ABSTRACT

Title of dissertation: THE STOCHASTIC NAVIER STOKES EQUATIONS FOR HEAT CONDUCTING, COMPRESSIBLE FLUIDS

Scott Andrew Smith, Doctor of Philosophy, 2016

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This dissertation is devoted to the equations of motion governing the evolution of a fluid or gas at the macroscopic scale. The classical model is a PDE description known as the Navier-Stokes equations. The behavior of solutions is notoriously complex, leading many in the scientific community to describe fluid mechanics using a statistical language. In the physics literature, this is often done in an ad-hoc manner with limited precision about the sense in which the randomness enters the evolution equation. The stochastic PDE community has begun proposing precise models, where a random perturbation appears explicitly in the evolution equation. Although this has been an active area of study in recent years, the existing literature is almost entirely devoted to incompressible fluids.

The purpose of this thesis is to take a step forward in addressing this statistical perspective in the setting of compressible fluids. In particular, we study the well posedness for the corresponding system of Stochastic Navier Stokes equations, satisfied by the density, velocity, and temperature. The evolution of the momentum involves a random forcing which is Brownian in time and colored in space. We allow for multiplicative noise, meaning that spatial correlations may depend locally on the fluid variables.

Our main result is a proof of global existence of weak martingale solutions to the Cauchy problem set within a bounded domain, emanating from large initial datum. The proof involves a mix of deterministic and stochastic analysis tools. Fundamentally, the approach is based on weak compactness techniques from the deterministic theory combined with martingale methods. Four layers of approximate stochastic PDE's are built and analyzed. A careful study of the probability laws of our approximating sequences is required. We prove appropriate tightness results and appeal to a recent generalization of the Skorohod theorem. This ultimately allows us to deduce analogues of the weak compactness tools of Lions and Feireisl, appropriately interpreted in the stochastic setting.

THE STOCHASTIC NAVIER STOKES EQUATIONS FOR HEAT CONDUCTING, COMPRESSIBLE FLUIDS

by

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Dedication

To my parents.

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Chapter 1: Preliminaries

1.1 Introduction

Our primary object of interest is a system of stochastic partial differential equations governing the evolution of a viscous, heat conducting compressible fluid (or gas) subject to random perturbations by noise. The macroscopic state of the fluid is described by a triple (ρ, u, θ) consisting of the scalar, nonnegative density ρ, an $\mathbb{R}^d$ valued velocity field u, and a scalar, nonnegative temperature θ. The system is written as follows:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla P = \operatorname{div} \mathcal{S} + \rho \sigma_k(\rho, \rho u, \rho\theta, x)\dot{\beta}_k \\ \partial_t(\rho\theta) + \operatorname{div}(\rho\theta u) - \operatorname{div}(\kappa(\theta)\nabla\theta) = \mathcal{S} : \nabla u - \theta p_\theta(\rho) \operatorname{div} u \\ \big(\rho(0), (\rho u)(0), (\rho\theta)(0)\big) = (\rho_0, m_0, \rho_0\theta_0). \end{cases} \tag{1.1}$$

The spatial dimension is denoted by d, and we focus throughout the thesis on the case $d \geq 3$. The fluid is assumed to be confined to a domain $D \subset \mathbb{R}^d$ which is smooth, bounded, and connected. At time zero, the system is described by a non-negative initial density ρ_0, momentum m_0, and a non-negative temperature θ_0. More precisely, we assume the following:

Hypothesis 1.1.1. *The initial data ρ_0, m_0 and θ_0 are deterministic. The initial density $\rho_0 \in L^\gamma_x$ is nonnegative and compatible with the initial momentum in the sense that $m_0 1_{\{\rho_0=0\}}$ vanishes. The initial kinetic energy is finite, meaning that*

$$\int_D \frac{1}{2}\frac{|m_0|^2}{\rho_0}dx < \infty. \tag{1.2}$$

The initial temperature θ_0 belongs to L^∞_x and there exists a positive constant θ_{min} such that $\theta_0 \geq \theta_{min}$.

Note in particular that we do not force the initial density to be bounded away from zero, and in principle we allow for initial regions of vaccum. The assumption that the initial data are deterministic is only for simplicity. More generally, one could take as initial datum a probability measure for the law of (ρ_0, m_0, θ_0) provided it concentrates on triples satisfying Hypothesis 1.1.1.

The main fluid mechanical inputs for the problem are the pressure $P(\rho,\theta)$, the heat conductivity coefficient $\kappa(\theta)$ and the viscous stress tensor $\mathcal{S}(u)$. We will impose the following structural hypotheses:

Hypothesis 1.1.2. *The pressure law takes the form $P(\rho,\theta) = p_m(\rho)+\theta p_\theta(\rho)$, where p_m, p_θ are non-decreasing functions in $C[0,\infty)\cap C^1(0,\infty)$ such that $p_m(0) = p_\theta(0) = 0$. Moreover, we assume p_m is convex and satisfies $p_m(\rho) \sim \rho^\gamma$, while $p_\theta(\rho) \sim \rho^\Gamma$, where $\gamma > \frac{d}{2}$ and $\Gamma = \frac{\gamma}{d}$.*

Hypothesis 1.1.3. *The heat conductivity coefficient is bounded strictly away from zero, belongs to $C^2[0,\infty)$ and satisfies $\kappa(\theta) \sim \theta^2$.*

Hypothesis 1.1.4. *The stress tensor $\mathcal{S}$ is given by the relation*

$$\mathcal{S}(u) = \mu(\nabla u + \nabla u^t) + \lambda \operatorname{div} u I, \tag{1.3}$$

where $\mu, \lambda > 0$ are positive viscosity coefficients which are independent of the density.

Hypotheses1.1.1, 1.1.2, and 1.1.3 are slightly simplified (for purpose of exposition) versions of the assumptions in the deterministic theory [11]. We should note, however, that the virtue of considering an unbounded heat conductivity coefficient κ was identified already by P.L. Lions in the monograph [21].

A large time $T > 0$ is fixed in advance, and (1.1) is posed as a stochastic evolution equation over the time interval $[0, T]$. The collection $\{\beta_k\}_{k\in\mathbb{N}}$ consists of independent, one dimensional Brownian motions, and $\sigma = \{\sigma_k\}_{k\in\mathbb{N}}$ are referred to as the noise coefficients. Formally, the noise is realized as a random series

$$\sum_{k\in\mathbb{N}} \rho\sigma_k\dot{\beta}_k, \tag{1.4}$$

where the noise coefficients σ_k may depend on the fluid variables $\rho, \rho u$ and $\rho\theta$. Moreover, this dependence may change with spatial variable $x \in D$. More preciesely, for the individual noise coefficients we impose the following:

Hypothesis 1.1.5. *For each $k \in \mathbb{N}$, the noise coefficient $\sigma_k : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}_+ \times D \to \mathbb{R}^d$ is globally Lipschitz continuous.*

Moreover, the collection of noise coefficients must satisfy a certain coloring hypothesis. By coloring, we mean that the statistics of the noise at different spatial points are correlated. Alternatively, one can think of this as an assumption ensuring the convergence of the series (1.4). However, this is only part of the truth since we require something even stronger. In fact, the coloring hypothesis will fall naturally out of the a formal a priori bounds derived later in this chapter. The precise hypothesis required is as follows:

Hypothesis 1.1.6. *The collection $\sigma = \{\sigma_k\}_{k\in\mathbb{N}}$ belongs to $\ell^2\big(\mathbb{N}; L_x^{\frac{2\gamma}{\gamma-1}}(L^\infty_{\rho,m,\alpha})\big)$.*

At a heuristic level, one can view the addition of noise to the system (1.1) as a macroscopic realization of a background noise acting indiscriminately on the individual fluid particles. Each fluid parcel feels the same realization of the random noise, resulting in non-neglible correlations between particles at different points in space. As a result, the noise does not average out at the macroscopic scale, and one arrives at a stochastic PDE rather than a deterministic PDE in the hydrodynamic limit.

From a modelling perspective, the noise in the system (1.1) can be thought of as a surrogate for sophisticated boundary effects. These are one of the mechanisms believed to drive a transition from laminar to turbulent compressible flow. In this respect, the system (1.1) is a basic model for turbulence in applications where compressibility effects are essential, such as high speed aeronautics.

From a purely mathematical point of view, the addition of a statistically unbiased noise can be viewed as a mechanism for capturing generic behavior of the underlying deterministic sytem of partial differential equations. In this respect, there is some hope that certain unruly solutions to the deterministic version of (1.1) could be wrapped into a probability zero event at the level of the stochastic equation. This is a very active topic of interest in the field of SPDE's.

1.2 Statement of the main result

Let us proceed by stating the main result of the dissertation, followed by a discussion of their context within the existing fluid mechanics and stochastic PDE literature. We begin by introducing our notion of solution to (1.1). These are weak solutions in both the deterministic and stochastic sense, and are usually referred to in the literature as weak martingale solutions. From the PDE point of view, this essentially means that the equation (1.1) is assumed to hold in the sense of distributions in the spatial variable, integrated in time. From the probabilistic point of view, this means that we have a flexibility in choosing the stochastic basis where the solution is built. In other words, we are free to design a suitable probability space and filtration, together with the accompanying collection of Brownian motions. One way to think about this is that we are effectively forgetting the probability space entirely, and only thinking about the solution in terms of its probability law.

In fact, more precisely, the temperature equation must be replaced by an inequality in order to obtain our results. Following [11], we introduce a class of test functions $\mathcal{D}_{temp}$. We write $\varphi \in \mathcal{D}_{temp}$ provided that φ is a non-negative function in $C^\infty\big([0,T] \times D\big)$ such that $\frac{\partial\varphi}{\partial n}(t,x) = 0$ for $(t,x) \in [0,T] \times \partial D$ and $\varphi(T,x) = 0$ for $x \in D$. In addition, it is useful to introduce the anti-derivative of the heat conductivity coefficient κ. Namely, we define $\mathcal{K}$ by the relation

$$\mathcal{K}(\theta) = \int_1^\theta \kappa(\theta)\mathrm{d}z. \tag{1.5}$$

Definition 1.2.1. *A triple* (ρ, u, θ) *is a weak martingale solution to* (1.1) *provided* *there exists a stochastic basis* $\big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, \mathbb{P}, \{\beta_k\}_{k\in\mathbb{N}}\big)$ *such that:*

1. *The quadruple* $(\rho, \rho u, \rho\theta, u)$ *belongs to* $L^2\big(\Omega\times[0,T]; \mathcal{P}; L^\gamma \times L^{\frac{2\gamma}{\gamma+1}} \times L^q \times [H_0^1]^d\big)$, *where* $\mathcal{P}$ *is the predictable* σ*-algebra generated by* $(\mathcal{F}_t)_{t=0}^T$ *and* $\frac{1}{q} = \frac{1}{\gamma} + \frac{1}{2} - \frac{1}{d}$. *Moreover, the pair* $(\rho, \rho u) : \Omega\times[0,T] \to \big[L^\gamma \times L^{\frac{2\gamma}{\gamma+1}}\big]_w$ *is a continuous stochastic process with* $\mathbb{P}$ *a.s. continuous sample paths.*

2. *Continuity Equation: For all* $\phi \in C_c^\infty(D)$ *and* $t \in [0,T]$

$$\int_D \rho(t)\phi \mathrm{d}x = \int_D \rho_0 \phi \mathrm{d}x + \int_0^t \int_D \rho u \cdot \nabla\phi \mathrm{d}x\mathrm{d}s \quad \mathbb{P} \text{ a.s.} \tag{1.6}$$

3. *Momentum Equation: For all* $\phi \in \big[C_c^\infty(D)\big]^d$ *and* $t \in [0,T]$

$$\begin{aligned}\int_D \rho u(t) \cdot \phi \mathrm{d}x &= \int_D m_0 \cdot \phi \mathrm{d}x + \int_0^t \int_D [\rho u \otimes u - \mathcal{S}(u)] : \nabla\phi \mathrm{d}x\mathrm{d}s \\ &+ \int_0^t \int_D P(\rho,\theta) \operatorname{div}\phi \mathrm{d}x\mathrm{d}s + \sum_{k\in\mathbb{N}} \int_0^t \int_D \rho\sigma_k(\rho, \rho u, \rho\theta, x) \cdot \phi \mathrm{d}x\mathrm{d}\beta_k(s) \quad \mathbb{P} \text{ a.s.}\end{aligned} \tag{1.7}$$

4. *Temperature Inequality: For all* $\varphi \in \mathcal{D}_{temp}$

$$\begin{aligned}&\int_0^T \int_D \rho\theta\big(\partial_t\varphi + u\cdot\nabla\varphi\big) + \mathcal{K}(\theta)\Delta\varphi \ \mathrm{d}x\mathrm{d}s + \int_D \rho_0\theta_0\varphi_0 \mathrm{d}x \\ &\le \int_0^T \int_D \big[\theta p_\theta(\rho) \operatorname{div} u - \mathcal{S}(u) : \nabla u\big]\varphi \mathrm{d}x\mathrm{d}s \quad \mathbb{P} \text{ a.s.}\end{aligned} \tag{1.8}$$

The main result of the dissertation is the following global existence theorem.

Theorem 1.2.2. *Assuming Hypotheses 1.1.1-1.1.6, there exists a weak martingale solution* (ρ, u, θ) *to the system* (1.1) *in the sense of the Definition 1.2.1. Moreover,*

$$\begin{aligned}&\mathbb{E}^{\mathbb{P}}\Big[|\sqrt{\rho}u|^{2p}_{L_t^\infty(L_x^2)} + |\rho|^{\gamma p}_{L_t^\infty(L_x^\gamma)} + |\rho\theta|^p_{L_t^\infty(L_x^1)}\Big] < \infty. \\ &\mathbb{E}^{\mathbb{P}}\Big[|u|^{2p}_{L_t^2(H_{0,x}^1)} + |\theta|^{2p}_{L_t^2(H_x^1)} + |\nabla \log\theta|^{2p}_{L_{t,x}^2}\Big] < \infty.\end{aligned} \tag{1.9}$$

for all $p \in [1,\infty)$.

1.3 Existing literature

The literature devoted to the deterministic, compressible system is extensive; the most fundamental for our work are the results of Lions [21] and Feireisl [13]. These results are discussed in detail below. In fact, since we are free to work with degenerate noise coefficients, Theorem 1.2.2 recovers the deterministic results if the noise coefficients vanish identically. We should mention however, that the results of [13] are no longer the most general in the deterministic setting. A recent breakthough of Bresch/Jabin [5] allows to significantly weaken the hypotheses on the fluid mechanical inputs, using an exciting new approach to the problem. However, in this thesis our viewpoint aligns more with the classical Lions/Feireisl perspective.

There has also been an intensive study of the incompressible, stochastic Navier Stokes equations. The works most relevant to our paper concern the construction of weak, martingale solutions. See for instance [2],[7],[15], [23]. There are also a few articles concerning the non-homogenous, incompressible system; see [24] and [8]. However, the literature concerning the stochastic, compressible system is rather scarce. Some results are available in dimension one, see [26]. The most relevant for our analysis is the work of Berthelin/Vovelle [3] on the one dimensional compressible Euler equations. The paper [12] studies the compressible Navier Stokes equations driven by a forcing of the form $\rho\dot{W}$. In this special case, one can change variables and work pathwise in ω. This technique is not generally available for other types of multiplicative noise.

The only existing literature on the compressible, stochastic Navier Stokes equations that is comparable to ours is the work of Breit/Hofmanova [4]. We became aware of these results during the late stages of the write up of the article [25], covering the special case of barotropic fluids. Our results were independently concieved and obtained. There are a number of similarities between their work and ours, but our hypotheses on the noise do not overlap, so neither result implies the other. As a result, there are also several differences in the approximating schemes. Finally, we should remark that our work is set on a bounded domain, rather than the torus.

Let us proceed with a review of a number of key ideas from the deterministic, compressible theory and the stochastic, incompressible theory.

1.3.1 Key ideas from the deterministic, compressible literature

The literature on the deterministic system is extensive, and we will not attempt to give a complete discussion of the current status of the field. Instead, we focus on the results that provide the guiding principles for our work. The seminal work of P.L. Lions [21] initiated a large data global existence theory for finite energy weak solutions. Let us give a very rough outline of the construction in the barotropic case (where we neglect the temperature equation). The proof splits into two parts; proving that the solution set is weakly compact and constructing several layers of approximating schemes. That is, suppose $\{(\rho_n, u_n)\}_{n\in\mathbb{N}}$ is a sequence of weak solutions (or well chosen approximate solutions) which are uniformly bounded in the natural energy space. The strategy is to show that if the initial data are stongly convergent, then the corresponding solutions must converge to a solution (ρ, u)

emanating from the limit point of the data. Since the pressure is a nonlinear function of the density, the only feasible way to proceed is by proving that the sequence of densities $\{\rho_n\}_{n\in\mathbb{N}}$ converge strongly to ρ. However, the continuity equation is driven by too rough of a velocity field to provide any control on the densities in a positive Sobolev space. Hence, this is a nontrivial task and the basic energy bounds alone are not enough. Nonetheless, Lions found a more subtle mechanism in the nonlinear structure that gives compactness.

To motivate the proof, recall the method for obtaining compactness of the density in the Di Perna/Lions [9] theory of the transport equation driven by a "rough" velocity field with bounded divergence. One starts with a convienient renormalization (meaning just a smooth function to be applied to the density), for instance $\beta(\rho) = \rho^2$, and renormalizes the equation at the level of both the approximation and the level of the limiting solution, known a priori to be a renormalized solution of the transport equation in its own right. If strong oscillations in the density sequence are present, the operations of composition with a nonlinear function and extraction of a weak limit do not generally commute. However, the renormalized form allows one to track the evolution of this "commutator" $\overline{\rho^2} - \rho^2$ and a Gronwall argument shows that if compression effects are limited, strong convergence of initial densities implies the "commutator" vanishes for all later times. Unfortunately, one cannot apply this method directly to the compressible Navier Stokes system because the known a priori bounds are not enough to rule out the possibility of extreme compression (or expansion). To proceed, Lions made the crucial observation that a sort of "monotoncity miracle" occurs for particular pressure laws, and in some sense, it suffices that the so called effective viscous pressure $P(\rho) - (2\mu + \lambda)\,\mathrm{div}\, u$ is "slightly well behaved" (in the sense of a certain weak continuity property), even if the divergence of the velocity field alone is potentially unbounded. The importance of this quantity had already been observed in a simpler context by D. Serre and others. Moreover, the evolution of this quantity is readily available upon taking the divergence, followed by the inverse laplacian on both sides of the momentum equation. By studying this quantity before and after a preliminary passage to the limit in the momentum equation, one is able to prove a subtle compactness result, known as the weak continuity of the effective viscous pressure, which is just barely enough to complete an analysis of a similar "commutator" as in the bounded divergence case, and hence conclude the strong convergence of the density.

The original work of Lions considered pressures of the form $P(\rho) = \rho^\gamma$ with γ large enough to ensure that the continuity equation could be renormalized. Several years later, Feireisl introduced in [10] some additional tools which, combined with Lions general strategy of proof, succeeded in weakening the hypothesis on γ in dimensions two and three, to what seems to be the critical level [1] $\gamma > \frac{d}{2}$. This is a nontrival task, since for low enough values of γ, one dips below the integrabililty required to classically renormalize the continuity equation. More importantly for this paper, with co-authors in [13], Feireisl developed a somewhat simplified (but still rather long) approximation scheme, based on a Galerkin appromation for the velocity, a vanishing viscosity regularization for the continuity equation, and an

[1] Below this level, one can just barely give a meaning to the flux term in the momentum equation, and Lions method seems to break down.

artifical pressure regularization.

1.3.2 Key ideas from the incompressible, stochastic literature

There is also a fairly developed literature concerning the stochastic Navier Stokes equations for incompressible fluids, which we will not review in much depth. Naturally, much more is known in dimension two, but the existence of weak solutions is known in any dimension. In this regard, the primary inspiration for our work is Flandoli's construction of weak martingale solutions in [14]; see also [7]. The main point we wish to emphasize is that these solutions are weak in both the analytic and the probabilistic sense.

To understand the virtue of flexibility in the choice of a stochastic basis, recall Leray's construction of weak solutions to the deterministic, incompressible Navier Stokes equations. The key point is that uniform bounds in $L_t^2(W_x^{1,2}) \cap C_t^{\alpha}(H_x^{-1})$ allow one to apply the Aubin/Lions lemma and obtain strong compactness in $L_t^2(L_x^2)$ (and hence weak stability of the flux term), leading to a straightforward (from a modern point of view) weak compactness theory. At a superficial level, in the stochastic case, there is an additional variable ω, and the possibility of "oscillations" in this variable may block the compactness upgrade from the space/time bounds. However, if one is content with only accessing the probability law of the solution, then there is a classical fix. Namely, if one can show that the sequence of Galerkin approxmations becomes uniformly concentrated (up to a set of very small probability) on $L_t^2(W_x^{1,2}) \cap C_t^{\alpha}(H_x^{-1})$, the Skorohod embedding theorem (for random variables on complete separable metric spaces) guarantees the existence of a new sequence of random variables (with the same probability distribution) on the unit interval, along with a limit point, for which the usual $L^2_{t,x}$ convergence holds pointwise. Essentially, under an appropriate change of variables one is able to convert information that only holds on average on the initial probability space, to information that holds in every state of the universe of a well chosen probability space. One could visualize this in one dimension by noting that given a sequence of bumps sliding back and forth across the unit interval on smaller and smaller measure sets, if we rearrange the sequence based on the distribution of mass, one converts the typical counterexample to "weak convergence implies pointwise convergence" into a pointwise converging sequence, without altering its probability law.

1.4 Outline of the proof

Let us now proceed to a discussion of the proof of Theorem 1.2.2, as this will occupy the remainder of the thesis. The starting point for our analysis are the formal energy estimates obtained in Section 1.6. In the stochastic, compressible framework, the kinetic and potential energy are random processes which fluctuate due to noise and grow according to an Ito correction term. To close an estimate on their moments (in ω) and obtain an a priori bound for the SPDE, one is lead to the trace class type summability condition for the coefficients σ, Hypothesis 1.1.6 mentioned above. With the formal estimates at hand, we proceed to a construction of approximating sequences.

The proof of Theorem 1.2.2 relies on a four level approximating scheme. Three of the levels are inspired by the theory of Feireisl [11] for the treatment of the deterministic Navier-Stokes equations for compressible, non-isentropic fluids. The lowest level uses a time splitting scheme, and is similar to the technique used by Berthelin/Vovelle [3] for the 1-d compressible Euler system. Each layer involves a compactness step and an identification procedure. The compactness step involves proving an appropriate tightness result and applying a recent generalization of the Skorohod theorem due to Jakubowski [17] and Vaart/Wellner [28] (Theorem 1.7.16). The identification procedure involves several ingredients. The first step is to use a martingale method to make a preliminary passage to the limit in the momentum equation, up to a modification in the pressure law. The second is to use this partial stability result to upgrade from strong to weak convergence of the densities. The last step is to use the strong convergence of the density to prove the convergence of the temperature away from vaccum, then use a renormalized limit in the vaccum regions.

We will proceed with a description of some difficulties encountered within each layer of the approximating scheme. Chapter 2 focuses on builiding solutions to the τ, n, and ϵ layer approximations, while Chapter 3 builds the δ layer approximation and passes the limit $\delta \to 0$ to complete the proof of Theorem 1.2.2. Let us proceed by mentioning some of the difficulties encounted within each layer.

1.4.1 τ, n, and ϵ Layers

In Section 2.1, we prove existence for the lowest level of our approximating scheme. Definition 2.1.3 introduces the notion of a τ layer approximation to the system (1.1). The main result of the section, Theorem 2.1.1, shows that for each fixed $(n, \epsilon, \delta, \tau)$, such approximations exist. This level of the scheme is a time splitting method. For the first τ units of time, the determinstic system evolves and the stochastic forcing is neglected. For the next τ units of time, the density is frozen, and the system evolves only through the noise. The evolution is sped up appropriately so there is consistency when the time splitting parameter is sent to zero (in Section 2.2).

This method was recently used in the context of the stochastic isentropic compressible Euler equations by Berthelin/Vovelle in [3]. In our setting, we have also the temperature equation and the splitting is a bit more subtle. At this stage, the temperature equation contains a damping term $\delta\theta^3$. Two of the terms in the equation, $\delta\theta^3$ and $\mathrm{div}(\kappa(\theta)\nabla\theta)$ run at speed 1 for all times, while the remaining terms run at twice the usual speed, but only when the noise is turned off. The main tools for the existence at this layer are Propositions 2.1.6 and 2.1.8. These use fixed point arguments to obtain a basic existence result for both the deterministic and the stochastic systems, respectively. Patching these two together, we obtain our τ layer approximation.

Section 2.2 is devoted to proving Theorem 2.2.1, an existence result for the second layer. Definition 2.1.3 introduces the notion of an n layer approximation (1.1), and the goal is to construct a sequence $\{(\rho_n, u_n, \theta_n)\}_{n\in\mathbb{N}}$ of these which obey the uniform bounds.

For each fixed $n \in \mathbb{N}$, we apply Theorem 2.1.1 and find a sequence $\left\{(\rho_{\tau,n}, u_{\tau,n}, \theta_{\tau,n})\right\}_{\tau>0}$ of τ layer approximations. A distinctive feature of the τ layer is that one requires stronger convergence in time than in other layers of the scheme. This is required to compensate for the high frequency switching between the two types of evolution when τ is small. More precisely, there is an oscillatory factor h^{τ}_{det} (turned on only when the stochastic forcing is at rest) which arises in the weak form, converging only weakly in time as $\tau \to 0$. To combat these oscillations, we need strong convergence in time of any terms multiplied by this factor.

We begin by proving two types of uniform bounds. The first type are analagous to the formal estimates obtained in Section 1.6. The second type of estimates are encoded in a tightness proof. The trick is to obtain estimates on the density and velocity which give stronger bounds on these quantities as a function of (t, x), at the cost of taking a weaker norm as a function of ω. Namely, these estimates are not in terms of $L^p(\Omega)$, but only in measure. This leads to some subtleties, and one needs a probabilistic bootstrapping procedure to deal with the coupling between the density and the velocity.

Using these estimates, we are able to prove Proposition 2.2.4, which yields a candidate limit point (ρ_n, u_n, θ_n), together with a new sequence $\left\{\tilde{\rho}_{\tau,n}, \tilde{u}_{\tau,n}, \tilde{\theta}_{\tau,n}\right\}_{\tau>0}$ of τ layer approximations with improved compactness properties. A caveat is that the new sequence is defined relative to a new probability space. However, there exists a sequence $\{\tilde{T}_\tau\}_{\tau>0}$ of measure preserving transformations (referred to by the author as recovery maps), which link the new and the old sequence by composition. These mappings allow us to preserve information as we change probability spaces; in particular, ensuring that the new sequence solves the same equations, obeys the same uniform bounds and is generally unaltered in any of its arguments besides ω. Our main tool is Theorem 1.7.2, a generalization of the classical result of Skorohod. It merges two recent extensions of the theorem; the first, due to Jakubowksi [17], permits random variables on a class of topological spaces and the other, due to by Vaart/Wellner [28], provides the recovery maps.

After Proposition 2.2.4 is established, it is easy to pass to the limit in the parabolic equation on the new probability space (Lemma 2.2.9). To pass to the limit in the momentum equation, we use a martingale method based on an appendix result 1.7.6, which provides a convienient characterization of a series of one dimensional stochastic integrals. This was developed in [6] as an alternative to the martingale representation theorem. This method is used systematically throughout the paper when passing to the limit in the momentum equation at each layer.

Section 2.3 is devoted to proving Theorem 2.3.1, the ϵ layer. For each $\epsilon > 0$ fixed, we can apply Theorem 2.2.1 to obtain a sequence $\big\{(\rho_{n,\epsilon}, u_{n,\epsilon}, \theta_{n,\epsilon})\big\}_{n\in\mathbb{N}}$ of n layer approximations satisfying the uniform bounds. Again, the section splits into a compactness step, Proposition 2.3.4 and an identification step, Lemmas 2.3.6 and 2.3.8. In this section, the spaces where the tightness Lemma 2.3.5 are proved become a bit more sophisticated. In particular, we must use certain Banach spaces endowed with their weak or weak-$\star$ topology. At this stage, and at all later compactness steps, the Jakubowski extension of the Skorohod theorem is essential.

As $n \to \infty$, one challenging term is $\epsilon \nabla u_n \nabla \rho_n$ in the momentum equation, which corrects for the vanishing viscosity regularization in the energy balance. To treat this difficulty, we adapt a technique of Feireisl in Lemma 2.3.6, and upgrade the convergence of the density. This allows us to use a martingale method again in Lemma 2.3.8 and complete our stability analysis at this layer. Another subtlety arises in the compactness analysis of the temperature equation. Namely, in order to apply the Aubin-Lions type Lemma from Feireisl [11] (Proposition 1.7.12), we require uniform $L_t^\infty(L_x^1)$ bounds on $\big\{(\rho_n + \delta)\theta_n\big\}_{n\in\mathbb{N}}$ which hold pointwise in ω. However, due to the presense of stochastic integrals in the total energy balance, we only have uniform $L^p\big(\Omega; L_t^\infty(L_x^1)\big)$ bounds for $p < \infty$. To solve this problem, we use the Skorohod theorem for random variables in the space $\big[L_t^1(C_c(D))\big]_*'$, which contains $L_t^\infty(L_x^1)$ with an isometric embedding.

1.4.2 The δ layer and completion of the proof

In Section 3.1 we build our main approximating scheme, a sequence $\big\{(\rho_\delta, u_\delta, \theta_\delta)\big\}_{\delta>0}$ of δ layer approximations to (1.1). The proof still splits broadly into two parts; a compactness step and an identification procedure. However, as $\epsilon \to 0$ we encounter several new difficulties related to the pressure $\nabla(\rho_\epsilon^\gamma + P(\rho_\epsilon, \theta_\epsilon) + \delta\rho_\epsilon^\beta)$. The first is that the basic energy bounds only provide moment estimates on the $L_{t,x}^1$ norm of the pressure. To obtain tightness of the pressure sequence, we must improve these bounds. In Proposition 3.2.4, we prove stochastic analogues of the integrability gains observed by Lions [21]. Namely, we show that our weak martingale solutions inherit additional integrability from the equation itself.

An even more serious difficulty is passing to the limit in the pressure, which requires strong convergence of the density. The compactness step is setup in a way that anticipates this problem. We start with a preliminary identification step, passing to the limit in the continuity equation (Lemma 3.1.5) and also the momentum equation (Lemma 3.1.6), modulo a possible change in the pressure law.

Next we improve upon our preliminary identification step and work towards the strong convergence of the density. In Lemma 3.1.7, we prove a stochastic analogue of the weak continuity of the effective viscous pressure, originally discovered by Lions [21]. Namely, the weak continuity holds after averaging out the contribution of the stochastic integral. In Lemma 3.2.13, this result is used together with techniques from the theory of renormalized solutions of the transport equation [9] to prove the strong convergence of the density and complete the δ layer existence proof.

At this stage, we have suceeded in constructing a sequence $\big\{(\rho_\delta, u_\delta, \theta_\delta)\big\}_{\delta>0}$ of δ layer approximations to the stochastic Navier Stokes equations which obey the uniform bounds (3.1). We now proceed to the proof of our main result, Theorem 1.2.2. As $\delta \to 0$, two regularizations in the temperature equation degenerate. In particular, we no longer have a free $L^p(\Omega; L^3_{t,x})$ bound on the sequence $\{\theta_\delta\}_{\delta>0}$. Instead, this has to be proved directly from the weak form at the δ layer. Additional difficulties lie in the strong convergence of the densities, due to the fact that the limiting density may not lie in $L^2_{t,x}$. As usual, we begin with a compactness step, and the statement of the theorem is rather technical, but natural if one anticipates again the difficulties with the pressure term.

The identification procedure faces a new difficulty at this stage, regarding the nonlinear compositions in the multiplicative noise. Namely, at each of the earlier stages in the analysis, we used the parameter δ to regularize the density and momentum before composing with the diffusion coefficient σ_k. This was crucial for checking the two parts in our key identifiation Lemma 1.7.6. In the δ layer existence proof, it allowed us to make a preliminary passage to the limit in the equation before proceeding to the proof of the strong convergence of the density. In this final step, this is simply not possible. Instead, we can only prove that the momentum process minus its drift is a martingale. Nonetheless, we show that this is in fact enough information to prove again a stochastic analogue of weak continuity of the effective viscous flux. We use the momentum martingale together with a regularization procedure to establish an averaged Itô product rule, which is in turn enough to establish the weak continuity. This is used again to prove convergence of the density. Finally, we analyze renormalized limits to address the temperature equation. We conclude the proof of our main result by passing the limit in the momentum equation, once and for all.

1.5 Notation

Here, and in what follows, we use the notation:

Functional spaces: (i) The shorthand notation $L_t^q(L_x^p), L_t^q(W_x^{k,p}), W_t^{k,q}(L_x^p)$ is used to denote the spaces $L^q\left([0,T];L^p(D)\right), L^q\left([0,T];W^{k,p}(D)\right), W^{k,q}\left([0,T];L^p(D)\right)$ respectively, where each space is understood to be endowed with its strong topology. Also, we use M_x to denote the finite, signed radon measures on D. The abbreviation $L_t^\infty(M_x)$ denotes $L^\infty\big([0,T];M_x\big)$. We will often use the same notation to denote scalar functions in $L_t^q(L_x^p)$ and vector valued functions(with d components) in $[L_t^q(L_x^p)]^d$, but the meaning will always be clear from the context. To emphasize when one of the spaces above is endowed with its weak topology, we write $[L_t^q(L_x^p)]_w, [L_t^q(W_x^{k,p})]_w$. Also, the abbreviation $C_t([L_x^p]_w)$ denotes the topological space of weakly continuous functions $f:[0,T]\to L^p(D)$. The space $W_{0,x}^{k,p}$ is the closure of the smooth compactly supported functions, $C_c^\infty(D)$, with respect to the $W^{k,p}(D)$ norm. Moreover, we denote $W_{0,x}^{1,2}$ as $H_{0,x}^1$.

Probability space: (ii) Given a probability space $(\Omega,\mathcal{F},\mathbb{P})$ and a Banach space E, let $L^p\big(\Omega;E\big)$ be the collection of equivalence classes of $\mathcal{F}$ measurable mappings $X:\Omega\to E$ such that the p^{th} moment of the E norm is finite. Again, we write $L_w^p\big(\Omega;E\big)$ when emphasizing that the space is endowed with its weak topology. To define the sigma algebra generated by various random variables, we use a restriction operator $r_t: C\big([0,T];E\big)\to C\big([0,t];E\big)$ which realizes a mapping $f:[0,T]\to E$ as a mapping $r_tf:[0,t]\to E$. The same notation is used for the restriction of an equivalence class $f\in L^p\big([0,T];E\big)$ to $r_tf\in L^p\big([0,t];E\big)$.

Operators and operations: (iii) We denote $\mathcal{A}=\nabla\Delta^{-1}$, understood to be well defined on compactly supported distributions in $\mathbb{R}^d$. The symbol $\mathcal{B}$ is reserved for the Bogovski operator, see the remarks preceding Lemma 1.7.16 for the definition of the operator, along with its basic properties. Given two $d\times d$ matrices A,B, $A:B$ denotes a Frobenius matrix product. The notation $A\lesssim B$ denotes inequality up to an insignificant constant. The notion of insignificance will be clear from the context.

1.6 Formal energy estimates

In this section, we derive the formal a priori bounds for regular solutions to the SPDE (1.1). In subsection 1.6.1, we derive the evolution of the total energy, which is the sum of internal and kinetic contributions. The evolution of the internal energy is obtained through the renormalized form of the continuity equation. Namely, given $\beta : \mathbb{R}_+ \to \mathbb{R}$, multiplying the continuity equation by $\beta'(\rho)$ yields:

$$\partial_t \beta(\rho) + \operatorname{div}(\beta(\rho)u) + [\rho\beta'(\rho) - \beta(\rho)] \operatorname{div} u = 0. \tag{1.10}$$

On the other hand, the evolution of the kinetic energy is obtained from the momentum equation and the Ito formula. Since the stochastic forcing is non-conservative, the total energy undergoes random fluctations produced by a stochastic integral. Moreover, since the noise is understood in the Ito sense, a non-negative correction term appears. As the stochastic integral is an unbounded stochastic process, it is not possible to obtain $L^\infty(\Omega)$ bounds for the total energy. However, the coloring hypothesis 1.1.6 leads to bounds in $L^p(\Omega)$ for every $p \in [1, \infty)$.

In subsections 1.6.2 and 1.6.3, we obtain further bounds on the temperature and the velocity. The basic tool is the following: for $H : \mathbb{R}_+ \to \mathbb{R}$, multiplying the temperature equation by $H'(\theta)$ and combining with the continuity equation yields:

$$\begin{aligned} \partial_t\big(\rho H(\theta)\big) + \operatorname{div}\big(\rho u H(\theta) - H'(\theta)\kappa(\theta)\nabla\theta\big) + H''(\theta)\kappa(\theta)|\nabla\theta|^2 \\ = H'(\theta)[\mathcal{S} : \nabla u - \theta p_\theta(\rho) \operatorname{div} u]. \end{aligned} \tag{1.11}$$

Recall that along ∂D, we assume u and $\frac{\partial \theta}{\partial n}$ vanish. Integrating over D and rearranging gives:

$$\begin{aligned}\int_D \big[H'(\theta)\mathcal{S} : \nabla u - H''(\theta)\kappa(\theta)|\nabla\theta|^2\big]\mathrm{d}x = \frac{d}{dt}\int_D \rho H(\theta)\mathrm{d}x \\ + \int_D \theta H'(\theta)p_\theta(\rho)\,\mathrm{div}\, u\mathrm{d}x.\end{aligned} \tag{1.12}$$

For concave H, the main difficulty in using this identity to control temperature and velocity gradients is the second term on the RHS of (1.12). Using a trick of Lions from [21], in subsection 1.6.2 we introduce an entropy to remove this term and obtain a preliminary bound on θ. In subsection 1.6.3, we revisit identity (1.12) to obtain an additional temperature estimate and a bound for the velocity. Finally, to convert a gradient estimate into a full H^1 bound, we will need the following variant of the Poincare inequality:

Lemma 1.6.1. *For all $M \in (0,\infty)$ and $\beta \in [1,\infty)$ there exists a positive constant $C_{M,\beta}$ with the following property.*

Each non-negative $f, g : D \to \mathbb{R}$ such that $\|g\|_{L^1_x} \geq M$ satisfies

$$\left\|f^\beta\right\|_{L^2_x} \leq C_{M,\beta}\left[\|\nabla(f^\beta)\|_{L^2_x} + \|g\|_{L^\gamma_x}^{\frac{\gamma\beta}{\gamma-1}}\left[\|fg\|_{L^1_x}^\beta + \|\nabla f\|_{L^2_x}^\beta\right]\right]. \tag{1.13}$$

The proof of Lemma 1.6.1 uses the method of Mellet/Vasseur [22], but tracking a bit more carefully the dependence of the estimate on $\|g\|_{L^\gamma}$. This is important for our purposes since this quantity will be a random process.

