ABSTRACT

Title of Dissertation: MULTIVARIATE LÉVY PROCESSES
FOR FINANCIAL RETURNS.

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We apply a signal processing technique known as independent component analysis (ICA) to multivariate financial time series. The main idea of ICA is to decompose the observed time series into statistically independent components (ICs). We further assume that the ICs follow the variance gamma (VG) process. The VG process is evaluated by Brownian motion with drift at a random time given by a gamma process. We build a multivariate VG portfolio model and analyze empirical results of the investment.

MULTIVARIATE LÉVY PROCESSES
FOR FINANCIAL RETURNS.

by

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DEDICATION

To My Parents

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Chapter 1

Introduction

The concept of modern portfolio theory was developed by Harry M. Markowitz in 1952. Markowitz measured the risk for various kinds of securities and developed methods which optimized the trade-off between the return and the risk. The theory shows that the portfolio risk comes from the covariance of the assets in the portfolio. Investors should not consider the prospects of a single security individually, but instead look at how each investment fits into the overall portfolio. Markowitz constructed a model which showed that the risk of a portfolio was lower than the average of the risks of each individual asset. He also gave quantitative evidence of the advantage of diversified investment. The theory was shown to be consistent with expected utility maximization for exponential utility with Gaussian returns.

Prices of assets are generally modelled as continuous functions of time. Black-Scholes [6] and Merton [31] built a stock price model using geometric Brownian motion. Discontinuities of stock price models have been considered as an additional orthogonal compound Poisson process [32, 3]. This type of model is called a jump-diffusion model. A purely discontinuous stock price process called the variance gamma (VG) process was proposed by Madan and Seneta in 1987 [28]. The VG process has no continuous component. The VG model has infinitely many jumps and finite variation.

The VG process $X_{VG}(t)$ is defined as follows:

$$X_{VG}(t) = \theta g(t) + \sigma W(g(t)),$$

where $g(t)$ is a gamma process with unit mean and variance ν, and $W(g)$ is Gaussian with zero mean and variance g. The risk free process for the VG stock price $S(t)$ at time t is modelled as follows,

$$S(t) = S(0)\frac{\exp(rt + X_{VG}(t))}{E[\exp(X_{VG}(t))]},$$

such that the mean rate of return equals to the continuous compounded interest rate r, that is, $E(S(t)) = S(0)\exp(rt)$.

As two stock price processes S_1 and S_2 are often driven by the same factors, we model the two-factor VG stock price model as follows. Denote

$$S_1(t) = \frac{\exp(m_1 t + B_1 X_1(t) + B_2 X_2(t))}{E[\exp(B_1 X_1(t) + B_2 X_2(t))]} \tag{1.1}$$

and

$$S_2(t) = \frac{\exp(m_2 t + C_1 X_1(t) + C_2 X_2(t))}{E[\exp(C_1 X_1(t) + C_2 X_2(t))]}, \tag{1.2}$$

where X_1 and X_2 are independent VG processes, m_1, m_2 are continuous compounded interest rates, and B_1, B_2, C_1, and C_2 are coefficients of the linear combination of the $\log$ of the stock prices, S_1 and S_2. Let

$$\begin{aligned} B(t) &= E[\exp(B_1 X_1(t) + B_2 X_2(t))] \\ C(t) &= E[\exp(C_1 X_1(t) + C_2 X_2(t))]. \end{aligned}$$

The quantities $B(t)$ and $C(t)$ are the convexity correction factors of S_1 and S_2. We may see that the mean rate of returns of S_1 and S_2 equal the continuous compounded interest rates m_1 and m_2, respectively.

We construct a multivariate VG portfolio which assumes that the asset returns are driven by independent VG components. We apply independent component analysis (ICA) to transform the observed daily return data into independent components. And these independent components are assumed to follow the zero mean VG process in our analysis. ICA is a method which transforms a set of observed multi-dimensional data into components that are as statistically independent as possible. We compare our model with the Gaussian portfolio. The observed data not only fits the VG model better, but the investment strategy built by VG also performs better.

Chapter 2 briefly presents the Markowitz portfolio theory. The VG process is introduced in Chapter 3. In Chapter 4, we construct the multivariate VG asset pricing model. We describe the independent component analysis in Chapter 5. In Chapter 6, we discuss the solution of multivariate portfolio problems. We implement independent component analysis in Chapter 7. Chapter 8 presents the results of the VG investment.

Chapter 2

Markowitz portfolio theory

The Markowitz portfolio theory deals with individual agents in portfolio selection. Probability theory and optimization theory are used to model the behavior of individual economic agents. Under rational behavior, the agents are assumed to seek for a balance between maximizing the return and minimizing the risk of the portfolios. Return is measured by the mean, and risk is measured by the variance of the portfolio. The solution will depend on the risk level that the investors would take. A probability distribution of security prices is assumed to be known in the Markowitz model later by followers. The variance is presented as a quadratic approximation to a utility function [29, 30].

2.1 Utility function

Portfolio theory was developed to solve any decision problems in investment in an uncertain future. Under the rules of rational behavior, one has to define an opportunity set and a preference function. The preference function is called the utility function. Investors prefer having more to having less, and are consistent with their choices. The growth rate of the utility function is always positive. The utility function gives each situation a value based on the expected return and the risk of the portfolio. A utility function U can represent an investor's attitude toward risk. The utility function conserves the order, that is, the utility of more return is higher than the utility of less return. The expected utility of an investor's wealth W is defined as follows[13, 14]:

$$E(U(W)) = \sum_i p_i U(w_i),$$

where p_i is the probability of obtaining the final wealth w_i. One commonly used utility function is called exponential utility function. The general form of the exponential utility function is

$$U(W) = 1 - \exp(-\eta W),$$

where W is the final wealth, and η is the risk aversion coefficient. Each investor has different degree of tolerance towards risk. The choice of how to adjust a portfolio is made by maximizing the investor's expected utility function.

2.2 Certainty equivalent

The utility of a risk-free portfolio is the rate of return on the portfolio. Investors can compare the utility value of a risky portfolio with the risk-free return rate when making a decision between a risky portfolio and a safe one. We may define a portfolio utility value as its certainty equivalent rate of return. In other words, the certainty equivalent CE of a portfolio is the rate that the risk-free portfolio has to provide with certainty to be considered as attractive as the risky portfolio[36]. Mathematically, we may write

$$U(CE) = E(U).$$

That is, the utility of the certainty equivalent equals the expected utility. Using the exponential utility function, we get

$$1 - \exp(-\eta \times CE) = E(U).$$

The optimal strategy of an investor is the one with the highest certainty equivalent. The certainty equivalent depends on the investor's attitude toward risk.

2.3 The Markowitz model

Let x_i be the proportion invested in security i in the Markowitz model, where $1 \leq i \leq n$, and R_i is the return for the ith security. Denote $\mu_i = E(R_i)$ as the expected return of security i. The return of the portfolio is $R'x = \sum_{i=1}^{n} R_i x_i$ with expected return $E(R'x) = \mu' x = \sum_{i=1}^{n} x_i \mu_i$, where $R = (R_1, R_2, \ldots, R_n)'$, $x = (x_1, x_2, \ldots, x_n)'$, and $\mu = (\mu_1, \mu_2, \ldots, \mu_n)'$. Let Σ be the variance-covariance matrix of the random vector R. Then the variance of the portfolio is $x'\Sigma x$. Investors are assumed to prefer return and avoid risk. Thus, given a fixed level of risk, the investors will select a portfolio which maximizes the return. Or equivalently, for a fixed level of return, the investors will select a portfolio which minimizes the risk. A portfolio which follows this condition is called an efficient portfolio. The set of all efficient portfolios is called the efficient frontier [29]. The mathematical formulation of the Markowitz model is

$$\begin{aligned}
&\text{maximize } E(R'x) \\
&\text{subject to } x'\Sigma x \leq \gamma \\
&\sum_{i=1}^{n} x_i = 1 \\
&x_i \geq 0,\ i = 1, \ldots, n
\end{aligned}$$

where γ is the maximum of risk level the investor would take. Equivalently, the model may be formulated as

$$\begin{aligned}
&\text{maximize } x'\Sigma x \\
&\text{subject to } E(R'x) \geq \kappa \\
&\sum_{i=1}^{n} x_i = 1 \\
&x_i \geq 0,\ i = 1, \ldots, n
\end{aligned}$$

where κ is the minimum return the investor would take.

2.4 Expected utility maximization for exponential utility and Gaussian returns

Let the certainty equivalent for wealth W follow the gaussian distribution with mean μ and variance σ^2 in the case of exponential utility. The density function $p(W)$ of wealth is

$$p(W) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(W-\mu)^2}{2\sigma^2}\right),$$

and the utility function is

$$U(W) = 1 - \exp(-\eta W), \tag{2.1}$$

where η is risk aversion coefficient. The certainty equivalent, CE, is defined by the equation

$$U(CE) = E(U),$$

or equivalently

$$1 - \exp(-\eta \times CE) = E(U).$$

The expected utility is evaluated as follows,

$$\begin{aligned} E(U) &= \int_{-\infty}^{\infty} U(W)p(W)dW \\ &= \int_{-\infty}^{\infty} (1-\exp(-\eta W))p(W)dW \\ &= 1-\int_{-\infty}^{\infty} \exp(-\eta W)p(W)dW \\ &= 1-\int_{-\infty}^{\infty} \exp(-\eta W)\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{(W-\mu)^2}{2\sigma^2}\right)dW. \end{aligned}$$

Make the change of variable for the above integral as follows,

$$\begin{aligned} z &= \frac{W-\mu}{\sigma} \\ W &= \mu+\sigma z \\ dW &= \sigma dz, \end{aligned}$$

We then have

$$\begin{aligned} E(U) &= 1-\int_{-\infty}^{\infty} e^{-\eta(\mu+\sigma z)}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{z^2}{2}\right)dz \\ &= 1-e^{-\eta\mu}\int_{-\infty}^{\infty} e^{\eta\sigma z}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{z^2}{2}\right)dz \\ &= 1-e^{-\eta\mu+\frac{\eta^2\sigma^2}{2}}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(z+\eta\sigma)^2}{2}\right)dz \\ &= 1-e^{-\eta\mu+\frac{\eta^2\sigma^2}{2}}. \end{aligned} \tag{2.2}$$

Combine equations 2.1 and 2.2, we get

$$1-e^{-\eta\times CE} = 1-e^{-\eta\times(\mu-\eta\frac{\sigma^2}{2})}.$$

Finally, we see that

$$CE = \mu-\eta\frac{\sigma^2}{2}.$$

Chapter 3

Long-tailed returns

While the Black-Scholes model remains widely used in the financial world, it has known biases. The financial data shows that the log returns of most assets are not normally distributed. We often find some skewness and excess kurtosis from the empirical distribution. Thus, more flexible distributions than normal are necessary. In order to model asset prices over time, we need more flexible stochastic processes as well.

Over a long period of time, there are many independent effects on asset prices. Thus, the independent and stationary increments properties of the Brownian motion seem reasonable for economic assumptions. We would like to work with processes which not only have the independent and stationary increments properties, but also can have skewness and excess kurtosis.

To keep the independent and stationary increments properties of a stochastic process, the distribution needs to be infinitely divisible. Such processes are called Lévy processes. In financial literature, models which are able to capture the characteristics of infinitely divisible distributions as well as the skewness and excess kurtosis were proposed. The first Lévy process financial model proposed was the symmetric variance gamma model by Madan and Seneta [28], and Madan and Milne [27]. The symmetric variance gamma model controls for kurtosis and volatility. Madan, Carr, and Chang

[8] generalized the process proposed in [28, 27] which addresses the skewness as well.

3.1 Lévy processes and infinitely divisible laws

We start with a few definitions in this section. General references on infinitely divisible distributions and Lévy processes are in [4, 5, 35].

Definition 3.1. *A stochastic process $X = \{X(t) : t \geq 0\}$ is said to be a Lévy process if the following conditions are satisfied.*

1. *X has independent increments: for any choice of $n \geq 1$ and $0 \leq t_0 < t_1 < \cdots < t_n$, $X(t_0)$, $X(t_1) - X(t_0)$, $\cdots$, $X(t_n) - X(t_{n-1})$ are independent random variables.*

2. *$X(0) = 0$ almost surely.*

3. *X has stationary increments: the distribution of $X(t+s) - X(s)$ does not depend on s.*

4. *X is stochastically continuous: for every $t \geq 0$ and $\epsilon > 0$,*
   $\lim\limits_{s \to t} P\left[|X(s) - X(t)| > \epsilon\right] = 0.$

5. *X is right continuous with left limits almost surely.*

Definition 3.2. *The characteristic function $\phi(u)$ of the distribution $F(x) = P(X \leq x)$ of a random variable X is denoted by*

$$\phi_X(u) = E[\exp(iuX)] = \int\limits_{-\infty}^{\infty} \exp(iux) dF(x).$$

Definition 3.3. *A probability distribution with a characteristic function $\phi(u)$ is infinitely divisible if, for any positive integer n,*

$$\phi_n(u) = \phi(u)^{1/n}$$

is also a characteristic function.

The following theorem gives the characteristic functions of infinitely divisible distributions. The representation is called the Lévy-Khintchine formula.

Theorem 3.4. **(i)** *If a probability distribution with a characteristic function $\phi(u)$ is infinitely divisible, and let $\psi(u) = \log \phi(u)$, then*

$$\psi(u) = i\gamma u - \frac{1}{2}\sigma^2 u^2 + \int_{-\infty}^{\infty} (\exp(iux) - 1 - iux1_{\{|x|<1\}})\nu(dx), \tag{3.1}$$

where $\gamma \in R$, $\sigma^2 \geq 0$, and ν is a measure on $R\backslash\{0\}$ with

$$\int_{-\infty}^{\infty} (1 \wedge x^2)\nu(dx) < \infty. \tag{3.2}$$

(ii) *The representation of $\psi(u)$ in (i) by γ, σ, and ν is unique.*

(iii) *Conversely, if $\gamma \in R$, $\sigma^2 \geq 0$, and ν is a measure satisfying (3.2), then there exists an infinitely divisible distribution whose log characteristic function is given by (3.1).*

The measure ν is called the Lévy measure of the process. When the measure has the form $\nu(dx) = k(x)dx$, we call $k(x)$ the Lévy density. In (3.1), we observe that the Lévy-Khintchine formula has three components: a linear deterministic component with the drift coefficient γ, a Brownian component with the diffusion coefficient σ, and a pure jump component. The Lévy measure describes the jump process; that is, $k(x)$

specifies the arrival rate of jumps of size x. If $\sigma^2 = 0$ and $\int\limits_{|x|\leq 1} \nu(dx) < \infty$, the Lévy process is a pure jump process with finite arrival rate of jumps. On the other hand, a pure jump Lévy process has infinite arrival rate if $\sigma^2 = 0$ and $\int\limits_{|x|\leq 1} \nu(dx) = \infty$; that is, the process has infinitely many jumps in any bounded interval.

If $\sigma^2 = 0$ and $\int\limits_{|x|\leq 1} |x|\nu(dx) < \infty$, then the process is of pure jump finite variation. In this case, the exponent of the Lévy-Khintchine formula can be written as

$$\psi(u) = i\gamma' u + \int_{-\infty}^{\infty} (\exp(iux) - 1)\nu(dx), \tag{3.3}$$

for some γ'. A finite variation Lévy process can be expressed as the difference of two increasing processes. The converse is also true.

3.2 The variance gamma process

The variance gamma (VG) process is a pure jump Lévy process with finite variation and infinite arrival rate of jumps. The VG process may be written as time-changed Brownian motion or as the difference of two independent gamma processes [17].

3.2.1 The VG process as time-changed Brownian motion

Although the VG process is a pure jump process, Geman, Madan, and Yor [16] generalized the result of Clark (1973) [10] and verified that it may be viewed as a continuous process with a stochastic clock.

Definition 3.5. *A real-valued Lévy process is called a subordinator if it has almost surely nondecreasing paths.*

A subordinator is a nondecreasing Lévy process, and thus has no diffusion part and has finite variation. Clark [10] gave examples of subordinated processes in which prices were evaluated by a geometric Brownian motion with its time given by another independent geometric Brownian motion. In general, the time process need not be independent of the price process. The VG process evaluates Brownian motion with drift at a random time change given by a gamma process. Let

$$Y(t;\sigma,\theta) = \theta t + \sigma W(t)$$

where $W(t)$ is a standard Brownian motion. The process $Y(t;\sigma,\theta)$ is a Brownian motion with drift θ and variance rate σ^2.

Definition 3.6. *The gamma process $G(t;\nu)$ is a Lévy process whose increments $G(t+h;\nu) - G(t;\nu) = g$ have the gamma density with mean h and variance νh:*

$$f_h(g) = \frac{g^{h/\nu-1}\exp(-g/\nu)}{\nu^{h/\nu}\Gamma(h/\nu)}.$$

Its characteristic function is:

$$\phi_g(u) = \left(\frac{1}{1-iu\nu}\right)^{h/\nu},$$

and for $x > 0$, its Lévy density is:

$$k_g(x) = \frac{\exp(-x/\nu)}{\nu x}.$$

The gamma process is a pure jump process with no drift. Note that $\int_0^\infty k_g(x)dx = \infty$ and $\int_0^\infty x k_g(x) < \infty$, so that the gamma process is of infinite activities and finite variation.

The VG process $X_{VG}(t;\sigma,\nu,\theta)$ is defined by

$$\begin{aligned} X_{VG}(t;\sigma,\nu,\theta) &= Y(G(t;\nu);\sigma,\theta) \\ &= \theta G(t;\nu) + \sigma W(G(t;\nu)) \end{aligned}$$

with the mean rate of g to be unity. The characteristic function of the VG process may be evaluated by conditioning on the gamma process. This is because, given $G(t;\nu)$, $X_{VG}(t)$ is Gaussian. Therefore

$$E[\exp(iuX_{VG}(t))|G(t;\nu)] = \exp(iu\theta G(t;\nu) - \frac{\sigma^2 u^2}{2} G(t;\nu)).$$

We then may obtain the characteristic function of the VG process $\phi_{X_{VG}}(u)$ by unconditioning the expectation. Thus,

$$\begin{aligned} \phi_{X_{VG}}(t;u) &= E[\exp(iuX_{VG})] \\ &= \left(\frac{1}{1 - iu\theta\nu + \frac{\sigma^2\nu}{2}u^2} \right)^{\frac{t}{\nu}}. \end{aligned} \tag{3.4}$$

The moments of VG process may be calculated by differentiating the characteristic function. We shall observe that θ controls skewness and ν controls kurtosis.

3.2.2 The VG process as a finite variation process

Consider two independent gamma processes G_p and G_n with mean rate μ_p, μ_n, and variance rate ν_p, ν_n, respectively. The VG process is a finite variation process and may be represented as the difference of two independent increasing processes, G_p and G_n. That is,

$$X_{VG}(t) = G_p(t) - G_n(t).$$

The characteristic functions of the two gamma processes are

$$\phi_{G_p}(u) = \left(\frac{1}{1 - iu\nu_p/\mu_p} \right)^{\mu_p^2 t/\nu_p},$$

and

$$\phi_{G_n}(u) = \left(\frac{1}{1 - iu\nu_n/\mu_n} \right)^{\mu_n^2 t/\nu_n}.$$

Let $\nu_p/\mu_p^2 = \nu_n/\mu_n^2 = \nu$. Then the characteristic function of the difference of the two gamma processes G_p and G_n is

$$\phi_{G_p - G_n}(u) = \left(\frac{1}{1 - iu\left(\frac{\nu_p}{\mu_p} - \frac{\nu_n}{\mu_n}\right) + u^2 \frac{\nu_p}{\mu_p}\frac{\nu_n}{\mu_n}} \right)^{t/\nu}. \tag{3.5}$$

Note that $\phi_{G_p-G_n}(u) = \phi_{X_{VG}}(u)$, by comparing the two equations (3.4) and (3.5). We then have

$$\begin{aligned}
\mu_p &= \frac{1}{2}\sqrt{\theta^2 + \frac{2\sigma^2}{\nu}} + \frac{\theta}{2}, \\
\mu_n &= \frac{1}{2}\sqrt{\theta^2 + \frac{2\sigma^2}{\nu}} - \frac{\theta}{2}, \\
\nu_p &= \mu_p^2 \nu, \\
\nu_n &= \mu_n^2 \nu.
\end{aligned}$$

The Lévy density of the VG model is then easily obtained from this representation via the Lévy density of the gamma process. The Lévy density of the VG process is

$$k_{VG}(x) = \begin{cases} \frac{1}{\nu|x|}\exp(-\frac{\mu_n}{\nu_n}|x|) & \text{for } x < 0 \\ \frac{1}{\nu|x|}\exp(-\frac{\mu_p}{\nu_p}|x|) & \text{for } x > 0 \end{cases}. \tag{3.6}$$

To express the VG Lévy density in terms of the original parameters, θ, σ, and ν, we have,

$$k_{VG}(x) = \frac{\exp(\theta x/\sigma^2)}{\nu|x|} \exp\left(-\frac{|x|}{\sigma}\sqrt{\frac{2}{\nu} + \frac{\theta^2}{\sigma^2}} \right). \tag{3.7}$$

3.3 Moments of the VG process

The moments of the VG process may be easily obtained by differentiating its characteristic function.

Theorem 3.7. [25] *If $E[|X|^k] < \infty$, then $\phi_X^{(k)}$ exists and*

$$E[X^k] = i^{-k}\phi_X^{(k)}(0).$$

Madan, Carr, and Chang [8] showed that

$$\begin{aligned}
E[X(t)] &= \theta t, \\
E[(X(t))^2] &= (\theta^2\nu + \sigma^2)t + \theta^2 t^2, \\
E[(X(t) - E[X(t)])^2] &= (\theta^2\nu + \sigma^2)t, \\
E[(X(t) - E[X(t)])^3] &= (2\theta^3\nu^2 + 3\sigma^2\theta\nu)t, \\
E[(X(t) - E[X(t)])^4] &= (3\sigma^4\nu + 12\sigma^2\theta^2\nu^2 + 6\theta^4\nu^3)t + \\
&\quad (3\sigma^4 + 6\sigma^2\theta^2\nu + 3\theta^4\nu^2)t^2.
\end{aligned}$$

Definition 3.8. *For any random variable X with $E|X|^4 < \infty$, define*

$$\begin{aligned}
s &= \frac{E[(X(t) - E[X(t)])^3]}{(E[(X(t) - E[X(t)])^2])^{3/2}}, \\
k &= \frac{E[(X(t) - E[X(t)])^4]}{(E[(X(t) - E[X(t)])^2])^2}.
\end{aligned}$$

The quantity s is known as the coefficient of skewness and is used as a measure of asymmetry, and k is known as kurtosis and is used to measure the peakedness of a distribution.

In particular, the coefficient of skewness s_{VG} and the coefficient of the kurtosis k_{VG} of the VG process are evaluated as follows:

$$\begin{aligned}
s_{VG} &= \frac{\theta\nu t(2\theta^2\nu + 3\sigma^2)}{(\sigma^2 + \theta^2\nu)^{3/2} t^{3/2}}, \\
k_{VG} &= 3 + \frac{3\nu}{t}\left(2 - \frac{\sigma^4}{(\theta^2\nu + \sigma^2)^2}\right).
\end{aligned}$$

We observe that in the VG process, skewness is zero if $\theta = 0$. Also, when $\theta = 0$, $k = 3(1 + \frac{\nu}{t})$. We see that the parameters θ and ν control skewness and kurtosis. If $\theta \neq 0$, ν affects skewness. θ also affects kurtosis.

3.4 Simulation of the VG process

Since a VG process may be seen either as a time-changed Brownian motion or as the difference of two independent gamma processes, a VG process can be simulated either as a time-changed Brownian motion or as the difference of two independent gamma processes.

3.4.1 Simulation of VG as a time-changed Brownian motion

A VG process X_{VG} with parameters θ, ν, and σ can be written as

$$X_{VG} = \theta g + \sigma W(g),$$

where $W(g) \stackrel{d}{=} g^{1/2}Z$, Z is a standard normal random variable and Z is independent of g. A sample path of the VG process may be obtained by simulating a gamma process g with shape parameter $1/\nu$, and scale parameter ν; and by independently simulating a standard Brownian motion, that is, by simulating normal random numbers with mean 0 and variance $\sigma^2 g$.

3.4.2 Simulation of VG as the difference of two gamma processes

A VG process $X_{VG}(t)$ can be written as the difference of two independent gamma processes $G_p(t)$ and $G_n(t)$, that is,

$$X_{VG}(t) = G_p(t) - G_n(t).$$

G_p and G_n are two independent gamma processes with mean rate μ_p, μ_n, and variance rate ν_p, ν_n, respectively. A sample path of VG process may be obtained by simulating G_p with the shape parameter $\frac{{\mu_p}^2}{\nu_p}$, and the scale parameter $\frac{\nu_p}{\mu_p}$; and G_n with the shape parameter $\frac{{\mu_n}^2}{\nu_n}$, and the scale parameter $\frac{\nu_n}{\mu_n}$.

3.5 The VG stock price model

The VG stock price is constructed by replacing the Brownian motion in the Black-Scholes geometric Brownian motion model by the VG process, and define the risk free process for the VG stock price $S(t)$ at time t, such that the mean rate of return equals the continuous compounded interest rate r. We have

$$S(t) = S(0)\frac{\exp(rt + X_{VG}(t))}{E[\exp(X_{VG}(t))]},$$

so that $E(S(t)) = S(0)\exp(rt)$. In particular,

$$\exp(-wt) = E[\exp(X_{VG}(t))] = \phi_{X_{VG}}(-i) = \exp\left(-\frac{t}{\nu}\ln\left(1-\theta\nu-\frac{\sigma^2\nu}{2}\right)\right).$$

Then $S(t) = S(0)\exp\left((r+w)t + X_{VG}(t)\right)$ for $w = 1/\nu\ln(1-\theta\nu-\sigma^2\nu/2)$. We call w the convexity correction factor. We may derive the characteristic function of the $\ln(S(t))$ from that of the $X_{VG}(t)$. Thus,

$$\begin{aligned}
\phi_{\ln(S(t))}(u) &= E[\exp(iu\ln(S(t)))] \\
&= e^{\left(iu(\ln(S(0))+rt+\frac{t}{\nu}\ln(1-\theta\nu-\frac{\sigma^2\nu}{2}))\right)}\phi_{X_{VG}}(u) \\
&= e^{\left(iu(\ln(S(0))+rt+\frac{t}{\nu}\ln(1-\theta\nu-\frac{\sigma^2\nu}{2}))\right)}\left(1-iu\theta\nu+\frac{\sigma^2\nu}{2}u^2\right)^{-\frac{t}{\nu}}. \qquad (3.8)
\end{aligned}$$

The VG density may be found by applying Fourier inversion to the characteristic function of the $\log$ of the stock price.

Chapter 4

Multivariate asset pricing

Two stock price processes are often driven by the same factors. In this chapter, we would like to investigate the $\log$ of the stock prices in terms of the linear combinations of two independent VG processes. The two independent VG processes are treated as the common factors of two stock prices.

