ABSTRACT

Title of dissertation: THE ADELIC DIFFERENTIAL GRADED ALGEBRA FOR SURFACES

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For any variety X/k, we consider the Beilinson–Huber adeles A_X as a differential graded k-algebra and examine the category $\underline{\mathrm{Mod}}_A^{dg}$ of differential graded A-modules. We characterize the modules associated to certain quasi-coherent sheaves and define an adelic Chern class $c(M)$ for modules which are graded free of rank 1. We study the intersection pairing in terms of a cup product and prove a version of the Bloch–Quillen formula that respects this cup product.

Fesenko [6] proved Serre duality and the Riemann–Roch theorem for surfaces using a topological duality on the adeles. On the other hand, Mattuck–Tate [18] and Grothendieck [12] provided proofs of the Riemann hypothesis for curves using the Riemann–Roch theorem for surfaces by studying the graph of the Frobenius morphism on the surface $S = C \times C$. Therefore the combined results of Fesenko and Mattuck–Tate–Grothendieck can be said to provide an adelic proof of the Riemann hypothesis for a curve C over a finite field. We apply the results of this thesis to the adelic intersection pairing, and state a version of the Hodge index theorem which implies the Riemann hypothesis for curves.

THE ADELIC DIFFERENTIAL GRADED ALGEBRA FOR SURFACES

by

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Dedication

For Charlie.

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Chapter 1: Introduction

1.1 Background and motivation

The adeles of function fields and the finite adeles of number fields provide a fundamental example of a local-to-global construction. For these two types of one-dimensional schemes, the adeles provide a common framework. The adeles of a surface were given by Parshin [21] and generalized to arbitrary noetherian schemes by Beilinson [1]. The excellent paper by Huber [14] provides a complete account of the construction, as follows: For an arbitrary quasi-coherent sheaf $\mathcal{F}$ on an arbitrary noetherian scheme X, Huber constructs an adele functor $A_X(\mathcal{F})$ into the category of cosimplicial groups. Rather than taking a restricted direct product over points (as in the classical construction), the product is over all tuples Δ of scheme points ordered by specialization. Such a tuple is called a **Parshin flag**, and the collection of Parshin flags forms a simplicial set (Definition 2.2.1). We recover the classical adeles as $A_X = A^1_X(\mathcal{O}_X)$ when $\dim X = 1$. The main theorem for adeles states:

Theorem 2.2.13 ([14, Theorem 4.2.3 and §5.2])**.** *Considering the cosimplicial group $A_X(\mathcal{F})$ as a chain complex via the Moore functor, we have an isomorphism*

$$H^n(A_X(\mathcal{F})) \simeq H^n(X, \mathcal{F})$$

for any noetherian scheme X, quasi-coherent sheaf $\mathcal{F}$, and n.

Fesenko [6] gave a relatively elementary proof of Serre duality and the Riemann–Roch theorem for surfaces using a topological duality on the higher adeles. On the other hand, Mattuck–Tate [18] and Grothendieck [12] provided proofs of the Riemann hypothesis for curves as corollaries to the Riemann–Roch theorem for surfaces, with the latter utilizing the forgotten[1] Hodge index theorem. Both methods come down to studying the graph of the Frobenius morphism on the surface $S = C \times C$. Therefore, the combined results of Fesenko and Mattuck–Tate–Grothendieck can be said to provide an adelic proof of the Riemann hypothesis for a curve C over a finite field.

At the end of [6], Fesenko poses a number of questions for further research:

> "Study functorial properties of the adelic complex with respect to morphisms of surfaces and their applications. Extend the argument in this paper to the case of a quasi-coherent sheaf $\mathcal{F}$ on [a surface] S and the associated adelic complex $\mathcal{A}_S(\mathcal{F})$... Find an adelic proof of the Noether formula and the Hodge index theorem." [6, p. 451]

The motivation for this thesis is to understand the adelic proof of the Riemann hypothesis alluded to above, while following the program laid out by Fesenko. Although our results are in the same spirit, we do not provide the proofs implied by the

[1] *In trying to comprehend the scope of [Mattuck–Tate's] method, I stumbled upon the following statement, a fact known since 1937... (as shown to me by J. P. Serre), but apparently not very well-known or used. [12, Translated]*

context of [6] (for example, our proof of the Hodge index theorem is Grothendieck's, and not the purely topological one intended in the question). Future work will be devoted to extending the results of this thesis to the more topological setting that the above question is posed in.

The main results of this thesis can be summarized as follows:

1. We study the category of differential graded modules over the adelic algebra and characterize certain quasi-coherent sheaves. Using this, we are able to provide a novel construction of Chern classes (Chapter 3).

2. We prove that the tensor product of modules corresponding to line bundles compute their proper intersection number via their length. For self-intersection, we construct a sort of projective resolution which does the same thing (Chapter 5).

3. We define the simplicial Milnor K-algebra, and prove that the Bloch–Quillen formula for surfaces is actually a ring isomorphism (Chapter 6).

1.2 The adelic algebra

Let X/k be a variety over an algebraically closed field. Our main object of interest, in slight contrast to earlier works, will be what we call the **adelic algebra**. This is the cosimplicial algebra A_X defined, in the notation of Huber [14], as

$$A_X^p = a(S_p^{(\mathrm{red})}(X), \mathcal{O}_X).$$

Here, $S_p^{(\mathrm{red})}(X)$ denotes the simplicial set of all *non-degenerate* Parshin flags. If $\dim X = n$, the top dimensional ring A_X^n is generally called the **ring of rational adeles** of X. We define the **degenerate adelic algebra** $A_{\deg,X}$ by $A^p_{\deg,X} = a(S_n(X), \mathcal{O}_X)$, where $S_p(X)$ denotes the set of *all* Parshin flags.

The so-called "monoidal" Dold–Kan correspondence [23] is the pair of functors

$$\{\text{cosimplicial } k\text{-algebras}\} \underset{\mathcal{K}}{\overset{\mathcal{C}}{\rightleftarrows}} \{\text{differential graded } k\text{-algebras}\}.$$

The Alexander–Whitney map, i.e. the cup product, is what facilitates between the products on both sides, and we simultaneously think of A_X as a **differential graded algebra**.

From now on fix a surface X/k and let $A = A_X$. We begin by studying the differential graded category $\underline{\mathrm{Mod}}_A^{dg}$ of right differential graded modules over A. We show,

Proposition 3.2.5. *Let A be the adelic algebra of a variety and $A(-)$ the associated adelic functor on quasi-coherent sheaves (Definition 2.2.8 and subsequent remark).*

1. *Let $L(D)$ be the line bundle corresponding to a Cartier divisor D. Since D is* locally principal, *$A(D) = A(L(D))$ is a* principal *differential graded A-module, in an appropriate sense of the word.*

2. *We have isomorphisms of differential graded A-modules $A(D) \otimes_A A(E) \simeq A(D+E)$, and $A(D) \simeq A(E)$ if and only if $D \sim_{\mathrm{rat}} E$.*

3. *Further, if we move to the category of differential graded (A, A)-bimodules we can define the internal hom object $\mathcal{H}om_A(M, N)$ and the dual $M^\vee =$*

$\mathcal{H}om_A(M, A)$ *(the objects $A(\mathcal{F})$ are (A, A)-bimodules). Then $A(D)^\vee \simeq A(-D)$.*

1.3 Intersection theory

Return to the case of a fixed surface X/k. Since objects in the differential graded category $\underline{\mathrm{Mod}}_A^{dg}$ are already chain complexes, we have a way of doing a simplified form of derived algebraic geometry. Let C, D be effective Cartier divisors intersecting properly, and I_C, I_D their associated ideals in A (here, ideal means a two-sided differential graded A-submodule). Then (§5.1)

$$C.D = \mathrm{length}_A\, A/I_C \otimes_A^{dg} A/I_D.$$

Fix a curve C, possibly singular. For divisors D that do not intersect C properly, $- \otimes_A^{dg} A/I_C$ does not "represent" the derived tensor $- \otimes^L_{\mathcal{O}_X} \mathcal{O}_C$. We can however construct an object $P_t \in \underline{\mathrm{Mod}}_A^{dg}$ as follows. We set $P_t = A[T]/T^2$ with $dT = t - T(t^{-1}dt)$ for a choice of generator t of $I_C = tA$. Then P_t satisfies property (P) in the sense of [25, Tag 09KK] and is an extension

$$0 \to A \to P_t \to A[1] \to 0$$

of differential graded A-modules, where $M[1]$ denotes the shifted module. The module P_t acts like a resolution of the A-module A_C. The choice of t also corresponds to a choice of divisor $E \sim C$ intersecting C properly. We have (§5.2)

Theorem 5.2.4. *There is an isomorphism of differential graded A-modules*

$$P_t \otimes_A^{dg} A/I_C \simeq A_C \oplus A_C(E|_C)[1]$$

By the Riemann–Roch theorem for the curve C, the Euler characteristic of the right side is C^2. Since the isomorphism is a morphism of differential graded modules (in particular a chain map), the Euler characteristic of the module on the left computes the self-intersection number of C. We conclude that a similar statement is true for $P_t \otimes_A^{dg} A/I_D$ for arbitrary divisors D.

1.4 K-groups of adeles

Let X/k be a curve or surface (possibly singular) with associated adelic algebra A. Our goal is to approach the K-theory of X via A, but only insofar as is needed to understand the Riemann–Roch theorem for surfaces.

Gorchinskii [11, 10] constructs adeles of K-theoretic sheaves by mimicking the Huber–Beilinson construction. Braunling [2] does this similarly for cycle modules. These constructions have the benefit that they are flasque resolutions of the original K-theoretic sheaf, by construction.

However, one must go to greater lengths to demonstrate a cup product. In fact, Braunling [2, Example 27] gives an example due to Gorchinskii of why the adeles of K-groups do not have a product on the level of cochains, without further refinement.

We instead use a simplified approach following Budylin [3]. For any covariant functor $K : \underline{\text{Ring}} \to \underline{\text{Ring}}$, the cosimplicial algebra $K(A)$ has a natural cup product structure, given by the Alexander–Whitney map. While we gain a product structure, *we cannot write $K(A)$ as a flasque resolution of sheaves.* However, for the purpose of understanding intersection theory on surfaces, particularly aimed at framing Riemann–Roch, we find this simplified construction preferable.

In fact, we do still have a Bloch–Quillen formula for a smooth surface X/k and associated adelic algebra A, proved by Budylin:

Theorem 1.4.1 ([3, Theorem 1]). *For X/k a smooth surface*

$$H^2(K_2^M(A)) \simeq CH^2(X).$$

Unlike Gorshinskii or Braunling's adeles of K-groups, this isomorphism must be built in a more ad hoc manner.

We refine this theorem by providing an interpretation of $K^M(A)$ and $H^\bullet(K^M(A))$ as differential graded rings.

Proposition 3.3.7. *Let A be any differential graded algebra which comes from a cosimplicial algebra. Let M be a differential graded (A, A)-bimodule. Suppose M is free and rank 1 as a graded left and right A-module. There is a natural construction of a Chern class $c(M) \in H^1(K_1^M(A))$.*

On the other hand, given a Cartier divisor D we may assign an adelic Cartier divisor $t_D \in A_{01}^\times \oplus A_{02}^\times \oplus A_{12}^\times$. Then these two notions agree:

$$c(A(D)) = \{\textit{class of } t_D \textit{ in } H^1(K_1^M(A))\}.$$

The cosimplicial group $K^M(A)$ defined by $(K^M(A))^n = K^M(A^n)$ is a differential graded $\mathbb{Z}$-algebra under the Alexander–Whitney map. If t_C and t_D are two adelic Cartier divisors, then their cup product $t_C \smile t_D$ is a well-defined element of $K^M(A^2)$. We show that under the canonical isomorphism given by the Bloch–Quillen formula above,

1. t_D maps to the class of D, and,

2. $t_C \smile t_D$ maps to the class of $C.D$.

Therefore,

Theorem 6.2.11. *Let X/k be a smooth surface. The map*

$$H^\bullet(K^M(A)) \xrightarrow{\phi} CH^\bullet(X)$$

is an isomorphism of rings.

1.5 Hodge index and Riemann hypothesis for curves

Finally, we apply the results of the thesis to the Hodge index theorem and Riemann hypothesis for curves. Our presentation follows [12], although we cannot add too much more other than frame the existing proofs in the language of differential graded A-modules.

Let $C/\mathbb{F}_q$ be a smooth projective curve of genus g. Of the two strategies Weil utilized in proving the Riemann hypothesis for C, the more geometric is to consider the diagonal embedding of C in its product $X = C \times C$ [27]. This is the graph Δ of the identity map, which has transverse intersection with the graph Γ of the (purely inseparable) Frobenius morphism. Therefore, this intersection number is exactly $\Delta.\Gamma = \# C(\mathbb{F}_q)$. The Riemann hypothesis in this case is easily seen to be equivalent to the inequality

$$|1 + q - \# C(\mathbb{F}_q)| \leq 2g\sqrt{q}. \tag{1.1}$$

We will derive (1.1) from a version of the Hodge index theorem. As we do not, at this time, have an adelic definition to replace ampleness, we satisfy ourselves with an ad hoc class of divisors. Call a divisor D **simplicially effective** if it is effective, and its self-intersection divisor is effective. A very ample divisor is clearly simplicially effective. We show, for the adelic intersection pairing,

Theorem 3.2 (baby Hodge index)**.** *Let D, E be divisors, with E simplicially effective. If $(D, E)_{\text{adelic}} = 0$, then $(D, D)_{\text{adelic}} \leq 0$.*

Applying this version of the Hodge index theorem to the case $X = C \times C$ with appropriately chosen divisors gives (1.1), as outlined in §3.

1.6 Notation

Rings and algebras are associative and have an identity. If A is an algebra, an (A, A)-bimodule is just called an A-bimodule. k is an algebraically closed field, although most results hold with appropriate modifications for an arbitrary perfect field.

If X is a variety, then $k(X)$ is the function field of X, and $\mathcal{K} = \mathcal{K}_X$ is the corresponding constant sheaf. We denote the category of quasi-coherent sheaves on X by $\underline{\text{Qco}}(X)$. We use $D \sim_{\text{rat}} E$ to denote rational equivalence of divisors.

If X is a noetherian scheme, we let A_X denote the *normalized*, *rational* differential graded algebra (Definition 2.2.8). We let $\hat{A}_X$ denote the *complete* differential graded algebra (Definition 2.2.18). When the context is clear we suppress the X and write A and $\hat{A}$. [§2.2] Δ denotes an arbitrary Parshin flag on a noetherian scheme, possibly degenerate. $S_\bullet(X)$ is the simplicial set of all Parshin flags. $S_I(X)$ is the simplicial set of all flags of type I. $A^\bullet(K, \mathcal{F})$ is the functor defined in Proposition 2.2.7 for a simplicial set $K \subset S_\bullet(X)$. We denote $A(\mathcal{F})$ for either the rational or complete (we will specify in the context, but often it doesn't matter) *reduced* adele functor from quasicoherent sheaves on X to A-modules. Using the notation of Huber [14], as a chain complex

$$A(\mathcal{F})^\bullet = a_{(\mathrm{red})}(X, \mathcal{F})^\bullet,$$

while for the complete adeles,

$$\hat{A}(\mathcal{F})^\bullet = \mathbb{A}_{(\mathrm{red})}(X, \mathcal{F})^\bullet.$$

Chapter 2: Background

2.1 Introduction

In this chapter, we lay out our main objects of study. First are the Beilinson–Huber adele functors $A_X(\mathcal{F})$, described in §2.2. The main result from their construction is Theorem 2.2.14, which states that for any noetherian scheme X and quasi-coherent sheaf $\mathcal{F}$,

$$H^n(A_X(\mathcal{F})) = H^n(X, \mathcal{F}).$$

In §2.3 we lay out basic facts about differential graded algebras and their modules. When applied to cosimplicial algebras, the Dold–Kan functor preserves their structure through the Alexander–Whitney map (i.e. cup product) and produces differential graded algebras and modules, as outlined in §2.4. We will study Beilinson–Huber adeles as differential graded algebras, together with their corresponding differential graded modules, in the next chapter.

2.1.1 Intuition for the flag simplicial structure of sheaves

To try and motivate the use of the simplicial set of Parshin flags, let us consider the most general case of arbitrary sheaves of abelian groups. Let X be a noetherian locally ringed space. A presheaf is a functor $\underline{\text{Top}}(X) \to \underline{\text{Ab}}$, where $\underline{\text{Top}}(X)$ is the category of open sets on X. On the other hand, much of the information of a sheaf is contained in its stalks, and we want to focus on how far the data at the stalks are from describing the sheaf. We can think of a sheaf as defined by sections of stalks at points, but there is extra gluing data. For example, the stalk functors $\underline{\text{Ab}}(X) \to \underline{\text{Ab}}$ given by $\mathcal{F} \mapsto \mathcal{F}_x$ assemble into a functor $\mathcal{F} \mapsto \prod_x \mathcal{F}_x$. However, we cannot construct an adjoint in the obvious way, as there is no way to reassemble the stalks (this is the gluing data). Still, *some* of the local information is captured in the following statement: A morphism of sheaves $\phi : \mathcal{F} \to \mathcal{G}$ is an isomorphism if and only if $\prod_x \phi_x$ is.

Now consider the category $\underline{\text{Ab}}^0(X)$ as follows. Objects are groups of the form $A = \prod_{x \in X} A_x$, for abelian groups A_x. Morphisms $A \to B$ are products of homomorphisms $A_x \to B_x$. Another way to think of such an object is as a functor $\underline{|X|} \to \underline{\text{Ab}}$, where $\underline{|X|}$ denotes the discrete category of points of X (objects are $x \in X$ and $\text{Hom}(x, y) = \varnothing$ if $x \neq y$). Again, we have a functor $F : \underline{\text{Ab}}(X) \to \underline{\text{Ab}}^0(X)$, but this time we can construct a right adjoint. Given $A = \prod_x A_x \in \underline{\text{Ab}}^0(X)$ we can define a presheaf GA as follows:[1]

$$GA(U) = \prod_{x \in U} A_x$$

with restriction maps the projections: if $V \subset U$, then $\prod_{x \in U} A_x \twoheadrightarrow \prod_{x \in V} A_x$.

Assertion 2.1.1. *This is in fact a sheaf, and it is flasque.*

Proof. First notice if $s \in GA(U)$, then $(s|_V)_x = s_x$ for all $x \in V$.

[1]This construction is the standard Godement resolution

(separated) If $V = \bigcup_i V_i$, let $s \in GA(V)$. Then $s|_{V_i} \in \prod_{x \in V_i} A_x$. If $s|_{V_i} = 0$, then $(s|_{V_i})_x = 0$ for all $x \in V_i$. But then $s_x = (s|_{V_i})_x = 0$, so since the V_i cover V, $s_x = 0$ for all $x \in \bigcup_i V_i = V$. Thus $s = 0$, so the presheaf is separated.

(sheaf axiom) Let $V = \bigcup_i V_i$, $\{s_i\} \in GA(V_i)$, with $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$. For each $x \in V$, choose $V_i \ni x$, and set $s_x = (s_i)_x$. If $V_j \ni x$, then $(s_j)_x = (s_j|_{V_i \cap V_j})_x = (s_i|_{V_i \cap V_j})_x = (s_i)_x$, so this is well defined. So $s = \prod_x s_x \in A$ is the required element, and the presheaf is a sheaf. □

Assertion 2.1.2. *We have constructed a right adjoint to F:* $\mathrm{Hom}_{\underline{\mathrm{Ab}}^0(X)}(FA, B) = \mathrm{Hom}_{\underline{\mathrm{Ab}}(X)}(A, GB)$.

Proof. ($\phi \mapsto F\phi$ injects) Let $A \xrightarrow{\phi} GB$ be a morphism of sheaves. For every $x \in X$, we get a map of stalks $A_x \xrightarrow{\phi_x} (GB)_x$. By construction, $(GB)_x = B_x$. Thus we get maps $A_x \xrightarrow{\phi_x} B_x$, which define maps $F\phi : FA \to B$. If $F\phi = 0$, then $\phi_x = 0$ for all x, so $\phi = 0$ since a map on sheaves is trivial if it is trivial on all stalks.

($\phi \mapsto F\phi$ surjects) Let $\psi : \prod_x A_x \to \prod_x B_x$, which is a collection of $\psi_x : A_x \to B_x$ for all x. We must define a sheaf homomorphism from A to GB. Take any $U \neq \varnothing$. For $s \in A(U)$, $s_x = (s, U) \in A_x$ for each $x \in U$. Thus we may define $\phi(s) = \prod_x \psi_x(s_x) \in \prod_{x \in U} B_x = GB(U)$, by the universal property of the product. This is clearly a sheaf homomorphism. □

Remark 2.1.3. It should be clear that F doesn't have a left adjoint in general.

The construction GFA for a sheaf A is nothing more than the first step in the Godement resolution [13, III.2.2]. However, whereas the Godement resolution procedes to the right by taking cokernels, we will diverge from the classical construction. We consider GFA to be the first step in a different cosimplicial resolution that takes into account the simplicial nature of Parshin flags on a scheme. It is important to point out that GFA is a member of a **triple** in the language of [26, 8.6.1], part of the more general construction of the canonical resolution [8.6.6, loc. cit.]. One obtains a triple (respectively, cotriple) from any adjoint pair and can define their canonical resolution from the associated cosimplicial (respectively, simplicial) sets. N.b. that we will *not* be using this resolution, as it is too general. The resulting simplicial set would be associated with the discrete space $|X|$ (in the form of the discrete category $\underline{|X|}$, above). As mentioned in the introduction of [11, p1190], by choosing the correct simplicial set one generally gains a key *product structure* (the adelic differential graded algebra §3.1) that does not exist for something as general as the Godement resolution.