1.6.1 Estimates for the total energy

Recall that by Hypothesis 1.1.2, the pressure decomposes as $P(\rho,\theta) = p_m(\rho)+\theta p_\theta(\rho)$. Let us define

$$P_m(\rho) = \int_1^\rho \frac{p_m(z)}{z^2} dz. \tag{1.14}$$

Set $\beta(\rho) = \rho P_m(\rho)$ in the renormalized form (1.10) and use the identity

$$\rho\big[(\rho P_m(\rho))' - P_m(\rho)\big] = p_m(\rho). \tag{1.15}$$

Combining with the temperature equation, we find the evolution of the internal energy:

$$\begin{aligned} &\partial_t\big(\rho(P_m(\rho)+\theta)\big) + \operatorname{div}\big(\rho u(P_m(\rho)+\theta) - \kappa(\theta)\nabla\theta\big) \\ &\qquad = \mathcal{S} : \nabla u - P(\rho,\theta)\operatorname{div} u. \end{aligned} \tag{1.16}$$

Using Ito's Formula together with the momentum equation in (1.1) yields:

$$\partial_t(\frac{1}{2}\rho|u|^2) + \operatorname{div}(\frac{1}{2}\rho u|u|^2) = (\operatorname{div}\mathcal{S} - \nabla P)\cdot u + \rho|\sigma_k|^2 + \rho\sigma_k \cdot u\dot{\beta}_k. \tag{1.17}$$

Combining (1.16) and (1.17), integrating over $D\times[0,t]$, and using the boundary conditions gives:

$$\begin{aligned} &\int_D \Big(\rho\theta(t) + \frac{1}{2}\rho|u|^2(t) + \rho P_m(\rho)(t)\Big)\mathrm{d}x = \int_D \Big(\rho_0\theta_0 + \frac{1}{2}\frac{|m_0|^2}{\rho_0} + \rho_0 P_m(\rho_0)\Big)\,\mathrm{d}x \\ &+ \sum_{k=1}^\infty \int_0^t \int_D \rho u\cdot\sigma_k(\rho,\rho u,\rho\theta,x)\mathrm{d}x\mathrm{d}\beta_k(s) + \frac{1}{2}\sum_{k=1}^\infty \int_0^t\int_D \rho|\sigma_k(\rho,\rho u,\rho\theta,x)|^2\mathrm{d}x\mathrm{d}s. \end{aligned} \tag{1.18}$$

Applying the Burkholder/Davis/Gundy inequality followed by Hölder and Hypothesis 1.1.6 yields for all $p \in [1, \infty)$

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]}\left|\sum_{k\in\mathbb{N}}\int_0^t\int_D \rho u\cdot\sigma_k \mathrm{d}x\mathrm{d}\beta_k(s)\right|^p\right] + \mathbb{E}\left[\left|\sum_{k\in\mathbb{N}}\int_0^T\int_D \rho|\sigma_k|^2\mathrm{d}x\mathrm{d}s\right|^p\right]\\
&\lesssim \mathbb{E}\left[\left|\sum_{k\in\mathbb{N}}\int_0^T\left(\int_D \rho u\cdot\sigma_k\mathrm{d}x\right)^2\mathrm{d}s\right|^{p/2}\right] + \mathbb{E}\left[\left|\sum_{k\in\mathbb{N}}\int_0^T\int_D \rho|\sigma_k|^2\mathrm{d}x\mathrm{d}s\right|^p\right]\\
&\lesssim \left(\sum_{k\in\mathbb{N}}\|\sigma_k\|^2_{L_x^{\frac{2\gamma}{\gamma-1}}(L^\infty_{\rho,m,\alpha})}\right)^{p/2}\mathbb{E}\left[\left|\int_0^T\|\rho u(s)\|^2_{L_x^{\frac{2\gamma}{\gamma+1}}}\mathrm{d}s\right|^{p/2}\right]\\
&+\left(\sum_{k\in\mathbb{N}}\|\sigma_k\|^2_{L_x^{\frac{2\gamma}{\gamma-1}}(L^\infty_{\rho,m,\alpha})}\right)^{p}\mathbb{E}\left[\left|\int_0^T\|\rho(s)\|_{L_x^\gamma}\mathrm{d}s\right|^p\right]\\
&\lesssim \mathbb{E}\left[|\rho|^{\frac{p}{2}}_{L_t^\infty(L_x^\gamma)}|\sqrt{\rho}u|^p_{L_t^\infty(L_x^2)}\right] + \mathbb{E}\left[|\rho|^p_{L_t^\infty(L_x^\gamma)}\right].
\end{aligned}
$$

Maximizing over $[0, T]$ and taking $L^p(\Omega)$ norms of both sides of the total energy identity yields:

$$
\begin{aligned}
&\mathbb{E}\left[\|\rho\theta\|^p_{L_t^\infty(L_x^1)} + \|\sqrt{\rho}u\|^{2p}_{L_t^\infty(L_x^2)} + \|\rho P_m(\rho)\|^p_{L_t^\infty(L_x^1)}\right]\\
&\lesssim \mathbb{E}\left[\|\rho_0\theta_0\|^p_{L_x^1} + \left\|\frac{m_0}{\sqrt{\rho_0}}\right\|^{2p}_{L_x^2} + \|\rho_0 P_m(\rho_0)\|^p_{L_x^1}\right]\\
&+\mathbb{E}\left[\|\rho\|^{\frac{p}{2}}_{L_t^\infty(L_x^\gamma)}\|\sqrt{\rho}u\|^p_{L_t^\infty(L_x^2)}\right] + \mathbb{E}\left[\|\rho\|^p_{L_t^\infty(L_x^\gamma)}\right] + 1.
\end{aligned}
$$

In view of Hypothesis 1.1.2, $\rho P_m(\rho) \sim \rho^\gamma$. By Young's inequality we deduce the following a priori bound:

$$
\mathbb{E}\left[\|\rho\theta\|^p_{L_t^\infty(L_x^1)} + \|\sqrt{\rho}u\|^{2p}_{L_t^\infty(L_x^2)} + \|\rho\|^{\gamma p}_{L_t^\infty(L_x^\gamma)}\right] \lesssim 1. \tag{1.19}
$$

The implicit constant in the inequality above depends only on p, T and the inputs described in Hypotheses 1.1.2-1.1.6.

1.6.2 Further bounds on the temperature

Begin by letting

$$P_\theta(\rho) = \int_1^\rho \frac{p_\theta(z)}{z^2} \mathrm{d}z. \tag{1.20}$$

Define the entropy $s(\rho,\theta) = \log(\theta) - P_\theta(\rho)$. Use the temperature renormalization $H(\theta) = \log(\theta)$ in (1.12) and the density renormalization $\beta(\rho) = \rho P_\theta(\rho)$ in (1.10). Taking the difference of these two equations yields the $\mathbb{P}$ a.s. inequality:

$$\int_D \theta^{-2}\kappa(\theta)|\nabla\theta|^2 \mathrm{d}x \le \frac{d}{dt}\int_D \rho s \mathrm{d}x. \tag{1.21}$$

The integral over $[0,T]$ of the RHS of inequality (1.21) can be controlled as follows:

$$\begin{aligned}
&\int_D \rho s(T)\mathrm{d}x - \int_D \rho_0 s_0 \mathrm{d}x \le \int_D \rho\log(\theta)(T)\mathrm{d}x - \int_D \rho_0 s_0 \mathrm{d}x \\
&\le \int_{\theta(T)\ge 1} \rho\log(\theta)(T)\mathrm{d}x + \int_D \rho_0 P_\theta(\rho_0)\mathrm{d}x - \int_{\theta_0<1} \rho_0\log(\theta_0)\mathrm{d}x \\
&\lesssim \|\rho\theta\|_{L_t^\infty(L_x^1)} + \int_D \rho_0^\Gamma \mathrm{d}x + \big|\log(\theta_{min}\wedge 1)\big|\|\rho_0\|_{L_x^1}.
\end{aligned}$$

In the last inequality, we used Hypothesis 1.1.1 to bound θ_0 from below and Hypothesis 1.1.2 to deduce $P_\theta(\rho_0) \sim \rho_0^{\Gamma-1}$. Finally, since $\Gamma = \frac{\gamma}{d}$, we can integrate (1.21) over $[0,T]$, take $L^p(\Omega)$ norms on both sides, and appeal to (1.19) to deduce:

$$\mathbb{E}\|\theta^{-1}\kappa^{\frac{1}{2}}(\theta)\nabla\theta\|_{L_{t,x}^2}^{2p} \lesssim 1. \tag{1.22}$$

Hypothesis 1.1.3 on the heat conductivity coefficient now implies

$$\mathbb{E}\big[\|\nabla\theta\|_{L_{t,x}^2}^{2p} + \|\nabla\log(\theta)\|_{L_{t,x}^2}^{2p}\big] \lesssim 1. \tag{1.23}$$

Finally, we seek an estimate on θ in L_x^2. For each $(\omega,t) \in \Omega\times[0,T]$, apply Lemma 1.6.1 (with $\beta = 1$) to $f(x) = \theta(t,x,\omega)$ and $g(x) = \rho(t,x,\omega)$. Conservation of mass together with Hypothesis 1.1.1 imply that the stochastic process $(\omega, t) \to \|\rho(t,\omega)\|_{L^1(D)}$ is deterministic and stationary through time. Hence, integrating over $[0,T]$ yields the following $\mathbb{P}$ a.s. inequality:

$$
\begin{aligned}
\|\theta\|_{L^2_{t,x}} &\lesssim \left[\|\nabla\theta\|_{L^2_{t,x}} + \|\rho\|^{\frac{\gamma}{\gamma-1}}_{L^\infty_t(L^\gamma_x)}\left(\|\rho\theta\|_{L^2_t(L^1_x)} + \|\nabla\theta\|_{L^2_{t,x}}\right)\right] \\
&\lesssim \left[1 + \|\nabla\theta\|^2_{L^2_{t,x}} + \|\rho\theta\|^2_{L^\infty_t(L^1_x)} + \|\rho\|^{\frac{2\gamma}{\gamma-1}}_{L^\infty_t(L^\gamma_x)}\right].
\end{aligned}
$$

Taking $L^p(\Omega)$ norms on both sides and using the bounds (1.19) for the total energy together with (1.23), we deduce for all $p \in [1,\infty)$:

$$
\mathbb{E}\left[\|\theta\|^{2p}_{L^2_t(H^1_x)}\right] \lesssim 1. \tag{1.24}
$$

1.6.3 Velocity estimates and an improved temperature bound

Apply (1.12) with $H(\theta) = \theta$ and $H_\sigma(\theta) = \theta^{1-\sigma}$, then integrate over $[0,T]$ to find the $\mathbb{P}$ a.s. inequality:

$$
\begin{aligned}
&\int_0^T \int_D \left(|\nabla u|^2 + \theta^{-(\sigma+1)}\kappa(\theta)|\nabla\theta|^2\right)\mathrm{d}x \\
&\lesssim \left\|\rho\theta^{1-\sigma}\right\|_{L^\infty_t(L^1_x)} + \|\rho\theta\|_{L^\infty_t(L^1_x)} + \left\|\theta[\theta^{-\sigma}+1]p_\theta(\rho)\operatorname{div} u\right\|_{L^1_{t,x}} \\
&\lesssim \|\rho\|_{L^\infty_t(L^\gamma_x)} + \|\rho\theta\|_{L^\infty_t(L^1_x)} \\
&+ \left[\|\theta\|_{L^2_t(L_x^{\frac{2d}{d-2}})} + \|\theta\|^{1-\sigma}_{L^2_t(L_x^{\frac{2d}{d-2}})}\right]\|p_\theta(\rho)\|_{L^\infty_t(L^d_x)}\|\operatorname{div} u\|_{L^2_{t,x}}.
\end{aligned}
$$

Applying Hypotheses (1.1.2) and (1.1.3) together with Young's inequality gives another $\mathbb{P}$ a.s. inequality:

$$
\|\nabla u\|^2_{L^2_{t,x}} + \|\nabla(\theta^{\frac{3-\sigma}{2}})\|^2_{L^2_{t,x}} \lesssim \left[1 + \|\rho\|^\gamma_{L^\infty_t(L^\gamma_x)} + \|\rho\theta\|_{L^\infty_t(L^1_x)} + \|\theta\|^{\frac{1}{2}-\frac{1}{d}}_{L^2_t(H^1_x)}\right].
$$

Taking $L^p(\Omega)$ norms on both sides gives:

$$\mathbb{E}\big[\|\nabla u\|_{L^2_{t,x}}^{2p} + \|\nabla(\theta^{\frac{3-\sigma}{2}})\|_{L^2_{t,x}}^{2p}\big] \lesssim 1.$$

Applying Lemma 1.6.1 with $\beta = (3-\sigma)/2$ and arguing as in the last section:

$$\mathbb{E}\big[\|\nabla u\|_{L^2_{t,x}}^{2p} + \|\theta^{\frac{3-\sigma}{2}}\|_{L^2_t(H^1_x)}^{2p}\big] \lesssim 1.$$

1.7 Auxilliary classical results

In this section, we recall a number of classical results which will be useful throughout the thesis.

1.7.1 Random variables on topological spaces and the Skorohod theorem

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(E, \tau, \mathcal{B}_\tau)$ be a topological space endowed with its Borel sigma algebra. A mapping $X : \Omega \to (E, \tau)$ is called an "E valued random variable" provided it is a measurable mapping between these spaces. Every E valued random variable induces a probability measure on $(E, \tau, \mathcal{B}_\tau)$ by pushforward, which we denote $\mathbb{P} \circ X^{-1}$. A sequence of probability measures $\{\mathbb{P}_n\}_{n\in\mathbb{N}}$ on $\mathcal{B}_\tau$ is said to be "tight" provided that for each $\xi > 0$ there exists a τ compact set K_ξ such that $\mathbb{P}_n(K_\xi) \geq 1 - \xi$ for all $n \in \mathbb{N}$.

A collection $\big(X_t\big)_{t=0}^{T}$ is an E valued stochastic process provided that for each t, X_t is an E valued random variable. An E valued stochastic process is progressively measurable with respect to the filtration $\left(\mathcal{F}^t\right)_{t=0}^T$ provided that for each $t \leq T$,

$$X\mid_{[0,t]} : \Omega \times [0,t] \to (E, \tau, \mathcal{B}_\tau)$$

is measurable with respect to the product sigma algebra $\mathcal{F}_t \times \mathcal{B}([0,t])$.

Definition 1.7.1. *A topological space (E, τ) is called a Jakubowski space provided there exists a countable sequence $\{G_k\}_{k\in\mathbb{N}} : E \to \mathbb{R}$ of τ continuous functionals which separate points in E.*

Our main interest in such spaces is the following fundamental result:

Theorem 1.7.2. *Let (E, τ) be a Jakubowski space. Suppose that $\{X_k\}_{k\in\mathbb{N}}$ is a sequence of E valued random variables on a sequence of probability spaces $\{\left(\Omega_k, \mathcal{F}_k, \mathbb{P}_k\right)\}_{k\in\mathbb{N}}$ such that $\left\{\mathbb{P}_k \circ X_k^{-1}\right\}_{k\in\mathbb{N}}$ is tight.*

Then there exists a new probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with an E valued random variable X and a sequence of "recovery" maps $\{\widetilde{T}_k\}_{k\in\mathbb{N}}$

$$\widetilde{T}_k : (\Omega, \mathcal{F}, \mathbb{P}) \to \left(\Omega_k, \mathcal{F}_k, \mathbb{P}_k\right)$$

with the following two properties:

1. *For each $k \in \mathbb{N}$, the measure $\mathbb{P}_k$ may be recovered from $\mathbb{P}$ by pushing forward $\widetilde{T}_k$.*

2. *The new sequence $\{\tilde{X}_k\}_{k\geq 1} := \{X_k \circ \widetilde{T}_k\}_{k\geq 1}$ converges pointwise to X (with respect to the topology τ).*

Proof. This result is a combination of the versions of the Skorohod theorem proved in [17] and [28]. It can be proved by modifying the proof in [17] in a very slight way. Namely, at the point in the proof where the classical Skorohod theorem for metric spaces is applied, one may apply the Skorohod theorem in [28] to obtain the recovery maps. $\square$

Remark 1.7.3. *It is straightforward to check that the following are examples of Jakubowski spaces: Polish spaces, dual spaces of separble Banach spaces B_{w*} endowed with the weak star topology, and $C_t(B_w)$ for reflexive Banach spaces B.*

Also, the class of Jakubowski spaces is closed under countable products. In particular, given a Jakubowski space (E, τ), E^∞ is also a Jakubowski space with respect to the τ product topology. Similarly, for finite products of different Jakubowski spaces.

We will need the following lemma in our analysis of the temperature equation.

Lemma 1.7.4. *The space $L^1(\mathbb{R}^d)$ is a Jakubowski space.*

Proof. It suffices to show that there is a contable family $\{e_i\}$ of functions in $L^\infty(\mathbb{R}^d)$ such that for any $f \in L^1(\mathbb{R}^d)$, the linear-functional

$$L_i(f) := \int_{\mathbb{R}^d} e_i \, f$$

vanishes for all i if and only if $f = 0$. Clearly if $f = 0$, $L_i(f) = 0$ regardless of the choice of $\{e_i\}$. In view of the Sobolev embedding $W^{d,1}(\mathbb{R}^d) \hookrightarrow C_0(\mathbb{R}^d) \subseteq L^\infty(\mathbb{R}^d)$ and the separability of $W^{d,1}(\mathbb{R}^d)$, we choose $\{e_i\}$ any countable dense subset of $W^{d,1}(\mathbb{R}^d)$. Suppose that $L_i(f) = 0$ for all i. Clearly the map $e_i \to L_i(f)$ is continuous in $W^{d,1}(\mathbb{R}^d)$ since

$$|L_i(f)| \leq \|f\|_{L^1} \|e_i\|_{L^\infty} \leq C \|f\|_{L^1} \|e_i\|_{W^{d,1}}.$$

Therefore by the density of $\{e_i\}$ in $W^{d,1}(\mathbb{R}^d)$ we conclude that

$$\int_{\mathbb{R}^d} \varphi f = 0 \quad \forall \varphi \in W^{d,1}(\mathbb{R}^d).$$

It follows that $f = 0$. Note that we can always normalize define $K_i = L_i/\|L_i\|$ so that K_i takes values in $[-1, 1]$. $\square$

1.7.2 Series of one dimensional stochastic integrals

By a stochastic basis, we mean a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ together with a a filtration $(\mathcal{F}_t)_{t=0}^T$ and a collection $\{\beta_k\}_{k=1}^\infty$ of $(\mathcal{F}_t)_{t=0}^T$ one dimensional Brownian motions.

Proposition 1.7.5. *Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}^t)_{t=0}^T, \{\beta_k\}_{k\in\mathbb{N}})$ be a stochastic basis endowed with a collection of $\{\mathcal{F}^t\}_{t=0}^T$ progressively measurable proceeses $\{f_k\}_{k=1}^\infty : \Omega \times [0, T] \to \mathbb{R}$, such that*

$$\sum_{k=1}^\infty \int_0^T \mathbb{E}^{\mathbb{P}} \left[f_k^2(s) \right] ds < \infty.$$

Then we may construct an $\{\mathcal{F}^t\}_{t=0}^T$ martingale $\{M_t\}_{t=0}^T$ with $\mathbb{P}$ a.s. continuous paths of the form

$$M_t = \sum_{k=1}^\infty \int_0^t f_k(s) d\beta_k(s).$$

The series above converges uniformly in time in probability and the quadratic variation process is given by

$$\langle M \rangle_2^t(\omega) = \sum_{k=1}^\infty \int_0^t f_k^2(s, \omega) ds$$

.

Proof. This is a consequence of the Kolomogorov Three Series theorem and the construction of the one dimensional stochastic integral. See Krylov [19] for more discussion. $\square$

The next lemma, taken from [6], provides a procedure for identifying a continuous, adapted process as a series of one dimensional stochastic integrals.

Lemma 1.7.6. *Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t=0}^T, \{\beta_k\}_{k=1}^\infty)$ be a stochastic basis endowed with a continuous $\{\mathcal{F}_t\}_{t=0}^T$ martingale $\{M_t\}_{t=0}^T$. Moreover, suppose the following are also $\{\mathcal{F}_t\}_{t=0}^T$ martingales*

1. $(\omega, t) \to M_t^2(\omega) - \sum_{k=1}^\infty \int_0^t f_k^2(\omega, s) ds$
2. $(\omega, t) \to M_t(\omega)\beta_t^k(\omega) - \int_0^t f_k(s) ds \quad$ *(for each $k \geq 1$)*

then the process $\{M_t\}_{t=0}^T$ may be identified as

$$M_t = \sum_{k=1}^\infty \int_0^t f_k(s) d\beta_k(s).$$

1.7.3 The space of weakly continuous functions in L_x^m

This section contains a useful tightness criterion for probability measures over the topological space $C_t([L_x^p]_w)$.

Lemma 1.7.7. *Let $\{f_n\}_{n=1}^\infty$ be a sequence in $L_t^\infty(L_x^m)$ with $1 < m < \infty$. Suppose that the following two criterion are met:*

1. $\sup_n |f_n|_{L_t^\infty(L_x^m)} < \infty$

2. *For all $\phi \in C_c^\infty(D)$ in a dense subset of L_x^m, the following sequence in C_t is equicontinuous*

$$\left\{ t \to \int_D f_n(x,t)\phi(x)dx \right\}_{n=1}^\infty .$$

Then there exists an $f \in C_t([L_x^m]_w)$ and a subsequence such that

$$f_{n_k} \to f \quad in \quad C_t([L_x^m]_w).$$

Proof. See the Appendix in Lions [20]. □

A straightforward application of the lemma above yields

Corollary 1.7.8. *For any positive M, integer k and $q > 1$, the following sets are compact in $C_t([L_x^p]_w)$*

$$\{f \in C_t([L_x^p]_w) \mid |f|_{L_t^\infty(L_x^m)} + |\partial_t f|_{L_t^q(W_x^{-k,p})} \le M\}$$

The tightness criterion can now be stated as follows

Lemma 1.7.9. *Let $\{f_n\}_{n=1}^\infty$ be a collection of $C_t([L_x^m]_w)$ valued random variables, each defined on a probability space $(\Omega_n, \mathcal{F}_n, \mathbb{P}_n)$ such that*

1.
$$\sup_n \mathbb{E}^{\mathbb{P}_n} |f_n|_{L_t^\infty(L_x^m)} < \infty$$

2. *For any $\phi \in C_c^\infty(D)$, there exists an integer k, $\gamma > 0$, and $p > \frac{1}{\gamma}$ such that*

$$\sup_n \mathbb{E}^{\mathbb{P}_n} \left[\left| \langle f_n(t) - f_n(s), \phi \rangle \right|^p \right] \le |\phi|_{C_x^k}^p |t-s|^{\gamma p}$$

for all $0 \le s, t \le T$. Then the sequence of induced measures $\{\mathbb{P}_n \circ f_n^{-1}\}_{n=1}^\infty$ are tight on $C_t([L_x^m]_w)$.

Proof. Enumerate a countable collection $\{\phi_j\}_{j=1}^{\infty}$ in $C_c^\infty(D)$ which is dense in $L_x^{m'}$. The second hypothesis of the lemma implies that for all $s < \gamma$ and $j \geq 1$

$$\sup_{n\geq 1} \mathbb{E}|\langle f_n, \phi_j\rangle|^p_{W_t^{s,p}} \lesssim |\phi_j|^p_{C^k}.$$

Choosing $\alpha > 0$ sufficiently small to apply the Sobolev embedding theorem gives

$$\sup_{n\geq 1} \mathbb{E}|\langle f_n, \phi_j\rangle|^p_{C_t^\alpha} \lesssim |\phi_j|^p_{C^k}.$$

Given a small number $\xi > 0$, define a set K_ξ by

$$K_\xi = \{f \in L_t^\infty(L_x^m) \mid |f|_{L_t^\infty(L_x^m)} \leq M\xi^{-1}\} \cap \bigcap_{j=1}^{\infty} \left\{ f \in L_t^\infty(L_x^m) \mid |\langle f, \phi_j\rangle|_{C_t^\alpha} \leq (2^j\xi^{-1})^{\frac{1}{p}}|\phi|_{C^k} \right\}.$$

Lemma 1.7.7 implies this set is sequentially compact in $C_t([L_x^m]_w)$. Sequential compactness and compactness are equivalent in $C_t([L_x^m]_w)$. Applying Chebyshev, then using the uniform bounds, we find that

$$\sup_{n\geq 1} \mathbb{P} \circ f_n^{-1}(K_\xi^c) \lesssim \xi.$$

□

1.7.4 Weak convergence upgrades

The following lemma is simple, but fundamental enough to state explicitly.

Lemma 1.7.10. *Let E, F be Banach spaces and use E_w to denote the space endowed with its weak topology. Let $T : E \to F$ be a bounded linear operator. Suppose the sequence $\{f_n\}_{n=1}^{\infty}$ converges to f in $C_t(E_w)$. Then $\{Tf_n\}_{n=1}^{\infty}$ converges to Tf in $L_t^q(F_w)$ for all $1 \leq q < \infty$.*

Lemma 1.7.11. *Make the same assumptions as in Lemma 1.7.10 above. In addition, assume T is compact. Then $\{Tf_n\}_{n=1}^{\infty}$ converges to Tf in $L_t^q(F)$ for all $1 \leq q < \infty$.*

Proof. Since bounded operators preserve weak convergence, for each $t \in [0, T]$ we have $Tf_n(t) \to Tf(t)$ weakly in F. If T is compact then the convergence is strong. Combining the uniform bounds in $C_t(E_w)$ with the Vitali convergence theorem gives both claims. □

The following proposition is proved in [11].

Proposition 1.7.12. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence converging to f in $D'_{t,x}$. Suppose there exists $\{g_n\}_{n=1}^{\infty}$ such that $\partial_t f_n \geq g_n$ in $D'_{t,x}$ for all $n \geq 1$.*

If $f_n \in L_t^2(L_x^q) \cap L_t^{\infty}(L_x^1)$ with $q < \frac{2d}{d+2}$ and $g_n \in L_t^1(W_x^{-m,r})$ with $m, r > 1$, uniformly in n, then $f_n \to f$ in $L_t^2(H_x^{-1})$.

Recall that M_x denotes the vector space of signed radon measures over D, equipped with the total variation norm. Also note that $L_t^{\infty}(M_x) = [L_t^1\big(C_c(D)\big)]^*$, hence this space carries a natural weak-$\star$ topology.

Lemma 1.7.13. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in $L_t^{\infty}(L_x^1)$. Then $\{f_n\}_{n=1}^{\infty}$ is weak-$\star$ compact in $L_t^{\infty}(M_x)$ if and only it is uniformly bounded in $L_t^{\infty}(L_x^1)$.*

Proof. Recall that every $g \in L^1(D)$ defines a radon measure μ_g with density g such that $|\mu_g|_{TV} = |g|_{L^1(D)}$. In particular, this implies that

$$|f_n|_{L_t^{\infty}(L_x^1)} = |f_n|_{L_t^{\infty}(M_x)}. \tag{1.25}$$

By Banach-Alaoglu, closed balls in $L_t^\infty(M_x)$ are weak-$\star$ compact. Hence, (1.25) implies bounded sequences in $L_t^\infty(L_x^1)$ are weak-$\star$ compact in $L_t^\infty(M_x)$.

On the other hand, if $\{f_n\}_{n=1}^\infty$ is weak-$\star$ compact in $L_t^\infty(M_x)$, then it is uniformly bounded in $L_t^\infty(M_x)$. Using 1.25 once more yields $\{f_n\}_{n=1}^\infty$ is uniformly bounded in $L_t^\infty(L_x^1)$. $\square$

Lemma 1.7.14. *Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space. Let $\{f_n\}_{n=1}^\infty$ in $L^p(\Omega, \mathcal{F}, \mu)$ converge weakly to $f \in L^p(\Omega, \mathcal{F}, \mu)$. Moreover, assume there is a convex function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\{\varphi(f_n)\}_{n=1}^\infty$ converges weakly to $\varphi(f)$ in $L^1(\Omega, \mathcal{F}, \mu)$. Denote by $\mathcal{C}$ the subset of $\mathbb{R}$ where φ is strictly convex.*

Then there is a full μ measure set Ω' such that $\{f_n(\omega)\}_{n=1}^\infty$ converges pointwise to $f(\omega)$ for all $\omega \in \mathcal{C} \cap \Omega'$.

1.7.5 Some tools from the deterministic, compressible theory

The following result is a consequence of the Div Curl lemma. Denote $\mathcal{R}_{ij} = \partial_{ij}\Delta^{-1}$, understood to be well defined on compactly supported distributions.

Lemma 1.7.15. *Let D be a smooth, bounded domain and η a smooth cutoff. Let B be a Banach space. Suppose $\{f_n\}_{n=1}^\infty$ converges to f in $C_t([L_x^p]_w)$ and $\{g_n\}_{n=1}^\infty$ converges to g in $C_t([L_x^q]_w)$. Also, assume the embedding $L_x^r \hookrightarrow B$ is compact, where $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} < 1$.*

Then the following convergence holds:

$$\eta\left(f_n \mathcal{R}_{ij}[\eta g_n] - g_n \mathcal{R}_{i,j}[\eta f_n]\right) \to \eta\left(f \mathcal{R}_{ij}[\eta g] - g \mathcal{R}_{i,j}[\eta f]\right)$$

weakly in $L_t^m(B)$ for all $1 \leq m < \infty$.

Proof. Combine the corresponding result in Feiresil [10] with Lemma 1.7.11(using the compact injection operator from L^r_x to B). $\square$

Next we collect some properties of the Bogovoski operator $\mathcal{B}$. Recall that classically, $\mathcal{B}[g]$ is defined to be the solution to the problem

$$\begin{cases} \operatorname{div} v = g & \text{in} \quad D \\ \\ v = 0 & \text{on} \quad \partial D \end{cases} \tag{1.26}$$

for $g \in L^p(D)$ such that $\int_D g dx = 0$. For our purposes, it is useful to have an extension of this operator to the negative Sobolev spaces. We recall a result from [16]. Define the space $L^p_0(D) = \{f \in L^p(D) \mid \int_D f dx = 0\}$. For $s \in [0,1]$ define $\widehat{W}^{s,p}(D) := W^{s,p}(D) \cap L^p_0(D)$. Furthermore, let $\widehat{W}^{-s,p}(D) := [\widehat{W}^{s,p'}(D)]'$.

Theorem 1.7.16. *Let $1 < p < \infty$ and $s \in [-1,1]$. Then there exists a bounded linear operator $\mathcal{B} : \widehat{W}^{s,p}(D) \to [W^{s+1,p}(D)]^d$ such that $\operatorname{div} \mathcal{B}[g] = g$ for all $g \in \widehat{W}^{s,p}(D)$.*

1.7.6 Lemmas on parabolic equations

The following lemma provides an energy equality for sufficiently integrable weak solutions to the parabolic Neumann problem driven by a rough velocity field.

Lemma 1.7.17. *Let $u \in L^2_t(W^{1,2}_x)$ and $p > d$. Suppose $\rho \in L^\infty_t(L^p_x)$ is a distribu-*

tional solution of

$$\begin{cases} \partial_t \rho - \epsilon \Delta \rho + \mathrm{div}(\rho u) = 0 & \textit{in} \quad D \times [0,T] \\ \frac{\partial \rho}{\partial n} = 0 & \textit{in} \quad \partial D \times [0,T] \\ \rho(x,0) = \rho_0(x) & \textit{in} \quad D \end{cases} \tag{1.27}$$

Then for all times $0 \le t \le T$*, the energy identity holds:*

$$\frac{1}{2}\int_D \rho^2(t)dx + \epsilon \int_0^t \int_D |\nabla \rho|^2 dxdt = \frac{1}{2}\int_D \rho_0^2 dx - \frac{1}{2}\int_0^t \int_D \mathrm{div}\, u\, \rho^2 dxdt.$$

We also need a variant of the usual $L_t^q(W_x^{2,p})$ estimates for the parabolic Neumann problem. A similar result is proved in the appendix to [3]. The lemma below states that by giving up the optimal exponent, one can retain a form of the usual estimate even if the solution. Recalling the splitting from section 2.1. Choose τ such that $\frac{T}{\tau}$ is an even integer and let $t_k = k\tau$ for $k = 0, ..., \frac{T}{\tau}$. Define h_{det}^{τ} by (2.1).

Lemma 1.7.18. *Let* $1 < p < \infty$ *and* $F \in L_{t,x}^p$*. Suppose that* ρ *solves*

$$\begin{cases} \partial_t \rho - \epsilon \Delta \rho = F & \textit{in} \quad D \times \bigcup_{k=0}^{\frac{T}{2\tau}-1} (t_{2k}, t_{2k+1}] \\ \partial_t \rho = 0 & \textit{in} \quad D \times \bigcup_{k=0}^{\frac{T}{2\tau}-1} (t_{2k+1}, t_{2k+2}] \\ \frac{\partial \rho}{\partial n} = 0 & \textit{on} \quad \partial D \times [0,T] \\ \rho(0,x) = \rho_0 & \textit{in} \quad D \end{cases} \tag{1.28}$$

Then for all $q < p$

$$|\partial_t \rho|_{L_{t,x}^q} + |\rho|_{L_t^q(W_x^{2,q})} \le C(\epsilon, d) \left(|\rho_0|_{W_x^{2,p}} + |F h_{det}^{\tau}|_{L_{t,x}^p} \right). \tag{1.29}$$

1.7.7 A weighted poincare inequality

The following variant of the Poincare inequality can be established with the Rellich lemma via the classical argument by contradiction(for instance).

Lemma 1.7.19. *For all $\beta \geq 1$ there exists a positive constant C_β such that for all non-negative $f : D \to \mathbb{R}$, the following inequality holds:*

$$|f|_{L^2(D)} \leq C\big[|\nabla f|_{L^2(D)} + |f^{\frac{1}{\beta}}|_{L^1(D)}\big]. \tag{1.30}$$

We will employ Lemma 1.7.19 to prove the following:

Lemma 1.7.20. *For all $M > 0$ and $\beta \geq 1$, there exists a positive constant $C_{M,\beta}$ such that for all non-negative $f, g : D \to \mathbb{R}$ with $|g|_{L^1(D)} \geq M$, the following inequality holds:*

$$|f^\beta|_{L^2(D)} \leq C_M\big(|\nabla(f^\beta)|_{L^2(D)} + |g|_{L^\gamma}^{\frac{\gamma\beta}{\gamma-1}}\big[|fg|^\beta_{L^1(D)} + |\nabla f|^\beta_{L^2(D)}\big]\big). \tag{1.31}$$

Proof. We will apply Lemma 1.7.19 to f^β. Hence, it it suffices to estimate $|f|_{L^1(D)}$. Towards this end, introduce a good set G as follows:

$$G = \{x \in D \mid g(x) > \frac{1}{2|D|}\int_D gdx\}.$$

First we check the following lower bound:

$$|G| \geq \big(\frac{|g|_{L^1_x}}{2|g|_{L^\gamma}}\big)^{\frac{\gamma}{\gamma-1}}. \tag{1.32}$$

This follows from the following decomposition

$$\int_D gdx = \int_G gdx + \int_{D\setminus G} gdx \leq |G|^{\frac{\gamma-1}{\gamma}}|g|_{L^\gamma_x} + \frac{1}{2}\frac{|D\setminus G|}{|G|}\int_D gdx. \tag{1.33}$$

Absorbing the last term into the LHS, then raising both sides to the power $\frac{\gamma}{\gamma-1}$ gives the claim. Next we write

$$
\begin{aligned}
&\int_D f dx = |G|^{-1}\big(|D|\int_G f dx + [|G|\int_D f dx - |D|\int_G f dx]\big) \\
&= |G|^{-1}\big(|D|\int_G f dx + \iint_{G\times D} |f(y)-f(x)|dxdy\big) \\
&\lesssim |G|^{-1}\big(2|D||g|_{L^1}^{-1}\int_G gf dx + |\nabla f|_{L^2}\big)
\end{aligned}
$$

Raising to the power β we find:

$$
|f|^\beta_{L^1(D)} \leq |G|^{-\beta}2^{\beta-1}\big[2^\beta|D|^\beta|g|^{-\beta}_{L^1}|fg|^\beta_{L^1(D)} + |\nabla f|^\beta_{L^2}\big]. \tag{1.34}
$$

Using our lower bound for $|G|$, we find:

$$
\begin{aligned}
&|f|^\beta_{L^1(D)} \leq \big(\frac{|g|_{L^\gamma_x}}{|g|_{L^1}}\big)^{\frac{\beta\gamma}{\gamma-1}}2^{\beta-1}\big[2^\beta|D|^\beta|g|^{-\beta}_{L^1}|fg|^\beta_{L^1(D)} + |\nabla f|^\beta_{L^2}\big] \\
&\leq C_{\beta,M}|g|_{L^\gamma}^{\frac{\gamma\beta}{\gamma-1}}\big[|fg|^\beta_{L^1(D)} + |\nabla f|^\beta_{L^2(D)}\big].
\end{aligned}
$$

Combining this estimate with the Lemma gives the claim. □

Chapter 2: Approximate Weak Martingale Solutions: τ, n, and ϵ Layers

2.1 τ Layer existence

In this section, we build the first layer of our approximating scheme, the τ layer. Each of the parameters n, ϵ, and δ are present in the notion of solution, Definition 2.1.3 below, but they are frozen in this section, so we only indicate dependence of the approximating sequence on τ, the time splitting parameter. We partition the time interval $[0, T]$ into $\frac{T}{\tau}$ time intervals of length τ, where $\frac{T}{\tau}$ is assumed to be an even integer. Denoting $t_j = j\tau$, we define the functions h^τ_{det} and h^τ_{st} via

$$h^\tau_{\text{det}}(s) = \sum_{j=0}^{\frac{T}{2\tau}-1} 1_{(t_{2j}, t_{2j+1}]}(s) = 1 - h^\tau_{\text{st}}(s). \tag{2.1}$$

The main result of this section is the following:

Theorem 2.1.1. *For each $\tau > 0$, there exists a τ layer approximation $(\rho_\tau, u_\tau, \theta_\tau)$ to 1.1 (in the sense of Definition 2.1.3 below).*

Now we give the precise definition of a τ layer approximation. There are three elements of our approximating scheme we must introduce: a finite dimensional space where the velocity evolves, a regularization of the multiplicative structure of the noise, and an artifical pressure.

To truncate the high modes of the velocity field, we introduce of collection $\{\Pi_n\}_{n\in\mathbb{N}}$ of linear operators satisfying the following:

Hypothesis 2.1.2. *For each $n \in \mathbb{N}$, Π_n is a bounded linear operator from $L^1(D;\mathbb{R}^d)$ to $C^3(D;\mathbb{R}^d) \cap C_c(D;\mathbb{R}^d)$ with a finite dimensional range. For all $p \in (1,\infty)$, $s = 0,1,2,3$, and $u \in W^{s,p}(D;\mathbb{R}^d)$,*

$$\lim_{n\to\infty} \|\Pi_n u - u\|_{W^{s,p}(D;\mathbb{R}^d)} = 0. \tag{2.2}$$

The collection $\{\Pi_n\}_{n\in\mathbb{N}}$ can be constructed using a wavelet expansion. For more details on wavelet expansions in domains, see [27]. These operators are accompanied by a collection of finite dimensional spaces $\{X_n\}_{n\in\mathbb{N}}$ defined by $X_n = \Pi_n\big(L^2(D;\mathbb{R}^d)\big)$.

In view of these remarks, let $C^+(D)$ be the cone of positive functions in $C(D)$ and η_δ a standard mollifer. Define the operator

$$\sigma_{k,\tau,n,\delta} : C^+(D) \times L^1(D;\mathbb{R}^d) \times L^1(D) \to X_n$$

by the relation

$$\sigma_{k,\tau,n,\delta}(\rho, m, \alpha) = \Pi_n \circ \sigma_k\big(\rho * \eta_\delta(\cdot), [(\rho \wedge \frac{1}{\tau})\frac{m}{\rho}] * \eta_\delta(\cdot), \alpha * \eta_\delta(\cdot), \cdot\big),$$

where (ρ, m, θ) are understood to be extended by zero outside of D to give a meaning to the convolution. For the remainder of this section, we will use the abbreviation $\sigma_{k,\tau}$.

Finally, the original pressure in the momentum equation will be replaced by an “artifical” one of the form $p_m(\rho) + \delta\rho^\beta + \theta p_\theta(\rho)$ for a sufficiently large power β.

For technical reasons which will be clear later, we require that

$$\beta > \max(d, 2\gamma, 4). \tag{2.3}$$

Definition 2.1.3. *A triple $(\rho_\tau, u_\tau, \theta_\tau)$ is defined to be a τ layer approximation to (1.1) provided there exists a stochastic basis $\big(\Omega_\tau, \mathcal{F}_\tau, (\mathcal{F}_t^\tau)_{t=0}^T, \mathbb{P}_\tau, \{\beta_k^\tau\}_{k=1}^n\big)$ such that:*

1. *For all $t \in [0,T]$, $\mathcal{F}_t^\tau = \sigma\big(\{\beta_k^\tau(s)\}_{k=1}^n : s \le t\big)$.*
2. *The quadruple $(\rho_\tau, \rho_\tau u_\tau, \rho_\tau\theta_\tau, u_\tau)$ belongs in $L^2\big(\Omega \times [0,T]; \mathcal{P}; L^\beta \times L^\beta \times L^q \times X_n\big)$, where $\mathcal{P}$ is the predictable σ-algebra generated by $(\mathcal{F}_\tau^t)_{t=0}^T$ and $\frac{1}{q} = \frac{1}{\gamma} + \frac{1}{2} - \frac{1}{d}$.*
3. *For all $\phi \in C^\infty(D)$ and $t \in [0,T]$, the following equality holds $\mathbb{P}_\tau$ a.s.*
$$\int_D \rho_\tau(t)\phi \mathrm{d}x = \int_D \rho_{0,\delta}\phi \mathrm{d}x + \int_0^t \int_D 2h_{det}^\tau(s)[\rho_\tau u_\tau \cdot \nabla\phi + \epsilon\rho_\tau\Delta\phi]\mathrm{d}x\mathrm{d}s. \tag{2.4}$$
4. *For all $\phi \in X_n$ and $t \in [0,T]$, the following equality holds $\mathbb{P}_\tau$ a.s.*
$$\begin{aligned}\int_D \rho_\tau u_\tau(t)\cdot\phi \mathrm{d}x &= \int_D m_{0,\delta}\cdot\phi \mathrm{d}x + \int_0^t\int_D 2h_{det}^\tau[\rho_\tau u_\tau \otimes u_\tau - \mathcal{S}(u_\tau)] : \nabla\phi \mathrm{d}x\mathrm{d}s \\ &+ \int_0^t\int_D 2h_{det}^\tau\big[(P(\rho_\tau,\theta_\tau) + \delta\rho_\tau^\beta)\operatorname{div}\phi - \epsilon\nabla u_\tau\nabla\rho_\tau\cdot\phi\big]\mathrm{d}x\mathrm{d}s \\ &+ \sum_{k=1}^n\int_0^t\int_D \sqrt{2}h_{st}^\tau\rho_\tau\sigma_{k,\tau}(\rho_\tau,\rho_\tau u_\tau,\rho_\tau\theta_\tau)\cdot\phi\mathrm{d}x\mathrm{d}\beta_k^\tau(s).\end{aligned} \tag{2.5}$$
5. *For all $\varphi \in C^\infty(D)$ with $\frac{\partial\varphi}{\partial n} = 0$ along ∂D, the following equality holds $\mathbb{P}_\tau$ almost surely:*
$$\begin{aligned}&\int_D \big(\rho_\tau(t)+\delta\big)\theta_\tau(t)\varphi\mathrm{d}x - \int_0^t\int_D \big(\delta\theta_\tau^3\varphi + K(\theta_\tau)\Delta\varphi\big)\mathrm{d}x\mathrm{d}s \\ &= \int_D \big(\rho_{0,\delta}+\delta\big)\theta_{0,\delta}\phi\mathrm{d}x + \int_0^t\int_D 2h_{det}^\tau\rho_\tau\theta_\tau u_\tau\cdot\nabla\varphi\mathrm{d}x\mathrm{d}s \\ &+ \int_0^t\int_D 2h_{det}^\tau\big[(1-\delta)\mathcal{S}(u_\tau):\nabla u_\tau - \theta_\tau p_\theta(\rho_\tau)\operatorname{div} u_\tau\big]\varphi\mathrm{d}x\mathrm{d}s.\end{aligned} \tag{2.6}$$

In the definition above, we have replaced the initial data (ρ_0, m_0, θ_0) by a triple $(\rho_{0,\delta}, m_{0,\delta}, \theta_{0,\delta})$ satisfying:

Hypothesis 2.1.4. *For each $\delta > 0$, $\rho_{0,\delta}$ and $\theta_{0,\delta}$ belong to $C^\infty(D)$ and obey the bounds*

$$\delta \leq \rho_{0,\delta} \leq \delta^{-\frac{1}{2\beta}} \qquad\qquad \underline{\theta} \leq \theta_{0,\delta} \leq |\theta_0|_{L^\infty}. \tag{2.7}$$

Both $\rho_{0,\delta}$ and $\theta_{0,\delta}$ are assumed to satisfy a Neumann boundary condition. The sequence $\big\{(\rho_{0,\delta}, \theta_{0,\delta})\big\}_{\delta>0}$ converges strongly to (ρ_0, θ_0) in $L^\gamma_x \times L^1_x$. Finally, we define $m_{0,\delta} = m_0 1_{\{\rho_{0,\delta} \geq \rho_0\}}$ and assume $|\{\rho_{0,\delta} < \rho_0\}| \to 0$.