4.1 Multivariate VG process

Let X_1 and X_2 be two independent VG processes with parameters θ_i, ν_i, and σ_i, for $i = 1, 2$. According to equation (3.3), the characteristic function of a VG process may be written as

$$\phi_{VG}(u) = \exp\left(\int_{-\infty}^{\infty} (\exp(iux) - 1)k(x)dx\right). \tag{4.1}$$

Here, we suppose both the absence of a continuous martingale and a deterministic drift rate. A deterministic drift rate may easily be added in applications, we are mainly concerned with the stochastic component. We then have the characteristic function ϕ_{X_1}, ϕ_{X_2} of X_1, X_2 as,

$$\phi_{X_1}(u) = \exp\,(\int_{-\infty}^{\infty} (\exp(iux_1) - 1)k_1(x_1)dx_1),$$

and

$$\phi_{X_2}(v) = \exp\,(\int_{-\infty}^{\infty} (\exp(ivx_2) - 1)k_2(x_2)dx_2),$$

where

$$k_i(x_i) = \frac{\exp\,(\theta_i x_i/{\sigma_i}^2)}{\nu_i |x_i|} \exp\left(-\frac{|x_i|}{\sigma_i}\sqrt{\frac{2}{\nu_i} + \frac{{\theta_i}^2}{{\sigma_i}^2}}\right),\ \text{for } i = 1, 2.$$

Definition 4.1. *The joint characteristic function $\Phi_{X_1,X_2}(u, v)$ of two random variables X_1 and X_2 is denoted by*

$$\Phi_{X_1,X_2}(u, v) = E[\exp(iuX_1 + ivX_2)].$$

Lemma 4.2. *Let X_1 and X_2 be two independent random variables. Let $\phi_{X_1}(u)$ and $\phi_{X_2}(v)$ be the characteristic functions of X_1 and X_2, respectively. The joint characteristic function $\Phi_{X_1,X_2}(u, v)$ of X_1 and X_2 is*

$$\begin{aligned} \Phi_{X_1,X_2}(u, v) &= E[\exp(iuX_1 + ivX_2)] \\ &= E[\exp(iuX_1)]E[\exp(iuX_2)] \qquad (4.2) \\ &= \phi_{X_1}(u)\phi_{X_2}(v). \qquad (4.3) \end{aligned}$$

Fact 4.3. *The Lévy-Khintchine representation of the joint characteristic function of two VG processes $X_1(t)$ and $X_2(t)$ is denoted as*

$$\Phi_{X_1,X_2}(u, v) = \exp\,(t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\exp(i(ux_1 + vx_2)) - 1)k(x_1, x_2)dx_1dx_2). \qquad (4.4)$$

Definition 4.4. *The indicator function of a subset A of a set X is a function from A into $\{1, 0\}$ defined as follows*

$$1_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases}$$

Theorem 4.5. *Let $X_1(t)$ and $X_2(t)$ be two independent VG processes with parameters θ_1, ν_1, σ_1 and θ_2, ν_2, σ_2, respectively. The characteristic function $\phi_{X_1}(u)$ of $X_1(t)$ is*

$$\phi_{X_1}(u) = \exp\,(t \int_{-\infty}^{\infty} (\exp(iux_1) - 1)k_1(x_1)dx_1),$$

and the characteristic function $\phi_{X_2}(u)$ of $X_2(t)$ is

$$\phi_{X_2}(v) = \exp\,(t \int_{-\infty}^{\infty} (\exp(ivx_2) - 1)k_2(x_2)dx_2),$$

where the Lévy densities $k_i(x_i)$ are

$$k_i(x_i) = \frac{\exp\left(\frac{\theta_i x_i}{\sigma_i{}^2}\right)}{\nu_i|x_i|} \exp\left(-\frac{\sqrt{\frac{2}{\nu_i} + \frac{\theta_i{}^2}{\sigma_i{}^2}}}{\sigma_i}|x_i|\right), \text{ for i = 1, 2.}$$

Then the joint characteristic function $\Phi_{X_1,X_2}(u,v)$ of X_1 and X_2 is

$$\Phi_{X_1,X_2}(u,v) = \exp\,(t \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\exp(i(ux_1 + vx_2)) - 1)k(x_1,x_2)dx_1dx_2),$$

where

$$k(x_1,x_2) = 1_{x_2=0}k_1(x_1) + 1_{x_1=0}k_2(x_2).$$

Proof. Let X_1 and X_2 be two independent VG processes. Then

$$\Phi_{X_1,X_2}(u,v)$$
$$= E[e^{(iuX_1+ivX_2)}]$$
$$= e^{(t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\exp(i(ux_1+vx_2))-1)k(x_1,x_2)dx_1dx_2)}$$
$$= E[e^{(iuX_1)}]E[e^{(iuX_2)}]$$
$$= \phi_{X_1}(u)\phi_{X_2}(v)$$
$$= e^{(t\int_{-\infty}^{\infty}(\exp(iux_1)-1)k_1(x_1)dx_1)}e^{(t\int_{-\infty}^{\infty}(\exp(ivx_2)-1)k_2(x_2)dx_2)}$$
$$= e^{(t((\int_{-\infty}^{\infty}(\exp(iux_1)-1)k_1(x_1)dx_1)+(\int_{-\infty}^{\infty}(\exp(ivx_2)-1)k_2(x_2)dx_2)))}$$
$$= e^{(t((\int_{-\infty}^{\infty}[\int_{-\infty}^{\infty}(\exp(i(ux_1+vx_2))-1)(1_{x_2=0}k_1(x_1))dx_2]dx_1)+(\int_{-\infty}^{\infty}(\exp(ivx_2)-1)k_2(x_2)dx_2)))}$$
$$= e^{(t(\int_{-\infty}^{\infty}[\int_{-\infty}^{\infty}(\exp(i(ux_1+vx_2))-1)(1_{x_2=0}k_1(x_1))dx_1+(\exp(ivx_2)-1)k_2(x_2)]dx_2))}$$
$$= e^{(t(\int_{-\infty}^{\infty}[\int_{-\infty}^{\infty}(\exp(i(ux_1+vx_2))-1)(1_{x_2=0}k_1(x_1))dx_1+\int_{-\infty}^{\infty}(\exp(i(ux_1+vx_2))-1)(1_{x_1=0}k_2(x_2))]dx_2))}$$
$$= e^{(t(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\exp(i(ux_1+vx_2))-1)(1_{x_2=0}k_1(x_1)+1_{x_1=0}k_2(x_2))dx_1dx_2))}.$$

We then have

$$k(x_1,x_2) = 1_{x_2=0}k_1(x_1) + 1_{x_1=0}k_2(x_2).$$

□

The function $k_i(x)$ specifies the arrival rate of jumps of size x, for $i =$ 1, 2. Since $X_1(t)$ and $X_2(t)$ are independent processes, $k_1(x_1)$ and $k_2(x_2)$ are also independent of each other. From Theorem 4.5, we see that almost surely the two independent processes $X_1(t)$ and $X_2(t)$ do not have joint jumps.

4.2 Two-factor VG stock price model

Let X_1 and X_2 be two independent VG processes. Let S_1 and S_2 be two stock prices given by the following processes:

$$S_1 = \frac{\exp(m_1 t + B_1 X_1(t) + B_2 X_2(t))}{E[\exp(B_1 X_1(t) + B_2 X_2(t))]} \tag{4.5}$$

and

$$S_2 = \frac{\exp(m_2 t + C_1 X_1(t) + C_2 X_2(t))}{E[\exp(C_1 X_1(t) + C_2 X_2(t))]}, \tag{4.6}$$

where m_1, m_2 are continuous compounded interest rates, and B_1, B_2, C_1, and C_2 are coefficients of the linear combination of the $\log$ of the stock prices, S_1 and S_2. Hence S_1 and S_2 are two factor VG processes. Let

$$B(t) = E[\exp(B_1 X_1(t) + B_2 X_2(t))], \tag{4.7}$$

$$C(t) = E[\exp(C_1 X_1(t) + C_2 X_2(t))]. \tag{4.8}$$

$B(t)$ and $C(t)$ are the convexity correction factors of S_1 and S_2. We may see that the mean rate of returns of S_1 and S_2 equal the continuous compounded interest rates m_1 and m_2, respectively.

Lemma 4.6. *Assume X_1 and X_2 are two independent VG processes with parameters θ_1, ν_1, σ_1, and θ_2, ν_2, σ_2, respectively. Let S_1 and S_2 be the two stock prices described in equations 4.5 and 4.6, respectively. We have*

$$\begin{aligned}
\ln S_1 &= m_1 t + B_1 X_1(t) + B_2 X_2(t) - \ln B(t), \\
\ln S_2 &= m_2 t + C_1 X_1(t) + C_2 X_2(t) - \ln C(t),
\end{aligned}$$

where

$$\begin{aligned}
B(t) &= E[\exp(B_1 X_1(t) + B_2 X_2(t))], \\
C(t) &= E[\exp(C_1 X_1(t) + C_2 X_2(t))],
\end{aligned}$$

m_1, m_2 are continuous compounded interest rates, and B_1, B_2, C_1, and C_2 are constant coefficients of the linear combination of $\ln S_1$ and $\ln S_2$. Then the joint characteristic function $\Phi_{\ln S_1, \ln S_2}(u, v)$ of $\ln S_1$ and $\ln S_2$ is

$$\begin{aligned}
\Phi_{\ln S_1, \ln S_2}(u, v) &= E[\exp(iu \ln S_1 + iv \ln S_2)] \\
&= \exp(i(m_1 u + m_2 v)) \cdot \\
&\quad [1 - \theta_1 \nu_1 B_1 - \frac{{\sigma_1}^2 \nu_1}{2} {B_1}^2]^{\frac{iut}{\nu_1}} \cdot [1 - \theta_2 \nu_2 B_2 - \frac{{\sigma_2}^2 \nu_2}{2} {B_2}^2]^{\frac{iut}{\nu_2}} \cdot \\
&\quad [1 - \theta_1 \nu_1 C_1 - \frac{{\sigma_1}^2 \nu_1}{2} {C_1}^2]^{\frac{ivt}{\nu_1}} \cdot [1 - \theta_2 \nu_2 C_2 - \frac{{\sigma_2}^2 \nu_2}{2} {C_2}^2]^{\frac{ivt}{\nu_2}} \cdot \\
&\quad [1 - i\theta_1 \nu_1 (uB_1 + vC_1) + \frac{{\sigma_1}^2 \nu_1}{2} (uB_1 + vC_1)^2]^{-\frac{t}{\nu_1}} \cdot \\
&\quad [1 - i\theta_2 \nu_2 (uB_2 + vC_2) + \frac{{\sigma_2}^2 \nu_2}{2} (uB_2 + vC_2)^2]^{-\frac{t}{\nu_2}}.
\end{aligned}$$

Proof. See A.1. □

Proposition 4.7. *Let $\Phi_{\ln S_1, \ln S_2}(u, v)$ be the joint characteristic function of $\ln S_1$ and $\ln S_2$ where S_1 and S_2 are defined in equations 4.5 and 4.6. Then the marginal characteristic functions $\phi_{\ln S_1}(u)$ of $\ln S_1$ and $\phi_{\ln S_2}(v)$ of $\ln S_2$ are $\Phi_{\ln S_1, \ln S_2}(u, 0)$ and $\Phi_{\ln S_1, \ln S_2}(0, v)$, respectively.*

Proof. See A.2. □

4.3 Moments of VG stock price process

If X_i are VG processes, from Theorem 3.7, we have

$$\begin{aligned}
E[X_i(t)] &= \theta_i t \\
E[{X_i}^2(t)] &= t^2 \theta_i^2 + t \theta_i^2 \nu_i + t \sigma_i^2 \\
&\quad \text{for } i = 1, 2.
\end{aligned}$$

Theorem 4.8. *X_1 and X_2 are two independent random variables, then*

$$Cov(X_1, X_2) = E(X_1X_2) - E(X_1)E(X_2) = 0.$$

Corollary 4.9. *If $X_1(t)$ and $X_2(t)$ are two independent VG processes, then*

$$E(X_1(t)X_2(t)) = \theta_1\theta_2 t^2$$

Proof. X_1 and X_2 are two independent VG processes, then

$$Cov(X_1(t), X_2(t)) = E(X_1(t)X_2(t)) - E(X_1(t))E(X_2(t)) = 0.$$

Thus,

$$\begin{aligned} E(X_1(t)X_2(t)) &= E(X_1(t))E(X_2(t)) \\ &= \theta_1\theta_2 t^2. \end{aligned}$$

□

Proposition 4.10. *Let $X_1(t)$ and $X_2(t)$ be two independent VG processes. If S_1 and S_2 are defined in equations 4.5 and 4.6, and $B(t)$ and $C(t)$ are defined in 4.7 and 4.8, respectively, then*

$$\begin{aligned} E(\ln S_1) &= m_1 t + B_1\theta_1 t + B_2\theta_2 t - \ln B(t) \\ E(\ln S_2) &= m_2 t + C_1\theta_1 t + C_2\theta_2 t - \ln C(t). \end{aligned}$$

Proof.

$$\begin{aligned} E(\ln S_1) &= E(m_1 t + B_1 X_1(t) + B_2 X_2(t) - \ln B(t)) \\ &= m_1 t + B_1\theta_1 t + B_2\theta_2 t - \ln B(t), \end{aligned}$$

and similarly,

$$E(\ln S_2) = m_2 t + C_1\theta_1 t + C_2\theta_2 t - \ln C(t).$$

□

Proposition 4.11. *Let $X_1(t)$ and $X_2(t)$ be two independent VG processes. Suppose S_1 and S_2 are defined in equations 4.5 and 4.6, respectively. Then*

$$\begin{aligned} Var(\ln S_1) &= B_1^2 t(\theta_1^2\nu_1 + \sigma_1^2) + B_2^2 t(\theta_2^2\nu_2 + \sigma_2^2) \\ Var(\ln S_2) &= C_1^2 t(\theta_1^2\nu_1 + \sigma_1^2) + C_2^2 t(\theta_2^2\nu_2 + \sigma_2^2). \end{aligned}$$

Proof. See A.3. □

Proposition 4.12. *Let $X_1(t)$ and $X_2(t)$ be two independent VG processes. Suppose S_1 and S_2 are defined in equations 4.5 and 4.6, respectively. Then*

$$Cov(\ln S_1, \ln S_2) = B_1C_1 t(\theta_1^2\nu_1 + \sigma_1^2) + B_2C_2 t(\theta_2^2\nu_2 + \sigma_2^2).$$

Proof. See A.4. □

We see that the variance of $\ln S_1$, the variance of $\ln S_2$, and the covariance of $\ln S_1$ and $\ln S_2$ are all proportional to t.

4.4 Conditional expectation of VG stock price process

Theorem 4.13. [34] *Suppose that X_1 and X_2 are bivariate normal random variables with mean μ_1, μ_2, and variance σ_1^2, σ_2^2, respectively. The correlation coefficient ρ of X_1 and X_2 is denoted as*

$$\rho = \frac{Cov(X_1, X_2)}{\sigma_1\sigma_2}.$$

Then $X_2|X_1$ is a normal distribution with mean $\mu_2 + \beta(X_1 - \mu_1)$, and variance $\sigma_2^2(1 - \rho^2)$, where $\beta = \rho\sigma_2/\sigma_1$.

Theorem 4.14. *If X_1 and X_2 are independent normally distributed random variables with means μ_1, μ_2, and variances σ_1^2, σ_2^2, respectively, then $X_1 + X_2$ is also a normally distributed random variable with mean $(\mu_1 + \mu_2)$ and variance $(\sigma_1^2 + \sigma_2^2)$.*

Let $X_1(t)$ and $X_2(t)$ be two independent processes. Suppose B_i and C_i are constants for $i = 0, 1, 2$. Suppose

$$\begin{aligned} Y_1(t) &= B_0 + B_1 X_1(t) + B_2 X_2(t), \\ Y_2(t) &= C_0 + C_1 X_1(t) + C_2 X_2(t). \end{aligned}$$

By Theorem 4.14, if X_1 and X_2 are Gaussian processes, then Y_1 and Y_2 are Gaussian processes as well. According to Theorem 4.13, the conditional expectation $E(Y_2|Y_1)$ is a linear function if X_1 and X_2 are normal random variables. We would like to investigate the conditional expectation $E(Y_2|Y_1)$ when X_1 and X_2 are two independent VG processes.

Theorem 4.15. *Suppose X_1 and X_2 are two independent VG processes that g_1 and g_2 are two independent gamma processes and that W_1 and W_2 are standard Brownian motions. Suppose*

$$\begin{aligned} X_1 &= \theta_1 g_1 + \sigma_1 W_1(g_1) \\ X_2 &= \theta_2 g_2 + \sigma_2 W_2(g_2) \end{aligned}$$

and

$$\begin{aligned} Y_1 &= B_0 + B_1 X_1(t) + B_2 X_2(t) \\ Y_2 &= C_0 + C_1 X_1(t) + C_2 X_2(t), \end{aligned}$$

where B_i and C_i are constants for $i = 0, 1, 2$. We then have

$$E(Y_2|Y_1) = E(\alpha_t|Y_1) + E(\beta_t|Y_1)Y_1,$$

where

$$\begin{aligned} \beta_t &= \frac{B_1 C_1 \sigma_1^2 g_1 + B_2 C_2 \sigma_2^2 g_2}{B_1^2 \sigma_1^2 g_1 + B_2^2 \sigma_2^2 g_2} \\ \alpha_t &= C_0 + C_1 \theta_1 t + C_2 \theta_2 t - \beta_t (B_0 + B_1 \theta_1 t + B_2 \theta_2 t). \end{aligned}$$

Proof. By Theorem 4.13,

$$E[Y_2|Y_1, g_1, g_2] = \alpha_t + \beta_t Y_1,$$

where

$$\beta_t = \frac{Cov(Y_1, Y_2|g_1, g_2)}{Var(Y_1|g_1, g_2)}.$$

$$\begin{aligned}
Cov(Y_1, Y_2|g_1, g_2) &= E(Y_1Y_2|g_1, g_2) - E(Y_1|g_1, g_2)E(Y_2|g_1, g_2), \\
Var(Y_1|g_1, g_2) &= E(Y_1^2|g_1, g_2) - (E(Y_1|g_1, g_2))^2.
\end{aligned}$$

$$\begin{aligned}
E(X_1|g_1, g_2) &= \theta_1 g_1, \\
E(X_2|g_1, g_2) &= \theta_2 g_2, \\
E(X_1^2|g_1, g_2) &= Var(X_1|g_1, g_2) + (E(X_1|g_1, g_2))^2 \\
&= \sigma_1^2 g_1 + \theta_1^2 g_1^2.
\end{aligned}$$

Similarly,

$$E(X_2^2|g_1, g_2) = \sigma_2^2 g_2 + \theta_2^2 g_2^2.$$

$$\begin{aligned}
E(Y_1Y_2|g_1, g_2) = {} & B_0C_0 + B_0C_1\theta_1 g_1 + B_1C_0\theta_1 g_1 + B_0C_2\theta_2 g_2 + \\
& B_2C_0\theta_2 g_2 + B_1C_2\theta_1\theta_2 g_1 g_2 + B_2C_1\theta_1\theta_2 g_1 g_2 + \\
& B_1C_1(\sigma_1^2 g_1 + \theta_1^2 g_1^2) + B_2C_2(\sigma_2^2 g_2 + \theta_2^2 g_2^2), \qquad (4.9)
\end{aligned}$$

$$\begin{aligned}
E(Y_1|g_1, g_2)E(Y_2|g_1, g_2) &= (B_0 + B_1\theta_1 g_1 + B_2\theta_2 g_2)(C_0 + C_1\theta_1 g_1 + C_2\theta_2 g_2) \\
&= B_0C_0 + B_0C_1\theta_1 g_1 + B_1C_0\theta_1 g_1 + B_0C_2\theta_2 g_2 + \\
&\quad B_2C_0\theta_2 g_2 + B_1C_2\theta_1\theta_2 g_1 g_2 + B_2C_1\theta_1\theta_2 g_1 g_2 + \\
&\quad B_1C_1\theta_1^2 g_1^2 + B_2C_2\theta_2^2 g_2^2. \qquad (4.10)
\end{aligned}$$

Subtract equation (4.9) by equation (4.10), we get

$$\begin{aligned} Cov(Y_1, Y_2|g_1, g_2) &= B_1C_1(\sigma_1^2 g_1 + \theta_1^2 g_1^2) + B_2C_2(\sigma_2^2 g_2 + \theta_2^2 g_2^2) - \\ & \quad B_1C_1\theta_1^2 g_1^2 + B_2C_2\theta_2^2 g_2^2 \\ &= B_1C_1\sigma_1^2 g_1 + B_2C_2\sigma_2^2 g_2. \end{aligned}$$

$$\begin{aligned} Var(Y_1|g_1, g_2) &= E(Y_1^2|g_1, g_2) - (E(Y_1|g_1, g_2))^2 \\ &= Var(Y_1|g_1, g_2) + (E(Y_1|g_1, g_2))^2 - (E(Y_1|g_1, g_2))^2 \\ &= B_1^2\sigma_1^2 g_1 + B_2^2\sigma_2^2 g_2. \end{aligned}$$

Hence,

$$\beta_t = \frac{B_1C_1\sigma_1^2 g_1 + B_2C_2\sigma_2^2 g_2}{B_1^2\sigma_1^2 g_1 + B_2^2\sigma_2^2 g_2}$$

$$\begin{aligned} \alpha_t &= E(Y_2) - \beta E(Y_1) \\ &= C_0 + C_1\theta_1 t + C_2\theta_2 t - \beta_t(B_0 + B_1\theta_1 t + B_2\theta_2 t). \end{aligned}$$

$$\begin{aligned} E(Y_2|Y_1) &= E(E(Y_2|Y_1, g_1, g_2)|Y_1) \\ &= E(\alpha_t + \beta_t Y_1|Y_1) \\ &= E(\alpha_t|Y_1) + E(\beta_t Y_1|Y_1) \\ &= E(\alpha_t|Y_1) + E(\beta_t|Y_1)Y_1. \end{aligned}$$

$\square$

We may see that when X_1 and X_2 are two independent VG processes, then the conditional expectation $E(Y_2|Y_1)$ is approaching to a linear function.

Chapter 5

Independent component analysis

Independent component analysis (ICA) is a technique for finding underlying factors for the observed multivariate data. The observed multivariate data are assumed to be linear or nonlinear combinations of some unknown latent variables. The mixing system is also to be determined by the technique. The latent variables are assumed statistically independent and nongaussian. Namely, the latent variables are the independent components of the observed data. The details of ICA can be found in [23], [1], and [7]. This chapter follows the guidelines of [23].

5.1 Second-order methods

One can treat ICA as an extension to the classical principal component analysis (PCA) and factor analysis (FA). Both PCA and FA are second-order methods. The second-order methods require only the information contained in the variances and the covariances of the observed data. The second-order methods often assume that the observed data are normally distributed so that the higher moments are irrelevant.

5.1.1 Principal component analysis

Principal components are linear combinations of random variables which have special properties associated with variances. The first principal component is the linear transformation with maximum variance [24, 26].

Suppose matrix C is the covariance matrix of the random vector X with n components. Furthermore, we assume that the mean of the vector X is zero. Let w be a n-component scalar column vector such that $w'w = 1$, and suppose $Y = w'X$. Denote y_1 as the first principal component of X. That is,

$$y_1 = \sum_{i=1}^{n} w_{i1} x_i = w_1' x.$$

The variance of $w'X$ is

$$E(w'X)^2 = E(w'XX'w) = w'Cw,$$

where $C = E(XX')$. To find the linear transformation $w'X$ with maximum variance, we need to find w which maximizes the variance of $w'X$ subject to the constraint $w'w = 1$. From the facts of linear algebra [15], the PCA problem is solved in terms of the unit-length eigenvectors $e_1, ..., e_n$ of the covariance matrix C. Let $d_1, ..., d_n$ be the corresponding eigenvalues of the eigenvectors $e_1, ..., e_n$ with $d_1 \geq d_2 \geq \cdots \geq d_n$. The solution that maximizes the variance of $w'X$ so that $w'w = 1$ is given by $w_1 = e_1$, and the first principal component of X is $y_1 = e_1'X$. To find the rest of the principal components, we need to find a combination $w'X$ that has maximum variance and is uncorrelated to the previously found principal components. For example, second principal component y_2 satisfies the condition

$$E(y_2 y_1) = E((w_2'X)(w_1'X)) = w_2' C w_1 = d_1 w_2' e_1 = 0.$$

The solution is given by $w_2 = e_2$. This procedure is continued, and it follows that $w_i = e_i$. Then the ith principal component is $y_i = e_i'X$.

5.1.2 Factor analysis

The general model for factor analysis is

$$x = As + n \tag{5.1}$$

where x is the observed data, s is the vector of factors that cannot be observed, A is a constant matrix, and n is the noise vector. Both s and n are assumed to be Gaussian. The dimension of s is assumed to be lower than the dimension of x. Factor analysis is a modification of PCA. However, factor analysis results are invariant under rescaling, but PCA results are not. Assume the covariance matrix $E(nn')$ of the noise n is known. The factors are found by processing PCA using the modified covariance matrix $E(xx') - E(nn')$ [2]. Thus the vector s is the vector of principal components of x without noise. Equation (5.1) does not uniquely define the factors. One must impose extra conditions to get a unique model.

5.2 Definition of ICA

Second-order methods often assume that the observed data are Gaussian, which is usually not true for the real data. ICA takes the higher-order moments into account. ICA is the process of finding the linear combination of a set of statistically independent vectors y that determines the multivariate observed data x. The vectors y are called the independent components, which are the estimates of the original factors and are mixed to form the observed data [23].

Assume we observe multivariate data $x_1, ..., x_n$. Furthermore, we assume that x is a linear function of the independent components,

$$x_i = a_{i1}s_1 + a_{i2}s_2 + ... + a_{in}s_n, \text{ for all } i.$$

Or equivalently, we use vector-matrix notation so that the above system of equations is written as

$$x = As, \tag{5.2}$$

where A is the unknown mixing matrix. For simplicity, we assume there are as many observed signals as their original sources. Thus A is a $n \times n$ square matrix. Without loss of generality, we center the observed data by subtracting the sample mean, so that the model is zero-mean. The goal of ICA is to find a demixing matrix W such that

$$\begin{aligned} y &= Wx \\ &= WAs. \end{aligned}$$

If $W = A^{-1}$, then $y = s$. It is hard to find the perfect separation $y = s$. In general, it's possible to find W such that $WA = PD$ where P is a permutation matrix, and D is a diagonal matrix.

5.3 Principles of ICA

The first assumption of ICA is that the components s_i are statistically independent. Independence among random variables is a much stronger requirement than lack of correlation. Independence implies zero covariance, but zero covariance does not always imply independence of random variables. In PCA and FA, the random variables are assumed to be Gaussian. For Gaussian random variables, independence and zero covariance are equivalent. For PCA and FA, it is enough to use zero covariance. For ICA, zero covariance is not enough. Note that, if two random variables s_1 and s_2 are independent, then any nonlinear transformations $g_1(s_1)$ and $g_2(s_2)$ are independent. ICA is a method for finding the demixing matrix W by decorrelation. That is, one finds the matrix W such that the components s_i and s_j are uncorrelated and the nonlinear transformation $g_1(s_i)$ and $g_2(s_j)$ are also uncorrelated.

The second assumption of ICA is that the independent components are nongaussian distributions. The observed data cannot be separated if the components are Gaussian. For example, assume that two independent components s_1 and s_2 are Gaussian. The joint density of s_1 and s_2 is

$$p(s_1, s_2) = \frac{1}{2\pi}\exp(-\frac{s_1^2 + s_2^2}{2}) = \frac{1}{2\pi}\exp(-\frac{\parallel s \parallel^2}{2}).$$

Assume also that the mixing matrix A is orthogonal. The joint density of the observed signals x_1 and x_2 is

$$p(x_1, x_2) = \frac{1}{2\pi}\exp(-\frac{\parallel A'x \parallel^2}{2})|\mathrm{det}A'| = \frac{1}{2\pi}\exp(-\frac{\parallel x \parallel^2}{2}).$$

The equality holds because of the orthogonality of the mixing matrix A. For an orthogonal matrix A, $||A'x||^2 = ||x||^2$ and $|\mathrm{det}A| = |\mathrm{det}A'| = 1$. The independent components and the observed signals have the same distribution. The ICA model is not well defined if $s_1, s_2, \cdots, s_n$ are jointly Gaussian. For Gaussian random variables, independence and zero covariance are equivalent. Also, the higher-order cumulants of Gaussian distribution are zero. ICA is a process that utilizes the information of higher-order cumulants. The information of higher-order cumulants is important.

5.4 Maximize the nongaussianity of ICA

One of the main assumptions of ICA is that the independent components are nongaussian. Thus we would like to maximize the nongaussianity of ICA. The Central Limit Theorem is one of the most important results in probability theory. It states that under certain conditions, the sum of a large number of independent random variables is approximately gaussian. Loosely speaking, the sum of independent random variables has a distribution that is more gaussian than the original random variables.

Recall the ICA model in equation 5.2. The goal is to find the latent variables s and the mixing matrix A. To find the independent components, let us consider a set of linear combinations of the multivariate observed data x. We write $y = Wx = WAs$. We see that y is a linear function of s. Under regularity conditions on WA and the distribution of s, the Central Limit Theorem suggests that $y = WAs$ is more Gaussian than s. However, if $y = WAs$ has only one nonzero component, then y equal to one of the independent components. That is, if $W = A^{-1}$, then $y = s$.

5.5 Objective function of ICA

Negentropy is introduced in information theory to measure nongaussianity. Negentropy is based on entropy. Entropy is a quantity which can be interpreted as the degree of information that the observation of the random variable gives. The larger the entropy, the more random is the variable. A fact in information theory indicates that the Gaussian variable has the largest entropy among all random variables of equal variance [12]. Thus, we can use Entropy to measure nongaussianity. Entropy H of a discrete random variable Y is defined as

$$H(Y) = -\sum_{i} P(Y = a_i) \log P(Y = a_i),$$

where a_i are the possible values of Y. The entropy H of continuous random variables and vectors is called differential entropy, and is defined as

$$H(y) = -\int f(y) \log f(y) dy,$$

where $f(y)$ is the density of the random vector y.

The negentropy J is defined as follows from the differential entropy:

$$J(y) = H(y_{gauss}) - H(y),$$

where y_{gauss} is a Gaussian random vector and the vectors y and y_{gauss} have the same covariance matrix. Since a gaussian random variable has the largest entropy among all the random variables with the same variance, negentropy is always nonnegative. Negentropy is zero if and only if the random vector y is gaussian. Negentropy is invariant under invertible linear transformation.

The mutual information I is a quantity which measures the dependence between random variables y_i, for $i = 1, ..., n$. The mutual information I is defined as

$$I(y_1, ..., y_n) = \sum_{i=1}^{n} H(y_i) - H(y).$$

The measure of mutual information is always nonnegative. It is zero if and only if the y_i are statistically independent. For an invertible linear transformation $y = Wx$, we have

$$I(y_1, ..., y_n) = \sum_{i} H(y_i) - H(x) - \log|\det W|.$$

In context, we assume the y_i are uncorrelated and of unit variance. That is $E(yy') = I = WE(xx')W'$. Observe that

$$\det I = 1 = \det(WE(xx')W') = (\det W)(\det E(xx'))(\det W'),$$

we see that $\det W$ is a constant since $\det E(xx')$ does not depend on W. With the constraint that y_i is unit variance, entropy and negentropy differ by a constant, and the sign. We have [11],

$$\begin{aligned} I(y_1, ..., y_n) &= J(y) - \sum_{i} J(y_i) \\ &= C - \sum_{i} J(y_i), \end{aligned} \tag{5.3}$$

where C is a constant. Equation 5.3 shows the relation between negentropy and mutual information. We also see that determining the matrix W which minimizes the mutual information is equivalent to maximizing the negentropy.

5.6 Approximation of the objective function

To be able to evaluate the negentropy efficiently using the definition of ICA, approximations were developed in [21], [11]. For the simplest case, the approximations are as follows:

$$J(y_i) \approx c\left(E(G(y_i)) - E(G(y_{gauss}))\right)^2,$$

where G is a non-quadratic function, c is a constant, and y_{gauss} is a standard gaussian random variable. For the ICA model, one needs to maximize J_G to find one independent component $y_i = w'x$, where

$$J_G(w) = \left(E(G(w'x)) - E(G(y_{gauss}))\right)^2,$$

with the constraint $E((w'x)^2) = 1$. Or equivalently,

$$\begin{array}{ll} \text{maximize} & \sum_{i=1}^{n} J_G(w_i), i = 1, ..., n \\ \text{such that} & E((w_i'x)(w_j'x)) = \delta_{ji}. \end{array} \tag{5.4}$$

Hyvärinen gives the following practical selections of the non-quadratic function G:

$$\begin{aligned} G_1(x) &= \frac{1}{c_1}\log\cosh(c_1 x), \\ G_2(x) &= \frac{1}{c_2}\exp(-c_2 x^2/2), \\ G_3(x) &= \frac{1}{4}x^4, \end{aligned}$$

with their derivative g_i, $i = 1, 2, 3$, as follows:

$$g_1(x) = \tanh(c_1 x), \tag{5.5}$$

$$g_2(x) = x \exp(-c_2 x^2/2), \tag{5.6}$$

$$g_3(x) = x^3, \tag{5.7}$$

where c_1 and c_2 are constants, with $1 \leq c_1 \leq 2$, and $c_2 \approx 1$.

