text{Gal}(\mathbb{E}/\mathbb{F}) \rightarrow 1$$

is exact.

Corollary 6.3.1 *Let \mathbb{E}/\mathbb{F} be an abelian field extension of \mathbb{F} and \mathbb{E}_1 and \mathbb{E}_2 be two abelian field extensions of \mathbb{F} inside \mathbb{E} with N^i ($i=1,2$) the corresponding Norm maps. Then $N^1(\mathbb{E}_1) = N^2(\mathbb{E}_2)$ if and only if $\mathbb{E}_1 = \mathbb{E}_2$. We assume that $\mu_N \subseteq \mathbb{F}$ where $N = |\mathbb{E}/\mathbb{F}|$.*

Proof.

Since $N^1(\mathbb{E}_1) = N^2(\mathbb{E}_2)$ we have from the above theorem that $|\text{Gal}(\mathbb{E}_1/\mathbb{F})| = |\text{Gal}(\mathbb{E}_2/\mathbb{F})| = n$. Therefore $|\mathbb{E}_i/\mathbb{F}| = n$ for $i = 1, 2$. By lemma 6.3.2 we obtain $(\mathbb{F}^* \cap \mathbb{E}_1^{*n})/\mathbb{F}^{*n} = (\mathbb{F}^* \cap \mathbb{E}_2^{*n})/\mathbb{F}^{*n}$. From results in Kummer theory ([11]) we know that $\mathbb{E} \rightarrow (\mathbb{E}^{*n} \cap \mathbb{F}^*)/\mathbb{F}^{*n}$ defines a one-one correspondence between subfields lying between \mathbb{F} and $\mathbb{F}[n]$ and the subgroups of $\mathbb{F}^*/\mathbb{F}^{*n}$ where $\mathbb{F}[n]$ is the maximal field extension of \mathbb{F} of exponent n . This gives $\mathbb{E}_1 = \mathbb{E}_2$. The converse is obvious. ■

Next we prove theorem 6.0.5.

Let \mathbb{E} be a field extension of \mathbb{F} with $|\mathbb{E}/\mathbb{F}| = n$. Let \mathbb{E}^1 be the maximal subextension of \mathbb{F} inside \mathbb{E} . We wish to prove that:

$$N_{\mathbb{F}}^{\mathbb{E}}(\mathbb{E}^*) = N_{\mathbb{F}}^{\mathbb{E}^1}(\mathbb{E}^{1*}).$$

One side is immediate: $N_{\mathbb{F}}^{\mathbb{E}}(\mathbb{E}^*) \subseteq N_{\mathbb{F}}^{\mathbb{E}^1}(\mathbb{E}^{1*})$ follows because of the transitivity of the Norm map.

We will use proposition 6.3.3. Let $|\mathbb{E}^1/\mathbb{F}| = d$. Then we have the non-canonical isomorphisms:

$$\mathbb{F}^*/N_{\mathbb{F}}^{\mathbb{E}^1}(\mathbb{E}^{1*}) \simeq \mathbb{F}^* \cap \mathbb{E}^{1*d}/\mathbb{F}^{*d} \quad (6.3)$$

and

$$\mathbb{F}^*/N_{\mathbb{F}}^{\mathbb{E}}(\mathbb{E}^*) \simeq (\mathbb{F}^* \cap \mathbb{E}^{*n})/\mathbb{F}^{*n}. \quad (6.4)$$

If $a \in \mathbb{E}^*$ satisfies $a^n \in \mathbb{F}^*$ then $\mathbb{F}[a]/\mathbb{F}$ is cyclic ($\mu_n \subset \mathbb{F}$) and hence $\mathbb{F}[a] \subseteq \mathbb{E}^1$. This gives $a \in \mathbb{E}^1$ and therefore $\mathbb{E}^{*n} \cap \mathbb{F}^* \subseteq \mathbb{E}^{1*n} \cap \mathbb{F}^*$. The other containment is obvious giving us $\mathbb{E}^{*n} \cap \mathbb{F}^* = \mathbb{E}^{1*n} \cap \mathbb{F}^*$. This allows us to write the relation (6.4) as:

$$\mathbb{F}^*/N_{\mathbb{F}}^{\mathbb{E}}(\mathbb{E}^*) \simeq \mathbb{F}^* \cap \mathbb{E}^{1*n}/\mathbb{F}^{*n}. \quad (6.5)$$