2.1.1 Machinery from the deterministic theory

For each $\rho \in C^+(D)$, define a multiplication type operator $\mathcal{M}[\rho] : X_n \to X_n^*$ as follows: for $u, \eta \in X_n$,

$$\langle \mathcal{M}[\rho]u, \eta\rangle = \int_D \rho(x)u(x) \cdot \eta(x)\mathrm{d}x.$$

The proof of the lemma below is left to the reader.

Lemma 2.1.5. *For each $\rho \in C^+(D)$, $\mathcal{M}[\rho] : X_n \to X_n^*$ is an invertible (linear) mapping.*

Let us also define a mapping $\mathcal{N} : C(D) \times X_n \times L^2(D) \to X_n^*$ by

$$\langle \mathcal{N}[\rho, u, \theta], \eta\rangle = \int_D \big[\rho u \otimes u - \mathcal{S}(u) + (P(\rho, \theta) + \delta\rho^\beta)I\big] : \nabla\eta - \epsilon\nabla u\nabla\rho \cdot \eta \;\mathrm{d}x.$$

Proposition 2.1.6. *Let $s < t$ be initial and final times and suppose initial data $\rho_{in} \in C^+(D)$, $u_{in} \in C^1(D; \mathbb{R}^d)$, and $\theta_{in} \in C^+(D)$ are given. Then there exists a* *unique pair*

$$(\rho, u, \theta) \in C\left((s,t]; C^2(D) \cap C^+(D)\right) \times C\left((s,t]; X_n\right) \times C\left((s,t]; C^2(D)\right),$$

solving the following system in $D \times [s,t]$:

$$\begin{cases} \partial_t \rho = 2\epsilon \Delta \rho - 2\operatorname{div}(\rho u) \\ u(S) = \mathcal{M}^{-1}[\rho(S)] \circ \left(m^*_{in} + \int_s^S 2\mathcal{N}\big[u(r), \rho(r), \theta(r)\big]\mathrm{d}r\right) \\ \partial_t\big((\rho+\delta)\theta\big) + \delta\theta^3 - \operatorname{div}(\kappa(\theta)\nabla\theta) + 2\operatorname{div}(\rho\theta u) \\ = 2(1-\delta)\mathcal{S} : \nabla u - 2\theta p_\theta(\rho)\operatorname{div} u \\ \frac{\partial \rho}{\partial n} = \frac{\partial \theta}{\partial n} = 0 \\ (\rho(s), \theta(s)) = (\rho_{in}, \theta_{in}) \end{cases} \tag{2.8}$$

The quantity $m^*_{in} \in X^*_n$ *is defined for* $\eta \in X_n$ *via the relation*

$$\langle m^*_{in}, \eta\rangle = \int_D \rho_{in} u_{in} \cdot \eta \mathrm{d}x.$$

If $u_{in} \in X_n$, *then* $u(s) = u_{in}$. *Moreover, the solution map is continuous from* $C^+(D) \times X_n \times L^2(D)$ *to* $C\left((s,t]; X_n\right) \times C\left((s,t]; L^2(D)\right)$.

We also require the following classical result on nonlinear parabolic equations.

Proposition 2.1.7. *Let* $\rho \in C^\infty(D)$ *be non-negative and* $\delta > 0$. *For all times* $s < t$ *and* $\theta_{in} \in L^2(D)$, *there exists a unique classical solution to*

$$\begin{cases} \partial_t\big((\rho+\delta)\theta\big) + \delta\theta^3 - \operatorname{div}(\kappa(\theta)\nabla\theta) = 0 \quad \textit{in} \quad D \times (s,t] \\ \theta(s) = \theta_{in}. \end{cases} \tag{2.9}$$

2.1.2 A classical SPDE result

Let $s < t$ and $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_r)_{r=s}^t, \{\beta_k\}_{k=1}^n)$ be a stochastic basis such that the filtration $(\mathcal{F}_r)_{r=s}^t$ is generated by the collection $\{\beta_k\}_{k=1}^n$. Let $\mathcal{P}$ be the predictable σ-algebra generated by $(\mathcal{F}_r)_{r=s}^t$. Suppose $\rho \in L^2(\Omega; \mathcal{F}_s; C_x^+)$ and $\theta \in L^2\big(\Omega \times [s,t]; \mathcal{P}; C^+(D)\big)$ are given. A stochastic process $u \in L^2\big(\Omega \times [s,t]; \mathcal{P}; X_n\big)$ is said to be a solution to the SPDE

$$\begin{cases} \partial_t u = \sum_{k=1}^n \sigma_{k,\tau}(\rho, \rho u, \rho\theta)\dot{\beta}_k(t) & \text{in} \quad (s,t] \times D \\ u(s) = u_{in} & \text{in} \quad D \end{cases} \tag{2.10}$$

provided that for all $S \in [s,t]$ the following equality (in X_n) holds $\mathbb{P}$ a.s.

$$u(S) = u_{in} + \sum_{k=1}^n \int_s^S \sigma_{k,\tau}\big(\rho, \rho u(r), \rho\theta(r)\big) \mathrm{d}\beta_k(r). \tag{2.11}$$

Proposition 2.1.8. *There exists a unique solution* $u \in L^2\big(\Omega; C\left([s,T]; X_n\right)\big)$ *to* (2.10) *in the sense of* (2.11).

Proof. The τ layer regularization of the noise coefficient ensures that the proposition can be established with the Cauchy-Lipschitz theorem in a classical way. □

2.1.3 Proof of Theorem 2.1.1

We are now prepared to establish an existence theorem for the lowest level of our scheme.

Proof. Let $\tau > 0$ be given. We will define the solution inductively. Namely, suppose that $(\rho_\tau, u_\tau, \theta_\tau)$ have been constructed to satisfy the continuity equation (2.4), the momentum equation (2.5), and the temperature equation (2.6) on the time interval

$[0, t_{2j}]$. To extend the solution to the interval $(t_{2j}, t_{2j+1}]$, apply Proposition 2.1.6 to find a unique triple (ρ, u, θ) satisfying the following system on $D \times (t_{2j}, t_{2j+1}]$:

$$\begin{cases} \partial_t \rho + 2\,\mathrm{div}(\rho u) - 2\epsilon \Delta \rho = 0 \\ u(t) = \mathcal{M}^{-1}[\rho(t)] \circ \Big(\rho_\tau u_\tau(t_{2j})^* + \int_{t_{2j}}^t 2\,\mathcal{N}\big[u(s), \rho(s), \theta(s)\big]\mathrm{d}s \Big) \\ \partial_t\big((\rho+\delta)\theta\big) + 2\,\mathrm{div}(\rho\theta u) + \delta\theta^3 - \Delta\mathcal{K}(\theta) \\ = 2(1-\delta)\mathcal{S} : \nabla u - 2\theta p_\theta(\rho)\,\mathrm{div}\, u \\ \frac{\partial \rho}{\partial n} = \frac{\partial \theta}{\partial n} = 0 \\ (\rho(t_{2j}), u(t_{2j}), \theta(t_{2j})) = (\rho_\tau(t_{2j}), u_\tau(t_{2j}), \theta_\tau(t_{2j})) \end{cases} \tag{2.12}$$

To extend the solution to the interval $(t_{2j+1}, t_{2j+2}]$ we appeal first to Proposition 2.1.7 to solve for the temperature and then to Propositions 2.1.8 to solve for the velocity. Observe that the evolution of the temperature does not involve the velocity, allowing us to decouple the two equations. In this manner, we find a unique triple (ρ, u, θ) satisfying the following system over $D \times (t_{2j+1}, t_{2j+2}]$:

$$\begin{cases} \partial_t \rho = 0 \\ \partial_t u = \sqrt{2}\sum_{k=1}^n \sigma_{k,\tau}(\rho, \rho u, \rho\theta)\dot{\beta}_k \\ \partial_t\big((\rho+\delta)\theta\big) + \delta\theta^3 = \mathrm{div}(\kappa(\theta)\nabla\theta) \\ (\rho(t_{2j+1}), u(t_{2j+1}), \theta(t_{2j+1})) = (\rho_\tau(t_{2j+1}), u_\tau(t_{2j+1}), \theta_\tau(t_{2j+1})) \end{cases} \tag{2.13}$$

Using the Ito Formula and the inductive hypothesis, one may check that (2.4)-(2.6) continue to hold for $t \in [t_{2j}, t_{2j+2}]$. The desired measurability, part 2 of Definition 2.1.3, follows from the continuity of the solution map to the deterministic problem (guaranteed by Proposition 2.1.6), together with the fact the that we obtain a stochastically strong solution during each time interval where the stochastic forcing evolves. $\square$

2.2 n Layer existence

In this section, we apply Theorem 2.1.1 to build the next layer of the approximating scheme, the n layer. Our goal is to establish the following:

Theorem 2.2.1. *For each $n \in \mathbb{N}$, there exists an n layer approximation (in the sense of Definition 2.2.2 below) (ρ_n, u_n, θ_n) of* (1.1) *relative to a stochastic basis $\big(\Omega_n, \mathcal{F}_n, (\mathcal{F}_t^n)_{t=0}^T, \mathbb{P}_n, \{\beta_k^n\}_{k=1}^n\big)$.*

Let us introduce the n layer regularization of the multiplicative noise structure. Define an operator $\sigma_{k,n,\delta} : L^1(D) \times L^1(D;\mathbb{R}^d) \times L^1(D) \to X_n$ via the relation

$$\sigma_{k,n,\delta}(\rho, m, \alpha) = \Pi_n \circ \sigma_k\big(\rho * \eta_\delta(\cdot), m * \eta_\delta(\cdot), \alpha * \eta_\delta(\cdot), \cdot\big). \tag{2.14}$$

For the remainder of this section, we will use the abbreviation $\sigma_{k,n}$.

Definition 2.2.2. *A triple (ρ_n, u_n, θ_n) is defined to be an n layer approximation to* (1.1) *provided there exists a stochastic basis $\big(\Omega_n, \mathcal{F}_n, (\mathcal{F}_t^n)_{t=0}^T, \mathbb{P}_n, \{\beta_k^n\}_{k=1}^n\big)$ such that*

1. *The quadruple $(\rho_n, \rho_n u_n, \theta_n, u_n)$ belongs in $L^2\big(\Omega \times [0,T]; \mathcal{P}; L^\beta \times L^\beta \times L^q \times X_n\big)$, where $\mathcal{P}$ is the predictable σ-algebra generated by $(\mathcal{F}_t^n)_{t=0}^T$ and $\frac{1}{q} = \frac{1}{\gamma} + \frac{1}{2} - \frac{1}{d}$.*

2. *For all $\phi \in C^\infty(D)$ and all times $t \in [0,T]$ the following equality holds $\mathbb{P}_n$ a.s.*

$$\int_D \rho_n(t)\phi \mathrm{d}x = \int_D \rho_{0,\delta}\phi \mathrm{d}x + \int_0^t \int_D [\rho_n u_n \cdot \nabla\phi + \epsilon\rho_n \Delta\phi] \mathrm{d}x \mathrm{d}s. \tag{2.15}$$

3. For all $\phi \in X_n$ and all times $t \in [0,T]$ the following equality holds $\mathbb{P}_n$ a.s.

$$\begin{aligned}
&\int_D \rho_n u_n(t) \cdot \phi \mathrm{d}x = \int_D m_{0,\delta} \cdot \phi \mathrm{d}x + \int_0^t \int_D [\rho_n u_n \otimes u_n - \mathcal{S}(u_n)] : \nabla \phi \mathrm{d}x \mathrm{d}s \\
&+ \int_0^t \int_D \left[(P(\rho_n, \theta_n) + \delta \rho_n^\beta) \operatorname{div} \phi - \epsilon \nabla u_n \nabla \rho_n \cdot \phi \right] \mathrm{d}x \mathrm{d}s \\
&+ \sum_{k=1}^n \int_0^t \int_D \rho_n \sigma_{k,n}(\rho_n, \rho_n u_n, \rho_n \theta_n) \cdot \phi \mathrm{d}x \mathrm{d}\beta_k^n(s).
\end{aligned} \tag{2.16}$$

4. For all $\varphi \in C^\infty(D)$ with $\frac{\partial \varphi}{\partial n} |_{\partial D} = 0$, the following equation holds $\mathbb{P}_n$ almost surely:

$$\begin{aligned}
&\int_D (\rho_n(t) + \delta) \theta_n(t) \varphi \mathrm{d}x = \int_D (\rho_{0,\delta} + \delta) \theta_{0,\delta} \varphi \mathrm{d}x \\
&+ \int_0^t \int_D [\rho_n \theta_n u_n - \kappa(\theta_n) \nabla \theta_n] \cdot \nabla \varphi \mathrm{d}x \mathrm{d}s \\
&+ \int_0^t \int_D [(1-\delta) \mathcal{S}(u_n) : \nabla u_n - \theta_n p_\theta(\rho_n) \operatorname{div} u_n - \delta \theta_n^3] \varphi \mathrm{d}x \mathrm{d}s.
\end{aligned} \tag{2.17}$$

For each n fixed we apply Theorem 2.1.1 to obtain a sequence of τ layer approximations $\{(\rho_{\tau,n}, u_{\tau,n}, \theta_{\tau,n})\}_{\tau>0}$. In Section 2.2.1, we prove a compactness result for this sequence and extract a candidate n layer approximation (ρ_n, u_n, θ_n) built on a convenient choice of probability space $(\Omega_n, \mathcal{F}_n, \mathbb{P}_n)$. In Section 2.2.2, we use the compactness result to verify (ρ_n, u_n, θ_n) is an n layer approximation in the sense of Definition 2.2.2.

2.2.1 $\tau \to 0$ Compactness step

A key tool in this section is a τ layer analogue of the renormalized temperature equation (1.10). For the convienience of the reader, we will now explain briefly how to derive this in the current context. For simplicitly of notation, we drop dependence on τ for the moment. The following computations can be justified using the further regularity properties proved in Lemma 7.4 of [11]. Begin by observing that the continuity and temperature equations can be written in the compact form:

$$\begin{cases} \partial_t\rho = 2h^\tau_{\mathrm{det}}\big[\operatorname{div}(\rho u) - \epsilon\Delta\rho\big] \\ \partial_t\big((\rho+\delta)\theta\big) + \delta\theta^3 + \operatorname{div}\big(2h^\tau_{\mathrm{det}}\rho u\theta - \kappa(\theta)\nabla\theta\big) \\ = 2h^\tau_{\mathrm{det}}[(1-\delta)\mathcal{S}(u) : \nabla u - \theta p_\theta(\rho)\operatorname{div} u]. \end{cases} \tag{2.18}$$

Let $H : \mathbb{R}_+ \to \mathbb{R}$, multiply by $H'(\theta)$, and use the parabolic equation to deduce:

$$\begin{aligned} &\big[\partial_t\big((\rho+\delta)\theta\big) + \operatorname{div}(2h^\tau_{\mathrm{det}}\rho u\theta - \kappa(\theta)\nabla\theta)\big]H'(\theta) \\ &= \partial_t\big((\rho+\delta)H(\theta)\big) + \operatorname{div}\big(2h^\tau_{\mathrm{det}}\rho u H(\theta) - \kappa(\theta)H'(\theta)\nabla\theta\big) \\ &+ 2\epsilon h^\tau_{\mathrm{det}}\Delta\rho\big[\theta H'(\theta) - H(\theta)\big] + \kappa(\theta)H''(\theta)|\nabla\theta|^2. \end{aligned}$$

Hence, we obtain the following renormalized form:

$$\begin{aligned} &\partial_t\big((\rho+\delta)H(\theta)\big) + \kappa(\theta)H''(\theta)|\nabla\theta|^2 + \delta\theta^3 H'(\theta) \\ &+ \operatorname{div}\big(2h^\tau_{\mathrm{det}}\rho u H(\theta) - \kappa(\theta)H'(\theta)\nabla\theta\big) \\ &= 2h^\tau_{\mathrm{det}}\big[(1-\delta)\mathcal{S}(u) : \nabla u - \theta p_\theta(\rho)\operatorname{div} u\big]H'(\theta) \\ &- 2\epsilon h^\tau_{\mathrm{det}}\Delta\rho[\theta H'(\theta) - H(\theta)]. \end{aligned} \tag{2.19}$$

The next lemma obtains estimates of the type derived in Section 1.6.

Lemma 2.2.3. *For all $p \geq 1$, there exists $C(p,n,\epsilon,\delta)$ such that:*

$$\begin{aligned} &\sup_{\tau>0}\mathbb{E}^{\mathbb{P}}\Big[\|\sqrt{\rho_\tau}u_\tau\|^{2p}_{L^\infty_t(L^2_x)} + \|\rho_\tau\|^{\beta p}_{L^\infty_t(L^\beta_x)} + \|(\rho_\tau+\delta)\theta_\tau\|^p_{L^\infty_t(L^1_x)}\Big] \leq C. \\ &\sup_{\tau>0}\mathbb{E}^{\mathbb{P}}\Big[\|h^\tau_{det}u_\tau\|^{2p}_{L^2_t(H^1_{0,x})} + \|\nabla(h^\tau_{det}\rho_\tau^{\frac{\beta}{2}})\|^{2p}_{L^2_{t,x}} + \|\theta_\tau\|^{3p}_{L^3_{t,x}}\Big] \leq C. \\ &\sup_{\tau>0}\mathbb{E}^{\mathbb{P}}\Big[\|\theta_\tau\|^{2p}_{L^2_t(H^1_x)} + \|\nabla\log(\theta_\tau)\|^{2p}_{L^2_{t,x}}\Big] \leq C. \end{aligned} \tag{2.20}$$

Proof. The first two lines of (2.20) may be obtained all at once. Indeed, we can apply the Ito formula to find the evolution of the total energy. For all $t \in [0,T]$:

$$\begin{aligned}
&\int_D \frac{1}{2}\rho_\tau |u_\tau|^2(t) + \rho_\tau P_m(\rho_\tau(t)) + \frac{\delta}{\beta-1}\rho_\tau^\beta(t) + (\rho_\tau+\delta)\theta_\tau(t)\mathrm{d}x \\
&+ \int_0^t \int_D \delta[\mathcal{S}(u_\tau) : \nabla u_\tau + \theta_\tau^3]\mathrm{d}x\mathrm{d}s \\
&+ \int_0^t \int_D 2\epsilon h_{\text{det}}^\tau (\frac{p_m'(\rho_\tau)}{\rho_\tau} + \delta\beta\rho_\tau^{\beta-2})|\nabla\rho_\tau|^2 \mathrm{d}x\mathrm{d}s \\
&= \sum_{k=1}^n \int_0^t \int_D \sqrt{2} h_{\text{st}}^\tau \rho_\tau u_\tau \cdot \sigma_{k,\tau}(\rho_\tau, \rho_\tau u_\tau, \rho_\tau\theta_\tau)\mathrm{d}x\mathrm{d}\beta_k^\tau(s) \\
&+ \sum_{k=1}^n \int_0^t \int_D h_{\text{st}}^\tau \rho_\tau |\sigma_{k,\tau}(\rho_\tau, \rho_\tau u_\tau, \rho_\tau\theta_\tau)|^2 \mathrm{d}x\mathrm{d}t + E_n(0).
\end{aligned} \tag{2.21}$$

Moreover, the sequence $\{E_n(0)\}_{n=1}^\infty$ satisfies the uniform bounds

$$\sup_n E_n(0) \leq E_\delta(0) = \frac{1}{2}\int_D \left[\frac{|m_{0,\delta}|^2}{\rho_{0,\delta}} + \rho_{0,\delta}P_m(\rho_{0,\delta}) + \rho_{0,\delta}\theta_{0,\delta}\right]\mathrm{d}x.$$

Using the same approach as in Section 1.6, the $L^p(\Omega; L_t^\infty)$ norms of the RHS of (2.21) can be estimated in terms of the $L^p(\Omega; L_t^\infty)$ norms of the LHS of (2.21). Indeed, the only additional fact needed is the L_x^p boundedness of the operators Π_n. This follows from Hypothesis 2.1.2 and the Banach/Steinhaus theorem.

To obtain the remaining estimates, use (2.19) with $H(\theta) = \log(\theta)$and integrate over $[0,T] \times D$ to find the following $\mathbb{P}$ a.s. inequality:

$$\begin{aligned}
\int_0^T \int_D \theta_\tau^{-2}\kappa(\theta_\tau)|\nabla\theta_\tau|^2 \mathrm{d}x\mathrm{d}s \leq\ & 2\int_0^T \int_D p_\theta(\rho_\tau)\operatorname{div}(h_{\text{det}}^\tau u_\tau)\mathrm{d}x\mathrm{d}s \\
&+ 2\int_0^T \int_D \epsilon\Delta(h_{\text{det}}^\tau \rho)(\log(\theta_\tau) - 1)\mathrm{d}x\mathrm{d}s \\
&+ \delta\int_0^T \int_D \theta_\tau^2 \mathrm{d}x\mathrm{d}s + \int_D \rho\log(\theta_\tau)(T)\mathrm{d}x \\
&- \int_D \rho_{0,\delta}\log(\theta_{0,\delta})\mathrm{d}x.
\end{aligned}$$

Integrating by parts, we observe that:

$$\int_0^T \int_D \epsilon\Delta(h^\tau_{\det}\rho)(\log(\theta_\tau)-1)\mathrm{d}x\mathrm{d}s = -\int_0^T \int_D \epsilon\nabla(h^\tau_{\det}\rho_\tau)\cdot\nabla(\log(\theta_\tau))\mathrm{d}x\mathrm{d}s$$
$$\le \gamma'\|\nabla\log(\theta_\tau)\|^2_{L^2_{t,x}} + C_{\gamma'}\|\epsilon\nabla(h^\tau_{\det}\rho_\tau)\|^2_{L^2_{t,x}}.$$

Hence, for any $\gamma' > 0$, there exists a $C_{\gamma'}$ such that

$$\int_0^T \int_D \theta^{-2}\kappa(\theta_\tau)|\nabla\theta_\tau|^2\mathrm{d}x\mathrm{d}s \le \gamma'\|\nabla\log(\theta)\|^2_{L^2_{t,x}} + \|p_\theta(\rho)\|_{L^2_t(L_x^{\frac{2d}{d+2}})}\|h^\tau_{\det}u\|_{L^2_t(H^1_x)}$$
$$+ C_\gamma\left(\|\epsilon\nabla(h^\tau_{\det}\rho)\|^2_{L^2_{t,x}} + \|\rho\theta\|_{L^\infty_t(L^1_x)} + \delta\|\theta\|^3_{L^3_{t,x}} + 1\right).$$

Observe that $p_\theta(\rho) \sim \rho^{\frac{\gamma}{d}}$ and $L^\gamma_x \hookrightarrow L_x^{\frac{2\gamma}{d+2}}$. Thus, we obtain:

$$\sup_{\tau>0}\mathbb{E}\left(\|\nabla\theta_\tau\|^{2p}_{L^2_{t,x}} + \|\nabla\log(\theta_\tau)\|^{2p}_{L^2_{t,x}}\right) \le C(p,n,\epsilon,\delta). \tag{2.22}$$

Finally, using the modified Poincare inequality (1.6.1) as in Section 1.6 yields the last line in (2.20). □

The next proposition is the main compactness step, yielding a candidate n layer approximation and a new sequence of τ layer approximations with improved compactness properties.

Proposition 2.2.4. *There exists a probability space $(\Omega_n, \mathcal{F}_n, \mathbb{P}_n)$, a collection of independent Brownian motions $\{\beta^k_n\}^n_{k=1}$, a limit point (ρ_n, u_n, θ_n), and a sequence of measurable maps $\{\widetilde{T}_\tau\}_{\tau>0}$ such that:*

1. *For all $\tau > 0$, $\widetilde{T}_\tau : (\Omega_n, \mathcal{F}_n, \mathbb{P}_n) \to (\Omega, \mathcal{F}, \mathbb{P})$ and $\mathbb{P} = (\widetilde{T}_\tau)_\#\mathbb{P}_n$.*
2. *The new sequence $\{(\tilde{\rho}_\tau, \tilde{u}_\tau, \tilde{\theta}_\tau)\}_{\tau>0}$ defined by $\left(\tilde{\rho}_\tau, \tilde{u}_\tau, \tilde{\theta}_\tau\right) = (\rho_\tau, u_\tau, \theta_\tau)\circ\widetilde{T}_\tau$ constitutes a τ layer approximation to (1.1) relative to the stochastic basis*

$\big(\Omega_n, \mathcal{F}_n, \mathbb{P}_n, (\widetilde{\mathcal{F}}^t_\tau)_{t=0}^T, \widetilde{W}_\tau\big)$, *where* $\widetilde{W}_\tau := W \circ \widetilde{T}_\tau$ *and* $(\widetilde{\mathcal{F}}^\tau_t)_{t=0}^T$ *is the filtration generated by* W_τ*. Moreover, the initial data are recovered in the sense that* $\tilde{\rho}_n(0) = \rho_{0,\delta}$, $\tilde{u}_n(0) = \mathcal{M}^{-1}[\rho_{0,\delta}]m_{0,\delta}$, *and* $\tilde{\theta}_n(0) = \theta_{0,\delta}$.

3. *The uniform bounds in Lemma 2.2.3 hold with* $\tilde{\rho}_\tau, \tilde{\theta}_\tau, \tilde{u}_\tau$ *replacing* $\rho_\tau, \theta_\tau, u_\tau$ *and* $\mathbb{P}_n$ *replacing* $\mathbb{P}$.

4. *As* $\tau \to 0$, *the following convergences hold pointwise* Ω_n :

$$\tilde{\rho}_\tau \to \rho_n \quad in \quad C_t(L^\beta_x) \cap L^\beta_t(W^{1,\beta}_x) \tag{2.23}$$

$$\tilde{u}_\tau \to u_n \quad in \quad C_t(X_n) \tag{2.24}$$

$$(\tilde{\rho}_\tau + \delta)\tilde{\theta}_\tau \to (\rho_n + \delta)\theta_n \quad in \quad C_t([L^2_x]_w) \tag{2.25}$$

$$\tilde{\theta}_\tau \to \theta_n \quad in \quad [L^2_t(H^1_x) \cap L^4_{t,x}]_w \tag{2.26}$$

$$\widetilde{W}_\tau \to W_n \quad in \quad \big[C_t\big]^n, \tag{2.27}$$

where $W = \{\beta^k\}_{k=1}^n$ *and* $W_n = \{\beta^k_n\}_{k=1}^n$.

The proof of Proposition 2.2.4 uses a tightness lemma. For each $\tau > 0$, define

$$Y_\tau = \big(\rho_\tau, u_\tau, (\rho_\tau + \delta)\theta_\tau, \theta_\tau, W\big),$$

where $W = \{\beta_k\}_{k=1}^n$ Observe that Y_τ induces a measure on the topological space

$$E = C_t(L^\beta_x) \cap L^\beta_t(W^{1,\beta}_x) \times C_t(X_n) \times C_t([L^2_x]_w) \times [L^2_t(H^1_x) \cap L^4_{t,x}]_w \times [C_t]^n.$$

Lemma 2.2.5. *The sequence of induced measures* $\big\{\mathbb{P} \circ (\rho_\tau, u_\tau, W)^{-1}\big\}_{\tau>0}$ *are tight on* $C_t(L^\beta_x) \cap L^\beta_t(W^{1,\beta}_x) \times \big[C_t(X_n)\big] \times [C_t]^n$.

Proof. Note that it suffices to show the tightness of each component separately. Tightness of $\mathbb{P} \circ W^{-1}$ is an immediate consequence of Arzela-Ascoli and the usual $L^2(\Omega; C_t^{1/3})$ bound on each one dimensional Brownian motion. Next we'll show

$$\lim_{M\to\infty} \sup_{\tau>0} \mathbb{P}\big(\|u_\tau\|_{C_t(X_n)} > M\big) = 0. \tag{2.28}$$

Multiplying and dividing by the density gives the pathwise upper bound

$$\sup_{t\in[0,T]} \int_D |u_\tau|^2 \mathrm{d}x \leq \|\rho_\tau^{-1}\|_{L^\infty_{t,x}} \sup_{t\in[0,T]} \int_D \rho_\tau |u_\tau|^2 \mathrm{d}x.$$

Also, note that if $X, Y : \Omega \to \mathbb{R}$ are two positive random variables, then

$$\mathbb{P}\Big(XY > M\Big) \leq \mathbb{P}\Big(X > \sqrt{M}\Big) + \mathbb{P}\Big(Y > \sqrt{M}\Big). \tag{2.29}$$

Combining these observations yields

$$\begin{aligned}\mathbb{P}\Big(\sup_{t\in[0,T]} \int_D |u_\tau|^2 \mathrm{d}x > M\Big) \leq \mathbb{P}\Big(\sup_{t\in[0,T]} \int_D \rho_\tau |u_\tau|^2 \mathrm{d}x > \sqrt{M}\Big)& \\ + \mathbb{P}\Big(\|\rho_\tau^{-1}\|_{L^\infty_{t,x}} > \sqrt{M}\Big)&.\end{aligned}$$

The first term is treated with the $L^2(\Omega)$ bounds for the kinetic energy implied by (2.20). To treat the second term, recall the splitting scheme from Section 2.1 defining the evolution of ρ_τ. On time intervals $(t_{2j}, t_{2j+1}]$, ρ_τ solves a divergence form parabolic equation with drift u_τ and remains constant on the intervals $(t_{2j+1}, t_{2j+2}]$. Iteratively apply the maximum principle then use the equivalence of the X_n and C_x^1 norms. This controls the second probability from above by

$$\begin{aligned}&\mathbb{P}\left(\|\rho_{0,\delta}^{-1}\|_{L^\infty_x} \exp\Big[\int_0^T h_{\mathrm{det}}^\tau(t) \|\operatorname{div} u_\tau(t)\|_{L^\infty_x} \mathrm{d}t\Big] > \sqrt{M}\right) \\ \leq &\mathbb{P}\left(\int_0^T \|h_{\mathrm{det}}^\tau(t) u_\tau(t)\|_{X_n} \mathrm{d}t > C_n^{-1} \log\Big[M \|\rho_{0,\delta}^{-1}\|_{L^\infty_x}^{-1}\Big]\right).\end{aligned}$$

Applying a Hölder (in time) and the $L^2_t(X_n)$ bounds on the velocity implied by (2.20), we can make this second probability uniformly arbitrarily small also. Hence, (2.28) is established.

We can now bootstrap (2.28) and prove the tightness of $\{\mathbb{P}\circ\rho_\tau^{-1}\}_{\tau>0}$ on $C_t(L^\beta_x)\cap L^\beta_t(W^{1,\beta}_x)$. To this end, we use Lemma 1.7.18 from the appendix. For simplicity, we will omit dependence of the estimate on the initial density, since it has been smoothed out already. Start by defining the exponent q via the interpolation condition $\frac{1}{q}=\frac{1}{2\beta}+\frac{1}{2(\beta+1)}$ to obtain the following estimate:

$$\begin{aligned}\|\partial_t\rho_\tau\|_{L^\beta_{t,x}}+\|\rho_\tau\|_{L^\beta_t(W^{2,\beta}_x)}&\lesssim\|h^\tau_{\mathrm{det}}\operatorname{div}(\rho_\tau u_\tau)\|_{L^q_{t,x}}\lesssim\|u_\tau\|_{C_t(X_n)}\|h^\tau_{\mathrm{det}}\rho_\tau\|_{L^q_t(W^{1,q}_x)}\\&\lesssim\|u_\tau\|_{C_t(X_n)}\|h^\tau_{\mathrm{det}}\rho_\tau\|^{\frac{1}{2}}_{L^{\beta+1}_{t,x}}\|\rho_\tau\|^{\frac{1}{2}}_{L^\beta_t(W^{2,\beta}_x)}.\end{aligned}$$

Applying Cauchy's inequality, we may close the estimate then interpolate once more to obtain

$$\begin{aligned}\|\partial_t\rho_\tau\|_{L^\beta_{t,x}}+\|\rho_\tau\|_{L^\beta_t(W^{2,\beta}_x)}&\lesssim\|u_\tau\|^2_{C_t(X_n)}\|h^\tau_{\mathrm{det}}\rho_\tau\|_{L^{\beta+1}_{t,x}}\lesssim\|u_\tau\|^2_{C_t(X_n)}\|h^\tau_{\mathrm{det}}\rho_\tau^\beta\|^{\frac{1}{\beta}}_{L^{\frac{\beta+1}{\beta}}_{t,x}}\\&\lesssim\|u_\tau\|^2_{C_t(X_n)}\|h^\tau_{\mathrm{det}}\rho^\beta_\tau\|^{\frac{\theta}{\beta}}_{L^\infty_t(L^1_x)}\|h^\tau_{\mathrm{det}}\rho^\beta_\tau\|^{\frac{1-\theta}{\beta}}_{L^1_t(L^{\frac{d}{d-2}}_x)}\\&\lesssim\|u_\tau\|^2_{C_t(X_n)}\|h^\tau_{\mathrm{det}}\rho_\tau\|^{\theta}_{L^\infty_t(L^\beta_x)}\|h^\tau_{\mathrm{det}}\rho^{\beta/2}_\tau\|^{\frac{1-\theta}{2\beta}}_{L^2_t(L^{\frac{2d}{d-2}}_x)}\\&\lesssim\|u_\tau\|^2_{C_t(X_n)}\|h^\tau_{\mathrm{det}}\rho_\tau\|^{\theta}_{L^\infty_t(L^\beta_x)}\|h^\tau_{\mathrm{det}}\rho^{\beta/2}_\tau\|^{\frac{1-\theta}{2\beta}}_{L^2_t(W^{1,2}_x)}.\end{aligned}\tag{2.30}$$

Note that θ is defined by the relation $\frac{\beta}{\beta+1}=\theta(1-\frac{2}{d})+(1-\theta)$. Bootstrapping this estimate once yields for all $r<\beta$

$$\|\partial_t\nabla\rho_\tau\|_{L^r_{t,x}}+\|\nabla\rho_\tau\|_{L^r_t(W^{2,r}_x)}\lesssim\|\nabla\operatorname{div}(\rho_\tau u_\tau)\|_{L^\beta_{t,x}}\lesssim\|u_\tau\|_{C_t(X_n)}\|\rho_\tau\|_{L^\beta_t(W^{2,\beta}_x)}.\tag{2.31}$$

Choosing r large enough to ensure the embedding $W^{1,r}_x\hookrightarrow L^\beta_x$ is compact, we may conclude from Arzela-Ascoli and Aubin-Lions the set

$$\{f \in L_t^\beta(W_x^{1,\beta}) \mid \|\partial_t f\|_{L_t^\beta(L_x^\beta)} + \|\partial_t \nabla f\|_{L_t^r(L_x^r)} + \|f\|_{L_t^\beta(W_x^{2,\beta})} \leq M\}.$$

is compact in $C_t(L_x^\beta) \cap L_t^\beta(W_x^{1,\beta})$. Combining (2.28) together with the uniform estimates (2.20) to control the RHS of (2.30) and (2.31), we obtain the desired tightness by Chebyshev for M large enough. Our final step is to show

$$\lim_{M\to\infty} \sup_\tau \mathbb{P}\left([u_\tau]_{C_t^{1/3}(X_n)} \geq M\right) = 0. \tag{2.32}$$

Note that the brackets indicate we are considering the Hölder seminorm, since the uniform norm has already been handled above. Recalling the operators introduced in Section 2.1.1, define the X_n^* valued processes $\left(I_\tau^D(t)\right)_{t=0}^T$ and $\left(I_\tau^S(t)\right)_{t=0}^T$ via

$$\begin{aligned} I_\tau^D(t) &= \int_0^t 2h_{\mathrm{det}}^\tau(r)\mathcal{N}\big[u_\tau(r),\rho_\tau(r)\big]\mathrm{d}r. \\ \langle I_\tau^S(t),\phi\rangle &= \sum_{k=1}^n \int_0^t \int_D \sqrt{2}h_{\mathrm{st}}^\tau(r)\tilde{\rho}_\tau(r)\sigma_{k,\tau}\big(\rho_\tau(r),\rho_\tau u_\tau(r),\rho_\tau\theta_\tau\big)\cdot\phi \mathrm{d}x\mathrm{d}x\beta_k(r) \end{aligned}$$

for $\phi \in X_n$. For each $s < t$ the momentum equation yields

$$\begin{aligned} u_\tau(t) - u_\tau(s) &= \mathcal{M}^{-1}[\rho_\tau(t)]\Big(I_\tau^D(t) - I_\tau^D(s) + I_\tau^S(t) - I_\tau^S(s)\Big) \\ &\quad + \big(\mathcal{M}^{-1}[\rho_\tau(t)] - \mathcal{M}^{-1}[\rho_\tau(s)]\big)\circ\Big(m_{0,\delta}^* + I_\tau^D(s) + I_\tau^S(s)\Big). \end{aligned} \tag{2.33}$$

Using Lemma 2.1.5 and the maximum principle, we obtain the $\mathbb{P}$ a.s. estimate

$$\begin{aligned} [u_\tau]_{C_t^{1/3}(X_n)} &\leq e^{C_nT\|\tilde{u}_\tau\|_{C_t(X_n)}}\Big[\|m_{0,\delta}^*\|_{X_n^*} + [I_\tau^D]_{C_t^{1/3}(X_n^*)} + [I_\tau^S]_{C_t^{1/3}(X_n^*)}\Big] \\ &\quad + e^{C_nT\|\tilde{u}_\tau\|_{C_t(X_n)}}\Big[\|\tilde{\rho}_\tau\|_{C_t^{1/3}(L_x^1)}\big(\|I_\tau^D\|_{C_t(X_n^*)} + \|I_\tau^S\|_{C_t(X_n^*)}\big)\Big]. \end{aligned} \tag{2.34}$$

In view of the estimates for the density above, this reduces the problem to controlling the probability that I_τ^S and I_τ^D have a large Holder norm. To estimate $\|I_\tau^D\|_{C_t^{1/3}(X_n^*)}$, note first that for $\rho \in L_x^1$ and $u \in X_n$

$$\|\mathcal{N}(\rho,u,\theta)\|_{X_n^*} \lesssim_n \left(\|\rho\|_{L_x^1}\|u\|_{X_n}^2 + \|u\|_{X_n} + \|P(\rho,\theta)\|_{L_x^1} + \delta\|\rho\|_{L_x^\beta}^\beta + \|u\|_{X_n}\|\rho\|_{W_x^{1,1}}\right).$$