2.1.2 Enriching the Godemont resolution from $\mathrm{Ab}^0(X)$ to $\mathrm{Ab}^{\leq 1}(X)$

Note that F is clearly not a full functor, although it is faithful. This is not surprising, as one cannot simply define a morphism of sheaves by defining it first on the stalks.

We can enrich our category in a way that it captures more data of the sheaf. Suppose X has a generic point, i.e., a unique $\eta \in X$ which is contained in every open set. Now consider the category $\underline{\mathrm{Ab}}^{\leq 1}(X)$, whose objects are abelian groups A_x for every $x \in X$, together with maps $A_x \xrightarrow{\partial_x} A_\eta$, such that ∂_η is the identity (intuitively, the natural maps on stalks). Write $\partial_0 = \partial_\eta$ and $\partial_1 = \prod_{x \neq \eta} \partial_x$, and we may assemble these data into a cosimplicial group, which as a cochain complex looks like:

$$A_\eta \oplus \prod_x A_x \xrightarrow{\partial_1 - \partial_0} \prod_x A_\eta.$$

We denote $A^0 = A_\eta \oplus \prod_x A_x$ and $A^1 = \prod_x A_\eta$. A morphism $A \xrightarrow{\phi} B$ is one which respects the simplicial structure: it is defined by maps $\phi_x : A_x \to B_x$. We also require that the maps be chain maps, so that the following commutes for all x:

$$\begin{tikzcd} A_x \arrow[r, "\partial_A"] \arrow[d, "\phi_x"'] & A_\eta \arrow[d, "\phi_\eta"] \\ B_x \arrow[r, "\partial_B"'] & B_\eta. \end{tikzcd} \tag{2.1}$$

Remark 2.1.4. For an arbitrary noetherian space X, $\underline{\mathrm{Ab}}^{\leq 1}(X)$ may be defined similarly, taking the finitely many generic points into account.

Remark 2.1.5. Via the forgetful functor $\underline{\mathrm{Ab}}^{\leq 1}(X) \to \underline{\mathrm{Ab}}^0(X)$, $A \mapsto A^0$, we have generalized the previous construction. Again, we have a functor $\underline{\mathrm{Ab}}(X) \to \underline{\mathrm{Ab}}^{\leq 1}(X)$ given by taking products over points, although this time we are keeping the data of the generic point. For a sheaf A, we again let $FA^0 = \prod_x A_x = A_\eta \oplus \prod_{x \neq \eta} A_x$, and let $FA^1 = \prod_{x \neq \eta} A_\eta$, where the boundary maps are the maps on stalks. We have two directed systems of open sets: those containing a fixed x and those containing η (i.e., all open sets). Here, the key property is that since $x \in \overline{\{\eta\}} = X$, the set $\{U \mid U \ni x\}$ is cofinal[2] in $\{U \mid U \ni \eta\}$. Thus there is a map $\partial_x : A_x \to A_\eta$, $(f, U) \mapsto (f|_V, V)$ for any $V \ni \eta$ (we can just use $V = U$).

Assertion 2.1.6. *Let A, B be sheaves, and suppose we have a morphism $FA \xrightarrow{\phi} FB$. That is, we have a collection of maps of stalks $\phi_x : A_x \to B_x$ satisfying (2.1).*

[2]More than cofinal: a sub directed system

Suppose also that the maps on stalks $B_x \to B_\eta$ are injections. Then there is a unique sheaf morphism $\psi : A \to B$ such that $F\psi = \phi$.

Remark 2.1.7. Thus if $B_x \to B_\eta$ is an injection for every sheaf B and $x \in X$, then F is a full functor.

Proof. Let U be an open set. We must create maps $A(U) \to B(U)$. Take $s \in A(U)$, and let $x \in U$. Then $\phi_x(s, U) \in B_x$, say $\phi_x(s, U) = (t_x, U_x)$ for some $U_x \ni x$ and $t_x \in B(U_x)$. The collection $\{U_x\}$ is an open cover of U.

The collection of sections $\{t_x\}$ agree on intersections. Suppose $x, y \in U$. We have $(t_x, U_x)|_{U_x \cap U_y} = (t_x|_{U_x \cap U_y}, U_x \cap U_y)$ and similarly for (t_y, U_y). These sections agree by commutativity of (2.1), as they are both $(\phi_\eta(s), U_x \cap U_y)$ in B_η, and we use fact that the map $B_x \to B_\eta$ is an injection.

By the sheaf axiom applied to B, we can assemble the $\{t_x\}$ into a unique section t of B on U. The association $s \mapsto t$, $A(U) \to B(U)$ is clearly independent of our initial choice t_x. Since it is defined on stalks, it satisfies the properties of a presheaf morphism, so is a sheaf morphism. $\square$

2.2 Parshin flags and the higher adeles

The adeles of quasi-coherent sheaves are generalizations of the objects of the category $\underline{\mathrm{Ab}}^{\leq 1}(X)$. In fact, objects in $\underline{\mathrm{Ab}}^{\leq 1}(X)$ form the grade 0 and 1 pieces of the big adeles (see Definition 2.2.3).

Let X be a noetherian scheme.

Definition 2.2.1. A **Parshin flag** is a chain of scheme points $(\eta_0, \ldots, \eta_l)$ such that $\eta_{i+1} \in \overline{\{\eta_i\}}$ for all i. We will call l its **length**. The flags of length 0 are simply the scheme points of X. A Parshin flag is **non-degenerate** (resp. degenerate) if its scheme points are distinct (resp. not distict). If X is equidimensional, a Parshin flag is **complete** if it is non-degenerate of length $\dim(X)$. Finally, we call a Parshin flag **smooth** if each point P_i is a smooth point of $\overline{\{P_{i-1}\}}$.

If X/k is a variety, then Parshin flags are tuples of irreducible closed subvarieties linearly ordered by inclusion. For low dimensional varieties, we label Parshin flags according to their codimension. An n-**flag** is a scheme point of codimension n. If X has dimension n, then the set of n-flags is the set of closed points of X, and there is a unique 0-flag since X is irreducible.

Let Δ be a Parshin flag and let $I \subset \{0, \ldots n\}$ be an ordered tuple of (not necessarily distinct) numbers $0 \leq i_0 \leq \cdots \leq i_p \leq n$. We will say Δ is a flag of **type** I, or an I-**flag**, if each $\overline{\{\eta_j\}}$ has codimension i_j.

We'll denote by $S_n(X)$ the set of all Parshin flags on X of length n. Denote by $S_n^{(\mathrm{red})}(X)$ the set of all non-degenerate Parshin flags on X. Finally, write $S_I(X)$ for the set of all Parshin flags on X of type I.

For example, if X is a curve then $S_{01}(X)$ is the set of all complete flags, i.e., the set $\{(\eta, x)\}_{x \in |X|}$ of pairs (η, x) with η the generic point and x an arbitrary closed point.

2.2.1 Huber's construction of the higher (rational) adeles

In this section, let X be a noetherian scheme. The following is a review of Huber [14, §1–4], together with the comment in §5 *loc. cit.* that the construction works for complete as well as rational adeles.

We begin with the simplified version of Huber's original construction, the rational adeles of a noetherian scheme. The difference between the rational adeles, defined in this section, and the complete adeles, defined in the next section, is the same as the difference between so-called valuation vectors and the classical adeles.

Definition 2.2.2. Let $\mathcal{F}$ be a quasi-coherent sheaf on X, and $\Delta = (P_0, \ldots, P_n)$ a Parshin flag on X. We define the **local factor** or **simplicial stalk** at Δ to be the stalk at the most generic point,

$$\mathcal{F}_\Delta = \mathcal{F}_{P_0}.$$

If $X = \operatorname{Spec} R$ is affine, and M is an R-module we define the **local factor** at Δ to be

$$M_\Delta = S_{\mathfrak{p}_0}^{-1} M.$$

Definition 2.2.3. The **big adeles** are the cosimplicial algebra C_X associated with

$$C_X^n = \prod_{\Delta \in S_n^{(\mathrm{red})}(X)} \mathcal{O}_{X,\Delta}.$$

The coboundaries are the diagonal maps, for $\Delta \in S_{n-1}^{(\mathrm{red})}(X)$,

$$d_i^n(s_\Delta) = \prod_{\substack{\Delta' \in S_n^{(\mathrm{red})}(X) \\ \delta_i^n \Delta' = \Delta}} s_\Delta$$

where we use the inclusions of stalks $\mathcal{O}_{X,P} \to \mathcal{O}_{X,Q}$ if $\Delta = (P, \ldots)$ and $\Delta' = (Q, \ldots)$.

The **degenerate big adeles** $C_{\deg,X}$ are defined similarly, except the product is over all flags $S_n(X)$. The degeneracy maps are the obvious products of identity maps.

Remark 2.2.4. We can consider the big adeles as a functor $\underline{\mathrm{Qco}}(X) \to \underline{\mathrm{CoSimp}}(\underline{\mathrm{Ab}})$, to the category of cosimplicial groups by replacing $\mathcal{O}_{X,\Delta}$ with $\mathcal{F}_\Delta$. In this case we will write the big adeles as $C_X(\mathcal{F})$ for a quasi-coherent sheaf $\mathcal{F}$, or $C(\mathcal{F})$ when the context is clear. Similarly, we define $\widehat{C}_X(\mathcal{F})$, $C_{\deg,X}(\mathcal{F})$, and $\widehat{C}_{\deg,X}(\mathcal{F})$.

Definition 2.2.5. Let $K_n \subset S_n(X)$ be any set of Parshin flags of length n. For any scheme point P, let

$$\widehat{P}K_n = \{\Delta \in S_{n-1}(X) \mid (P, \Delta) \in K_n\}.$$

That is, $\widehat{P}K_n$ is the set of all flags in K_n with the most generic point P removed. If $\Delta' = (P, \ldots) \in K_n$, then $\delta_0^n \Delta' \in \widehat{P}K_n$; therefore $\delta_0^n K_n = \bigcup_P \widehat{P}K_n$.

Definition 2.2.6. Let $\mathcal{F}$ be a quasi-coherent sheaf, P a scheme point. Let $j : \operatorname{Spec} \mathcal{O}_{X,P} \to X$ be the inclusion. For any $\mathcal{O}_{X,P}$ module M, we let $[M]_P$ denote the push-forward sheaf $j_*(\widetilde{M})$.

Proposition 2.2.7 (Huber, Proposition 5.2.1 [14])**.** *Let $K_n \subset S_n(X)$ be any set of Parshin flags of length n. There is a unique subfunctor A_X of C_X which is additive, exact, commutes with direct limits, and satisfies the inductive construction*

1. *If $n = 0$, then $A_X(K_0, \mathcal{F}) = C_X(K_0, \mathcal{F})$*

2. If $n > 0$ and $\mathcal{F}$ is coherent, then

$$A_X(K_n, \mathcal{F}) = \prod_{P \in X} A(\widehat{P} K_n, [\mathcal{F}_P]_P).$$

Definition 2.2.8. Let X be a noetherian scheme, $S_\bullet(X)$ the simplicial set of all Parshin flags, and $S_\bullet^{\mathrm{red}}(X)$ the set of all non-degenerate Parshin flags. The **adelic algebra** A_X associated with X is the cosimplicial algebra

$$A_X^n = A(S_n^{\mathrm{red}}(X), \mathcal{O}_X).$$

The **degenerate adelic algebra** is the cosimplical algebra $A_{\mathrm{deg},X}$ defined by $A_{\mathrm{deg},X}^n = A(S_n(X), \mathcal{O}_X)$.

We sometimes call this the **associated cosimplicial algebra**. In the terminology of [14], A_X is called the cosimplicial group **reduced rational adeles**.

Example 2.2.9. Let $X = \operatorname{Spec} R$ for a DVR R with field of fractions K. Then $A_X^0 = K \oplus R$, $A_X^1 = K$, and $A_X^n = 0$ for $n > 1$. The adeles in this case agree with the big adeles. There is no restricted direct product as there are only finitely many points.

Example 2.2.10. Let $X = \operatorname{Spec} R$ for a Dedekind domain R with field of fractions K. Then $A_X^0 = K \oplus \prod_{\mathfrak{p}} R_{\mathfrak{p}}$, the product over all nonzero prime ideals of X. Looking at the induction step $n = 1$ in Proposition 2.2.7, we have $P = (0)$, the generic point, and $\mathcal{F} = \mathcal{O}_X = \widetilde{R}$. In this case, $[\mathcal{F}_P]_P = \widetilde{K}$, the constant sheaf on X. This is not coherent, so we write it as a direct limit of coherent sheaves and apply the first property of the construction (Proposition 2.2.7, 1). The result is that $A_X^1 = \prod_{\mathfrak{p}}' K$, the restricted direct product of the fields K with respect to the subrings $R_{\mathfrak{p}}$. This is exactly the classical definition of the ring of valuation vectors of R.

Remark 2.2.11. We can also consider the the adeles as a functor $\underline{\mathrm{Qco}}(X) \to \underline{\mathrm{CoSimp}}(\underline{\mathrm{Ab}})$. In this case, we write $A_X(\mathcal{F})$ for a quasi-coherent sheaf $\mathcal{F}$, or $A(\mathcal{F})$ when the context is clear. We similarly define the degenerate adele functor $A_{\mathrm{deg},X}(\mathcal{F})$.

Lemma 2.2.12 (Huber, [14]). *Let A be the adele functor above. Let X be a reduced noetherian scheme.*

1. *If $X = \operatorname{Spec} R$ is affine, and $s \in R$, $s \neq 0$, then for an R-module M,*

$$A(K, \widetilde{s^{-1}M}) = s^{-1}A(K, \widetilde{M}).$$

2. *Let C be the big adele functor. Then the inclusions $A^n(\mathcal{F}) \to C^n(\mathcal{F})$ form a morphism of cosimplicial groups.*

Proof. 1. [14, Lemma 3.1.4]

2. This is the statement of [14, Proposition 2.2.4, Proposition 2.3.3].

□

From a cosimplicial group, it is a standard construction to produce a cochain complex (See §2.4 for how this applies to the adelic algebra A_X). Applying this to the cosimplicial groups $A(\mathcal{F})$, we obtain the fundamental property of the adeles.

Theorem 2.2.13 (Beilinson–Huber). *Let $\mathcal{F}$ be a quasi-coherent sheaf, and $A(\mathcal{F})$ its rational adeles as defined above. Then for all n,*

$$H^n(A(\mathcal{F})) = H^n(X, \mathcal{F}).$$

Proof. Huber proves this for the complete adeles in [14, Theorem 4.2.3]. For the rational adeles, see §5.2 *loc. cit.* □

In fact, Huber shows that the association $U \mapsto A^n(U, \mathcal{F}|_U)$ is a complex of sheaves, and proves

Proposition 2.2.14 (Huber, Proposition 4.2.2 [14])**.** *Let X be a noetherian scheme and let $\underline{A}_X(\mathcal{F})^\bullet$ be the complex of sheaves associated with $U \mapsto A_X(U, \mathcal{F}|_U)^\bullet$ (rational or complete). Then each $\underline{A}_X(\mathcal{F})^\bullet$ is flasque, and $\mathcal{F} \to \underline{A}_X(\mathcal{F})^\bullet$ is a resolution of the sheaf $\mathcal{F}$.*

2.2.2 Huber's construction of the higher complete adeles

In this section, X is a noetherian scheme. The construction of this section mirrors that of the previous section, except that it replaces localization functors with completion functors. The following is a review of Huber [14, §1–4].

Proposition 2.2.15 (Huber, Proposition 2.1.1 [14])**.** *Let $K_n \subset S_n(X)$ be any set of Parshin flags of length n. There is a unique functor $\hat{A}_X$ which is additive, exact, commutes with direct limits, and satisfies the inductive construction*

1. *If $n = 0$ and $\mathcal{F}$ is coherent, then $\hat{A}_X(K_0, \mathcal{F}) = \prod_{P \in X} \varprojlim_l \mathcal{F}_P / \mathfrak{m}_P^l \mathcal{F}_P$.*

2. *If $n > 0$ and $\mathcal{F}$ is coherent, then*

$$\hat{A}_X(K_n, \mathcal{F}) = \prod_{P \in X} \varprojlim_l \hat{A}\left(\widehat{P}K_n, [\mathcal{F}_P / \mathfrak{m}_P^l \mathcal{F}_P]_P\right).$$

Definition 2.2.16. Let $\mathcal{F}$ be a quasi-coherent sheaf on X and $\Delta = (P_0, \ldots, P_n)$ a Parshin flag on X. We define the **complete local factor**, **complete simplicial stalk**, or **Beilinson completion** at Δ to be,

$$\widehat{\mathcal{F}}_\Delta = \hat{A}_X(\{\Delta\}, \mathcal{F}).$$

Let $X = \operatorname{Spec} R$ be affine, and M an R-module. Following Huber [14], for each prime ideal $\mathfrak{p}$, let $S_\mathfrak{p}^{-1}$ denote the localization functor at $\mathfrak{p}$. Let $C_\mathfrak{p}$ denote the completion functor at $\mathfrak{p}$. We define the **complete local factor** at Δ to be

$$\widehat{M_\Delta} = C_{\mathfrak{p}_0} S_{\mathfrak{p}_0}^{-1} \cdots C_{\mathfrak{p}_p} S_{\mathfrak{p}_p}^{-1} R \otimes_R M.$$

The two definitions agree for affine schemes, in the sense that if $\mathcal{F} = \widetilde{M}$, then $\widehat{\mathcal{F}}_\Delta = \widehat{M_\Delta}$ ([14, Proposition 3.2.1]).

Definition 2.2.17. The **big complete adeles** are the cosimplicial algebra $\widehat{C}_X$ associated with

$$\widehat{C}_X^n = \prod_{\Delta \in S_n^{(\mathrm{red})}(X)} \widehat{\mathcal{O}}_{X,\Delta}.$$

The coboundaries are the diagonal maps, for $\Delta \in S_{n-1}^{(\mathrm{red})}(X)$,

$$d_i^n(s_\Delta) = \prod_{\substack{\Delta' \in S_n^{(\mathrm{red})}(X) \\ \delta_i^n \Delta' = \Delta}} s_\Delta$$

where we use the map of completed stalks $\widehat{\mathcal{O}}_{X,P} \to \widehat{\mathcal{O}}_{X,Q}$ if $\Delta = (P, \ldots)$ and $\Delta' = (Q, \ldots)$.

The **degenerate complete big adeles** $\widehat{C}_{\deg,X}$ are defined similarly, except the product is over all flags $S_n(X)$. The degeneracy maps are the obvious products of identity maps.

Similarly to the rational adeles, we define the following:

Definition 2.2.18. The **complete adelic algebra** $\hat{A}_X$ associated with X is the cosimplicial algebra

$$\hat{A}_X^n = \hat{A}(S_n^{\mathrm{red}}(X), \mathcal{O}_X).$$

The **degenerate complete adelic algebra** is the cosimplical algebra $\hat{A}_{\mathrm{deg},X}$ defined by $\hat{A}^n_{\mathrm{deg},X} = \hat{A}(S_n(X), \mathcal{O}_X)$. We have inclusions of cosimplicial algebras $\hat{A}_X \to \widehat{C}_X$ and $\hat{A}_{\mathrm{deg},X} \to \widehat{C}_{\mathrm{deg},X}$ ([14, Theorem 2.4.1]).

Remark 2.2.19. We can also consider the the complete adeles as a functor $\underline{\mathrm{Qco}}(X) \to \underline{\mathrm{CoSimp}}(\underline{\mathrm{Ab}})$. In this case, we write the complete adeles as $\hat{A}_X(\mathcal{F})$ for a quasi-coherent sheaf $\mathcal{F}$, or $\hat{A}(\mathcal{F})$ when the context is clear. We similarly define $\hat{A}_{\mathrm{deg},X}(\mathcal{F})$.

The complete adeles enjoy the same main theorem as the rational adeles.

Theorem 2.2.20 (Huber, Theorem 4.2.3 [14]). *Let $\mathcal{F}$ be a quasi-coherent sheaf, and $\hat{A}(\mathcal{F})$ the associated complete adeles as defined above. Then for all n,*

$$H^n(\hat{A}(\mathcal{F})) = H^n(X, \mathcal{F}).$$

Remark 2.2.21. It also follows that the inclusions $A_X \to \hat{A}_X$ and $A_X(\mathcal{F}) \to \hat{A}_X(\mathcal{F})$ are quasi-isomorphisms.

Example 2.2.22. Returning to Example 2.2.9, let $X = \operatorname{Spec} R$ for a DVR R with field of fractions K and maximal ideal $\mathfrak{m}$. Then $A^0_X = K \oplus \widehat{R}$, where $\widehat{R}$ denotes the $\mathfrak{m}$-adic completion of R. If $\widehat{K}$ denotes the field of fractions of $\widehat{R}$, $A^1_X = \widehat{K}$. For all $n > 1$, $A^n_X = 0$.