To complete the proof we have a lemma:

Lemma 6.3.3 *Let $n = dk$. Define a homomorphism of multiplicative groups $\phi : \mathbb{F}^* \cap \mathbb{E}^{1*d} \rightarrow \mathbb{F}^* \cap \mathbb{E}^{1*n}$ by $\phi(x) = x^k$. Then ϕ induces an isomorphism:*

$$\mathbb{F}^* \cap \mathbb{E}^{1*d}/\mathbb{F}^{*d} \xrightarrow{\phi} \mathbb{F}^* \cap \mathbb{E}^{1*n}/\mathbb{F}^{*n}.$$

We see that from this lemma and relations (6.3) and (6.5) we obtain:

$$\mathbb{F}^*/N_{\mathbb{F}}^{\mathbb{E}}(\mathbb{E}^*) \simeq \mathbb{F}^*/N_{\mathbb{F}}^{\mathbb{E}^1}(\mathbb{E}^{1*}).$$

Since $N_{\mathbb{F}}^{\mathbb{E}}(\mathbb{E}^*) \subseteq N_{\mathbb{F}}^{\mathbb{E}^1}(\mathbb{E}^{1*})$, this gives us theorem 6.0.5 stated in the beginning of this chapter.

Corollary 6.3.2 *We have defined the symbol τ for the case $(p, n) = 1$ and so we cannot use it for the case $p|n$. If we replace the symbol τ with the classical Norm-residue symbol (which we have used throughout this thesis in previous chapters)*

then the proof of theorem 6.0.5 remains exactly the same. This amounts to proving theorem 6.0.5 for the case $p|n$ assuming the existence of n^{th} Norm-residue symbol on field \mathbb{F} . This uses local class field theory but gives a different proof of theorem 6.0.5 (For classical proof refer [20] Page No. 168).

We proceed to prove lemma 6.3.3.

Proof.

Clearly, ϕ is well-defined, since if $x = a^d, a \in \mathbb{F}^*$, then $\phi(x) = a^n \in \mathbb{F}^{*n}$.

Let $x \in \mathbb{F}^* \cap \mathbb{E}^{1*d}, x = y^d, y \in \mathbb{E}^*$ such that $\phi(x) \in \mathbb{F}^{*n}$. Then:

$$\begin{aligned} x^k &\in \mathbb{F}^{*n} \\ \Rightarrow y^n &\in \mathbb{F}^{*n} \\ \Rightarrow y &\in \mathbb{F}^* \end{aligned}$$

The last line follows because $\mu_n \subset \mathbb{F}^*$.

Therefore ϕ is injective.

We only need that ϕ is surjective. That forms the contents of the next lemma.

In the proof so far we have not used that fact that \mathbb{E}^1 is an abelian extension of \mathbb{F} . Note that we use this fact in the next lemma. ■

Lemma 6.3.4 *Let \mathbb{E}^1/\mathbb{F} be an extension with $|\mathbb{E}^1/\mathbb{F}| = d$. Let $x \in \mathbb{E}^{1*}$ such that $x^n = a \in \mathbb{F}^*$ where $d|n$. Then, in fact, $x^d \in \mathbb{F}^*$. We are assuming $\mu_n \subset \mathbb{F}$.*

Proof. Let $|\mathbb{F}[x]/\mathbb{F}| = e$ and let $G = \text{Gal}(\mathbb{F}[x]/\mathbb{F})$. Then G is a quotient group of $(\mathbb{Z}/n\mathbb{Z})^*$ and is hence abelian.

Let $\alpha \in G$. Then $\alpha(x) = x\zeta^{a_\alpha}$ where ζ is a primitive n^{th} root of unity.

Define $\psi : G \rightarrow \mathbb{Z}/n\mathbb{Z}$ by $\psi(\alpha) = a_\alpha$.