Applying Hölder's inequality in time yields for all $s < t$

$$
\begin{aligned}
&\int_s^t \|\mathcal{N}\big(\rho_\tau(r), u_\tau(r), \theta_\tau(r)\big)\|_{X_n^*}\mathrm{d}r \\
&\lesssim (t-s)^{1-\frac{1}{\beta}} \left[\|\rho_\tau\|_{L_t^\infty(L_x^1)}\|u_\tau\|_{C_t(X_n)}^2 + \big\|P(\rho_\tau, \theta_\tau)\big\|_{L_t^\infty(L_x^1)} + \|\rho_\tau\|_{L_t^\infty(L_x^\beta)}^\beta\right] \\
&+ (t-s)^{1-\frac{1}{\beta}} \left[\|u_\tau\|_{C_t(X_n)}\|\rho_\tau\|_{L_t^\beta(W_x^{1,1})}\right].
\end{aligned}
$$

Certainly $1-\frac{1}{\beta} > \frac{1}{3}$. Hence, we may combine (2.28), (2.30), and the uniform bounds (2.20) to obtain

$$
\lim_{M\to\infty} \sup_\tau \mathbb{P}\left(\|I_\tau^D\|_{C_t^{1/3}(X_n^*)} > M\right) = 0. \tag{2.35}
$$

To estimate $\|I_\tau^S\|_{C_t^{1/3}(X_n^*)}$, fix a $\phi \in X_n$. Apply the BDG inequality, the boundedness of the projections, and the summability Hypotheses 1.1.6 for the noise coefficients to obtain for all $p \geq 2$

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\left[\left|\int_s^t \int_D \sqrt{2} h_{\mathrm{st}}^\tau(r)\rho_\tau(r)\sigma_{k,\tau}\big(\rho_\tau(r), \rho_\tau u_\tau(r), \rho_\tau\theta_t(r)\big)\cdot\phi\mathrm{d}x\mathrm{d}\beta_k(r)\right|^p\right] \\
&\lesssim \mathbb{E}^{\mathbb{P}}\left[\left|\int_s^t \left(\int_D \rho_\tau(r)\sigma_{k,\tau}(\rho_\tau(r), \rho_\tau u_\tau(r))\cdot\phi\mathrm{d}x\right)^2\mathrm{d}r\right|^{\frac{p}{2}}\right] \\
&\lesssim \|\sigma_k\|^p_{L_x^{\frac{\gamma}{\gamma-1}}(L^\infty_{\rho,m})} \mathbb{E}^{\mathbb{P}}\left[\left|\int_s^t \|\rho_\tau(r)\|_{L_x^\gamma}^2\mathrm{d}r\right|^{p/2}\right] \lesssim (t-s)^{p/2}\sup_\tau \mathbb{E}^{\mathbb{P}}\big[\|\rho_\tau\|^p_{L_t^\infty(L_x^\gamma)}\big].
\end{aligned}
$$

This yields for all $s < \frac{1}{2}$, $p \geq 2$ and $\phi \in X_n$

$$
\sup_{\tau>0} \mathbb{E}^{\mathbb{P}}\big[\|\langle I_\tau^S, \phi\rangle\|^p_{W_t^{s,p}}\big] \leq \sup_{\tau>0} \mathbb{E}^{\mathbb{P}}\big[\|\rho_\tau\|^p_{L_t^\infty(L_x^\gamma)}\big].
$$

Choose s, p such that $W_t^{s,p} \hookrightarrow C_t^{1/3}$. Since X_n is a finite dimensional, the uniform bounds (2.20) imply

$$
\lim_{M\to\infty} \sup_\tau \mathbb{P}\left(\|I_\tau^S\|_{C_t^{1/3}(X_n^*)} > M\right) = 0. \tag{2.36}
$$

Starting with the identity (2.34) and using (2.35), (2.36) and some elementary estimates similar to (2.29) give the tightness of the laws $\{\mathbb{P} \circ u_\tau^{-1}\}_{\tau>0}$ on $C_t(X_n)$ by Arzela-Ascoli. $\square$

Lemma 2.2.6. *The sequence of laws $\{\mathbb{P} \circ Y_\tau^{-1}\}_{\tau>0}$ are tight on E.*

Proof. The tightness of $\{\mathbb{P}\circ(\rho_\tau, u_\tau, W)^{-1}\}_{\tau>0}$ on $C_t(L_x^\beta)\cap L_t^\beta(W_x^{1,\beta})\times C_t(X_n)\times[C_t]^n$ has been established in the previous lemma. In the course of these arguments, the following useful fact is proved:

$$\lim_{M\to\infty}\sup_{\tau>0}\mathbb{P}(\|u_\tau\|_{C_t(X_n)} + \|\rho_\tau\|_{L^\infty_{t,x}} \geq M) = 0. \tag{2.37}$$

To prove the tightness of $\{\mathbb{P}\circ\big((\rho_\tau+\delta)\theta_\tau, \theta_\tau\big)^{-1}\}_{\tau>0}$ on $C_t([L_x^2]_w)\times[L_t^2(H_x^1)\cap L^4_{t,x}]_w$, we require some further estimates. Use the renormalized form (2.19) with $H(\theta) = \frac{1}{2}\theta^2$ and integrate over D to find:

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\int_D(\rho_\tau+\delta)\theta_\tau^2\mathrm{d}x + \int_D\big(\kappa(\theta_\tau)|\nabla\theta_\tau|^2 + \delta\theta_\tau^4\big)\mathrm{d}x \\ = 2h^\tau_{\det}\int_D\big[(1-\delta)S(u_\tau):\nabla u_\tau\theta_\tau - \theta_\tau^2 p_\theta(\rho_\tau)\operatorname{div}u_\tau - \frac{1}{2}\epsilon\Delta\rho_\tau\theta_\tau^2\big]\mathrm{d}x.\end{aligned}$$

Integrating by parts, we find that for any small $\gamma'>0$,

$$\int_D\Delta\rho_\tau\theta_\tau^2\mathrm{d}x = -\int_D\nabla\rho_\tau\cdot\nabla\theta_\tau\theta_\tau\mathrm{d}x \leq \gamma'\int_D\theta_\tau^2|\nabla\theta_\tau|^2\mathrm{d}x + \int_D|\nabla\rho|^2\mathrm{d}x. \tag{2.38}$$

Using that $\kappa(\theta)\sim\theta^2$, the first term above can be absorbed into the LHS of the estimate after integrating over $[0,T]$. Hence, we obtain the following $\mathbb{P}$ a.s. inequality:

$$\begin{aligned}\|\sqrt{\rho_\tau+\delta}\theta_\tau\|^2_{L_t^\infty(L_x^2)} + \delta\|\theta_\tau\|^4_{L^4_{t,x}} \lesssim & 1 + \|u_\tau\|^2_{C_t(X_n)}\|\theta_\tau\|_{L^1_{t,x}} \\ & + \|u_\tau\|_{C_t(X_n)}\|\rho_\tau\|^{\frac{\gamma}{d}}_{L^\infty_{t,x}}\|\theta_\tau\|^2_{L^2_{t,x}} + \|\nabla\rho_\tau\|^2_{L^2_{t,x}}.\end{aligned}$$

Using (2.37) and the $L^p(\Omega; L_t^2(H_x^1))$ bounds on θ_τ, ρ_τ, we find that:

$$\lim_{M\to\infty} \sup_{\tau>0} \mathbb{P}\big(\|\theta_\tau\|_{L^4_{t,x}}^4 + \|\sqrt{\rho_\tau+\delta}\theta_\tau\|^2_{L_t^\infty(L_x^2)} \geq M\big) = 0. \tag{2.39}$$

Using once more the $L^p(\Omega; L_t^2(H_x^1))$ bounds on θ_τ, we can apply (2.39) and Banach-Alagolu to deduce the tightness of $\{\mathbb{P}\circ\theta_\tau^{-1}\}_{\tau>0}$ on $[L_t^2(H_x^1) \cap L^4_{t,x}]_w$.

Our final task is to prove the tightness of $\{\mathbb{P}\circ\big((\rho_\tau+\delta)u_\tau\big)^{-1}\}_{\tau>0}$ on $C_t([L_x^2]_w)$. Towards this end, recall that for any $p>1$ and $M>0$, the set

$$\big\{f \in L_t^\infty(L_x^2) \mid \|f\|_{L_t^\infty(L_x^2)} + \|\partial_t f\|_{L_t^p(W_x^{-2,p})} \leq M\big\} \tag{2.40}$$

is compact in $C_t([L_x^2]_w)$. See Corollary 1.7.8 for instance. We will show that for $p = \frac{4}{3}$, the induced measures above become uniformly concentrated on such sets, up to small probability. Start by noting the inequality:

$$\|(\rho_\tau+\delta)\theta_\tau\|_{L_t^\infty(L_x^2)} \leq \|\sqrt{\rho_\tau+\delta}\|_{L^\infty_{t,x}}\|\sqrt{\rho_\tau+\delta}\theta_\tau\|_{L_t^\infty(L_x^2)}.$$

Hence, (2.37) and (2.39) imply:

$$\lim_{M\to\infty} \sup_{\tau>0} \mathbb{P}\big(\|(\rho_\tau+\delta)\theta_\tau\|_{L_t^\infty(L_x^2)} \geq M\big) = 0. \tag{2.41}$$

Next, write the temperature equation as follows:

$$\begin{aligned}\partial_t\big((\rho_\tau+\delta)\theta_\tau\big) &= -2h^\tau_{\text{det}}\operatorname{div}(\rho_\tau u_\tau \theta) - \delta\theta_\tau^3 + \Delta\mathcal{K}(\theta_\tau)\\ &\quad + 2h^\tau_{\text{det}}[(1-\delta)\mathcal{S}(u) : \nabla u_\tau - \theta p_\theta(\rho_\tau)\operatorname{div} u_\tau].\end{aligned}$$

Estimating each term on the RHS gives:

$$\begin{aligned}
\|h^\tau_{\det}\operatorname{div}(\rho_\tau\theta_\tau u_\tau)\|_{L^2_t(W^{-1,\frac{2d}{d-2}}_x)} &\le C\|\rho_\tau\theta_\tau u_\tau\|_{L^2_t(L^{\frac{2d}{d-2}}_x)} \le C\|\rho_\tau u_\tau\|_{L^\infty_{t,x}}\|\theta_\tau\|_{L^2_t(L^{\frac{2d}{d-2}}_x)} \\
\|-\delta\theta_\tau^3 + 2(1-\delta)h^\tau_{\det}S(u_\tau):\nabla u_\tau\|_{L^{\frac{4}{3}}_{t,x}} &\le C\Big(\|\theta_\tau\|^3_{L^4_{t,x}} + \|u_\tau\|^2_{C_t(X_n)}\Big) \\
\|h^\tau_{\det}\theta_\tau p_\theta(\rho_\tau)\operatorname{div} u_\tau\|_{L^2_t(L^{\frac{2d}{d-2}}_x)} &\le C\|\theta_\tau\|_{L^2_t(L^{\frac{2d}{d-2}}_x)}\|\rho_\tau\|^{\gamma/d}_{L^\infty_{t,x}}\|u_\tau\|_{C_t(X_n)}. \\
\|\Delta K(\theta_\tau)\|_{L^{\frac{4}{3}}_t(W^{-2,\frac{4}{3}}_x)} &\le C\|\theta_\tau\|^3_{L^4_{t,x}}.
\end{aligned}$$

Applying (2.37) and Lemma 2.2.3, we find that

$$\lim_{M\to\infty}\sup_{\tau>0}\mathbb{P}\Big(\|\partial_t\big((\rho_\tau+\delta)\theta_\tau\big)\|_{L^{\frac{4}{3}}_t(W^{-2,\frac{4}{3}}_x)} \ge M\Big) = 0. \tag{2.42}$$

□

Next we apply the tightness result above together with a version of the Skorohod Theorem 1.7.2 to complete our compactness step.

Proof of Proposition 2.2.4. : Note that (E,τ) is a Jakubowski space. Hence, in view of Lemma 2.2.6, we may apply the Jakubowski/Skorohod theorem 1.7.2 to the sequence $\{Y_\tau\}_{\tau>0}$ in order to obtain a probability space $(\Omega_n,\mathcal{F}_n,\mathbb{P}_n)$, a sequence of measurable maps $\{\widetilde{T}_\tau\}_{\tau>0}$, and a limit point Y such that $Y_\tau\circ\widetilde{T}_\tau$ converges pointwise to Y. In components, we write $Y=(\rho_n,u_n,T_n,\theta_n,W_n)$. Noting that $(\tilde{\rho}_\tau+\delta)\tilde{u}_\tau\to(\rho_n+\delta)u_n$ in $D'_{t,x}$ pointwise in Ω_n, we identify $T_n=(\rho_n+\delta)\theta_n$. This yields parts 1 and 4 of Proposition 2.2.4. The uniform energy bounds in Part 3 now follow from the fact that $\widetilde{T}_\tau$ pushes forward $\mathbb{P}_n$ to $\mathbb{P}$ together with the estimates obtained already in Lemma 2.2.3. Finally, using the explicit relationship between $\tilde{\rho}_\tau,\tilde{u}_\tau,\tilde{\theta}_\tau$ and $\rho_\tau,u_\tau,\theta_\tau$, we are able to check that that the equation is preserved, that is, Part

2 holds. Similarly, Part 3 can be deduced from the bounds in Lemma 2.2.3. For more details on this last point, see [1]. $\square$

2.2.2 $\tau \to 0$ Identification step

Lemma 2.2.7. *For all $\omega \in \Omega_n$, $\tilde{\theta}_\tau(\omega) \to \theta_n(\omega)$ in $L^3_{t,x}$.*

Proof. Let us fix $\omega \in \Omega_n$ and mostly omit dependence of $\tilde{\theta}_\tau, \theta_n$ on this parameter within the context of this proof. By Part 4 of Proposition 2.2.4, $\tilde{\theta}_\tau \to \theta_n$ weakly in $L^4_{t,x}$. Hence, it suffices to check that $\tilde{\theta}_\tau^2 \to \theta_n^2$ weakly in $L^1_{t,x}$.

Towards this end, we will use Part 4 of Proposition 2.2.4 several more times. First observe that $(\tilde{\rho}_\tau + \delta)\tilde{\theta}_\tau \to (\rho_n + \delta)\theta_n$ in $C_t([L^2_x]_w)$. Moreover, applying a standard compactness upgrade, Lemma 1.7.11, we deduce that $(\tilde{\rho}_\tau + \delta)\tilde{\theta}_\tau \to (\rho_n + \delta)\theta_n$ strongly in $L^2_t(H^{-1}_x)$. Since $\tilde{\theta}_\tau \to \theta_n$ weakly in $L^2_t(H^{-1}_x)$, we obtain $(\tilde{\rho}_\tau + \delta)\tilde{\theta}_\tau^2 \to (\rho_n + \delta)\theta_n^2$ in $D'_{t,x}$. Moreover, there exists a $\hat{q} > 1$ and $C(\omega)$ such that

$$\sup_{\tau > 0} \|(\tilde{\rho}_\tau(\omega) + \delta)\tilde{\theta}_\tau(\omega)^2\|_{L^{\hat{q}}_{t,x}} \leq C(\omega).$$

This implies that $(\tilde{\rho}_\tau + \delta)\tilde{\theta}_\tau^2 \to (\rho_n + \delta)\theta_n^2$ in $L^q_{t,x}$ for a $q > 1$. Finally, since $(\tilde{\rho}_\tau + \delta)^{-1} \to (\rho_n + \delta)^{-1}$ in $L^{q'}_{t,x}$, we deduce that $\tilde{\theta}_\tau^2 \to \theta_n^2$ weakly in $L^1_{t,x}$. $\square$

Next we define a filtration $(\mathcal{F}^n_t)_{t=0}^T$ via $\mathcal{F}^n_t = \sigma(r_t X_n)$ where

$$X_n = \big(\rho_n, \rho_n u_n, u_n, W_n, \rho_n \theta_n\big)$$

and $r_t : E_T \to E_t$, where

$$E_s = C\big([0,s]; L^\beta\big) \cap L^\beta\big([0,s]; W^{1,\beta}_x\big) \times C\big([0,s]; [L^2]_w \times X_n \times \mathbb{R}^n\big)$$
$$\times L^2\big([0,s]; L^q\big),$$

and $\frac{1}{q} = \frac{1}{\gamma} + \frac{1}{2} - \frac{1}{d}$.

Lemma 2.2.8. *The triple* (ρ_n, u_n, θ_n) *satisfies* (2.15) *and* (2.16) *of Definition 2.2.2.*

Proof. By Part 2, $\tilde{\rho}_\tau, \tilde{u}_\tau$ satisfy (on Ω_n) the same parabolic equation as ρ_τ, u_τ. Using the pointwise convergences for the density and velocity, together with the observation that $h^\tau_{\text{det}} \to \frac{1}{2}$ weakly in $L^p\big([0,T]\big)$ for all $p \geq 1$, we may pass to the limit in the weak form and deduce that ρ_n, u_n satisfy the parabolic equation (2.15).

Next we verify the momentum equation (2.16) holds. Let $\phi \in X_n$ and define the continuous, $(\mathcal{F}^n_t)_{t=0}^T$ adapted process $\big(M^n_\phi(t)\big)_{t=0}^T$ via

$$
\begin{aligned}
M^n_\phi(t) &= \int_D \rho_n u_n(t) \cdot \phi \mathrm{d}x - \int_0^t \int_D \rho_n u_n \otimes u_n : \nabla\phi \mathrm{d}x\mathrm{d}s \\
&\quad + \int_0^t \int_D \big[\mathcal{S}(u_n) + (P(\rho_n, \theta_n) + \delta\rho_n^\beta)I\big] : \nabla\phi \mathrm{d}x\mathrm{d}s - \epsilon \int_0^t \int_D \nabla u_n \nabla\rho_n \phi \mathrm{d}x\mathrm{d}s.
\end{aligned}
$$

Similarly, we define the process $\big(\widetilde{M}^\phi_\tau(t)\big)_{t=0}^T$ in terms of $\tilde{\rho}_\tau, \tilde{u}_\tau, \tilde{\theta}_\tau$ and an additional oscillating factor $2h^\tau_{\text{det}}$. Let us check that

$$
\sup_{\tau>0} \mathbb{E}^{\mathbb{P}_n}\big[\sup_{t\in[0,T]} |\widetilde{M}^\phi_\tau(t)|^4\big] \leq C(n, \epsilon, \delta). \tag{2.43}
$$

Indeed, estimating each term we find that

$$
\begin{aligned}
\mathbb{E}^{\mathbb{P}_n}\left[\sup_{t\in[0,T]} \big|\widetilde{M}^\phi_\tau(t)\big|^4\right] &\lesssim \left(\mathbb{E}^{\mathbb{P}_n}\|\sqrt{\tilde{\rho}_\tau}\tilde{u}_\tau\|^4_{L^\infty_t(L^2_x)}\right)^{\frac{1}{2}} \left(\mathbb{E}^{\mathbb{P}_n}\|\tilde{\rho}_\tau\|^2_{L^\infty_t(L^\beta_x)}\right)^{\frac{1}{2}} \\
&\quad + \left(\mathbb{E}^{\mathbb{P}_n}\big(\|\sqrt{\tilde{\rho}_\tau}\tilde{u}_\tau\|^{12}_{L^\infty_t(L^2_x)}\right)^{\frac{1}{3}} \left(\mathbb{E}^{\mathbb{P}_n}\|\tilde{\rho}_\tau\|^6_{L^\infty_t(L^\beta_x)}\right)^{\frac{1}{3}} \cdot \left(\mathbb{E}^{\mathbb{P}_n}\|h^\tau_{\text{det}}\tilde{u}_\tau\|^{12}_{L^2_t(H^1_x)}\right)^{\frac{1}{3}} \\
&\quad + \mathbb{E}^{\mathbb{P}_n}\|h^\tau_{\text{det}} u_\tau\|^8_{L^2_t(H^2_x)} + \mathbb{E}^{\mathbb{P}_n}\big[\|\tilde{\rho}_\tau\|^{4\beta}_{L^\infty_t(L^{4\beta}_x)} + \|\tilde{\rho}_\tau\|^{4\gamma}_{L^\infty_t(L^{4\gamma}_x)}\big] \\
&\quad + \left(\mathbb{E}^{\mathbb{P}_n}\|\tilde{\theta}_\tau\|^8_{L^2_t(H^1_x)}\right)^{\frac{1}{2}} \mathbb{E}^{\mathbb{P}_n}\left(\|\tilde{\rho}_\tau\|^{\frac{8\gamma}{d}}_{L^\infty_t(L^\gamma_x)}\right)^{\frac{1}{2}} \\
&\quad + \mathbb{E}\left(\|h^\tau_{\text{det}}\tilde{u}_\tau\|^8_{L^2_t(H^1_x)}\right)^{\frac{1}{2}} \mathbb{E}\left(\|h^\tau_{\text{det}}\nabla\tilde{\rho}_\tau\|^8_{L^2_{t,x}}\right)^{\frac{1}{2}}.
\end{aligned} \tag{2.44}
$$

Our plan is to check the criterion laid forth in Lemma 1.7.6, in order to identify

$$M_n^\phi(t) = \sum_{k=1}^n \int_0^t \int_D \rho_n \sigma_{k,n}(\rho_n, \rho_n u_n, \rho_n \theta_n) \cdot \phi \mathrm{d}x \mathrm{d}\beta_k(s). \tag{2.45}$$

This implies the momentum equation (2.16) holds. Let us fix in advance two arbitrary times $s < t$ and a continuous functional $\gamma \in E_s$ which will be used repeatedly below. Note that in order to verify a process $\big(N_t\big)_{t=0}^T$ is a $\big(\mathcal{F}_t^n\big)_{t=0}^T$ martingale on $(\Omega_n, \mathcal{F}_n, \mathbb{P}_n)$, it suffices to show

$$\mathbb{E}^{\mathbb{P}_n}\Big[\gamma\big(r_s\rho_n, r_s u_n, r_s(\rho_n u_n), r_s W_n, r_s(\rho_n\theta_n)\big)\big(N_t - N_s\big)\Big] = 0.$$

We start by using the Levy Characterization to verify $\big(\beta_k^n(t)\big)_{t=0}^T$ is an $(\mathcal{F}_n^t)_{t=0}^T$ Brownian Motion. Applying the pointwise convergences (2.23)-(2.27) together with the uniform bounds, we find that

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}_n}\Big[\gamma\big(r_s\rho_n, r_s u_n, r_s(\rho_n u_n), r_s W_n, r_s(\rho_n\theta_n)\big)\Big(\beta_k^n(t) - \beta_k^n(s)\Big)\Big] \\
&= \lim_{\tau\to 0} \mathbb{E}^{\mathbb{P}_n}\Big[\gamma\big(r_s\tilde{\rho}_\tau, r_s\tilde{u}_\tau, r_s(\tilde{\rho}_\tau\tilde{u}_\tau), r_s\tilde{W}_\tau, r_s(\tilde{\rho}_\tau\tilde{\theta}_\tau)\big)\Big(\tilde{\beta}_k^\tau(t) - \tilde{\beta}_k^\tau(s)\Big)\Big].
\end{aligned}$$

Using Part 1 of Proposition 2.2.4 with a change of variables, then the martingale property of β_k, we deduce

$$\begin{aligned}
&\lim_{\tau\to 0} \mathbb{E}^{\mathbb{P}_n}\Big[\gamma\big(r_s\tilde{\rho}_\tau, r_s\tilde{u}_\tau, r_s(\tilde{\rho}_\tau\tilde{u}_\tau), r_s\tilde{W}_\tau, r_s(\tilde{\rho}_\tau\tilde{\theta}_\tau)\big)\Big(\tilde{\beta}_k^\tau(t) - \tilde{\beta}_k^\tau(s)\Big)\Big] \\
&= \lim_{\tau\to 0} \mathbb{E}^{\mathbb{P}}\Big[\gamma\big(r_s\rho_\tau, r_s u_\tau, r_s(\rho_\tau u_\tau), r_s, r_s(\rho_\tau\theta_\tau)\big)\Big(\beta_k(t) - \beta_k(s)\Big)\Big] = 0.
\end{aligned}$$

Similarly, one verifies

$$\mathbb{E}^{\mathbb{P}_n}\Big[\gamma\big(r_s\rho_n, r_s u_n, r_s\rho_n u_n, r_s W_n, r_s(\rho_n\theta_n)\big)\big(\beta_k^n(t)^2 - \beta_k^n(s)^2 - t + s\big)\Big] = 0.$$

Next we check that $\big(M_t^n(\phi)\big)_{t=0}^T$ is an $(\mathcal{F}_t^n)_{t=0}^T$ martingale with quadratic variation

$$\sum_{k=1}^n \int_0^t \left(\int_D \rho_n \sigma_{k,\tau,n,\delta}(\rho_n, \rho_n u_n) \cdot \phi \mathrm{d}x\right)^2 \mathrm{d}s.$$

Recall that $h_{\mathrm{det}}^\tau \to \frac{1}{2}$ weakly. Hence, by using (2.23)-(2.27) together with the uniform bounds ; followed by (2.5) of Definition 4, we obtain:

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}_n}\left[\gamma\big(r_s\rho_n, r_s u_n, r_s(\rho_n u_n), r_s W_n, r_s(\rho_n\theta_n)\big)\Big(M_t^n(\phi) - M_s^n(\phi)\Big)\right] \\
&\lim_{\tau\to 0}\mathbb{E}^{\mathbb{P}_n}\left[\gamma\big(r_s\tilde{\rho}_\tau, r_s\tilde{u}_\tau, r_s(\tilde{\rho}_\tau\tilde{u}_\tau), r_s\tilde{W}_\tau, r_s(\tilde{\rho}_\tau\tilde{\theta}_\tau)\big)\Big(\tilde{M}_t^\tau(\phi) - \tilde{M}_s^\tau(\phi)\Big)\right] \\
&= \lim_{\tau\to 0}\mathbb{E}^{\mathbb{P}}\left[\gamma\big(r_s\rho_\tau, r_s u_\tau, r_s(\rho_\tau u_\tau), r_s W_\tau, r_s(\rho_\tau\theta_\tau)\big)\Big(M_t^\tau(\phi) - M_s^\tau(\phi)\Big)\right] = 0
\end{aligned}$$

Arguing similarly, we find

$$\mathbb{E}^{\mathbb{P}_n}\left[\gamma\left(M_t^n(\phi)\beta_k^n(t) - M_s^n(\phi)\beta_k^n(s) - \int_s^t\int_D \rho_n\sigma_{k,\tau,n,\delta}(\rho_n, u_n)\cdot\phi\mathrm{d}x\mathrm{d}r\right)\right] = 0.$$

A similar analysis of the quadratic variation allows us to appeal to Lemma 1.7.6 and complete the proof. □

Lemma 2.2.9. *The triple* (ρ_n, u_n, θ_n) *satisfies equation* (2.15) *of Definition 2.2.2.*

Proof. By part 2 of Proposition 2.2.4, for all $\varphi \in C^\infty(D)$ with $\frac{\partial\varphi}{\partial n}\mid_{\partial D} = 0$, the following equality holds $\mathbb{P}_n$ almost surely:

$$\begin{aligned}
&\int_D (\tilde{\rho}_\tau(t) + \delta)\tilde{\theta}_\tau(t)\varphi\mathrm{d}x - \int_0^t\int_D \big(\delta\tilde{\theta}_\tau^3 + K(\tilde{\theta}_\tau)\Delta\varphi\big)dxds = \int_D (\rho_{0,\delta} + \delta)\theta_{0,\delta}\varphi\mathrm{d}x \\
&+ \int_0^t\int_D 2h_{\mathrm{det}}^\tau \tilde{\rho}_\tau\tilde{\theta}_\tau\tilde{u}_\tau\cdot\nabla\varphi\mathrm{d}x\mathrm{d}s \\
&+ \int_0^t\int_D 2h_{\mathrm{det}}^\tau[(1-\delta)\mathcal{S}(\tilde{u}_\tau) : \nabla\tilde{u}_\tau - \theta_\tau p_\theta(\rho_\tau)\operatorname{div} u_\tau]\varphi\mathrm{d}x\mathrm{d}s.
\end{aligned}$$

We will pass to the limit pointwise in $\omega \in \Omega_n$, proceeding term by term in the equality above from left to right. For the first term, use that $(\tilde{\rho}_\tau + \delta)\tilde{\theta}_\tau \to (\rho_n + \delta)\theta_n$

in $C_t([L^2_x]_w)$. For the next two terms, use that $\tilde{\theta}_\tau \to \theta_n$ in $L^4_{t,x}$. This is sufficent since $\kappa(\theta) \sim \theta^2$ implies $\mathcal{K}(\theta) \sim \theta^3$. The regularized data don't depend on n, and can be left alone.

For the next three terms, recall that $h^\tau_{\text{det}} \to \frac{1}{2}$ weakly in L^p_t for any $p \in [1, \infty)$. For the flux term, use Lemma 2.2.7 together with Proposition 2.2.4 to deduce that $\tilde{\rho}_\tau \tilde{u}_\tau \tilde{\theta}_\tau \to \rho_n u_n \theta_n$ in $L^3_t(L^p_x)$ where $\frac{1}{p} = \frac{1}{\beta} + \frac{1}{3}$. Note that $p > 1$ since $\beta > 4$, so we find that $2h^\tau_{\text{det}} \tilde{\rho}_\tau \tilde{u}_\tau \tilde{\theta}_\tau \to \rho_n u_n \theta_n$ in $D'_{t,x}$. For the next term, the $C_t(X_n)$ convergence of the velocity clearly implies $2(1-\delta) h^\tau_{\text{det}} \mathcal{S}(\tilde{u}_\tau) : \nabla \tilde{u}_\tau \to \mathcal{S}(u_n) : \nabla u_n$ in $D'_{t,x}$. For the last term, note that $p_\theta(\rho) \sim \rho^{\frac{\gamma}{d}}$ implies $p_\theta(\tilde{\rho}_\tau) \to p_\theta(\rho_n)$ in $C_t(L^q_x)$ for any $q < \frac{\beta d}{\gamma}$. Thus, $\tilde{\theta}_\tau p_\theta(\tilde{\rho}_\tau) \operatorname{div} \tilde{u}_\tau \to \theta_n p_\theta(\rho_n) \operatorname{div} u_n$ in $L^4_t(L^r_x)$ provided $\frac{1}{r} > \frac{1}{4} + \frac{\gamma}{\beta d}$. Since $\beta > \gamma$ and $d \geq 3$, we can ensure $r > 1$. Hence, we find that $2h^\tau_{\text{det}} \tilde{\theta}_\tau p_\theta(\tilde{\rho}_\tau) \operatorname{div} \tilde{u}_\tau \to \theta_n p_\theta(\rho_n) \operatorname{div} u_n$ in $L^4_t(L^{r-}_x)$ in $D'_{t,x}$. This completes the proof. $\square$

2.3 ϵ Layer existence

This section is devoted to the ϵ layer existence theory; sending $n \to \infty$ our goal is to prove:

Theorem 2.3.1. *For every $\epsilon > 0$, there exists an ϵ layer approximation (in the sense of Definition 2.3.2 below) $\rho_\epsilon, u_\epsilon, \theta_\epsilon$ to* (1.1)*, relative to a stochastic basis* $\big(\Omega_\epsilon, \mathcal{F}_\epsilon, (\mathcal{F}^t_\epsilon)_{t=0}^T, \mathbb{P}_\epsilon, \{\beta^\epsilon_k\}_{k \in \mathbb{N}}\big)$.

Moreover, for all $p \geq 1$, there exists C_p independent of ϵ, δ such that

$$\sup_{\epsilon>0} \mathbb{E}^{\mathbb{P}_\epsilon}\left[\|\sqrt{\rho_\epsilon}u_\epsilon\|^{2p}_{L^\infty_t(L^2_x)} + \|\rho_\epsilon\|^{\gamma p}_{L^\infty_t(L^\gamma_x)} + \|\delta^{\frac{1}{\beta}}\rho_\epsilon\|^{\beta p}_{L^\infty_t(L^\beta_x)} + \|(\rho_\epsilon+\delta)\theta_\epsilon\|^p_{L^\infty_t(L^1_x)}\right] \leq C_p.$$
$$\sup_{\epsilon>0} \mathbb{E}^{\mathbb{P}_\epsilon}\left[\|\delta^{\frac{1}{2}}u_\epsilon\|^{2p}_{L^2_t(H^1_{0,x})} + \|\epsilon^{\frac{1}{2}}\nabla(\rho_\epsilon^{\frac{\gamma}{2}} + \delta^{\frac{1}{2}}\rho_\epsilon^{\frac{\beta}{2}})\|^{2p}_{L^2_t(L^2_x)} + \|\delta^{\frac{1}{3}}\theta_\epsilon\|^{3p}_{L^3_{t,x}}\right] \leq C_p. \tag{2.46}$$

Moreover, there exists $C'(p,\delta)$ such that

$$\sup_{n\geq 1} \mathbb{E}^{\mathbb{P}_n}\left[\|\theta_n\|^{2p}_{L^2_t(H^1_x)}\right] \leq C'(p,\delta). \tag{2.47}$$

For each $k \geq 1$ and $\delta > 0$, define an operator $\sigma_{k,\delta} : L^1(D) \times L^1(D;\mathbb{R}^d) \times L^1(D) \to C(D)$ via

$$\sigma_{k,\delta}(\rho, m, \alpha) = \sigma_k(\rho * \eta_\delta(\cdot), m * \eta_\delta(\cdot), \alpha * \eta_\delta, \cdot).$$

Following Feireisl [11], we define a set $\mathcal{R}$ of admissible renormalizations of the temperature equation. Namely, $\mathcal{R}$ consists of non-increasing real valued functions $h \in C^2[0,\infty)$ which satisfy $h(0) = 1$, $\lim_{z\to\infty} h(z) = 0$ and $h''(z)h(z) \geq 2(h'(z))^2$ for all $z \geq 0$. Moreover, it is useful to introduce the potential

$$\mathcal{K}_h(\theta) = \int_0^\theta \kappa(z)h(z)\mathrm{d}z.$$

Definition 2.3.2. *A triple $(\rho_\epsilon, u_\epsilon, \theta_\epsilon)$ is an ϵ layer approximation to (1.1) provided there exists a stochastic basis $\left(\Omega_\epsilon, \mathcal{F}_\epsilon, (\mathcal{F}^t_\epsilon)_{t=0}^T, \mathbb{P}_\epsilon, \left\{\beta^\epsilon_k\right\}_{k\in\mathbb{N}}\right)$ such that*

1. *The quadruple $(\rho_\epsilon, \rho_\epsilon u_\epsilon, \rho_\epsilon\theta_\epsilon, u_\epsilon)$ belongs in $L^2\big(\Omega \times [0,T]; \mathcal{P}; L^\beta \times L^{\frac{2\beta}{\beta+1}} \times L^q \times [H^1_0]^d\big)$, where $\mathcal{P}$ is the predictable σ-algebra generated by $(\mathcal{F}^t_\epsilon)_{t=0}^T$ and $\frac{1}{q} = \frac{1}{\gamma} + \frac{1}{2} - \frac{1}{d}$.*

2. For all $\phi \in C^\infty(D)$ and all $t \in [0,T]$, the following equality holds $\mathbb{P}_\epsilon$ a.s.

$$\int_D \rho_\epsilon(t)\phi \mathrm{d}x = \int_D \rho_{0,\delta}\mathrm{d}x + \int_0^t \int_D [\rho_\epsilon u_\epsilon \cdot \nabla\phi + \epsilon\rho_\epsilon \Delta\phi]\mathrm{d}x\mathrm{d}s. \tag{2.48}$$

3. For all $\phi \in \left[C_c^\infty(D)\right]^d$ and all $t \in [0,T]$, the following equality holds $\mathbb{P}_\epsilon$ a.s.

$$\begin{aligned}
&\int_D \rho_\epsilon u_\epsilon(t)\cdot\phi \mathrm{d}x = \int_D m_{0,\delta}\cdot\phi\mathrm{d}x + \int_0^t \int_D [\rho_\epsilon u_\epsilon \otimes u_\epsilon - \mathcal{S}(u_\epsilon)] : \nabla\phi \mathrm{d}x\mathrm{d}s \\
&+ \int_0^t \int_D \left[(P(\rho_\epsilon,\theta_\epsilon) + \delta\rho_\epsilon^\beta)\operatorname{div}\phi - \epsilon\nabla u_\epsilon \nabla\rho_\epsilon \cdot \phi\right]\mathrm{d}x\mathrm{d}s \\
&+ \sum_{k\in\mathbb{N}} \int_0^t \int_D \rho_\epsilon \sigma_{k,\delta}(\rho_\epsilon, \rho_\epsilon u_\epsilon, \rho_\epsilon\theta_\epsilon)\cdot\phi \mathrm{d}x\mathrm{d}\beta_k^\epsilon(s).
\end{aligned} \tag{2.49}$$

4. For all non-negative $\varphi \in C^\infty\big([0,T]\times D\big)$ with $\frac{\partial\varphi}{\partial n}|_{\partial D} = 0$ and all $h \in \mathcal{R}$, the inequality below holds $\mathbb{P}_\epsilon$ a.s.

$$\begin{aligned}
&\int_0^T \int_D \left[(\rho_\epsilon + \delta)H(\theta_\epsilon)\partial_t\varphi + \rho_\epsilon H(\theta_\epsilon)u_\epsilon\cdot\nabla\varphi + \mathcal{K}_h(\theta_\epsilon)\Delta\varphi - \delta\theta_\epsilon^3 H'(\theta_\epsilon)\varphi\right]\mathrm{d}x\mathrm{d}t \\
&\leq \int_0^T \int_D h(\theta_\epsilon)[\theta_\epsilon p_\theta(\rho_\epsilon)\operatorname{div} u_\epsilon - \mathcal{S}(u_\epsilon) : \nabla u_\epsilon]\varphi \mathrm{d}x\mathrm{d}t \\
&+ \epsilon\int_0^T \int_D \nabla\rho_\epsilon \cdot \nabla\left[(H(\theta_\epsilon) - \theta_\epsilon h(\theta_\epsilon))\varphi\right]\mathrm{d}x\mathrm{d}s \\
&+ \int_0^T \int_D h'(\theta_\epsilon)\kappa(\theta_\epsilon)|\nabla\theta_\epsilon|^2\mathrm{d}x\mathrm{d}t - \int_D \rho_\epsilon(0+)H(\theta_\epsilon)\varphi(0)\mathrm{d}x.
\end{aligned} \tag{2.50}$$

2.3.1 $n \to \infty$ Compactness step

Let us begin with the following uniform energy bounds:

Lemma 2.3.3. *For all $p \geq 1$, there exists C_p independent of n, ϵ, δ such that*

$$\sup_{n\in\mathbb{N}} \mathbb{E}^{\mathbb{P}_n}\left[\|\sqrt{\rho_n}u_n\|^{2p}_{L^\infty_t(L^2_x)} + \|\rho_n\|^{\gamma p}_{L^\infty_t(L^\gamma_x)} + \|\delta^{\frac{1}{\beta}}\rho_n\|^{\beta p}_{L^\infty_t(L^\beta_x)} + \|(\rho_n+\delta)\theta_n\|^{p}_{L^\infty_t(L^1_x)}\right] \leq C_p.$$
$$\sup_{n\in\mathbb{N}} \mathbb{E}^{\mathbb{P}_n}\left[\|\delta^{\frac{1}{2}}u_n\|^{2p}_{L^2_t(H^1_{0,x})} + \|\epsilon^{\frac{1}{2}}\nabla(\rho_n^{\frac{\gamma}{2}} + \delta^{\frac{1}{2}}\rho_n^{\frac{\beta}{2}})\|^{2p}_{L^2_t(L^2_x)} + \|\delta^{\frac{1}{3}}\theta_n\|^{3p}_{L^3_{t,x}}\right] \leq C_p. \tag{2.51}$$

Moreover, there exists $C(p,\delta)$ such that

$$\sup_{n\geq 1} \mathbb{E}^{\mathbb{P}_n}\left[\|\theta_n\|^{2p}_{L^2_t(H^1_x)}\right] \leq C(p,\delta). \tag{2.52}$$

Proof. An application of Ito's formula yields for all $t \in [0,T]$:

$$\begin{aligned}
&\int_D \frac{1}{2}\rho_n|u_n|^2(t) + \rho_n P_m(\rho_n(t)) + \frac{\delta}{\beta-1}\rho_n^\beta(t) + (\rho_n+\delta)\theta_n(t)\mathrm{d}x \\
&+ \int_0^t\int_D \delta[\mathcal{S}(u_n):\nabla u_n + \theta_n^3]\mathrm{d}x\mathrm{d}s \\
&+ \int_0^t\int_D \epsilon\Big(\frac{p_m'(\rho_n)}{\rho_n} + \delta\beta\rho_n^{\beta-2}\Big)|\nabla\rho_n|^2\mathrm{d}x\mathrm{d}s \\
&= \sum_{k=1}^n \int_0^t\int_D \rho_n u_n\cdot\sigma_{k,n}(\rho_n,\rho_n u_n,\rho_n\theta_n)\mathrm{d}x\mathrm{d}\beta_k^n(s) \\
&+ \sum_{k=1}^n \int_0^t\int_D \rho_n|\sigma_{k,n}(\rho_n,\rho_n u_n,\rho_n\theta_n)|^2\mathrm{d}x\mathrm{d}t + E_n(0).
\end{aligned}$$

By Hypothesis 2.1.2 and the Banach/Steinhaus Theorem,

$$\sup_{n\in\mathbb{N}} \|\Pi_n\|_{\mathcal{L}(L^{\frac{2\gamma}{\gamma+1}}, L^{\frac{2\gamma}{\gamma+1}})} < \infty. \tag{2.53}$$

Hence, the first part of the lemma is obtained in the same way as the formal estimates Section 1.6.