5.7 Preprocessing the data for ICA

Preprocessing the data will make the ICA algorithm simpler. It reduces the computational cost significantly.

5.7.1 Centering

Centering of the observed data means that the mean of the observed data is subtracted from the observed data. This procedure makes x a zero-mean variable. This also implies that s is also zero-mean. This can be verified by taking the expectation to both of the equation 5.2. We see that

$$E(x) = AE(s).$$

This procedure simplifies the ICA algorithms.

5.7.2 Whitening

The strategy of whitening the observed data vector x is to linearly transform the observed data into a new vector $\tilde{x}$. The expectation of

$$E(\tilde{x}\tilde{x}') = I.$$

In other words, the components of $\tilde{x}$ are uncorrelated, and the variances are equal to unity. Or equivalently, the covariance matrix of $\tilde{x}$ is the identity matrix.

One way to whiten the data is to use the eigenvalue decomposition of the covariance matrix C of x. That is,

$$C = E(xx') = EDE',$$

where E is the orthogonal matrix of the eigenvectors of matrix C, and D is diagonal matrix of the corresponding eigenvalues. We denote $D = diag(d_1, ..., d_n)$, and that $D^{-\frac{1}{2}} = diag(d_1^{-\frac{1}{2}}, ..., d_n^{-\frac{1}{2}})$. Whitening gives us

$$\tilde{x} = ED^{-\frac{1}{2}}E'x. \tag{5.8}$$

It follows that

$$E(\tilde{x}\tilde{x}') = I.$$

From equations 5.2 and 5.8, we then have

$$\tilde{x} = ED^{-\frac{1}{2}}E'x = ED^{-\frac{1}{2}}E'As = \tilde{A}s.$$

Note that

$$E(\tilde{x}\tilde{x}') = \tilde{A}E(ss')\tilde{A}' = \tilde{A}\tilde{A}' = I,$$

so that the new mixing matrix $\tilde{A}$ is an orthogonal matrix. Estimating the orthogonal matrix $\tilde{A}$ is computationally simpler than estimating the original mixing matrix A.

5.8 The FastICA algorithm

The FastICA algorithm was developed in $[20, 22]$. FastICA is a fixed point iteration which is used to solve for the optimization problem defined in (5.4). The algorithm consists of the following steps:

1. Assign an initial weight vector w_0 randomly.

2. Let $w^+ = E(xg(w'x)) - E(\tilde{g}(w'x))w$.

3. Let $w_1 = w^+ / \parallel w^+ \parallel$.

4. If $|1 - w_0' w_1| > \epsilon$, go back to 2,

where g is defined in equations 5.5, and 5.6, and $\tilde{g}$ is derivative of g. The convergence means that the dot product of the old and new values of w is close to 1.

The derivation of FastICA is as follows. Using the Lagrange multiplier λ, the optimization problem defined in (5.4) can be written as

$$F(w) = E(xg(w'x)) - \lambda w = 0. \tag{5.9}$$

Applying Newton's method to solve equation 5.9, we get the Jacobian matrix $JF(w)$ as

$$JF(w) = E(xx'\tilde{g}(w'x)) - \lambda I. \tag{5.10}$$

Since the data is whitened, in [20], the following approximation is applied to make the Jacobian matrix diagonal: $E(xx'\tilde{g}(w'x)) \approx E(xx')E(\tilde{g}(w'x)) = E(\tilde{g}(w'x))I$. The approximative Newton iteration is then

$$w^+ = w - \left(E(xg(w'x)) - \lambda w\right) / \left(E(\tilde{g}(w'x)) - \lambda\right). \tag{5.11}$$

Multiply both sides of equation 5.11 by $\lambda - E(\tilde{g}(w'x))$ to obtain the FastICA iteration.

Chapter 6

Solution of multivariate portfolio problems

6.1 Gaussian portfolio

Suppose we invest y dollars in a zero cost cash flow with a Gaussian distribution for the investment horizon of length h with mean $(\mu - r)h$. We may write the zero cost cash flow accessed as X where

$$X = (\mu - r)h + Z.$$

Note that Z is a zero mean random vector.

We assume the Gaussian parameters are for the holding period h of length 1. We also assume that μ and r have been adjusted for the length of the period, which we take to be unity in what follows. Denote the covariance matrix of Z as follows:

$$E[ZZ'] = \Sigma.$$

The final period wealth is

$$W = yX,$$

where X is normally distributed with mean $\mu - r$ and covariance matrix Σ. We consider the certainty equivalent for the wealth W in the case of exponential utility. The exponential utility function is

$$U(W) = 1 - \exp(-\eta W), \tag{6.1}$$

where η is the risk aversion coefficient. The certainty equivalent, CE, is defined by the equation

$$U(CE) = E(U),$$

or equivalently

$$1 - \exp(-\eta \times CE) = E(U).$$

We choose the investment vector y to maximize expected exponential utility for the risk aversion η. The objective is therefore that of maximizing

$$1 - e^{-\eta y'(\mu - r)} E[e^{-\eta y' Z}].$$

The expectation is then given by

$$E[e^{-\eta y' Z}] = \exp\left(\frac{\eta^2}{2} y' \Sigma y\right).$$

It follows that the certainty equivalent is

$$CE = y'(\mu - r) - \frac{\eta}{2} y' \Sigma y. \tag{6.2}$$

We take the first derivatives of equation 6.2 with respect to the components of y, and set them equal to zero. Then the solution of y is given by

$$y = \Sigma^{-1} \frac{\mu - r}{\eta}. \tag{6.3}$$

6.2 Exponential utility and investment in zero cost VG cash flows

Suppose we invest y dollars in a zero cost cash flow with a VG distribution for the investment horizon of length h with mean $(\mu - r)h$. We may write the zero cost cash flow accessed as X

$$X = (\mu - r)h + \theta(g-1) + \sigma W(g), \tag{6.4}$$

where g is gamma distributed with unit mean and variance ν, and $W(g)$ is Gaussian with zero mean and variance g. The density of g is

$$f(g) = \frac{g^{\frac{1}{\nu}-1}e^{-\frac{g}{\nu}}}{\nu^{\frac{1}{\nu}}\Gamma(\frac{1}{\nu})}, \quad g > 0.$$

We suppose the VG parameters are for the holding period h as the unit period. We also suppose that μ and r have been adjusted for the length of the period and take this to be unity in what follows.

The final period wealth is

$$W = yX.$$

We employ the exponential utility and write

$$U(W) = 1 - \exp(-\eta W), \tag{6.5}$$

where η is the coefficient of risk aversion. The certainty equivalent CE solves

$$E(U(W)) = 1 - \exp(-\eta CE).$$

The goal of the investment is to maximize the expected utility function. The expected utility is

$$\begin{aligned} E(U(W)) &= E(1 - \exp(-\eta W)) \\ &= 1 - E(\exp(-y\eta X)). \end{aligned} \tag{6.6}$$

To determine y which maximizes the expected utility is equivalent to minimizing the following expression with respect to y:

$$E(\exp(-y\eta X)).$$

Theorem 6.1. *Suppose we invest y dollars in a zero cost cash flow with a VG distribution described in equation 6.4 for the investment horizon of length h. And suppose that we employ the exponential utility function as in equation 6.5. The optimal solution for the investment is*

$$\begin{aligned}\tilde{y} &= \left(\frac{\theta}{\sigma^2} - \frac{1}{(\mu - r - \theta)\nu}\right) \\ &\quad + \operatorname{sign}(\mu - r)\sqrt{\left(\frac{\theta}{\sigma^2} - \frac{1}{(\mu - r - \theta)\nu}\right)^2 + \frac{2(\mu - r)}{(\mu - r - \theta)\nu\sigma^2}}\end{aligned}$$

where $\tilde{y} = \eta y$ and η is the risk aversion coefficient.

Proof. To find the optimal solution for the investment, our goal is to maximize the expected utility function as in equation 6.6. It is equivalent to minimizing

$$E(\exp(-y\eta X))$$

over y.

$$\begin{aligned}&E(\exp(-y\eta X)) \\ &= \exp(-y\eta(\mu - r - \theta))E\left(\exp\left(-\left(y\eta\theta - \frac{y^2\eta^2\sigma^2}{2}\right)g\right)\right) \\ &= \exp\left(-y\eta(\mu - r - \theta) - \frac{1}{\nu}\ln\left(1 + \nu\left(y\eta\theta - \frac{y^2\eta^2\sigma^2}{2}\right)\right)\right).\end{aligned}$$

Minimizing the above expression is equivalent to maximizing

$$z(y) = y\eta(\mu - r - \theta) + \frac{1}{\nu}\ln\left(1 + \nu\left(y\eta\theta - \frac{y^2\eta^2\sigma^2}{2}\right)\right).$$

Suppose $\alpha, \beta \in R$ and $\alpha < 0 < \beta$. Let

$$q(y) = 1 + \nu\left(y\eta\theta - \frac{y^2\eta^2\sigma^2}{2}\right),$$

and $q(\alpha) = q(\beta) = 0$. The function $q(y) > 0$ for $y \in (\alpha, \beta)$ and q is differentiable on (α, β) and continuous on $[\alpha, \beta]$. We have

$$\begin{aligned} z(0) &= 0 \\ \lim_{y \to \alpha^+} z(y) &= -\infty \\ \lim_{y \to \beta^-} z(y) &= -\infty \end{aligned}$$

so that a maximum of z(y) exists on the interval (α, β). The first order condition with respect to y leads to

$$z'(y) = \eta(\mu - r - \theta) + \frac{\eta\theta - \eta^2\sigma^2 y}{1 + \nu\eta\theta y - \nu\eta^2\sigma^2 y^2/2}.$$

Furthermore, assume y_1 and y_2 are two roots for $z'(y) = 0$, and $y_1 < 0$, $y_2 > 0$. That is, $z'(y_1) = z'(y_2) = 0$. Setting $z'(y) = 0$, we obtain

$$\begin{aligned} & (\mu - r - \theta)\left(1 + \nu\eta\theta y - \frac{\nu\eta^2\sigma^2}{2} y^2\right) + \theta - \eta\sigma^2 y \\ = \ & \mu - r + ((\mu - r - \theta)\nu\theta - \sigma^2)\eta y - (\mu - r - \theta)\frac{\nu\eta^2\sigma^2}{2} y^2 \end{aligned} \tag{6.7}$$

Observe that $z'(0) > 0$ if $\mu > r$. We have $z(y_1) < 0$ and $z(y_2) > 0$. According to the mean value theorem, y_2 is the root which gives the optimal solution. Similarly, if $\mu < r$, then $z'(0) < 0$. We have $z(y_1) > 0$ and $z(y_2) < 0$ so that y_1 gives the optimal solution in this condition. Let $\tilde{y} = y\eta$ and solve for this magnitude, noting that y is then $\tilde{y}/\eta$. Hence we rewrite equation 6.7 as

$$\begin{aligned} & \tilde{y}^2 - 2\frac{(\mu - r - \theta)\nu\theta - \sigma^2}{(\mu - r - \theta)\nu\sigma^2}\tilde{y} - \frac{2(\mu - r)}{(\mu - r - \theta)\nu\sigma^2} \\ = \ & \tilde{y}^2 - 2\left(\frac{\theta}{\sigma^2} - \frac{1}{(\mu - r - \theta)\nu}\right)\tilde{y} - \frac{2(\mu - r)}{(\mu - r - \theta)\nu\sigma^2} \\ = \ & 0. \end{aligned}$$

Hence we have

$$\begin{aligned}\tilde{y} &= \left(\frac{\theta}{\sigma^2} - \frac{1}{(\mu - r - \theta)\nu}\right) \\ &\quad + \text{sign}(\mu - r)\sqrt{\left(\frac{\theta}{\sigma^2} - \frac{1}{(\mu - r - \theta)\nu}\right)^2 + \frac{2(\mu - r)}{(\mu - r - \theta)\nu\sigma^2}}.\end{aligned}$$

□

When y is positive, we have a long position. When y is negative, a short position is taken.

6.3 Multivariate VG portfolio

We take an investment horizon of length h and wish to study optimal portfolio for investment in a vector of assets with zero cost excess returns or financed returns over this period of $R - rh$. We suppose all parameters are adjusted for the time horizon and take this to be unity in what follows.

Let the vector y denote the dollar investment in the collection of assets. We suppose the mean excess return is $\mu - r$ and hence that

$$R - r = \mu - r + x,$$

where x is the zero mean random asset return vector.

Our structural assumption is that there exist a vector of independent zero mean VG random variables s of the same dimension as x and a matrix A such that

$$x = As.$$

The law of s_i is that of

$$s_i = \theta_i(g_i - 1) + \sigma_i W_i(g_i),$$

where the $W_i's$ are independent Brownian motions, and the g_i are gamma variates with unit mean and variance ν_i.

The strategy for estimating this structure is to organize the expectation

$$E[xx'] = I.$$

This is possible by applying the whiting technique described in section 5.7.2. Under the assumption that the observed data x is whitened, we apply independent component analysis to get the mixing matrix A. We then construct the data for the independent components s by

$$s = A^{-1}x.$$

The VG parameters can be estimated on these series by univariate methods.

Theorem 6.2. *Let the vector y denote the dollar investment in the collection of assets. We suppose the mean excess return is $\mu - r$ and the zero cost excess return is $R - r$, hence that*

$$R - r = \mu - r + x,$$

where x is the zero mean random asset return vector and assume that $E[xx'] = I$. Let

$$x = As$$

and assume the law of s_i is

$$s_i = \theta_i(g_i - 1) + \sigma_i W_i(g_i),$$

where A is the mixing matrix, the $W_i's$ are independent Brownian motions, and the g_i are gamma variates with unit mean and variance ν_i. Denote

$$\zeta = A^{-1}\frac{\mu - r}{\eta} - \frac{\theta}{\eta}$$

and

$$y = \frac{1}{\eta} A^{-1} \tilde{y},$$

where $y = (y_1, y_2, \cdots, y_n)'$*,* $\tilde{y} = (\tilde{y_1}, \tilde{y_2}, \cdots, \tilde{y_n})'$*, and* η *is the risk aversion coefficient. Then the solution of* $\tilde{y_i}$*, for* $i = 1, 2, \cdots, n$*, is given by*

$$\begin{aligned} \tilde{y_i} &= \frac{|\zeta_i|\theta_i\nu_i - sign(\zeta_i)\frac{\sigma_i^2}{\eta}}{|\zeta_i|\sigma_i^2\nu_i} \pm \\ &\quad \frac{\sqrt{\left(|\zeta_i|\theta_i\nu_i - sign(\zeta_i)\frac{\sigma_i^2}{\eta}\right)^2 + 2\left(|\zeta_i| + sign(\zeta_i)\frac{\theta_i}{\eta}\right)|\zeta_i|\sigma_i^2\nu_i}}{|\zeta_i|\sigma_i^2\nu_i} \\ &= \frac{\theta_i}{\sigma_i^2} - \frac{1}{\eta\zeta_i\nu_i} \pm \sqrt{\left(\frac{\theta_i}{\sigma_i^2} - \frac{1}{\eta\zeta_i\nu_i}\right)^2 + 2\frac{\zeta_i + \frac{\theta_i}{\eta}}{\zeta_i\sigma_i^2\nu_i}}, \end{aligned} \tag{6.8}$$

Proof. We choose the investment vector y to maximize expected exponential utility for the risk aversion coefficient η. The objective is therefore that of maximizing

$$1 - e^{-\eta y'(\mu - r)} E[e^{-\eta y' x}] = 1 - e^{-\eta y'(\mu - r)} E[e^{-\eta y' A s}].$$

The expectation is then given by

$$E[e^{-\eta y' A s}] = \exp\left(\sum_{i=1}^{n} \eta(y'A)_i\theta_i - \frac{1}{\nu_i}\ln\left(1 + \theta_i\nu_i\eta(y'A)_i - \frac{\sigma_i^2\nu_i}{2}\eta^2(y'A)_i^2\right)\right).$$

It follows that the certainty equivalent is

$$CE = y'(\mu - r) + \sum_{i=1}^{n}(-y'A)_i\theta_i + \frac{1}{\eta\nu_i}\ln\left(1 + \theta_i\nu_i\eta(y'A)_i - \frac{\sigma_i^2\nu_i}{2}\eta^2(y'A)_i^2\right).$$

We may write equivalently

$$CE = \eta(y'A)\left(A^{-1}\frac{\mu - r}{\eta} - \frac{\theta}{\eta}\right) + \sum_{i=1}^{n}\frac{1}{\eta\nu_i}\ln\left(1 + \theta_i\nu_i\eta(y'A)_i - \frac{\sigma_i^2\nu_i}{2}\eta^2(y'A)_i^2\right).$$

Now define

$$\begin{aligned} \tilde{y}' &= \eta y' A, \\ \zeta &= A^{-1}\frac{\mu - r}{\eta} - \frac{\theta}{\eta}, \end{aligned}$$

and write

$$\begin{aligned} CE &= \sum_{i=1}^{n} \left[\zeta_i \tilde{y}_i + \frac{1}{\eta \nu_i} \ln \left(1 + \theta_i \nu_i \tilde{y}_i - \frac{\sigma_i^2 \nu_i}{2} \tilde{y}_i{}^2 \right) \right] \\ &= \sum_{i=1}^{n} \psi(\tilde{y}_i). \end{aligned}$$

We have additive functions in the vector $\tilde{y}_i$ and these may be solved for using univariate methods in closed form. We then determine

$$y = \frac{1}{\eta} A^{-1} \tilde{y}.$$

First observe that the argument of the logarithm is positive only in a finite interval for $\tilde{y}_i$. Hence the CE maximization problem has an interior solution for $\tilde{y}_i$.

The first order condition yields

$$\psi'(\tilde{y}_i) = \zeta_i + \frac{\frac{\theta_i}{\eta} - \frac{\sigma_i^2}{\eta} \tilde{y}_i}{1 + \theta_i \nu_i \tilde{y}_i - \frac{\sigma_i^2 \nu_i}{2} \tilde{y}_i{}^2} = 0.$$

It is clear that

$$\psi'(0) = \zeta_i + \frac{\theta_i}{\eta}$$

and the optimal value for $\tilde{y}_i$ is positive when $\psi'(0) > 0$ and negative otherwise.

We may write the condition as

$$|\zeta_i| + \frac{\operatorname{sign}(\zeta_i) \left(\frac{\theta_i}{\eta} - \frac{\sigma_i^2}{\eta} \tilde{y}_i \right)}{1 + \theta_i \nu_i \tilde{y}_i - \frac{\sigma_i^2 \nu_i}{2} \tilde{y}_i{}^2} = 0.$$

The argument of the logarithm must be positive and so we write

$$|\zeta_i| \left(1 + \theta_i \nu_i \tilde{y}_i - \frac{\sigma_i^2 \nu_i}{2} \tilde{y}_i{}^2 \right) + \operatorname{sign}(\zeta_i) \left(\frac{\theta_i}{\eta} - \frac{\sigma_i^2}{\eta} \tilde{y}_i \right) = 0.$$

We may rewrite this expression as the quadratic

$$\left(|\zeta_i| + \operatorname{sign}(\zeta_i) \frac{\theta_i}{\eta} \right) + \left(|\zeta_i| \theta_i \nu_i - \operatorname{sign}(\zeta_i) \frac{\sigma_i^2}{\eta} \right) \tilde{y}_i - \frac{|\zeta_i| \sigma_i^2 \nu_i}{2} \tilde{y}_i{}^2 = 0,$$

or equivalently that

$$\frac{|\zeta_i|\sigma_i^2\nu_i}{2}\tilde{y_i}^2 - \left(|\zeta_i|\theta_i\nu_i - \text{sign}(\zeta_i)\frac{\sigma_i^2}{\eta}\right)\tilde{y}_i - \left(|\zeta_i| + \text{sign}(\zeta_i)\frac{\theta_i}{\eta}\right) = 0.$$

The solution for $\tilde{y}_i$ is given by

$$\begin{aligned}
\tilde{y}_i &= \frac{|\zeta_i|\theta_i\nu_i - \text{sign}(\zeta_i)\frac{\sigma_i^2}{\eta}}{|\zeta_i|\sigma_i^2\nu_i} \\
&\pm \frac{\sqrt{\left(|\zeta_i|\theta_i\nu_i - \text{sign}(\zeta_i)\frac{\sigma_i^2}{\eta}\right)^2 + 2\left(|\zeta_i| + \text{sign}(\zeta_i)\frac{\theta_i}{\eta}\right)|\zeta_i|\sigma_i^2\nu_i}}{|\zeta_i|\sigma_i^2\nu_i} \\
&= \frac{\theta_i}{\sigma_i^2} - \frac{1}{\eta\zeta_i\nu_i} \pm \sqrt{\left(\frac{\theta_i}{\sigma_i^2} - \frac{1}{\eta\zeta_i\nu_i}\right)^2 + 2\frac{\zeta_i + \frac{\theta_i}{\eta}}{\zeta_i\sigma_i^2\nu_i}}.
\end{aligned}$$

□

We take the positive or the negative root depending on the sign of $\left(\zeta_i + \frac{\theta_i}{\eta}\right)$. When y is positive we have a portfolio in which one is taking a long position. If y is negative then the short position is taken. If we wish to describe the portfolio weights we may normalize the vector y by the sum of its entries.

Chapter 7

Implementation of ICA

7.1 ICA in finance

There are many factors that drive the movements of asset returns. It is not unusual to assume that a set of different asset returns are driven by some common factors. ICA is a process which takes a set of multivariate observed data, x, and extracts from them a new set of statistically independent components, s. ICA assumes that the observed data vectors x are the result of a mixing process

$$x_i(t) = \sum_{j=1}^{n} a_{ij} s_j(t).$$

$s_j(t)$ are assumed to be statistically independent. They can be sources from wide range which affect the asset returns. Using matrix notation, the model can be written as

$$x = As,$$

where A is the unknown mixing matrix.

Another key assumption of ICA is that the independent components s are nongaussian. Option pricing theory was introduced by Fischer Black and Myron Scholes [18]. In order to value options, Black and Scholes derived a partial differential equation via a hedging argument. The Black-Scholes model evolves a stock price p from the geometric Brownian motion model. The stock price model is given by

$$dp = \mu p dt + \sigma p dw,$$

where μ denotes the drift rate, σ is the volatility, and w is the standard Brownian motion. While the Black-Scholes model remains the most widely used in the financial world, it has known biases, such as volatility smiles. The variance gamma model proposed in [8] replaces the diffusion process in the Black-Scholes model by a pure jump process. Bakshi and Chen show the importance of a jump component in modelling stock prices and argues that a diffusion model has difficulties in explaining smile effects [3]. The VG process is a nongaussian process. While ICA requires no knowledge about the distributions of the independent components s, we assume that s follows the nongaussian VG process. We then estimate VG parameters on these series by univariate methods described in [9]. We use ICA to decompose the multivariate stock return data into statistically independent components. We hope to investigate the common factors for the multivariate stock price returns. The VG model provides the information of higher order statistics. The Gaussian model gives only second order statistics. Non-gaussian models require ICA algorithms to process the information of higher order statistics.

7.2 Description of the data

We create three portfolios, with five, ten, and twenty stocks, respectively. We choose five, ten, and twenty stocks for each portfolio at random from the S&P 500 to perform our analysis. First, we get the time series stock price data for these five chosen stocks for fifteen years, starting from 1990. We apply ICA to the multivariate financial time series. The goal is to decompose the observed multivariate time series into a linear combination of statistically independent components. We use daily adjusted closing prices from five companies in the S&P 500. For the five stock portfolio, we choose 3M Company, Boeing Company, International Bus. Machines (IBM), Johnson & Johnson, McDonald's Corp., and Merck & Co. We take the first 1000 time series data since January 1990 for our first analysis. We then move forward one month to get the second set of 1000 time series data for our second time period analysis. We repeat the same methodology for 125 time periods of our analysis. The same procedures are applied to the ten and the twenty stock portfolios. For the ten stock portfolio, we choose 3M Company, Boeing Company, International Bus. Machines (IBM), Johnson & Johnson, McDonald's Corp., Merck & Co, Bausch & Lomb, Du Pont (E.I.), FedEx Corporation, Merck & Co., and Wal-Mart Stores. For the twenty stock portfolio, we choose the above ten companies plus Goodyear Tire & Rubber, PepsiCo Inc., McGraw-Hill, Ford Motor Co., Phizer, Apple Computer, Lockheed Martin Corp, Caterpillar Inc., Colgate-Palmolive, and Xerox Corp. We include the figures for the fiftieth time period of each of the three portfolios to demonstrate the visual results.

We are interested in daily relative returns. We transform the nonstationary stock price $p(t)$ to the stationary relative daily returns as follows,

$$x(t) = \frac{p(t) - p(t-1)}{p(t-1)}.$$

Figure 7.1 displays the relative daily returns for the fiftieth time period of the five stock portfolio.

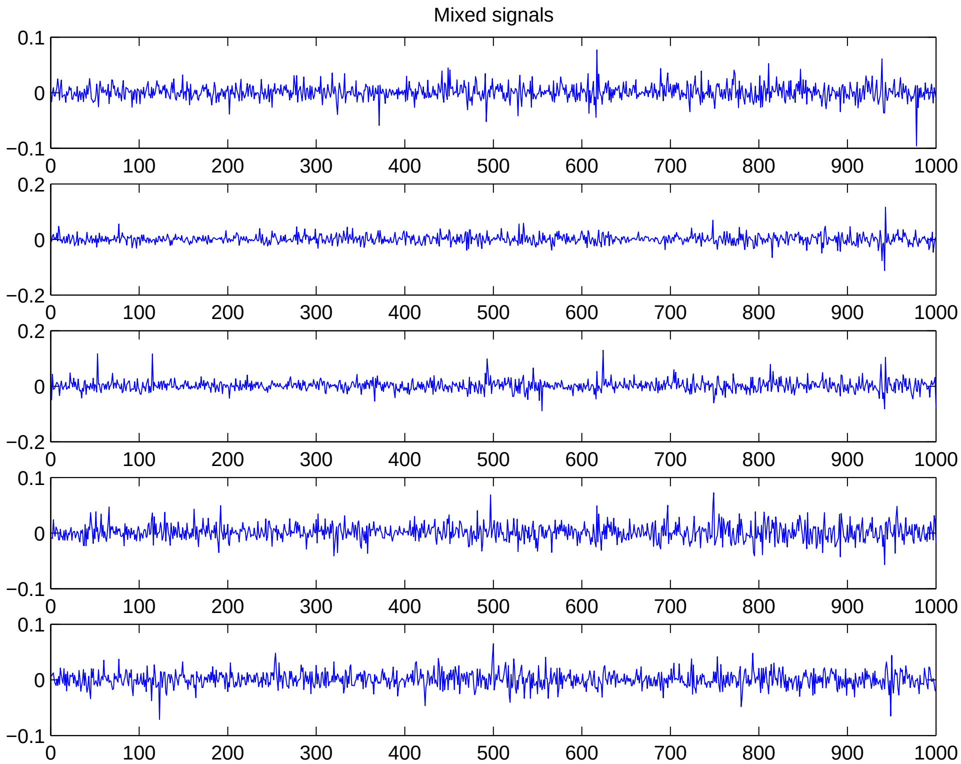

Figure 7.1: The Daily Returns for 5 Stocks of the 50th Time Period. (The five stocks are 3M, Boeing, IBM, Johnson & Johnson, and McDonald's.)

7.3 Preprocessing the data

Before we apply the ICA algorithms, it is useful to do some preprocessing for the observed data. The preprocessing procedures make the ICA estimation simpler. The preprocessing procedures consist of three steps: we obtain the relative daily returns as described in the previous section, subtract the mean of each stock, then whiten the data. We are using the FastICA [19] algorithm to process the data.

7.3.1 Centering

Centering of the observed data means that the mean of the observed data is subtracted from the observed data. This procedure makes x a zero-mean variable. This also implies that the independent components s are also zero-mean. This can be verified by taking the expectation to both of the equation 5.2. We see that

$$E(x) = AE(s).$$

This procedure simplified the ICA algorithms.

7.3.2 Whitening

The strategy of whitening the observed data vector x is to linearly transform the observed data into a new vector $\tilde{x}$ so that

$$E(\tilde{x}\tilde{x}') = I.$$

In other words, the components of $\tilde{x}$ are uncorrelated, and the variances are equal to unity. Or equivalently, the covariance matrix of $\tilde{x}$ is the identity matrix.

We use the eigenvalue decomposition of the covariance matrix C of x to whiten the data. That is,

$$C = E(xx') = EDE',$$

where E is the orthogonal matrix of the eigenvectors of matrix C, and D is the diagonal matrix of the corresponding eigenvalues. We denote $D = diag(d_1, ..., d_n)$, so that $D^{-\frac{1}{2}} = diag(d_1^{-\frac{1}{2}}, ..., d_n^{-\frac{1}{2}})$.