It is easy to check that ψ is a homomorphism. If $a_\alpha = 0$ then $\alpha(x) = x$ and therefore α fixes the whole of $\mathbb{F}[x]$ and is the trivial element. This proves that ψ is injective. We obtain that G is cyclic of order e . Let ρ be a generator of G so that $a_\rho = n/e$. Therefore $\rho(x^e) = (\rho(x))^e = (x\zeta^{n/e})^e = x^e$. This gives us that G fixes x^e and therefore $x^e \in \mathbb{F}^*$. Since $e|d$ we obtain $x^d \in \mathbb{F}^*$. ■

6.4 Recapitulation

In this section we briefly go over the principal ideas involved in the proof of theorems 6.0.4 and 6.0.5. We also compare our proof with the proof given in [20].

We start with a p -adic field \mathbb{F} with the residue field $k_{\mathbb{F}}$ such that $|k_{\mathbb{F}}| = q$. Therefore \mathbb{F} has $q - 1$ roots of unity (Hensel's lemma). We consider an extension \mathbb{E}/\mathbb{F} such that $|\mathbb{E}/\mathbb{F}| = n$ divides $q - 1$. We consider \mathbb{E} as a cartan subgroup, $T_{\mathbb{E}}$ of $\mathrm{GL}(n, \mathbb{F})$ and consider orbital integrals over the pull back of $T_{\mathbb{E}}$ in the n -fold covering group of $\mathrm{GL}(n, \mathbb{F})$. We use the following tools in our analysis:

- Construction of covering groups, calculation of commutators for the tori in covering groups (this uses the transfer map in K-theory when we deal with elliptic tori). This calculation essentially gives us the center of the cartan subgroups in covering groups and is the main ingredient in proving proposition 6.3.1 which is used to prove theorems 6.0.4 and 6.0.5. One should note that the corresponding calculation with the linear group will not yield anything useful to us.
- The fact that $\mathrm{Cent}_{\tilde{G}}(\tilde{T}_{\mathbb{E}}^n) = \tilde{T}_{\mathbb{E}}$. This follows from the fact that $\mathrm{Cent}_G(T_{\mathbb{E}}) = T_{\mathbb{E}}$ i.e $T_{\mathbb{E}}$ is a *maximal* torous in G .

- Orbital Integrals on $\tilde{T}_{\mathbb{E}}$ are conjugation invariant functions on regular semi-simple elements of $\tilde{T}_{\mathbb{E}}$ and that semi-simple elements in an algebraic group have closed conjugacy classes.

We use above methods to compare orbital integrals on \tilde{G} and \tilde{G}_1 and obtain a sufficient condition (proposition 6.3.1) for an element of \mathbb{F}^* to fall inside the subgroup $N(\mathbb{E}^*)$. This condition in turn leads to the theorem 6.0.4 if \mathbb{E}/\mathbb{F} is abelian and to theorem 6.0.5 in the general case.

We are not able to say from our methods that σ in theorem 6.0.4 is the Artin map constructed in [20]. In [20], the Artin map is constructed by using methods of group cohomology and class formations. These methods yield all the properties of Artin map as well as the fact that the kernel of the Artin map is $N(\mathbb{E}^*)$. However, from our methods we are able to prove that the map we have constructed has kernel $N(\mathbb{E}^*)$. Our proof is direct in the sense that we start with an element of \mathbb{F}^* and determine a necessary condition for it to be in $N(\mathbb{E}^*)$. This is very different from the approach of [20]. In [20], a direct proof is given in the case when \mathbb{E}/\mathbb{F} is unramified or cyclic and totally ramified without using class formations. Both proofs are very different from one another and use filtrations, exact sequences, etc. In contrast the proof obtained here is independent of the fact that \mathbb{E}/\mathbb{F} is unramified or totally ramified and is analytic in nature.

Our analysis raises some important questions:

1. Why should we expect a completely arithmetic result like theorem 6.0.3 to follow from our methods which are mostly of harmonic analysis on non-linear groups or rather what is the philosophical cause to explain the kind of result we have obtained from the methods we have used? Comparing

the proof given in [20] it is natural to ask what does analysis on non-linear groups has to do with the algebraic methods in the proof of this theorem?

2. Is there some more general object or technique which when understood can answer the above question?

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