The second part of the lemma can be proved with the same technique as in Lemma 2.2.3. The δ dependence of the constant arises from the $L^p(\Omega; L^2_{t,x})$ bound for $\operatorname{div} u_n$. $\square$

Next we establish the following compactness result:

Proposition 2.3.4. *There exists a probability space $(\Omega_\epsilon, \mathcal{F}_\epsilon, \mathbb{P}_\epsilon)$, a collection of independent Brownian motions $\{\beta_k^\epsilon\}_{k\in\mathbb{N}}$, limit points $\big(\rho_\epsilon, u_\epsilon, \theta_\epsilon, \overline{\sqrt{\rho_\epsilon}u_\epsilon}\big)$, and a sequence of measurable maps $\{\widetilde{T}_n\}_{n\in\mathbb{N}}$ such that*

1. *For each $n \in \mathbb{N}$, $\widetilde{T}_n : (\Omega_\epsilon, \mathcal{F}_\epsilon, \mathbb{P}_\epsilon) \to (\Omega_n, \mathcal{F}_n, \mathbb{P}_n)$ and $(\widetilde{T}_n)_\#\mathbb{P}_\epsilon = \mathbb{P}_n$.*

2. *The new sequence $\{(\tilde{\rho}_n, \tilde{u}_n, \tilde{\theta}_n)\}_{n\in\mathbb{N}}$ defined by $\Big(\tilde{\rho}_n, \tilde{u}_n, \tilde{\theta}_n\Big) = (\rho_n, u_n, \theta_n) \circ \widetilde{T}_n$ constitutes an n layer approximation relative to the stochastic basis $\big(\widetilde{\Omega}_\epsilon, \widetilde{\mathcal{F}}_\epsilon, \tilde{\mathbb{P}}_\epsilon, (\widetilde{\mathcal{F}}_n^t)_{t=0}^T, \widetilde{W}_n\big)$, where $\widetilde{W}_n := W_n \circ \widetilde{T}_n$ and $\widetilde{\mathcal{F}}_n^t = \widetilde{T}_n^{-1} \circ \mathcal{F}_n^t$.*

3. *The uniform bounds in Lemma 2.3.3 hold with ρ_n, θ_n, u_n replaced by $\tilde{\rho}_n, \tilde{\theta}_n, \tilde{u}_n$ and $\mathbb{P}_n$ replaced by $\mathbb{P}_\epsilon$.*

4. *The following convergences hold pointwise on Ω_ϵ*

$$\tilde{\rho}_n \to \rho_\epsilon \quad \text{in} \quad C_t\big([L_x^\beta]_w\big) \cap L_t^2(L_x^2) \tag{2.54}$$

$$\tilde{u}_n \to u_\epsilon \quad \text{in} \quad [L_t^2(H_{0,x}^1)]_w \tag{2.55}$$

$$\tilde{\rho}_n\tilde{u}_n \to \rho_\epsilon u_\epsilon \quad \text{in} \quad C_t([L_x^{\frac{2\beta}{\beta+1}}]_w) \tag{2.56}$$

$$\tilde{\theta}_n \to \theta_\epsilon \quad \text{in} \quad [L_t^2(H_x^1) \cap L_{t,x}^3]_w \tag{2.57}$$

$$(\tilde{\rho}_n + \delta)\tilde{\theta}_n \to (\rho_\epsilon + \delta)\theta_\epsilon \quad \text{in} \quad [L_t^1(C_c(D))]_*' \tag{2.58}$$

$$\widetilde{W}_n \to W_\epsilon \quad \text{in} \quad [C_t]^\infty, \tag{2.59}$$

5. *The following additional convergences hold*

$$\tilde{u}_n \to u_\epsilon \quad in \quad L^p_w\big(\Omega_\epsilon; L^2_t(H^1_{0,x})\big) \tag{2.60}$$

$$\tilde{\rho}_n \to \rho_\epsilon \quad in \quad L^p_w\big(\Omega_\epsilon; L^2_t(W^{1,2}_x)\big) \tag{2.61}$$

For each $n \in \mathbb{N}$ define a random variable

$$Y_n = \big(\rho_n, u_n, \Pi_n(\rho_n u_n), \theta_n, (\rho_n + \delta)\theta_n, \{\beta^n_k\}_{k\in\mathbb{N}}\big).$$

Our convention is that given a topological vector space G, a finite sequence $\{x_k\}_{k=1}^n$ is viewed as an element of G^∞ where $x_j = 0$ for $j \geq n$. Observe that Y_n induces a measure on the topological space E, where

$$E = C_t([L^\beta_x]_w) \cap L^2_{t,x} \times [L^2_t(H^1_{0,x})]_w \times C_t([L_x^{\frac{2\beta}{\beta+1}}]_w)$$
$$\times [L^2_t(H^1_x) \cap L^3_{t,x}]_w \times [L^1_t(C_c(D))]'_* \times [C_t]^\infty.$$

Lemma 2.3.5. *The sequence of induced measures* $\big\{\mathbb{P}_n \circ Y_n^{-1}\big\}_{n\in\mathbb{N}}$ *is tight on* E.

Proof. It suffices to consider each component of Y_n separately. The tightness of $\big\{\mathbb{P}_n \circ (u_n, \rho_n\theta_n)^{-1}\big\}_{n\in\mathbb{N}}$ follows immediately from the uniform bounds and Banach Alaoglu. To treat the collection of SBM, note

$$\sup_{k,n:\, k\leq n} \mathbb{E}^{\mathbb{P}_n}\big[\|\beta^n_k\|^2_{C_t^{\frac{1}{3}}}\big] < \infty. \tag{2.62}$$

For each $M > 0$, the set

$$\prod_{j=1}^{\infty}\big\{f \in C_t \mid \|f\|_{C_t^{\frac{1}{3}}} \leq M2^j\big\} \tag{2.63}$$

is compact in $[C_t]^\infty$ by Arzela-Ascoli and Tychnoff. Choosing $M > 0$ appropriately and summing a geometric series gives the desired tightness of $\big\{\mathbb{P}_n \circ W_n^{-1}\big\}_{n\in\mathbb{N}}$. Recall that $\rho_n(0) = \rho_{0,\delta}$ by Part 2 of Proposition 2.2.4. Since $\beta > d$ we may choose a $\theta \in (0,1)$ and define a $q > 2$ by the relation $\frac{1}{q} = \theta(\frac{1}{\beta} + \frac{1}{2} - \frac{1}{d}) + (1-\theta)(\frac{1}{2\beta} + \frac{1}{2})$. Maximal regularity results for parabolic equations and interpolation give the $\mathbb{P}_n$ a.s. inequality

$$\begin{aligned} \|\partial_t \rho_n\|_{L^q_t(W^{-1,q}_x)} + \epsilon \|\rho_n\|_{L^q_t(W^{1,q}_x)} &\lesssim \|\rho_{0,\delta}\|_{L^\beta_x} + \|\rho_n u_n\|_{L^q_{t,x}} \\ &\lesssim 1 + \|\rho_n u_n\|^{\theta}_{L^2_t(L_x^{\frac{2^*\beta}{\beta+2^*}})} \|\rho_n u_n\|^{1-\theta}_{L^\infty_t(L_x^{\frac{2\beta}{\beta+1}})} \\ &\lesssim 1 + \|\rho_n\|^{\frac{1}{2}(1+\theta)}_{L^\infty_t(L^\beta_x)} \|u_n\|^{1-\theta}_{L^2_t(L^{2^*}_x)} \|\sqrt{\rho_n} u_n\|^{1-\theta}_{L^\infty_t(L^2_x)}. \end{aligned} \tag{2.64}$$

Hence, the LHS is uniformly controlled in $L^2(\Omega_n)$ in view of the uniform bounds and Hölder (in ω). Corollary 1.7.8 and Aubin-Lions imply the set

$$\left\{ f \in L^\infty_t(L^\beta_x) \cap L^2_{t,x} \mid \|\partial_t f\|_{L^q_t(W^{-1,q}_x)} + \|f\|_{L^q_t(W^{1,q}_x)} + \|f\|_{L^\infty_t(L^\beta_x)} \leq \sqrt{M} \right\}.$$

is compact in $C_t([L^\beta_x]_w) \cap L^2_{t,x}$, for each $M > 0$. Using the uniform bounds on $\{\rho_n\}_{n\in\mathbb{N}}$ in $L^2(\Omega_n; L^\infty_t(L^\beta_x))$ gives the tightness of $\left\{\mathbb{P}_n \circ \rho_n^{-1}\right\}_{n\in\mathbb{N}}$ for an appropriate choice of $M > 0$. To address the sequence $\mathbb{P}_n \circ \Pi_n(\rho_n u_n)^{-1}$, let $s < t$ and $\phi \in X_n$ be arbitrary and use the momentum equation (2.16) to decompose $\langle \rho_n u_n(t) - \rho_n(s) u_n(s), \phi\rangle_{L^2_x}$ into three terms: the stochastic integrals, the energy correction, and the rest. To estimate the stochastic integrals, we use the BDG inequality together with the stability of the projection operators, Hypothesis 2.1.2, via

$$\begin{aligned} &\mathbb{E}^{\mathbb{P}_n}\left[\left(\sum_{k=1}^n \int_s^t \int_D \rho_n \sigma_{k,n,\delta}(\rho_n, \rho_n u_n, \rho_n\theta_n)\cdot \phi \mathrm{d}x \mathrm{d}\beta^k_n\right)^p\right] \\ &\lesssim \mathbb{E}^{\mathbb{P}_n}\left[\left(\sum_{k=1}^\infty \int_s^t (\int_D \rho_n \sigma_{k,n,\delta}(\rho_n, \rho_n u_n, \rho_n\theta_n)\cdot\phi \mathrm{d}x)^2 \mathrm{d}r\right)^{\frac{p}{2}}\right] \\ &\lesssim \|\phi\|^p_{L^\infty_x} \left(\sum_{k=1}^\infty \|\sigma_k\|^2_{L^{\gamma'}_x(L^\infty_{\rho,m})}\right)^{\frac{p}{2}} \mathbb{E}^{\mathbb{P}_n}\left[\left(\int_s^t \|\rho_n(r)\|^2_{L^\gamma_x}\mathrm{d}r\right)^{\frac{p}{2}}\right] \\ &\lesssim (t-s)^{\frac{p}{2}} \|\phi\|^p_{L^\infty_x} \mathbb{E}^{\mathbb{P}_n}\left[\|\rho_n\|^p_{L^\infty_t(L^\gamma_x)}\right]. \end{aligned} \tag{2.65}$$

To estimate the energy correction, we use (2.64) to obtain the following inequality

$$\left|\epsilon \int_s^t \int_D \nabla u_n \nabla \rho_n \cdot \phi \mathrm{d}x \mathrm{d}s\right| \lesssim \|\phi\|_{L_x^\infty} \int_s^t \|\nabla u_n(r)\|_{L_x^2} \|\nabla \rho_n(r)\|_{L_x^2} \mathrm{d}r$$

$$\lesssim (t-s)^{\frac{1}{2}-\frac{1}{q}} \|\phi\|_{L_x^\infty} \|u_n\|_{L_t^2(H_{0,x}^1)} \|\nabla \rho_n\|_{L_{t,x}^q} \quad \mathbb{P}_n \quad \text{a.s.}$$

To treat the remaining terms, Hölder and Sobolev yield the $\mathbb{P}_n$ a.s. inequality

$$\begin{aligned} &\int_s^t \int_D \left[\rho_n u_n \otimes u_n - 2\mu \nabla u + (\rho_n^\gamma + \delta \rho_n^\beta - \operatorname{div} u_n) I\right] : \nabla \phi \mathrm{d}x \mathrm{d}r \\ &\lesssim (t-s)^{\frac{1}{2}} \|\nabla \phi\|_{L^\infty} \left[\|u_n\|_{L_t^2(W_x^{1,2})} (\|\sqrt{\rho_n} u_n\|_{L_t^\infty(L_x^2)} \|\rho_n\|_{L_t^\infty(L_x^\gamma)}^{1/2} + 1)\right] \\ &+ (t-s)^{\frac{1}{2}} \|\nabla \phi\|_{L^\infty} \left[\|\rho_n\|_{L_t^\infty(L_x^\gamma)}^\gamma + \|\rho_n\|_{L_t^\infty(L_x^\beta)}^\beta\right]. \end{aligned} \tag{2.66}$$

Indeed, the only additional fact needed is the $\mathbb{P}_n$ a.s. inequality

$$\int_s^t \int_D \theta_n p_\theta(\rho_n) \operatorname{div} \varphi \mathrm{d}x \mathrm{d}s \lesssim \|\varphi\|_{W_x^{1,\infty}} \int_s^t \|\theta(r)\|_{L^{\frac{2d}{d-2}}} \|\rho(r)^{\frac{\gamma}{d}}\|_{L^{\frac{2d}{d+2}}} \mathrm{d}r$$

$$\lesssim (t-s)^{\frac{1}{2}} \|\rho_n\|_{L_t^\infty(L_x^\gamma)}^{\frac{\gamma}{d}} \|\theta_n\|_{L_t^2(H_x^1)}.$$

Combining (2.65)-(2.66) and using the uniform bounds together with (2.64) we obtain for all k and $p \geq 1$

$$\sup_{n \geq k} \mathbb{E}^{\mathbb{P}_n}\left[\left|\langle \rho_n u_n(t) - \rho_n u_n(s), \phi_k \rangle_{L_x^2}\right|^p\right] \leq C \|\phi_k\|_{C_x^1}^p |t-s|^{p(\frac{1}{2}-\frac{1}{q})}. \tag{2.67}$$

Combining this observation with the Sobolev embedding theorem for fractional sobolev spaces(in time), for any $\alpha < \frac{1}{2} - \frac{1}{q}$ there exists a p such that

$$\sup_{n \geq k} \mathbb{E}^{\mathbb{P}_n}\left[\left|\langle \rho_n u_n, \phi_k \rangle_{C_t^\alpha}\right|^p\right] \lesssim \|\phi_k\|_{C_x^1}^p. \tag{2.68}$$

For each $M > 0$, define the set K_M via

$$K_M = \prod_{j=1}^\infty \left\{f \in C_t \mid \ \|f\|_{C_t^\alpha} \leq M^{\frac{1}{p}} 2^{\frac{j}{p}} \|\phi_j\|_{C_x^1}\right\}.$$

In view of Arzela-Ascoli and Tychonoff, K_M is a compact set, and Chebyshev yields

$$\begin{aligned}\mathbb{P}_n\left(Y_n \notin K_M\right) &\leq \sum_{k=1}^{n} \mathbb{P}_n\left(\langle \rho_n u_n, \phi_k \rangle_{C_t^\alpha} \geq M^{\frac{1}{p}} 2^{\frac{k}{p}} \|\phi_k\|_{C_x^1}\right) \\ &\leq M^{-1} \sum_{k=1}^{n} 2^{-k} \|\phi_k\|_{C_x^1}^{-p} \sup_{n\geq k} \mathbb{E}^{\mathbb{P}_n}\left[\left|\langle \rho_n u_n, \phi_k \rangle_{C_t^\alpha}\right|^p\right] \leq M^{-1}.\end{aligned}$$

This implies the tightness of the projected momentum sequence, as desired.

Next we note that for all $M > 0$, Lemma 1.7.13 implies the ball of radius M in $L_t^\infty(L_x^1)$ is compact with respect to the weak-$\star$ topology on in $L_t^\infty(M_x)$. Hence, by Chebyshev and the uniform bounds from Lemma 2.3.3, we obtain the tightness of $\left\{\mathbb{P}_n \circ \left((\rho_n + \delta)\theta_n\right)^{-1}\right\}_{n\in\mathbb{N}}$ in $[L_t^\infty(M_x)]_{w-\star}$. Similarly, using Banach-Alaoglu and the uniform bounds in Lemma 2.3.3, we obtain the tightness of $\left\{\mathbb{P}_n \circ \theta_n^{-1}\right\}_{n\in\mathbb{N}}$ on $[L_t^2(H_x^1) \cap L_{t,x}^3]_w$. This completes the proof.

$\square$

Now we can complete the proof of our compactness step.

Proof of Proposition 2.3.4. : Note that $E \times F$ is a Jakubowski space, so we may apply Theorem 1.7.2 in order to obtain a sequence of maps $\{\tilde{T}_n\}_{n=1}^\infty$

$$\widetilde{T}_n : (\Omega_\epsilon, \mathcal{F}_\epsilon, \mathbb{P}_\epsilon) \to (\Omega_n, \mathcal{F}_n, \mathbb{P}_n)$$

and a limiting random variable $X_\epsilon = (\rho_\epsilon, m_\epsilon, u_\epsilon, W_\epsilon)$. Moreover, the properties listed in Theorem 1.7.2 imply directly Part 1 of the Proposition and guarantee that

$$\begin{aligned}&(\tilde{\rho}_n, \tilde{u}_n, \tilde{\rho}_n \tilde{u}_n, \widetilde{W}_n,) = (\rho_n, u_n, \rho_n u_n, W_n,) \circ \widetilde{T}_n \\ &= \left(\rho_n \circ \widetilde{T}_n,\ u_n \circ \widetilde{T}_n,\ \rho_n \circ \widetilde{T}_n\, u_n \circ \widetilde{T}_n,\ \{\tilde{\beta}_k^n \circ \widetilde{T}_n\}_{k=1}^n, \right) \qquad (2.69) \\ &= \tilde{X}_n \circ \widetilde{T}_n \to X_\epsilon.\end{aligned}$$

The limit is understood to hold $\tilde{\mathbb{P}}_\epsilon$ almost surely in each of the topologies where the tightness was proven. In particular, we obtain the pointwise convergences (2.54), (2.55), and (2.59).

It may be checked with a regularization argument that the energy functional remains measurable with respect to the new topology introduced in this section. Hence, we may recover the uniform bounds (2.51) from the prior probability space and combine these with Banach-Alaogolu theorem to obtain (**??**)-(2.61) □

2.3.2 $n \to \infty$ Identification step

Next we define a filtration $(\mathcal{F}_t^\epsilon)_{t=0}^T$ via $\mathcal{F}_t^\epsilon = \sigma(r_t X_\epsilon)$ where $X_\epsilon = \big(\rho_\epsilon, \rho_\epsilon u_\epsilon, W_\epsilon, u_\epsilon, \rho_\epsilon \theta_\epsilon\big)$ and $r_t : E_T \to E_t$

$$\begin{aligned} E_s &= C\big([0,s];[L^\beta]_w\big) \cap L^2\big([0,s];H^1(D)\big) \times C\big([0,s];[L^{\frac{2\beta}{\beta+1}}]_w \times \mathbb{R}^\infty\big) \\ &\times L^2\big([0,s];H_0^1\big) \times L^2\big([0,s];L^q(D)\big), \end{aligned}$$

where $\frac{1}{q} = \frac{1}{\gamma} + \frac{1}{2} - \frac{1}{d}$.

Lemma 2.3.6. *The pair* $(\rho_\epsilon, u_\epsilon)$ *satisfies the parabolic equation,* (2.48) *of Definition 2.3.2. Moreover, we have the following convergence upgrade: for all* $p \geq 1$,

$$\lim_{n\to\infty} \mathbb{E}^{\mathbb{P}_\epsilon}\big[\|\tilde{\rho}_n - \rho_\epsilon\|^p_{L_t^2(W_x^{1,2})}\big] = 0 \tag{2.70}$$

Proof. In view of Part 2 of Proposition 2.3.4, $(\tilde{\rho}_n, \tilde{u}_n)$ satisfy the same parabolic equation on the new probability space Ω_ϵ almost surely with respect to $\mathbb{P}_\epsilon$. Appealing to the pointwise convergences (2.54)-(2.55), we may pass to the limit in the weak form $\mathbb{P}_\epsilon$ a.s. and obtain the same equation for $(\rho_\epsilon, u_\epsilon)$. To prove the convergence upgrade, begin by appealing to Lemma 1.7.17 and obtain the following energy identity for all $t \in [0,T]$, $\mathbb{P}_\epsilon$ a.s.

$$\int_D \tilde{\rho}_n^2(t)\mathrm{d}x + \epsilon \int_0^t \int_D |\nabla \tilde{\rho}_n|^2 \mathrm{d}x\mathrm{d}t = \int_D \rho_{0,\delta}^2 \mathrm{d}x - \int_0^t \int_D \operatorname{div} \tilde{u}_n \tilde{\rho}_n^2 \mathrm{d}x\mathrm{d}t. \tag{2.71}$$

$$\int_D \rho_\epsilon^2(t)\mathrm{d}x + \epsilon \int_0^t \int_D |\nabla \rho_\epsilon|^2 \mathrm{d}x\mathrm{d}t = \int_D \rho_{0,\delta}^2 \mathrm{d}x - \int_0^t \int_D \operatorname{div} u_\epsilon \rho_\epsilon^2 \mathrm{d}x\mathrm{d}t. \tag{2.72}$$

Using again (2.54)-(2.55), we can pass limits on the RHS of (2.71) and conclude from the LHS of (2.72) that $\tilde{\mathbb{P}}_\epsilon$ a.s.

$$\lim_{n\to\infty} \epsilon \int_0^T \int_D |\nabla \tilde{\rho}_n|^2 \mathrm{d}x\mathrm{d}t = \epsilon \int_0^T \int_D |\nabla \rho_\epsilon|^2 \mathrm{d}x\mathrm{d}t. \tag{2.73}$$

Combining this observation with the pointwise convergence (2.54) and the uniform bounds, one obtains

$$\lim_{n\to\infty} \|\tilde{\rho}_n\|_{L^p\left(\Omega_\epsilon; L_t^2(W_x^{1,2})\right)} = \|\rho_\epsilon\|_{L^p\left(\Omega_\epsilon; L_t^2(W_x^{1,2})\right)}.$$

Hence, we may upgrade the weak convergence (2.56) and obtain (2.70) as desired.

□

Lemma 2.3.7. *For $\mathbb{P}_\epsilon$ almost all $\omega \in \Omega_\epsilon$, $\tilde{\theta}_n(\omega) \to \theta_\epsilon(\omega)$ in $L^q_{t,x}$ for each $q < 3$. Moreover, $\rho_\epsilon, u_\epsilon$ and θ_ϵ satisfy the renormalized temperature inequality,* (3.57).

Proof. The basic strategy of the proof is the same as in Lemma 2.2.7. Namely, by Proposition 2.3.4, for all $\omega \in \Omega_\epsilon$, $\tilde{\theta}_n(\omega) \to \theta_\epsilon(\omega)$ weakly in $L^3_{t,x}$. Hence, to prove the Lemma, it suffices to prove $\tilde{\theta}_n^2(\omega) \to \theta_\epsilon^2(\omega)$ in $D'_{t,x}$ for arbitary $\omega \in \Omega_\epsilon$.

As in Lemma 2.2.7, we will begin by proving that $(\tilde{\rho}_n(\omega)+\delta)\theta_n(\omega) \to (\rho_\epsilon(\omega)+\delta)\theta_\epsilon(\omega)$ strongly in $L^2_t(H_x^{-1})$. However, this is not as simple as in Lemma 2.2.7

because we no longer know that $(\tilde{\rho}_n(\omega)+\delta)\theta_n(\omega) \to (\rho_\epsilon(\omega)+\delta)\theta_\epsilon(\omega)$ in $C_t([L_x^2]_w)$. Instead our approach will be to verify the Hypotheses of Proposition 1.7.12.

Fix $\omega \in \Omega_\epsilon$. In the language of Proposition 1.7.12, let $f_n = (\tilde{\rho}_n(\omega)+\delta)\tilde{\theta}_n(\omega)$, $f = (\rho_\epsilon(\omega)+\delta)\theta_\epsilon(\omega)$ and

$$\begin{aligned} g_n =& -\operatorname{div}(\tilde{\rho}_n(\omega)\tilde{u}_n(\omega)\tilde{\theta}_n(\omega)) + \Delta\mathcal{K}(\tilde{\theta}_n(\omega)) - \delta\tilde{\theta}_n(\omega)^3 \\ &+ (1-\delta)\mathcal{S}(\tilde{u}_n(\omega)) : \nabla\tilde{u}_n(\omega) - \tilde{\theta}_n(\omega)p_\theta(\tilde{\rho}_n(\omega))\operatorname{div}\tilde{u}_n(\omega). \end{aligned}$$

The temperature equation implies $\partial_t f_n \leq g_n$ in $D'_{t,x}$. By Proposition 2.3.4, $(\tilde{\rho}_n(\omega)+\delta)\tilde{\theta}_n(\omega)$ in $L_t^\infty(M_x)$, so in particular $f_n \to f$ in $D'_{t,x}$. Moreover, observe that for all $n \geq 1$, $(\tilde{\rho}_n(\omega)+\delta)\tilde{\theta}_n(\omega) \in L_t^\infty(L_x^1)$. Indeed, this follows from the fact that $(\rho_n(\omega)+\delta)\theta_n(\omega) \in L_t^\infty(L_x^1)$ for all $\omega \in \Omega_n$ together with the explicit representation $(\tilde{\rho}_n+\delta)\tilde{\theta}_n = (\rho_n+\delta)\theta_n \circ \widetilde{T}_n$. Hence, by Lemma 1.7.13, the weak-$\star$ compactness of $\{(\rho_n(\omega)+\delta)\theta_n(\omega)\}_{n=1}^\infty$ implies the sequence $\{f_n\}_{n=1}^\infty$ is uniformly bounded in $L_t^\infty(L_x^1)$.

It remains to verify that $\{f_n\}_{n=1}^\infty$ is uniformly bounded in $L_t^2(L_x^{\frac{2d}{d+2}})$ and $\{g_n\}_{n=1}^\infty$ is uniformly bounded in $L_t^1(W_x^{-k,p})$ for some $p > 1$. For this purpose, we use the pointwise convergences in Proposition 2.3.4 to select a constant $C(\omega)$ independent of $n \geq 1$ such that

$$\sup_{n\geq 1}\left[\|\tilde{\rho}_n(\omega)\|_{L_t^\infty(L_x^\gamma)} + \|\tilde{u}_n(\omega)\|_{L_t^2(H_x^1)}\right] \leq C(\omega).$$

$$\sup_{n\geq 1}\left[\|\tilde{\theta}_n(\omega)\|_{L_t^2(H_x^1)} + \|\tilde{\theta}_n(\omega)\|_{L_{t,x}^3}\right] \leq C(\omega).$$

Thus, we find that $f_n = (\tilde{\rho}_n+\delta)\tilde{\theta}_n \in L_t^2(L_x^r)$ for $\frac{1}{r} = \frac{1}{\gamma} + \frac{1}{2} - \frac{1}{d}$, uniformly in $n \geq 1$. Moreover, $\frac{1}{r} < \frac{1}{2} + \frac{1}{d}$ since $\gamma > \frac{d}{2}$.

Next we will estimate each term in the definition of g_n. For simplicity of notation, we drop dependence on ω. Let $\frac{1}{p} = 1 + \frac{1}{\gamma} - \frac{2}{d}$, then

$$\begin{aligned}
&\|\operatorname{div}(\tilde{\rho}_n \tilde{\theta}_n \tilde{u}_n)\|_{L_t^1(W_x^{-1,q})} \leq \|\tilde{\rho}_n \tilde{\theta}_n \tilde{u}_n\|_{L_t^1(L_x^q)} \\
&\leq \|\tilde{\rho}_n\|_{L_t^\infty(L_x^\gamma)} \|\tilde{\theta}_n\|_{L_t^2(H_x^1)} \|\tilde{u}_n\|_{L_t^2(H_x^1)}.
\end{aligned}$$

Since $\mathcal{K}(\theta) \sim \theta^3$, we find that

$$\begin{aligned}
&\|\Delta \mathcal{K}(\theta_\tau)\|_{L_t^1(W_x^{-2,1})} + \delta \|\tilde{\theta}_n^3\|_{L_{t,x}^1} + 2(1-\delta)|S(\tilde{u}_n) : \nabla \tilde{u}_n|_{L_{t,x}^1} \\
&\leq C\big[\|\tilde{\theta}_n\|_{L_{t,x}^3}^3 + \|\tilde{u}_n\|_{L_t^2(H_x^1)}^2\big].
\end{aligned}$$

Finally, since $p_\theta(\rho) \sim \rho^{\frac{\gamma}{d}}$,

$$\|\tilde{\theta}_n p_\theta(\tilde{\rho}_n) \operatorname{div} \tilde{u}_n\|_{L_{t,x}^1} \leq C \|\tilde{\theta}_n\|_{L_t^2(L_x^{\frac{2d}{d-2}})} \|\tilde{\rho}_n\|_{L_t^\infty(L_x^\gamma)}^{\gamma/d} \|\tilde{u}_n\|_{L_t^2(H_x^1)}.$$

Choose a $p > 1$ such that $L_{t,x}^1 + L_t^1(W_x^{-2,1}) + L_t^1(W_x^{-1,q}) \hookrightarrow L_t^1(W_x^{-3,p})$.

With these observations at hand, we may apply Proposition 1.7.12 and deduce $(\tilde{\rho}_n + \delta)\tilde{\theta}_n \to (\rho_\epsilon + \delta)\theta_\epsilon$ in $L_t^2(H_x^{-1})$. Moreover, by Proposition 2.3.4, $\tilde{\theta}_n \to \theta_\epsilon$ in $L_t^2(H_x^1)$. Hence, $(\tilde{\rho}_n + \delta)\tilde{\theta}_n^2 \to (\rho_\epsilon + \delta)\theta_\epsilon^2$ in $D'_{t,x}$. This completes the proof of the strong convergence of the temperature.

Finally, to pass the limit in the temperature equation, work pointwise in Ω_ϵ and follow exactly the arguments in [11], page 174. □

Lemma 2.3.8. *The pair $(\rho_\epsilon, u_\epsilon)$ satisfies the energy corrected momentum equation 2.49 from Definition 2.3.2.*

Proof. Our task is to verify the weak form of the momentum equation (2.49) holds with respect to each test function $\phi \in [C_c^\infty(D)]^d$. The first observation is that it suffices to verify (2.49) for $\phi \in \cup_{n=1}^\infty X_n$. After this is proven, if ϕ is a general test function, then $\{\Pi_n\phi\}_{n=1}^\infty$ converges to ϕ in C_x^2 by (2.1.2) and Sobolev embeddings. Hence, a density argument completes the proof.

We follow the same general strategy as in the $\tau \to 0$ step. Indeed, first note that the exact same proof works in order to check that $\{\beta_k^\epsilon\}_{k=1}^\infty$ is a collection of $\{\mathcal{F}_\epsilon^t\}_{t=0}^T$ independent Brownian motions. Next, for each $\phi \in \cup_{n=1}^\infty X_n$ we introduce a continuous $\{\mathcal{F}_\epsilon^t\}_{t=0}^T$ adapted stochastic process $\{M_t^\epsilon(\phi)\}_{t=0}^T$ defined by the relation

$$\begin{aligned}
M_t^\epsilon(\phi) &= \int_D \rho_\epsilon u_\epsilon(t)\cdot\phi\mathrm{d}x - \int_D m_{0,\delta}\cdot\phi\mathrm{d}x \\
&- \int_0^t\int_D [\rho_\epsilon u_\epsilon \otimes u_\epsilon - 2\mu\nabla u_\epsilon - \lambda \operatorname{div} u_\epsilon I] : \nabla\phi\mathrm{d}x\mathrm{d}s \\
&- \int_0^t\int_D \left[(P(\rho_\epsilon,\theta_\epsilon) + \delta\rho_\epsilon^\beta)\operatorname{div}\phi - \epsilon\nabla u_\epsilon\nabla\rho_\epsilon\cdot\phi\right]\mathrm{d}x\mathrm{d}s.
\end{aligned}$$

Our goal is it is straightforward to implement the method in Lemma **??** and identify

$$M_t^\epsilon(\phi) = \sum_{k=1}^\infty \int_0^t\int_D \sigma_{k,\delta}(\rho_\epsilon,\rho_\epsilon u_\epsilon,\rho_\epsilon\theta_\epsilon)\cdot\phi dx d\beta_k^\epsilon(s).$$

We will sketch the main points of the argument. In view of the pointwise convergences (2.54)-(2.55), the following limits hold $\tilde{\mathbb{P}}_\epsilon$ a.s. for all $t\in[0,T]$

$$\begin{aligned}
&\lim_{n\to\infty}\int_D \tilde\rho_n\tilde u_n(t)\cdot\phi\mathrm{d}x = \int_D \rho_\epsilon u_\epsilon(t)\cdot\phi\mathrm{d}x \\
&\lim_{n\to\infty}\int_0^t\int_D [2\mu\nabla\tilde u_n + \lambda\operatorname{div}\tilde u_n I] : \nabla\phi\mathrm{d}x\mathrm{d}s = \int_0^t\int_D [2\mu\nabla u_\epsilon + \lambda\operatorname{div} u_\epsilon]\, I : \nabla\phi\mathrm{d}x\mathrm{d}s.
\end{aligned}$$

Noting the compact embedding $L^{\frac{2\gamma}{\gamma+1}} \hookrightarrow W_x^{-1,2}$, we may upgrade (2.56) with Lemma 1.7.10 and obtain $\mathbb{P}_\epsilon$ a.s. $\tilde\rho_n\tilde u_n \to \rho_\epsilon u_\epsilon$ in $L_t^2(W_x^{-1,2})$. Combining with (2.55) we have a weak/strong pairing and obtain $\mathbb{P}_\epsilon$ a.s. for all $t\in[0,T]$

$$\lim_{n\to\infty}\int_0^t\int_D \tilde\rho_n\tilde u_n\otimes\tilde u_n : \nabla\phi\mathrm{d}x\mathrm{d}t = \int_0^t\int_D \rho_\epsilon u_\epsilon\otimes u_\epsilon : \nabla\phi\mathrm{d}x\mathrm{d}t. \tag{2.74}$$

Combining (2.70), the strong convergence upgrade for the density, with the weak convergence of the velocity (2.55), yields(along a subsequence) $\mathbb{P}_\epsilon$ a.s. for all $t \in [0,T]$

$$\lim_{n\to\infty}\int_0^t\int_D \epsilon\nabla\tilde{u}_n\nabla\tilde{\rho}_n\cdot\phi \mathrm{d}x\mathrm{d}s = \int_0^t\int_D \epsilon\nabla u_\epsilon\nabla\rho_\epsilon\cdot\phi \mathrm{d}x\mathrm{d}s. \tag{2.75}$$

Recalling the interpolation argument in the proof of the $\tau \to 0$ tightness Lemma 2.2.6, we may use the uniform bounds (**??**) to obtain further: for all $p \geq 2$

$$\sup_n \mathbb{E}^{\mathbb{P}_\epsilon}\|\tilde{\rho}_n\|^p_{L_t^{\beta+1}(L_x^{\beta+1})} < \infty. \tag{2.76}$$

Combining this observation with (2.54) gives $\mathbb{P}_\epsilon$ a.s. for all $t \in [0,T]$

$$\lim_{n\to\infty}\int_0^t\int_D \left(\tilde{\rho}_n^\gamma + \delta\tilde{\rho}_n^\beta\right)\operatorname{div}\phi\mathrm{d}x\mathrm{d}s = \int_0^t\int_D \left(\rho_\epsilon^\gamma + \delta\rho_\epsilon^\beta\right)\operatorname{div}\phi\mathrm{d}x\mathrm{d}s. \tag{2.77}$$

The next fact required is that $P(\tilde{\rho}_n, \tilde{\theta}_n) + \delta\tilde{\rho}_n^\beta \to P(\rho_\epsilon, \theta_\epsilon) + \delta\rho_\epsilon^\beta$ strongly in $L^1_{t,x}$, $\mathbb{P}_\epsilon$ almost surely. This follows from Lemma 2.3.6, Lemma 2.3.7, and the pointwise(in ω) uniform control of $\tilde{\theta}_n p_\theta(\tilde{\rho}_n)$ in $L_t^2(L_x^r)$ for $\frac{1}{r} > \frac{1}{2} - \frac{1}{d} + \frac{d}{\gamma}$. Moreover, the fourth moments of the pressure contribution to the weak form can be estimated as in (2.44), from $\tau \to 0$ identification of the momentum martingale.