Whitening gives us

$$\tilde{x} = ED^{-\frac{1}{2}}E'x. \tag{7.1}$$

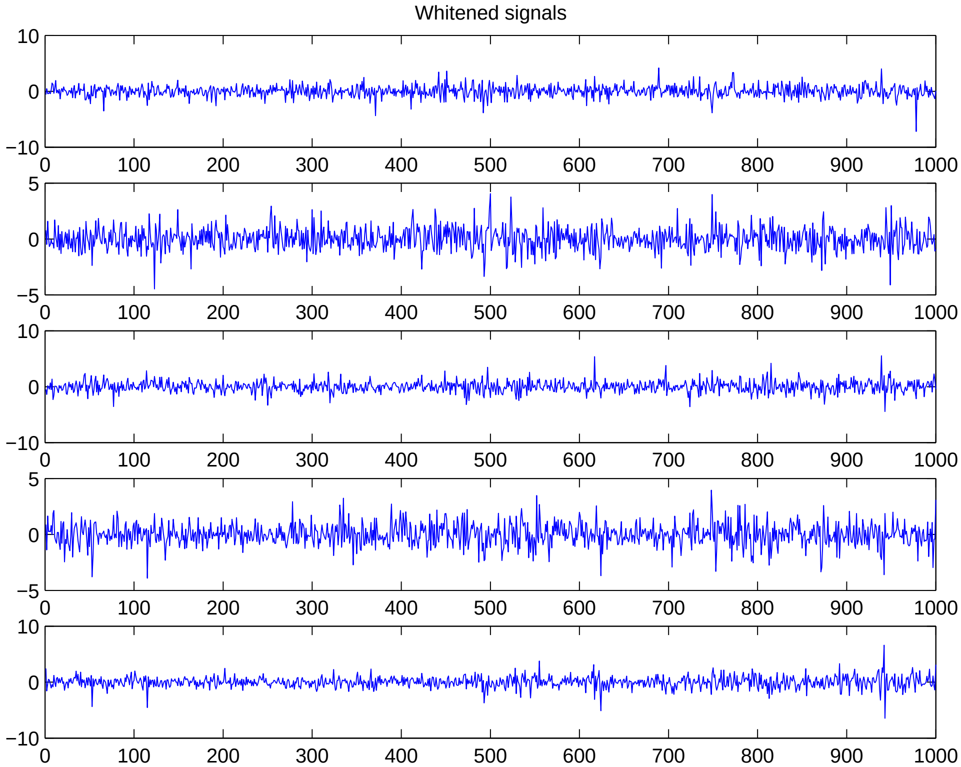

Figure 7.2: The Whitened Daily Returns for 5 Stocks of the 50th Time Period

It follows that

$$E(\tilde{x}\tilde{x}') = I.$$

From equations 5.2 and 7.1, we then have

$$\tilde{x} = ED^{-\frac{1}{2}}E'x = ED^{-\frac{1}{2}}E'As = \tilde{A}s.$$

Note that

$$E(\tilde{x}\tilde{x}') = \tilde{A}E(ss')\tilde{A}' = \tilde{A}\tilde{A}' = I,$$

we see that the new mixing matrix $\tilde{A}$ is an orthogonal matrix. Estimating the orthogonal matrix $\tilde{A}$ is computationally simpler than estimating the original mixing matrix A. Figure 7.2 displays the whitened signals of the observed data of the fiftieth time period for the five stock portfolio.

7.4 Independent VG components

We take an investment horizon of length h and wish to study optimal portfolio for investment in a vector of assets with zero cost excess returns or financed returns over this period of $R - rh$. We suppose all parameters are adjusted for the time horizon and take this to be unity in what follows.

We suppose the mean excess return is $\mu - r$ and hence that

$$R - r = \mu - r + x,$$

where x is the zero mean random asset return vector.

Our structural assumption is that there exist a vector of independent zero mean VG random variables s of the same dimension as x and a matrix A such that

$$x = As.$$

The law of s_i is that of

$$s_i = \theta_i(g_i - 1) + \sigma_i W_i(g_i),$$

where the $W_i's$ are independent Brownian motions, and the g_i are gamma variates with unit mean and variance ν_i.

We apply the FastICA algorithm described earlier to obtain the statistically independent VG random variables s. We choose the nonlinear function $G(x) = \frac{1}{4}x^4$ which corresponds to the fourth power as in kurtosis. We then construct the data of s by

$$s = A^{-1}x,$$

and estimate VG parameters on these series by univariate methods. Figure 7.3 gives the signals of the independent components of the fiftieth time periods for the five stock portfolio. In each time period of our analysis, we assume the observed data is the linear combination of independent components.

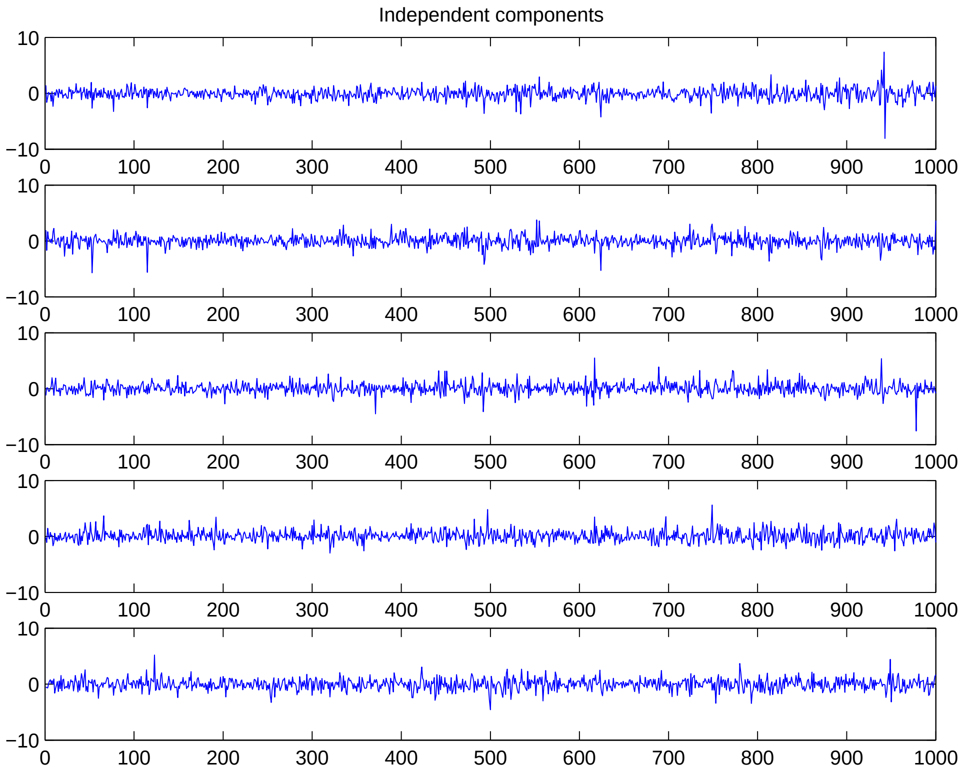

Figure 7.3: The Signal of 5 ICs of the 50th Time Period

7.5 The statistical estimation of VG parameters

For each underlying asset, we form the time series of daily relative returns described earlier. We apply ICA to the preprocessed time series data. We assume the independent components s follow the nongaussian VG process. We then estimate the VG parameters ν, σ, and θ from the independent components s. Using maximum likelihood estimation to evaluate the VG parameters is computationally inefficient. The Fourier inversion needs to be applied to each data point to find the density. To find the parameters, one needs an optimization algorithm for the inversions.

The fast Fourier transform (FFT) is used to invert the characteristic function once for each parameter setting. The method obtains the probability density function at the pre-specified values for s. We take .25 as the integration spacing. The density of s then has the spacing of $8\pi/N$, where N is a power of 2. We take $N = 16384$. We bin s into 100 cells. Recall that

$$s = \theta(g-1) + \sigma W(g)$$

where W is Brownian motion, and g is a gamma variate with unit mean and variance ν. The characteristic function of s is

$$\begin{aligned}\phi(u) &= E(e^{ius}) \\ &= \exp\left(-iu\theta - \frac{1}{\nu}\ln\left(1 - iu\theta\nu + \frac{\sigma^2\nu}{2}u^2\right)\right).\end{aligned}$$

The density function of s is

$$f(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ius}\phi(u)du.$$

By FFT, we get $f(s)$.

With the density evaluated at the pre-specified points, we bin the vector s by counting the number of observations n_i at each pre-specified point z_i. We find the parameter estimates which maximize the likelihood of the binned data. That is, we maximize

$$L = \sum_i n_i \ln f(z_i; \sigma, \nu, \theta).$$

7.6 Chi-square goodness-of-fit test

We apply the chi-square goodness-of-fit test to our binned data. The data is divided into k cells. Let O_i denote the observed number of data for cell i, and E_i denote the expected number of data for cell i, assuming the data have a VG distribution. The test statistic is given by [33]

$$\chi^2 = \sum_{i=1}^{k}\frac{(O_i - E_i)^2}{E_i}.$$

Let c be the number of parameters in the model. The test statistic follows approximately a chi-square distribution with degrees of freedom $d = k - c - 1$. We reject the hypothesis if

$$\chi^2 > \chi^2_{\alpha,d}$$

where $\chi^2_{\alpha,d}$ is the chi-square percent point function with degrees of freedom d, and a level of significant α. In our model, $k = 100$, $c = 3$, and we use $\alpha = .01$. The three parameters are σ, ν, and θ. Thus $\chi^2_{.01,96} = 131.14$. We assess the goodness of fit of the VG model for the independent components of each time period by calculating the chi-square goodness-of-fit test.

Tables 7.1 to 7.5 give the mean, the standard deviation, the minimum, and the maximum of the VG parameters for the five independent components over the 125 time periods of the five stock portfolio. The complete results of the parameter estimates for the five independent components of each time period are given at the end of Chapter 8. The summary of the ICs over the 125 time periods for the ten stock portfolio and the twenty stock portfolio are given in appendix B. Table 7.6 shows the chi-square statistic of the half annually investment for the twenty stock portfolio at the first and the second time periods.

Figures 7.4 to 7.8 demonstrate the estimated density obtained from the maximum likelihood estimation for the five independent components of the five stock portfolio at the 50th time period, where the circles denote the binned data, the solid line denotes the VG process, and the dash-dot line denotes the Gaussian process.

Statistical Estimation			
	σ	ν	θ
mean	0.9573	0.5986	-0.0022
standard deviation	0.0189	0.1322	0.1368
minimum	0.9128	0.2585	-0.4088
maximum	1.0274	1.1564	0.3656

Table 7.1: The First Independent Component

Statistical Estimation			
	σ	ν	θ
mean	0.9639	0.5579	0.0190
standard deviation	0.0198	0.1612	0.1792
minimum	0.9162	0.1535	-0.4552
maximum	1.0532	1.0816	0.6411

Table 7.2: The Second Independent Component

Statistical Estimation			
	σ	ν	θ
mean	0.9738	0.5318	0.0464
standard deviation	0.0144	0.1414	0.1485
minimum	0.9373	0.2296	-0.3358
maximum	1.0028	0.9340	0.4834

Table 7.3: The Third Independent Component

Statistical Estimation

	σ	ν	θ
mean	0.9806	0.4232	0.0287
standard deviation	0.0112	0.1277	0.2205
minimum	0.9386	0.1332	-0.4790
maximum	0.9962	0.7287	0.7248

Table 7.4: The Fourth Independent Component

Statistical Estimation

	σ	ν	θ
mean	0.9833	0.3634	0.0079
standard deviation	0.0100	0.1270	0.2876
minimum	0.9569	0.0977	-0.8482
maximum	1.0056	0.6856	0.8324

Table 7.5: The Fifth Independent Component

12/93			06/94		
ICs	χ^2(VG)	χ^2 (Gauss)	ICs	χ^2(VG)	χ^2 (Gauss)
1st	136.2482	2.2776e+031	1st	70.8492	3.1820e+009
2nd	84.0193	1.3617e+005	2nd	0.5841	1.7578e+006
3rd	0.6055	8.9314e+005	3rd	68.3350	4.9341e+010
4th	76.2664	6.7052e+005	4th	120.2310	3.5071e+032
5th	129.2998	3.6943e+011	5th	79.1577	2.7214e+012
6th	77.0964	2.1348e+013	6th	0.3245	151.6225
7th	82.0128	5.5169e+007	7th	0.7302	623.8396
8th	82.6927	9.9940e+005	8th	63.8533	6.1312e+023
9th	0.4872	322.3800	9th	83.6321	3.6894e+005
10th	91.3730	8.7305e+008	10th	0.9306	7.3505e+004
11th	0.1550	20.9039	11th	0.6507	5.4653e+004
12th	3.5015	4.9703e+004	12th	88.8289	7.0872e+008
13th	0.3264	669.6947	13th	0.1151	6.2102
14th	0.1895	131.6822	14th	0.1701	31.7546
15th	0.1885	5.9786	15th	0.1483	3.7325
16th	0.1855	42.4625	16th	1.9673	6.5628e+003
17th	0.2498	93.5152	17th	0.2638	15.7981
18th	0.3312	8.5085e+003	18th	0.1599	2.7688
19th	0.1024	0.1734	19th	0.1163	3.9716
20th	0.1268	0.1675	20th	0.0895	0.1658

Table 7.6: χ^2 of the 1st and the 2nd Time Periods

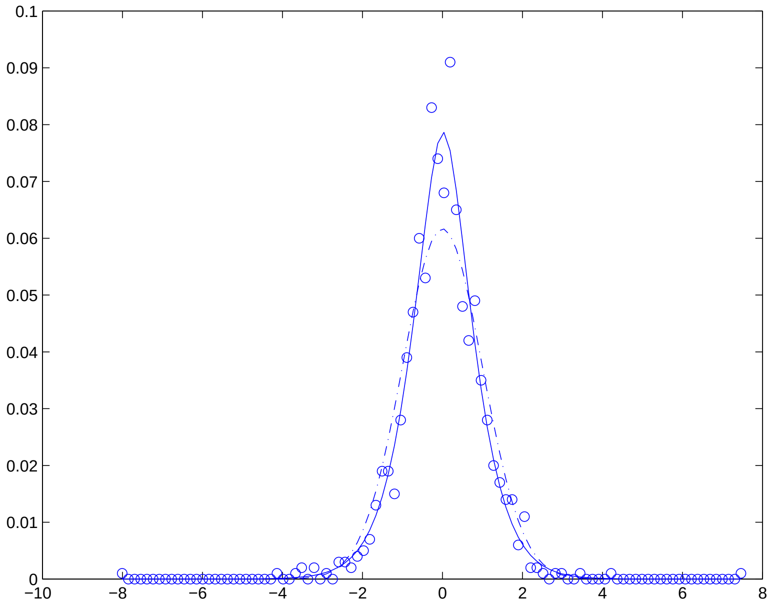

Figure 7.4: 1st IC Density Fit of the 50th Time Period

7.7 Arbitrary horizon growth

In the previous sections, we assume the VG parameters are for the the holding period h as the unit period. We now wish to incorporate an investment arbitrary horizon h. Let X_d be the daily VG with parameters ν_d, σ_d, θ_d. We postulate a scaling law and say that the distribution of X_h is that of $\sqrt{h}X_d$. We are not summing the independent increments but instead using a scaling law. Under this hypothesis the characteristic function of X_h is

$$
\begin{aligned}
E(e^{iuX_h}) &= E(e^{iu\sqrt{h}X_d}) \\
&= \left(\frac{1}{1 - iu\sqrt{h}\theta_d\nu_d + \frac{\sigma_d^2\nu_d}{2}u^2 h}\right)^{\frac{1}{\nu_d}} \\
&= \left(\frac{1}{1 - iu\theta_h\nu_h + \frac{\sigma_h^2\nu_h}{2}u^2}\right)^{\frac{1}{\nu_h}}.
\end{aligned}
$$

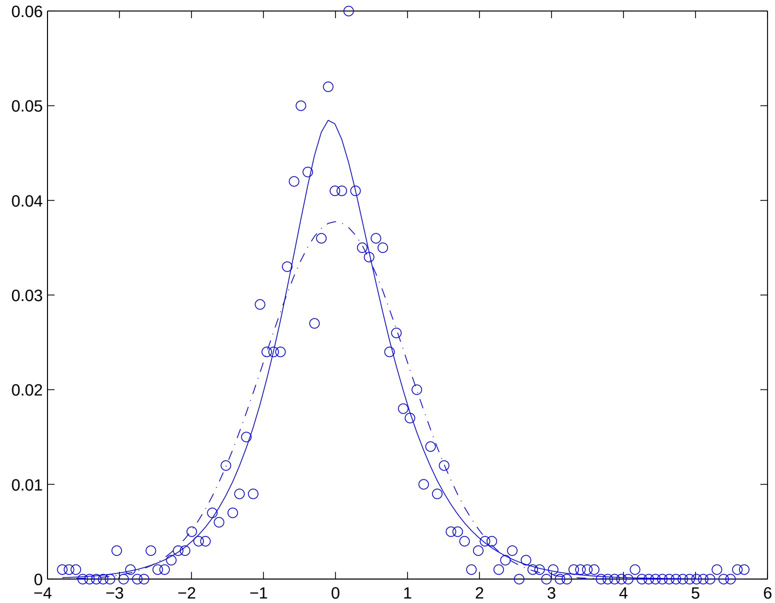

Figure 7.5: 2nd IC Density Fit of the 50th Time Period

Hence X_h is VG with parameters

$$\begin{aligned} \nu_h &= \nu_d, \\ \sigma_h &= \sigma_d \sqrt{h}, \\ \theta_h &= \theta_d \sqrt{h}. \end{aligned}$$

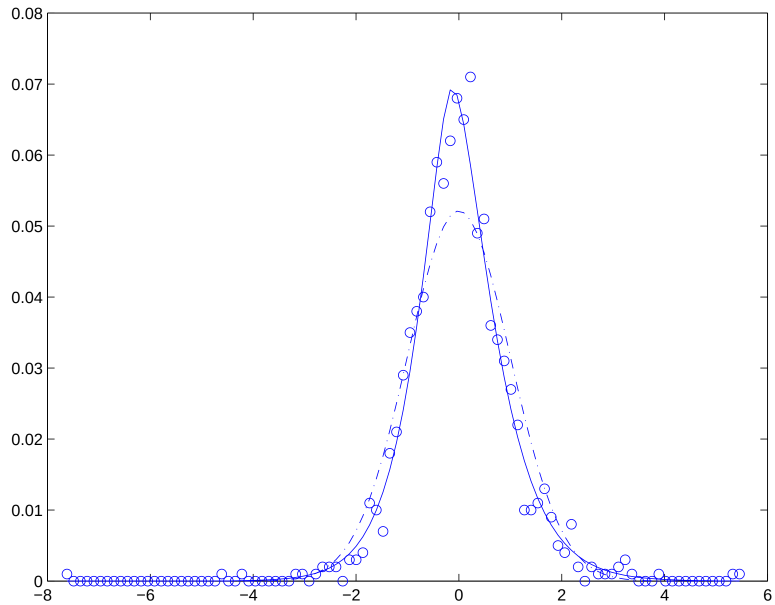

Figure 7.6: 3rd IC Density Fit of the 50th Time Period

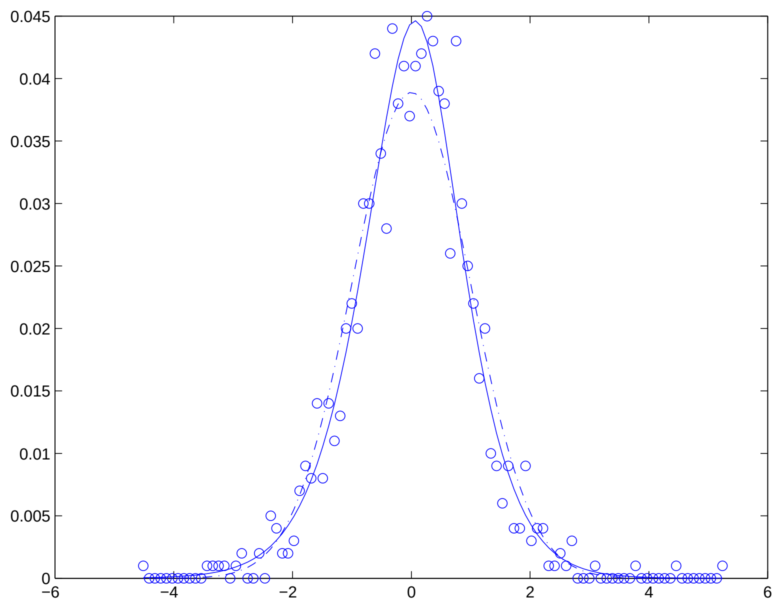

Figure 7.7: 4th IC Density Fit of the 50th Time Period

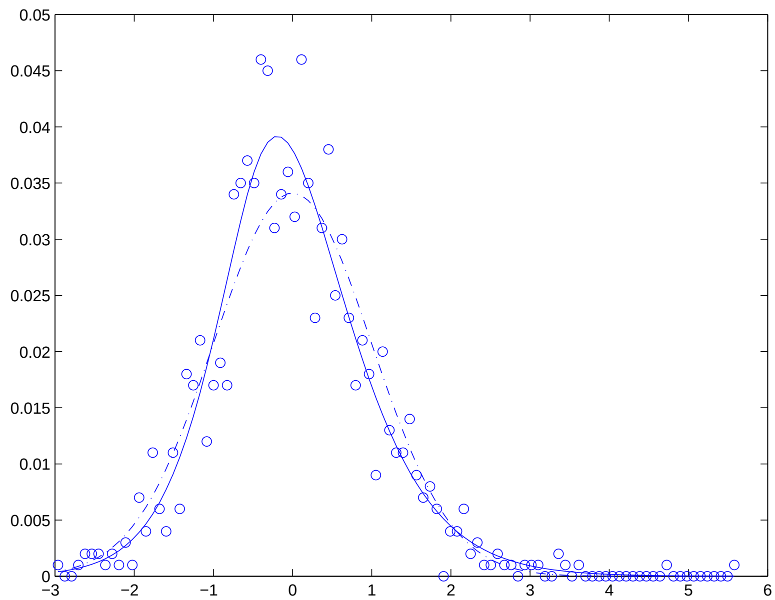

Figure 7.8: 5th IC Density Fit of the 50th Time Period

Chapter 8

Conclusion

We have derived the solution of the multivariate portfolio problem by assuming that the risky assets follow either the VG process or the Gaussian process. We chose five stocks from the S&P 500 to include in our risky portfolio. We analyzed the data using the relative daily return $x(t)$. Note that

$$x(t) = \frac{p(t) - p(t-1)}{p(t-1)}$$

where $p(t)$ is the stock price for time t. The five stocks chosen were 3M Company, Boeing Company, International Bus. Machines, Johnson & Johnson, McDonald's Corp., Merck & Co, with ticker symbols mmm, ba, ibm, jnj, and mcd, respectively. We took the first 1000 time series data since January 1990 for our first analysis. We then moved forward one month to get the second set of 1000 time series data for our second time period analysis. We repeated the same methodology for 125 time periods of our analysis from January 1990 to May 2004. Thus, we had 125 different 5 by 1000 matrices x_i, $i = 1, 2 \ldots, 125$ of the relative daily returns. Our structural assumption was that there exists a vector of independent zero mean VG random variables s of the same dimension as x, and a matrix A such that

$$x = As.$$

The law of s_i is that of

$$s_i = \theta_i(g_i - 1) + \sigma_i W_i(g_i),$$

where the $W_i's$ are independent Brownian motions, and the g_i are gamma variates with unit mean and variance ν_i. We applied independent component analysis to get the mixing matrix A. We then constructed the data for the independent components s by

$$s = A^{-1}x.$$

The VG parameters, σ, ν, and θ were then estimated on these series by univariate methods for each time period. We performed the chi-square goodness-of-fit test for each independent component. After we obtained the VG parameter values, we used equation 6.8 to compute the vector of dollars, y, invested in each stock of the VG process. We also computed dollar amounts invested for the Gaussian process using equation 6.3. We repeated the analysis for the 125 investments.

At the end of each investment time period, we invested an amount of money y according to our analysis. When an element of y is positive, we take a long position. When an element of y is negative, a short position is taken. We forward the investment for one month, and calculate the cash flow CF at the end of the month for each time period. The formula is as follows:

$$CF = y \cdot \left(\frac{p(t+21) - p(t)}{p(t)} - r \right)$$

where $p(t)$ is the initial price of the investment, $p(t+21)$ is the price at the maturity, and r is the 3-month treasury bill monthly interest rate. Note that we used $p(t+21)$ as the maturity price, because there are 21 trading days in a month on average. The same procedures were applied to the ten stock portfolio and the twenty stock portfolio. The results of cash flow for the five stock portfolio were shown at the end of the chapter. The plots of cumulated cash flow of both VG and Gaussian processes were also given.

The the Sharpe ratio, the certainty equivalent and the gain-loss ratio of both processes were calculated.

The Sharpe ratio is a measure of the risk, but not the amount of the return, you take by pursuing the return. Generally speaking, it doesn't represent the relative performance of an individual asset, but it represents the risk-adjusted return of the entire portfolio of many assets, and higher values are considered better [14].

The formula is as follows:

$$\text{Sharpe ratio} = \frac{\text{mean of cash flow}}{\text{standard deviation of cash flow}}.$$

The numerator represents the net return realized by taking the risk, and the denominator represents the risk. The formula as a whole measures how efficiently you have gained the return by taking the risk. The larger the value, the more efficient it is considered.

The general form of the exponential utility function is

$$U(CF) = 1 - \exp(-\eta CF)$$

where η is the risk aversion coefficient. Note that

$$U(CE) = E(U).$$

Using the exponential utility function, we get

$$\begin{aligned} 1 - \exp(-\eta \times CE) &= E(U) \\ &= \frac{1}{n}\sum_{i=1}^{n}(1 - e^{-\eta CF_i}) \end{aligned}$$

where $n = 125$ in our analysis. We then have

$$CE = -\frac{1}{\eta}\ln\left(1 - \frac{1}{n}\sum_{i=1}^{n}(1 - e^{-\eta CF_i})\right).$$

Performance Measures

	VG	Gauss
Sharpe Ratio	0.2548	0.2127
CE $(\eta = .0005)$	47.6883	0.0230
Gain-Loss Ratio	2.3909	1.4536

Table 8.1: 5 Stock Portfolio

Performance Measures

	VG	Gauss
Sharpe Ratio	0.1995	0.2106
CE $(\eta = .0005)$	117.2754	0.0306
Gain Loss Ratio	4.3234	1.4845

Table 8.2: 10 Stock Portfolio

The gain-loss ratio is a measure between the positive cash flows and the negative cash flows of all the investment time periods. Let CF^+ and CF^- be vectors which include all the positive elements and all the negative elements of the cash flows of the entire investment time periods, respectively. The formula of the gain-loss ratio is as follows,

$$\text{gain-loss ratio} = \frac{E[CF^+]}{|E[CF^-]|}.$$

Tables 8.1 to 8.3 present the three performance measures, the Sharpe ratio, the certainty equivalent, and the gain-loss ratio of both the VG and the Gaussian processes for the three portfolios.

Figures 8.1 to 8.6 plot the cumulated cash flows through the 125 investment time periods of our analysis for the VG and the Gaussian processes of the three portfolios.

Performance Measures		
	VG	Gauss
Sharpe Ratio	0.2910	0.3513
CE $(\eta = .0005)$	282.6599	0.1204
Gain Loss Ratio	3.7869	1.9502

Table 8.3: 20 Stock Portfolio

The complete tables of the estimated independent VG components parameters, the dollar amounts invested, and the cash flow of both the VG and the Gaussian processes of each time period for the five stock portfolio, are provided at the end of the chapter.