Example 2.2.23. Returning to Example 2.2.10, let $X = \operatorname{Spec} R$ for a Dedekind domain R with field of fractions K. Then $A_X^0 = K \oplus \prod_{\mathfrak{p}} \widehat{R}_{\mathfrak{p}}$, where $\widehat{R}_{\mathfrak{p}}$ denotes the $\mathfrak{p}$-adic completion of R, and the product is over all nonzero prime ideals $\mathfrak{p}$ of R. If $\widehat{K}_{\mathfrak{p}}$ denotes the field of fractions of $\widehat{R}_{\mathfrak{p}}$, then a similar argument to Example 2.2.10 shows that $A_X^1 = \prod'_{\mathfrak{p}} \widehat{K}_{\mathfrak{p}}$, the restricted direct product with respect to the subrings $\widehat{R}_{\mathfrak{p}}$. This is exactly the classical adele ring of R.

Definition 2.2.24. We recursively define a field K to be an n**-local field** $(n > 0)$ if it is the field of fractions of a complete DVR with an $(n-1)$-local residue field; a 0-local field is just an arbitrary field. Local fields are 1-local fields.

For more information on the theory of n-local fields, together with applications to higher adeles, see [7]. The following is clear from the definitions.

Proposition 2.2.25. *Let X/k be a variety of dimension n. Let $\Delta = (P_0, \ldots, P_n)$ be a complete smooth Parshin flag on X (Definition 2.2.1). Then the local factor $A_{X,\Delta}$ is an n-local field.*

If P_2 is not a smooth point of $V = \overline{\{P_1\}}$, the local factor will split into a finite product over the formal branches of V passing through P_2. For more in the case of a surface, see Parshin [22, §1].

2.3 Differential graded algebras and their modules

We have seen that the simplicial set of Parshin flags on a scheme (more generally any noetherian space) gives rise to a cosimplicial group of adeles. In fact, this is just one example among many. Gorchinskii [11] provides a similar construction for K-theoretic sheaves. Later, we will give a construction (§4.2.2) of S-adeles for surfaces. Thus, it is important to get a better sense of the category over which we are working.

Recall that there is a fundamental correspondence between cosimplicial groups and cochain complexes called the Dold–Kan correspondence [26]. The extra fact we will mention is that if one additionally has a monoid in the category of cosimplicial groups, i.e., a cosimplicial algebra, then the normalization functor of Dold–Kan will respect the structure and produce a *differential graded (co)algebra*. This is done via the Alexander–Whitney morphism. We refer the reader to [23] for more information. We include a section reviewing the monoidal Dold–Kan correspondence in §2.4.

In this section, we will review basic facts about differential graded algebras and their modules. Although most of it is well-known, our construction of Chern classes (§2.5) seems to be novel. Later, we will make simplifying assumptions using the fact that our differential graded algebra A_X actually comes from a cosimplicial algebra. I.e., A_X is in the image of the Dold–Kan correspondence. Further, we might frequently use the fact that our boundary operators are always created from maps of stalks (the prototypical example being the map $\mathcal{F}_x \to \mathcal{F}_y$ for scheme points $x \in \overline{\{y\}}$ and a sheaf $\mathcal{F}$). The boundary maps of the differential graded algebra are created from alternating sums of these inclusions.

To motivate this section, we give some basic results concerning the adeles of a variety that we will prove later (Proposition 3.2.5):

Proposition 2.3.1. *Let X/k be a smooth variety, $A = A_X$ its differential graded algebra of adeles. Let $K = A(\mathcal{K})$ be the adeles associated with its constant sheaf $\mathcal{K}$. We have an inclusion $A \to K$ of differential graded k-algebras.*

To each divisor D is associated a two-sided fraction ideal $A(D) \subset K$ (Definition 3.2.2). This ideal $A(D)$ is principal, generated by an element of degree 0. We have:

1. *$A(-D) \simeq A(D)^\vee = \mathcal{H}om_A^{dg}(A(D), A)$ as differential graded A-bimodules.*

2. *$A(D+E) \simeq A(D) \otimes_A^{dg} A(E)$ as differential graded A-bimodules.*

3. *$A(D) \simeq A(E)$ as differential graded A-modules if and only if $D \sim E$.*

This is of course an exact mirror of the fundamental correspondence between invertible sheaves and divisors. The key difference is that since A is an "affinization of $\mathcal{O}_X$", these statements are true as algebras and modules, without reference to underlying sheaves (although, all modules can be sheafifed if needed into $\mathcal{O}_X$-modules).

2.3.1 Definitions

Recall a **differential graded algebra** is a $\mathbb{Z}$-graded algebra $A = \oplus_n A^n$ with linear derivations $d : A^n \to A^{n+1}$ making A into a cochain complex, and such that the derivations satisfy the Leibniz rule

$$d(ab) = da\, b + (-1)^{|a|} a\, db$$

for any homogenous element $a \in A^n$. Here $|a| = n$ denotes the degree of a. A right (resp. left) **differential graded module** for $A^\bullet$ is a chain complex $M^\bullet$ such that the $\oplus_n M^n$ is a right (resp. left) graded $\oplus_n A^n$-module, and such that multiplication satsifies the Leibniz rule for $d(ma)$ (resp. $d(am)$).

All differential graded A-modules are assumed to be right modules (unless otherwise specified), which is required to ensure Hom is itself an A-module (Definition 2.3.7).

Definition 2.3.2. Let A be a differential graded algebra. An **ideal** I of A is a differential graded right A-submodule of A which is simultaneously a differential graded left A-submodule. In other words, it is a two-sided ideal (in the usual sense) which is closed under d. We similarly define right (resp. left) ideals as differential graded right (resp. left) A-submodules.

2.3.2 The differential category of differential graded A-modules

Throughout this section let M, N be differential graded A-modules. We have three structures that morphisms can preserve: A-linearity, grading, and the differential. As it turns out, there are situations where we will have morphisms which preserve some but not all of these structures. This is further complicated by the fact that most A-modules we consider are actually A-bimodules.

Definition 2.3.3. We will write $\underline{\mathrm{Mod}}_A$ for the category of right A-modules. The morphisms in this category are denoted Hom_A.

We will write $\underline{\mathrm{Mod}}_A^{gr}$ for the category of graded right A-modules. Morphisms in this category, Hom_A^{gr} are graded A-linear maps, i.e., A-linear maps $\phi : M \to N$ satisfying $\phi(M^i) \subset N^i$.

Definition 2.3.4. The group of **degree n homomorphisms** is

$$\mathrm{GrHom}_A^n(M, N) = \{f \in \mathrm{Hom}_A(M, N) \mid f(M^i) \subset N^{i+n}\}.$$

For example, $\mathrm{GrHom}_A^0(M, N)$ is the group $\mathrm{Hom}_A^{gr}(M, N)$ of graded A-linear homomorphisms. If $a \in A^n$ and M is a differential graded A-bimodule, then left multiplication by a is an example of such a morphism. That is, if we put $\mu_a(m) = am$, then $\mu_a \in \mathrm{GrHom}_A^n(M, M)$.

The collection of degree n homomorphisms assembles into a cochain complex with boundary maps: if $\phi \in \mathrm{GrHom}_A^n(M, N)$ then

$$d\phi = d_N \phi + (-1)^{|\phi|} \phi d_M$$

where $|\phi| = n$. This is the internal hom in the category of cochain complexes. As it stands, $\mathrm{GrHom}_A^\bullet$ has no extra structure except as a cochain complex, just as $\mathrm{Hom}_A(M, N)$ has no extra structure beyond an abelian group. If A is a differential graded algebra over a *commutative* differential graded algebra k (as is the case when A is the adele ring of a scheme X over a commutative ring k; k itself is a differential graded algebra with trivial grading), or if A is itself (graded) commutative, then $\mathrm{GrHom}_A^\bullet(M, N)$ does gain some k- or A-linear structure and becomes a differential graded module. Otherwise, $\mathrm{GrHom}_A^\bullet$ is just a differential graded module over $\mathbb{Z}$, where $\mathbb{Z}$ is thought of as a differential graded ring concentrated in degree 0.

Definition 2.3.5. Let M, N be differential graded A-modules. A **differential graded morphism** is an A-linear morphism which is also a chain map. That is, $\phi : M \to N$ satisfies $\phi d_M = d_N \phi$ and $\phi(M^i) \subset N^i$. Denote the set of such morphisms $\mathrm{Hom}_A^{dg}(M, N)$.

In other words, ϕ is a differential graded homomorphism if it is a 0-cocycle in $\mathrm{GrHom}_A^{\bullet}(M, N)$:

$$Z^0(\mathrm{GrHom}_A^{\bullet}(M, N)) = \mathrm{Hom}_A^{dg}(M, N).$$

Definition 2.3.6. We write $\underline{\mathrm{Mod}}_{(A,d)}$ for the category of differential graded A-modules, whose morphisms are differential graded homomorphisms.

In fact, it will turn out that this is not the category we want to work in. First, the condition of being a chain map is too strong. Second, the category $\underline{\mathrm{Mod}}_{(A,d)}$ does not have internal Hom objects.

Definition 2.3.7. Let M, N be differential graded right A-modules. We write $\mathcal{H}om_A^{dg}(M, N) = \mathrm{GrHom}_A^{\bullet}(M, N)$ endowed with the structure of a differential graded k-module as above.

Remark 2.3.8. In this definition it is *required* that we take right A-modules. A simple calculation shows $\mathrm{GrHom}_A^{\bullet}(M, N)$ is a complex only when using right modules and right module homomorphisms, as d acts on the left, in a sense.

Definition 2.3.9. Let k be a commutative ring. A category $\mathcal{C}$ is a **differential graded category** over k if the Hom sets have the structure of a differential graded k-module. That is, for any objects M, N, and P, $\mathrm{Hom}_{\mathcal{C}}(M, N)$ is a differential graded k-module, and the compositions $\mathrm{Hom}_{\mathcal{C}}(M,N)\times\mathrm{Hom}_{\mathcal{C}}(N,P)\to\mathrm{Hom}_{\mathcal{C}}(M,P)$ are differential graded homomorphisms over k.

Finally, we have:

Definition 2.3.10. Let k be a commutative ring and let A be a differential graded k-module. We write $\underline{\mathrm{Mod}}_A^{dg}$ for the category of differential graded A-modules, whose homomorphisms are the differential graded k-modules $\mathcal{H}om_A^{dg}$ above. Then $\underline{\mathrm{Mod}}_A^{dg}$ is a differential graded category over k.

Definition 2.3.11. We define $\underline{\mathrm{Mod}}_{(A,A)}$ to be the category of (A,A)-bimodules, which we abbreviate and just call A-bimodules. We denote its Hom sets by $\mathrm{Hom}_{(A,A)}$.

Similarly, $\underline{\mathrm{Mod}}_{(A,A)}^{gr}$ and $\underline{\mathrm{Mod}}_{(A,A)}^{dg}$ are the categories of graded and differential graded A-bimodules. We denote the respective hom sets $\mathrm{Hom}_{(A,A)}^{gr}$ and $\mathrm{Hom}_{(A,A)}^{dg}$.

Remark 2.3.12. In practice we will be considering modules M which are differential graded A-bimodules. If M,N are both differential graded A-bimodules, then defining $(a\phi)(m)=\phi(ma)$ and $(\phi a)(m)=\phi(m)a$ makes $\mathrm{GrHom}_A^{\bullet}(M,N)$ into a differential graded A-bimodule. In this case it is actually an **internal hom**, and we will denote it by $\mathcal{H}om_{(A,A)}^{dg}(M,N)$, or $\mathcal{H}om_A(M,N)$ if the context is clear.

Example 2.3.13. We continue the example of $X=\operatorname{Spec} R$ for a DVR R (Example 2.2.9 and Example 2.2.22). Let A be the adelic algebra of X. This is a cosimplicial algebra, and we may simultaneously think of it as a differential graded algebra (See §2.4). Let M be a differential graded A-module. This means that for every n, M^n is an A^0-module, where $A^0=K\oplus R$. Thus for every n, we have a K-module M_0^n and an R-module M_1^n. Conversely, these data always assemble into a differential graded A-module with $M^n = M_0^n \oplus M_1^n$.

2.3.3 Graded free and graded projective modules

In this section, we follow [25, Tag 09JZ].

Definition 2.3.14. Let M be a differential graded A-module. Define M to be **free** if it is free as an A-module. Thus,

$$M \simeq \bigoplus_i A[k_i]$$

as graded A-modules for some integers k_i. Define M to be **graded-free** if it is free as a graded A-module. Equivalently, $k_i = 0$ for all i in the above expression. Define M to be **dg-free** if it is graded-free and the above isomorphism holds as differential graded A-modules.

Remark 2.3.15 (**Warning**). Again, *free* means as A-modules, thus a free differential graded A-module of rank one is not isomorphic to A in $\underline{\mathrm{Mod}}_{(A,d)}$.

Example 2.3.16. Let X/k be a surface and $A = A_X$ its associated differential graded algebra (See §3.1). We need to be careful about the concept of freeness for a module M. Let $I_D \subset A$ be the ideal associated with some effective Cartier divisor D (Definition 3.2.2). Then A is a free, graded-free, and dg-free object in the categories $\underline{\mathrm{Mod}}_A$, $\underline{\mathrm{Mod}}_A^{gr}$, and $\underline{\mathrm{Mod}}_A^{dg}$. More generally, the situation is illustrated in Table 2.1.

In particular, I_D becomes dg-free exactly when $D \sim_{\mathrm{rat}} 0$ (Proposition 3.2.5(3)), and $I_D[k]$ becomes graded-free exactly when $k = 0$. Thus, the three types of freeness convey different geometric information.

Table 2.1: Types of freeness

	$\underline{\mathrm{Mod}}_A$	$\underline{\mathrm{Mod}}_A^{gr}$	$\underline{\mathrm{Mod}}_A^{dg}$
A	free	graded-free	dg-free
I_D	free	graded-free	not dg-free
$I_D[k]$	free	not graded-free	not dg-free

Definition 2.3.17. In any of the above freeness conditions, M has a well defined **rank**, defined as the rank of M as an A-module.

The notion of projective objects in $\underline{\mathrm{Mod}}_A^{dg}$ is a subtle one, as the category is already, in a way, derived.

Definition 2.3.18. Let P be a differential graded A-module. Define P to be **graded projective** if it is projective as a graded A-module.

See the discussion in [25, Tag 09JZ]. Another type of projectivity in differential graded categories, called property (P), is discussed in [25, Tag 09KK]. We will return to property (P) when discussing intersection theory (Definition 5.3.1).

2.3.4 Duality

The definition of the dual in $\underline{\mathrm{Mod}}_A^{dg}$ mirrors duality $\mathcal{E}^\vee$ of locally free $\mathcal{O}_X$-modules. Note that A is highly non-commutativity in general, therefore the dual is only an endofunctor within the category of A-bimodules.

Definition 2.3.19. Let M be a right differential graded A-module. Define

$$M^\vee = \mathcal{H}om_A^{dg}(M, A).$$

Since A is also a left A-module, $M^\vee$ is a left differential graded A-module. If M is an A-bimodule, then $M^\vee$ is as well.

Remark 2.3.20. By Definition 2.3.7, $\mathcal{H}om_A^{dg}$ is the set of *right* A-module homomorphisms. Therefore if M is a left module, we will not consider $M^\vee$ unless M is also a right module. This is an obstruction to defining the double dual $(M^\vee)^\vee$, since $M^\vee$ is not a right A-module. If we only cared about the graded A-module structure, one could go through the trouble of defining $N^\vee$ for graded left A-modules as left graded A-module homomorphisms. Or, one could go through the opposite ring A^{opp} and corresponding opposite category. The A-modules coming from quasi-coherent sheaves will be bimodules, so we avoid this.

We saw that multiplication by A^n is a degree n homomorphism. In particular, if $a \in A^n$ and μ_a is left multiplication by a, then $\mu_a \in \mathrm{End}_A^{gr}(M)$ if and only if $a \in A^0$. Even then it is not a *differential* graded homomorphism unless $a \in H^0(A)$ as well.

Instead, multiplication defines a map

$$\mu : A \to \mathcal{H}om_A^{dg}(A, A) = A^\vee.$$

Proposition 2.3.21. *The map μ is an isomorphism of differential graded A-bimodules.*

Proof. It is a homomorphism of differential graded A-bimodules since $(d^\vee \mu_a)(b) = da\, b + (-1)^{|a|} a\, db - (-1)^{|\mu_a|} a\, db$, and $|a| = |\mu_a|$. Each $\mathrm{GrHom}_A^n(A, A)$ is isomorphic to

A^n as A^0-modules, so it is clearly surjective. Injectivity is from $\mathrm{Ann}_A(A) = (0)$. $\square$

Mirroring the fact for commutative rings and $\mathcal{O}_X$-modules,

Proposition 2.3.22. *Let M, N be (right) differential graded A-modules. Suppose M, N are free (as in §2.3.3) and finitely generated. Then*

$$\mathcal{H}om_A^{dg}(M, N) \simeq M^\vee \otimes_A^{dg} N$$

as graded k-bimodules. If further M, N are differential graded A-bimodules, then the isomorphism is as differential graded A-bimodules.

Remark 2.3.23. Although we will write $M \otimes_A^{gr} N$ and $M \otimes_A^{dg} N$ to denote the tensor product in the categories $\underline{\mathrm{Mod}}_A^{gr}$ and $\underline{\mathrm{Mod}}_A^{dg}$, in fact these are simply the tensor products as A-modules together with additional structure of a differential:

$$d_{M \otimes N} = d_M \otimes \mathrm{id}_N + \mathrm{id}_M \otimes d_N.$$

2.4 The monoidal Dold–Kan correspondence

The construction of the differential graded algebra A_X is a special case of the Dold–Kan correspondence over the category of algebras. This is called the **monoidal Dold–Kan correspondence**. We include it here for completeness, but stress that we do not require the full corrspondence. In particular, we will not use the reverse functor $\mathcal{K}$ in the correspondence; we require only the Moore, $\mathcal{C}$, and normalized Moore, $\mathcal{N}$, functors.

See [23] for more information. Our presentation is based on [5, Ch. 7].

The following holds for more general abelian categories, but we work with $\underline{\mathrm{Alg}}_k$, the category of k-algebras.

Definition 2.4.1. Given cosimplicial groups M, N, their tensor product is the cosimplicial group $M \otimes_k N$ with $(M \otimes_k N)^n = M^n \otimes_k N^n$.

Let A be a cosimplicial k-algebra, which is a functor $\underline{\Delta} \to \underline{\mathrm{Alg}}_k$ from the simplex category. Equivalently, it is a monoid in the category of cosimplicial k-modules, i.e., a functor $\underline{\Delta} \to \underline{\mathrm{Mod}}_k$ with a natural transformation of functors $A \otimes_k A \to A$.

We represent the maps in a cosimplicial algebra as

$$A^0 \underset{\sigma}{\overset{\delta}{\rightleftarrows}} A^1 \underset{\sigma}{\overset{\delta}{\rightleftarrows}} A^2 \cdots$$

remembering that each arrow actually represents a collection of maps. We label them as $\delta_i^n : A^{n-1} \to A^n$ and $\sigma_i^n : A^{n+1} \to A^n$ with $0 \le i \le n$. A morphism $f : A \to B$ of cosimplicial algebras is a transformation of functors and therefore a map making the following commute:

$$\begin{tikzcd}
A^0 \arrow[r, shift left, "\delta"] \arrow[d, "f"] & A^1 \arrow[l, shift left, "\sigma"] \arrow[r, shift left, "\delta"] \arrow[d, "f"] & A^2 \arrow[l, shift left, "\sigma"] \arrow[d, "f"] & \cdots \\
B^0 \arrow[r, shift left, "\delta"] & B^1 \arrow[l, shift left, "\sigma"] \arrow[r, shift left, "\delta"] & B^2 \arrow[l, shift left, "\sigma"] & \cdots
\end{tikzcd}$$

which means that $f(A^n) \subset B^n$, and,

$$f(\delta_i^n(a)) = \delta_i^n(f(a)), \quad \text{and,} \quad f(\sigma_i^n(a)) = \sigma_i^n(f(a)).$$

Definition 2.4.2. A cosimplicial group is a functor from $\underline{\Delta} \to \underline{\mathrm{Ab}}$. A **cosimplicial A-module** is cosimplicial group M with an of action A satisfying the obvious identities, but with cosimplicial morphisms.

Definition 2.4.3. The **Moore functor** $\mathcal{C}$ takes a cosimplicial group M and produces a cochain complex in the natural way. Namely, $(\mathcal{C}(M))^n = M^n$, and

$$d(m) = \sum_{i=0}^{n+1} (-1)^i \delta_i^{n+1}(m)$$

for $m \in M^n$.

The **normalized Moore functor** $\mathcal{N}$ is the reduced cochain complex associated with $\mathcal{C}$. That is, for a cosimplicial group M, $\mathcal{N}(M)^0 = M^0$ and

$$\mathcal{N}(M)^n = \bigcap_{i=0}^{n-1} \ker \sigma_i^{n-1}$$

with differentials inherited from $\mathcal{C}(M)$.