These remarks allow us to proceed as in the $\tau \to 0$ step and conclude that $\{\widetilde{M}_t^\epsilon(\phi)\}_{t=0}^T$ is an $\{\mathcal{F}_\epsilon^t\}_{t=0}^T$ martingale. Next we will check that it has the proper quadratic variation. The shorthand $\langle .\,, .\rangle$ is used for $L_x^q \times L_x^{q'}$ pairing in the remainder of the proof. Our claim is that for each $t \in [0,T]$, the following limit holds in $L^1(\Omega_\epsilon)$:

$$\sum_{k=1}^n \left|\left\langle \tilde{\rho}_n\sigma_{k,n,\delta}(\tilde{\rho}_n, \tilde{\rho}_n\tilde{u}_n, \tilde{\rho}_n\tilde{\theta}_n), \varphi\right\rangle\right|^2_{L^2[0,t]} \to \sum_{k=1}^\infty \left|\left\langle \rho_\epsilon\sigma_{k,\delta}(\rho_\epsilon, \rho_\epsilon u_\epsilon, \rho_\epsilon\theta_\epsilon), \varphi\right\rangle\right|^2_{L^2[0,t]}. \tag{2.78}$$

Towards this end, fix a $t \in [0,T]$ and begin also with $k \geq 1$ fixed. We claim that the following limit holds in $L^2(\Omega_\epsilon)$:

$$\left\langle \tilde{\rho}_n(t)\sigma_{k,n,\delta}(\tilde{\rho}_n(t), \tilde{\rho}_n\tilde{u}_n(t), \tilde{\rho}_n\tilde{\theta}_n(t)), \phi \right\rangle \to \left\langle \rho_\epsilon(t)\sigma_{k,\delta}(\rho_\epsilon(t), \rho_\epsilon u_\epsilon(t), \rho_\epsilon\theta_\epsilon(t)), \phi \right\rangle. \tag{2.79}$$

First we will check that the limit above holds pointwise in Ω_ϵ. Begin by noting the $\mathbb{P}_\epsilon$ almost sure $L^2_t(L^q_x)$ convergence of $\{\tilde{\rho}_n\tilde{\theta}_n\}_{n=1}^\infty$ to $\rho_\epsilon\theta_\epsilon$ together with the compactness of the mollification operator. Now since $\tilde{\rho}_n \to \rho_\epsilon$ in $C_t(L^\beta_x)$ with $\mathbb{P}_\epsilon$ probability one, it is enough to check that $\sigma_{k,n,\delta}(\tilde{\rho}_n(t), \tilde{\rho}_n\tilde{u}_n(t), \tilde{\rho}_n\tilde{\theta}_n(t)) \to \sigma_{k,\delta}(\rho_\epsilon(t), \rho_\epsilon u_\epsilon(t), \rho_\epsilon\theta_\epsilon(t))$ in $L_x^{\frac{\beta}{\beta-1}}$ with $\mathbb{P}_\epsilon$ probability one. To see this, note that $\sigma_{k,\delta} : L^\beta_x \times [L_x^{\frac{2\beta}{\beta+1}}]^d \to L_x^{\frac{\beta}{\beta-1}}$ is a compact operator. Also recall that the operators $\{\Pi_n\}_{n=1}^\infty$ converge to the identity, pointwise in $L_x^{\frac{\beta}{\beta-1}}$ by Hypothesis (2.1.2). Since $\sigma_{k,n,\delta} = \Pi_n \circ \sigma_{k,\delta}$ the pointwise convergences (2.54) and (2.56) of the density and momentum yield the desired pointwise convergence in $\widetilde{\Omega}_\epsilon$. To upgrade the limit to $L^2(\Omega_\epsilon)$, apply the Banach-Steinhaus theorem to the sequence $\{\Pi_n\}_{n=1}^\infty$ together with the uniform bounds on $\{\tilde{\rho}_n\}_{n=1}^\infty$ in $L^3(\Omega_\epsilon; C_t(L^\beta_x))$. Now to treat the full summation in k, observe the inequality

$$\begin{aligned}
&\left\| \sum_{k=1}^n \left| \left\langle \tilde{\rho}_n\sigma_{k,n,\delta}(\tilde{\rho}_n, \tilde{\rho}_n\tilde{u}_n, \tilde{\rho}_n\tilde{\theta}_n), \varphi \right\rangle \right|^2_{L^2[0,t]} - \sum_{k=1}^\infty \left| \left\langle \rho_\epsilon\sigma_{k,\delta}(\rho_\epsilon, \rho_\epsilon u_\epsilon, \rho_\epsilon\theta_\epsilon), \varphi \right\rangle \right|^2_{L^2[0,t]} \right\|_{L^1(\Omega_\epsilon)} \\
&\leq \sum_{k=1}^\infty \left\| \left\langle \tilde{\rho}_n\sigma_{k,n,\delta}(\tilde{\rho}_n, \tilde{\rho}_n\tilde{u}_n, \tilde{\rho}_n\tilde{\theta}_n) - \rho_\epsilon\sigma_{k,\delta}(\rho_\epsilon, \rho_\epsilon u_\epsilon, \rho_\epsilon\theta_\epsilon), \varphi \right\rangle \right\|^2_{L^2(\Omega_\epsilon\times[0,T])} \\
&+ \sum_{k=n}^\infty \left\| \left\langle \tilde{\rho}_n\sigma_{k,n,\delta}(\tilde{\rho}_n, \tilde{\rho}_n\tilde{u}_n, \tilde{\rho}_n\tilde{\theta}_n), \varphi \right\rangle \right\|^2_{L^2(\Omega_\epsilon\times[0,T])}.
\end{aligned}$$

For k fixed, each term in the sequences inside the sums above are dominated(up to a constant) by $|\sigma_k|^2_{L_x^{\frac{2\gamma}{\gamma-1}}(L^\infty_{\rho,m,\alpha})}$, uniformly in n. This upper bound is absolutely summable by Hypothesis 1.1.6. For the first term, the pointwise(in k) convergence to zero follows from the argument above, so the dominated convergence theorem for sequences gives the claim. For the second term, use the pointwise bound again and the convergence of $1_{k>n}$ towards zero. $\square$

2.3.3 Concluding the proof

Proof of Theorem 2.3.1. For each $\epsilon > 0$, we obtain an ϵ layer approximation $(\tilde{\rho}_\epsilon, \tilde{u}_\epsilon)$ using our compactness step, Proposition 2.3.4 together with Lemmas 2.3.6 and 2.3.8. To obtain the uniform bounds, use the weak and weak-$\star$ $L^p(\Omega)$ convergences in the Proposition 2.3.4 to treat all the terms which don't involve the temperature. To treat the uniform bounds for the temperature use the lower semicontinuity of the $L^p(\Omega; L^2_t(H^1_x) \cap L^3_{t,x})$ with respect to weak convergence. Finally, note that by another lower-semicontinuity argument,

$$\mathbb{E}^{\mathbb{P}_\epsilon} \|(\rho_\epsilon + \delta)\theta_\epsilon\|^p_{L^\infty_t(M_x)} \le C_p. \tag{2.80}$$

Since the total variation norm and the L^1_x norm agree for absolutely continuous measures, we find that for any $q > 1$, we have the continuous embedding $L^q_{t,x} \cap L^\infty_t(M_x) \hookrightarrow L^\infty_t(L^1_x)$. Since $(\rho_\epsilon(\omega) + \delta) \in L^\infty_t(L^\gamma_x)$ with $\gamma > \frac{3}{2}$ and $\theta_\epsilon(\omega) \in L^3_{t,x}$ for all $\omega \in \Omega_\epsilon$, we deduce that

$$\mathbb{E}^{\mathbb{P}_\epsilon} \|(\rho_\epsilon + \delta)\theta_\epsilon\|^p_{L^\infty(L^1_x)} \le C_p. \tag{2.81}$$

$\square$

Chapter 3: Proof of the Main Result: $\epsilon, \delta \to 0$

3.1 δ Layer Existence

This section is devoted to the δ layer existence theory; sending $\epsilon \to 0$ our goal is to prove:

Theorem 3.1.1. *For every $\delta > 0$, there exists a δ layer approximation(in the sense of Definition 3.1.2 below) $\rho_\delta, u_\delta, \theta_\delta$ to (1.1), relative to a stochastic basis $\big(\Omega_\delta, \mathcal{F}_\delta, (\mathcal{F}_\delta^t)_{t=0}^T, \mathbb{P}_\delta, \{\beta_k^\delta\}_{k\in\mathbb{N}}\big)$. Moreover, for all $p \geq 1$, there exists a constant $C_p > 0$ independent of δ such that*

$$\begin{aligned}
&\sup_{\delta>0} \mathbb{E}^{\mathbb{P}_\delta}\left[\|\sqrt{\rho_\delta}u_\delta\|_{L_t^\infty(L_x^2)}^{2p} + \|\rho_\delta\|_{L_t^\infty(L_x^\gamma)}^{\gamma p} + \|\delta^{\frac{1}{\beta}}\rho_\delta\|_{L_t^\infty(L_x^\beta)}^{\beta p} + |(\rho_\delta+\delta)\theta_\delta|_{L_t^\infty(L_x^1)}^{p}\right] \leq C_p.\\
&\sup_{\delta>0} \mathbb{E}^{\mathbb{P}_\delta}\left[\|\delta^{\frac{1}{2}}u_\delta\|_{L_t^2(H_{0,x}^1)}^{2p} + \|\delta^{\frac{1}{3}}\theta_\delta\|_{L_{t,x}^3}^{3p}\right] \leq C_p.
\end{aligned} \tag{3.1}$$

Recall that $\mathcal{R}$ is the collection of admissable renormalizations defined in the previous section.

Definition 3.1.2. *A triple $(\rho_\delta, u_\delta, \theta_\delta)$ is a δ layer approximation to (1.1) provided there exists a stochastic basis $\big(\Omega_\delta, \mathcal{F}_\delta, (\mathcal{F}_\delta^t)_{t=0}^T, \mathbb{P}_\delta, \{\beta_k^\delta\}_{k\in\mathbb{N}}\big)$ such that*

1. *The quadruple $(\rho_\delta, \rho_\delta u_\delta, \rho_\delta\theta_\delta, u_\delta)$ belongs in $L^2\big(\Omega\times[0,T]; \mathcal{P}; L^\beta \times L^{\frac{2\beta}{\beta+1}} \times L^1 \times$*

$[H_0^1]^d\big)$, where $\mathcal{P}$ is the predictable σ-algebra generated by $(\mathcal{F}_\delta^t)_{t=0}^T$ and $\frac{1}{q} = \frac{1}{\gamma} + \frac{1}{2} - \frac{1}{d}$.

2. *For all $\phi \in C^\infty(D)$ and all $t \in [0,T]$, the following equality holds $\mathbb{P}_\delta$ a.s.*

$$\int_D \rho_\delta(t)\phi \mathrm{d}x = \int_D \rho_{0,\delta} \mathrm{d}x + \int_0^t \int_D \rho_\delta u_\delta \cdot \nabla\phi \mathrm{d}x\mathrm{d}s. \tag{3.2}$$

3. *For all $\phi \in \left[C_c^\infty(D)\right]^d$ and all $t \in [0,T]$, the following equality holds $\mathbb{P}_\delta$ a.s.*

$$\begin{aligned}&\int_D \rho_\delta u_\delta(t) \cdot \phi \mathrm{d}x = \int_D m_{0,\delta} \cdot \phi \mathrm{d}x + \int_0^t \int_D [\rho_\delta u_\delta \otimes u_\delta - \mathcal{S}(u_\delta)] : \nabla\phi \mathrm{d}x\mathrm{d}s \\ &+ \int_0^t \int_D \left[(P(\rho_\delta,\theta_\delta) + \delta\rho_\delta^\beta) \operatorname{div}\phi\right] \mathrm{d}x\mathrm{d}s \\ &+ \sum_{k=1}^\infty \int_0^t \int_D \rho_\delta \sigma_{k,\delta}(\rho_\delta, \rho_\delta u_\delta, \rho_\delta\theta_\delta) \cdot \phi \mathrm{d}x\mathrm{d}\beta_k^\delta(s).\end{aligned} \tag{3.3}$$

4. *For all $\varphi \in \mathcal{D}_{temp}$ and all $h \in \mathcal{R}$, the inequality below holds $\mathbb{P}_\delta$ a.s.*

$$\begin{aligned}&\int_0^T \int_D (\rho_\delta + \delta) H(\theta_\delta)\partial_t\varphi + \rho_\delta H(\theta_\delta) u_\delta \cdot \nabla\varphi + \mathcal{K}_h(\theta_\delta)\Delta\varphi - \delta\theta_\delta^3 H'(\theta_\delta)\varphi \mathrm{d}x\mathrm{d}t \\ &\leq \int_0^T \int_D h(\theta_\delta)[\theta_\delta p_\theta(\rho_\delta) \operatorname{div} u_\delta - \mathcal{S}(u_\delta) : \nabla u_\delta]\varphi \mathrm{d}x\mathrm{d}t \\ &+ \int_0^T \int_D h'(\theta_\delta)\kappa(\theta_\delta)|\nabla\theta_\delta|^2 dxdt - \int_D \rho_\delta^0 H(\theta_\delta)\varphi(0)\mathrm{d}x.\end{aligned} \tag{3.4}$$

3.1.1 $\epsilon \to 0$ Compactness step

The main goal of this subsection is to prove the following compactness result:

Proposition 3.1.3. *There exists a probability space $(\Omega_\delta, \mathcal{F}_\delta, \mathbb{P}_\delta)$, a collection of independent Brownian motions $\{\beta_\delta^k\}_{k\in\mathbb{N}}$, limit points*

$$\left(\rho_\delta, u_\delta, \theta_\delta, \overline{\sqrt{\rho_\delta} u_\delta}, \overline{p_m(\rho_\delta)} + \delta\overline{\rho_\delta^\beta}, \overline{\theta_\delta\, p_\theta(\rho_\delta)}\right),$$

and a sequence of measurable maps $\{\widetilde{T}_\epsilon\}_{\epsilon>0}$ *such that:*

1. *For each* $\epsilon > 0$, $\widetilde{T}_\epsilon : (\Omega_\delta, \mathcal{F}_\delta, \mathbb{P}_\delta) \to (\Omega_\epsilon, \mathcal{F}_\epsilon, \mathbb{P}_\epsilon) \to$ *and* $(\widetilde{T}_\epsilon)_{\#}\mathbb{P}_\delta = \mathbb{P}_\epsilon$.

2. *The new sequence* $\left\{(\tilde{\rho}_\epsilon, \tilde{u}_\epsilon, \tilde{\theta}_\epsilon)\right\}_{\epsilon>0}$ *defined by* $\left(\tilde{\rho}_\epsilon, \tilde{u}_\epsilon, \tilde{\theta}_\epsilon\right) = \left(\rho_\epsilon, u_\epsilon, \tilde{\theta}_\epsilon\right) \circ \widetilde{T}_\epsilon$ *constitutes an* ϵ *layer approximation relative to the stochastic basis* $\big(\Omega_\delta, \mathcal{F}_\delta, \mathbb{P}_\delta, (\widetilde{\mathcal{F}}^t_\epsilon)_{t=0}^T, \widetilde{W}_\epsilon\big)$, *where* $\widetilde{W}_\epsilon := W_\epsilon \circ \widetilde{T}_\epsilon$ *and* $\widetilde{\mathcal{F}}^t_\epsilon = \widetilde{T}_\epsilon^{-1} \circ \mathcal{F}^t_\epsilon$.

3. *The uniform bounds in 2.46 hold with* $\rho_\epsilon, u_\epsilon, \theta_\epsilon$ *replaced by* $\tilde{\rho}_\epsilon, \tilde{u}_\epsilon, \tilde{\theta}_\epsilon$ *and* $\mathbb{P}_\epsilon$ *replaced by* $\mathbb{P}_\delta$.

4. *The following convergences hold pointwise on* Ω_δ*:*

$$\tilde{\rho}_\epsilon \to \rho_\delta \quad in \quad C_t\big([L^\beta_x]_w\big) \tag{3.5}$$

$$\tilde{u}_\epsilon \to u_\delta \quad in \quad [L^2_t(H^1_{0,x})]_w \tag{3.6}$$

$$\tilde{\rho}_\epsilon \tilde{u}_\epsilon \to \rho_\delta u_\delta \quad in \quad C_t([L_x^{\frac{2\beta}{\beta+1}}]_w) \tag{3.7}$$

$$p_m(\tilde{\rho}_\epsilon) + \delta\overline{\tilde{\rho}_\epsilon^\beta} \to \overline{p_m(\rho_\delta)} + \delta\overline{\rho_\delta^\beta} \quad in \quad [L^{1+\beta^{-1}}_{t,x}]_w \tag{3.8}$$

$$\tilde{\theta}_\epsilon p_\theta(\tilde{\rho}_\epsilon) \to \overline{\theta_\delta p_\theta(\rho_\delta)} \quad in \quad [L^2_t(L^q_x)]_w \tag{3.9}$$

$$\tilde{\theta}_\epsilon \to \theta_\delta \quad in \quad [L^2_t(H^1_x) \cap L^3_{t,x}]_w \tag{3.10}$$

$$(\tilde{\rho}_\epsilon + \delta)\tilde{\theta}_\epsilon \to (\rho_\delta + \delta)\theta_\delta \quad in \quad [L^\infty_t(M_x)]_{w*} \tag{3.11}$$

$$\widetilde{W}_\epsilon \to W_\delta \quad in \quad [C_t]^\infty, \tag{3.12}$$

   where $\frac{1}{q} = \frac{1}{2} - \frac{1}{d} + \frac{\gamma}{\beta d}$.

5. The following additional convergences hold

$$\sqrt{\tilde{\rho}_\epsilon}\tilde{u}_\epsilon \to \overline{\sqrt{\rho_\delta}u_\delta} \quad in \quad L^p_{w^*}\big(\Omega_\delta; L^\infty_t(L^2_x)\big) \tag{3.13}$$

$$\tilde{u}_\epsilon \to u_\delta \quad in \quad L^p_w\big(\Omega_\delta; L^2_t(H^1_{0,x})\big) \tag{3.14}$$

$$\tilde{\rho}_\epsilon \to \rho_\delta \quad in \quad L^p_{w^*}(\Omega_\delta; L^\infty_t(L^\beta_x)) \cap L^p_w\big(\Omega_\delta; L^2_t(W^{1,2}_x)\big) \tag{3.15}$$

$$\tilde{\rho}_\epsilon \operatorname{div} \tilde{u}_\epsilon \to \overline{\rho_\delta \operatorname{div} u_\delta} \quad in \quad L^p_w(\Omega_\delta; L^2_t(L_x^{\frac{2\beta}{\beta+2}})) \tag{3.16}$$

$$\tilde{\rho}_\epsilon \log \tilde{\rho}_\epsilon \to \overline{\rho_\delta \log \rho_\delta} \quad in \quad L^p_{w^*}\big(\Omega_\delta; L^\infty_t(L^2_x)\big). \tag{3.17}$$

To prove the tightness, we need the following integrability gains:

Proposition 3.1.4. *For every $p \geq 1$, there exists $C_p > 0$ such that*

$$\sup_{\epsilon>0} \mathbb{E}^{\mathbb{P}_\epsilon}\left[\left|\int_0^T \int_D \rho_\epsilon P(\rho_\epsilon, \theta_\epsilon) + \delta\rho_\epsilon^{\beta+1} \mathrm{d}x\mathrm{d}t\right|^p\right] \leq C_p. \tag{3.18}$$

Proof. For regular domains D, one can define a sort of "inverse divergence", known as the Bogovski operator $\mathcal{B}$. The properties of $\mathcal{B}$ are recalled in appendix lemma 1.7.16. Define the following "random test function"

$$\varphi_\epsilon = \mathcal{B}[\rho_\epsilon - \frac{1}{|D|}\int_D \rho_\epsilon \mathrm{d}x]. \tag{3.19}$$

The parabolic equation and the dirichlet boundary condition for the velocity yields the $\tilde{\mathbb{P}}_\epsilon$ a.s. equality

$$\partial_t \varphi_\epsilon = \epsilon \nabla \rho_\epsilon - \mathcal{B}[\operatorname{div}(\rho_\epsilon u_\epsilon)]. \tag{3.20}$$

Since the weak form of the momentum equation is stated in terms of deterministic test functions, "testing" φ_ϵ requires an appeal to a version of the Ito product rule. The equality below can be justified with a somewhat lengthy, but straightforward regularization argument(which we omit) in the spirit of [18] or [6]. For all times $t \in [0,T]$ we have $\mathbb{P}_\epsilon$ a.s.

$$\begin{aligned}\int_D \rho_\epsilon u_\epsilon(t)\cdot\varphi_\epsilon(t)dx = &\int_D m_{0,\delta}\cdot\varphi_\epsilon(0)dx\\ &+\int_0^t\int_D \rho_\epsilon u_\epsilon\cdot\partial_t\varphi_\epsilon + [\rho_\epsilon u_\epsilon\otimes u_\epsilon - 2\mu\nabla u_\epsilon]:\nabla\varphi_\epsilon \mathrm{d}x\mathrm{d}s\\ &+\int_0^t\int_D (P(\rho_\epsilon,\theta_\epsilon)+\delta\rho_\epsilon^\beta - \lambda\operatorname{div} u_\epsilon)I:\nabla\varphi_\epsilon - \epsilon\nabla u_\epsilon\nabla\rho_\epsilon\cdot\varphi_\epsilon\mathrm{d}x\mathrm{d}s\\ &+\sum_{k=1}^\infty\int_0^t\int_D \rho_\epsilon\sigma_{k,\delta}(\rho_\epsilon,\rho_\epsilon u_\epsilon.\rho_\epsilon\theta_\epsilon)\cdot\varphi_\epsilon\mathrm{d}x\mathrm{d}\tilde{\beta}_k^\epsilon(s).\end{aligned} \tag{3.21}$$

For our purposes, we will use the identity above at time $t = T$. By definition of the Bogovski operator, we have

$$\int_0^T\int_D(\rho_\epsilon^\gamma+\delta\rho_\epsilon^\beta)I:\nabla\varphi_\epsilon\mathrm{d}x\mathrm{d}s = \int_0^T\int_D(\rho_\epsilon^\gamma+\delta\rho_\epsilon^\beta)(\rho_\epsilon - \frac{1}{|D|}\int_D\rho_\epsilon\mathrm{d}x)\mathrm{d}x\mathrm{d}s. \tag{3.22}$$

We can now rearrange and obtain

$$\begin{aligned}\int_0^T\int_D P(\rho_\epsilon,\theta_\epsilon)\rho_\epsilon+\delta\rho_\epsilon^{\beta+1}\mathrm{d}x\mathrm{d}t = &\int_D\big[\rho_\epsilon u_\epsilon(T)\cdot\varphi_\epsilon(T) - m_{0,\delta}\cdot\varphi_\epsilon(0)\big]\mathrm{d}x\\ &+\int_0^T\int_D[2\mu\nabla u_\epsilon+\lambda\operatorname{div} u_\epsilon I - \rho_\epsilon u_\epsilon\otimes u_\epsilon]:\nabla\varphi_\epsilon\mathrm{d}x\mathrm{d}s\\ &+\int_0^T\int_D(P(\rho_\epsilon,\theta_\epsilon)+\delta\rho_\epsilon^\beta)\oint_D\rho_\epsilon+\epsilon\nabla u_\epsilon\nabla\rho_\epsilon\cdot\varphi_\epsilon\mathrm{d}x\\ &+\int_0^T\int_D\big[\mathcal{B}[\operatorname{div}(\rho_\epsilon u_\epsilon)] - \epsilon\nabla\rho_\epsilon\big]\cdot\rho_\epsilon u_\epsilon\mathrm{d}x\mathrm{d}s\\ &-\sum_{k=1}^\infty\int_0^T\int_D\rho_\epsilon\sigma_k(\rho_\epsilon,\rho_\epsilon u_\epsilon,\rho_\epsilon\theta_\epsilon)\cdot\varphi_\epsilon\mathrm{d}x\mathrm{d}\tilde{\beta}_k^\epsilon(s).\end{aligned} \tag{3.23}$$

We proceed by estimating the p^{th} moments on both sides of this equality. In view of Theorem 1.7.16, the β constraints (2.3), and the Sobolev embedding $W_x^{1,\beta}\hookrightarrow L_x^\infty$, we obtain

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{P}_\epsilon}\left[\left|\int_D (\rho_\epsilon u_\epsilon(T) - m_{0,\delta})\varphi_\epsilon(T)\mathrm{d}x\right|^p\right] \\
&\lesssim \mathbb{E}^{\mathbb{P}_\epsilon}\big[\big(\|\rho_\epsilon u_\epsilon\|^p_{L_t^\infty(L_x^{\frac{2\gamma}{\gamma+1}})} + \|m_{0,\delta}\|^p_{L_x^{\frac{2\gamma}{\gamma+1}}}\big)\|\rho_\epsilon\|^p_{L_t^\infty(L_x^\beta)}\big] \\
&\lesssim \mathbb{E}^{\mathbb{P}_\epsilon}\big[\|\rho_\epsilon u_\epsilon\|^{2p}_{L_t^\infty(L_x^{\frac{2\gamma}{\gamma+1}})} + \|m_{0,\delta}\|^{2p}_{L_x^{\frac{2\gamma}{\gamma+1}}}\big]^{\frac{1}{2}}\mathbb{E}^{\mathbb{P}_\epsilon}[\|\rho_\epsilon\|^{2p}_{L_t^\infty(L_x^\beta)}]^{\frac{1}{2}}.
\end{aligned}
\tag{3.24}
$$

Using Theorem 1.7.16 and (2.3), we obtain

$$
\begin{aligned}
\mathbb{E}^{\mathbb{P}_\epsilon}\left[\left|\int_0^T\int_D 2\mu\nabla u_\epsilon + \lambda \operatorname{div} u_\epsilon : \nabla\varphi_\epsilon \mathrm{d}x\mathrm{d}s\right|^p\right] &\lesssim \mathbb{E}^{\mathbb{P}_\epsilon}\big[\|u_\epsilon\|^p_{L_t^2(W_x^{1,2})}\|\nabla\varphi_\epsilon\|^p_{L_t^2(L_x^2)}\big] \\
&\lesssim \mathbb{E}^{\mathbb{P}_\epsilon}\big[\|u_\epsilon\|^{2p}_{L_t^2(W_x^{1,2})}\big]^{\frac{1}{2}}\mathbb{E}^{\mathbb{P}_\epsilon}\big[\|\rho_\epsilon\|^{2p}_{L_t^\infty(L_x^\beta)}\big]^{\frac{1}{2}}.
\end{aligned}
\tag{3.25}
$$

Note that (2.3) implies the embedding $L_x^\beta \hookrightarrow L_x^{\frac{d\beta}{2\beta-d}}$ so applying Theorem 1.7.16 yields

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{P}_\epsilon}\left[\left|\int_0^T\int_D \rho_\epsilon u_\epsilon \otimes u_\epsilon : \nabla\varphi_\epsilon \mathrm{d}x\mathrm{d}s\right|^p\right] \\
&\lesssim \mathbb{E}^{\mathbb{P}_\epsilon}\left[\left|\int_0^T \|\rho_\epsilon\|_{L_x^\beta}\|u_\epsilon\|^2_{L^{\frac{2d}{d-2}}}\|\nabla\varphi_\epsilon\|_{L^{\frac{d\beta}{2\beta-d}}}\mathrm{d}s\right|^p\right] \\
&\lesssim \mathbb{E}^{\mathbb{P}_\epsilon}\left[\left|\int_0^T \|\rho_\epsilon\|^2_{L_x^\beta}|u_\epsilon|^2_{L^{\frac{2d}{d-2}}}\mathrm{d}s\right|^p\right] \lesssim \mathbb{E}^{\mathbb{P}_\epsilon}[\|\rho_\epsilon\|^{2p}_{L_t^\infty(L_x^\beta)}|u_\epsilon|^{2p}_{L_t^2(W_x^{1,2})}] \\
&\lesssim \mathbb{E}^{\mathbb{P}_\epsilon}[\|\rho_\epsilon\|^{4p}_{L_t^\infty(L_x^\beta)}]^{\frac{1}{2}}\mathbb{E}^{\mathbb{P}_\epsilon}[u_\epsilon|^{4p}_{L_t^2(W_x^{1,2})}]^{\frac{1}{2}}.
\end{aligned}
\tag{3.26}
$$

Applying Hölder yields

$$
\mathbb{E}^{\mathbb{P}_\epsilon}\left[\left|\int_0^T (\oint_D \rho_\epsilon(s)dx)\int_D(\delta\rho_\epsilon^\beta + \rho_\epsilon^\gamma)\mathrm{d}x\mathrm{d}s\right|^p\right] \lesssim \mathbb{E}^{\mathbb{P}_\epsilon}[\delta^p\|\rho_\epsilon\|^{(\beta+1)p}_{L_t^\infty(L_x^\beta)} + \|\rho_\epsilon\|^{(\gamma+1)p}_{L_t^\infty(L_x^\gamma)}]. \tag{3.27}
$$

Defining r by the relation $\frac{1}{r} = \frac{1}{2} + \frac{1}{d} - \frac{1}{\beta}$ and applying Theorem 1.7.16, then using

Hölder, the embedding $L_x^{\frac{d\beta}{2\beta-d}} \hookrightarrow L_x^\beta$, and (2.3), we obtain

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}_\epsilon}\left[\left|\int_0^T\int_D \rho_\epsilon u_\epsilon\cdot\mathcal{B}[\operatorname{div}(\rho_\epsilon u_\epsilon)]\mathrm{d}x\mathrm{d}t\right|^p\right]\\
&\lesssim \mathbb{E}^{\mathbb{P}_\epsilon}\left[\left|\int_0^T \|\rho_\epsilon\|_{L_x^\beta}\|u_\epsilon\|_{L_x^{\frac{2d}{d-2}}}|\mathcal{B}[\operatorname{div}(\rho_\epsilon u_\epsilon)]|_{L_x^r}\mathrm{d}t\right|^p\right]\\
&\lesssim \mathbb{E}^{\mathbb{P}_\epsilon}\left[\left|\int_0^T \|\rho_\epsilon\|_{L_x^\beta}\|u_\epsilon\|_{L_x^{\frac{2d}{d-2}}}\|\rho_\epsilon u_\epsilon\|_{L^r}\mathrm{d}t\right|^p\right] \qquad (3.28)\\
&\lesssim \mathbb{E}^{\mathbb{P}_\epsilon}\left[\left|\int_0^T \|\rho_\epsilon\|_{L_x^\beta}\|u_\epsilon\|^2_{L_x^{\frac{2d}{d-2}}}\|\rho_\epsilon\|_{L^{\frac{\beta d}{2\beta-d}}}\mathrm{d}t\right|^p\right]\\
&\lesssim \mathbb{E}^{\mathbb{P}_\epsilon}[\|\rho_\epsilon\|^{4p}_{L_t^\infty(L_x^\beta)}]^{\frac{1}{2}}\mathbb{E}^{\mathbb{P}_\epsilon}[\|u_\epsilon\|^{4p}_{L_t^2(W_x^{1,2})}]^{\frac{1}{2}}.
\end{aligned}$$

Using again the Sobolev embedding of $W_x^{1,\beta}\hookrightarrow L_x^\infty$, we can estimate the energy correction as follows:

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}_\epsilon}\left[\left|\int_0^T\int_D \epsilon\nabla u_\epsilon\nabla\rho_\epsilon\cdot\varphi_\epsilon\mathrm{d}x\mathrm{d}t\right|^p\right] \qquad (3.29)\\
&\lesssim \epsilon^{\frac{p}{2}}\mathbb{E}^{\mathbb{P}_\epsilon}[\|u_\epsilon\|^{3p}_{L_t^2(W_x^{1,2})}]^{\frac{1}{3}}\mathbb{E}^{\mathbb{P}_\epsilon}[\ |\sqrt{\epsilon}\nabla\rho_\epsilon\|^{3p}_{L_t^2(L_x^2)}]^{\frac{1}{3}}\mathbb{E}^{\mathbb{P}_\epsilon}[\|\rho_\epsilon\|^{3p}_{L_t^\infty(L_x^\beta)}]^{\frac{1}{3}}.
\end{aligned}$$

For the artificial viscosity, we use Hölder followed by (2.3) to obtain

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}_\epsilon}\left[\left|\int_0^T\int_D \epsilon\rho_\epsilon u_\epsilon\cdot\nabla\rho_\epsilon\mathrm{d}x\mathrm{d}t\right|^p\right]\\
&\lesssim \epsilon^{p/2}\mathbb{E}^{\mathbb{P}_\epsilon}[\|\sqrt{\epsilon}\nabla\rho_\epsilon\|^p_{L_t^2(L_x^2)}\|\tilde\rho_\epsilon\|^p_{L_t^\infty(L_x^\beta)}\|\tilde u_\epsilon\|^p_{L_t^2(W_x^{1,2})}] \qquad (3.30)\\
&\lesssim \epsilon^{p/2}\mathbb{E}^{\mathbb{P}_\epsilon}[\|\sqrt{\epsilon}\nabla\tilde\rho_\epsilon\|^{3p}_{L_t^2(L_x^2)}]^{\frac{1}{3}}\mathbb{E}^{\mathbb{P}_\epsilon}[\|\rho_\epsilon\|^{3p}_{L_t^\infty(L_x^\beta)}]^{\frac{1}{3}}\mathbb{E}^{\mathbb{P}_\epsilon}[\|u_\epsilon\|^{3p}_{L_t^2(W_x^{1,2})}]^{\frac{1}{3}}.
\end{aligned}$$

Finally, we use the BDG inequality, the summability Hypothesis 1.1.6, (2.3), and the Sobolev embedding of $W_x^{1,\beta}\hookrightarrow L_x^\infty$ to estimate the series of stochastic integrals as follows:

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{P}_\epsilon}\left[\left|\sum_{k=1}^{\infty}\int_0^T\int_D \rho_\epsilon\sigma_{k,\delta}(\rho_\epsilon,\rho_\epsilon u_\epsilon)\cdot\varphi_\epsilon \mathrm{d}x\mathrm{d}\tilde{\beta}_k^\epsilon(s)\right|^p\right] \\
&\lesssim \mathbb{E}^{\mathbb{P}_\epsilon}\left[\left|\sum_{k=1}^{\infty}\int_0^T(\int_D \rho_\epsilon\sigma_{k,\delta}(\rho_\epsilon,\rho_\epsilon u_\epsilon,\rho_\epsilon\theta_\epsilon)\cdot\varphi_\epsilon \mathrm{d}t\right|^{\frac{p}{2}}\right] \\
&\lesssim \left[\sum_{k=1}^{\infty}\|\sigma_k\|^2_{L_x^{\gamma'}(L^\infty_{\rho,m,\alpha})}\right]^{\frac{p}{2}}\mathbb{E}^{\mathbb{P}_\epsilon}\left[\left|\int_0^T\|\rho_\epsilon\|^2_{L_x^\gamma}\|\varphi_\epsilon\|^2_{L_x^\infty}\mathrm{d}t\right|^{p/2}\right] \\
&\lesssim \mathbb{E}^{\mathbb{P}_\epsilon}\left[\left|\int_0^T\|\rho_\epsilon\|^2_{L_x^\gamma}\|\rho_\epsilon\|^2_{L_x^\beta}dt\right|^{p/2}\right]\lesssim \mathbb{E}^{\mathbb{P}_\epsilon}[\|\rho_\epsilon\|^{2p}_{L_t^\infty(L_x^\beta)}].
\end{aligned}
\tag{3.31}
$$

Hence, appealing to the uniform bounds, we may close each estimate and obtain (3.72) as claimed.

□

To finish the proof of 3.1.3, follow the approach in the previous section. Namely, prove an similar tightness result (using the improved estiamtes on the pressure obtained above) and then appeal to the Jakubowski/Skorohod theorem 1.7.2. The details are very similar, so we omit them.

3.1.2 Preliminary identification step

Next we define a filtration $(\mathcal{F}_t^\delta)_{t=0}^T$ via $\mathcal{F}_t^\epsilon=\sigma(r_tX_\delta)$ where $X_\delta=\big(\rho_\delta,\rho_\delta u_\delta,W_\delta,u_\delta,\rho_\delta\theta_\delta\big)$ and $r_t: E_T\to E_t$

$$
\begin{aligned}
E_s &= C\big([0,s];[L^\beta]_w\big)\times C\big([0,s];[L^{\frac{2\beta}{\beta+1}}]_w\times\mathbb{R}^\infty\big) \\
&\times L^2\big([0,s];H_0^1\big)\times L^2\big([0,s];L^q(D)\big),
\end{aligned}
$$

where $\frac{1}{q}=\frac{1}{\gamma}+\frac{1}{2}-\frac{1}{d}$.

The following two lemmas follow in a straightforward way from our compactness step, using arguments similar to the $n\to\infty$ identification step.

Lemma 3.1.5. *The pair (ρ_δ, u_δ) satisfies the continuity equation, 3.2 of Definition 3.1.2.*

Lemma 3.1.6. *The pair $(\tilde{\rho}_\delta, \tilde{u}_\delta)$ satisfies the momentum equation* (3.3) *from Definition 3.1.2, with a modified pressure law $\overline{p_m(\rho_\delta)} + \overline{\theta_\delta\, p_\theta(\rho_\delta)} + \delta\overline{\rho_\delta^\beta}$.*

3.1.3 Strong convergence of the density

Now to proceed to the proof of the strong convergence of the density. The first step is the following weak continuity result:

Lemma 3.1.7. *Let $K \subset\subset D$ be arbitrary, then the weak continuity of the effective viscous pressure holds on average, that is:*

$$\begin{aligned}
&\lim_{\epsilon\to 0} \mathbb{E}^{\mathbb{P}_\delta}\left[\int_0^T \int_K \left((2\mu+\lambda)\operatorname{div}\tilde{u}_\epsilon - P_m(\tilde{\rho}_\epsilon) - \tilde{\theta}_\epsilon p_\theta(\tilde{\rho}_\epsilon) - \delta\tilde{\rho}_\epsilon^\beta\right)\tilde{\rho}_\epsilon \mathrm{d}x\mathrm{d}t\right] \\
&= \mathbb{E}^{\mathbb{P}_\delta}\left[\int_0^T \int_K \left((2\mu+\lambda)\operatorname{div} u_\delta - \overline{P_m(\rho_\delta)} - \overline{\theta_\delta p_\theta(\rho_\delta)} - \delta\overline{\rho_\delta^\beta}\right)\rho_\delta \mathrm{d}x\mathrm{d}t\right].
\end{aligned}$$

Proof. Recall that $\mathcal{A} = \nabla\Delta^{-1}$, where the inverse laplacian is understood to be well defined on compactly supported distributions in $\mathbb{R}^d$. Let η be a bump function supported in D. Define the following two random test functions: $\tilde{\varphi}_\epsilon = \eta\mathcal{A}\left[\eta\tilde{\rho}_\epsilon\right]$ and $\varphi_\delta = \eta\mathcal{A}\left[\eta\rho_\delta\right]$. Using the parabolic equation for $\tilde{\rho}_\epsilon$ driven by $\tilde{u}_\epsilon$ and the transport equation for ρ_δ driven by u_δ, we may check that

$$\partial_t\tilde{\varphi}_\epsilon = \eta\mathcal{A}\circ\operatorname{div}(\eta(\epsilon\nabla\tilde{\rho}_\epsilon - \tilde{\rho}_\epsilon\tilde{u}_\epsilon)) + \eta\mathcal{A}\left[\nabla\eta\cdot(\tilde{\rho}_\epsilon\tilde{u}_\epsilon - \epsilon\nabla\tilde{\rho}_\epsilon)\right]$$

$$\partial_t\varphi_\delta = -\eta\mathcal{A}\circ\operatorname{div}(\eta\rho_\delta u_\delta) + \eta\mathcal{A}\left[\nabla\eta\cdot\rho_\delta u_\delta\right]$$

Using the momentum equation for $(\tilde{\rho}_\epsilon, \tilde{u}_\epsilon)$ and (ρ_δ, u_δ) we may use the Ito product rule twice(see the remarks in Proposition 3.2.4 regarding justification) to find the evolution of $\tilde{\rho}_\epsilon \tilde{u}_\epsilon \cdot \tilde{\varphi}_\epsilon$ and $\rho_\delta u_\delta \cdot \varphi_\delta$. The first application yields the $\mathbb{P}_\delta$ a.s. equality

$$
\begin{aligned}
\int_D \tilde{\rho}_\epsilon \tilde{u}_\epsilon(T) \cdot \tilde{\varphi}_\epsilon(T) \mathrm{d}x = &\int_D m_{0,\delta} \cdot \tilde{\varphi}_\epsilon(0) \mathrm{d}x \\
&+ \int_0^T \int_D [\tilde{\rho}_\epsilon \tilde{u}_\epsilon \cdot \partial_t \tilde{\varphi}_\epsilon + [\tilde{\rho}_\epsilon \tilde{u}_\epsilon \otimes \tilde{u}_\epsilon - 2\mu \nabla \tilde{u}_\epsilon] : \nabla \tilde{\varphi}_\epsilon] \mathrm{d}x \mathrm{d}t \\
&+ \int_0^T \int_D (-\lambda \operatorname{div} \tilde{u}_\epsilon + P(\tilde{\rho}_\epsilon, \tilde{\theta}_\epsilon) + \delta \tilde{\rho}_\epsilon^\beta) I] : \nabla \tilde{\varphi}_\epsilon] - \epsilon \nabla \tilde{u}_\epsilon \nabla \tilde{\rho}_\epsilon \cdot \tilde{\varphi}_\epsilon \mathrm{d}x \mathrm{d}s \\
&+ \sum_{k=1}^\infty \int_0^T \int_D \tilde{\rho}_\epsilon \sigma_{k,\delta}(\tilde{\rho}_\epsilon, \tilde{\rho}_\epsilon \tilde{u}_\epsilon, \tilde{\rho}_\epsilon \tilde{\theta}_\epsilon) \cdot \varphi_\epsilon \mathrm{d}x \mathrm{d}\tilde{\beta}_k^\delta(s).
\end{aligned}
$$

The second application yields the $\tilde{\mathbb{P}}_\delta$ a.s. equality

$$
\begin{aligned}
\int_D \rho_\delta u_\delta(T) \cdot \varphi_\delta(T) \mathrm{d}x = &\int_D m_{0,\delta} \cdot \hat{\varphi}_\delta(0) \mathrm{d}x \\
&+ \int_0^T \int_D [\rho_\delta u_\delta \cdot \partial_t \varphi_\delta + [\rho_\delta u_\delta \otimes u_\delta - 2\mu \nabla u_\delta] \mathrm{d}x \mathrm{d}t \\
&+ \int_0^T \int_D [(-\lambda \operatorname{div} u_\delta + \overline{\rho_\delta^\gamma} + \delta \overline{P(\rho_\delta, \theta_\delta)}) I] : \nabla \varphi_\delta] \mathrm{d}x \mathrm{d}t \\
&+ \sum_{k=1}^\infty \int_0^T \int_D \rho_\delta \sigma_{k,\delta}(\rho_\delta, \rho_\delta u_\delta) \cdot \varphi_\delta \mathrm{d}x \mathrm{d}\tilde{\beta}_k^\delta(s).
\end{aligned}
$$

Note that

$$
\begin{aligned}
&[\tilde{\rho}_\epsilon \tilde{u}_\epsilon \otimes \tilde{u}_\epsilon - 2\mu \nabla \tilde{u}_\epsilon + (-\lambda \operatorname{div} \tilde{u}_\epsilon + \tilde{\rho}_\epsilon^\gamma + \delta \tilde{\rho}_\epsilon^\beta) I] : \nabla \tilde{\varphi}_\epsilon \\
&= [\tilde{\rho}_\epsilon \tilde{u}_\epsilon \otimes \tilde{u}_\epsilon - 2\mu \nabla \tilde{u}_\epsilon + (-\lambda \operatorname{div} \tilde{u}_\epsilon + \tilde{\rho}_\epsilon^\gamma + \delta \tilde{\rho}_\epsilon^\beta) I] : \nabla \eta \otimes \mathcal{A}[\eta \tilde{\rho}_\epsilon] \\
&+ \eta \tilde{\rho}_\epsilon \tilde{u}_\epsilon \otimes \tilde{u}_\epsilon : \nabla \mathcal{A}[\eta \tilde{\rho}_\epsilon] - 2\mu \eta \nabla \tilde{u}_\epsilon : \nabla \mathcal{A}[\eta \tilde{\rho}_\epsilon] + \eta(\tilde{\rho}_\epsilon^\gamma + \delta \tilde{\rho}_\epsilon^\beta - \lambda \operatorname{div} \tilde{u}_\epsilon) \tilde{\rho}_\epsilon.
\end{aligned}
$$

Moreover, integrating by parts twice(justifying on a smooth approximation) reveals

$$
\begin{aligned}
\int_0^T \int_D -2\mu \eta \nabla \tilde{u}_\epsilon : \nabla \mathcal{A}[\eta \tilde{\rho}_\epsilon] \mathrm{d}x \mathrm{d}s = &\int_0^T \int_D -2\mu \eta^2 \operatorname{div} \tilde{u}_\epsilon \rho_\epsilon \mathrm{d}x \mathrm{d}s \\
&+ \int_0^T \int_D \tilde{u}_\epsilon \cdot [\mathcal{A}(\eta \tilde{\rho}_\epsilon) \nabla \eta - \nabla \eta \eta \rho_\epsilon] \mathrm{d}x \mathrm{d}s
\end{aligned}
$$