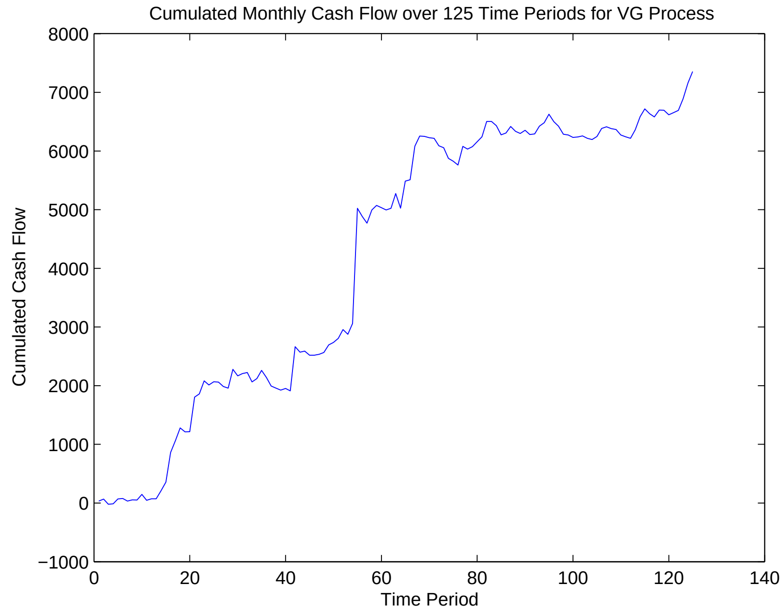

Figure 8.1: VG Cumulated Cash Flows (5 Stock Portfolio)

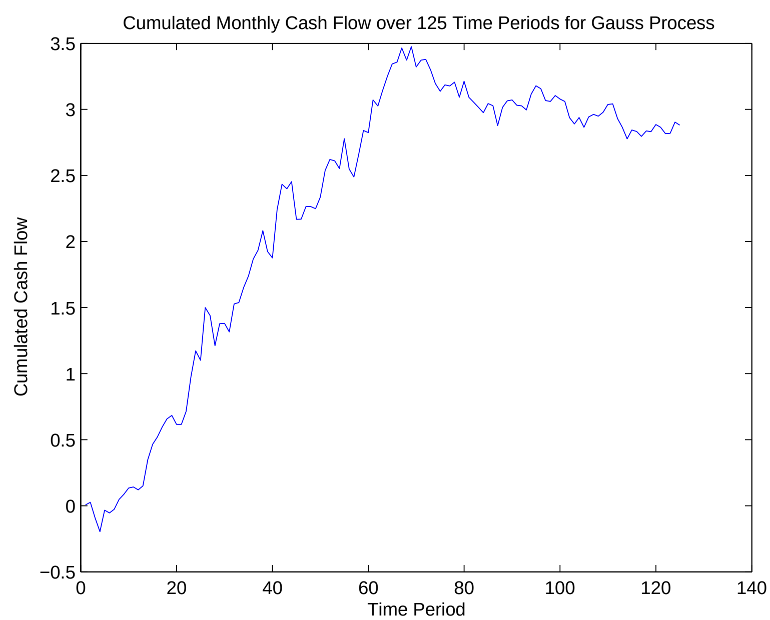

Figure 8.2: Gaussian Cumulated Cash Flows (5 Stock Portfolio)

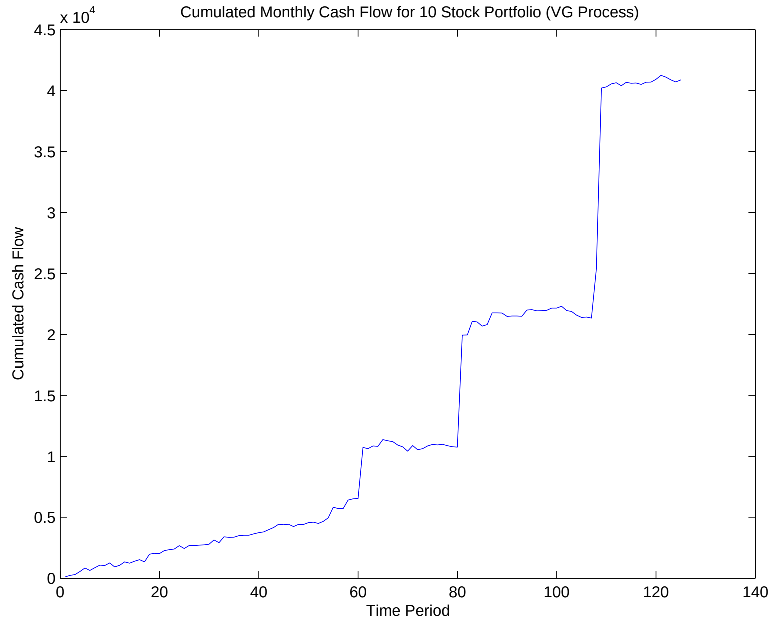

Figure 8.3: VG Cumulated Cash Flows (10 Stock Portfolio)

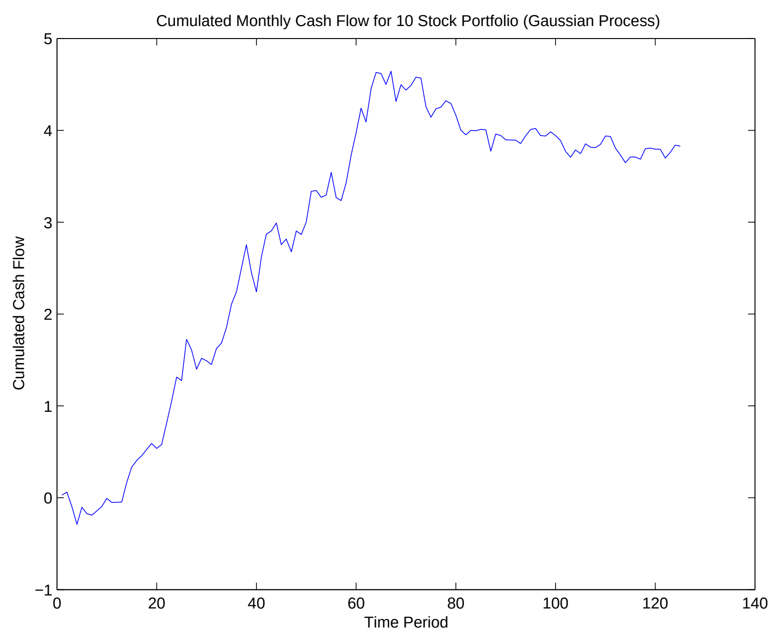

Figure 8.4: Gaussian Cumulated Cash Flows (10 Stock Portfolio)

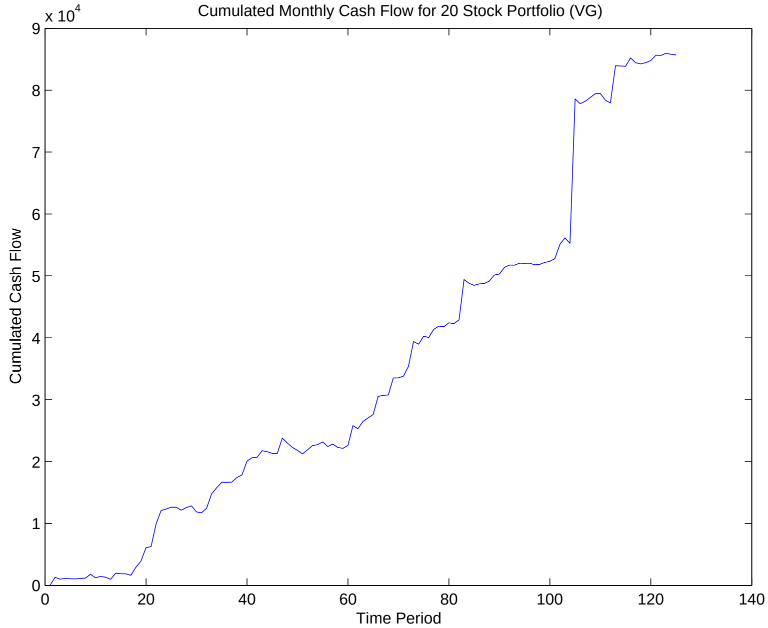

Figure 8.5: VG Cumulated Cash Flows (20 Stock Portfolio)

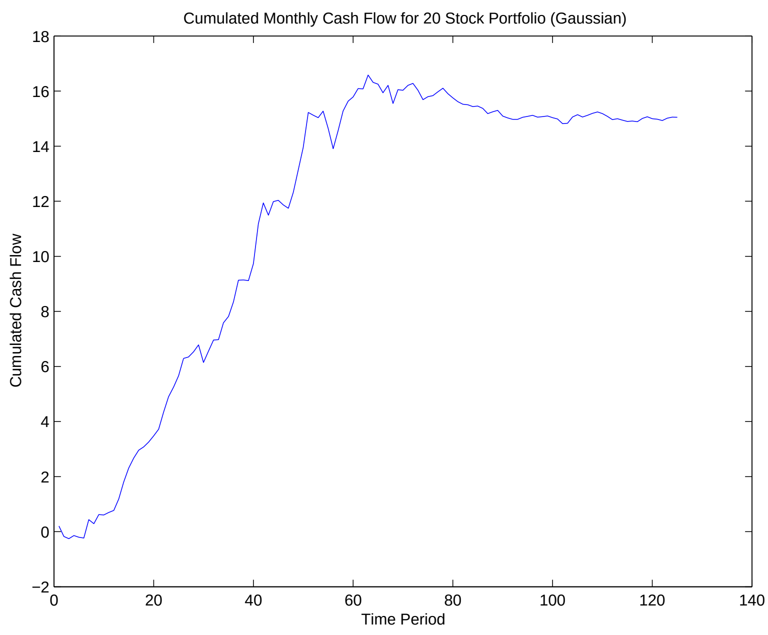

Figure 8.6: Gaussian Cumulated Cash Flows (20 Stock Portfolio)

The following tables present the statistical estimation of the three independent components parameters of the entire 125 investment periods. The dollar amounts invested, and the monthly cash flow in each investment time period of both the VG and the Gaussian processes are listed. The three-month treasury bill interest rates are also given. The investment is re-balanced monthly from December 1993 to April 2004. The cumulated cash flow plots and the performance measures provide evidence which shows that the VG process investment is a better strategy than the Gaussian process one.

Statistical Estimation					12/93
	σ	ν	θ		
1st IC	0.9755	0.5494	-0.0610		
2nd IC	0.9600	0.8693	-0.0951		
3rd IC	0.9522	0.6331	0.1421		
4th IC	0.9874	0.2340	-0.2867		
5th IC	0.9827	0.2359	0.3936		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG) $\times 10^3$	-0.1602	-0.8040	-1.1543	0.8404	0.2606
\$ invest(Gauss)	1.1532	-0.1344	-0.0627	0.6714	0.7154
cash flow(VG)	-2.4589	-25.5172	80.7858	-21.7710	2.3236
cash flow(Gauss)	0.0177	-0.0043	0.0044	-0.0174	0.0064
% T-bill rate	3.08				

Statistical Estimation					01/94
	σ	ν	θ		
1st IC	0.9506	0.6438	0.1489		
2nd IC	0.9737	0.5574	-0.0767		
3rd IC	0.9493	0.7548	0.1651		
4th IC	0.9820	0.2653	-0.3815		
5th IC	0.9870	0.2153	0.3003		
ticker	mmm	ba	ibm	jnj	mcd
$ invest(VG)	847.59	845.24	136.13	-600.33	-67.99
$ invest(Gauss)	1.2860	-0.1291	-0.1140	0.6438	0.7624
cash flow(VG)	-9.3426	53.0208	-1.9441	-3.5361	-3.1967
cash flow(Gauss)	-0.0142	-0.0081	0.0016	0.0038	0.0358
% T-bill rate	3.02				

Statistical Estimation					02/94
	σ	ν	θ		
1st IC	0.9497	0.6805	-0.1559		
2nd IC	0.9746	0.5765	0.0836		
3rd IC	0.9525	0.7566	-0.1569		
4th IC	0.9899	0.2379	0.2570		
5th IC	0.9828	0.2649	-0.3495		
ticker	mmm	ba	ibm	jnj	mcd
$ invest(VG)	863.84	713.51	159.90	440.92	377.11
$ invest(Gauss)	1.1829	-0.1144	-0.2067	0.7042	0.8384
cash flow(VG)	-54.6490	-20.8176	10.3848	-30.6657	7.7397
cash flow(Gauss)	-0.0748	0.0033	-0.0134	-0.0490	0.0172
% T-bill rate	3.21				

Statistical Estimation					03/94
	σ	ν	θ		
1st IC	0.9521	0.7658	0.1764		
2nd IC	0.9580	0.7400	0.1226		
3rd IC	0.9724	0.5561	-0.0618		
4th IC	0.9885	0.2540	0.2572		
5th IC	0.9837	0.2595	-0.3534		
ticker	mmm	ba	ibm	jnj	mcd
$ invest(VG)	-120.61	-261.66	-475.38	512.79	352.34
$ invest(Gauss)	1.0771	-0.1771	-0.1237	0.5991	0.8329
cash flow(VG)	8.3864	14.8688	-11.2115	22.2099	-26.6283
cash flow(Gauss)	-0.0749	0.0101	-0.0029	0.0259	-0.0629
% T-bill rate	3.52				

Statistical Estimation					04/94
	σ	ν	θ		
1st IC	0.9743	0.5504	0.0809		
2nd IC	0.9564	0.7110	0.1066		
3rd IC	0.9613	0.7730	0.0457		
4th IC	0.9879	0.2386	0.2766		
5th IC	0.9820	0.2518	0.3653		
ticker	mmm	ba	ibm	jnj	mcd
$ invest(VG)	527.21	-330.21	-160.14	388.84	489.08
$ invest(Gauss)	0.9892	-0.2619	-0.1502	0.6395	0.7863
cash flow(VG)	53.3611	-6.5945	-14.4564	43.7086	5.6442
cash flow(Gauss)	0.1001	-0.0052	-0.0136	0.0719	0.0091
% T-bill rate	3.74				

Statistical Estimation					05/94
	σ	ν	θ		
1st IC	0.9560	0.7142	0.0221		
2nd IC	0.9631	0.7875	0.1147		
3rd IC	0.9738	0.5584	-0.0584		
4th IC	0.9904	0.2488	0.2654		
5th IC	0.9845	0.2890	0.3399		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG)	-93.28	-323.32	-455.26	430.67	454.84
\$ invest(Gauss)	1.0964	-0.4266	-0.1187	0.6295	0.6513
cash flow(VG)	1.0259	-25.2269	17.9238	-5.3709	18.8745
cash flow(Gauss)	-0.0121	-0.0333	0.0047	-0.0078	0.0270
% T-bill rate	4.19				

Statistical Estimation					06/94
	σ	ν	θ		
1st IC	0.9738	0.5785	0.0601		
2nd IC	0.9745	0.8860	0.0463		
3rd IC	0.9443	0.7123	0.1907		
4th IC	0.9830	0.3161	0.3281		
5th IC	0.9904	0.2469	-0.2634		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG) $\times 10^3$	2.0852	-0.3664	-0.3995	-0.4946	0.1564
\$ invest(Gauss)	0.9348	-0.3616	-0.1329	0.5438	0.6907
cash flow(VG)	-7.3217	15.6604	-7.7635	-37.8312	-5.3603
cash flow(Gauss)	-0.0033	0.0155	-0.0026	0.0416	-0.0237
% T-bill rate	4.18				

Statistical Estimation					07/94
	σ	ν	θ		
1st IC	0.9696	0.5820	-0.0539		
2nd IC	0.9758	0.8130	0.0931		
3rd IC	0.9630	0.7296	-0.0663		
4th IC	0.9861	0.2782	-0.2886		
5th IC	0.9716	0.2476	-0.4962		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG) $\times 10^3$	0.0124	-1.1764	-0.0952	0.3103	0.3805
\$ invest(Gauss)	1.0468	-0.3103	-0.0524	0.6640	0.8430
cash flow(VG)	1.1812	42.3877	-9.5108	19.6183	-32.5044
cash flow(Gauss)	0.0994	0.0112	-0.0052	0.0420	-0.0720
% T-bill rate	4.39				

Statistical Estimation					08/94
	σ	ν	θ		
1st IC	0.9368	0.6966	0.2085		
2nd IC	0.9588	0.6718	-0.0949		
3rd IC	0.9767	0.5177	0.0053		
4th IC	0.9864	0.2708	-0.2788		
5th IC	0.9766	0.2807	-0.4193		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG)	89.66	-324.67	-908.14	352.74	425.60
\$ invest(Gauss)	1.2969	-0.2604	-0.0583	0.6487	0.8256
cash flow(VG)	0.6672	5.8528	-23.2961	1.4317	11.2660
cash flow(Gauss)	0.0097	0.0047	-0.0015	0.0026	0.0219
% T-bill rate	4.50				

Statistical Estimation					09/94
	σ	ν	θ		
1st IC	0.9380	0.6936	0.2344		
2nd IC	0.9553	0.4332	0.1502		
3rd IC	0.9627	0.6923	0.0728		
4th IC	0.9880	0.2844	-0.2797		
5th IC	0.9792	0.2875	-0.3966		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG) $\times 10^3$	1.3783	-0.0015	0.5991	0.5715	0.3280
\$ invest(Gauss)	1.2863	-0.2124	-0.0011	0.6537	0.9228
cash flow(VG)	1.4260	-0.0026	40.8956	63.1721	-8.7092
cash flow(Gauss)	0.0013	-0.0004	-0.0001	0.0723	-0.0245
% T-bill rate	4.64				

Statistical Estimation					10/94
	σ	ν	θ		
1st IC	0.9628	0.7337	-0.0848		
2nd IC	0.9610	0.7040	-0.1173		
3rd IC	0.9561	0.4192	-0.1427		
4th IC	0.9761	0.3281	-0.3833		
5th IC	0.9885	0.3022	0.2802		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG) $\times 10^3$	1.2074	-0.1119	0.3176	-0.3981	-0.4982
\$ invest(Gauss)	1.2198	-0.2033	-0.0294	0.7342	0.9646
cash flow(VG)	-56.1421	-4.0487	-4.5500	-2.3106	-35.1258
cash flow(Gauss)	-0.0567	-0.0074	0.0004	0.0043	0.0680
% T-bill rate	4.96				

Statistical Estimation 11/94

	σ	ν	θ		
1st IC	0.9368	0.6914	-0.2114		
2nd IC	0.9785	0.5114	-0.1345		
3rd IC	0.9779	0.3288	0.3666		
4th IC	0.9633	0.6493	-0.0718		
5th IC	0.9864	0.2965	-0.2881		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG)	-314.44	517.82	-427.37	-509.39	903.46
\$ invest(Gauss)	0.8462	-0.1271	-0.0790	0.6468	0.8983
cash flow(VG)	3.8039	33.7604	-2.5014	-1.7764	-6.1491
cash flow(Gauss)	-0.0102	-0.0083	-0.0005	0.0023	-0.0061
% T-bill rate	5.25				

Statistical Estimation 12/94

	σ	ν	θ		
1st IC	0.9675	0.7251	0.0782		
2nd IC	0.9606	0.7042	-0.1243		
3rd IC	0.9782	0.4874	0.1053		
4th IC	0.9773	0.3318	-0.3824		
5th IC	0.9878	0.3138	0.2622		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG) $\times 10^3$	1.3838	-0.5300	-0.5013	-0.6937	-0.0748
\$ invest(Gauss)	0.9882	-0.1136	-0.0313	0.6921	0.8631
cash flow(VG)	6.8446	9.3562	-1.7627	-15.7371	-0.6639
cash flow(Gauss)	0.0049	0.0020	-0.0001	0.0157	0.0077
% T-bill rate	5.64				

Statistical Estimation					01/95
	σ	ν	θ		
1st IC	0.9632	0.6338	0.0839		
2nd IC	0.9765	0.4871	0.1158		
3rd IC	0.9642	0.7404	0.1228		
4th IC	0.9878	0.3451	0.2745		
5th IC	0.9816	0.2667	0.3944		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG) $\times 10^3$	1.0415	-0.0823	-0.0879	0.5262	0.6036
\$ invest(Gauss)	0.9267	-0.2316	-0.1584	0.6055	0.9820
cash flow(VG)	28.8789	1.4782	-0.7888	16.4938	93.3628
cash flow(Gauss)	0.0257	0.0042	-0.0014	0.0190	0.1519
% T-bill rate	5.81				

Statistical Estimation					02/95
	σ	ν	θ		
1st IC	0.9641	0.6377	-0.0632		
2nd IC	0.9644	0.8546	-0.1133		
3rd IC	0.9739	0.4776	0.1606		
4th IC	0.9814	0.3117	0.3694		
5th IC	0.9901	0.2962	0.2173		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG) $\times 10^3$	2.0387	-0.4788	0.1865	0.4856	-0.0271
\$ invest(Gauss)	0.9040	-0.1530	-0.1200	0.5096	1.0066
cash flow(VG)	118.8279	-27.4549	22.0231	35.159	-1.3470
cash flow(Gauss)	0.0527	-0.0088	-0.0142	0.0369	0.0500
% T-bill rate	5.80				

Statistical Estimation					03/95
	σ	ν	θ		
1st IC	0.9406	0.7024	0.2313		
2nd IC	0.9585	0.6513	-0.0458		
3rd IC	0.9752	0.4782	-0.1599		
4th IC	0.9940	0.3377	-0.1640		
5th IC	1.0033	0.2017	0.3355		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG) $\times 10^3$	-0.0229	-0.9092	5.7354	0.0808	1.1834
\$ invest(Gauss)	1.1032	-0.1095	0.0755	0.4570	0.9634
cash flow(VG)	-1.3435	-139.2297	667.1110	3.4050	-26.3376
cash flow(Gauss)	0.0647	-0.0168	0.0088	0.0193	-0.0214
% T-bill rate	5.73				

Statistical Estimation					04/95
	σ	ν	θ		
1st IC	0.9449	0.6215	0.0530		
2nd IC	0.9671	0.6643	-0.0538		
3rd IC	0.9755	0.4757	-0.1693		
4th IC	0.9928	0.3530	-0.1506		
5th IC	0.9823	0.2425	0.3886		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG) $\times 10^3$	-0.4721	-2.1155	3.1264	0.2132	0.9143
\$ invest(Gauss)	1.0669	0.0365	0.1785	0.5228	1.0153
cash flow(VG)	4.8334	-9.2199	145.5187	5.7679	58.1586
cash flow(Gauss)	-0.0109	0.0002	0.0083	0.0141	0.0646
% T-bill rate	5.67				

Statistical Estimation					05/95
	σ	ν	θ		
1st IC	0.9435	0.6179	0.0453		
2nd IC	0.9761	0.4902	0.1773		
3rd IC	0.9810	0.3240	0.3449		
4th IC	0.9690	0.6694	0.0553		
5th IC	0.9946	0.3237	-0.1008		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG) $\times 10^3$	-0.0154	0.3554	-2.2729	1.8230	-1.8628
\$ invest(Gauss)	0.8896	0.0597	0.2161	0.6033	1.0423
cash flow(VG)	0.1416	45.8981	4.4832	171.4126	-8.2759
cash flow(Gauss)	-0.0082	0.0077	-0.0004	0.0567	0.0046
% T-bill rate	5.70				

Statistical Estimation					06/95
	σ	ν	θ		
1st IC	0.9710	0.6991	-0.0526		
2nd IC	0.9449	0.6750	-0.1762		
3rd IC	0.9717	0.4721	0.2030		
4th IC	0.9829	0.3043	-0.3526		
5th IC	0.9960	0.3438	-0.1060		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG) $\times 10^3$	2.1204	-0.6769	-0.2552	2.4010	0.9930
\$ invest(Gauss)	0.8955	0.2957	0.2488	0.6843	1.1472
cash flow(VG)	-18.0059	8.2771	-24.9783	-57.7309	25.6875
cash flow(Gauss)	-0.0076	-0.0036	0.0243	-0.0165	0.0297
% T-bill rate	5.50				

Statistical Estimation					07/95
	σ	ν	θ		
1st IC	0.9809	0.7467	0.0912		
2nd IC	0.9649	0.5980	0.0660		
3rd IC	0.9733	0.4640	-0.2002		
4th IC	0.9962	0.3845	0.0795		
5th IC	1.0056	0.2003	-0.2684		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG) $\times 10^3$	-0.5978	0.0614	-1.8050	2.1152	0.7413
\$ invest(Gauss)	1.0393	0.2767	0.2796	0.5963	1.2115
cash flow(VG)	37.2686	1.0174	-10.0934	-23.6855	-1.2130
cash flow(Gauss)	-0.0648	0.0046	0.0016	-0.0067	-0.0020
% T-bill rate	5.47				

Statistical Estimation					08/95
	σ	ν	θ		
1st IC	0.9642	0.6407	0.0492		
2nd IC	0.9720	0.4375	0.2278		
3rd IC	0.9690	0.7058	0.0955		
4th IC	0.9822	0.2900	-0.3511		
5th IC	0.9951	0.3585	-0.0663		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG) $\times 10^3$	3.8092	-0.4714	0.7311	9.1693	0.9487
\$ invest(Gauss)	0.8740	0.1251	0.2884	0.6305	1.2071
cash flow(VG)	14.1148	-46.8998	-93.3282	737.1091	-23.3421
cash flow(Gauss)	0.0032	0.0124	-0.0368	0.050	-0.0297
% T-bill rate	5.41				

Statistical Estimation					09/95
	σ	ν	θ		
1st IC	0.9752	0.4572	0.2254		
2nd IC	0.9814	0.7535	-0.1102		
3rd IC	0.9818	0.3138	-0.3358		
4th IC	0.9650	0.6715	0.0448		
5th IC	0.9969	0.3542	0.0556		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG) $\times 10^3$	-0.4785	0.3076	0.1856	-0.0790	2.2536
\$ invest(Gauss)	0.7520	0.3518	0.1937	0.7397	1.1345
cash flow(VG)	-12.1475	-24.4708	1.1365	-6.0505	95.8706
cash flow(Gauss)	0.0191	-0.0280	0.0012	0.0567	0.0483
% T-bill rate	5.26				

Statistical Estimation					10/95
	σ	ν	θ		
1st IC	0.9658	0.6584	-0.0566		
2nd IC	1.0532	0.7748	0.0538		
3rd IC	0.9728	0.4212	-0.2433		
4th IC	0.9846	0.3289	-0.3187		
5th IC	0.9953	0.3362	0.0383		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG) $\times 10^3$	1.9370	-0.3084	-1.3527	0.2527	-0.8125
\$ invest(Gauss)	0.7830	0.2557	0.1773	0.7437	1.1530
cash flow(VG)	273.9084	-33.7270	25.9058	15.5813	-58.5204
cash flow(Gauss)	0.1107	0.0280	-0.0034	0.0459	0.0830
% T-bill rate	5.30				

Statistical Estimation					11/95
	σ	ν	θ		
1st IC	0.9624	0.3212	-0.4088		
2nd IC	0.9413	0.5760	0.2207		
3rd IC	0.9627	0.6393	-0.0816		
4th IC	0.9794	0.3185	0.3558		
5th IC	0.9960	0.3664	0.0340		
ticker	mmm	ba	ibm	jnj	mcd
$ invest(VG)	-412.15	-968.36	559.89	-140.71	355.89
$ invest(Gauss)	1.0209	0.5678	0.2411	0.6999	1.2690
cash flow(VG)	-12.2332	-58.3592	-24.3407	-5.4349	32.0511
cash flow(Gauss)	0.0303	0.0342	-0.0105	0.0270	0.1143
% T-bill rate	5.35				

Statistical Estimation					12/95
	σ	ν	θ		
1st IC	0.9965	0.5218	0.0262		
2nd IC	0.9598	0.5960	0.1542		
3rd IC	0.9676	0.6212	-0.1226		
4th IC	0.9815	0.2489	0.3628		
5th IC	0.9755	0.3625	-0.3293		
ticker	mmm	ba	ibm	jnj	mcd
$ invest(VG)	607.79	61.06	436.91	-328.1702	-441.73
$ invest(Gauss)	0.9710	0.4956	0.1897	0.6214	1.2082
cash flow(VG)	-30.2880	0.6188	61.1164	3.2807	17.6524
cash flow(Gauss)	-0.0484	0.0050	0.0265	-0.0062	-0.0483
% T-bill rate	5.16				

Statistical Estimation					01/96
	σ	ν	θ		
1st IC	0.9600	0.6807	-0.1383		
2nd IC	0.9696	0.2908	-0.3957		
3rd IC	0.9614	0.6851	0.0711		
4th IC	0.9952	0.4517	-0.0038		
5th IC	0.9810	0.2509	-0.3798		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG) $\times 10^3$	-1.2242	0.3178	-0.3749	0.1097	0.4637
\$ invest(Gauss)	0.9003	0.5226	0.2789	0.7163	1.0872
cash flow(VG)	-51.9611	16.3247	-68.3071	11.4705	88.8624
cash flow(Gauss)	0.0382	0.0268	0.0508	0.0749	0.2083
% T-bill rate	5.02				

Statistical Estimation					02/96
	σ	ν	θ		
1st IC	0.9664	0.6962	0.0484		
2nd IC	0.9641	0.2806	-0.4552		
3rd IC	0.9621	0.6532	-0.1321		
4th IC	0.9751	0.3115	0.3881		
5th IC	0.9942	0.4753	0.0277		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG) $\times 10^3$	3.2643	0.6935	0.7561	0.2930	0.8170
\$ invest(Gauss)	0.9120	0.7163	0.4075	0.8598	1.4042
cash flow(VG)	-38.3911	35.7859	-37.1179	-3.6066	-33.0775
cash flow(Gauss)	-0.0107	0.0370	-0.0200	-0.0106	-0.0568
% T-bill rate	4.87				

Statistical Estimation					03/96
	σ	ν	θ		
1st IC	0.9638	0.6375	-0.1546		
2nd IC	0.9947	0.5152	0.0343		
3rd IC	0.9831	0.2687	0.3172		
4th IC	0.9617	0.6453	0.0995		
5th IC	0.9589	0.3210	-0.4508		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG)	-544.19	935.35	-225.75	-517.53	81.95
\$ invest(Gauss)	0.8760	0.8101	0.3447	0.8331	1.2604
cash flow(VG)	1.2157	-61.3034	15.8514	24.6694	-7.0362
cash flow(Gauss)	-0.0020	-0.0531	-0.0242	-0.0397	-0.1082
% T-bill rate	4.96				

Statistical Estimation					04/96
	σ	ν	θ		
1st IC	0.9672	0.6686	-0.0456		
2nd IC	0.9574	0.2975	0.4721		
3rd IC	0.9658	0.6400	-0.1082		
4th IC	0.9817	0.3206	-0.3150		
5th IC	0.9952	0.4721	-0.0209		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG) $\times 10^3$	3.2135	0.7310	0.2210	0.3804	0.8077
\$ invest(Gauss)	0.8014	0.6897	0.1966	0.8303	1.0830
cash flow(VG)	253.0302	16.4611	9.8582	28.6827	12.0655
cash flow(Gauss)	0.0631	0.0155	0.0088	0.0626	0.0162
% T-bill rate	4.99				

Statistical Estimation					05/96
	σ	ν	θ		
1st IC	0.9630	0.5985	0.1221		
2nd IC	0.9621	0.3067	-0.4393		
3rd IC	0.9638	0.6117	0.0402		
4th IC	0.9817	0.3316	0.3019		
5th IC	0.9961	0.4582	-0.0108		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG) $\times 10^3$	5.2965	0.6929	1.1423	0.3698	-0.1278
\$ invest(Gauss)	0.8617	0.7084	0.2382	0.9470	0.9971
cash flow(VG)	-24.6926	25.9944	-110.6701	-1.0030	-0.7204
cash flow(Gauss)	-0.0040	0.0266	-0.0231	-0.0026	0.0056
% T-bill rate	5.02				

Statistical Estimation					06/96
	σ	ν	θ		
1st IC	0.9648	0.6241	0.0541		
2nd IC	0.9600	0.2956	0.4656		
3rd IC	0.9646	0.6246	-0.1035		
4th IC	0.9818	0.3513	0.3053		
5th IC	0.9944	0.4742	-0.0053		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG) $\times 10^3$	2.2083	0.0329	0.6288	0.1700	-0.7820
\$ invest(Gauss)	0.7585	0.9157	0.1009	0.9114	0.9959
cash flow(VG)	34.7862	-0.5172	-54.0781	2.8453	54.5939
cash flow(Gauss)	0.0119	-0.0144	-0.0087	0.0153	-0.0695
% T-bill rate	5.11				