Proposition 2.4.4. *Both $\mathcal{N}$ and $\mathcal{C}$ are additive, exact covariant functors. The natural inclusion $\mathcal{N} \to \mathcal{C}$ induces a quasi-isomorphism $\mathcal{N}(M) \to \mathcal{C}(M)$ for all M.*

Proof. [23, §2.1] □

Definition 2.4.5. Let M, N be cosimplicial groups. The **Alexander–Whitney** map is the map $M^p \otimes_k N^q \to (M \otimes_k N)^{p+q}$ given by

$$AW(a \otimes b) = \delta_n^n \circ \cdots \circ \delta_{p+1}^{p+1}(a) \otimes \delta_0^n \circ \cdots \circ \delta_0^{q+1}(b)$$

where $n = p + q$.

There is an inverse to $\mathcal{N}$ called $\mathcal{K}$ that sets up a (so-called weak monoidal) equivalence of categories between cosimplicial algebras and differential graded algebras:

Theorem 2.4.6 (Monoidal Dold–Kan correspondence)**.** *Let A be a cosimplicial algebra. Then we have a cosimplicial morphism $\mu : A \otimes_k A \to A$. Composing with the Alexander–Whitney map gives*

$$\mathcal{C}(A) \otimes_k \mathcal{C}(A) \xrightarrow{AW} \mathcal{C}(A \otimes_k A) \xrightarrow{\mathcal{C}(\mu)} \mathcal{C}(A).$$

Under this multiplication, $\mathcal{C}(A)$ is a differential graded algebra. For any cosimplicial A-module M, $\mathcal{C}(M)$ is a differential graded A-module.

Further, the Alexander–Whitney map respects normalized Moore functor. That is, we also get a multiplication

$$\mathcal{N}(A) \otimes_k \mathcal{N}(A) \xrightarrow{AW} \mathcal{N}(A \otimes_k A) \xrightarrow{\mathcal{N}(\mu)} \mathcal{N}(A)$$

turning $\mathcal{N}(A)$ into a differential graded subalgebra of $\mathcal{C}(A)$.

Together with $\mathcal{K}$ and a shuffle map, this sets up a (weak monoidal) equivalence of categories between cosimplicial k-algebras and differential graded k-algebras.

Proof. [23, Theorem 1.1(1)] □

2.5 Differential graded algebras of cosimplicial type

In the previous section, we saw how a cosimplicial algebra A becomes a differential graded algebra, with the Alexendary–Whitney map facilitating between the two product structures. To distinuish the two, in this section, juxtaposition denotes the product in the differential graded algebra, and $\cdot$ denotes the commutative product in the rings A^n.

The material in this section does not seem to appear in the literature, although we have been inspired by [15, 16], which in turn is inspired by connections with differential geometry.

Definition 2.5.1. A differential graded algebra A is of **cosimplicial type** if A is a differential graded subalgebra of $\mathcal{C}(A')$ for some cosimplicial algebra A'. In other words, A is in the image of the Moore functor.

Remark 2.5.2. The differential graded algebra of adeles is of cosimplicial type.

For the rest of this section, A is a differential graded algebra of cosimplicial type.

Lemma 2.5.3. *Let A be a differential graded algebra of cosimplicial type. For all $a \in A^0$ and $b \in A^1$ we have a simple commutator relation*

$$ba - ab = b \cdot da \tag{2.2}$$

and an associativity rule $(ab) \cdot c = a(b \cdot c)$ for $a \in A^0$ and $b, c \in A^p$.

Proof. Simple computation, $ba - ab = b \cdot \delta_0^1(a) - \delta_1^1(a) \cdot b = b \cdot da$. The associativity is $a(b \cdot c) = \delta_p^p \circ \cdots \circ \delta_1^1(a) \cdot b \cdot c = (ab) \cdot c$. $\square$

2.5.1 Chern class

Let A be a differential graded algebra of cosimplicial type. We will define Chern classes of certain differential graded A-modules, which we will use later in Proposition 3.3.7.

Definition 2.5.4. Let M be a differential graded A-bimodule. We call M **invertible** if M is free of rank 1 as both a left and right A-module. By a **generator** we mean an element $t \in M^0$ which generates M as a right A-module.

Remark 2.5.5. The generator need not generate M as a left A-module.

If M is invertible, then $M^\vee$ is a invertible, and $M \otimes M^\vee \simeq A$ as differential graded A-bimodules. Thus the set of isomorphism classes of invertible A-modules forms a group.

Definition 2.5.6. Let M be an invertible A-module with generator t. Then there exists a unique $\theta_t \in A^1$ such that $dt = t\theta_t$. Define the associated **Chern class** to be

$$c(M,t) = 1_{A^1} + \theta_t \in A^1$$

where 1_{A^1} denotes the unit element in the ring A^1.

We may change t by any unit $u \in (A^0)^\times$ to obtain a new generator and a new Chern class. Doing so will give (since $|t| = 0$)

$$d(tu) = (dt)u + t(du) = tu[u^{-1}\theta_t u + \theta_u]$$

where we define $\theta_u = u^{-1}du$ for $u \in (A^0)^\times$, mimicking the definition of θ_t. Thus if we write a^u for the action of $A^\times$ on A by conjugation, $a \mapsto u^{-1}au$, then we get a cocycle

$$\theta_{tu} = (\theta_t)^u + \theta_u. \tag{2.3}$$

If M is an invertible A-module with generator t, we get a commutator relation

$$at = t(a + \theta_t \cdot a). \tag{2.4}$$

Definition 2.5.7. Since A is a cosimplicial object, applying the units functor to each A^n produces a cosimplicial object in the category of abelian groups. We denote this object, by abuse of notation, as $A^\times$. As a cochain complex with differential $d^\times(u) = \prod_{i=0}^n (\delta_i^n)^\times$, $A^\times$ is

$$(A^0)^\times \xrightarrow{d^\times} (A^1)^\times \xrightarrow{d^\times} \cdots$$

Proposition 2.5.8. *Let M, N be invertible A-modules.*

1. *Let s, t be generators of M, N respectively. Then $M \otimes_A N$ is invertible with generator $s \otimes t$, and*

$$c(M \otimes_A N, s \otimes t) = c(M, s) \cdot c(N, t).$$

2. *For any generator t, $c(M, t) \in (A^1)^\times$.*

3. *If s, t are distinct generators of M as a right A-module, then $c(M, s)$ and $c(M, t)$ are cohomologous in $A^\times$. Therefore we have a well-defined element*

$$c(M) \in H^1(A^\times).$$

Proof. 1. The first assertion is clear; note that nothing is assumed commutative, so we must use the fact that since M is invertible, for every $a \in A$, there exists a unique $b \in A$ such that $at = tb$; a similar statement holds for N and s. By

Lemma 2.5.3,

$$\begin{aligned} d_{M\otimes N}(s\otimes t) &= s\theta_s \otimes t + s\otimes t\theta_t \\ &= s\otimes \theta_s t + s\otimes t\theta_t \\ &= s\otimes (t\theta_s + dt\cdot\theta_s) + s\otimes t\theta_t \\ &= s\otimes (t\theta_s + (t\theta_t)\cdot\theta_s) + s\otimes t\theta_t \\ &= (s\otimes t)(\theta_s + \theta_t + \theta_s\cdot\theta_t). \end{aligned}$$

Rewriting $\theta_{s\otimes t} = \theta_s + \theta_t + \theta_s \cdot \theta_t$ as Chern classes gives the result.

2. Follows from (1), since $M \otimes_A M^\vee \simeq A$, $c(A,1) = 1_{A^1}$, and we may choose generators t, $t^\vee$ such that $t\otimes t^\vee \mapsto 1$ under this isomorphism. Therefore $c(M,t)\cdot c(M^\vee, t^\vee) = 1_{A^1}$.

3. Choose two generators of M; since they differ by a unit $u \in (A^0)^\times$, we may call them t and tu. Note that A is itself an invertible A-module. Units $u \in (A^0)^\times$ are generators, and for all $a \in A^1$, $a^u = a\cdot c(A,u)$ by Lemma 2.5.3. Applying this to equation (2.2), we obtain

$$c(M,tu) = c(M,t)\cdot c(A,u).$$

Finally,

$$c(A,u) = 1_{A^1} + u^{-1}du = 1_{A^1} + \delta_1^1(u^{-1})\cdot[\delta_0^1(u) - \delta_1^1(u)].$$

Therefore $c(A,u) = d^\times(u)$; thus the Chern classes differ by a coboundary.

□

Corollary 2.5.9. *The isomorphism classes of invertible A-modules, call it* $\mathrm{Pic}(A)$, *is a group under $\otimes$. The association $M \mapsto c(M)$ is a homomorphism* $\mathrm{Pic}(A) \to H^1(A^\times)$.

Definition 2.5.10. Let M be a differential graded A-bimodule. Suppose M is a free graded A-module of rank r, on both the left and the right. Then $\bigwedge^r M$ is an invertible A-module and we may define its Chern class to be $c(M) = c(\bigwedge^r M)$.

Conjecture 2.5.11. *We may define higher Chern classes as follows. Repeat the construction of $A^\times$, but with the Milnor algebra $K_\bullet^M$, to get a differential graded ring $K_\bullet^M(A)$. Suppose M is a free graded A-module of rank r, on both the left and the right, say with right generators $\vec{t} \in (M^0)^r$. There exists a matrix $\Theta \in \mathrm{M}_r(A^1)$ such that*

$$d\vec{t} = \vec{t}\Theta.$$

Similar to the above construction of Chern classes, $I + \Theta \in \mathrm{GL}_r A^1$. Suppose further that $I + \Theta$ is upper triangular. Then the higher Chern classes can be defined as the symmetric polynomials in the diagonal terms, where the ring operations happen in $K_\bullet^M(A)$. The nth symmetric polynomial of the diagonal terms gives a well-defined class in $H^n(K_n^M(A))$.

Finally, we state a result which will be needed later:

Proposition 2.5.12. *Let M, N be invertible A-modules with generators t, s.*

1. *There is a canonical isomorphism*

$$\mathrm{Hom}_A^{dg}(M, N) \simeq \{n \in N \mid \theta_n = \theta_t\}.$$

Therefore there exists a nontrivial $\phi \in \mathrm{Hom}_A^{dg}(M, N)$ *if and only if there is* $s' \in N$ *with* $|t| = |s'|$ *and* $\theta_t = \theta_{s'}$*. In this case,* ϕ *is defined by* $\phi(t) = s'$*.*

2. *If* $\theta_t = \theta_s$ *then* $M \simeq N[k]$ *as differential graded* A*-modules with isomorphism* $t \mapsto s$*. If* $|t| = |s|$ *as well, then in fact* $M \simeq N$*.*

Proof. Both follow directly from the definitions. $\square$

Chapter 3: Adeles of varieties and adelic Cartier divisors

Let X/k be a variety with adelic algebra $A = A_X$. In this chapter, we describe the structure of differential graded A-modules of the form $A(\mathcal{F})$ for quasi-coherent sheaves $\mathcal{F}$ (§3.1). The case of invertible sheaves is particularly simple, as we essentially get "principal fractional ideals" (§3.2). Finally, we define adelic Cartier divisors as certain 1-cocycles in the cosimplicial group $U(A)$, where U is the units functor (§3.3, compare with Definition 2.5.7).

3.1 $\mathcal{O}_X$-modules, quasi-coherent sheaves, and their A_X-modules

If X is a noetherian scheme, then $A_X = A_X(\mathcal{O}_X)$ and $A_{\mathrm{deg},X} = A_{\mathrm{deg},X}(\mathcal{O}_X)$ are differential graded algebras of cosimplicial type (Definition 2.5.1).

Proposition 3.1.1. *For any quasi-coherent sheaf $\mathcal{F}$, $A_X(\mathcal{F})$ is a cosimplicial module over A_X. This is similarly true for $A_{\mathrm{deg},X}(\mathcal{F})$ and $A_{\mathrm{deg},X}$.*

Further, via the Moore and/or normalized Moore functors, we can consider A_X as a differential graded k-algebra. In this case, $A_X(\mathcal{F})$ is a differential graded A-module.

Proof. This is purely formal and follows from functoriality. The definition of a morphism f of $\mathcal{O}_X$-modules translates into f being cosimplicial. The cosimplicial groups are just stalks, and the differentials are inclusions of stalks. The last part follows from the monoidal Dold–Kan correspondence (Theorem 2.4.6). □

Remark 3.1.2. The previous proposition holds if we choose $A_X(-)$ to be any exact, additive subfunctor of the big adele functor $C_X(-)$ (Definition 2.2.3). We have a cosimplicial algebra $A_X = A_X(\mathcal{O}_X)$. Then for every $\mathcal{O}_X$-module $\mathcal{F}$, $A_X(\mathcal{F})$ is a cosimplicial module over A_X. This holds similarly for the complete adeles $\hat{A}_X$.

Example 3.1.3. Consider the case of the adeles of $X = \operatorname{Spec} R$ for a DVR R with field of fractions K and maximal ideal $\mathfrak{m}$ (Example 2.2.9). Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let η be the generic point of X. Then $\mathcal{F}$ is given by the data of an R-module $M = \mathcal{F}_\mathfrak{m}$ and a K-module $V = \mathcal{F}_\eta$ such that $V \simeq M \otimes_R K$.

The adeles of X are, as a complex, $A_X = [K \oplus R \to K]$. Write an element of A_X as $a = (a_0, a_1, a_{01})$. The cup product on A_X is given by

$$ab = (a_0 b_0, a_1 b_1, a_0 b_{01} + a_{01} b_1).$$

The adeles of $\mathcal{F}$ are, as a complex, $A_X(\mathcal{F}) = [V \oplus M \to V]$. Therefore an element of $A_X(\mathcal{F})$ is a tuple $m = (m_0, m_1, m_{01})$ with $m_0, m_{01} \in V$ and $m_1 \in M$. The right module structure over A_X is given by

$$ma = (m_0 a_0, m_1 a_1, m_0 a_{01} + m_{01} a_1).$$

In certain circumstances one can define $A(\mathcal{F})$ for an arbitrary $\mathcal{O}_X$-module $\mathcal{F}$.

Example 3.1.4. Return to the situation in the previous example. Let $M \neq 0$ be an R-module. Define an $\mathcal{O}_X$-module by $\mathcal{F}(X) = M$ and $\mathcal{F}(\{\eta\}) = 0$ where η is the generic point. Then $\mathcal{F}$ is not a quasi-coherent sheaf. However, the construction of the adeles still goes through (although we have not defined it). The stalks are $\mathcal{F}_{\mathfrak{m}} = M$ and $\mathcal{F}_\eta = 0$, so

$$A(\mathcal{F}) : 0 \oplus M \to 0.$$

We may therefore ask, among all differential graded A-modules, what property characterizes those of the form $A(\mathcal{F})$ for quasi-coherent sheaves $\mathcal{F}$?

The bad property of $A(\mathcal{F})$ in the above example is that $A_{01}(\mathcal{F}) = 0$, when we would expect it to be nonzero since $A_1(\mathcal{F}) \neq 0$. Looking at how A acts on the *right*, we see that this is a consequence of $A(\mathcal{F})$ not being an induced module on the right (in the following sense):

Definition 3.1.5. Define a differential graded A-module M to be **induced** if there exists an A^0-module N such that $M \simeq N \otimes_{A^0} A$ as differential graded A-modules.

Among all differential graded A-modules, one property characterizing those of the form $A(\mathcal{F})$ for quasi-coherent sheaves $\mathcal{F}$ is that they are induced.

Proposition 3.1.6. *Let $A = A_X$ be the adelic algebra for a variety X/k. Let $\mathcal{F}$ be a quasi-coherent sheaf, and $A(\mathcal{F})$ the associated differential graded A-module. Then*

$$A(\mathcal{F}) \simeq A(\mathcal{F})^0 \otimes_{A^0} A.$$

That is, $A(\mathcal{F})$ is an induced module.

We first have a lemma.

Lemma 3.1.7. *Fix an index $I = (i_0, \dots, i_p)$. Then canonically for any A^0-module M,*

$$M_i \otimes_{A^0} A_I \simeq \begin{cases} M_i \otimes_{A_i} A_I & i = i_0 \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Let e_i denote the vector in A^0 whose ith coordinate is the unit element 1_{A_i} of the ring A_i, and whose jth coordinate is zero for $j \neq i$. By the definition of the Alexander–Whitney product (Definition 2.4.5), together with the fact that $e_{i_0} \in A^0$ acts as the left identity on A_I and $A_i e_j = \delta_{ij} A_i$,

$$A_i \otimes_{A^0} A_I \simeq \begin{cases} A_I & i = i_0 \\ 0 & \text{otherwise.} \end{cases}$$

The result then follows from the canonical isomorphism

$$(M_i \otimes_{A_i} A_i) \otimes_{A^0} A_I \simeq M_i \otimes_{A_i} (A_i \otimes_{A^0} A_I).$$

□

Proof of Proposition 3.1.6. It suffices to prove it when $X = \operatorname{Spec} R$ is affine. Let K be the simplicial set of Parshin flags on X. Let $N = A(\mathcal{F})^0 = \prod_{\mathfrak{p}} M_{\mathfrak{p}}$, the product over all scheme points of X.

We get $N_I \simeq N^0 \otimes_{A^0} A_I$. By the lemma, it suffices to prove

$$M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} A(\overline{\mathfrak{p}}K, \mathcal{O}_X) \simeq A(\overline{\mathfrak{p}}K, \widetilde{M}),$$

where $\overline{\mathfrak{p}}K$ is the simplicial subset of Parshin flags in K beginning with $\mathfrak{p}$. But this follows from quasi-coherence since for all $U \subset V$, we have

$$\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(V) \simeq \mathcal{F}(V)$$

taking the appropriate limits and products over directed systems of open sets containing Parshin flags will give the result.

Therefore we get an isomorphism as graded A-modules. The fact that it is a differential graded A-module isomorphism is clear. $\square$

Finally, we illustrate how being induced on the right differs from being induced on the left. The product structure on the adelic algebra A is highly noncommutative, as the following example illustrates. The left module structure reflects flags "from the top down", while the right module structure reflects flags "from the bottom up".

Example 3.1.8. Return to Example 3.1.3. Let U be the open set $\{\eta\}$, and let $j : U \to X$ be the inclusion. Given a K-vector space V, we get an $\mathcal{O}_U$-module $\mathcal{F} = \widetilde{V}$. The sheaf $j_!\mathcal{F}$ which extends $\mathcal{F}$ outside U by zero is a canonical example of a $\mathcal{O}_X$-module which is not quasi-coherent. Its stalks are $(j_!\mathcal{F})_{\mathfrak{m}} = 0$ and $(j_!\mathcal{F})_\eta = V$, and we may again construct a corresponding A-module

$$A(j_!\mathcal{F}) : V \oplus 0 \to 0.$$

This is an example of an A-module which is induced (on the right), but not induced on the left, as

$$0 = A^1(j_!\mathcal{F}) \not\simeq A^1 \otimes_{A^0} A^0(j_!\mathcal{F}) = A_{01} \otimes_{A_1} V.$$

Quasi-coherent sheaves will be induced on both the left and the right, as the same proof above works on the left.

3.2 Invertible A_X-modules

Let X/k be a variety, $A = A_X$ its adelic algebra. Let $K = A(\mathcal{K})$ be the adeles associated with its constant sheaf $\mathcal{K}$. We have an inclusion $A \to K$ of differential graded k-algebras.

Analogous to the situation of a Dedekind domain, Cartier divisors on X define adeles $A(D)$ which are, in a sense, fractional ideals of K. Most importantly, since Cartier divisors are locally principal, these ideals are actually graded-free (Definition 2.3.14).

Proposition 3.2.1. *Let X/k be a variety, and $A = A_X$ its differential graded algebra of adeles. Let $\mathcal{F}$ be a locally free sheaf. Then $A(\mathcal{F})$ is a graded-free A-module (Definition 2.3.14).*

Proof. Since $\mathcal{F}$ is locally free, each stalk $\mathcal{F}_x$ is a free $\mathcal{O}_{X,x}$-module. X is connected, so the stalks have constant rank. Therefore, we may find a basis β for $A(\mathcal{F})^0$ as an A^0-module.

Since $\mathcal{F}$ is quasi-coherent, $A(\mathcal{F})$ is an induced A-module. Thus, every element of $A(\mathcal{F})^n$ can be uniquely written as ma where $m \in \beta$ and $a \in A^n$. This means $A(\mathcal{F})$ is free as a graded A-module, with basis β. □

In the case of invertible $\mathcal{O}_X$-modules we define the following.

Definition 3.2.2. To each Cartier divisor D, we have canonically defined invertible $\mathcal{O}_X$-module $L(D) \subset \mathcal{K}$. We define the adeles of D to be $A(D) = A(L(D))$. By functoriality this is an ideal $A(D) \subset K$ (Definition 2.3.2). By Proposition 3.2.1, this ideal $A(D)$ is principal, generated by an element of degree 0. If we call this element t_D, then we have $t_D \in K^{\times}$.

Remark 3.2.3. Any element generating $A(D)$ as a right A-module also generates $A(D)$ as a left A-module, and vice versa.

Definition 3.2.4. It follows from the definition and the previous proposition that $A(D)$ is an invertible A-module (Definition 2.5.4). Let t be a choice of generator. Then we call t a **local parameter** for D. If $D = -V$ for closed codimension 1 subvariety V, then we call t a local parameter for V; it is a generator for the ideal $I_V = A(D) \subset A$.

This is an adelic formalization of the intuitive notion of a local parameter or local defining function for a divisor.