Also note that $\tilde{\varphi}_\epsilon(0) = \varphi_\delta(0)$. Taking expectation (so that the stochastic integrals vanish) of both Ito product rules above yields two fundamental identities:

$$\begin{aligned}&\mathbb{E}^{\mathbb{P}_\delta}\left[\int_0^T\int_D \eta^2[(2\mu+\lambda)\operatorname{div}\tilde{u}_\epsilon - \rho_\epsilon^\gamma - \delta\tilde{\rho}_\epsilon^\beta]\tilde{\rho}_\epsilon \mathrm{d}x\mathrm{d}s\right] \\ &= I^0 + I_1^{A,\epsilon} + I_2^{A,\epsilon} + I_1^{C,\epsilon} + I_2^{C,\epsilon} + I_3^{C,\epsilon} + I_1^{P,\epsilon} + I_2^{P,\epsilon}.\end{aligned} \tag{3.32}$$

$$\mathbb{E}^{\mathbb{P}_\delta}\left[\int_0^T\int_D \eta^2[(2\mu+\lambda)\operatorname{div} u_\delta - \overline{\rho_\delta^\gamma} - \delta\overline{P(\rho_\delta,\theta_\delta)}]\rho \mathrm{d}x\mathrm{d}s\right] = I^0 + I_1^C + I_2^C + I_3^C + I_1^P + I_2^P. \tag{3.33}$$

Our labeling convention should be interpreted as follows. The terms $I_1^{A,\epsilon}, I_2^{A,\epsilon}$ are "artificial " and will tend to zero as $\epsilon \to 0$, $I_1^{C,\epsilon}, I_2^{C,\epsilon}, I_3^{C,\epsilon}$ are lower order""cutoff" terms arising due to the localization of the estimate, and $I_1^{P,\epsilon}, I_2^{P,\epsilon}$ are the principal terms arising irregardless of the boundary conditions. More precisely, the contribution at the ϵ layer yields

$$I^0 = \mathbb{E}^{\mathbb{P}_\delta}\left[\int_D \eta m_{0,\delta}\cdot\mathcal{A}[\eta\rho_{0,\delta}]\mathrm{d}x\right] \tag{3.34}$$

$$I_1^{A,\epsilon} = \mathbb{E}^{\mathbb{P}_\delta}\left[\int_0^T\int_D \epsilon\eta\tilde{\rho}_\epsilon\tilde{u}_\epsilon\cdot\mathcal{A}[\operatorname{div}(\eta\nabla\tilde{\rho}_\epsilon) - \nabla\eta\cdot\nabla\tilde{\rho}_\epsilon]\mathrm{d}x\mathrm{d}s\right] \tag{3.35}$$

$$I_2^{A,\epsilon} = -\mathbb{E}^{\mathbb{P}_\delta}\left[\int_0^T\int_D \epsilon\nabla\tilde{u}_\epsilon\nabla\tilde{\rho}_\epsilon\cdot\tilde{\varphi}_\epsilon\mathrm{d}x\mathrm{d}s\right] \tag{3.36}$$

$$I_1^{C,\epsilon} = \mathbb{E}^{\mathbb{P}_\delta}\left[\int_0^T\int_D [\tilde{\rho}_\epsilon\tilde{u}_\epsilon\otimes\tilde{u}_\epsilon - 2\mu\nabla\tilde{u}_\epsilon] : \nabla\eta\otimes\mathcal{A}\left[\eta\tilde{\rho}_\epsilon\right]\mathrm{d}x\mathrm{d}s\right] \tag{3.37}$$

$$+\,\mathbb{E}^{\mathbb{P}_\delta}\left[\int_0^T\int_D (-\lambda\operatorname{div}\tilde{u}_\epsilon + \tilde{\rho}_\epsilon^\gamma + \delta\tilde{\rho}_\epsilon^\beta)I] : \nabla\eta\otimes\mathcal{A}\left[\eta\tilde{\rho}_\epsilon\right]\mathrm{d}x\mathrm{d}s\right] \tag{3.38}$$

$$I_2^{C,\epsilon} = \mathbb{E}^{\mathbb{P}_\delta}\left[\int_0^T\int_D \tilde{\rho}_\epsilon\tilde{u}_\epsilon\cdot\mathcal{A}[\nabla\eta\cdot\tilde{\rho}_\epsilon\tilde{u}_\epsilon]\mathrm{d}x\mathrm{d}s\right] \tag{3.39}$$

$$I_3^{C,\epsilon} = \mathbb{E}^{\mathbb{P}_\delta}\left[\int_0^T\int_D \tilde{u}_\epsilon\cdot[\mathcal{A}(\eta\tilde{\rho}_\epsilon)\nabla\eta - \nabla\eta\eta\tilde{\rho}_\epsilon]\mathrm{d}x\mathrm{d}s\right] \tag{3.40}$$

$$I_1^{P,\epsilon} = \mathbb{E}^{\mathbb{P}_\delta}\left[\int_0^T\int_D \eta\left[\tilde{\rho}_\epsilon\tilde{u}_\epsilon\otimes\tilde{u}_\epsilon : \nabla\mathcal{A}[\eta\tilde{\rho}_\epsilon] - \tilde{\rho}_\epsilon\tilde{u}_\epsilon\cdot\mathcal{A}\circ\operatorname{div}(\eta\tilde{\rho}_\epsilon\tilde{u}_\epsilon)\right]\mathrm{d}x\mathrm{d}s\right] \tag{3.41}$$

$$I_2^{P,\epsilon} = -\mathbb{E}^{\mathbb{P}_\delta}\left[\int_D \tilde{\rho}_\epsilon\tilde{u}_\epsilon(T)\cdot\tilde{\varphi}_\epsilon(T)\mathrm{d}x\right] \tag{3.42}$$

In the limit as $\epsilon \to 0$ we expect to obtain the following contribution

$$I_1^C = \mathbb{E}^{\mathbb{P}_\delta}\left[\int_0^T \int_D [\rho_\delta u_\delta \otimes u_\delta - 2\mu \nabla u_\delta] : \nabla \eta \otimes \mathcal{A}\left[\eta \rho_\delta\right] \mathrm{d}x \mathrm{d}s\right] \tag{3.43}$$

$$+ \mathbb{E}^{\mathbb{P}_\delta}\left[\int_0^T \int_D (-\lambda \operatorname{div} u_\delta + \rho_\delta^\gamma + \delta \rho_\delta^\beta) I] : \nabla \eta \otimes \mathcal{A}\left[\eta \rho_\delta\right] \mathrm{d}x \mathrm{d}s\right] \tag{3.44}$$

$$I_2^C = \mathbb{E}^{\mathbb{P}_\delta}\left[\int_0^T \int_D \rho_\delta u_\delta \cdot \mathcal{A}[\nabla \eta \cdot \rho_\delta u_\delta] \mathrm{d}x \mathrm{d}s\right] \tag{3.45}$$

$$I_3^C = \mathbb{E}^{\mathbb{P}_\delta}\left[\int_0^T \int_D u_\delta \cdot [\mathcal{A}(\eta \rho_\delta) \nabla \eta - \nabla \eta \eta \rho_\delta] \mathrm{d}x \mathrm{d}s\right] \tag{3.46}$$

$$I_1^P = \mathbb{E}^{\mathbb{P}_\delta}\left[\int_0^T \int_D \eta \left[\rho_\delta u_\delta \otimes u_\delta : \nabla \mathcal{A}[\eta \rho_\delta] - \rho_\delta u_\delta \cdot \mathcal{A} \circ \operatorname{div}(\eta \rho_\delta u_\delta)\right] \mathrm{d}x \mathrm{d}s\right] \tag{3.47}$$

$$I_2^P = -\mathbb{E}^{\mathbb{P}_\delta}\left[\int_D \rho_\delta u_\delta(T) \cdot \varphi_\delta(T) \mathrm{d}x\right] \tag{3.48}$$

First note that by the uniform bounds combined with an interpolation argument we obtain the estimate

$$I_1^{A,\epsilon} + I_2^{A,\epsilon} \lesssim \sqrt{\epsilon} \mathbb{E}^{\tilde{\mathbb{P}}_\delta}\left[|\sqrt{\epsilon} \nabla \rho_\epsilon|_{L^2_{t,x}}^2\right]^{\frac{1}{2}} \left(\mathbb{E}^{\tilde{\mathbb{P}}_\delta}\left[|\tilde{\rho}_\epsilon \tilde{u}_\epsilon|_{L^2_{t,x}}^2\right]^{\frac{1}{2}} + \mathbb{E}^{\tilde{\mathbb{P}}_\delta}\left[|u_\epsilon|_{L^2_t(W^{1,2}_x)}^2\right]^{\frac{1}{2}}\right) \to 0. \tag{3.49}$$

To treat the remaining integrals, note that by the uniform bounds and the Vitali convergence theorem, it suffices to establish the relevant $\mathbb{P}_\delta$ convergence, so the analysis essentially reduces to the same arguments as in the deterministic framework (by design). We recall them here for the convienience of the reader.

Starting with the first cutoff term, note that for all $q \geq 1$, $\mathcal{A} : L^\beta_x \to L^q_x$ is compact. Hence we may combine the pointwise convergence of the density in $C_t([L^\beta_x]_w)$ with appendix result Theorem 1.7.16 to obtain $\mathcal{A}[\eta\rho_\epsilon] \to \mathcal{A}[\eta\tilde{\rho}_\delta]$ strongly in $L^q_{t,x}$. In view of a similar argument in Lemma 2.3.8, we have $\rho_\epsilon u_\epsilon \otimes u_\epsilon \to \tilde{\rho}_\delta \tilde{u}_\delta \otimes \tilde{u}_\delta$ in $L^2_t(L_x^{\frac{\beta d}{\beta(d-1)+d}})$. Combining these two observations, the $[L^2_t(H^1_x)]_w$ convergence of the velocity and the $[L^1_{t,x}]_w$ convergence of the pressure, we end up with a product of a weakly converging sequence and a strongly converging sequence and conclude $I_1^{C,\epsilon} \to I_1^C$. A similar argument also yields $I_2^{P,\epsilon} \to I_2^P$.

Next note that $\mathcal{A} : L_x^{\frac{2\beta}{\beta+1}} \to L_x^r$ is compact for $\frac{1}{r} > \frac{1}{2} + \frac{1}{2\beta} - \frac{1}{d}$. We may now use the pointwise convergence of the momentum together with Theorem 1.7.16 to conclude that $\mathcal{A}[\nabla\eta \cdot \rho_\epsilon u_\epsilon] \to \mathcal{A}[\nabla\eta \cdot \tilde{\rho}_\delta \tilde{u}_\delta]$ strongly in $L_t^m(L_x^r)$ for all $m \geq 1$ and r satsifying the relation above. We obtain another weak times strong where the exponents match up appropriately since $\beta > d$, allowing us to conclude $I_2^{C,\epsilon} \to I_2^C$. To treat the final cutoff term, simply argue as in the passage to the limit in the flux term of the continuity equation and obtain $I_3^{C,\epsilon} \to I_3^C$.

The treatment of the principle term $I^{P,1}$ is the nontrival part, but we have built most of the work into an appendix result based on the Div Curl lemma. Working componentwise we may write

$$I_1^{P,\epsilon} = \sum_{i,j=1}^{d} \mathbb{E}^{\tilde{\mathbb{P}}_\delta}\left[\int_0^T \int_D u_\epsilon^i \big(\eta\rho_\epsilon \partial_{ij}\Delta^{-1}(\eta\rho_\epsilon u_\epsilon^j) - \eta\rho_\epsilon u_\epsilon^j \partial_{ij}\Delta^{-1}(\eta\rho_\epsilon)\big) dxds\right] \tag{3.50}$$

In view of the pointwise covergences of the density and momentum in the relevant weak spaces, we may appeal to Lemma 1.7.15 with $p = \beta$ and $q = \frac{2\beta}{\beta+1}$, making use of the compact embedding $L^{\frac{2\beta}{3+\beta}} \hookrightarrow W_x^{-1,2}$ for $\beta > \frac{3}{2}d$ in order to conclude that $\mathbb{P}_\delta$ a.s.

$$\big(\eta\rho_\epsilon \partial_{ij}\Delta^{-1}(\eta\rho_\epsilon u_\epsilon^j) - \eta\rho_\epsilon u_\epsilon^j \partial_{ij}\Delta^{-1}(\eta\rho_\epsilon)\big) \to \big(\eta\tilde{\rho}_\delta \partial_{ij}\Delta^{-1}(\eta\tilde{\rho}_\delta \tilde{u}_\delta^j) - \eta\tilde{\rho}_\delta \tilde{u}_\delta^j \partial_{ij}\Delta^{-1}(\eta\tilde{\rho}_\delta)\big)$$

strongly in $L_t^2(W_x^{-1,2})$. Appealing once more to the uniform bounds and Vitali we find $I_1^{P,\epsilon} \to I_1^P$. $\square$

We will now apply Lemma 3.1.7 to deduce the following strong convergence result for the density.

Lemma 3.1.8. *The sequence of densities* $\{\tilde{\rho}_\epsilon\}_{\epsilon>0}$ *converges strongly to* ρ_δ *in the sense that for all* $p \geq 1$ *and* $r < \beta + 1$

$$\lim_{\epsilon\to 0} |\tilde{\rho}_\epsilon - \rho_\delta|_{L^p\left(\tilde{\Omega}_\delta; L^r_{t,x}\right)} = 0. \tag{3.51}$$

Proof. Our plan is to establish

$$\overline{\rho_\delta \log \rho_\delta} = \rho_\delta \log(\rho_\delta) \text{ a.e. in } \Omega_\delta \times [0,T] \times D.$$

Using the renormalized form for ρ_δ, Hardy's inequality, and rearranging, we find for an arbitrary time cutoff ψ:

$$\begin{aligned}
&\int_0^T \int_D \psi_t \mathbb{E}^{\mathbb{P}_\delta}[\rho_\delta \log \rho_\delta - \overline{\rho_\delta \log \rho_\delta}] \mathrm{d}x\mathrm{d}s \\
&\leq \liminf_{\epsilon\to 0} \mathbb{E}^{\mathbb{P}_\delta} \left[\int_0^T \int_K \psi \rho_\delta(\overline{P(\rho_\delta,\theta_\delta)} + \delta\overline{\rho_\delta{}^\beta}) - \psi\rho_\epsilon(P(\rho_\epsilon,\theta_\epsilon) + \delta\rho_\epsilon^\beta)\mathrm{d}x\mathrm{d}s\right] + \overline{R}_K(\psi). \\
&= \liminf_{\epsilon\to 0} \mathbb{E}^{\mathbb{P}_\delta} \left[\int_0^T \int_K \psi\left(\rho_\delta \overline{P(\rho_\delta,\theta_\delta)} - \rho_\epsilon P(\rho_\epsilon,\theta_\epsilon)\right) + \psi\delta\left(\rho_\delta\overline{\rho_\delta{}^\beta} - \rho_\epsilon^{\beta+1}\right)\mathrm{d}x\mathrm{d}s\right] \\
&+ \overline{R}_K(\psi),
\end{aligned}$$

with

$$\overline{R}_K(\psi) = \mathbb{E}^{\mathbb{P}_\delta}\left[\int_0^T \int_{D\setminus K} \psi[\rho_\delta \operatorname{div} u_\delta - \overline{\rho_\delta \operatorname{div} u_\delta}]\mathrm{d}x\mathrm{d}r\right].$$

Now,

$$\overline{P(\rho_\delta,\theta_\delta)} = \overline{p_m(\rho_\delta)} + \overline{\theta_\delta p_\theta(\rho_\delta)},$$

and

$$\overline{\theta_\epsilon p_\theta(\rho_\epsilon)} = \theta_\delta \overline{p_\theta(\rho_\delta)}, \;\; \overline{\theta_\epsilon p_\theta(\rho_\epsilon)\rho_\epsilon} = \theta_\delta \overline{p_\theta(\rho_\epsilon)\rho_\epsilon}. \tag{3.52}$$

In particular, as p_θ is a nondecreasing function of the density we get

$$\overline{\theta_\epsilon p_\theta(\rho_\epsilon)\rho_\epsilon} = \theta_\delta \overline{p_\theta(\rho_\epsilon)} \geq \theta_\delta \overline{p_\theta(\rho_\epsilon)}\rho_\epsilon = \overline{\theta_\epsilon p_\theta(\rho_\epsilon)}\rho_\delta. \tag{3.53}$$

In addition, as the function $z \to \delta z^\beta$ is increasing, we get

$$\liminf_{\epsilon\to 0} \mathbb{E}^{\mathbb{P}_\delta}\left[\int_0^T \int_K \left(\rho_\epsilon^{\beta+1} - \rho_\delta \overline{\rho_\delta{}^\beta}\right) \mathrm{d}x\mathrm{d}s\right] \geq 0. \tag{3.54}$$

Taking into consideration (3.52), (3.53),(3.54) and noting that for any Lebesgue point $s \in [0,T]$ of the function $s \to \mathbb{E}^{\mathbb{P}}\left[\rho_\delta \log \rho_\delta - \overline{\rho_\delta \log \rho_\delta}]dx\right](s)$, we may choose a sequence of test functions that approximate $1_{[0,s]}(t)$, so that their time derivatives approximate the negative of a dirac mass centered at the point s, we conclude that the following estimate holds almost everywhere in time

$$\mathbb{E}^{\mathbb{P}_\delta}\left[\int_D [\overline{\rho_\delta \log \rho_\delta} - \rho_\delta \log \rho_\delta]\mathrm{d}x\right](s) \leq \mathbb{E}^{\mathbb{P}_\delta}\left[\int_0^T \int_{D\setminus K} \psi\left[\rho_\delta \operatorname{div} u_\delta - \overline{\rho_\delta \operatorname{div} u_\delta}\right]\mathrm{d}x\mathrm{d}t\right]. \tag{3.55}$$

Choosing K close enough to D and taking into consideration the convexity of the function $\rho \log \rho$ we conclude that

$$\overline{\rho_\delta \log \rho_\delta} = \rho_\delta \log \rho_\delta \ \text{ a.e.} \quad \text{in } \tilde{\Omega}_\delta \times D.$$

□

We are now in a position to use the strong convergence of the density to deduce the strong convergence of the temperature.

Lemma 3.1.9. *For $\mathbb{P}_\epsilon$ almost all $\omega \in \Omega_\delta$ and $q < 3$, $\tilde{\theta}_\epsilon(\omega) \to \theta_\delta(\omega)$ in $L^q_{t,x}$. Moreover, ρ_δ, u_δ and θ_δ satisfy the renormalized temperature inequality,* (3.4).

Proof. The proof of the strong convergence claim follows the argument in Lemma 2.3.7. The only additional detail is to explain how to pass from the renormalized form to an inequality in $\mathcal{D}'_{t,x}$ directly on $\partial_t\big[(\tilde{\rho}_\epsilon+\delta)\tilde{\theta}_\epsilon\big]$. Towards this end, for each $m \in \mathbb{N}$, introduce the renormalization $h_m(z) = (1+z)^{-\frac{1}{m}}$. It is straightforward to verify $h_m \in \mathcal{R}$. Using the recovery maps, we may transfer the renormalized temperature inequality to the new probability space to find for each $\epsilon > 0$ and $m \in \mathbb{N}$, the $\mathbb{P}_\delta$ a.s. inequality

$$
\begin{aligned}
&\int_0^T \int_D (\tilde{\rho}_\epsilon+\delta)H_m(\tilde{\theta}_\epsilon)\partial_t\varphi + \tilde{\rho}_\epsilon H_m(\tilde{\theta}_\epsilon)\tilde{u}_\epsilon \cdot \nabla\varphi + \mathcal{K}_m(\tilde{\theta}_\epsilon)\Delta\varphi - \delta\tilde{\theta}_\epsilon^3 h_m(\tilde{\theta}_\epsilon)\varphi \mathrm{d}x\mathrm{d}t \\
&\leq \int_0^T \int_D h_m(\tilde{\theta}_\epsilon)[\tilde{\theta}_\epsilon p_\theta(\tilde{\rho}_\epsilon)\operatorname{div}\tilde{u}_\epsilon - \mathcal{S}(\tilde{u}_\epsilon) : \nabla\tilde{u}_\epsilon]\varphi \mathrm{d}x\mathrm{d}t \\
&\epsilon \int_0^T \int_D \nabla\tilde{\rho}_\epsilon \cdot \nabla\big[(H_m(\tilde{\theta}_\epsilon) - \tilde{\theta}_\epsilon h_m(\tilde{\theta}_\epsilon))\varphi\big]\mathrm{d}x\mathrm{d}s \\
&+ \int_0^T \int_D h_m'(\tilde{\theta}_\epsilon)\kappa(\tilde{\theta}_\epsilon)|\nabla\tilde{\theta}_\epsilon|^2 dxdt - \int_D \tilde{\rho}_\epsilon(0+)H_m(\theta_\epsilon)\varphi(0)\mathrm{d}x,
\end{aligned}
\tag{3.56}
$$

where $H_m(\theta) = \int_0^\theta h_m(z)dz$ and $\mathcal{K}_m(\theta) = \int_0^\theta \kappa(z)h_m(z)dz$. Therefore, sending $m \to \infty$ yields the inequality

$$
\begin{aligned}
&\int_0^T \int_D (\tilde{\rho}_\epsilon+\delta)\tilde{\theta}_\epsilon\partial_t\varphi + \tilde{\rho}_\epsilon\tilde{\theta}_\epsilon\tilde{u}_\epsilon \cdot \nabla\varphi + \mathcal{K}(\tilde{\theta}_\epsilon)\Delta\varphi - \delta\tilde{\theta}_\epsilon^3\varphi \mathrm{d}x\mathrm{d}t \\
&\leq \int_0^T \int_D [\tilde{\theta}_\epsilon p_\theta(\tilde{\rho}_\epsilon)\operatorname{div}\tilde{u}_\epsilon - \mathcal{S}(\tilde{u}_\epsilon) : \nabla\tilde{u}_\epsilon]\varphi \mathrm{d}x\mathrm{d}t \\
&- \int_D \tilde{\rho}_\epsilon(0+)\theta_\epsilon\varphi(0)\mathrm{d}x.
\end{aligned}
\tag{3.57}
$$

Finally, passing to the limit in the renormalized temperature equation follows the same arguments as in the proof of Lemma 2.3.7. The only additional detail is to check that the term arising from the vanishing viscosity regularization tends to zero as $\epsilon \to 0$. Given an $H \in \mathcal{R}$, this term contributes

$$\int_0^T \int_D \epsilon \nabla \tilde{\rho}_\epsilon \cdot \nabla(\beta(\tilde{\theta}_\epsilon)\varphi) \mathrm{d}x \mathrm{d}s, \tag{3.58}$$

where $\beta(\theta) = \theta H'(\theta) - H(\theta)$. As we mentioned in the proof of Lemma 2.3.7, β' is globally bounded on $\mathbb{R}$. This yields the inequality

$$\mathbb{E}^{\mathbb{P}_\delta}\left[\epsilon \|\nabla \tilde{\rho}_\epsilon \cdot \nabla\big(\beta(\tilde{\theta}_\epsilon)\varphi\big)\|_{L^1_{t,x}}\right] \leq \epsilon^{\frac{1}{2}}\left[\mathbb{E}^{\mathbb{P}_\delta}\|\nabla(\epsilon^{\frac{1}{2}}\tilde{\rho}_\epsilon)\|^2_{L^2_{t,x}}\right]^{\frac{1}{2}}\left[\mathbb{E}^{\mathbb{P}_\delta}\|\tilde{\theta}_\epsilon\|^2_{L^2_t(H^1_x)}\right]^{\frac{1}{2}}. \tag{3.59}$$

In view of the uniform bounds guaranteed by Proposition 3.1.3, this term tends to zero. □

3.1.4 Concluding the proof

Proof of Theorem 3.1.1. For each $\delta > 0$ we apply Proposition 3.1.3 to construct a candidate δ layer approximation $(\rho_\delta, u_\delta, \theta_\delta)$. Combing the Lemma 3.1.5 and Lemma 3.1.6 with the strong convergence of the density we are able to identify

$$\overline{P(\rho_\delta, \theta_\delta)} + \delta \overline{\rho_\delta{}^\beta} = P(\rho_\delta, \theta_\delta) + \delta \rho_\delta^\beta$$

and complete the identification procedure. The uniform bounds can be argued as in the proof of Theorem 2.3.1. □

3.2 $\delta \to 0$

3.2.1 Further estimates

As $\delta \to 0$, the bounds based on the total energy are stable. However, the bounds obtained from the renormalized form of the temperature equation degenerate and need to be re-derived using the δ layer temperature inequality. In particular, the following estimates can be proved.

Lemma 3.2.1. *For all $\sigma \in (0,1)$ and $p \in [1,\infty)$, there exists a constant $C_{\sigma,p} > 0$ such that*

$$\sup_{\delta>0} \mathbb{E}^{\mathbb{P}_\delta}\left[\|\theta_\delta\|_{L_t^2(H_x^1)}^{2p} + \|\nabla \log(\theta_\delta)\|_{L_{t,x}^2}^{2p} + \|u_\delta\|_{L_t^2(H_x^1)}^{2p}\right] \leq C_p.$$
$$\sup_{\delta>0} \mathbb{E}^{\mathbb{P}_\delta}\left[\|\theta_\delta^{(3-\sigma)/2}\|_{L_t^2(H_x^1)}\right] \leq C_p.$$

Proof. The proof of this Lemma is essentially the same as in the formal estimates section, so we will only sketch the proof. The main difference is that we must use the δ layer temperature inequality (3.4), which a priori holds only in a weak, renormalized form.

Begin by using the renormalization H_m defined by:

$$H_m(\theta) = \int_1^\theta \frac{1}{1+mz}\mathrm{d}z. \tag{3.60}$$

Derive the analogue of (1.12) and multiply by a factor of $m \in \mathbb{N}$ to obtain

$$\mathbb{E}^{\mathbb{P}_\delta}\Big(\int_0^T \int_D \frac{\kappa(\theta_\delta)|\nabla\theta_\delta|^2}{(\frac{1}{m}+\theta_\delta)^2}\mathrm{d}x\mathrm{d}s\Big)^p \lesssim 1 + \mathbb{E}\|p_\theta(\rho_\delta)\operatorname{div} u_\delta\|_{L_{t,x}^1}^p + \mathbb{E}\|\delta^{\frac{1}{2}}\theta_\delta\|_{L_{t,x}^2}^{2p}. \tag{3.61}$$

Applying Hölder yields

$$\mathbb{E}\|\delta^{\frac{1}{2}}\theta_\delta\|_{L_{t,x}^2}^{2p} \leq \delta^{\frac{1}{3}}\mathbb{E}\|\delta^{\frac{1}{3}}\theta_\delta\|_{L_{t,x}^3}^{3p} \lesssim 1.$$

Using the renormalized continuity equation and sending $m \to \infty$ yields

$$\mathbb{E}\|\nabla\theta_\delta\|_{L_{t,x}^2}^{2p} \leq C_p. \tag{3.62}$$

Using the modified Poincare inequality from the formal estimates section gives:

$$\mathbb{E}\|\theta_\delta\|_{L_t^2(H_x^1)}^{2p} \le C_p. \tag{3.63}$$

The remaining estimates are obtained in an analogous way using the renormalizations $\tilde{H}_m$ and $\hat{H}_m$ defined by

$$\tilde{H}_m(\theta) = \int_1^\theta \frac{\mathrm{d}z}{(1+z)^{\frac{1}{m}}}.$$
$$\hat{H}_m(\theta) = \int_1^\theta \frac{\mathrm{d}z}{\frac{1}{m}+z}.$$

A limited amount of additional details are provided in [11]. □

Observe that the bounds above do not control the $L^p(\Omega; L^3_{t,x})$ norm of the $\{\theta_\delta\}_{\delta>0}$, which is essential for passing to the limit in the temperature inequality. Instead, this estimate must be derived directly from the weak form of the δ layer renormalized temperature inequality. For this purpose, we adapt the approach of Feirisel [11] to the stochastic case. Namely, we begin by proving estimates away from vaccum (Lemma 3.2.2), then we bootstrap these bounds and use a random test function approach to get estimates in the low density regions (Proposition 3.2.3)).

Lemma 3.2.2. *For all $p \in [1,\infty)$ there exists a constant $C_p > 0$ such that for all $\nu < 1$*

$$\sup_{\delta>0} \mathbb{E}^{\mathbb{P}_\delta}\left[\|\theta_\delta 1_{\{\rho_\delta \ge \nu\}}\|_{L^3_{t,x}}^p\right] \le \nu^{-p} C_p. \tag{3.64}$$

Proof. For each $\sigma \in (0,1)$ let $r(\sigma) = \frac{(3-\sigma)d}{d-2}$. By interpolation, there exists $\alpha(\sigma) \in (0,1)$ and $q(\sigma) > 1$ such that for all $f \in L_t^\infty(L_x^1) \cap L_t^{3-\sigma}(L_x^{r(\sigma)})$, the following interpolation inequality holds:

$$\|f\|_{L^{q(\sigma)}_{t,x}} \le \|f\|_{L_t^\infty(L_x^1)}^{\alpha(\sigma)} \|f\|_{L_t^{3-\sigma}(L_x^{r(\sigma)})}^{1-\alpha(\sigma)}.$$

Choosing σ sufficiently small, we can guarantee $q(\sigma) > 3$, leading to the following chain of inequalities:

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}_\delta}\left[\|\theta_\delta 1_{\{\rho_\delta\geq\nu\}}\|^p_{L^{q(\sigma)}_{t,x}}\right] \leq \mathbb{E}^{\mathbb{P}_\delta}\left[\|\theta_\delta 1_{\{\rho_\delta\geq\nu\}}\|^{\alpha p}_{L^\infty_t(L^1_x)}\|\theta_\delta 1_{\{\rho_\delta\geq\nu\}}\|^{(1-\alpha)p}_{L^{3-\sigma}_t(L^{r(\sigma)}_x)}\right] \\
&\leq \nu^{-\alpha p}\mathbb{E}^{\mathbb{P}_\delta}\left[\|\rho_\delta\theta_\delta\|^{\alpha p}_{L^\infty_t(L^1_x)}\|\theta_\delta\|^{(1-\alpha)p}_{L^{3-\sigma}_t(L^{r(\sigma)}_x)}\right] \\
&\leq \nu^{-\alpha p}\mathbb{E}^{\mathbb{P}_\delta}\left[\|\rho_\delta\theta_\delta\|^p_{L^\infty_t(L^1_x)}\right]^\alpha \mathbb{E}^{\mathbb{P}_\delta}\left[\|\theta_\delta\|^p_{L^{3-\sigma}_t(L^{r(\sigma)}_x)}\right]^{1-\alpha}.
\end{aligned}$$

By the uniform $L^p(\Omega_\delta; H^1)$ bounds in $\{\theta_\delta^{\frac{3-\sigma}{2}}\}_{\delta>0}$ from Lemma 3.2.1 and the Sobolev embedding theorem, this concludes the proof. $\square$

Now we improve on our initial estimate by adapting a method from [11] to obtain estimates on the temperature near the vaccum.

Proposition 3.2.3. *For all $p \in [1,\infty)$ there exists a constant $C_p > 0$ such that*

$$\sup_{\delta>0}\mathbb{E}^{\mathbb{P}_\delta}\left[\|\theta_\delta\|^p_{L^3_{t,x}}\right] \leq C_p. \tag{3.65}$$

Proof. For each level $\nu > 0$, we introduce a sequence of random variables $\{X^\nu_\delta\}_{\delta>0}$, where $X^\nu_\delta : \Omega_\delta \to \mathbb{R}$ is defined by:

$$X^\nu_\delta(\omega) = \inf_{t\in[0,T]}\lambda\{x \in D \mid \rho_\delta(t,x,\omega) \geq \nu\}.$$

Here, λ denotes the Lebesgue measure on $\mathbb{R}^d$. We will begin by proving there exists a $\nu_0 > 0$ such that for all $\nu \leq \nu_0$, $X^\nu_\delta > 0$ a.s. with respect to $\mathbb{P}_\delta$. Moreover, for all $p \geq 1$, there exists a constant $C_{p,\nu} > 0$ such that the following uniform bound holds:

$$\sup_{\delta>0}\mathbb{E}^{\mathbb{P}_\delta}\left[|X^\nu_\delta|^{-p}\right] \leq C_{p,\nu}. \tag{3.66}$$

To establish the claim above, begin by applying Holder in space, then minimizing in time to deduce the following $\mathbb{P}_\delta$ a.s. inequality:

$$\inf_{t\in[0,T]} \|\rho_\delta(t)1_{\{\rho_\delta(t)\geq\nu\}}\|_{L^1_x} \leq \left[X^\nu_\delta\right]^{1-\frac{1}{\gamma}} \|\rho_\delta\|_{L^\infty_t(L^\gamma_x)}. \tag{3.67}$$

On the other hand, conservation of mass implies that $\mathbb{P}_\delta$ almost surely,

$$\inf_{t\in[0,T]} \|\rho_\delta(t)1_{\{\rho_\delta(t)\geq\nu\}}\|_{L^1_x} \geq \|\rho^0_\delta\|_{L^1_x} - \nu\lambda(D). \tag{3.68}$$

Combining (3.67) and (3.68) yields the following lower bound for X^ν_δ:

$$X^\nu_\delta \geq \|\rho_\delta\|^{-\frac{\gamma}{\gamma-1}}_{L^\infty_t(L^\gamma_x)}[\|\rho^0_\delta\|_{L^1_x} - \nu\lambda(D)]^{\frac{\gamma}{\gamma-1}}.$$

By the strong convergence of the initial densities, Hypothesis 2.1.4, we can choose a small enough ν_0 to ensure $X^\nu_\delta > 0$ a.s. with respect to $\mathbb{P}_\delta$. Hence, inverting the lower bound above and applying the uniform $L^p(\Omega_\delta; L^\infty_t(L^\gamma_x))$ bounds on $\{\rho_\delta\}_{\delta>0}$ yields (3.66).

The next step of the proof is to construct a suitable random test function. For each $\nu < \nu_0$, let $B_\nu : \mathbb{R}_+ \to [-1, 0]$ be a smooth function such that $B_\nu(z) = 0$ for $z \leq \nu$ and $B_\nu(z) = -1$ for $z \geq 2\nu$. For each $\delta > 0$, construct $\eta^\delta_\nu : [0,T]\times D\times\Omega_\delta \to \mathbb{R}$ such that for each $(t,\omega) \in [0,T]\times\Omega_\delta$, the function $x \to \eta^\delta_\nu(t,x,\omega)$ solves the following Neumann problem:

$$\begin{cases} \Delta\eta = B_\nu(\rho_\delta(t,\omega)) - \lambda^{-1}(D)\int_D B_\nu(\rho_\delta(t,\omega,x))\mathrm{d}x & \text{in} \quad D \\ \frac{\partial\eta}{\partial n} = 0 & \text{on} \quad \partial D \\ \int_D \eta \, dx = 0 \end{cases} \tag{3.69}$$

Let $\psi : [0,T] \to [0,1]$ be smooth and compactly supported. For each $\delta > 0$ and $\nu > \nu_0$, we may now define our random test function $\varphi_\nu^\delta : [0,T] \times D \times \Omega_\delta \to \mathbb{R}$ by setting

$$\varphi_\nu^\delta(t,x,\omega) = \psi(t)\big[\eta_\nu^\delta(t,x,\omega) - \inf_{(t,x)\in[0,T]\times D} \eta_\nu^\delta(t,x,\omega)\big]. \tag{3.70}$$

Applying directly the argument in [11], we find that for each $\omega \in \Omega_\delta$, the function $(t,x) \to \varphi_\delta^\eta(t,x,\omega)$ is a valid test function in (3.4) for the renormalized form satisfied by $(\rho_\delta, \theta_\delta, u_\delta)$ in the state $\omega \in \Omega_\delta$. Taking a sequence of renormalizations $\{H_k\}_{k=1}^\infty$ which converge to the identity, then taking $L^p(\Omega_\delta)$ norms we find that:

$$\mathbb{E}^{\mathbb{P}_\delta}\Big|\int_0^T \int_D \psi\theta_\delta^3 1_{\{\rho_\delta \geq \nu\}} \mathrm{d}x\mathrm{d}t\Big|^{p/2} \lesssim \big[\mathbb{E}^{\mathbb{P}_\delta}|X_\delta^\nu|^{-p}\big]^{\frac{1}{2}}\Big(\sum_{k=1}^{6} I_k^p\Big)^{\frac{1}{2}}. \tag{3.71}$$

The terms I_1^p to I_6^p are given by:

$$I_1^p = \mathbb{E}^{\mathbb{P}_\delta}\|\theta_\delta 1_{\{\rho_\delta \geq \nu\}}\|_{L^3_{t,x}}^{3p}$$

$$I_2^p = \mathbb{E}^{\mathbb{P}_\delta}\|\rho_\delta\theta_\delta u_\delta \nabla\varphi_\nu^\delta\|_{L^1_{t,x}}^p$$

$$I_3^p = \mathbb{E}^{\mathbb{P}_\delta}\|\delta\theta_\delta^3\varphi_\nu^\delta\|_{L^1_{t,x}}^p$$

$$I_4^p = \mathbb{E}^{\mathbb{P}_\delta}\|\theta_\delta p_\theta(\rho_\delta)\operatorname{div} u_\delta\varphi_\nu^\delta\|_{L^1_{t,x}}^p$$

$$I_5^p = \mathbb{E}^{\mathbb{P}_\delta}\|(\rho_{0,\delta} + \delta)\theta_{0,\delta}\varphi_\nu^\delta\|_{L^1_{t,x}}^p.$$

$$I_6^p = \mathbb{E}^{\mathbb{P}_\delta}\|(\rho_\delta + \delta)\theta_\delta\partial_t\varphi_\nu^\delta\|_{L^1_{t,x}}^p$$

Begin by applying Lemma 3.2.2 to control I_1^p. The remaining terms can now be estimated as follows:

$$I_2^p \leq \mathbb{E}^{\mathbb{P}_\delta}\left[\|\rho_\delta\|^{4p}_{L_t^\infty(L_x^\gamma)}\right]^{\frac{1}{4}}\mathbb{E}^{\mathbb{P}_\delta}\left[\|\theta_\delta\|^{4p}_{L_t^2(L_x^{\frac{2d}{d-2}})}\right]^{\frac{1}{4}}\mathbb{E}^{\mathbb{P}_\delta}\left[\|u_\delta\|^{4p}_{L_t^2(H_x^1)}\right]^{\frac{1}{4}}\mathbb{E}^{\mathbb{P}_\delta}\left[\|\nabla_x\eta_\nu^\delta\|^{4p}_{L_{t,x}^\infty}\right]^{\frac{1}{4}}.$$

$$I_3^p + I_5^p \leq \left(1+\mathbb{E}^{\mathbb{P}_\delta}\|\delta^{\frac{1}{3}}\theta_\delta\|^{3p}_{L_{t,x}^3}\right)\|\varphi_\nu^\delta\|^p_{L^\infty([0,T]\times D\times\Omega_\delta)}.$$

$$I_4^p \leq \left[\mathbb{E}^{\mathbb{P}_\delta}\|\theta_\delta\|^{3p}_{L_t^2(L_x^{\frac{2d}{d-2}})}\right]^{\frac{1}{3}}\left[\mathbb{E}^{\mathbb{P}_\delta}\|\rho_\delta\|^{\frac{3p\gamma}{d}}_{L_t^\infty(L_x^\gamma)}\right]^{\frac{1}{3}}\left[\mathbb{E}^{\mathbb{P}_\delta}\|u_\delta\|^{3p}_{L_t^2(H_x^1)}\right]^{\frac{1}{3}}\|\varphi_\nu^\delta\|^p_{L^\infty([0,T]\times D\times\Omega_\delta)}.$$

Combining classical results on the Neumann problem with the uniform bounds in Lemma 3.2.1, each of these terms $I_1^p - I_5^p$ are controlled uniformly in $\delta > 0$. Finally, use the renormalized form of the continuity equation as in [11] to estimate I_6^p. □

In addition, the following integrability gains can be proved for the $\{\rho_\delta\}_{\delta>0}$. The approach is entirely analogous to the $\epsilon \to 0$ section, so we omit the proof.