Statistical Estimation					07/96
	σ	ν	θ		
1st IC	0.9515	0.5795	0.1089		
2nd IC	0.9635	0.6194	-0.1537		
3rd IC	0.9806	0.3277	0.3249		
4th IC	0.9606	0.2676	0.4926		
5th IC	0.9939	0.5279	-0.0286		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG)	933.46	-144.49	110.67	-416.80	-209.82
\$ invest(Gauss)	0.6635	0.8259	0.0712	0.7976	0.9629
cash flow(VG)	44.4536	-9.3351	22.7744	-25.7861	-13.8004
cash flow(Gauss)	0.0316	0.0534	0.0146	0.0493	0.0633
% T-bill rate	5.17				

Statistical Estimation					08/96
	σ	ν	θ		
1st IC	0.9993	0.6632	-0.0348		
2nd IC	0.9553	0.5502	-0.1078		
3rd IC	0.9634	0.2555	0.4834		
4th IC	0.9815	0.3253	-0.3140		
5th IC	0.9941	0.5090	-0.0200		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG) $\times 10^3$	-0.9896	4.3197	-0.9535	0.5252	0.3304
\$ invest(Gauss)	0.6549	1.0019	0.2058	0.8067	1.1004
cash flow(VG)	-39.3138	11.3071	-108.1622	-19.8800	-3.2552
cash flow(Gauss)	0.0260	0.0026	0.0234	-0.0305	-0.0108
% T-bill rate	5.09				

Statistical Estimation					09/96
	σ	ν	θ		
1st IC	0.9559	0.5476	-0.0957		
2nd IC	0.9551	0.2086	0.6411		
3rd IC	0.9541	0.6900	-0.1459		
4th IC	0.9924	0.3300	0.2920		
5th IC	0.9948	0.5033	-0.0504		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG) $\times 10^3$	1.3090	0.2199	0.1145	-0.2177	-0.5303
\$ invest(Gauss)	0.7245	1.0631	0.3152	0.9238	1.0746
cash flow(VG)	31.8112	13.0300	4.9533	-7.2435	13.6923
cash flow(Gauss)	0.0176	0.0630	0.0136	0.0307	-0.0277
% T-bill rate	5.15				

Statistical Estimation					10/96
	σ	ν	θ		
1st IC	0.9516	0.5623	-0.1184		
2nd IC	0.9676	0.2909	0.4056		
3rd IC	0.9500	0.6530	-0.1518		
4th IC	0.9814	0.3177	0.2902		
5th IC	0.9953	0.5111	-0.0527		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG)	974.18	295.57	199.40	-167.75	-551.06
\$ invest(Gauss)	0.6478	1.1367	0.4359	0.8718	0.9987
cash flow(VG)	128.4532	-14.5618	28.3986	3.5531	-6.6285
cash flow(Gauss)	0.0854	-0.0560	0.0621	-0.0185	0.0120
% T-bill rate	5.01				

Statistical Estimation 11/96

	σ	ν	θ		
1st IC	0.9513	0.6518	0.1579		
2nd IC	0.9531	0.5923	-0.1040		
3rd IC	0.9951	0.5351	-0.0811		
4th IC	0.9842	0.3054	0.2354		
5th IC	0.9631	0.3006	0.4107		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG) $\times 10^3$	-1.5457	-0.4868	-0.6481	0.5452	-0.8183
\$ invest(Gauss)	0.8660	1.0939	0.5406	0.7218	0.9590
cash flow(VG)	11.7898	-41.0659	-49.5529	-18.3301	-22.6917
cash flow(Gauss)	-0.0066	0.0923	0.0413	-0.0243	0.0266
% T-bill rate	5.03				

Statistical Estimation 12/96

	σ	ν	θ		
1st IC	0.9519	0.5609	-0.1258		
2nd IC	0.9939	0.5750	0.0844		
3rd IC	0.9411	0.5586	0.2863		
4th IC	0.9951	0.3045	-0.2250		
5th IC	0.9856	0.3493	0.3401		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG) $\times 10^3$	1.1972	-0.7262	0.5797	-0.1267	2.0727
\$ invest(Gauss)	0.8414	0.9797	0.8265	0.8180	0.8937
cash flow(VG)	48.6458	-32.6798	28.2523	-4.2939	-186.8711
cash flow(Gauss)	0.0342	0.0441	0.0403	0.0277	-0.0806
% T-bill rate	4.87				

Statistical Estimation					01/97
	σ	ν	θ		
1st IC	0.9498	0.5726	0.1307		
2nd IC	0.9928	0.5491	0.0920		
3rd IC	0.9380	0.5836	-0.2688		
4th IC	0.9866	0.2939	-0.2508		
5th IC	0.9618	0.3443	-0.3916		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG) $\times 10^3$	0.4947	-0.5441	0.1622	0.3023	-2.2746
\$ invest(Gauss)	0.7309	1.2635	0.7509	0.8678	0.8362
cash flow(VG)	1.9432	-17.8778	-22.0774	63.9700	-61.7505
cash flow(Gauss)	0.0029	0.0415	-0.1022	0.1836	0.0227
% T-bill rate	5.05				

Statistical Estimation					02/97
	σ	ν	θ		
1st IC	0.9502	0.5674	-0.1255		
2nd IC	0.9666	0.3529	0.3707		
3rd IC	0.9861	0.3389	-0.2631		
4th IC	0.9386	0.5567	0.2838		
5th IC	0.9880	0.5173	-0.1395		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG) $\times 10^3$	0.7189	0.2882	1.3355	-0.1095	-0.2884
\$ invest(Gauss)	0.6436	1.2175	0.5329	1.0866	0.8067
cash flow(VG)	33.1001	-13.978	-68.2270	8.4703	6.2271
cash flow(Gauss)	0.0296	-0.0591	-0.0272	-0.0840	-0.0174
% T-bill rate	5.00				

Statistical Estimation					03/97
	σ	ν	θ		
1st IC	0.9393	0.5775	0.2725		
2nd IC	0.9557	0.5469	0.1022		
3rd IC	0.9826	0.4669	0.2134		
4th IC	0.9774	0.4591	0.2833		
5th IC	0.9862	0.3329	0.2502		
ticker	mmm	ba	ibm	jnj	mcd
$ invest(VG)	351.71	78.64	203.08	907.26	579.92
$ invest(Gauss)	0.6342	1.0967	0.5335	1.1765	0.6529
cash flow(VG)	-23.3376	-5.0231	-0.1228	-1.6798	59.6511
cash flow(Gauss)	-0.0421	-0.0700	-0.0003	-0.0022	0.0672
% T-bill rate	5.14				

Statistical Estimation					04/97
	σ	ν	θ		
1st IC	0.9595	0.6040	-0.0722		
2nd IC	0.9353	0.5674	-0.2919		
3rd IC	0.9843	0.3685	-0.2334		
4th IC	0.9847	0.3527	0.2455		
5th IC	0.9780	0.4806	-0.2676		
ticker	mmm	ba	ibm	jnj	mcd
$ invest(VG) $\times 10^3$	1.3932	0.2695	-0.6072	-0.5980	0.2055
$ invest(Gauss)	0.5444	0.9140	0.5893	1.1399	0.9144
cash flow(VG)	138.3567	3.7658	-158.1432	-39.8164	15.8034
cash flow(Gauss)	0.0541	0.0128	0.1535	0.0759	0.0703
% T-bill rate	5.17				

Statistical Estimation					05/97
	σ	ν	θ		
1st IC	0.9622	0.6286	0.0934		
2nd IC	0.9301	0.5798	0.3140		
3rd IC	0.9678	0.3823	0.3012		
4th IC	0.9836	0.3330	0.2344		
5th IC	0.9956	0.4015	0.0971		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG) $\times 10^4$	-0.0715	-0.1048	0.1224	1.3423	0.1213
\$ invest(Gauss)	0.5329	0.9236	0.6673	1.1243	1.0096
cash flow(VG) $\times 10^3$	-0.0694	-0.1319	0.0332	1.0162	-0.0964
cash flow(Gauss)	0.0517	0.1162	0.0181	0.0851	-0.0802
% T-bill rate	5.13				

Statistical Estimation					06/97
	σ	ν	θ		
1st IC	0.9634	0.4257	-0.1852		
2nd IC	0.9378	0.5923	0.2765		
3rd IC	0.9925	0.5753	-0.1801		
4th IC	0.9826	0.4555	0.2698		
5th IC	0.9878	0.3400	0.2377		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG)	834.84	-356.98	-197.35	766.94	431.60
\$ invest(Gauss)	0.7644	1.0202	0.7238	1.3084	0.9289
cash flow(VG)	-14.5242	14.8738	-30.3363	-62.1657	-1.7837
cash flow(Gauss)	-0.0133	-0.0425	0.1113	-0.1061	-0.0038
% T-bill rate	4.92				

Statistical Estimation					07/97
	σ	ν	θ		
1st IC	0.9256	0.5654	-0.3384		
2nd IC	0.9630	0.4258	-0.1797		
3rd IC	0.9901	0.3283	0.1851		
4th IC	0.9959	0.5603	-0.1531		
5th IC	0.9840	0.4310	0.2740		
ticker	mmm	ba	ibm	jnj	mcd
$ invest(VG)	-698.08	177.15	500.06	729.06	-752.31
$ invest(Gauss)	0.6822	0.8797	0.8815	1.2684	0.8520
cash flow(VG)	42.6174	9.4812	19.9140	-16.5439	-37.8863
cash flow(Gauss)	-0.0416	0.0471	0.0351	-0.0288	0.0429
% T-bill rate	5.07				

Statistical Estimation					08/97
	σ	ν	θ		
1st IC	0.9217	0.5506	0.3656		
2nd IC	0.9666	0.4261	-0.1418		
3rd IC	0.9957	0.5842	0.1184		
4th IC	0.9832	0.4343	-0.2715		
5th IC	0.9883	0.3064	0.1870		
ticker	mmm	ba	ibm	jnj	mcd
$ invest(VG) $\times 10^3$	1.2112	-0.4863	-0.7929	0.4204	1.0216
$ invest(Gauss)	0.5570	0.8389	0.8902	1.1025	0.8041
cash flow(VG)	-62.9949	38.1367	70.8346	-11.6791	-102.4763
cash flow(Gauss)	-0.0290	-0.0658	-0.0795	-0.0306	-0.0807
% T-bill rate	5.13				

Statistical Estimation					09/97
	σ	ν	θ		
1st IC	0.9234	0.5464	-0.3784		
2nd IC	0.9681	0.4141	0.1249		
3rd IC	0.9812	0.4188	0.2805		
4th IC	0.9894	0.3234	-0.1947		
5th IC	0.9978	0.5308	-0.1081		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG) $\times 10^3$	-0.6487	-0.3782	-0.8928	2.6576	-0.6162
\$ invest(Gauss)	0.4849	0.7596	0.8570	1.1077	0.6780
cash flow(VG)	-48.2144	10.4880	33.6709	14.4590	-10.9161
cash flow(Gauss)	0.0360	-0.0211	-0.0323	0.0060	0.0120
% T-bill rate	4.97				

Statistical Estimation					10/97
	σ	ν	θ		
1st IC	0.9666	0.3700	-0.1381		
2nd IC	0.9329	0.5560	-0.3280		
3rd IC	0.9994	0.5353	0.1118		
4th IC	0.9899	0.3058	-0.1914		
5th IC	0.9788	0.4206	0.2927		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG) $\times 10^3$	0.6880	-2.3596	-0.7761	-0.2766	-0.1311
\$ invest(Gauss)	0.6219	0.7893	0.7236	0.9989	0.6533
cash flow(VG)	0.6465	102.5013	-65.5614	-25.4123	4.3704
cash flow(Gauss)	0.0006	-0.0343	0.0611	0.0918	-0.0218
% T-bill rate	4.95				

Statistical Estimation					11/97
	σ	ν	θ		
1st IC	0.9366	0.3989	0.0812		
2nd IC	0.9751	0.4885	-0.1452		
3rd IC	0.9813	0.4345	0.2798		
4th IC	0.9921	0.3441	0.1697		
5th IC	0.9569	0.3506	0.3380		
ticker	mmm	ba	ibm	jnj	mcd
$ invest(VG)	-169.33	-257.01	756.33	423.93	376.80
$ invest(Gauss)	0.5241	0.5192	0.6383	0.9393	0.4822
cash flow(VG)	16.8600	-3.4002	-14.1754	17.8124	12.6990
cash flow(Gauss)	-0.0522	0.0069	-0.0120	0.0395	0.0163
% T-bill rate	5.15				

Statistical Estimation					12/97
	σ	ν	θ		
1st IC	0.9467	0.5392	0.1431		
2nd IC	0.9369	0.6020	-0.2595		
3rd IC	0.9910	0.3148	0.1066		
4th IC	0.9960	0.5563	-0.0041		
5th IC	0.9651	0.2555	0.5050		
ticker	mmm	ba	ibm	jnj	mcd
$ invest(VG) $\times 10^3$	0.7122	-1.0243	0.3509	-0.4528	0.1600
$ invest(Gauss)	0.4159	0.4429	0.5382	1.0776	0.6104
cash flow(VG)	-16.6166	147.3747	20.3835	-18.0911	-4.5302
cash flow(Gauss)	-0.0097	-0.0637	0.0313	0.0431	-0.0173
% T-bill rate	5.16				

Statistical Estimation					01/98
	σ	ν	θ		
1st IC	0.9451	0.5083	0.1702		
2nd IC	0.9411	0.3775	-0.0132		
3rd IC	0.9809	0.5199	-0.1218		
4th IC	0.9696	0.3256	-0.4104		
5th IC	0.9920	0.3126	0.1434		
ticker	mmm	ba	ibm	jnj	mcd
$ invest(VG)	449.78	-54.74	669.22	334.43	965.55
$ invest(Gauss)	0.3104	0.2930	0.5860	1.2111	0.4919
cash flow(VG)	27.0630	-9.1450	-35.0441	6.7702	51.8003
cash flow(Gauss)	0.0187	0.0489	-0.0307	0.0245	0.0264
% T-bill rate	5.09				

Statistical Estimation					02/98
	σ	ν	θ		
1st IC	0.9376	0.4028	-0.0414		
2nd IC	0.9788	0.4916	0.0983		
3rd IC	0.9465	0.5261	0.1733		
4th IC	0.9696	0.3132	0.4269		
5th IC	0.9946	0.2890	0.1358		
ticker	mmm	ba	ibm	jnj	mcd
$ invest(VG)	-920.06	-358.14	-199.18	108.78	987.38
$ invest(Gauss)	0.3666	0.2649	0.6208	1.2828	0.5938
cash flow(VG)	-61.0836	-17.0101	2.1939	7.5196	138.4850
cash flow(Gauss)	0.0243	0.0126	-0.0068	0.0887	0.0833
% T-bill rate	5.11				

Statistical Estimation					03/98
	σ	ν	θ		
1st IC	0.9396	0.6043	0.2298		
2nd IC	0.9488	0.4802	-0.1768		
3rd IC	0.9903	0.5414	0.0054		
4th IC	0.9931	0.3083	0.0940		
5th IC	0.9635	0.2597	0.5124		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG) $\times 10^3$	-0.3904	-0.1988	-0.2394	0.2562	1.9732
\$ invest(Gauss)	0.5323	0.2948	0.5520	1.4130	0.7722
cash flow(VG)	-12.5971	2.2735	-36.3999	-17.9907	214.4314
cash flow(Gauss)	0.0172	-0.0034	0.0839	-0.0992	0.0839
% T-bill rate	5.03				

Statistical Estimation					04/98
	σ	ν	θ		
1st IC	0.9408	0.4099	-0.0608		
2nd IC	0.9872	0.3763	-0.1672		
3rd IC	0.9950	0.3091	0.1043		
4th IC	0.9552	0.5250	0.1772		
5th IC	0.9907	0.3920	-0.1225		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG) $\times 10^3$	0.0848	0.5475	-1.1454	0.2489	-1.5968
\$ invest(Gauss)	0.5731	0.3422	0.6045	1.2039	0.8367
cash flow(VG)	-0.3560	-53.0828	-53.0903	2.2464	25.9246
cash flow(Gauss)	-0.0024	-0.0332	0.0280	0.0109	-0.0136
% T-bill rate	5.00				

Statistical Estimation					05/98
	σ	ν	θ		
1st IC	0.9381	0.4343	0.0722		
2nd IC	0.9485	0.5036	-0.1847		
3rd IC	0.9907	0.4802	0.0139		
4th IC	0.9926	0.3042	0.0858		
5th IC	0.9712	0.2819	0.4170		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG) $\times 10^3$	-0.3060	0.3908	0.1709	0.2604	2.5993
\$ invest(Gauss)	0.5565	0.1710	0.6387	1.1212	0.8436
cash flow(VG)	41.1714	-42.8633	-24.7830	15.1476	193.4907
cash flow(Gauss)	-0.0749	-0.0187	-0.0926	0.0652	0.0628
% T-bill rate	5.03				

Statistical Estimation					06/98
	σ	ν	θ		
1st IC	0.9524	0.5708	0.1028		
2nd IC	0.9403	0.3827	0.0631		
3rd IC	0.9896	0.4892	0.0276		
4th IC	0.9915	0.3092	-0.1166		
5th IC	0.9628	0.2845	0.4581		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG) $\times 10^3$	1.5101	5.1775	8.1817	0.0988	-2.7000
\$ invest(Gauss)	0.3501	0.0928	0.6331	1.2658	0.8691
cash flow(VG) $\times 10^3$	-0.0334	0.6111	1.6635	0.0004	-0.2779
cash flow(Gauss)	-0.0077	0.0109	0.1287	0.0053	0.0895
% T-bill rate	4.99				

Statistical Estimation					07/98
	σ	ν	θ		
1st IC	0.9515	0.5789	-0.0875		
2nd IC	0.9396	0.3782	-0.0585		
3rd IC	0.9939	0.3829	0.0158		
4th IC	0.9955	0.2797	-0.1948		
5th IC	0.9772	0.1503	-0.4505		
ticker	mmm	ba	ibm	jnj	mcd
$ invest(VG)	604.45	89.37	-822.04	527.383	570.43
$ invest(Gauss)	0.1898	0.2730	0.7089	1.1504	1.2075
cash flow(VG)	-27.8152	-21.3255	-11.0227	-0.9435	-77.5261
cash flow(Gauss)	-0.0087	-0.0651	0.0095	-0.0021	-0.1641
% T-bill rate	4.96				

Statistical Estimation					08/98
	σ	ν	θ		
1st IC	0.9516	0.5364	-0.1006		
2nd IC	0.9191	0.3447	0.0419		
3rd IC	0.9791	0.4722	0.0989		
4th IC	0.9613	0.1332	0.7248		
5th IC	0.9912	0.2660	-0.2134		
ticker	mmm	ba	ibm	jnj	mcd
$ invest(VG)	713.78	0.59	840.83	732.99	867.30
$ invest(Gauss)	0.1465	-0.0120	0.7259	1.1115	1.0017
cash flow(VG)	-62.5029	-0.0398	-37.5096	4.7816	-18.7382
cash flow(Gauss)	-0.0128	0.0008	-0.0324	0.0073	-0.0216
% T-bill rate	4.94				

Statistical Estimation					09/98
	σ	ν	θ		
1st IC	0.9583	0.5746	0.0641		
2nd IC	0.9205	0.4461	0.0089		
3rd IC	0.9511	0.2998	0.1363		
4th IC	0.9822	0.5149	-0.0371		
5th IC	0.9780	0.1494	0.5437		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG) $\times 10^3$	2.2691	0.4141	0.6327	-4.0792	-7.7670
\$ invest(Gauss)	0.0610	-0.0878	0.6832	1.1354	0.8395
cash flow(VG)	429.1849	-1.6488	72.0346	-305.6420	30.9252
cash flow(Gauss)	0.0115	0.0003	0.0778	0.0851	-0.0033
% T-bill rate	4.74				

Statistical Estimation					10/98
	σ	ν	θ		
1st IC	0.9230	0.4939	-0.0025		
2nd IC	0.9430	0.3270	0.1530		
3rd IC	0.9570	0.5923	0.1102		
4th IC	0.9847	0.5146	0.0127		
5th IC	0.9620	0.1059	0.8324		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG)	-792.50	285.19	-191.57	324.13	-225.90
\$ invest(Gauss)	0.2889	-0.0345	0.6563	1.0325	0.8342
cash flow(VG)	50.3140	68.9404	-25.5936	8.9358	-24.4640
cash flow(Gauss)	-0.0183	-0.0083	0.0877	0.0285	0.0903
% T-bill rate	4.08				

Statistical Estimation					11/98
	σ	ν	θ		
1st IC	0.9231	0.5077	0.0202		
2nd IC	0.9625	0.5970	0.0955		
3rd IC	0.9547	0.3835	0.2292		
4th IC	0.9825	0.5343	0.0309		
5th IC	0.9644	0.0977	-0.8482		
ticker	mmm	ba	ibm	jnj	mcd
$ invest(VG)	540.41	-91.35	-548.29	122.75	248.60
$ invest(Gauss)	0.2246	0.0091	0.7562	1.0797	0.8788
cash flow(VG)	-38.2737	23.4296	-25.9234	-7.4825	9.2341
cash flow(Gauss)	-0.0159	-0.0023	0.0358	-0.0658	0.0326
% T-bill rate	4.44				

Statistical Estimation					12/98
	σ	ν	θ		
1st IC	0.9161	0.5766	-0.0933		
2nd IC	0.9712	0.6634	0.0788		
3rd IC	0.9679	0.4947	0.178		
4th IC	0.9568	0.3816	0.2219		
5th IC	0.9699	0.1741	-0.3864		
ticker	mmm	ba	ibm	jnj	mcd
$ invest(VG)	-623.75	-889.90	333.73	-730.51	-482.54
$ invest(Gauss)	0.0534	-0.1843	0.8756	1.0291	0.8845
cash flow(VG)	30.9404	-60.7815	55.8893	-6.2787	-58.4730
cash flow(Gauss)	-0.0026	-0.0126	0.1466	0.0088	0.1072
% T-bill rate	4.42				

Statistical Estimation					01/99
	σ	ν	θ		
1st IC	0.9193	0.6241	-0.0745		
2nd IC	0.9881	0.7366	-0.0071		
3rd IC	0.9580	0.3870	0.2122		
4th IC	0.9772	0.4879	0.1401		
5th IC	0.9877	0.2452	0.3193		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG) $\times 10^3$	-0.0752	1.2318	0.3442	0.4222	1.0542
\$ invest(Gauss)	0.1292	-0.1746	0.9603	0.9159	0.9168
cash flow(VG)	-8.3411	52.3752	-40.8898	30.5433	-6.8866
cash flow(Gauss)	0.0143	-0.0074	-0.1141	0.0663	-0.0060
% T-bill rate	4.34				

Statistical Estimation					02/99
	σ	ν	θ		
1st IC	0.9207	0.6269	0.0624		
2nd IC	0.9733	0.4517	0.1383		
3rd IC	0.9884	0.4766	-0.0162		
4th IC	0.9890	0.7264	-0.0287		
5th IC	0.9860	0.2575	-0.3323		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG) $\times 10^3$	0.6929	0.5366	0.3882	-2.2408	3.9343
\$ invest(Gauss)	0.1429	-0.0654	0.7533	0.9840	0.8364
cash flow(VG)	-41.4175	-14.2534	-11.9190	-121.0981	442.0010
cash flow(Gauss)	-0.0085	0.0017	-0.0231	0.0532	0.0940
% T-bill rate	4.45				

Statistical Estimation					03/99
	σ	ν	θ		
1st IC	0.9409	0.6583	0.0461		
2nd IC	0.9892	0.7054	0.0302		
3rd IC	0.9598	0.4246	0.2186		
4th IC	0.9778	0.4912	0.1015		
5th IC	0.9825	0.2294	0.3802		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG) $\times 10^3$	-1.3942	-0.4538	0.3776	-0.2620	1.1886
\$ invest(Gauss)	-0.0052	-0.1934	0.7097	1.0681	0.8612
cash flow(VG)	-190.7332	-69.9513	9.6040	-25.0965	27.2054
cash flow(Gauss)	-0.0007	-0.0298	0.0181	0.1023	0.0197
% T-bill rate	4.48				

Statistical Estimation					04/99
	σ	ν	θ		
1st IC	0.9128	0.6479	0.1407		
2nd IC	0.9868	0.4901	0.0698		
3rd IC	0.9925	0.7959	0.0502		
4th IC	0.9789	0.5709	-0.2168		
5th IC	0.9860	0.2875	0.2884		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG) $\times 10^3$	3.8410	0.2495	0.0063	0.0370	-0.8306
\$ invest(Gauss)	0.1118	-0.0870	0.6536	1.0282	0.9162
cash flow(VG)	356.1563	26.2141	2.2313	-2.0343	76.6390
cash flow(Gauss)	0.0104	-0.0091	0.2316	-0.0566	-0.0845
% T-bill rate	4.28				

Statistical Estimation					05/99
	σ	ν	θ		
1st IC	0.9888	0.5098	-0.0138		
2nd IC	0.9397	0.6858	0.0654		
3rd IC	0.9524	0.5228	0.1356		
4th IC	0.9865	0.6160	0.0850		
5th IC	0.9812	0.2572	0.3694		
ticker	mmm	ba	ibm	jnj	mcd
$ invest(VG)	-383.99	512.85	521.44	-217.17	863.95
$ invest(Gauss)	0.2563	-0.1060	0.7859	0.8040	0.7432
cash flow(VG)	-2.1076	-20.6058	35.2344	12.1466	0.4293
cash flow(Gauss)	0.0014	0.0043	0.0531	-0.0450	0.0004
% T-bill rate	4.51				

Statistical Estimation					06/99
	σ	ν	θ		
1st IC	0.9506	0.6154	-0.2709		
2nd IC	0.9162	0.6670	-0.1292		
3rd IC	0.9861	0.6162	-0.1034		
4th IC	0.9882	0.4560	0.0599		
5th IC	0.9799	0.2673	0.3736		
ticker	mmm	ba	ibm	jnj	mcd
$ invest(VG) $\times 10^3$	1.9189	1.1877	2.0791	0.5740	9.4543
$ invest(Gauss)	0.2714	-0.1515	0.7752	0.7720	0.6947
cash flow(VG)	-13.1049	47.1504	62.9420	44.8717	428.6694
cash flow(Gauss)	-0.0019	-0.0060	0.0235	0.0603	0.0315
% T-bill rate	4.59				

Statistical Estimation					07/99
	σ	ν	θ		
1st IC	0.9479	0.6233	0.2561		
2nd IC	0.9176	0.6742	-0.1393		
3rd IC	0.9882	0.5959	0.1097		
4th IC	0.9822	0.4458	-0.1067		
5th IC	0.9761	0.2666	0.4046		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG) $\times 10^3$	0.6888	0.1451	0.5737	-0.3300	-1.4191
\$ invest(Gauss)	0.3463	-0.1294	0.7087	0.7949	0.7187
cash flow(VG)	54.3003	0.9094	-28.6690	4.8703	140.9414
cash flow(Gauss)	0.0273	-0.0008	-0.0354	-0.0117	-0.0714
% T-bill rate	4.60				

Statistical Estimation					08/99
	σ	ν	θ		
1st IC	0.9387	0.6909	0.0043		
2nd IC	0.9807	0.5023	0.1271		
3rd IC	0.9907	0.6636	0.0888		
4th IC	0.9620	0.4943	0.2049		
5th IC	0.9712	0.2194	-0.4943		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG)	471.22	-418.34	-153.16	316.13	-400.71
\$ invest(Gauss)	0.4167	-0.1652	0.7243	0.7783	0.6524
cash flow(VG)	7.2029	23.0545	-8.3439	0.9847	-27.1185
cash flow(Gauss)	0.0064	0.0091	0.0395	0.0024	0.0442
% T-bill rate	4.76				

Statistical Estimation					09/99
	σ	ν	θ		
1st IC	0.9420	0.7537	-0.0011		
2nd IC	0.9795	0.5005	0.1448		
3rd IC	0.9655	0.5271	0.1801		
4th IC	0.9885	0.5542	0.1114		
5th IC	0.9685	0.2125	-0.5466		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG)	-327.89	-319.05	213.99	-145.71	-751.76
\$ invest(Gauss)	0.3762	-0.1701	0.8235	0.6147	0.6354
cash flow(VG)	29.0168	2.2313	-38.6876	-4.3347	-10.3713
cash flow(Gauss)	-0.0333	0.0012	-0.1489	0.0183	0.0088
% T-bill rate	4.73				

Statistical Estimation					10/99
	σ	ν	θ		
1st IC	0.9545	0.5715	-0.2450		
2nd IC	0.9509	0.8689	0.0733		
3rd IC	0.9884	0.5595	0.1090		
4th IC	0.9791	0.4572	-0.1175		
5th IC	0.9620	0.2338	-0.5633		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG) $\times 10^3$	-0.7759	0.0586	0.0689	0.3563	1.0177
\$ invest(Gauss)	0.3276	-0.1689	0.6954	0.6666	0.6575
cash flow(VG)	-106.9764	-0.3142	-8.7973	21.5518	84.1729
cash flow(Gauss)	0.0452	0.0009	-0.0888	0.0403	0.0544
% T-bill rate	4.88				

Statistical Estimation					11/99
	σ	ν	θ		
1st IC	0.9405	0.7585	-0.0287		
2nd IC	0.9381	0.5098	-0.1861		
3rd IC	0.9882	0.5456	0.0934		
4th IC	0.9770	0.4873	0.1530		
5th IC	0.9717	0.2387	-0.4878		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG) $\times 10^3$	0.3817	-0.9621	-1.6962	-0.4961	-0.5745
\$ invest(Gauss)	0.3466	-0.2015	0.6388	0.6311	0.6207
cash flow(VG)	-25.6653	100.1656	-286.8018	52.5396	31.0325
cash flow(Gauss)	-0.0233	0.0210	0.1080	-0.0668	-0.0335
% T-bill rate	5.07				