Proposition 3.2.5. *Let D, E be Cartier divisors with adeles $A(D)$ and $A(E)$.*

1. *$A(-D) \simeq A(D)^{\vee}$ as differential graded A-bimodules.*
2. *$A(D+E) \simeq A(D) \otimes_A^{dg} A(E)$ as differential graded A-bimodules.*
3. *$A(D) \simeq A(E)$ as differential graded A-modules if and only if $D \sim E$.*

Proof of (1). Recall (Definition 2.3.19) that $A(D)^{\vee} = \mathcal{H}om_A(A(D), A)$, where $\mathcal{H}om_A$ denotes the internal hom of right A-linear homomorphisms in the category $\underline{\mathrm{Mod}}^{dg}_{(A,A)}$ (Definition 2.3.4). Since $A(-D)$ is graded-free of rank 1, say by $t = t_{-D} = t_D^{-1} \in K^0$, we can define

$$A(-D) \to A(D)^\vee \tag{3.1}$$

$$t \mapsto (\phi_t : b \mapsto tb) \tag{3.2}$$

which by freeness extends uniquely to a right graded A-module homomorphism, $\phi_a(b) = ab$ for $a \in A(-D)$. Note that since $A(D)$ is generated on the *left* (so that $A(D)^\vee$ is a *right* module) by t^{-1}, we actually do get $\phi_a(A(D)) \subset A$, so the map is well-defined. It is injective since t is a unit in K, so $\operatorname{Ann}_A A(D) = (0)$. Since $A(D) = (t_D)$, then $\phi : A(D) \to A$ is defined by the image of t_D. Then let $a = \phi(t_D)t_D^{-1} \in A(-D)$. It follows that $\phi = \phi_a$ since $\phi_a(t_D b) = \phi_a(t_D)b = \phi(t_D)t_D^{-1}t_D b = \phi(t_D)b = \phi(t_D b)$. The map is therefore an isomorphism. □

Proof of (2). Let $A(D)A(E)$ denote the smallest submodule of K containing $\{ab \mid a \in A(D), b \in A(E)\}$. Then

$$A(D)A(E) = \left\{ \sum_i a_i b_i \mid a_i \in A(D), b_i \in A(E) \right\}$$

and $A(D)A(E) = A(D+E)$. We must prove that $A(D)A(E)$ is in fact a differential graded A-module, as a submodule of K. Leibnitz is automatic as it is inherited from K, so we must show it is closed under d. Closure under d follows from the fact that $d\left(\sum_i a_i b_i\right) = \sum_i da_i \cdot b_i + a_i \cdot db_i$ for $a_i \in A(D)^i$, together with the observations that 1) the a_i generate $A(D)$ as a group, 2) $A(D), A(E)$ are closed under d, and 3) $A(D)$ is a right A-module and $A(E)$ is a left A-module.

Now define $A(D) \otimes_A A(E) \to A(D)A(E)$ by $a_i \otimes b_i \mapsto a_i b_i$. The map is surjective since $t_D \otimes t_E \mapsto t_D t_E = t_{D+E}$, which generates $A(D)A(E) = A(D+E)$.

To show it is injective, note if $a \in A(D)$, then $a = t_D a'$; we find for $a' \in A$

$$\sum_i a_i \otimes b_i = \sum_i t_D a_i' \otimes b_i = \sum_i t_D \otimes a_i' b_i = t_D \otimes \sum_i a_i' b_i.$$

so each element in $A(D) \otimes_A A(E)$ is represented by $t_D \otimes b$ for $b \in A(E)$. If $t_D \otimes b \mapsto t_D b = 0$, then $b = 0$, since $t_D \in K^\times$. $\square$

We investigate the proof of (3) in the next section.

3.2.1 Rational equivalence and fractional ideals

In this section, we examine the relationship between rational equivalence of divisors D, E and their associated "fractional ideals" $A(D)$ and $A(E)$.

Remark 3.2.6. For arbitrary D, E, we always have an isomorphism $A(D) \simeq A(E)$ as left/right graded A-modules. Both are freely generated in degree 0 of rank 1. Thus, isomorphism in $\underline{\mathrm{Mod}}_A^{gr}$ is not enough to distinguish rational equivalence, although it is enough to distinguish rank.

In fact, we have multiple conditions which preserve rational equivalence.

Proposition 3.2.7. *Let D, E be Cartier divisors on X/k, $A(D), A(E) \subset K$ their associated fractional ideals. Then*

$$\begin{aligned} D \sim_{\mathrm{rat}} E &\iff A(D) \simeq A(E) \quad \textit{as } A\textit{-bimodules} \\ &\iff A(D) \simeq A(E) \quad \textit{as right differential graded } A\textit{-modules} \\ &\iff A(D) \simeq A(E) \quad \textit{as left differential graded } A\textit{-modules} \end{aligned}$$

The fact that there are multiple conditions follows from the following useful lemma.

Lemma 3.2.8. *Let A be the adelic algebra of a variety. Then*

$$H^0(K) = Z(K) = k(X)$$

where $Z(K)$ denotes the center of K.

Proof. The equality $H^0(K) = k(X)$ is true by construction, without appealing to the Beilinson–Huber theorem (Theorem 2.2.14).

We will prove the second equality for a surface X; generalizing to an arbitrary variety follows the same proof, but with messier subscripts. Let $t \in Z(K)$. First, we show $t \in K^0$. For arbitrary $a \in K$ we have

$$(at - ta)_{ij} = a_{ij} \cdot (t_j - t_i) - t_{ij} \cdot (a_j - a_i), \tag{3.3}$$

for distinct $i, j \in \{0, 1, 2\}$. Define a by $a_j = 1 \in K_j$ and $a_I = 0$ otherwise; by the above $t_{ij} = 0$. Therefore the coordinate of t in K^1 is zero. From this we get

$$(at - ta)_{012} = a_{012} \cdot (t_2 - t_0) - t_{012} \cdot (a_2 - a_0).$$

Again choose $a_0 = 1 \in k(X)$ and $a_I = 0$ otherwise; we get $t_{012} = 0$. Therefore $t \in K^0$. Finally, again apply (3.3) with $a_{ij} = 1$ to see that $t_i = t_j$ for all i, j. Therefore $t \in H^0(K)$. □

Part 3 of Proposition 3.2.5 will follow from:

Lemma 3.2.9. *Let $\phi : A \to K$ be a nontrivial right A-module homomorphism. Suppose*

- *ϕ is also a left A-module homomorphism, or,*

- *ϕ is a differential graded homomorphism.*

Then the image $\phi(A)$ is the fractional ideal of a divisor rationally equivalent to zero.

Proof. 1. ϕ is defined by the image of 1_A, say $\phi(1_A) = t$, and $\phi(A) = tA \subset K$. If ϕ is also a left A-module homomorphism, then for all $a \in A$,

$$at = a\phi(1_A) = \phi(a \cdot 1_A) = \phi(1_A \cdot a) = ta.$$

It follows that $t \in Z(K)$, i.e., $t \in k(X)$. The ideal tA generated by such a t is $A_{-\operatorname{div} t}$.

2. Since ϕ is a chain map, it sends $H^0(A)$ into $H^0(K)$, so sends $1_A \in H^0(A)$ to $H^0(K) = k(X)$. Again, it follows that $\phi(1_A) = t \in k(X)$, and $\phi(A) = tA = A_{-\operatorname{div} t}$.

□

The proposition now follows, say by appealing to previous results about the tensor product.

3.3 Adelic Cartier divisors

In this section we define adelic Cartier divisors on a surface X as 1-cocycles in the unit group $U(A)$ (compare with Definition 2.5.7), and show they correspond to Cartier divisors on X (Proposition 3.3.5).

Lemma 3.3.1 (Units in A_{012})**.** *Let X/k be a smooth surface, and $A = A_X$ or $\hat{A}_X$ its rational or complete differential graded algebra. There is an exact sequence*

$$1 \to A_{12}^{\times} \to A_{012}^{\times} \to \operatorname{Div} X \to 1.$$

Proof. The map $A^\times_{012} \to \operatorname{Div} X$ is given by taking valuations at each coordinate; since no coordinate is zero, we may take valuations, and sum of valuations lands in $\operatorname{Div} X$ by the adelic condition on A_{012}. The kernel is clearly $A^\times_{12}$. $\square$

Lemma 3.3.2 (Units in $\hat{A}_{12}$)**.** *Let X/k be a smooth surface, and $\hat{A}$ its complete differential graded algebra. Then*

$$\hat{A}_{12} \simeq \prod_C \hat{A}_{12,C}$$

the product over all curves (not necessarily smooth) C on X. Further, a choice of uniformizer $t = t_C$ for every C sets up isomorphisms

$$\hat{A}_{12,C} \simeq \mathbb{A}_C[[t]].$$

Thus

$$\hat{A}^\times_{12,C} = \{a \in \hat{A}_{12,C} \mid \pi_C(a) \in \mathbb{A}^\times_C\} \simeq \mathbb{A}^\times_C + t\mathbb{A}_C[[t]].$$

Proof. See [6]. $\square$

Proposition 3.3.3. *Let X/k be a surface (not necessarily smooth). Let $A = A(\mathcal{O}_X)$ be the (rational or complete) differential graded algebra of adeles, and $K = A(\mathcal{K}_X)$ its "field of fractions". Let U denote the unit functor on rings $R \mapsto R^\times$. We can apply U to the cosimplicial algebras A and K and obtain an inclusion of cosimplicial groups $U(A) \to U(K)$. Then there is a long exact sequence*

$$1 \to k^\times \to k(X)^\times \to H^0(U(K)/U(A)) \to H^1(U(A)) \to 1$$

and an isomorphism $H^2(U(A)) \simeq H^1(U(K)/U(A))$.

Proof. We have an exact sequence

$$1 \to U(A) \to U(K) \to U(K)/U(A) \to 1$$

which is

$$\begin{tikzcd}
1 \arrow[r] & A_0^\times \times A_1^\times \times A_2^\times \arrow[r] \arrow[d, "d"] & A_0^\times \times A_{01}^\times \times A_{02}^\times \arrow[r] \arrow[d, "d"] & A_{01}^\times/A_1^\times \times A_{02}^\times/A_2^\times \arrow[r] \arrow[d, "d"] & 1 \\
1 \arrow[r] & A_{01}^\times \times A_{02}^\times \times A_{12}^\times \arrow[r] \arrow[d, "d"] & A_{01}^\times \times A_{02}^\times \times A_{012}^\times \arrow[r] \arrow[d, "d"] & A_{012}^\times/A_{12}^\times \arrow[r] \arrow[d, "d"] & 1 \\
1 \arrow[r] & A_{012}^\times \arrow[r] & A_{012}^\times \arrow[r] & 1 \arrow[r] & 1.
\end{tikzcd}$$

Trivially, $H^0(U(A)) = k^\times$, $H^0(U(K)) = k(X)^\times$, and $H^i(U(K)) = 0$ for $i = 1, 2$. This gives the desired long exact sequence.

□

Remark 3.3.4. We use $U(A)$ to denote the complex associated with the units functor $R \mapsto R^\times$. We refrain from denoting this complex $A^\times$, since this represents the units of A as an algebra, which does not coincide with $U(A)$. We refrain from denoting it $K_1(A)$, since $K_1(A_{02})$ is not necessarily $A_{02}^\times$ (See [3]).

Proposition 3.3.5. *Let X/k be a surface (not necessarily smooth). Denote by $U(A)$ the complex in Proposition 3.3.3. We have a chain map*

$$\begin{tikzcd}
A_0^\times \times A_1^\times \times A_2^\times \arrow[r, "d"] \arrow[d, "\pi^0"] & A_{01}^\times \times A_{02}^\times \times A_{12}^\times \arrow[d, "\pi^1"] \\
k(X)^\times \arrow[r, "\mathrm{div}"] & \mathrm{Div}\, X.
\end{tikzcd}$$

The image under π of $Z^1(U(A))$, the group of 1-cocycles, is the group of locally principal divisors on X. If $\pi^1(a) \in \operatorname{im} \mathrm{div}$, then $a \in \operatorname{im} d$. That is, π induces an isomorphism

$$H^1(U(A)) \simeq \mathrm{Pic}(X).$$

Proof. Defining

$$\pi^0(a_0, a_1, a_2) = a_0$$

$$\pi^1(a_{01}, a_{02}, a_{12}) = \textstyle\sum_Y \operatorname{ord}_Y(a_{01}) \cdot Y$$

clearly gives a chain map.

Certainly, π^1 surjects. For any Y, choose any $f \in k(X)^\times$ such that $\operatorname{ord}_Y f = 1$, and set $a_{01,Y} = f$ and $a_{01,Z} = 1$ for $Z \neq Y$. This does not require Y to be locally principal. However, if it is, then we may simultaneously choose local defining functions $f_x \in k(X)^\times$ for every $x \in Y$. Set $a_{02,x} = f_x$ if $x \in Y$, and 1 otherwise. Then $a_{01}/a_{02} \in A_{12}^\times$ by construction, so $a = (a_{01}, a_{02}, a_{02}a_{01}^{-1})$ is a 1-cocycle. Conversely, it is easy to see that $a \in A_{01}^\times \cap (A_{02}^\times \cdot A_{12}^\times)$ is exactly the condition of local principality.

Let $a \in Z^1(U(A))$, and suppose $\pi^1(a) = \operatorname{div} f$ for some $f \in k(X)^\times$. Let $b = (b_0, b_1, b_2) \in U(A^0)$ with $b_0 = f^{-1}$, $b_{1,Y} = f^{-1}$, $b_{1,Z} = 1$ for $Z \neq Y$, and $b_2 = 1$. Consider the 1-cocycle $a\,db$. Since $(a\,db)_{01} = 1$, we have $a\,db = (1, c, c)$ for some $c \in A_{12}^\times \cap A_{02}^\times$. But $A_{12}^\times \cap A_{02}^\times = A_2^\times$ (to see this, use the fact that $A_{12} \cap A_{02} = A_2$[6] and consider vanishing along curves), so $a\,db$ is a coboundary; therefore so is a. $\square$

Remark 3.3.6. A similar statement should hold for integral schemes over k.

Proposition 3.3.7. *Let M be an invertible A-module (Definition 2.5.4). Then we may assign an adelic Chern class $c(M) \in H^1(U(A))$ (Definition 2.5.6).*

On the other hand, given a Cartier divisor D, we may assign an adelic Cartier divisor $t_D \in A_{01}^\times \times A_{02}^\times \times A_{12}^\times$. Then

$$c(A(D)) = \{\textit{class of } t_D \textit{ in } H^1(U(A))\}.$$

Proof. Follows from the definitions. □

Chapter 4: Adeles of curves and surfaces

In this chapter we will examine the adeles of curves (§4.1) and surfaces (§4.2). We will construct the S-adeles of both, generalizing the construction from global class field theory to surfaces.

4.1 The structure for curves

Let C/k be a curve, not necessarily smooth. Let A be either the rational or complete adelic algebra. There are two classes of 0-flags (see the discussion after Definition 2.2.1): closed points (type 1) and the generic point (type 0). A complete flag corresponds to a pair (x, η) consisting of a closed point and the generic point. The simplicial structure of A is illustrated by the diagram below.

$$\begin{array}{ccccc} & & A_{01} & & \\ & \diagup & & \diagdown & \\ A_0 & & & & A_1 \\ & \diagdown & & \diagup & \\ & & k & & \end{array} \tag{4.1}$$

As a complex, A is $A_0 \oplus A_1 \to A_{01}$.

Now let $A = \hat{A}_C$ be the complete adele ring. The top degree algebra, $A^1 = \mathbb{A}_C$, is exactly the classical adele ring of C (cf. Example 2.2.23). Every closed point x provides a completed local ring $\widehat{\mathcal{O}}_{C,x}$. The field of fractions of $\widehat{\mathcal{O}}_{C,x}$ is denoted $\widehat{K}_x$.

Via a choice of local uniformizer $s \in \widehat{\mathfrak{m}}_x$ for x, we have noncanonical isomorphisms $\widehat{\mathcal{O}}_{C,x} \simeq k[[s]]$ and $\widehat{K}_x \simeq k((s))$.

Since complete flags are pairs (x, η), the adeles are the subalgebra of $\prod_x \widehat{K}_x$ for which only finitely many points have poles, i.e., the restricted direct product $\prod'_x \widehat{K}_x$ with respect to the subalgebra $\mathbb{O}\mathbb{A}_C = \prod_x \widehat{\mathcal{O}}_{C,x}$.

As a ring (with coordinate-wise multiplication) with no extra differential graded structure, $\mathbb{A}_C$ itself not particularly interesting; in fact it distinguishes C in no way. This is because the completions of the local rings $\mathcal{O}_{X,x}$ are all isomorphic to $k[[s]]$; in other words, all smooth curves are locally analytically isomorphic to affine space $\mathbb{A}^1_k$ (more generally every smooth variety is locally analytically isomorphic to $\mathbb{A}^n_k$). The adelic condition is independent of C itself and so $\mathbb{A}_C$ only depends on the cardinality of k. In the concrete example of $\overline{\mathbb{F}}_p$, there are countably many geometric points in both C and $\mathbb{P}^1$, and so their adele rings are the same.

Therefore, the structure of A as a cosimplicial group or as a differential graded algebra is vital. We have a number of ways of representing it as such, the easiest is as a chain complex,

$$k(C) \oplus \mathbb{O}\mathbb{A}_C \to \mathbb{A}_C$$

where $k(C) \subset \mathbb{A}_C$ via the diagonal embedding of the inclusions $k(C) \hookrightarrow \widehat{K}_x$, and $\mathbb{O}\mathbb{A}_C = \prod_{x \in C} \widehat{\mathcal{O}}_{X,x}$. The differential is $(a_0, a_1) \mapsto a_1 - a_0$. The multiplication in this ring is given by

$$ab = (a_0 b_0, a_1 b_1, a_0 b_{01} + a_{01} b_1) \tag{4.2}$$

for $a_0, b_0 \in k(C)$, $a_1, b_1 \in \mathbb{O}\mathbb{A}_C$, and $a_{01}, b_{01} \in \mathbb{A}_C$.

More generally for any divisor D, the associated differential graded A-module $A(D)$ (Definition 3.2.2) is, as a complex,

$$k(C) \oplus \mathbb{A}_C(D) \to \mathbb{A}_C$$

where $\mathbb{A}_C(D)$ is the module associated with the divisor $D = \sum_x n_x x$,

$$\mathbb{A}_C(D) = \prod_{x \in C} s_x^{-n_x} \widehat{\mathcal{O}}_{X,x}$$

for any choice of local uniformizers s_x generating $\mathfrak{m}_x$. The right action of A is given by

$$ma = (m_0 a_0, m_1 a_1, m_0 a_{01} + m_{01} a_1)$$

for a as above, $m_0 \in k(C)$, $m_1 \in \mathbb{A}_C(D)$, and $m_{01} \in \mathbb{A}_C$.

Using the local uniformizers s_x above, we may define a generator of the invertible A-module $A(D)$ (Definition 2.5.4). Let $t_1 = \prod_x s_x^{-n_x} \in K_1^\times$. Define

$$t = (1, \textstyle\prod_x s_x^{-n_x}, 0)$$

which is an element of K_C^0. By the above description of $\mathbb{A}_C(D)$ and the multiplication (4.2), t generates $A(D)$ as a right A-module. We can compute $t^{-1} = (1, \prod_x s_x^{n_x}, 0)$, so (Definition 2.5.6)

$$\theta_t = t^{-1} dt = (0, 0, \textstyle\prod_x s_x^{-n_x} - 1)\,.$$

Therefore $c(A(D), t) = 1_{\mathbb{A}_C} + \theta_t = \prod_x s_x^{-n_x}$, considered as an element of $\mathbb{A}_C$.

4.1.1 Riemann–Roch and Serre duality for curves

The topological proof of Riemann–Roch and Serre duality for curves via the complete adeles goes back at least to Tate's thesis [24]. For now, we refer to [6, §0] for discussion; note there is a proof of Serre duality for Gorenstein curves as well.

Proposition 4.1.1 (Adelic Riemann–Roch for curves, [6, §0])**.** *Let C/k be a curve, not necessarily smooth, and $\hat{A}_C$ the complete adelic algebra. For every divisor on C, we have*

$$\chi(\hat{A}_C(D)) - \chi(\hat{A}_C) = \deg D.$$

If C is further a Gorenstein curve, then $H^0(\hat{A}_C(D)) \simeq H^0(\hat{A}_C(K - D))$ for any canonical divisor K.

In [6, §0], Fesenko considers more generally an arbitrary perfect field k. It is also clear from the proof that the statement is also true for the rational adeles A_C.

4.1.2 Quasi-isomorphisms and S-adeles

By the theorem of Beilinson–Huber (Theorem 2.2.13), $H^i(A) \simeq H^i(X, \mathcal{O}_X)$ for all i. If the adelic complex were to be analogous to singular cohomology, then we might be interested in finding an analog to finite CW complexes or finite triangulations. Recall that the Huber adeles $A(K_n, \mathcal{F})$ are defined for any subset $K_n \subset S(X)_n$ of the set of all Parshin flags (Proposition 2.2.7).

Question 4.1.2. How close can we get to a *finite* set and still capture the cohomology of X? Can we pick $K_\bullet$ independent of $\mathcal{F}$? How small can we make $K_\bullet$ if we only care about $\mathcal{F} = \mathcal{O}_X$?