Statistical Estimation					12/99
	σ	ν	θ		
1st IC	0.9372	0.7654	0.0158		
2nd IC	0.9382	0.5421	-0.1807		
3rd IC	0.9740	0.4847	0.1718		
4th IC	0.9911	0.5825	0.1094		
5th IC	0.9748	0.2308	-0.4683		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG)	-475.16	-288.28	-279.12	-29.21	-672.64
\$ invest(Gauss)	0.2276	-0.2917	0.7909	0.5317	0.5706
cash flow(VG)	-16.3852	-68.8424	-22.8887	1.1316	73.7100
cash flow(Gauss)	0.0078	-0.0697	0.0649	-0.0206	-0.0625
% T-bill rate	5.23				

Statistical Estimation					01/00
	σ	ν	θ		
1st IC	0.9484	0.8237	-0.0169		
2nd IC	0.9907	0.6227	0.0971		
3rd IC	0.9373	0.4834	-0.2014		
4th IC	0.9768	0.2083	-0.4790		
5th IC	0.9753	0.4505	-0.1634		
ticker	mmm	ba	ibm	jnj	mcd
$ invest(VG)	507.03	93.84	367.06	424.54	482.10
$ invest(Gauss)	0.2698	-0.1557	0.7110	0.4006	0.3939
cash flow(VG)	-49.4030	-20.1365	-9.7340	-56.8218	-45.3004
cash flow(Gauss)	-0.0263	0.0334	-0.0189	-0.0536	-0.0370
% T-bill rate	5.34				

Statistical Estimation					02/00
	σ	ν	θ		
1st IC	1.0068	0.7735	-0.0733		
2nd IC	0.9349	0.5221	0.2016		
3rd IC	0.9749	0.4946	0.1152		
4th IC	0.9873	0.5800	-0.1337		
5th IC	0.9807	0.2250	-0.4129		
ticker	mmm	ba	ibm	jnj	mcd
$ invest(VG)	-399.85	519.24	-404.34	-165.83	478.91
$ invest(Gauss)	0.2450	-0.2746	0.6842	0.2727	0.2312
cash flow(VG)	8.4694	-9.2596	15.7342	0.5486	-62.4161
cash flow(Gauss)	-0.0052	0.0049	-0.0266	-0.0009	-0.0301
% T-bill rate	5.57				

Statistical Estimation					03/00
	σ	ν	θ		
1st IC	0.9444	0.7653	0.0416		
2nd IC	0.9774	0.5088	-0.1379		
3rd IC	0.9893	0.6925	0.1093		
4th IC	0.9436	0.5115	-0.1798		
5th IC	0.9818	0.2284	0.3981		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG)	-240.16	-253.01	66.24	-596.09	-426.86
\$ invest(Gauss)	0.2546	-0.2990	0.6767	0.3190	0.1760
cash flow(VG)	-6.7108	16.6120	-1.0548	-13.6169	-60.1358
cash flow(Gauss)	0.0071	0.0196	-0.0108	0.0073	0.0248
% T-bill rate	5.72				

Statistical Estimation					04/00
	σ	ν	θ		
1st IC	0.9458	0.7409	0.0382		
2nd IC	0.9348	0.5080	0.2600		
3rd IC	0.9915	0.4370	0.0830		
4th IC	0.9815	0.5181	-0.1507		
5th IC	0.9922	0.3018	-0.0699		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG) $\times 10^3$	0.1713	-0.1383	-0.2457	3.9910	-0.0806
\$ invest(Gauss)	0.2288	-0.2256	0.6544	0.3659	0.3468
cash flow(VG)	-4.0029	-11.3341	12.8763	324.3831	-4.5988
cash flow(Gauss)	-0.0053	-0.0185	-0.0343	0.0297	0.0198
% T-bill rate	5.67				

Statistical Estimation					05/00
	σ	ν	θ		
1st IC	0.9476	0.7629	-0.0370		
2nd IC	0.9388	0.5034	0.2441		
3rd IC	0.9908	0.4233	0.0742		
4th IC	0.9774	0.5981	0.1516		
5th IC	0.9918	0.3321	-0.1152		
ticker	mmm	ba	ibm	jnj	mcd
$ invest(VG)	-152.70	-166.69	-497.03	-955.85	-223.69
$ invest(Gauss)	0.1212	-0.2405	0.6736	0.3485	0.3140
cash flow(VG)	6.9980	-5.2766	-64.6177	-20.5675	37.2947
cash flow(Gauss)	-0.0056	-0.0076	0.0876	0.0075	-0.0523
% T-bill rate	5.92				

Statistical Estimation					06/00
	σ	ν	θ		
1st IC	0.9351	0.5372	-0.2783		
2nd IC	0.9566	0.8558	-0.0187		
3rd IC	0.9721	0.3763	0.1674		
4th IC	0.9904	0.3295	-0.1215		
5th IC	0.9852	0.5171	0.0806		
ticker	mmm	ba	ibm	jnj	mcd
$ invest(VG)	-688.54	-168.54	-838.46	849.09	524.49
$ invest(Gauss)	0.0901	-0.2514	0.7479	0.3300	0.2247
cash flow(VG)	-21.4080	-28.8709	85.0298	24.1077	-16.7687
cash flow(Gauss)	0.0028	-0.0431	-0.0758	0.0094	-0.0072
% T-bill rate	5.74				

Statistical Estimation					07/00
	σ	ν	θ		
1st IC	0.9468	0.5281	-0.2412		
2nd IC	0.9554	0.8058	0.0253		
3rd IC	0.9950	0.4118	0.0655		
4th IC	0.9835	0.6187	-0.1241		
5th IC	0.9950	0.3057	-0.1057		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG) $\times 10^3$	0.0774	1.2824	-0.0359	1.4153	0.5599
\$ invest(Gauss)	0.0965	-0.1688	0.6858	0.3469	0.2224
cash flow(VG)	7.4422	-2.7316	-4.4043	50.6348	31.5273
cash flow(Gauss)	0.0093	0.0004	0.0841	0.0124	0.0125
% T-bill rate	6.14				

Statistical Estimation					08/00
	σ	ν	θ		
1st IC	0.9594	0.8619	-0.0155		
2nd IC	0.9501	0.5286	-0.2249		
3rd IC	0.9952	0.4266	0.0622		
4th IC	0.9833	0.5933	0.1334		
5th IC	0.9956	0.2856	0.1193		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG) $\times 10^3$	0.1875	-0.5807	-1.6920	1.4459	-1.6701
\$ invest(Gauss)	0.1373	-0.1806	0.6513	0.3891	0.2239
cash flow(VG)	-22.7645	-136.8968	-1.4096	-71.2329	317.7448
cash flow(Gauss)	-0.0167	-0.0426	0.0005	-0.0192	-0.0426
% T-bill rate	6.28				

Statistical Estimation					09/00
	σ	ν	θ		
1st IC	0.9555	0.7593	-0.0257		
2nd IC	0.9504	0.4949	0.2331		
3rd IC	0.9726	0.5534	0.1775		
4th IC	0.9935	0.4203	-0.0545		
5th IC	0.9946	0.2863	-0.1287		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG) $\times 10^3$	-0.5876	1.0522	-1.3068	0.0767	2.4601
\$ invest(Gauss)	0.0333	-0.0650	0.6344	0.3103	0.1065
cash flow(VG)	-53.0773	60.2865	115.4632	3.7011	133.6639
cash flow(Gauss)	0.0030	-0.0037	-0.0561	0.0150	0.0058
% T-bill rate	6.18				

Statistical Estimation					10/00
	σ	ν	θ		
1st IC	0.9768	0.5658	0.1575		
2nd IC	0.9780	0.6223	0.0539		
3rd IC	0.9533	0.7041	0.0424		
4th IC	0.9957	0.4272	0.0062		
5th IC	0.9910	0.3036	0.1445		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG)	657.80	-987.26	59.19	261.87	238.33
\$ invest(Gauss)	0.0358	-0.0092	0.4481	0.3894	0.1761
cash flow(VG)	27.1933	-46.9369	-7.3843	-6.9297	35.1090
cash flow(Gauss)	0.0015	-0.0004	-0.0559	-0.0103	0.0259
% T-bill rate	6.29				

Statistical Estimation					11/00
	σ	ν	θ		
1st IC	0.9769	0.5503	-0.1680		
2nd IC	0.9578	0.7300	0.0538		
3rd IC	0.9816	0.5885	0.0388		
4th IC	0.9691	0.3197	0.2352		
5th IC	0.9928	0.3002	0.1255		
ticker	mmm	ba	ibm	jnj	mcd
$ invest(VG)	-708.50	739.77	0.2357	606.81	-353.82
$ invest(Gauss)	0.0092	0.0372	0.3560	0.3537	0.2119
cash flow(VG)	-131.5019	11.3902	-0.0287	23.8372	22.5860
cash flow(Gauss)	0.0017	0.0006	-0.0433	0.0139	-0.0135
% T-bill rate	6.36				

Statistical Estimation					12/00
	σ	ν	θ		
1st IC	0.9766	0.5747	0.0170		
2nd IC	0.9693	0.3501	0.0983		
3rd IC	0.9825	0.5813	0.0348		
4th IC	0.9874	0.2685	0.2315		
5th IC	0.9855	0.4452	-0.0927		
ticker	mmm	ba	ibm	jnj	mcd
$ invest(VG)	150.95	281.64	-707.45	457.43	749.98
$ invest(Gauss)	0.1306	-0.0038	0.3295	0.4460	0.2288
cash flow(VG)	-5.0566	-33.9400	-161.6828	-23.4025	68.4012
cash flow(Gauss)	-0.0044	0.0005	0.0753	-0.0228	0.0209
% T-bill rate	5.94				

Statistical Estimation					01/01
	σ	ν	θ		
1st IC	0.9571	0.6513	0.0647		
2nd IC	0.9821	0.5240	0.1143		
3rd IC	0.9738	0.6483	0.0315		
4th IC	0.9756	0.5055	0.2439		
5th IC	0.9855	0.2473	-0.3220		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG)	346.65	-238.71	-245.50	603.06	-278.14
\$ invest(Gauss)	0.1441	-0.0902	0.4303	0.3306	0.3060
cash flow(VG)	11.8784	-10.5698	-14.3631	4.6864	40.9837
cash flow(Gauss)	0.0049	-0.0040	0.0252	0.0026	-0.0451
% T-bill rate	5.29				

Statistical Estimation					02/01
	σ	ν	θ		
1st IC	0.9708	0.2868	-0.1104		
2nd IC	0.9694	0.6221	0.0555		
3rd IC	0.9866	0.4579	0.0939		
4th IC	0.9826	0.5499	0.0469		
5th IC	0.9885	0.3032	0.2048		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG) $\times 10^3$	1.1195	-0.0127	-0.0500	-2.8849	-0.4623
\$ invest(Gauss)	0.1229	-0.0091	0.4539	0.3349	0.2253
cash flow(VG)	-72.4531	1.1064	11.8306	125.6027	45.3285
cash flow(Gauss)	-0.0080	0.0008	-0.1073	-0.0146	-0.0221
% T-bill rate	5.01				

Statistical Estimation					03/01
	σ	ν	θ		
1st IC	0.9619	0.6127	0.0600		
2nd IC	0.9827	0.5081	0.1272		
3rd IC	0.9734	0.6058	0.0215		
4th IC	0.9798	0.4613	-0.2137		
5th IC	0.9855	0.2486	-0.2961		
ticker	mmm	ba	ibm	jnj	mcd
$ invest(VG)	295.96	-259.83	-259.86	-930.34	278.28
$ invest(Gauss)	0.1482	-0.0354	0.4034	0.3295	0.1256
cash flow(VG)	30.5568	-25.5930	-76.0216	-17.4437	5.9258
cash flow(Gauss)	0.0153	-0.0035	0.1180	0.0062	0.0027
% T-bill rate	4.54				

Statistical Estimation					04/01
	σ	ν	θ		
1st IC	0.9721	0.2585	-0.1679		
2nd IC	0.9845	0.4377	-0.0946		
3rd IC	0.9669	0.4771	0.0599		
4th IC	0.9758	0.5756	-0.0192		
5th IC	0.9876	0.3486	-0.1711		
ticker	mmm	ba	ibm	jnj	mcd
$ invest(VG)	-17.31	532.16	-538.36	-957.41	-198.05
$ invest(Gauss)	0.1756	0.0087	0.4174	0.2759	0.1184
cash flow(VG)	-1.1029	67.5252	-12.8733	-88.2804	-3.2051
cash flow(Gauss)	0.0112	0.0011	0.0100	0.0254	0.0019
% T-bill rate	3.97				

Statistical Estimation					05/01
	σ	ν	θ		
1st IC	0.9726	0.2715	-0.1538		
2nd IC	0.9860	0.4604	0.0834		
3rd IC	0.9765	0.5635	0.0061		
4th IC	0.9662	0.4786	0.0363		
5th IC	0.9875	0.3836	-0.1608		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG) $\times 10^3$	-3.5007	0.4514	0.2520	1.3250	-1.5845
\$ invest(Gauss)	0.1942	0.0217	0.4090	0.3684	0.1561
cash flow(VG)	87.3523	-44.6144	-6.3708	68.7891	-50.9861
cash flow(Gauss)	-0.0048	-0.0021	-0.0103	0.0191	0.0050
% T-bill rate	3.70				

Statistical Estimation					06/01
	σ	ν	θ		
1st IC	0.9759	0.4973	0.2412		
2nd IC	0.9564	0.1535	0.4440		
3rd IC	0.9761	0.5523	0.0231		
4th IC	0.9879	0.3927	0.0856		
5th IC	0.9640	0.5478	0.0678		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG) $\times 10^3$	-0.1318	0.7468	0.1256	1.1697	0.4237
\$ invest(Gauss)	0.1288	-0.0299	0.3674	0.4227	0.1854
cash flow(VG)	6.6458	-67.6298	-12.2245	16.7511	-15.4030
cash flow(Gauss)	-0.0065	0.0027	-0.0358	0.0061	-0.0067
% T-bill rate	3.57				

Statistical Estimation					07/01
	σ	ν	θ		
1st IC	0.9587	0.5782	0.0948		
2nd IC	0.9734	0.6014	0.0300		
3rd IC	0.9808	0.5237	-0.1518		
4th IC	0.9952	0.3159	0.0141		
5th IC	0.9874	0.3896	-0.1747		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG)	-81.16	-401.69	359.23	262.72	-464.39
\$ invest(Gauss)	0.1605	-0.0908	0.2940	0.4862	0.1309
cash flow(VG)	4.4755	17.0982	1.4214	-2.1138	-11.4898
cash flow(Gauss)	-0.0088	0.0039	0.0012	-0.0039	0.0032
% T-bill rate	3.59				

Statistical Estimation					08/01
	σ	ν	θ		
1st IC	0.9732	0.6281	0.0260		
2nd IC	0.9763	0.4562	-0.2463		
3rd IC	0.9939	0.2296	-0.0687		
4th IC	0.9862	0.4073	0.0919		
5th IC	0.9634	0.5427	0.0478		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG) $\times 10^3$	-2.2472	0.6435	-0.9126	1.5621	-0.1380
\$ invest(Gauss)	0.1462	-0.1082	0.2982	0.5475	0.2029
cash flow(VG)	336.7749	-254.3927	88.0805	-43.3923	5.7885
cash flow(Gauss)	-0.0219	0.0428	-0.0288	-0.0152	-0.0085
% T-bill rate	3.44				

Statistical Estimation					09/01
	σ	ν	θ		
1st IC	0.9740	0.6261	0.0075		
2nd IC	0.9502	0.7378	-0.0174		
3rd IC	0.9664	0.2359	0.3095		
4th IC	0.9877	0.4499	0.1009		
5th IC	0.9863	0.4112	-0.1701		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG) $\times 10^3$	0.8807	-3.3011	0.5592	0.1332	-1.3492
\$ invest(Gauss)	0.0899	-0.2274	0.2996	0.5929	0.1867
cash flow(VG)	139.4116	-101.7648	63.7209	16.0659	-56.3906
cash flow(Gauss)	0.0142	-0.0070	0.0341	0.0715	0.0078
% T-bill rate	2.69				

Statistical Estimation					10/01
	σ	ν	θ		
1st IC	0.9395	0.6017	0.0722		
2nd IC	0.9859	0.4511	0.1582		
3rd IC	0.9638	0.6171	-0.0062		
4th IC	0.9918	0.2603	-0.0895		
5th IC	0.9883	0.4264	0.0982		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG) $\times 10^3$	0.7207	-0.1664	0.0133	2.3163	0.1026
\$ invest(Gauss)	0.1732	-0.2016	0.3805	0.6940	0.2256
cash flow(VG)	67.4199	-4.5570	1.0687	81.6898	-1.7943
cash flow(Gauss)	0.0162	-0.0055	0.0305	0.0245	-0.0039
% T-bill rate	2.20				

Statistical Estimation					11/01
	σ	ν	θ		
1st IC	0.9575	0.7013	0.0029		
2nd IC	0.9945	0.3118	0.1240		
3rd IC	0.9779	0.6516	-0.0239		
4th IC	0.9856	0.4000	0.1784		
5th IC	0.9887	0.4053	0.0536		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG) $\times 10^3$	-0.0461	0.2344	0.1557	-0.5798	2.7775
\$ invest(Gauss)	0.2258	-0.2248	0.3536	0.6322	0.2175
cash flow(VG)	-1.4648	26.1773	10.1792	11.7509	-173.8142
cash flow(Gauss)	0.0072	-0.0251	0.0231	-0.0128	-0.0136
% T-bill rate	1.91				

Statistical Estimation					12/01
	σ	ν	θ		
1st IC	0.9592	0.6944	0.0082		
2nd IC	0.9737	0.6597	-0.0204		
3rd IC	0.9833	0.5017	0.1416		
4th IC	0.9891	0.4049	-0.1627		
5th IC	0.9939	0.3432	0.1150		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG) $\times 10^3$	-0.4012	-0.8317	1.7413	-5.1082	-1.7641
\$ invest(Gauss)	0.3452	-0.1780	0.3908	0.5614	0.2063
cash flow(VG)	33.3628	-28.1381	-192.1211	150.8831	-40.4956
cash flow(Gauss)	-0.0287	-0.0060	-0.0431	-0.0166	0.0047
% T-bill rate	1.72				

Statistical Estimation					01/02
	σ	ν	θ		
1st IC	0.9722	0.6672	0.0231		
2nd IC	0.9567	0.6712	-0.0028		
3rd IC	0.9846	0.5181	0.1245		
4th IC	0.9874	0.4027	-0.1799		
5th IC	0.9945	0.3624	0.1052		
ticker	mmm	ba	ibm	jnj	mcd
$ invest(VG)	-400.18	-129.86	827.04	184.61	-519.05
$ invest(Gauss)	0.2588	-0.1263	0.3779	0.5171	0.2212
cash flow(VG)	-44.5096	-16.4626	-79.4390	5.6059	-2.0592
cash flow(Gauss)	0.0288	-0.0160	-0.0363	0.0157	0.0009
% T-bill rate	1.68				

Statistical Estimation					02/02
	σ	ν	θ		
1st IC	0.9866	0.7961	-0.0110		
2nd IC	0.9695	0.2692	0.2688		
3rd IC	0.9893	0.4595	-0.0789		
4th IC	0.9866	0.3927	0.1913		
5th IC	0.9787	0.6318	-0.0303		
ticker	mmm	ba	ibm	jnj	mcd
$ invest(VG)	204.05	599.58	-211.79	-27.64	-689.45
$ invest(Gauss)	0.2831	-0.1217	0.3272	0.4871	0.1484
cash flow(VG)	-6.7502	10.5369	-9.5949	-2.2827	-5.8304
cash flow(Gauss)	-0.0094	-0.0021	0.0148	0.0402	0.0013
% T-bill rate	1.76				

Statistical Estimation					03/02
	σ	ν	θ		
1st IC	0.9614	0.7049	0.0379		
2nd IC	0.9732	0.6497	0.0046		
3rd IC	0.9659	0.2648	0.2580		
4th IC	0.9837	0.3735	0.2100		
5th IC	0.9898	0.4376	-0.0658		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG) $\times 10^3$	0.0862	-0.2769	0.6867	0.4679	1.1865
\$ invest(Gauss)	0.2377	-0.1104	0.3251	0.5286	0.1652
cash flow(VG)	7.4790	23.9721	-109.0739	-10.1830	46.6579
cash flow(Gauss)	0.0206	0.0096	-0.0516	-0.0115	0.0065
% T-bill rate	1.83				

Statistical Estimation					04/02
	σ	ν	θ		
1st IC	0.9827	0.5755	-0.1731		
2nd IC	0.9808	0.8022	0.0606		
3rd IC	0.9572	0.6494	0.0074		
4th IC	0.9845	0.4028	0.1802		
5th IC	0.9945	0.4012	0.0616		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG) $\times 10^3$	-2.2700	-0.2823	0.0186	-0.6843	0.5209
\$ invest(Gauss)	0.2782	-0.1272	0.2005	0.5835	0.1140
cash flow(VG)	-37.0408	-10.4176	-0.7665	20.3997	34.8350
cash flow(Gauss)	0.0045	-0.0047	-0.0083	-0.0174	0.0076
% T-bill rate	1.75				

Statistical Estimation					05/02
	σ	ν	θ		
1st IC	0.9764	0.7348	-0.0392		
2nd IC	0.9577	0.6236	-0.0045		
3rd IC	0.9713	0.3004	0.1891		
4th IC	0.9861	0.3884	-0.1862		
5th IC	0.9879	0.4612	-0.0857		
ticker	mmm	ba	ibm	jnj	mcd
$ invest(VG)	351.47	99.27	-309.18	-157.76	952.06
$ invest(Gauss)	0.3068	-0.0749	0.1837	0.5629	0.0820
cash flow(VG)	-15.2033	-1.2427	54.4062	20.9277	-39.6325
cash flow(Gauss)	-0.0133	0.0009	-0.0323	-0.0747	-0.0034
% T-bill rate	1.76				

Statistical Estimation					06/02
	σ	ν	θ		
1st IC	0.9674	0.3735	-0.1779		
2nd IC	0.9862	0.5004	-0.1135		
3rd IC	0.9735	0.4880	-0.0980		
4th IC	0.9772	0.6761	0.0334		
5th IC	0.9810	0.3801	-0.2402		
ticker	mmm	ba	ibm	jnj	mcd
$ invest(VG)	-329.52	280.89	-311.18	-112.72	225.23
$ invest(Gauss)	0.4089	-0.0905	0.1190	0.3823	0.0229
cash flow(VG)	7.6337	0.0689	-2.9563	9.9863	-56.1376
cash flow(Gauss)	-0.0095	-0.0000	0.0011	-0.0339	-0.0057
% T-bill rate	1.73				

Statistical Estimation					07/02
	σ	ν	θ		
1st IC	0.9819	0.6323	-0.1609		
2nd IC	0.9517	0.4344	-0.1614		
3rd IC	0.9638	0.6134	-0.0271		
4th IC	0.9829	0.4385	0.2248		
5th IC	0.9750	0.5797	0.0101		
ticker	mmm	ba	ibm	jnj	mcd
$ invest(VG)	-57.09	-345.82	-388.93	30.99	-46.46
$ invest(Gauss)	0.4463	0.0110	0.0314	0.2261	-0.1118
cash flow(VG)	-3.5169	45.9025	-62.4230	3.9809	-4.8164
cash flow(Gauss)	0.0275	-0.0015	0.0050	0.0290	-0.0116
% T-bill rate	1.71				

Statistical Estimation					08/02
	σ	ν	θ		
1st IC	0.9746	0.6581	0.0122		
2nd IC	0.9815	0.6215	0.1610		
3rd IC	0.9479	0.3968	0.2121		
4th IC	0.9622	0.5420	0.0106		
5th IC	0.9789	0.4080	-0.2703		
ticker	mmm	ba	ibm	jnj	mcd
$ invest(VG)	-233.92	-274.91	-17.30	-212.97	-6.69
$ invest(Gauss)	0.5079	-0.0266	0.1332	0.3234	-0.0794
cash flow(VG)	13.7907	9.1534	125.3355	-21.8949	10.6011
cash flow(Gauss)	0.0596	-0.0015	0.0091	0.0231	-0.0122
% T-bill rate	1.65				

Statistical Estimation					09/02
	σ	ν	θ		
1st IC	0.9759	0.6738	-0.0200		
2nd IC	0.9520	0.4145	-0.1224		
3rd IC	0.9617	0.5292	-0.0006		
4th IC	0.9837	0.5539	-0.1359		
5th IC	0.9772	0.4028	0.2755		
ticker	mmm	ba	ibm	jnj	mcd
$ invest(VG)	97.13	-74.33	507.00	-275.73	143.29
$ invest(Gauss)	0.4201	0.0120	0.0367	0.2905	-0.1646
cash flow(VG)	13.7907	9.1534	125.3355	-21.8949	10.6011
cash flow(Gauss)	0.0596	-0.0015	0.0091	0.0231	-0.0122
% T-bill rate	1.66				

Statistical Estimation					10/02
	σ	ν	θ		
1st IC	0.9721	0.6478	0.1801		
2nd IC	0.9498	0.4065	-0.1793		
3rd IC	0.9855	0.5395	-0.1237		
4th IC	0.9749	0.6274	-0.0228		
5th IC	0.9777	0.4031	0.2710		
ticker	mmm	ba	ibm	jnj	mcd
$ invest(VG) $\times 10^3$	-0.2249	2.1505	-0.6374	0.1926	1.0039
$ invest(Gauss)	0.4842	-0.1294	0.0343	0.3139	-0.2318
cash flow(VG)	-4.7565	131.2729	-88.4831	5.4994	-16.3973
cash flow(Gauss)	0.0102	-0.0079	0.0048	0.0090	0.0038
% T-bill rate	1.61				

Statistical Estimation					11/02
	σ	ν	θ		
1st IC	0.9510	0.4469	0.1759		
2nd IC	0.9786	0.6985	0.0132		
3rd IC	0.9646	0.4726	0.0097		
4th IC	0.9829	0.5641	-0.1088		
5th IC	0.9784	0.4126	-0.2487		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG) $\times 10^3$	-1.6224	9.5002	4.1960	0.5445	-0.3709
\$ invest(Gauss)	0.4884	-0.1037	0.0416	0.3311	-0.2705
cash flow(VG)	93.5497	92.5669	-233.8221	-46.1194	62.8395
cash flow(Gauss)	-0.0282	-0.0010	-0.0023	-0.0280	0.0458
% T-bill rate	1.25				

Statistical Estimation					12/02
	σ	ν	θ		
1st IC	0.9502	0.4256	0.1803		
2nd IC	0.9842	0.6371	0.0658		
3rd IC	0.9744	0.7031	-0.0284		
4th IC	0.9649	0.4218	0.1367		
5th IC	0.9835	0.4528	0.1929		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG)	607.60	321.80	30.42	-3.46	210.69
\$ invest(Gauss)	0.5721	-0.0367	-0.0177	0.2337	-0.4280
cash flow(VG)	13.0422	-17.8119	-0.5102	0.0605	-9.0216
cash flow(Gauss)	0.0123	0.0020	0.0003	-0.0041	0.0183
% T-bill rate	1.21				

Statistical Estimation					01/03
	σ	ν	θ		
1st IC	0.9855	0.6179	-0.0635		
2nd IC	0.9666	0.5004	-0.0715		
3rd IC	0.9721	0.6072	0.0583		
4th IC	0.9501	0.3510	0.2031		
5th IC	0.9848	0.4659	0.1686		
ticker	mmm	ba	ibm	jnj	mcd
$ invest(VG)	-517.07	742.40	34.10	340.45	188.65
$ invest(Gauss)	0.5819	-0.0856	0.0010	0.2119	-0.4426
cash flow(VG)	-2.2185	-64.6024	0.0663	-4.3226	-22.3076
cash flow(Gauss)	0.0025	0.0075	0.0000	-0.0027	0.0523
% T-bill rate	1.19				

Statistical Estimation					02/03
	σ	ν	θ		
1st IC	0.9874	0.5903	-0.0472		
2nd IC	0.9664	0.4909	-0.0716		
3rd IC	0.9716	0.6370	0.0397		
4th IC	0.9495	0.3516	0.2165		
5th IC	0.9843	0.4647	0.1666		
ticker	mmm	ba	ibm	jnj	mcd
$ invest(VG)	-539.77	705.59	34.57	306.13	184.86
$ invest(Gauss)	0.6260	-0.1339	0.0171	0.1751	-0.5846
cash flow(VG)	-25.3797	-45.5739	1.0511	22.3248	14.9466
cash flow(Gauss)	0.0294	0.0087	0.0005	0.0128	-0.0473
% T-bill rate	1.19				

Statistical Estimation					03/03
	σ	ν	θ		
1st IC	0.9706	0.7382	-0.0195		
2nd IC	0.9525	0.4189	-0.1647		
3rd IC	0.9636	0.4300	-0.1503		
4th IC	0.9818	0.4732	0.1734		
5th IC	0.9837	0.5709	0.0761		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG)	237.89	-528.97	430.91	56.92	-32.06
\$ invest(Gauss)	0.7063	-0.1690	0.0028	0.1673	-0.5184
cash flow(VG)	-16.1689	-15.5000	11.8498	-0.5075	-3.5393
cash flow(Gauss)	-0.0480	-0.0050	0.0001	-0.0015	-0.0572
% T-bill rate	1.15				

Statistical Estimation					04/03
	σ	ν	θ		
1st IC	0.9539	0.3867	0.1627		
2nd IC	0.9677	0.3836	-0.0769		
3rd IC	0.9768	0.6610	-0.0326		
4th IC	0.9827	0.4938	-0.1229		
5th IC	0.9810	0.5992	0.1066		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG) $\times 10^3$	-0.3951	0.3503	1.0382	-1.0656	0.1893
\$ invest(Gauss)	0.5097	-0.2042	-0.0113	0.2215	-0.4213
cash flow(VG)	-14.2382	36.7383	48.2191	49.5464	23.2808
cash flow(Gauss)	0.0184	-0.0214	-0.0005	-0.0103	-0.0518
% T-bill rate	1.15				

Statistical Estimation					05/03
	σ	ν	θ		
1st IC	0.9539	0.4424	-0.1342		
2nd IC	0.9878	0.6447	-0.0315		
3rd IC	0.9707	0.3924	0.0687		
4th IC	0.9835	0.5243	-0.0996		
5th IC	0.9749	0.6156	0.0013		
ticker	mmm	ba	ibm	jnj	mcd
$ invest(VG)	985.97	7.80	-27.32	363.00	803.62
$ invest(Gauss)	0.5286	-0.1466	-0.0383	0.1599	-0.3073
cash flow(VG)	24.0423	1.1704	1.6452	-6.7737	201.3197
cash flow(Gauss)	0.0129	-0.0220	0.0023	-0.0030	-0.0770
% T-bill rate	1.09				

Statistical Estimation					06/03
	σ	ν	θ		
1st IC	0.9810	0.4371	0.0055		
2nd IC	0.9824	0.6555	0.0894		
3rd IC	0.9541	0.4125	0.1122		
4th IC	0.9857	0.5389	0.0576		
5th IC	0.9747	0.6219	0.0088		
ticker	mmm	ba	ibm	jnj	mcd
$ invest(VG) $\times 10^3$	0.8592	0.0172	2.2416	-0.6504	-0.7319
$ invest(Gauss)	0.5909	-0.0962	-0.1281	0.0839	-0.2173
cash flow(VG)	79.7754	-0.8794	26.9750	3.5455	26.9801
cash flow(Gauss)	0.0549	0.0049	-0.0015	-0.0005	0.0080
% T-bill rate	0.94				

Statistical Estimation					07/03
	σ	ν	θ		
1st IC	0.9510	0.4561	-0.1227		
2nd IC	0.9719	0.5607	0.1740		
3rd IC	0.9734	0.6629	-0.0028		
4th IC	0.9886	0.6924	0.0453		
5th IC	0.9883	0.4351	0.1696		
ticker	mmm	ba	ibm	jnj	mcd
$ invest(VG)	61.93	-722.25	-31.72	234.30	-411.05
$ invest(Gauss)	0.6654	-0.1688	-0.0971	0.0991	-0.2640
cash flow(VG)	1.3944	-61.3367	0.5678	-11.0640	-14.2862
cash flow(Gauss)	0.0150	-0.0143	0.0017	-0.0047	-0.0092
% T-bill rate	0.92				

Statistical Estimation					08/03
	σ	ν	θ		
1st IC	0.9510	0.4784	-0.1073		
2nd IC	0.9882	0.7066	-0.0482		
3rd IC	0.9707	0.5444	0.1742		
4th IC	0.9738	0.7287	0.0055		
5th IC	0.9753	0.3744	0.0594		
ticker	mmm	ba	ibm	jnj	mcd
$ invest(VG)	74.92	77.23	-679.40	-341.52	137.96
$ invest(Gauss)	0.6452	-0.0889	-0.1271	-0.0044	-0.2256
cash flow(VG)	-1.2155	-2.4571	-61.1269	1.1171	10.6172
cash flow(Gauss)	-0.0105	0.0028	-0.0114	0.0000	-0.0174
% T-bill rate	0.97				

Statistical Estimation					09/03
	σ	ν	θ		
1st IC	0.9692	0.4047	-0.0539		
2nd IC	0.9512	0.4599	-0.0936		
3rd IC	0.9864	0.6944	-0.0210		
4th IC	0.9823	0.5885	-0.0364		
5th IC	0.9828	0.6514	0.0383		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG) $\times 10^3$	0.9874	0.0080	-1.3523	1.3647	-0.1770
\$ invest(Gauss)	0.6064	-0.1061	-0.0634	0.0644	-0.2363
cash flow(VG)	74.0468	0.4006	19.9509	22.4189	-0.3832
cash flow(Gauss)	0.0455	-0.0053	0.0009	0.0011	-0.0005
% T-bill rate	0.96				

Statistical Estimation					10/03
	σ	ν	θ		
1st IC	0.9512	0.4766	0.0795		
2nd IC	0.9590	0.7626	0.0846		
3rd IC	0.9868	0.7383	0.0260		
4th IC	0.9670	0.5342	0.2011		
5th IC	0.9753	0.3685	0.0305		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG)	405.73	-186.73	49.16	782.03	-169.82
\$ invest(Gauss)	0.6748	-0.1109	-0.0030	-0.0433	-0.2117
cash flow(VG)	9.9123	-15.9050	0.3201	12.1588	-9.4962
cash flow(Gauss)	0.0165	-0.0094	-0.0000	-0.0007	-0.0118
% T-bill rate	0.94				

Statistical Estimation					11/03
	σ	ν	θ		
1st IC	0.9516	0.4794	-0.0768		
2nd IC	0.9870	0.7493	-0.0098		
3rd IC	1.0028	0.7810	-0.0770		
4th IC	0.9692	0.4011	0.1125		
5th IC	0.9818	0.6189	-0.0391		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG) $\times 10^3$	-0.7121	-0.1391	-0.3764	-0.3954	1.1365
\$ invest(Gauss)	0.6868	-0.0440	-0.0458	-0.0203	-0.2493
cash flow(VG)	-64.6814	-10.9019	-17.3443	2.8264	13.2021
cash flow(Gauss)	0.0624	-0.0034	-0.0021	0.0001	-0.0029
% T-bill rate	0.95				

Statistical Estimation					12/03
	σ	ν	θ		
1st IC	0.9511	0.5317	-0.0796		
2nd IC	0.9735	0.6597	-0.0947		
3rd IC	0.9768	0.9340	0.0230		
4th IC	0.9750	0.3956	0.0739		
5th IC	0.9824	0.6856	-0.0139		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG) $\times 10^3$	-1.1087	1.1391	0.3775	-0.5918	0.1091
\$ invest(Gauss)	0.6885	-0.0282	-0.0641	0.0368	-0.2189
cash flow(VG)	29.0052	12.8269	28.4598	-34.0580	-0.0834
cash flow(Gauss)	-0.0180	-0.0003	-0.0048	0.0021	0.0002
% T-bill rate	0.91				

Statistical Estimation					01/04
	σ	ν	θ		
1st IC	0.9485	0.5265	-0.0749		
2nd IC	0.9569	0.8490	0.0780		
3rd IC	0.9902	0.8307	-0.0302		
4th IC	0.9678	0.4248	-0.1047		
5th IC	0.9804	0.6440	-0.0074		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG) $\times 10^3$	0.6557	-1.1604	-0.0209	0.3671	0.9850
\$ invest(Gauss)	0.6982	-0.0713	-0.0590	0.1741	-0.1898
cash flow(VG)	-28.6543	-41.5892	0.6791	5.1949	102.9030
cash flow(Gauss)	-0.0305	-0.0026	0.0019	0.0025	-0.0198
% T-bill rate	0.90				

Statistical Estimation					02/04
	σ	ν	θ		
1st IC	1.0274	1.1564	0.0279		
2nd IC	0.9462	0.5264	0.0886		
3rd IC	0.9879	0.7720	0.0081		
4th IC	0.9698	0.4648	-0.0807		
5th IC	0.9833	0.5918	-0.0253		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG) $\times 10^3$	0.6728	-1.1624	-0.0796	-1.1576	-1.1178
\$ invest(Gauss)	0.6591	-0.0065	-0.0748	0.2922	-0.0116
cash flow(VG)	20.4522	109.6453	3.4834	83.5690	-19.5208
cash flow(Gauss)	0.0200	0.0006	0.0033	-0.0211	-0.0002
% T-bill rate	0.94				

Statistical Estimation					03/04
	σ	ν	θ		
1st IC	0.9412	0.4499	0.1591		
2nd IC	0.9950	1.0816	0.0857		
3rd IC	0.9872	0.7264	0.0266		
4th IC	0.9694	0.4799	0.0528		
5th IC	0.9817	0.5164	0.0041		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG) $\times 10^3$	1.6205	1.6682	0.2377	-0.0548	0.0346
\$ invest(Gauss)	0.7580	-0.0668	-0.1626	0.3039	-0.0740
cash flow(VG)	133.3004	146.3336	-5.2179	-3.9926	-1.4014
cash flow(Gauss)	0.0624	-0.0059	0.0036	0.0222	0.0030
% T-bill rate	0.95				

Statistical Estimation					04/04
	σ	ν	θ		
1st IC	0.9607	0.9400	0.0430		
2nd IC	0.9407	0.4342	-0.1197		
3rd IC	0.9651	0.5057	0.0358		
4th IC	0.9833	0.5265	-0.0014		
5th IC	0.9832	0.5698	-0.0703		
ticker	mmm	ba	ibm	jnj	mcd
\$ invest(VG) $\times 10^3$	-5.1623	0.4588	0.6105	-2.0190	-0.3986
\$ invest(Gauss)	0.9517	-0.0565	-0.1678	0.1678	-0.1729
cash flow(VG)	224.5537	17.3253	-10.9775	-63.8887	27.6037
cash flow(Gauss)	-0.0414	-0.0021	0.0030	0.0053	0.0120
% T-bill rate	0.96				

Appendix A

Proofs of some results

A.1 Proof of Lemma 4.6

Let X_1 and X_2 be two independent VG processes. Then

$$\begin{aligned}
&\Phi_{\ln S_1, \ln S_2}(u,v) \\
&= E[e^{(iu \ln S_1 + iv \ln S_2)}] \\
&= E[e^{(iu(m_1 t + B_1 X_1(t) + B_2 X_2(t) - \ln B(t)) + iv(m_2 t + C_1 X_1(t) + C_2 X_2(t) - \ln C(t)))}] \\
&= E[e^{(iu(m_1 t - \ln B(t)) + iv(m_2 t - \ln C(t)))} e^{i(uB_1 + vC_1)X_1(t)} e^{i(uB_2 + vC_2)X_2(t)}] \\
&= e^{(iu(m_1 t - \ln B(t)) + iv(m_2 t - \ln C(t)))} E[e^{i(uB_1 + vC_1)X_1(t)}] E[e^{i(uB_2 + vC_2)X_2(t)}] \\
&= e^{i(m_1 u + m_2 v)t} B(t)^{-iu} C(t)^{-iv} \phi_{X_1}(uB_1 + vC_1) \phi_{X_2}(uB_2 + vC_2),
\end{aligned}$$

where

$$\begin{aligned}
B(t) &= E[\exp(B_1 X_1(t) + B_2 X_2(t))] \\
&= E[\exp(B_1 X_1(t))] E[\exp(B_2 X_2(t))] \\
&= E[\exp(i(-iB_1) X_1(t))] E[\exp(i(-iB_2) X_2(t))] \\
&= (1 - \theta_1 \nu_1 B_1 - \frac{{\sigma_1}^2 \nu_1}{2} {B_1}^2)^{-\frac{t}{\nu_1}} (1 - \theta_2 \nu_2 B_2 - \frac{{\sigma_2}^2 \nu_2}{2} {B_2}^2)^{-\frac{t}{\nu_2}},
\end{aligned}$$

$$
\begin{aligned}
C(t) &= E[\exp(C_1X_1(t)+C_2X_2(t))] \\
&= E[\exp(C_1X_1(t))]E[\exp(C_2X_2(t))] \\
&= E[\exp(i(-iC_1)X_1(t))]E[\exp(i(-iC_2)X_2(t))] \\
&= (1-\theta_1\nu_1C_1-\frac{{\sigma_1}^2\nu_1}{2}{C_1}^2)^{-\frac{t}{\nu_1}}(1-\theta_2\nu_2C_2-\frac{{\sigma_2}^2\nu_2}{2}{C_2}^2)^{-\frac{t}{\nu_2}}, \\
\phi_{X_1}(uB_1+vC_1) &= [1-i\theta_1\nu_1(uB_1+vC_1)+\frac{{\sigma_1}^2\nu_1}{2}(uB_1+vC_1)^2]^{-\frac{t}{\nu_1}}, \\
\phi_{X_2}(uB_2+vC_2) &= [1-i\theta_2\nu_2(uB_2+vC_2)+\frac{{\sigma_2}^2\nu_2}{2}(uB_2+vC_2)^2]^{-\frac{t}{\nu_2}}.
\end{aligned}
$$

A.2 Proof of Proposition 4.7

Assume that

$$
\begin{aligned}
B(t) &= E[\exp(B_1X_1(t)+B_2X_2(t))] \\
C(t) &= E[\exp(C_1X_1(t)+C_2X_2(t))].
\end{aligned}
$$

According to the definition of characteristic function, we calculate $\phi_{\ln S_1}(u)$ and $\phi_{\ln S_2}(v)$ as follows:

$$
\begin{aligned}
&\phi_{\ln S_1}(u) \\
&= E[e^{iu\ln S_1}] \\
&= e^{(iu(m_1t-\ln B(t)))}E[e^{iuB_1X_1}]E[e^{iuB_2X_2}] \\
&= e^{ium_1t}\cdot[1-\theta_1\nu_1B_1-\frac{{\sigma_1}^2\nu_1}{2}{B_1}^2]^{\frac{iut}{\nu_1}}\cdot[1-\theta_2\nu_2B_2-\frac{{\sigma_2}^2\nu_2}{2}{B_2}^2]^{\frac{iut}{\nu_2}}\cdot \\
&[1-i\theta_1\nu_1(uB_1)+\frac{{\sigma_1}^2\nu_1}{2}(uB_1)^2]^{-\frac{t}{\nu_1}}\cdot \\
&[1-i\theta_2\nu_2(uB_2)+\frac{{\sigma_2}^2\nu_2}{2}(uB_2)^2]^{-\frac{t}{\nu_2}} \\
&= \Phi_{\ln S_1,\ln S_2}(u,0).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\phi_{\ln S_2}(v) \\
&= E[e^{iv \ln S_2}] \\
&= e^{(iv(m_2 t - \ln C(t)))} E[e^{ivC_1 X_1}] E[e^{ivC_2 X_2}] \\
&= e^{ivm_2 t} \cdot [1 - \theta_1 \nu_1 C_1 - \frac{{\sigma_1}^2 \nu_1}{2} {C_1}^2]^{\frac{ivt}{\nu_1}} \cdot [1 - \theta_2 \nu_2 C_2 - \frac{{\sigma_2}^2 \nu_2}{2} {C_2}^2]^{\frac{ivt}{\nu_2}} \cdot \\
&[1 - i\theta_1 \nu_1 (vC_1) + \frac{{\sigma_1}^2 \nu_1}{2} (vC_1)^2]^{-\frac{t}{\nu_1}} \cdot \\
&[1 - i\theta_2 \nu_2 (vC_2) + \frac{{\sigma_2}^2 \nu_2}{2} (vC_2)^2]^{-\frac{t}{\nu_2}} \\
&= \Phi_{\ln S_1, \ln S_2}(0, v).
\end{aligned}
$$

A.3 Proof of Proposition 4.11

$X_1(t)$ and $X_2(t)$ are two independent VG processes.

Let

$$
\begin{aligned}
B(t) &= E[\exp(B_1 X_1(t) + B_2 X_2(t))] \\
C(t) &= E[\exp(C_1 X_1(t) + C_2 X_2(t))].
\end{aligned}
$$

The definitions of the variance of $\ln S_1$ and the variance of $\ln S_2$ are

$$Var(\ln S_1) = E^2(\ln S_1) - (E(\ln S_1))^2$$

$$Var(\ln S_2) = E^2(\ln S_2) - (E(\ln S_2))^2.$$

We calculate the above equations term by term. We first calculate the term $E^2(\ln S_1)$, then we calculate the term $(E(\ln S_1))^2$, to get the variance of $\ln S_1$. The variance of $\ln S_2$ is obtained using the same method.

$$
\begin{aligned}
E^2(\ln S_1) &= E(m_1^2 t^2 + B_1^2 X_1^2(t) + B_2^2 X_2^2(t) + \ln^2 B(t) + \\
&\quad 2m_1 t B_1 X_1(t) + 2m_1 t B_2 X_2(t) - 2m_1 t \ln B(t) + \\
&\quad 2B_1 B_2 X_1(t) X_2(t) - 2B_1 X_1(t) \ln B(t) - 2B_2 X_2 \ln B(t)) \\
&= m_1^2 t^2 + B_1^2 E(X_1^2(t)) + B_2^2 E(X_2^2(t)) + \ln^2 B(t) + \\
&\quad 2m_1 t B_1 E(X_1(t)) + 2m_1 t B_2 E(X_2(t)) - \\
&\quad 2m_1 t \ln B(t) + 2B_1 B_2 E(X_1(t) X_2(t)) - \\
&\quad 2B_1 E(X_1(t)) \ln B(t) - 2B_2 E(X_2(t)) \ln B(t) \\
&= m_1^2 t^2 + B_1^2 (t^2 \theta_1^2 + t\theta_1^2 \nu_1 + t\sigma_1^2) + B_2^2 (t^2 \theta_2^2 + t\theta_2^2 \nu_2 + t\sigma_2^2) + \\
&\quad \ln^2 B(t) + 2m_1 t B_1 \theta_1 t + 2m_1 t B_2 \theta_2 t - 2m_1 t \ln B(t) + \\
&\quad 2B_1 B_2 \theta_1 \theta_2 t^2 - 2B_1 \theta_1 t \ln B(t) - 2B_2 \theta_2 t \ln B(t) \qquad \text{(A.1)} \\
(E(\ln S_1))^2 &= (m_1 t + B_1 \theta_1 t + B_2 \theta_2 t - \ln B(t))^2 \\
&= m_1^2 t^2 + B_1^2 \theta_1^2 t^2 + B_2^2 \theta_2^2 t^2 + \ln^2 B(t) + \\
&\quad 2m_1 B_1 \theta_1 t^2 + m_1 B_2 \theta_2 t^2 - 2m_1 t \ln B(t) + \\
&\quad 2B_1 B_2 \theta_1 \theta_2 t^2 - 2B_1 \theta_1 t \ln B(t) - 2B_2 \theta_2 t \ln B(t). \qquad \text{(A.2)}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
Var(\ln S_1) &= E^2(\ln S_1) - (E(\ln S_1))^2 \\
&= B_1^2 t(\theta_1^2 \nu_1 + \sigma_1^2) + B_2^2 t(\theta_2^2 \nu_2 + \sigma_2^2),
\end{aligned}
$$

similar,

$$
\begin{aligned}
Var(\ln S_2) &= E^2(\ln S_2) - (E(\ln S_2))^2 \\
&= C_1^2 t(\theta_1^2 \nu_1 + \sigma_1^2) + C_2^2 t(\theta_2^2 \nu_2 + \sigma_2^2).
\end{aligned}
$$

A.4 Proof of Proposition 4.12

Denote

$$\begin{aligned}
B(t) &= E[\exp(B_1X_1(t) + B_2X_2(t))] \\
C(t) &= E[\exp(C_1X_1(t) + C_2X_2(t))].
\end{aligned}$$

Note that

$$Cov(\ln S_1, \ln S_2) = E(\ln S_1 \ln S_2) - E(\ln S_1)E(\ln S_2).$$

We calculate the above equation term by term. We first calculate $E(\ln S_1 \ln S_2)$, then we calculate $E(\ln S_1)E(\ln S_2)$.

$$\begin{aligned}
E(\ln S_1 \ln S_2) &= E(m_1m_2t^2 + m_2tB_1X_1(t) + m_2tB_2X_2(t) + \\
&\quad m_1tC_1X_1(t) + m_1tC_2X_2(t) - \\
&\quad m_2t\ln B(t) - m_1t\ln C(t) + \\
&\quad B_1C_1X_1^2(t) + B_2C_2X_2^2(t) + \\
&\quad B_1C_2X_1(t)X_2(t) + B_2C_1X_1(t)X_2(t) - \\
&\quad B_1X_1(t)\ln C(t) - B_2X_2(t)\ln C(t) - \\
&\quad C_1X_1(t)\ln B(t) - C_2X_2(t)\ln B(t) + \ln B(t)\ln C(t)) \\
&= m_1m_2t^2 + m_2tB_1\theta_1t + m_2tB_2\theta_2t - m_2t\ln B(t) + \\
&\quad m_1tC_1\theta_1t + m_1tC_2\theta_2t - m_1t\ln C(t) + \\
&\quad B_1C_1(t^2\theta_1^2 + t\theta_1^2\nu_1 + t\sigma_1^2) + B_2C_2(t^2\theta_2^2 + t\theta_2^2\nu_2 + t\sigma_2^2) + \\
&\quad B_1C_2(\theta_1\theta_2t^2) + B_2C_1(\theta_1\theta_2t^2) - \\
&\quad B_1\theta_1t\ln C(t) - B_2\theta_2t\ln C(t) - C_1\theta_1t\ln B(t) - \\
&\quad C_2\theta_2t\ln B(t) + \ln B(t)\ln C(t), \qquad \text{(A.3)}
\end{aligned}$$

$$
\begin{aligned}
E(\ln S_1)E(\ln S_2) &= (m_1 t + B_1\theta_1 t + B_2\theta_2 t - \ln B(t)) \cdot \\
&\quad (m_2 t + C_1\theta_1 t + C_2\theta_2 t - \ln C(t)) \\
&= m_1 m_2 t^2 + m_2 B_1\theta_1 t^2 + m_2 B_2\theta_2 t^2 - m_2 t\ln B(t) + \\
&\quad m_1 C_1\theta_1 t^2 + m_1 C_2\theta_2 t^2 - m_1 t\ln C(t) + \\
&\quad B_1 C_1\theta_1^2 t^2 + B_2 C_2\theta_2^2 t^2 + B_1 C_2\theta_1\theta_2 t^2 + B_2 C_1\theta_1\theta_2 t^2 - \\
&\quad B_1\theta_1 t\ln C(t) - B_2\theta_2 t\ln C(t) - C_1\theta_1 t\ln B(t) - \\
&\quad C_2\theta_2 t\ln B(t) + \ln B(t)\ln C(t).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
Cov(\ln S_1, \ln S_2) &= E(\ln S_1 \ln S_2) - E(\ln S_1)E(\ln S_2) \\
&= B_1 C_1 t(\theta_1^2\nu_1 + \sigma_1^2) + B_2 C_2 t(\theta_2^2\nu_2 + \sigma_2^2).
\end{aligned}
$$

Appendix B

Summary of the ICs

B.1 Summary of the ICs for the ten stock portfolio

Statistical Estimation				1st IC
	σ	ν	θ	
mean	0.9536	0.9589	0.0267	
standard deviation	0.0918	1.0040	0.1457	
minimum	0.6919	0.1817	-0.3180	
maximum	1.2941	4.3327	0.4709	

Statistical Estimation				2nd IC
	σ	ν	θ	
mean	0.9441	0.5801	0.0072	
standard deviation	0.0313	0.2575	0.1397	
minimum	0.8202	0.1573	-0.3278	
maximum	1.0832	2.2201	0.2927	

Statistical Estimation			3rd IC
	σ	ν	θ
mean	0.9551	0.5397	-0.0129
standard deviation	0.0253	0.1792	0.1487
minimum	0.8687	0.1351	-0.4299
maximum	1.0193	1.4800	0.4654

Statistical Estimation			4th IC
	σ	ν	θ
mean	0.9608	0.5349	0.0005
standard deviation	0.0234	0.1464	0.1464
minimum	0.8782	0.1511	-0.4262
maximum	0.9936	0.9730	0.3313

Statistical Estimation			5th IC
	σ	ν	θ
mean	0.9744	0.5017	0.0008
standard deviation	0.0176	0.1662	0.1625
minimum	0.8916	0.1692	-0.4142
maximum	0.9957	1.2870	0.4178

Statistical Estimation			6th IC
	σ	ν	θ
mean	0.9795	0.4449	-0.0015
standard deviation	0.0148	0.1320	0.1771
minimum	0.9323	0.1040	-0.5119
maximum	0.9981	0.7839	0.4302

Statistical Estimation			7th IC
	σ	ν	θ
mean	0.9847	0.4644	-0.0114
standard deviation	0.0121	0.1479	0.1726
minimum	0.9431	0.2356	-0.4819
maximum	1.0031	0.9420	0.4377

Statistical Estimation			8th IC
	σ	ν	θ
mean	0.9893	0.3833	0.0035
standard deviation	0.0086	0.1180	0.1905
minimum	0.9587	0.1882	-0.4882
maximum	1.0042	0.7215	0.5514

Statistical Estimation			9th IC
	σ	ν	θ
mean	0.9926	0.3769	0.0211
standard deviation	0.0077	0.1207	0.1755
minimum	0.9594	0.1171	-0.3997
maximum	1.0066	0.7199	0.5339

Statistical Estimation			10th IC
	σ	ν	θ
mean	0.9940	0.3317	0.0190
standard deviation	0.0061	0.1007	0.2046
minimum	0.9737	0.1525	-0.4104
maximum	1.0041	0.6409	0.4736

B.2 Summary of the ICs for the twenty stock portfolio

Statistical Estimation				1st IC
	σ	ν	θ	
mean	0.8529	0.5696	0.0186	
standard deviation	0.0985	0.4999	0.1710	
minimum	0.4577	0.1667	-0.4093	
maximum	1.1271	4.3020	0.3856	

Statistical Estimation				2nd IC
	σ	ν	θ	
mean	0.9112	0.5790	0.0360	
standard deviation	0.0584	0.4863	0.1627	
minimum	0.7239	0.1530	-0.3214	
maximum	1.1879	3.6313	0.6654	

Statistical Estimation				3rd IC
	σ	ν	θ	
mean	0.9251	0.5911	-0.0086	
standard deviation	0.0417	0.4828	0.1745	
minimum	0.7941	0.1280	-0.3846	
maximum	1.1488	3.5823	0.7255	

Statistical Estimation			4th IC
	σ	ν	θ
mean	0.9318	0.4954	-0.0021
standard deviation	0.0356	0.2043	0.1440
minimum	0.8000	0.1283	-0.3116
maximum	0.9911	1.2827	0.3706

Statistical Estimation			5th IC
	σ	ν	θ
mean	0.9465	0.5207	0.0063
standard deviation	0.0539	0.2798	0.1506
minimum	0.8385	0.1372	-0.3860
maximum	1.4760	2.0024	0.3770

Statistical Estimation			6th IC
	σ	ν	θ
mean	0.9484	0.5182	0.0020
standard deviation	0.0249	0.3273	0.1380
minimum	0.8387	0.1174	-0.3228
maximum	1.0315	2.7857	0.3476

Statistical Estimation			7th IC
	σ	ν	θ
mean	0.9561	0.5112	-0.0068
standard deviation	0.0378	0.3829	0.1446
minimum	0.8730	0.1796	-0.5975
maximum	1.2522	3.4784	0.2912

Statistical Estimation			8th IC
	σ	ν	θ
mean	0.9562	0.4869	0.0224
standard deviation	0.0487	0.2601	0.1712
minimum	0.8625	0.1555	-0.5710
maximum	1.4399	2.6289	0.5990

Statistical Estimation			9th IC
	σ	ν	θ
mean	0.9637	0.5150	-0.0036
standard deviation	0.0317	0.3257	0.1607
minimum	0.8797	0.1614	-0.4516
maximum	1.2173	3.2210	0.3733

Statistical Estimation			10th IC
	σ	ν	θ
mean	0.9682	0.4662	0.0013
standard deviation	0.0163	0.1504	0.1329
minimum	0.9151	0.1269	-0.3277
maximum	0.9952	0.8334	0.3439

Statistical Estimation			11th IC
	σ	ν	θ
mean	0.9735	0.4480	-0.0190
standard deviation	0.0167	0.1748	0.1834
minimum	0.9166	0.1483	-0.6892
maximum	0.9975	1.3679	0.4830

Statistical Estimation			12th IC
	σ	ν	θ
mean	0.9749	0.4400	-0.0124
standard deviation	0.0163	0.1619	0.1830
minimum	0.9290	0.0623	-0.5340
maximum	0.9975	0.9178	0.5830

Statistical Estimation			13th IC
	σ	ν	θ
mean	0.9833	0.4325	-0.0183
standard deviation	0.0123	0.1334	0.1607
minimum	0.9446	0.1017	-0.4703
maximum	0.9982	0.7997	0.4769

Statistical Estimation			14th IC
	σ	ν	θ
mean	0.9859	0.4193	0.0068
standard deviation	0.0136	0.1254	0.1530
minimum	0.9303	0.1254	-0.6305
maximum	1.0010	0.7448	0.4425

Statistical Estimation			15th IC
	σ	ν	θ
mean	0.9878	0.3901	0.0103
standard deviation	0.0111	0.1197	0.1921
minimum	0.9423	0.1291	-0.6446
maximum	1.0040	0.7392	0.6230

Statistical Estimation			16th IC
	σ	ν	θ
mean	0.9911	0.3625	0.0022
standard deviation	0.0090	0.1188	0.1946
minimum	0.9382	0.1313	-0.8125
maximum	1.0026	0.8130	0.4107

Statistical Estimation			17th IC
	σ	ν	θ
mean	0.9922	0.3384	-0.0043
standard deviation	0.0081	0.1062	0.1921
minimum	0.9492	0.1550	-0.6747
maximum	1.0039	0.6938	0.4749

Statistical Estimation			18th IC
	σ	ν	θ
mean	0.9942	0.2922	-0.0101
standard deviation	0.0075	0.0876	0.1754
minimum	0.9559	0.0977	-0.5490
maximum	1.0039	0.6450	0.4873

Statistical Estimation			19th IC
	σ	ν	θ
mean	0.9961	0.2406	-0.0128
standard deviation	0.0055	0.0984	0.1970
minimum	0.9520	0.0382	-0.4711
maximum	1.0039	0.5366	1.0925

Statistical Estimation			20th IC
	σ	ν	θ
mean	0.9959	0.1585	0.0240
standard deviation	0.0082	0.0837	0.3192
minimum	0.9412	0.0000	-0.7904
maximum	1.0025	0.5296	1.9422

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