In this section, we will examine the case of a curve, where we answer affirmatively that we can pick a finite set of Parshin flags that compute the cohomology of

$\mathcal{O}_X$. In fact, we may pick only those Parshin flags supported at a single point. We will examine the case of a surface in Section 4.2.2.

Definition 4.1.3 (S-adeles of a curve)**.** Let $S \subset |C|$ be an arbitrary nonempty set of closed points on C. Define a cosimplicial group B^S of dimension 1 by

$$B^S_{01} = \prod_{x \in S}{}' K_x,$$

and

$$B^S_1 = \prod_{x \in S} \mathcal{O}_x,$$

and

$$B^S_0 = \{a \in k(C) \mid a \text{ is regular outside } S\},$$

with boundary maps $\delta^1_0(a_0, a_1) = a_1$, $\delta^1_1(a_0, a_1) = a_0$, so $d^0(a_0, a_1) = a_1 - a_0$. B^S is a differential graded algebra with the usual Alexander–Whitney product.

Remark 4.1.4. B^S is *not* the Huber–Beilinson adeles of a restricted simplicial set. Restricting the set would not change B_0. Instead, it is a cosimplicial subalgebra of $a(K, \mathcal{O}_X)$ where K is the set of all Parshin flags supported at S.

Proposition 4.1.5. *There is a differential graded algebra A^S that fits into a diagram*

$$A \xleftarrow{i} A^S \xrightarrow{\pi} B^S$$

where the maps are quasi-isomorphisms.

Proof. As a cosimplicial group, A^S is

$$A^S_{01} = \prod_{x \in S}{}' K_x \oplus \prod_{x \notin S} \mathcal{O}_x,$$

$$A^S_1 = A_1, \quad \text{and,} \quad A^S_0 = C^S_0$$

with boundary map $d(a_0, a_1) = a_1 - a_0$. The product is the usual one.

The maps i and π are the inclusion and projections, therefore are homomorphisms, therefore induce chain maps since all the boundary maps are compositions of inclusions and subtractions. The proposition then follows from the following lemmas. $\square$

Lemma 4.1.6. *The inclusion $A^S \xrightarrow{i} A$ is a quasi-isomorphism if $S \neq \varnothing$.*

Proof. By the snake lemma, we must prove that the vertical right hand map is an isomorphism:

$$\begin{tikzcd}
0 \arrow[r] & A_0^S \oplus A_1^S \arrow[r, "i"] \arrow[d, "d"] & A_0 \oplus A_1 \arrow[r] \arrow[d, "d"] & A_0/A_0^S \arrow[r] \arrow[d, "\bar{d}"] & 0 \\
0 \arrow[r] & A_{01}^S \arrow[r, "i"] & A_{01} \arrow[r] & A_{01}/A_{01}^S \arrow[r] & 0.
\end{tikzcd}$$

We have $A_{01}^S \cap A_0 = A_0^S$ by definition. We must show $A_{01} = A_{01}^S + A_0$.

Because of the Riemann–Roch *inequality*, we can "move" poles into S, as we will demonstrate. Let $a \in A_{01}$, and consider its pole of highest order. We may assume it is at some $Q \notin S$ and has order m. By the Riemann–Roch inequality, $\dim nP + mQ > 0$ for $n \gg 0$, so we can reduce the order of the pole by one by subtracting an appropriate multiple of an element of $L(nP + mQ) \subset A_0$ without introducing any poles outside of S. Proceeding by induction and the fact that there are only finitely many poles, we end up in A_{01}^S.

$\square$

Lemma 4.1.7. *The projection $A^S \xrightarrow{\pi} B^S$ is a quasi-isomorphism.*

Proof. Since $A_0^S = B_0^S$, the kernel of $\pi : A^{S,0} \to B^{S,0}$ is $\prod_{x \notin S} \mathcal{O}_x$ so this follows formally from the snake lemma, without Riemann–Roch:

$$\begin{tikzcd}
0 \arrow[r] & \prod_{x\notin S} \mathcal{O}_x \arrow[r] \arrow[d] & A^{S,0} \arrow[r, "\pi"] \arrow[d, "d"] & B^{S,0} \arrow[r] \arrow[d, "\bar{d}"] & 0 \\
0 \arrow[r] & \left(\prod'_{x\notin S} K_x\right) \cap A^{S,1} \arrow[r] & A^{S,1} \arrow[r, "\pi"] & B^{S,1} \arrow[r] & 0.
\end{tikzcd}$$

The left vertical map is an equality. □

Thus the proposition is proved.

Example 4.1.8. Let $C = \mathbb{P}^1$, $S = \{\infty\}$. Then $\mathcal{O}_\infty \simeq k[[w]]$, $K_\infty \simeq k((w))$, $C \setminus S = \operatorname{Spec} k[z]$ with $w = \frac{1}{z}$, so $B_0^S = k\left[\frac{1}{w}\right]$. Therefore B^S is, as a complex:

$$k\left[\tfrac{1}{w}\right] \oplus k[[w]] \to k((w))$$

$$(a, b) \mapsto a - b$$

This is clearly surjective, and the kernel is k. Thus, $H^1(C, \mathcal{O}_C) = 0$, $H^0(C, \mathcal{O}_C) \simeq k$.

Example 4.1.9 (Weierstrass gaps)**.** Let C be a curve of genus g, $S = \{P\}$. Then $B_0^S = \bigcup_{n\geq 0} L(nP)$, and B^S as a complex is

$$\bigcup_{n\geq 0} L(nP) \oplus k[[s_P]] \to k((s_P)).$$

Since the cokernel has dimension g, there are exactly g pole orders in $k((s_P))$ that cannot be created with elements of $k(C)$ without introducing poles at other points. These are called *Weierstrass gaps.*

4.2 The structure for surfaces

Let X/k be a surface, not necessarily smooth. In this section, points refer to closed points, curves refer to irreducible dimension one (sub)varieties, not necessarily smooth. We use x to denote an arbitrary point of X. We use y and z to denote distinct curves of X.

Throughout this section, let A be either the rational or complete adelic algebra.

4.2.1 Simplicial structure, adelic Cartier and Weil divisors

The simplicial structure of A is illustrated by the diagram below.

$$\begin{tikzcd}
& A_{012} \arrow[dl, no head] \arrow[d, no head] \arrow[dr, no head] & \\
A_{01} \arrow[d, no head] \arrow[dr, no head] & A_{02} \arrow[dl, no head] \arrow[dr, no head] & A_{12} \arrow[dl, no head] \arrow[d, no head] \\
A_0 \arrow[dr, no head] & A_1 \arrow[d, no head] & A_2 \arrow[dl, no head] \\
& k &
\end{tikzcd} \tag{4.3}$$

There are three classes of 0-flags (see the discussion after Definition 2.2.1): closed points (type 2), curves, possibly singular (type 1), and the generic point (type 0). There are three classes of 1-flags: A point on a curve (type 01), a point and the generic point (type 02), and a curve and the generic point (type 01). Complete flags correspond to a triple (x, y, η) of a point $x \in y$ on a curve, and the generic point η (type 012); the horizontal direct sums of the top three rows correspond to (from the top down) A^2, A^1, and A^0. The edges correspond to inclusions, and describe the boundary maps δ_i^j. For example, δ_1^1 is the sum of the inclusions $A_0 \to A_{01}$, $A_0 \to A_{02}$, and $A_1 \to A_{12}$.

Remark 4.2.1. Note this diagram really illustrates the augmented differential graded algebra, with k in degree -1. There is an argument to be made for using the augmented algebra throughout this thesis, but we forgo the extra formality. There is an obvious generalization of this diagram to higher dimensions.

The participating rings A_I fit nicely into a lattice structure, but only when X is projective:

Proposition 4.2.2 (Fesenko [6]). *Let X/k be a projective surface, and $A = A_X$ or $\hat{A}_X$ be its rational or complete algebra. Then*

$$A_{ij} \cap A_{jk} = A_j$$

for $i, j = 0, 1, 2$.

Proof. See [4, Theorem §2]. This is a nontrivial fact and is not true for an affine surface. For more see [4], which has comments on the difficulty of generalizing to higher dimensions. $\square$

As a chain complex, we write the adelic algebra A as

$$A_0 \oplus A_1 \oplus A_2 \to A_{01} \oplus A_{02} \oplus A_{12} \to A_{012}.$$

For any divisor D, we get the A-module $A_X(D)$ (Definition 3.2.2) which as a complex is

$$A_0 \oplus A_1(D) \oplus A_2(D) \to A_{01} \oplus A_{02} \oplus A_{12}(D) \to A_{012}.$$

Both are submodules of the differential graded algebra $K = A(\mathcal{K})$,

$$A_0 \oplus A_{01} \oplus A_{02} \to A_{01} \oplus A_{02} \oplus A_{012} \to A_{012}.$$

It is easy to see that the cohomology of the modules $A(D)$ can be computed in a number of ways, for example [6, §2]

$$\begin{aligned}
H^0(A(D)) &\simeq A_0 \cap A_{12}(D) = A_0 \cap A_1(D) = A_0 \cap A_2(D) = A_1(D) \cap A_2(D) \\
H^1(A(D)) &\simeq (A_{01} \cap (A_{12}(D) + A_{02}))/(A_1(D) + A_0) \\
&\simeq (A_{12}(D) \cap (A_{01} + A_{02}))/(A_1(D) + A_2(D)) \\
&\simeq (A_{02} \cap (A_{12}(D) + A_{01}))/(A_2(D) + A_0) \\
H^2(A(D)) &\simeq A_{012}/(A_{12}(D) + A_{01} + A_{02}).
\end{aligned}$$

Theorem 4.2.3 ([6, §1])**.** *Let X/k be a smooth surface, and $A = \hat{A}_X$ the* complete *adelic algebra. As illustrated in (4.3), all participating rings are subrings of A_{012}. Then A_{012} is the restricted direct product of subrings $\mathbf{A}_y$ for every curve y, with respect to subrings $\mathbf{OA}_y$. We have noncanonical isomorphisms*

$$\begin{aligned}
\mathbf{A}_y &\simeq \mathbb{A}_y((t)) \\
\mathbf{OA}_y &\simeq \mathbb{A}_y[[t]]
\end{aligned}$$

where $\mathbb{A}_y$ is adele ring of the curve y.

Remark 4.2.4. These isomorphisms are also homeomorphisms for a suitable topology on A_{012} and $\mathbb{A}_y$ [6, §1].

Similarly, the rational adele ring is a restricted direct product of subrings $\mathbf{A}_y$ for every curve y with respect to subrings $\mathbf{OA}_y$. This follows from the inductive description of the adeles in Proposition 2.2.7(2). However, neither $\mathbf{A}_y$ nor $\mathbf{OA}_y$ are so nicely represented as Laurent series or power series.

4.2.2 S-adeles of a surface

In this section, X/k is a smooth projective surface. Let S be an arbitrary set of curves on X.

Definition 4.2.5. The algebra of **S-adeles** $A^S = A^S_X$ is the cosimplicial algebra corresponding to

$$A^S_{012} = \prod_{y\in S}{}' \mathbf{A}_y \times \prod_{y\notin S} \mathbf{OA}_y.$$

That is, to give the cosimplicial structure to A^S means defining A^S_I for all other indices I, and we set $A^S_I = A_I \cap A^S_{012}$. As a complex, A^S is

$$A^S_0 \oplus A_1 \oplus A_2 \to A^S_{01} \oplus A^S_{02} \oplus A_{12} \to A^S_{012}$$

which is simultaneously a subcomplex and differential graded subalgebra of A.

More generally for a divisor $D = \sum_y n_y y$, we define the S-adeles associated to D to be the cosimplicial group corresponding to

$$A^S_{012}(D) = \prod_{y\in S}{}' \mathbf{A}_y \times \prod_{y\notin S} \mathbf{A}_y^{n_y} = A^S_{012} + A_{12}(D),$$

with $A^S_I(D) = A_I \cap A^S_{012}(D) = A_I(D) \cap A^S_{012}$. We get a submodule of $A_X(D)$,

$$A^S_0(D) \oplus A_1(D) \oplus A_2(D) \to A^S_{01}(D) \oplus A^S_{02}(D) \oplus A_{12}(D) \to A^S_{012}(D).$$

$A^S_X(D)$ is clearly a differential graded A^S_X-module. The inclusion i,

$$\begin{tikzcd}
A^S_0(D) \oplus A_1(D) \oplus A_2(D) \arrow[r, "d^{S,0}"] \arrow[d, "i"] & A^S_{01}(D) \oplus A^S_{02}(D) \oplus A_{12}(D) \arrow[r, "d^{S,1}"] \arrow[d, "i"] & A^S_{012}(D) \arrow[d, "i"] \\
A_0 \oplus A_1(D) \oplus A_2(D) \arrow[r, "d^0"] & A_{01} \oplus A_{02} \oplus A_{12}(D) \arrow[r, "d^1"] & A_{012}
\end{tikzcd}$$

has cokernel

$$A_0/A_0^S(D) \to A_{01}/A_{01}^S(D) \oplus A_{02}/A_{02}^S(D) \to A_{012}/A_{012}^S(D).$$

Lemma 4.2.6. *We have*

1. $H^0(\operatorname{coker} i) = 0$,

2. $H^2(\operatorname{coker} i) = 0$ *if* $S \neq \varnothing$.

Proof. The first follows simply from $A_0^S(D) = A_0 \cap A_{01}^S(D)$. For the second, we must show $A_{012} = A_{01} + A_{02} + A_{012}^S$. Since $H^2(X, L(E)) \simeq A_{012}/(A_{01} + A_{02} + A_{12}(E))$, we need only find a divisor E supported on S with trivial H^2. In fact, we may pick any $E \neq 0$ supported in S; then $H^2(X, L(nE)) \simeq H^0(X, L(K - nE))$, and we may choose n sufficiently large such that $\deg((K - nE)|_y) < 0$ for some curve y in X.

□

Remark 4.2.7. The proof of (2) appeals to Riemann–Roch for curves, Serre duality for the surface, and Huber's theorem. If we want to avoid all this we can add the following requirement to S:

Condition: every point of X is the transverse intersection of curves in S.

This allows us to move poles into S at each point; in fact $A_{012} = A_{02} + A_{012}^S$.

Definition 4.2.8. We say that the set S **supports an ample divisor** if there exists an ample divisor H such that $\operatorname{supp} H \subset S$.

Then we have:

Proposition 4.2.9. *If S supports an ample divisor, then the inclusion $i : A^S \to A$ is a quasi-isomorphism.*

First we have a useful lemma.

Lemma 4.2.10. *If S supports an ample divisor H, then*

$$A^S_{012} = \sum_n A_{12}(nH) + A^S_{01} + A^S_{02}.$$

Proof. This follows from the cohomological criterion for ampleness, as we will show. Let $\gamma \in A^S_{012}$. Then $\gamma \in A_{12}(D)$ for some D such that $\operatorname{supp} D \subset S$. For $n \gg 0$, $H^2(X, L(nH)) = 0$, so using the isomorphism

$$H^2(X, L(nH)) \simeq \frac{A_{012}}{A_{01} + A_{02} + A_{12}(nH)}$$

we can write γ as a coboundary

$$a + b + c = \gamma$$

for $(a, b, c) \in A_{01} \oplus A_{02} \oplus A_{12}(nH)$. Thus $a + b \in A_{12}(D) + A_{12}(nH)$. We can write $A_{12}(D) + A_{12}(nH) = A_{12}(nH + D')$ for some D', $\operatorname{supp} D' \subset S$, so $a + b \in A_{12}(nH + D')$, and therefore $a + b$ is an element of $A_{12}(nH + D') \cap (A_{01} + A_{02})$.

Again, for $n' > n \gg 0$, $H^1(X, L(n'H + D')) = 0$. We have the isomorphism

$$H^1(X, L(n'H + D')) \simeq \frac{A_{12}(n'H + D') \cap (A_{01} + A_{02})}{A_1(n'H + D') + A_2(n'H + D')}.$$

Therefore we can write $a + b$ as a coboundary:

$$a' + b' = a + b$$

for $(a', b') \in A_1(n'H + D') \oplus A_2(n'H + D') \subset A^S_{01} \oplus A^S_{02}$. Thus we have written $\gamma = a' + b' + c$ as the sum required in the statement of the lemma. $\square$

Remark 4.2.11. More generally for an arbitrary divisor D, we have

$$A_{012}^S(D) = \sum_n A_{12}(nH) + A_{01}^S(D) + A_{02}^S(D).$$

Proof of Proposition 4.2.9. If D is any divisor such that $\operatorname{supp} D \subset S$, then by definition $A_{0*}^S(D) = A_{0*}^S$ for $* = 1, 2, 12$. Since the map i above is the identity on $A_*(D)$ for $* = 1, 2, 12$, we see that we get the *same* cokernel for all such D.

Now let H be an ample divisor supported on S. For every n, by Remark 4.2.11 we get a long exact sequence in cohomology containing the sequence

$$H^1(A(nH)) \to H^1(\operatorname{coker} i) \to H^2(A^S(nH)).$$

Notice, again, that as n varies, the middle term remains the same.

Now let $n \to \infty$. The rightmost term is eventually 0 by the lemma, and the leftmost term is eventually 0 since H is ample. Thus $H^1(\operatorname{coker} i) = 0$. $\square$

Remark 4.2.12. This same proof will also show that $i : A^S(D) \to A(D)$ is a quasi-isomorphism for any divisor D.

Chapter 5: Intersection theory of the modules $A_X(D)$

In this chapter, we investigate what the modules $A(D)$ for Cartier divisors D say about intersection theory on a surface. We obtain a global analog (Proposition 5.1.1) of the usual local statement that the intersection number of two effective divisors intersecting properly at a point x is the length of $\mathcal{O}_{X,x}/(f, g)$, where f and g are local defining functions for the divisors. In the case of the self-intersection of a curve C, we construct a sort of "projective resolution" $P \to A(-C)$ (§5.3) and show that this resolution computes the self-intersection number (§5.2).

Through this chapter, X/k is a smooth surface, and $A = A_X$ is the associated adelic algebra (rational or complete).

5.1 Proper intersection

Let D, E be effective Cartier divisors of a surface X/k intersecting properly. We have associated ideals $I_D, I_E \subset A$ corresponding to $A(-D)$ and $A(-E)$ (Definition 3.2.2). Properness for irreducible divisors means distinctness, so this is reflected in the fact that,

$$(I_D)_{12} + (I_E)_{12} = A_{12}, \text{ and,}$$

$$(I_D)_1 + (I_E)_1 = A_1.$$

On the other hand,

$$(I_D)_2 + (I_E)_2 = \bigoplus_{x \in |X|} (f_x, g_x)$$

where f_x, g_x are equations defining D, E near x. Thus if we let $I = I_D + I_E$, then

$$A/I = 0 \oplus 0 \oplus \bigoplus_{x \in D \cap E} \mathcal{O}_{X,x}/(f_x, g_x) \to 0 \oplus 0 \oplus 0 \to 0.$$

A acts on A/I through its quotient A_2, which in turn acts through its quotient $\bigoplus_{x \in D \cap E} \mathcal{O}_{X,x}$. We see A/I has finite length as a module for any of these rings, and its length is clearly the intersection number:

$$D.E = \operatorname{length}_A A/I$$

since $\operatorname{length}_A A/I = \sum_x \operatorname{length}_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x}/(f_x, g_x) = \sum_{x \in D \cap E} \operatorname{length}_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x}/(f_x, g_x)$. Nothing is happening here other than formally gathering the local intersection data into the global object, the differential graded ideal $I = I_D + I_E$.

Note also that we have $A/(I_D + I_E) \simeq A/I_D \otimes_A^{dg} A/I_E$ as differential graded A-bimodules, with straightforward proof mimicking the case of commutative rings. This discussion proves

Proposition 5.1.1. *For any Cartier divisors D, E intersecting properly,*

$$D.E = \operatorname{length}_A A/I_D \otimes_A^{dg} A/I_E.$$

5.2 Self intersection

Let C be a curve contained in X, with associated adelic algebra A_C. The inclusion $j : C \to X$ induces a projection $j^{\#} : A \to A_C$. Its kernel is the ideal $I \subset A$ associated with C.

Now let C' be a second distinct curve on X with associated ideal I'. Then $A/I \otimes_A A/I'$ computes the intersection of C and C' via its length (§5.1.1). However, $A/I \otimes_A A/I \simeq A/I$, which is not a finite length A-module.

Definition 5.2.1. Let $t \in A$ be a local parameter for C (Definition 3.2.4), so $I = tA = A(-C)$ is the ideal associated with C. Define a differential graded A-module P_t as follows. As a graded A-module, $P_t = A[T]/T^2$ where T is a formal parameter of degree -1. The differential structure on P_t is defined by

$$dT = t - T\theta_t$$

where $\theta_t \in A^1$ is as in Definition 2.5.6.

Remark 5.2.2. We can call $P_t = A[T]/T^2$ the deformation module/ring of the ideal tA. This seems to play a role similar to the local theory, see Theorem 5.2.4.

Remark 5.2.3. A choice of local parameter t also defines a divisor $E \sim_{\mathrm{rat}} C$ as follows. By assumption, $t \in A^0$, and multiplying t by an element of $(A^0)^\times$, we may assume t is of the form

$$t = (1, t_1, t_2).$$

The element $t_1 \in A_1$ has a coordinate at C which is some rational function $f \in k(X)$.

Then div $f = C - E$ for some divisor E intersecting C properly, since t is a generator of I. Further, $t' = (1, j^{\#}(t_2/t_1))$ is a local equation in A_C for $E|_C$.

Theorem 5.2.4. *There is an isomorphism of differential graded A-modules*

$$P_t \otimes_A^{dg} A/I_C \simeq A_C \oplus A_C(E|_C)[1].$$

Proof. We have a surjective quasi-isomorphism

$$P_t \to A_C$$

defined by sending both t and T to 0. This is exactly the same as the exact sequence

$$0 \to I \to A \to A_C \to 0$$

where $P_t \simeq [I \to A]$.

Apply the functor $- \otimes_A A/I$ to the A-module P_t. Since $t \mapsto 0$, we see that the differential becomes

$$dT = -Tj^{\#}(t^{-1}dt).$$

The only nontrivial component of $j^{\#}(t^{-1}dt)$ is the 12 component, where

$$(t^{-1}dt)_{12} = t_1^{-1}(t_2 - t_1) = \frac{t_2}{t_1} - 1.$$

So $(\overline{t^{-1}dt})_{01} = j^{\#}(t_2/t_1) - 1$. But $t' = (1, j^{\#}(t_2/t_1))$ is a local equation in A_C for $E|_C$, and since $\theta_{t'} = t'_1 - 1$, by Proposition 2.5.12(2), we get a splitting (!) of (5.1) after tensoring. □

Remark 5.2.5. The above construction is analogous to taking the following projective resolution:

$$0 \to \mathcal{I}_C \to \mathcal{O}_X \to \mathcal{O}_C \to 0$$

and applying $-\otimes \mathcal{O}_C$ to get the complex of sheaves $[\mathcal{I}_C|_C \to \mathcal{O}_C]$. Computing the cohomology computes the higher Tor groups, which computes the intersection. The difference is that the complex of sheaves $[\mathcal{I}_C \to \mathcal{O}_X]$ is an actual object P_t in the category of differential graded A-modules.

By the adelic statement of Riemann–Roch for the curve (Proposition 4.1.1), we have

$$\chi(A_C \oplus A_C(D)[1]) = \deg D.$$

Therefore

$$C^2 = \chi(P_t \otimes_A^{dg} A/I_C).$$

Combining the results for proper and self-intersection of divisors, it is easy to derive the following:

Corollary 5.2.6. *Let C and P_t be as above, and let D be any effective divisor with associated ideal I_D. Then*

$$C.D = \chi(P_t \otimes_A^{dg} A/I_D).$$

5.3 Property (P) and P-resolutions

We provide an interpretation of the module P_t in the previous section. The material in this section is from the Stacks project's chapter on differential graded algebras [25, Tag 09JD], which constructs from $\underline{\mathrm{Mod}}_{(A,d)}$ a homotopy category $K(\underline{\mathrm{Mod}}_{(A,d)})$, followed by a derived category $D(A, d)$ in the natural way. The category $D(A, d)$ is simply the category of differential graded A-modules with quasi-isomorphisms inverted.

Definition 5.3.1. [25, Tag 09KK] A differential graded A-module P **has property (P)** if it has a filtration $P \supset \cdots \supset F_1P \supset F_0P \supset 0$ such that $P = \bigcup_i F_iP$ and each successive quotient $F_{i+1}P/F_iP$ is isomorphic, as a differential graded A-module, to $A[k]$ for $k \in \mathbb{Z}$.

Remark 5.3.2. Technically speaking, the condition from [25] is that each quotient $F_{i+1}P/F_iP$ is isomorphic to a direct sum of copies of $A[k]$. However, we will just be considering coherent sheaves, which correspond to finitely generated A-modules. In this case, we can arrange to have each quotient in the filtration be a single copy of $A[k]$.

Example 5.3.3. Clearly dg-free (Definition 2.3.14) differential graded A-modules have property (P). Remember that if $I_D \subset A$ is the ideal corresponding to an effective Cartier divisor D, then I_D is free as a graded A-module, but *not* dg-free unless $D \sim_{\text{rat}} 0$ (Proposition 3.2.7). Thus, "locally free" objects do not generally satisfy property (P).

In the category $\underline{\text{Mod}}_{(A,d)}$, dg-free A-modules are not projective, as the following example shows:

Example 5.3.4. Let F be the dg-free A-module generated by a formal parameter X, so that $F = XA$. We must have $dX = 0$ by dg-freeness. Further, any differential graded homomorphism $\phi : xA \to M$ must satisfy $d\phi(x) = 0$; in general we might have $Z^0(M) = 0$, in which case $\text{Hom}_A^{dg}(F, M) = 0$.

Example 5.3.5. Construct the free A-module P generated by formal symbols X, Y. Define a differential graded structure on P by setting $dX = Y$ and $dY = 0$, and define the degrees as $|X| = 0$ and $|Y| = 1$. Then P is isomorphic to $A \oplus A$ with differential

$$d(a, b) = (da, a - db),$$

the isomorphism being given by $Xa + Yb \mapsto (a, b)$.

Proposition 5.3.6. *We have,*

1. *For any differential graded A-module M, $\mathrm{Hom}_A^{dg}(P, M) = M$ as abelian groups.*

2. *P satisfies property (P).*

Proof. The first part is clear. For the second, we can write P as an extension in $\underline{\mathrm{Mod}}_A^{dg}$,

$$0 \to N \to P \to A \to 0$$

with $P \to A$ defined by $Xa + Yb \mapsto a$. The kernel N is isomorphic to $A[-1]$; for example we can choose the isomorphism $A[-1] \to N$, $1 \mapsto Y$. $\square$

More generally, for any graded-free A-module M of rank 1, choose a generator t. Then P is also an extension

$$0 \to M[-1] \to P \to M \to 0$$

with $P \to M$ defined by $Xa + Yb \mapsto t \cdot a + dt \cdot b$.

Proposition 5.3.7. *The category $\underline{\mathrm{Mod}}_A^{dg}$ has enough (P) objects. That is, for every differential graded A-module M, there exists a surjective quasi-isomorphism $P \to M$ such that P has property (P).*

Proof. See [25, Tag 09KP] $\square$

Let P_t be the module defined in Definition 5.2.1. We have an exact sequence

$$0 \to A \to P_t \xrightarrow{\pi} A[1] \to 0 \tag{5.1}$$

given by $\pi : T \mapsto t$. Thus P_t satisfies property (P). The map $P_t \to A/I_C$ given in the previous section is an example of a (P) resolution of the module A/I_C.

Chapter 6: The simplicial Milnor K-algebra

As mentioned in the introduction, Gorchinskii [11, 10] and Braunling [2] have constructed adelic resolutions of K-theoretic sheaves. Our presentation is based on Budylin [3], who also defines adelic Chern classes for rank 2 bundles. The adelic Bloch–Quillen formula is due to Budylin (Theorem 6.2.8). Our contribution is to interpret the intersection pairing as a cup product and prove that Budylin's map is a ring homomorphism (Theorem 6.2.11).

In §6.1 we define the cosimplicial ring $K^M(A)$, and endow it with a canonical map to the Gersten resolution (§6.2). Finally, we relate the cup product in $K^M(A)$ with the numerical intersection pairing (§1).

We should also mention the paper by Osipov [20], which has a similar definition of complete K-adeles and constructs a Gysin map in the relative situation of a surface mapping to a curve.

Throughout this chapter, X/k is a surface (not necessarily smooth) with associated adelic algebra $A = A_X$.

6.1 The Milnor K-theory of A_X

In this section we will define the Milnor ring $K^M(A)$, a cosimplicial object in $\underline{\text{Ring}}^{gr}$, the category of graded rings. Under the monoidal Dold–Kan correspondence (Section 2.4), it is a differential graded ring.

Normally, the Milnor K-groups of non-fields are not well-behaved. However, in the case of a local ring with infinite residue field, Nesterenko and Suslin [19] have shown the Milnor K-groups exhibit some desirable properties. Since A_X is built out of products of local rings, all of which contain an infinite field, it should not be too surprising that we can transfer these results over to the cosimplicial algebra A_X.

Let R be a commutative ring and define the Milnor ring $K^M(R)$ to be the quotient of the tensor algebra $T(R) = \bigotimes_n (R^\times)^{\otimes n}$ by the ideal generated by elements of the form $a \otimes (1-a)$ with $a, 1-a \in R^\times$. Then $K^M(R)$ is a graded ring in the usual way. Further, K^M is a covariant functor $\underline{\text{Ring}} \to \underline{\text{Ring}}^{gr}$. We denote elements as (a, b) for $a, b \in R^\times$ (and (a, b, c), etc.).

Lemma 6.1.1. *If $R = A_{012}$, then $K^M(R)$ is graded commutative. In other words, we have skew-symmetry $(a, b) = -(b, a)$ in $K_2^M(R)$ (in general, $(a_1, \ldots, a_n) = (-1)^{\operatorname{sgn}\sigma}(a_{\sigma(1)}, \ldots, a_{\sigma(n)})$).*

Proof. Let $R = A_{012}$. We follow [19] to prove $(a, -a) = 0$ for all $a \in R^\times$. Here we do not have local rings. However, each local factor (See §2.2.2) R_Δ of A_{012} contains an infinite field, and the product of all these fields has no adelic restriction. That is, $S = \prod_{x,C} k \subset A_{012}$ and $S^\times = \prod_{x,C} k^\times$.

If $a \in R^\times$ and $1 - a \notin R^\times$, then we can find an infinite family $b \in S^\times$ such that $ab, 1 - ab \in R^\times$. We just need to avoid the case that $a_{x,C} \equiv 1 \bmod t_C$, so we can choose $b_{x,C} \in k^\times$ such that $a_{x,C} \cdot b_{x,C} \not\equiv 1 \bmod t_C$.

□

More generally, the same proof shows:

Proposition 6.1.2. *Let X/k be a noetherian scheme over an infinite field. Let R be any ring A_I appearing in the Huber–Beilinson adeles. Then R has many units (in the terminology of [19]).*

Proof. Every ring R contains a ring of the form $S = \prod_\Delta k$ (an *unrestricted* product), which has many units, and there is an inclusion $S^\times = \prod_\Delta k^\times \hookrightarrow R^\times$. □

Definition 6.1.3. Let X/k be a surface or a curve, with associated (rational or complete, reduced or nonreduced) algebra $A = A_X$. This is a cosimplicial object in the category of algebras. Therefore by functoriality we can form the **simplicial Milnor K-algebra** $K^M(A)$, a cosimplicial object in the category of abelian groups. In other words, $(K^M(A))^n = K^M(A^n)$, with face maps

$$K^M(\delta_i^n) : K^M(A^{n-1}) \to K^M(A^n)$$

and degeneracy maps

$$K^M(\sigma_i^n) : K^M(A^{n+1}) \to K^M(A^n).$$

Proposition 6.1.4. *The simplicial Milnor K-algebra $K^M(A)$ is a differential graded $\mathbb{Z}$-algebra. $K^M(A)$ is graded commutative, that is, $ab = (-1)^{|a||b|}ba$ for all homogeneous elements $a, b \in K^M(A)$. Let $a, b \in K_1^M(A^1)$ be elements of degree 1. They* *have coordinates a_{ij}, b_{ij} for distinct $i, j \in \{0, 1, 2\}$. Then from the definition of the cup product (§2.4), we have*

$$ab = (a_{01}, b_{12}) \in K_2^M(A^2).$$

Proof. This follows from the monoidal Dold–Kan correspondence (§2.4), while graded commutativity follows from Lemma 6.1.1. □

Remark 6.1.5. In Definition 2.5.7 and §3.3 we defined adelic Cartier divisors as cocycles in a cosimplicial group of units. The group $U(A)$ sits inside the simplicial Milnor K-algebra via $U = K_1^M$. By Proposition 3.3.5, we have for any smooth surface

$$H^1(K_1^M(A)) \simeq \mathrm{Pic}(X).$$

6.2 The relationship between $K^M(A_X)$ and the Gersten complex

We establish the relationship between the Milnor K-ring and the Gersten resolution. A similar relationship is mentioned in Osipov [20] for complete adeles. Gorchinskii [11, Theorem 1.1] has a more robust construction of adeles of K-theoretic sheaves. We follow Budylin [3], who proves the Bloch–Quillen formula for $K^M(A)$ (Theorem 6.2.8).

Let X be a smooth surface. Recall that for every curve C (not necessarily smooth) and closed point $x \in C$, we have boundary maps in Milnor K-theory which we denote

$$K_2(k(X)) \xrightarrow{\partial_C} k(C)^\times \xrightarrow{\partial_x} \mathbb{Z}.$$

These maps are the tame symbol and ord map. Further, let $\pi_{x,C} : K_2^M(A_{012}) \to K_2(k(X))$ be the map induced from the projection $a \mapsto a_{x,C}$.

Lemma 6.2.1. *Let $Z_0(X) = \bigoplus_x \mathbb{Z}$ denote the group of dimension 0 algebraic cycles on X. Then there is a unique well-defined map $\partial : K_2^M(A_{012}) \to Z_0(X)$ such that the following commutes for every Parshin flag $x \in C$:*

$$\begin{tikzcd} K_2^M(A_{012}) \arrow[r, "\pi_{x,C}"] \arrow[d, "\partial"] & K_2(k(X)) \arrow[d, "\partial_x \circ \partial_C"] \\ Z_0(X) \arrow[r, "\pi_x"] & \mathbb{Z}. \end{tikzcd}$$

Remark 6.2.2. This is, essentially, proved in [3] (note that in this proof, Budylin uses Parshin's tame symbol). We provide a proof using the more familiar maps from Milnor K-theory. Further, Budylin uses the common type of numerical adelic condition ([3, Lemma 12]) we wish to avoid, as it is not tractable above dimension two.

Proof. We may consider symbols of the form (f, g) for $f, g \in A_{012}^\times$. If such a map exists it must be unique via the splitting $\mathbb{Z} \to Z_0(X)$ for each choice of x, so we must only show that it is well-defined, i.e., that the following sum over all flags $x \in C$ is finite:

$$\sum \partial_x \circ \partial_C(f_{x,C}, g_{x,C}).$$

It is a simple observation that $A_{012}^\times = A_{01}^\times \cdot A_{12}^\times$; the observation here is simply that an adele has finitely many poles along curves, and we may choose our local uniformizers as elements of A_{01}. Compare this with Lemma 3.3.1.

By linearity we may therefore consider three cases.

($f, g \in A_{12}^{\times}$) In this case, f and g are units along every curve C. Therefore $\partial_C(f_{x,C}, g_{x,C}) = 1$, so

$$\partial_x \circ \partial_C(f_{x,C}, g_{x,C}) = 0 \quad \text{for all } x \text{ and } C.$$

($f, g \in A_{01}^{\times}$) The adeles f, g have associated Weil divisors D, E. If $C \notin |D| \cup |E|$, then $\partial_C(f, g) = 0$ as f, g are units along C. Therefore for any fixed x, we have a representation as a finite sum

$$\sum \partial_x \circ \partial_C(f_{x,C}, g_{x,C}) = \sum_{C \in |D| \cup |E|} \partial_x \circ \partial_C(f_{x,C}, g_{x,C}).$$

For any of the finitely many curves $C \in |D| \cup |E|$, $\partial_C(f_{x,C}, g_{x,C})$ is a rational function on C, since $f, g \in A_{01}$. Therefore there are only finitely many points $x \in C$ with nonzero residue.

($f \in A_{01}^{\times}$, $g \in A_{12}^{\times}$) By linearity, we may reduce to the case of a single curve C for which $\operatorname{ord}_C f_C = 1$, and $\operatorname{ord}_D f_D = 0$ for all $D \neq C$. Then $\partial_D(f_{x,D}, g_{x,D}) = 1$ for all x and all $D \neq C$. On the other hand, for $x \in C$, we have $\partial_C(f_{x,C}, g_{x,C}) = g_{x,C}|_C$, a rational function along C for which only finitely many points x have zeros and poles. □

Lemma 6.2.3. *The following diagram commutes.*

$$\begin{tikzcd} K_2^M(A_{01}) \arrow[r] \arrow[d, "\sum_C \partial_C"] & K_2^M(A_{012}) \arrow[d, "\partial"] \\ \bigoplus_C k(C)^{\times} \arrow[r] & \bigoplus_x \mathbb{Z}. \end{tikzcd}$$

Proof. We may index the set of all flags $\{x \in C\}$ as curves first. □

The lemma shows that tame symbols of type 01 agree with the Gersten complex. To complete the proof, we need to know what happens to tame symbols of types 02 and 12. Since ∂ is trivial on $K_2^M(A_{12})$ (see the proof of Lemma 6.2.1), that only leaves the case of type 02. Intuitively, this is a statement about the commutativity of the intersection pairing in terms of tame symbols, as in Kresch [17].

Lemma 6.2.4. *The map ∂ on the image of $K_2^M(A_{02})$ in $K_2^M(A_{012})$ is trivial.*

Proof. We may as usual consider symbols of the form (f, g) for $f,\ g \in K_2^M(A_{02})$. Further, by definition of A_{02}, we may fix x and consider just the coordinates at x. In this case, $f,\ g \in K_2(k(X))$, and we are showing that the following map is zero

$$K_2(k(X)) \xrightarrow{\sum \partial_C} \bigoplus_{C \ni x} k(C)^\times \xrightarrow{\partial_x} \mathbb{Z}.$$

Consider $k(X)$ as the field of fractions of the UFD $\mathcal{O}_{X,x}$. Factoring f and g, we may assume both are prime and are reduced to three cases:

((f, g), f, g relatively prime) Let C and D be the curves defined by f and g respectively. Then $\partial_x \circ \partial_C(f, g) = \operatorname{ord}_x g^{-1}|_C$ and $\partial_x \circ \partial_D(f, g) = \operatorname{ord}_x f|_D$. So we are reduced to showing $\operatorname{ord}_x f|_D = \operatorname{ord}_x g|_C$. However, both compute the length of $A/(f+g)A$, so they are equal. This is essentially the easy (proper) case in the proof of Fulton [9, Theorem 2.4].

((f, f), f arbitrary) Unlike the global case in [9, Theorem 2.4], here we have a trivial statement. At one fixed point on a curve, the self intersection of the curve with itself looks trivial. So if f describes the curve C near x, then $\partial_x \circ \partial_C(f, f) = \operatorname{ord}_x(-f/f) = 0$.

((f, u), u a unit) Then $\partial_x \circ \partial_C(f, u) = \operatorname{ord}_x(u^{-1}) = 0$. $\square$

Remark 6.2.5. The proofs given could probably be made more general for normal surfaces; since X is smooth we give the shortest proof. Further, we are hopeful that the proof as stated should extend in some way to codimension two cycles on an arbitrary variety.

Remark 6.2.6. Budylin [3] cites Parshin reciprocity as the reason that $\partial : K_2^M(A_{012}) \to Z_0(X)$ factors through the group of 2-coboundaries. However, this seems to be an error, as Parshin reciprocity along curves requires the surface to be projective, a hypothesis which is not stated in Theorem 1 *loc. cit.* In fact, Parshin reciprocity along curves is not really required, as such coboundaries become rational equivalences.

Lemma 6.2.7. *Let X/k be a surface. There exists a chain map*

$$\begin{tikzcd}
K_2^M(A^0) \arrow[r] \arrow[d] & K_2^M(A^1) \arrow[r] \arrow[d] & K_2^M(A^2) \arrow[d, "\partial"] \\
K_2(k(X)) \arrow[r] & \bigoplus_C k(C)^\times \arrow[r] & \bigoplus_x \mathbb{Z}
\end{tikzcd}$$

from $K^M(A)$ to the second Gersten complex of X.

Expanding the groups $K_2^M(A^0)$, $K_2^M(A^1)$, $K_2^M(A^2)$, the chain map looks like

$$\begin{tikzcd}
K_2^M(A_0) \oplus K_2^M(A_1) \oplus K_2^M(A_2) \arrow[r] \arrow[d] & K_2^M(A_{01}) \oplus K_2^M(A_{02}) \oplus K_2^M(A_{12}) \arrow[r] \arrow[d] & K_2^M(A_{012}) \arrow[d, "\partial"] \\
K_2(k(X)) \arrow[r] & \bigoplus_C k(C)^\times \arrow[r] & \bigoplus_x \mathbb{Z}.
\end{tikzcd}$$

The map in degree 0 is

$$K_2^M(A_0) \oplus K_2^M(A_1) \oplus K_2^M(A_2) \to K_2(k(X))$$

$$(a, b, c) \mapsto a$$

since $A_0 = k(X)$ by definition and $K_2^M(k(X)) = K_2(k(X))$.

The map in degree 1 is

$$\begin{aligned} K_2^M(A_{01}) \oplus K_2^M(A_{02}) \oplus K_2^M(A_{12}) &\to \bigoplus_C k(C)^\times \\ (a, b, c) &\mapsto \sum_C \partial_C(a). \end{aligned}$$

This is well-defined by the adelic condition on A_{01}.

The map in degree 2 is the map ∂ defined in Lemma 6.2.3.

By the previous lemmas, we have a chain map. Recall the boundary maps of the Gersten complex are 1) the sum of all tame symbols $K_2(k(X)) \to K_1(k(C))$, and 2) the sum of all ord maps $K_1(k(C)) \to K_0(k(x))$.

Theorem 6.2.8 (Budylin, [3]). *Let X/k be a smooth surface, $A = A_X$ its rational differential graded algebra, and $K^M(A)$ the associated Milnor K-ring. Then via the chain map in Lemma 6.2.7,*

$$H^2(K_2^M(A)) \simeq CH^2(X).$$

Finally, we show that the cup product describes the intersection pairing.

Proposition 6.2.9. *Let s, t be cocycles in $(A^1)^\times$. They define classes in $H^1(K_1^M(A^1))$ and in turn give Weil divisors C, D on X (Proposition 3.3.5). Their cup product st defines an element of $K_2^M(A^2)$. Then*

$$\partial(st) = C.D$$

within $CH^2(X)$.

Proof. For the definition of the product on $K^M(A)$, refer to Proposition 6.1.4.

It suffices to prove the proposition for effective prime divisors by linearity. Proper intersection is a direct calculation as follows.

Recall that we write $s = (s_{01}, s_{02}, s_{12}) \in A_{01}^\times \times A_{02}^\times \times A_{12}^\times$ (and similarly for t). The cup product st is the element of $K_2^M(A_{012})$ whose coordinate at x, E is the symbol $((s_{01})_{x,E}, (t_{12})_{x,E}) \in K_2(k(X))$ for all flags $x \in E$.

To say $t_{02} \in A_{02}^\times$ means we have, for every point x, a local uniformizer $t_x \in k(X)^\times$ describing the divisor D; the coordinate $(t_{02})_{x,E} = t_x$ is constant with respect to the curve E. We have $(t_{01})_C \in \mathcal{O}_{X,C}^\times$, call this element u. So for all $x \in C$, $(t_{12})_{x,C} = \frac{(t_{02})_{x,C}}{(t_{01})_{x,C}} = t_x u^{-1}$.

Since $\mathrm{ord}_E(s_{01}) = 0$ for $E \neq C$ and t_{12} is a unit along all curves, the only terms contributing to ∂ come from C:

$$\partial(st) = \sum_{x \in C} \partial_x(\partial_C((s_{01})_{x,C}, (t_{12})_{x,C}).$$

Each term simplifies to $\partial_x((t_{12})_{x,C}|_C)$. Then

$$\partial(st) = \sum_{x \in C} \partial_x(u^{-1} t_x|_C).$$

The element $\sum_{x \in C} \partial_x(u|_C)$ is a rational equivalence on X. The remaining sum $\sum_{x \in C} \partial_x(t_x|_C)$ gives the local intersection multiplicities at the points $x \in |C| \cap |D|$, therefore its class in $CH^2(X)$ agrees with $D.C$.

Suppose we do not have proper intersection, and suppose C is irreducible and effective. We want to show that $\partial(s^2) = C^2$ in $CH^2(X)$. The point is that an adelic divisor contains a global description of C (see Remark 5.2.3). Define $f = (s_{01})_C \in k(X)^\times$. Then $\operatorname{div} f = C - E$ for some divisor E intersecting C properly.

We did not choose f, rather it is part of the data of s; elements cohomologous to s will have a different function f.

Then similar to the previous case, only terms from C will contribute. Now, $(s_{12})_{x,C} = \frac{(s_{02})_{x,C}}{(s_{01})_{x,C}}$ but $(s_{01})_{x,C}$ is not a unit as before. We get

$$\partial(ss) = \sum_{x \in C} \partial_x \circ \partial_C((s_{01})_{x,C}, (s_{02})_{x,C}(s_{01})^{-1}_{x,C}) = \sum_{x \in C} \partial_x \circ \partial_C((s_{01})_{x,C}, (s_{02})_{x,C})$$

since $\partial_x \circ \partial_C((s_{01})_{x,C}, (s_{01})_{x,C}) = 0$. The remaining terms describe the proper intersection of E with C, and we reduce to the case from before. $\square$

Remark 6.2.10. One way to think about this cup product is to recall that for any variety, $A_{01}^\times$ captures the data of the Weil divisors, as it is a restricted direct product over them. On the other hand, $A_{02}^\times$ in a sense captures the data of the Cartier divisors, as it a restricted direct product over points of codimension greater than 1. Then $A_{12}^\times$ mediates between the two, and the cup product description therefore resembles the process of intersecting a Cartier divisor with a subvariety, $D.[D']$ as in Fulton [9, Chapter 2].

Finally, we can define a new graded ring by $R^0 = \mathbb{Z}$, $R^n = H^n(K_n^M(A^n))$ for $n = 1, 2$, and $R^n = 0$ for $n > 2$. In other words, we take diagonal elements of the graded ring $K^M(A)$, and take cohomology to get a graded ring $R = H^\bullet(K^M(A))$.

Theorem 6.2.11. *Let X/k be a smooth surface. The map*

$$H^\bullet(K^M(A)) \xrightarrow{\phi} CH^\bullet(X)$$

is an isomorphism of rings.

Proof. This is a combination of Theorem 6.2.8, Proposition 6.2.9, and Proposition 3.3.5. □

Appendix: Riemann–Roch for surfaces and the Hodge index theorem

We review the adelic proof of the Riemman–Roch theorem, and show how the Hodge index theorem and Riemann hypothesis for curves follow. The results of this section are classical, and follow Mattuck–Tate [18] and Grothendieck [12]. Our proof of the Riemann–Roch theorem follows Fesenko [6], with an interpretation of the intersection pairing as in the previous chapter.

1 Parshin's adelic intersection pairing and Parshin reciprocity

We review Parshin's intersection pairing on the adeles via the tame symbol. The main source for this is Parshin [22].

In this section, X/k is a smooth projective surface, and $A = \hat{A}_X$ are the complete adeles of X.

We follow Parshin, but consider the intersection pairing as a cup product on the cosimplicial K-ring (via the Alexander–Whitney product). The main difference is that while Parshin's pairing takes values in $\mathbb{Z}$, we consider the Chern classes as elements in $H^\bullet(K^M(A))$.

Since we use the cup product, our definition differs slightly from Parshin's. Parshin defines the intersection number as (the symbol $(\cdot,\cdot)_{x,y}$ is defined in Definition

1.3)

$$\sum_{x,y}(t_{02}, s_{01})_{x,y}$$

with $t_{02} \in A_{02}^{\times}$, $s_{01} \in A_{01}^{\times}$. Thinking simplicially, we view these instead as Chern classes represented by elements of $(A^1)^{\times}$. In this case, the Alexander–Whitney product pairs 01 with 12, and 02 is ignored. Our definition is, for $t, s \in (A^1)^{\times}$,

$$ts = (t_{01}, s_{12}) \in K_2^M(A_{012}).$$

To obtain the intersection *number* we therefore have

$$\sum_{x,y}(t_{01}, s_{12})_{x,y}.$$

Note that (t_{01}, s_{12}) and (t_{02}, s_{01}) are not equal as elements of $K_2^M(A)$. However, by Parshin reciprocity the above sums agree.

Definition 1.1. Let K be either a global field $k(C)$ of a curve, or a local field K_x. We denote by ∂_x the valuation associated with a point $x \in C$.

Let K be either a global field $k(X)$ or 2-local field $K_{x,y}$. We denote by ∂_y the tame symbol associated with a nonsingular curve $y \subset X$.

Both maps are the first two cases of residue maps in Milnor K-theory, that is, $\partial_x : K_1(K) \to K_0(\overline{K})$ and $\partial_y : K_2^M(K) \to K_1(\overline{K})$.

The composition is important, in the case of a complete, smooth Parshin flag on a surface:

Definition 1.2. Let K be either a global field $k(X)$ of a surface or a 2-local field $K_{x,y}$. Both are the local factors corresponding to a complete, smooth Parshin flag on a surface.

We denote by $(\cdot,\cdot)_{x,y}$ the composition

$$(f,g)_{x,y} = \partial_x \circ \partial_y(f,g)$$

which is a symbol $(\cdot,\cdot)_{x,y} : K_2(K) \to K_0(k(x))$.

Definition 1.3. The symbol $(\cdot,\cdot)_{x,y} = \partial_x \circ \partial_y$ is defined, more generally, when y is a singular curve on a surface. See [22].

Theorem 1.4 (Parshin reciprocity)**.** *Let X/k be a smooth projective surface, and A its (rational or complete) adelic differential graded algebra.*

1. *If $a, b \in A_{01}^{\times}$, then $\sum_{x,y}(a,b)_{x,y} = 0$.*

2. *If $a, b \in A_{02}^{\times}$, then $\sum_{x,y}(a,b)_{x,y} = 0$.*

Proof. The first follows, essentially, from Weil reciprocity for projective curves: for a fixed y, $\sum_{x,y}(a,b)_{x,y} = 0$. See [22] for the rest. $\square$

Note that we also have: if $a, b \in A_{12}^{\times}$, then $\sum_{x,y}(a,b)_{x,y} = 0$. In fact, $(a,b)_{x,y} = 0$ for all x, y, as a and b have no poles along divisors, and therefore no residues. We conclude,

Corollary 1.5. *The residue map $K_2^M(A) \to K_0(k)$ is trivial on coboundaries.*

Proof. Recall $K_2^M(A)$ as a complex is

$$K_2^M(A_{01}) \oplus K_2^M(A_{02}) \oplus K_2^M(A_{12}) \xrightarrow{d} K_2^M(A_{012}).$$

$\square$

Definition 1.6 (Parshin's intersection pairing)**.** Let $s, t \in (A^1)^\times$ represent adelic Cartier divisors. We define their intersection number by

$$[s,t] = -\sum_{x,y} (s_{01}, t_{12})_{x,y}.$$

Equivalently, we can consider s, t as 1-cocycles in the differential graded ring $K^M(A)$. Then their product st is a 2-cocycle, and an element of $K_2^M(A)$. Applying the tame map $\partial : K_2^M(A) \to K_0(k)$ gives the intersection number: $[s,t] = \partial(st)$. In other words, the intersection number fits into a diagram

$$K(A) \otimes_{\mathbb{Z}} K(A) \xrightarrow{\text{cup}} K(A) \xrightarrow{\partial} K(k).$$

Remark 1.7. This definition differs from Parshin's original definition [22, §2], but both agree by Parshin reciprocity along curves (Theorem 1.4(1)):

$$\begin{aligned} -\sum_{x,y} (s_{01}, t_{12})_{x,y} &= -\sum_{x,y} (s_{01}, t_{02})_{x,y} + (s_{01}, t_{01})_{x,y} \\ &= \sum_{x,y} (t_{02}, s_{01})_{x,y}. \end{aligned}$$

Remark 1.8. Since $t_{12} \in A_{12}$, it has no poles, and therefore the intersection pairing simplifies drastically,

$$(s_{01}, t_{12})_y = (-1)^{0 \cdot \operatorname{ord}_y s_{01}} \frac{s^0_{01,y}}{t_{12,y}^{\operatorname{ord}_y s_{01}}},$$

so for a smooth Parshin flag $x \in y$,

$$(s_{01}, t_{12})_{x,y} = \partial_x((s_{01}, t_{12})_y) = \operatorname{ord}_y s_{01} \cdot \partial_x(t_{12,y}) = \operatorname{ord}_y s_{01} \cdot v_x(t_{02,x}/t_{01,y}). \tag{1}$$

Proposition 1.9. *Let X/k be a smooth projective surface. Let C, D be Cartier divisors, and $s, t \in (A^1)^\times$ their associated adelic Cartier divisors. Then the adelic* *intersection pairing agrees with the usual one. That is,*

$$C.D = [s,t] = -\sum_{x,y}(s_{01}, t_{12})_{x,y}.$$

This proposition is partially proved in [22]. The book [8] contains some details, but was never completed.

Proof. We show that the pairing satisfies the properties of the usual intersection pairing that defines it uniquely. Note that the sum is actually finite, by the adelic property of A_{012}.

(Linearity) Follows from linearity of the tame symbol.

(Symmetry) The main challenge is the symmetry of the pairing, which follows from Parshin reciprocity (Prop. 1.4) and skew-symmetry of the tame symbol. Since $s_x s_y^{-1} \in K_{x,y}$ has no poles, the first equality in:

$$\begin{aligned}
0 &= \sum_{x\in y}(s_y s_x^{-1}, t_x t_y^{-1})_{x,y} \\
&= \sum_{x\in y}\left[(s_y, t_x t_y^{-1})_{x,y} - (s_x, t_x)_{x,y} - (t_y, s_y)_{x,y} - (t_y, s_x s_y^{-1})_{x,y}\right] \\
&= [s,t] - [t,s]
\end{aligned}$$

is immediate. The other two follow from (a) skew-symmetry and bilinearity of the tame symbol, and (b) Parshin reciprocity, respectively.

Note that we have used all three properties of Parshin reciprocity. Only property (1) in Theorem 1.4 requires the surface to be projective.

(Trivial on $D \sim 0$) Suppose t is a trivial adelic Cartier divisor. In other words, if we represent t as an element of $(K^\times/A^\times)^0$, then it is in the image of the map

$(K^\times)^0 \to (K^\times/A^\times)^0$, i.e., t is represented by an element (f,f,f) with $f \in A_0^\times$. Then the image of t under the boundary map $(K^\times/A^\times)^0 \to (A^1)^\times$ is $(f,f,1)$, thus $t_{12} = 1$ and

$$(s, t_{12})_{x,y} = (s,1)_{x,y} = 0$$

for all $s \in A_{012}^\times$. Therefore $[s,t] = 0$ for all s.

(Normalization) Let C, D be two smooth curves intersecting properly in X. By construction,

$$\partial_x(t_{02,x}) = \begin{cases} 1 & x \in D \\ 0 & x \notin D. \end{cases}$$

Also, $t_{01,C} \in k(C)^\times$ and $\operatorname{ord}_C s_{01,x} = 1$ if and only if $x \in C$. By (1),

$$-(s_{01}, t_{12})_{x,C} = \begin{cases} 1 & x \in C \cap D \\ 0 & \text{otherwise} \end{cases}$$

and the symbol is trivial for all $y \neq C$.

It follows that $[s,t]$ computes the intersection pairing. □

2 Riemann–Roch for surfaces

Theorem 2.1 ([6, Theorem §3]). *Let X/k be a smooth projective surface; fix a Weil differential d with associated canonical divisor K. Let $A = \hat{A}_X$ be the complete adeles. Then the pairing from d sets up isomorphisms*

$$\operatorname{Hom}_{k,\text{cts}}(H^i(A(D)), k) \simeq H^{2-i}(A(K-D)) \quad \textit{for } 0 \le i \le 2.$$

Proof. This proof is due entirely to Fesenko [6, Theorem §3].

We will do the case of the middle cohomology $i = 1$ as it is the most interesting, for the rest see [6]. By following the argument that goes back to Tate's thesis [24], one can show that since A_{012} is the restricted direct product of self-dual additive groups (Theorem 4.2.3), it follows that A_{012} itself is self-dual (further, this is a topological self-duality). The pairing given by the Weil differential sets up the duality. It follows that, for any closed subspace $B \subset A_{012}$,

$$\mathrm{Hom}_{k,\mathrm{cts}}(B, k) \simeq A_{012}/B^{\perp}.$$

From the lattice structure of A (Proposition 4.2.2), we have

$$H^1(A(D)) \simeq (A_{01} \cap (A_{12}(D) + A_{02}))/(A_1(D) + A_0)$$

therefore,

$$\mathrm{Hom}_{k,\mathrm{cts}}(H^1(A(D)), k) \simeq (A_1(D)^{\perp} + A_0^{\perp})/(A_{01}^{\perp} + (A_{12}(D)^{\perp} \cap A_{02}^{\perp})).$$

Since X is projective, Parshin reciprocity is exactly the statements

$$A_{01}^{\perp} = A_{01}, \quad \text{and,} \quad A_{02}^{\perp} = A_{02}$$

while $A_0^{\perp} = A_{01} + A_{02}$ follows by an argument in [6, Theorem §2(5)]. By definition, $A_{12}(D)^{\perp} = A_{12}(K - D)$, while applying Proposition 4.2.2 together with standard properties of $\perp$ gives

$$A_1(D)^{\perp} = A_{01} + A_{12}(K - D).$$

Putting this all together, we get the group which is canonically isomorphic to

$$(A_{12}(K - D) \cap (A_{01} + A_{02}))/(A_1(K - D) + A_2(K - D)) \simeq H^1(A(K - D)).$$

$\square$

From this form of Serre duality, it follows that

Theorem 2.2 ([6, Theorem §4]). *Let X/k be a smooth projective curve, and A its complete adeles. We can either define the cohomological intersection pairing*

$$[D, E] = \chi(A) - \chi(A(D)) - \chi(A(E)) + \chi(A(D + E))$$

or the adelic intersection pairing

$$[D, E] = \sum_{x \in y} (s_{01}, t_{12})_{x,y}.$$

Then both pairings agree with the usual intersection pairing, and in both cases, we have

$$\chi(A(D)) - \chi(A) = \frac{1}{2}[D, D - K].$$

Proof. The equality is a tautology using the cohomological definition, so one must show that the cohomological pairing agrees with the usual intersection pairing. This follows from Bertini's theorem and a standard argument, see [6, Theorem §4(1–3)]

For the proof that the adelic intersection pairing agrees with the usual one, see the previous section. □

3 Hodge index and the Riemann Hypothesis for curves

Let $C/\mathbb{F}_q$ be a smooth projective curve of genus g. Following Grothendieck [12], we will derive the Riemann hypothesis for C (Theorem 3.3) from a version of the Hodge index theorem. As we do not, at this time, have an adelic definition to replace ampleness, we satisfy ourselves with an ad hoc class of divisors.

Definition 3.1. Call a divisor D **simplicially effective** if it is effective, and its self-intersection divisor is effective. An equivalent condition is that

$$l(D) > 1$$

where $l(D)$ is the dimension of the linear space $\{f \in k(X) \mid \operatorname{div} f \geq -D\}$ associated with D.

Any two such divisors D, E clearly satisfy, regardless of whether they intersect properly,

$$[D, E] > 0$$

(in particular $[D, D] > 0$) and take the role of a very ample divisor H. Clearly every very ample divisor is simplicially effective.

Theorem 3.2 (baby Hodge index). *Let D, E be divisors, with E simplicially effective. If $[D, E] = 0$, then $[D, D] \leq 0$.*

Proof. This proof follows Grothendieck [12, Proposition 2.1] and the ensuing discussion.

Since $\dim_k H^1(A(D)) \geq 0$, we have the Riemann–Roch inequality

$$l(D) + l(K - D) \geq \frac{1}{2}[D, D - K] + \chi(A).$$

Now assume D is any divisor such that $[D, D] > 0$. We obtain the asymptotic statement

$$l(nD) + l(K - nD) = \Omega(n^2).$$

But $[-D, -D] > 0$ as well, so we obtain

$$l(-nD) + l(K + nD) = \Omega(n^2).$$

Suppose that both $l(nD)$ and $l(-nD)$ remain bounded. Then $l(K-nD)\to\infty$ and $l(K+nD)\to\infty$ as $n\to\infty$. But then $l(2K)=l(K-nD+K+nD)\to\infty$, which is absurd.

It follows that $\limsup_n l(nD)=\infty$, possibly replacing D by $-D$. Now, let E be a simplicially effective divisor, and let D be any divisor such that $[D,E]=0$. Suppose $[D,D]>0$. Then $l(nD)>0$ for $n\gg 0$ or $n\ll 0$. But then nD is simplicially effective for such n, so

$$0<[nD,E]=n[D,E]=0.$$

This contradiction proves the theorem. □

Finally we derive:

Theorem 3.3 (Riemann hypothesis for a curve)**.** *Let $C/\mathbb{F}_q$ be a smooth projective curve. Then*

$$|1+q-\#\,C(\mathbb{F}_q)|\le 2g\sqrt{q}. \tag{2}$$

Proof. We work with the base changed curve $\overline{C}=C\otimes_{\mathbb{F}_q}\operatorname{Spec}\overline{\mathbb{F}}_q$ and still call it C. Consider the diagonal embedding of C in its product $X=C\times C$. This is the graph Δ of the identity map, which has transverse intersection with the graph Γ of the (purely inseparable) Frobenius morphism. Therefore this intersection number is exactly $\Delta.\Gamma=\#\,C(\mathbb{F}_q)$.

Choose an origin $(P,P)\in X$ and let $y=P\times C$ and $z=C\times P$ be the axes. Then $E=y+z$ is simplicially effective (in fact ample). By the theorem,

$\langle \cdot, \cdot \rangle = -[\cdot, \cdot]$ is positive definite on $E^\perp \subset \operatorname{Num} X \otimes \mathbb{R}$, so by Cauchy–Schwarz,

$$|\langle \operatorname{pr}_{E^\perp} D, \operatorname{pr}_{E^\perp} D' \rangle| \leq \sqrt{\| \operatorname{pr}_{E^\perp} D \| \| \operatorname{pr}_{E^\perp} D' \|}$$

for all divisors D, D'. Approximations $\Delta \approx y + z$ and $\Gamma \approx y + qz$ follow the intuition that $N = \Delta.\Gamma \approx 1 + q$. Plugging in the error terms $D = \Delta - (y + z)$ and $D' = \Gamma - (y + qz)$, using the adjunction formula to compute the self-intersections $[\Delta, \Delta]$ and $[\Gamma, \Gamma]$, gives (2). $\square$

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