Proposition 3.2.4. *For all $p \geq 1$ and $\kappa < \min(\frac{2\gamma}{d} - 1, \frac{\gamma}{2})$, there exists a constant $C_{p,\kappa}$ such that*

$$\sup_{\delta>0}\mathbb{E}^{\mathbb{P}_\delta}\left[\Big|\int_0^T\int_D \rho_\delta^\kappa P(\rho_\delta,\theta_\delta)\mathrm{d}x\mathrm{d}t\Big|^p\right] \leq C_{p,\kappa}. \tag{3.72}$$

3.2.2 $\delta \to 0$ Compactness step

In preparation for the compactness analysis of the temperature equation, we define a sequence of temperature renormalizations as in [11].

$$\mathcal{K}_m(\theta) = \int_0^\theta k(z)h_m(z)\,\mathrm{d}z, \quad h_m(z) = \frac{1}{(1+z)^{\frac{1}{m}}}.$$

We also introduce a sequence of cutoffs T_k defined by

$$T_k(z) = kT\left(\frac{z}{k}\right), \; z \in \mathbb{R}, \; k \in \mathbb{N},$$

with T being a smooth concave function on $\mathbb{R}$ such that

$$T(z) = \begin{cases} z \text{ for } z \leq 1 \\ \\ 2 \text{ for } z \geq 3. \end{cases}$$

Finally, we introduce a sequence of $\{L_k\}_{k=1}^{\infty}$ by setting Introduce the function L_k as

$$L_k(z) = \begin{cases} z \ln z, & \text{for } 0 \leq z < k, \\ \\ z \ln k + z \int_k^z \frac{T_k(z)}{Z^2} \mathrm{d}s, & \text{for } z \geq k, . \end{cases}$$

The main compactness result for the $\delta \to 0$ step is the following:

Proposition 3.2.5. *There exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a collection of independent Brownian motions $\{\beta_k\}_{k=1}^{\infty}$, a sequence of measurable maps $\{T_\delta\}_{\delta>0}$*

$$T_\delta : (\Omega, \mathcal{F}, \mathbb{P}) \to (\Omega_\delta, \mathcal{F}_\delta, \mathbb{P}_\delta),$$

and limit points

$$\Big(\rho, u, \overline{\theta}, \overline{\sqrt{\rho} u}, \overline{p_m(\rho)}, \overline{\theta\, p_\theta(\rho)}, \{\overline{L_k(\rho)}\}_{k=1}^{\infty}, \{\overline{T_k(\rho) \operatorname{div} u}\}_{k=1}^{\infty}, \Big)$$

$$\Big(\{\overline{(\rho T_k'(\rho) - T_k(\rho)) \operatorname{div} u}\}_{k=1}^{\infty}, \{\overline{T_k(\rho)\theta p_\theta(\rho)}\}_{k=1}^{\infty}, \{\overline{\mathcal{K}_m(\theta)}\}_{m=1}^{\infty}\Big)$$

such that the following hold:

1. *The measure $\mathbb{P}_\delta$ satisfies $(T_\delta)_{\#}\mathbb{P} = \mathbb{P}_\delta$.*

2. *The new sequence $\{(\tilde{\rho}_\delta, \tilde{u}_\delta, \tilde{\theta}_\delta)\}_{\delta>0}$ defined by $\Big(\tilde{\rho}_\delta, \tilde{u}_\delta, \tilde{\theta}_\delta\Big) = (\rho_\delta, u_\delta, \theta_\delta) \circ T_\delta$ constitutes a δ layer approximation relative to the stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, (\widetilde{\mathcal{F}}_\delta^t)_{t=0}^T, \widetilde{W}_\delta)$, where $\widetilde{W}_\delta := W_\delta \circ \widetilde{T}_\delta$ and $\widetilde{\mathcal{F}}_\delta^t = T_\delta^{-1} \circ \mathcal{F}_\delta^t$.*

3. *The uniform bounds in Lemma 3.2.1 and Proposition 3.2.3 hold with $\rho_\delta, u_\delta, \theta_\delta$ replaced by $\tilde{\rho}_\delta, \tilde{u}_\delta, \tilde{\theta}_\delta$ and $\mathbb{P}_\delta$ replaced by $\mathbb{P}$.*

4. The following convergences hold pointwise on Ω:

$$\tilde{\rho}_\delta \to \rho \quad in \quad C_t([L_x^\gamma]_w) \tag{3.73}$$

$$\tilde{u}_\delta \to u \quad in \quad [L_t^2(H_{0,x}^1)]_w \tag{3.74}$$

$$\tilde{\rho}_\delta \tilde{u}_\delta \to \rho u \quad in \quad C_t([L_x^{\frac{2\gamma}{\gamma+1}}]_w) \tag{3.75}$$

$$p_m(\tilde{\rho}_\delta) \to \overline{p_m(\rho)} \quad in \quad [L_{t,x}^{q_1}]_w \tag{3.76}$$

$$\tilde{\theta}_\delta p_\theta(\tilde{\rho}_\delta) \to \overline{\theta p_\theta(\rho)} \quad in \quad [L_{t,x}^{q_2}]_w \cap L_t^2(H_x^{-1}) \tag{3.77}$$

$$T_k(\tilde{\theta}_\delta)\tilde{\theta}_\delta p_\theta(\tilde{\rho}_\delta) \to \overline{\theta p_\theta(\rho) T_k(\rho)} \quad in \quad [L_{t,x}^{q_2}]_w \cap L_t^2(H_x^{-1}) \tag{3.78}$$

$$\tilde{\theta}_\delta \to \theta \quad in \quad [L_t^2(H_x^1) \cap L_{t,x}^3]_w \tag{3.79}$$

$$\tilde{\rho}_\delta \tilde{\theta}_\delta \to \rho\theta \quad in \quad [L_t^\infty(M_x)]_{w*} \tag{3.80}$$

$$\widetilde{W}_\delta \to W \quad in \quad [C_t]^\infty \tag{3.81}$$

In addition, we have

$$T_k(\tilde{\rho}_\delta) \to \overline{T_k(\rho)} \quad in \quad [C_t]([L_x^{\frac{4\gamma d}{2\gamma-d}}]_w) \tag{3.82}$$

$$L_k(\tilde{\rho}_\delta) \to \overline{L_k(\rho)} \quad in \quad [C_t]([L_x^q]_w) \tag{3.83}$$

$$(\tilde{\rho}_\delta T_k'(\tilde{\rho}_\delta) - T_k(\tilde{\rho}_\delta)) \operatorname{div} \tilde{u}_\delta \to \overline{(\rho T_k'(\rho) - T_k(\rho)) \operatorname{div} u} \quad in \quad L_t^\infty(L_x^2) \tag{3.84}$$

$$\mathcal{K}_m(\tilde{\theta}_\delta) \to \overline{\mathcal{K}_m(\theta)} \quad in \quad [L_{t,x}^1]_w \tag{3.85}$$

5. *The following additional convergences hold*

$$\sqrt{\tilde{\rho}_\delta}\tilde{u}_\delta \to \overline{\sqrt{\rho}u} \quad in \quad L^p_{w^*}\big(\Omega_\delta; L^\infty_t(L^2_x)\big) \tag{3.86}$$

$$\tilde{\rho}_\delta \to \rho \quad in \quad L^p_{w^*}(\Omega_\delta; L^\infty_t(L^\gamma_x)) \cap L^p_w\big(\Omega_\delta; L^2_t(W^{1,2}_x)\big) \tag{3.87}$$

$$T_k(\tilde{\rho}_\delta)\operatorname{div}\tilde{u}_\delta \to \overline{T_k(\rho)\operatorname{div}u} \quad in \quad L^p_w(\Omega_\delta; L^2_t(L_x^{\frac{2\gamma}{\gamma+2}})) \tag{3.88}$$

$$\tilde{\rho}_\delta \log\tilde{\rho}_\delta \to \overline{\rho\log\rho} \quad in \quad L^p_{w^*}\big(\Omega_\delta; L^\infty_t(L^2_x)\big) \tag{3.89}$$

The proof follows a similar line of argument as in the previous sections. The only key difference is to obtain a tightness result for the renormalizations of the temperature equation. Towards this end, we define for each $\delta > 0$ the $[L^1_{t,x}]^\infty$ valued random variable $Z_\delta = \{\mathcal{K}_m(\theta_\delta)\}_{m=1}^\infty$. Using the $L^p(\Omega; L^3_{t,x})$ estimates from Proposition 3.2.3, we will establish the following:

Lemma 3.2.6. *The sequence of induced measures $\{\mathbb{P}_\delta \circ Z_\delta^{-1}\}_{\delta>0}$ is tight on $\big[(L^1_{t,x})_w\big]^\infty$.*

Proof. By Proposition 3.2.3, there exists a constant C such that

$$\sup_{\delta>0}\mathbb{E}^{\mathbb{P}_\delta}\|\theta_\delta\|^3_{L^3_{t,x}} \leq C. \tag{3.90}$$

Moreover, by Hypothesis 1.1.3, there exists another constant D such that for all $m \geq 1$ and $\theta \geq 0$,

$$\mathcal{K}_m(\theta) \leq D\theta^{3-\frac{1}{m}}. \tag{3.91}$$

Fix an $\epsilon > 0$. For each $m \geq 1$ define the set $E^\epsilon_m \subset L^1_{t,x}$ by

$$E^\epsilon_m = \{f \in L^1_{t,x} \mid \ \|f\|_{L^{\frac{3m}{3m-1}}_{t,x}} \leq \epsilon^{-1}2^m\}.$$

Since every sequence in E_m^ϵ is uniformly bounded and uniformly integrable in $L^1_{t,x}$, we may conclude that E_m^ϵ a compact set in $(L^1_{t,x})_w$. In addition, define

$$E^\epsilon = \prod_{m=1}^\infty E_m^\epsilon.$$

By Tychnoff's theorem, E^ϵ a compact set in $\left[(L^1_{t,x})_w\right]^\infty$. Applying Chebyshev, we find that

$$\mathbb{P}_\delta\big(Z_\delta \notin E_\epsilon\big) \leq \sum_{m=1}^\infty \mathbb{P}_\delta\big(\mathcal{K}_m(\theta_\delta) \notin E_m^\epsilon\big) \leq \epsilon \sum_{m=1}^\infty 2^{-m} \mathbb{E}^{\mathbb{P}_\delta} \|\mathcal{K}_m(\theta_\delta)\|_{L_{t,x}^{\frac{3m}{3m-1}}}.$$

Hence, applying inequalities (3.90) and (3.91) yields

$$\mathbb{P}_\delta\big(Z_\delta \notin E_\epsilon\big) \leq D\epsilon \sum_{m=1}^\infty 2^{-m} \mathbb{E}^{\mathbb{P}_\delta} \|\theta_\delta\|_{L^3_{t,x}}^{3-\frac{1}{m}} \leq CD\epsilon.$$

Since $\epsilon > 0$ was arbitary and C, D were fixed in advance, this completes the proof.

□

3.2.3 $\delta \to 0$ Preliminary limit passage

Next we define a filtration $(\mathcal{F}_t)_{t=0}^T$ via $\mathcal{F}_t = \sigma(r_t X)$ where $X = \big(\rho, \rho u, W, u, \rho\overline{\theta}\big)$ and $r_t : E_T \to E_t$

$$E_s = C\big([0,s];[L^\gamma]_w\big) \times C\big([0,s];[L^{\frac{2\gamma}{\gamma+1}}]_w \times \mathbb{R}^\infty\big)$$
$$\times L^2\big([0,s];H_0^1\big) \times L^2\big([0,s];L^q(D)\big),$$

where $\frac{1}{q} = \frac{1}{\gamma} + \frac{1}{2} - \frac{1}{d}$.

Lemma 3.2.7. *The pair (ρ, u) satisfies the continuity equation 1.6 of Definition 1.2.1.*

Proof. In view of the strong convergence of the initial density (the initial data are assumed to be deterministic) this immediately follows from the pointwise convergences in Proposition 3.2.5. $\square$

At this point, we cannot make the same preliminary passage to the limit in the momentum equation as in the $\epsilon \to 0$ step. Recall that this preliminary passage to the limit was crucial for establishing the averaged weak continuity of the effective viscous pressure. Instead, one proves a partial result in this direction(Lemma 3.2.8), which essentially amounts to a preliminary passage to the limit in each of the terms besides the stochastic integrals. Indeed, as $\delta \to 0$, passing to the limit in the stochastic integrals is even more difficult than passing to the limit in the pressure. This lemma turns out to be enough to prove the strong convergence of the density and temperature. We are then able to return to the task of passing to the limit in the stochastic integrals at the very end of the proof.

Lemma 3.2.8. *For all $\phi \in C_c^\infty(D)$, the process $(M_t(\phi))_{t=0}^T$ defined by*

$$M_t(\phi) = \int_D \rho u(t)\cdot\phi \mathrm{d}x - \int_D m_0\cdot\phi - \int_0^t \int_D [\rho u\otimes u - \mathcal{S}(u)] : \nabla\phi + [\overline{p_m(\rho)} + \overline{\theta p_m(\rho)}] \operatorname{div}\phi \mathrm{d}x\mathrm{d}s.$$

is a continuous, $(\mathcal{F}^t)_{t=0}^T$ martingale satisfying for all $p \geq 1$

$$\mathbb{E}\left[\sup_{t\in[0,T]} |M_t(\phi)|^p\right] < \infty. \tag{3.92}$$

Proof. Introduce the continuous process $\{M_t^\delta(\phi)\}_{t=0}^T$ defined by

$$\begin{aligned}\widetilde{M_t^\delta}(\phi) &= \int_D \tilde\rho_\delta \tilde u_\delta(t)\cdot\phi \mathrm{d}x - \int_D m_{0,\delta}\cdot\phi \mathrm{d}x \\ &\quad - \int_0^t \int_D [\tilde\rho_\delta\tilde u_\delta\otimes\tilde u_\delta - 2\mu\nabla\tilde u_\delta - \lambda \operatorname{div}\tilde u_\delta I] : \nabla\phi + P(\tilde\rho_\delta, \tilde\theta_\delta)\operatorname{div}\phi \mathrm{d}x\mathrm{d}s.\end{aligned}$$

We may use our compactness step along with Hypothesis 1.1.1 to establish for all $t \in [0,T]$ the convergence $\widetilde{M}_t^\delta(\phi) \to M_t(\phi)$ almost surely with respect to $\mathbb{P}$. Indeed, the only additional steps (other than what was required at the ϵ layer) are noting

$$\mathbb{E}\left[|\int_0^T \int_D \delta \tilde{\rho}_\delta^\beta \mathrm{d}x|^p\right] \leq \delta^{\frac{\beta p}{\beta+\kappa}} \mathbb{E}\left[|\int_0^T \int_D \tilde{\rho}_\delta^{\beta+\kappa} \mathrm{d}x\mathrm{d}s|^{\frac{\beta p}{\beta+\kappa}}\right] \to 0.$$

and also that the strong convergence of the intial data holds by Hypothesis 1.1.1. The estimate above also leads to the uniform bounds

$$\sup_{\delta>0} \mathbb{E}\left[\sup_{t\in[0,T]} |\widetilde{M}_t^\delta(\phi)|^p\right] < \infty. \tag{3.93}$$

This information is enough in order to use our usual procedure and verify that $\{M_t(\phi)\}_{t\geq 0}$ is a continuous $\{\mathcal{F}^t\}_{t=0}^T$ martingale. Additionally, we may check the convergence in $L^p_{w^*}(\Omega; L^\infty[0,T])$, so the estimate (3.92) follows. □

We now want to work towards establishing the weak continuity of the effective viscous pressure. At the ϵ layer, we chose a test function and began by applying the Ito formula to find the evolution of $\rho_\delta u_\delta \cdot \varphi_\delta$. Since we have not identified our martingale as a stochastic integral, this is slightly less straightforward. Instead, we check only that our desired identity holds in $\mathbb{P}$ expectation. Let us now define the following two random test functions:

$$\tilde{\varphi}_\delta(t,x,\omega) = \eta\mathcal{A}\left[\eta T_k(\tilde{\rho}_\delta)\right]$$

$$\varphi(t,x,\omega) = \eta\mathcal{A}\left[\eta\overline{T_k(\rho)}\right]$$

By the Di Perna Lions commutator lemmas, we may verify the following identity:

$$\partial_t\tilde{\varphi}_\delta = -\eta\mathcal{A}\circ\operatorname{div}(\eta T_k(\tilde{\rho}_\delta)\tilde{u}_\delta) + \eta\mathcal{A}\left[\nabla\eta\cdot T_k(\tilde{\rho}_\delta)\tilde{u}_\delta\right] - \eta\mathcal{A}[\eta(\tilde{\rho}_\delta T_k'(\tilde{\rho}_\delta) - T_k(\tilde{\rho}_\delta))\operatorname{div}\tilde{u}_\delta] \tag{3.94}$$

Sending $\delta \to 0$ and using the Skorohod step we find

$$\partial_t \varphi = -\eta \mathcal{A} \circ \operatorname{div}(\eta \overline{T_k(\rho)} u) + \eta \mathcal{A}\left[\nabla \eta \cdot \overline{T_k(\rho)} u\right] - \eta \mathcal{A}[\eta \overline{(\rho T_k'(\rho) - T_k(\rho)) \operatorname{div} u}] \quad (3.95)$$

We now establish the following averaged Ito product rule.

Lemma 3.2.9. *(Two Averaged Ito Product Rules) Define φ_δ and φ as above, then the following two averaged Ito product rules hold $\mathbb{P}$ a.s. for all times $t \in [0, T]$.*

For the test function $\tilde{\varphi}_\delta$ defined above we have:

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}} \int_D \tilde{\rho}_\delta \tilde{u}_\delta(t) \cdot \tilde{\varphi}_\delta(t) \mathrm{d}x = \int_D m_{0,\delta} \cdot \tilde{\varphi}_\delta(0) \mathrm{d}x \\
&+ \mathbb{E}^{\mathbb{P}} \int_0^t \int_D [\tilde{\rho}_\delta \tilde{u}_\delta \otimes \tilde{u}_\delta - 2\mu \nabla \tilde{u}_\delta] : \nabla \tilde{\varphi}_\delta] \mathrm{d}x \mathrm{d}s \\
&+ \mathbb{E}^{\mathbb{P}} \int_0^t \int_D [(-\lambda \operatorname{div} \tilde{u}_\delta + P(\tilde{\rho}_\delta, \tilde{\theta}_\delta) + \delta \tilde{\rho}_\delta^\beta) I] : \nabla \tilde{\varphi}_\delta] \mathrm{d}x \mathrm{d}s + \mathbb{E}^{\mathbb{P}} \int_0^t \int_D [\tilde{\rho}_\delta \tilde{u}_\delta \cdot \partial_t \tilde{\varphi}_\delta] \mathrm{d}x \mathrm{d}s.
\end{aligned}$$

For φ defined above we have:

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}} \left[\int_D \rho u(t) \cdot \varphi(t) \mathrm{d}x\right] = \int_D m_0 \cdot \varphi(0) \mathrm{d}x \\
&+ \mathbb{E}^{\mathbb{P}} \int_0^t \int_D [\rho u \otimes u - 2\mu \nabla u] : \nabla \varphi \mathrm{d}x \mathrm{d}s \\
&+ \mathbb{E}^{\mathbb{P}} \int_0^t \int_D [(-\lambda \operatorname{div} u + \overline{\rho^\gamma}) I] : \nabla \varphi + \rho u \cdot \partial_t \varphi \mathrm{d}x \mathrm{d}s.
\end{aligned}$$

Proof. The first identity can be proved in the same way as at the ϵ layer. To proceed to the second, denote by ξ_κ the standard mollifier (localized at scale κ) and $\xi_{\kappa,x}$ the standard mollifier centered at the point x. Extending $(\rho, u, \overline{T_k(\rho)}, \overline{P(\rho,\theta)}, \overline{\rho^\beta})$

by zero outside of D, we may define the quantities g_κ, φ_κ via

$$g_\kappa(x,s) = \int_{\mathbb{R}^d} [\rho u \otimes u(s) - 2\mu \nabla u(s) + (-\lambda \operatorname{div} u(s) + \overline{P(\rho,\theta)} + \delta\overline{\rho^\beta})I] : \nabla \xi_\kappa(x-y)\mathrm{d}y$$

$$\varphi_\kappa(x,t) = (\varphi(t) * \xi_\kappa)(x)$$

$$(\rho u)_\kappa(x,t) = (\rho u(t) * \xi_\kappa)(x)$$

By definition of the process $\{M_t(\xi_{\kappa,x})\}_{t\geq 0}$ we obtain for all $\kappa > 0$ and $x \in D$, the following equality holds $\mathbb{P}$ a.s.

$$(\rho u)_\kappa(x,t) = m_{0,\kappa}(x) + \int_0^t g_\kappa(x,s)\mathrm{d}s + M_t(\xi_{\kappa,x})$$

By Lemma **??** the process $\{M_t(\xi_{\kappa,x})\}_{t=0}^T$ is a martingale satisfying enough bounds to give a meaning to the stochastic integral below. Applying the classical Ito product rule for continuous one dimensional martingales, we obtain for each $x \in D$, the following equality holds $\mathbb{P}$ a.s.

$$(\rho u)_\kappa(x,t) \cdot \varphi_\kappa(x,t) = m_{0,\kappa}(x) \cdot \varphi_\kappa(0) + \int_0^t \big[\varphi_\kappa(x,s) \cdot g_\kappa(x,s) + \partial_t \varphi_\kappa \cdot (\rho u)_\kappa(x,s)\big]\mathrm{d}s$$
$$+ \int_0^t \varphi_\kappa(x,s)\mathrm{d}M_s(\xi_{\kappa,x}).$$

Note the estimate

$$\mathbb{E}\left[\int_0^T \phi_\kappa(x,s)^2 \mathrm{d}M(\xi_{\kappa,x})_s\right] \leq \mathbb{E}\left[\sup_{t\in[0,T]} |\phi_k(x,s)|^4\right]^{1/2} \mathbb{E}\left[\langle M(\xi_{\kappa,x})\rangle_T^2\right]^{1/2}.$$

By the definition of quadratic variation(or the Doob Meyer Decomposition for continuous martingales) and the uniform bounds on the fourth moments of $M_t(\xi_{\kappa,x})$, the second moment of the quadratic variation is controlled. The other term is estimated using the expression for $\partial_t \phi_{k,x}$ implied by the equation above. Hence, the stochastic integral above is a martingale(rather than just a local martingale), and hence has mean zero. Taking expectation and integrating over D yields

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\left[\int_D (\rho u)_\kappa(x,t)\cdot\varphi_\kappa(x,t)\mathrm{d}x\right] = \int_D m_{0,\kappa}(x)\cdot\varphi_\kappa(0)\mathrm{d}x \\
&+\mathbb{E}^{\mathbb{P}}\left[\int_0^t\int_D \left[\varphi_\kappa(x,s)\cdot g_\kappa(x,s)+\partial_t\varphi_\kappa\cdot(\rho u)_\kappa(x,s)\right]\mathrm{d}x\right].
\end{aligned}
$$

Letting $\kappa\to 0$ and appealing to standard properties of mollifiers, we obtain the result. □

Using the averaged Ito product rule and our compactness result, it is straightforward to follow the arguments in the previous section and deduce the weak continuity of the effective viscous flux.

Lemma 3.2.10. *Let $K\subset\subset D$ be arbitrary, then the weak continuity of the effective viscous pressure holds on average, that is:*

$$
\begin{aligned}
&\lim_{\delta\to 0}\mathbb{E}^{\mathbb{P}}[\int_0^T\int_K \big((2\mu+\lambda)\operatorname{div}\tilde{u}_\delta - p_m(\tilde{\rho}_\delta)-\tilde{\theta}_\delta p_\theta(\tilde{\rho}_\delta)\big)T_k(\tilde{\rho}_\delta)\mathrm{d}x\mathrm{d}t] \\
&=\mathbb{E}^{\mathbb{P}}[\int_0^T\int_K \big((2\mu+\lambda)\operatorname{div}u-\overline{p_m(\rho)}-\overline{\theta p_\theta(\rho)}\big)\overline{T_k(\rho)}\mathrm{d}x\mathrm{d}t].
\end{aligned}
$$

3.2.4 Strong convergence of the density

Since γ may be close to $\frac{d}{2}$, the limiting density ρ may not belong to $L_t^\infty(L_x^2)$ for all $\omega\in\Omega$. Hence, it is not a priori clear that the continuity equation for ρ may be renormalized(which is crucial for the proof of strong convergence). In [11], this issue is addressed via a thorough analysis of the so-called oscillations defect measure:

$$
\operatorname{osc}_p[\tilde{\rho}_\delta\to\rho](\mathcal{O})=\sup_{k\geq 1}\big(\limsup_{\delta\to 0}\iint_O |T_k(\tilde{\rho}_\delta)-T_k(\rho)|^p\mathrm{d}x\mathrm{d}s\big). \tag{3.96}
$$

In our framework, the quantity above is random. If we could show that for some $p > 2$ and all $\mathcal{O} \subset [0,T] \times D$, $\mathrm{osc}_p[\rho_\delta \to \rho](\mathcal{O}) < \infty$ $\mathbb{P}$ almost surely, we could appeal directly to the results in [11] and deduce that ρ is a renormalized solution of the transport equation $\mathbb{P}$ almost surely. This would be the case, for instance, if we could show that

$$\mathbb{E}^{\mathbb{P}}\left[\mathrm{osc}_p[\rho_\delta \to \rho](\mathcal{O})\right] < \infty.$$

However, it seems that this would require proving the weak continuity of the effective viscous pressure, Lemma 3.2.10, in the $\mathbb{P}$ almost sure sense, rather than in expectation only. It is our point of view that this is most likely not even true based on the information at this stage in the proof. The issue is the contribution of the stochastic integrals, which seems to lead only to a convergence in probability law, not $\mathbb{P}$ almost sure convergence. Effectively, the way to cure this problem is to reverse the order of operations. Namely, in the notion of oscillations defect measure, one should take an expectation prior to passing the limit supremum in δ. This remark is made precise through the following Lemma. The proof follows closely the method in [11].

Lemma 3.2.11. *Let $K \subset\subset D$ by given. There exists a positive constant C_K such that*

$$\sup_{k\geq 1} \limsup_{\delta\to 0} \mathbb{E}^{\mathbb{P}} \int_0^T \int_K |T_k(\tilde{\rho}_\delta) - T_k(\rho)|^{\gamma+1} \mathrm{d}x\mathrm{d}t \leq C_K. \tag{3.97}$$

Proof. In view of our convexity and growth hypothesis 1.1.2 on P_m, it follows that for all $y, z \geq 0$,

$$|T_k(z) - T_k(y)|^{\gamma+1} \lesssim (p_m(z) - p_m(y))[T_k(z) - T_k(y)].$$

Combining this with observation with Lemma 3.2.10 and the monotonicity of p_θ yields:

$$
\begin{aligned}
&\limsup_{\delta\to 0+} \mathbb{E}^{\mathbb{P}} \int_0^T \int_K |T_k(\tilde\rho_\delta) - T_k(\rho)|^{\gamma+1} \mathrm{d}x\mathrm{d}t \\
&\leq \lim_{\delta\to 0+} \mathbb{E}^{\mathbb{P}} \int_0^T \int_K (p_m(\tilde\rho_\delta) - p_m(\rho))[T_k(\tilde\rho_\delta) - T_k(\rho)] \mathrm{d}x\mathrm{d}t \\
&\leq \lim_{\delta\to 0+} \mathbb{E}^{\mathbb{P}} \int_0^T \int_K (p_m(\tilde\rho_\delta) - p_m(\rho))[T_k(\tilde\rho_\delta) - T_k(\rho)] \mathrm{d}x\mathrm{d}t \\
&+ \mathbb{E}^{\mathbb{P}} \int_0^T \int_K (\overline{p_m(\rho)} - p_m(\rho))[T_k(\tilde\rho_\delta) - T_k(\rho)] \mathrm{d}x\mathrm{d}t \\
&= \lim_{\delta\to 0+} \mathbb{E}^{\mathbb{P}} \int_0^T \int_K p_m(\tilde\rho_\delta)T_k(\tilde\rho_\delta) - \overline{p_m(\rho)T_k(\rho)} \mathrm{d}x\mathrm{d}t \\
&\leq \left| \limsup_{\delta\to 0} \mathbb{E}^{\mathbb{P}} \int_K \left[\operatorname{div} \tilde u_\delta T_k(\tilde\rho_\delta) - \operatorname{div} \tilde u_\delta \overline{T_k(\rho)}\right] \mathrm{d}x\mathrm{d}t \right|,
\end{aligned}
$$

Finally, using the uniform estimates for $\{\tilde u_\delta\}_{\delta>0}$ in $L^2(\Omega; L^2_t(H^1_x))$ guaranteed by Proposition 3.2.5, it follows that:

$$
\begin{aligned}
&\limsup_{\delta\to 0+} \mathbb{E}^{\mathbb{P}} \int_0^T \int_K \left[\operatorname{div} u_\delta T_k(\rho_\delta) - \operatorname{div} u_\delta \overline{T_k(\rho)}\right] \mathrm{d}x\mathrm{d}t \\
&= \limsup_{\delta\to 0+} \mathbb{E}^{\mathbb{P}} \int_0^T \int_K \left[T_k(\tilde\rho_\delta) - T_k(\rho) + T_k(\rho) - \overline{T_k(\rho)}\right] \operatorname{div} \tilde u_\delta \,\mathrm{d}x\mathrm{d}t \\
&\leq 2 \limsup_{\delta\to 0} \mathbb{E}^{\mathbb{P}} \left(\|\operatorname{div} \tilde u_\delta\|_{L^2((0,T)\times D)} \|T_k(\tilde\rho_\delta) - T_k(\rho)\|_{L^2((0,T)\times K)}\right) \\
&\leq \tilde\epsilon \limsup_{\delta\to 0} \mathbb{E}^{\mathbb{P}} \left(\|T_k(\tilde\rho_\delta) - T_k(\rho)\|^{\gamma+1}_{L^{\gamma+1}((0,T)\times K)}\right) + C_{\tilde\epsilon},
\end{aligned}
$$

where $\tilde\epsilon$ may be chosen arbitrarily small. Combining these inequalities and bringing the last term back to the LHS of the estimate gives the claim. □

Using Lemma 3.2.11, we can now check that ρ is a renormalized solution of the continuity equation driven by u, $\mathbb{P}$ almost surely.

Lemma 3.2.12. *Extend ρ, u by zero outside outside D. Let $\beta \in C^1(\mathbb{R}_+)$ and suppose that β' is compactly supported. Then we have the $\mathbb{P}$ almost sure identity:*

$$\partial_t \beta(\rho) + \operatorname{div}(\beta(\rho)u) + [\beta'(\rho)\rho - \beta(\rho)] \operatorname{div} u = 0, \ \textit{in } \mathcal{D}'((0,T) \times \mathbb{R}^3). \tag{3.98}$$

Proof. Start by renormalizing the continuity equation for $\tilde{\rho}_\delta$ with T_k to obtain the $\mathbb{P}$ almost sure identity:

$$\partial_t T_k(\tilde{\rho}_\delta) + \operatorname{div}(T_k(\tilde{\rho}_\delta)\tilde{u}_\delta) = -[T_k'(\tilde{\rho}_\delta)\tilde{\rho}_\delta - T_k(\tilde{\rho}_\delta)] \operatorname{div} \tilde{u}_\delta \text{ in } \mathcal{D}'((0,T) \times \mathbb{R}^3). \tag{3.99}$$

Send $\delta \to 0$ and apply $\mathbb{P}$ a.s. convergences in Proposition 3.2.5 to pass the limit in each term above. Noting that the equation satisfied by $\overline{T_k(\rho)}$ can be renormalized(due to the unlimited integrability), we find that $\mathbb{P}$ almost surely,

$$\begin{aligned}
&\partial_t \beta(\overline{T_k(\rho)}) + \operatorname{div}(\beta(\overline{T_k(\rho)}u) + \left[\beta'(\overline{T_k(\rho)})(\overline{T_k(\rho)}) - \beta(\overline{T_k(\rho)})\right] \operatorname{div} u \\
&= -\beta'(\overline{T_k(\rho)})\overline{[T_k'(\rho)\rho - T_k(\rho)] \operatorname{div} u} \text{ in } \mathcal{D}'((0,T) \times \mathbb{R}^3).
\end{aligned} \tag{3.100}$$

By assumption, we may choose $M > 0$ such that β' is supported on $[0, M]$. Passing $k \to \infty$ on both sides of the identity above and using the $\mathbb{P}$ almost sure $L^1_{t,x}$ strong convergence of $\overline{T_k(\rho)}$ towards ρ, we see that the proof of the Lemma will be complete as soon as we establish:

$$\lim_{k \to \infty} \| \overline{[T_k'(\rho)\rho - T_k(\rho)] \operatorname{div} u} \ 1_{\{\overline{T_k(\rho)} \geq M\}} \|_{L^1(\Omega \times [0,T] \times K)} = 0, \tag{3.101}$$

where $K \subset\subset D$ is arbitary and fixed in advance. Let us now prove the convergence (3.101). Using a lower-semicontinuity argument together with the uniform bound for $\{\tilde{u}_\delta\}_\delta$ in $L^2(\Omega \times [0,T] \times D)$, we see that it suffices to prove:

$$\lim_{k \to \infty} \liminf_{\delta \to 0} \| [T_k'(\tilde{\rho}_\delta)\tilde{\rho}_\delta - T_k(\tilde{\rho}_\delta)] \ 1_{\{\overline{T_k(\rho)} \geq M\}} \|_{L^2(\Omega \times [0,T] \times K)} = 0. \tag{3.102}$$

Observe that as $k \to \infty$, we have $T_k'(\tilde{\rho}_\delta)\tilde{\rho}_\delta - T_k(\tilde{\rho}_\delta) \to 0$ in $L^1(\Omega \times [0,T] \times D)$, uniformly in δ. Interpolating, we see that it is enough to have the following uniform bound:

$$\sup_{k\geq 1} \liminf_{\delta\to 0} \|[T_k'(\tilde{\rho}_\delta)\tilde{\rho}_\delta - T_k(\tilde{\rho}_\delta)]\ 1_{\{\overline{T_k(\rho)}\geq M\}}\|_{L^{\gamma+1}(\Omega\times[0,T]\times K)} < \infty. \tag{3.103}$$

Using the fact that $T_k'(z)z \leq T_k(z)$, adding and subtracting $T_k(\rho)$, then appealing to lower-semicontinuity arguments once more yields:

$$\begin{aligned}
&\sup_{k\geq 1} \liminf_{\delta\to 0} \|[T_k'(\tilde{\rho}_\delta)\tilde{\rho}_\delta - T_k(\tilde{\rho}_\delta)]\ 1_{\{\overline{T_k(\rho)}\geq M\}}\|_{L^{\gamma+1}(\Omega\times[0,T]\times K)} \\
&\lesssim \sup_{k\geq 1} \liminf_{\delta\to 0} \|[T_k(\tilde{\rho}_\delta)]\ 1_{\{\overline{T_k(\rho)}\geq M\}}\|_{L^{\gamma+1}(\Omega\times[0,T]\times K)} \\
&\lesssim 1 + \sup_{k\geq 1} \liminf_{\delta\to 0} \|[T_k(\tilde{\rho}_\delta) - T_k(\rho)]\|_{L^{\gamma+1}(\Omega\times[0,T]\times K)}.
\end{aligned}$$

The proof is now complete in view of Lemma 3.2.11. □

Using Lemma 3.2.11 together with our renormalization Lemma 3.2.12, we may now obtain the following strong convergence of the density.

Lemma 3.2.13. *The sequence of densities $\{\rho_\delta\}_{\delta>0}$ converges strongly to ρ in the sense that for all $p \geq 1$ and $r < \gamma + \kappa$*

$$\lim_{\delta\to 0} \|\tilde{\rho}_\delta - \rho\|_{L^p\left(\Omega; L^r_{t,x}\right)} = 0. \tag{3.104}$$

Proof. Since ρ is a renormalized solution, the proof now follows along the lines of the arguments in the previous section and [11]. The role of the higher integrability of ρ in the case $\gamma > 2$ is now replaced by the use of Lemma 3.2.11. The treatment of the temperature part of the pressure does not differ substantially from the arguments given at the $\epsilon \to 0$ construction of the δ layer. □

3.2.5 Strong convergence of the temperature: away from vaccum

With the strong convergence of the density at hand, we may now deduce the following strong convergence of the temperature. Recall that $\overline{\theta}$ is the limit extracted in Proposition 3.2.5.

Lemma 3.2.14. *For all $q < 3$, $\omega \in \Omega$, we have $\tilde{\theta}_\delta(\omega)1_{\{\rho(\omega)>0\}} \to \overline{\theta}(\omega)1_{\{\rho(\omega)>0\}}$ in $L^q_{t,x}$.*

Proof. Following [11], it suffices to verify that $\sqrt{\rho}(\omega)\tilde{\theta}_\delta(\omega) \to \sqrt{\rho}(\omega)\theta(\omega)$ in $L^2_{t,x}$, then use the $L^3_{t,x}$ weak convergence guaranteed by Proposition 3.2.5.

Following the same strategy as in Lemma 3.1.9, one verifies that $(\tilde{\rho}_\delta(\omega) + \delta)\tilde{\theta}^2_\delta(\omega) \to \rho(\omega)\theta^2(\omega)$ in $\mathcal{D}'_{t,x}$. Using the uniform estimates, the convergence actually holds in $L^r_{t,x}$ for some $r > 1$. Finally, use the strong convergence, uniform bounds, and the decomposition $\rho\tilde{\theta}^2_\delta = (\rho - \tilde{\rho}_\delta)\tilde{\theta}^2_\delta + \rho\tilde{\theta}^2_\delta$ to deduce $\rho(\omega)\tilde{\theta}_\delta(\omega)^2 \to \rho(\omega)\theta(\omega)^2$ in $L^1_{t,x}$. This completes the proof. □

3.2.6 Defining the limiting temperature: renormalized limits

By Prop 3.2.5, we know that for all $m \geq 1$ and $\omega \in \Omega$, $\mathcal{K}_m(\tilde{\theta}_\delta)(\omega) \to \overline{\mathcal{K}_m(\theta)}(\omega)$ weakly in $L^1_{t,x}$. Since weak convergence is order preserving, in view of the definition of $\mathcal{K}_m$, we have that for each $\omega \in \Omega$, the sequence $\{\overline{\mathcal{K}_m(\theta)}(\omega)\}_{m=1}^\infty$ is monotone. Hence, by the monotone convergence theorem, there exists $\overline{\mathcal{K}}(\omega)$ such that for all $\omega \in \Omega$, we have the convergence $\overline{\mathcal{K}}_m(\omega) \to \overline{\mathcal{K}}(\omega)$ strongly in $L^1_{t,x}$.

We are now in a position to define the limiting temperature θ via the relation

$\theta(\omega) = \mathcal{K}^{-1} \circ \overline{\mathcal{K}}(\omega)$. Let us observe that, away from the vaccum, this definition is consistent with our initial limit $\overline{\theta}$. Indeed, Lemma 3.2.14 implies that $1_{\rho>0}\mathcal{K}_m(\tilde{\theta}_\delta)$ converges to $1_{\rho>0}\mathcal{K}_m(\overline{\theta})$ strongly in $L^1_{t,x}$, pointwise in Ω. Therefore, $\mathcal{K}_m(\overline{\theta})1_{\rho>0} = 1_{\rho>0}\overline{\mathcal{K}_m(\theta)}$ and passing $m \to \infty$ gives $\mathcal{K}(\overline{\theta})1_{\rho>0} = \overline{\mathcal{K}}1_{\rho>0}$. Since $\mathcal{K}$ is invertible, it follows that $\overline{\theta}1_{\rho>0} = \theta 1_{\rho>0}$. In particular, it follows that the definition of $(\mathcal{F}_t)_{t=0}^T$ is unchanged after replacing the role of $\overline{\theta}$ by θ, so this will still define our limiting filtration.

To summarize, we have the following Corollary of Lemma 3.2.14.

Corollary 3.2.15. *For all $q < 3$, $\omega \in \Omega$, we have the following strong convergence away from vaccum:*

$$\tilde{\theta}_\delta(\omega)1_{\{\rho(\omega)>0\}} \to \theta(\omega)1_{\{\rho(\omega)>0\}} \quad in \quad L^q_{t,x}.$$

3.2.7 Conclusion of the proof

Proof of Theorem 1.2.2.

Lemma 3.2.16. *The quantities ρ, θ, u satisfy the temperature inequality.*

Proof. Use the δ layer renormalized temperature inequality satisfied by $\tilde{\rho}_\delta, \tilde{\theta}_\delta, \tilde{u}_\delta$ with the renormalization H_m defined above. Sending $m \to \infty$ and using Corollary 3.2.15 together with the arguments from Lemma 3.1.9 yields:

$$\int_0^T \int_\Omega \rho\theta\partial_t\varphi + \rho\theta\mathbf{u}\cdot\nabla\varphi + \overline{\mathcal{K}(\theta)}\Delta\varphi \mathrm{d}x\mathrm{d}t \leq -\int_0^T \int_\Omega \mathbb{S} : \nabla\mathbf{u}\varphi\mathrm{d}x\mathrm{d}t +$$
$$\int_0^T \int_\Omega \varphi\left(\theta p_\theta(\rho) div\mathbf{u}\right)\mathrm{d}x\mathrm{d}t - \int_\Omega \rho_0\theta_0\varphi(0)\mathrm{d}x, \tag{3.105}$$

By the definition of θ, it follows that $\mathcal{K}(\theta)(\omega) = \overline{\mathcal{K}}(\omega)$, pointwise in Ω. This completes the lemma. $\square$

Arguing as in the previous sections, we find.

Lemma 3.2.17. *The pair (ρ, u) satisfies the momentum equation 1.7 of Definition 1.2.1.*

We may now complete the proof of our main result.

Proof. For each $\phi \in C_c^\infty(D)$ we introduce the continuous $(\mathcal{F}^t)_{t=0}^T$ adapted process $(M_t(\phi))_{t=0}^T$ defined by

$$M_t(\phi) = \int_D \rho u(t)\cdot\phi dx - \int_D m_0\cdot\phi dx - \int_0^t \int_D [\rho u \otimes u - \mathcal{S}(u)] : \nabla\varphi + [p_m(\rho) + \theta p_\theta(\rho)] \operatorname{div}\phi \, dxds.$$

Recall that since $\overline{\theta}$ and θ agree away from vaccum, $(\mathcal{F}^t)_{t=0}^T$ remains the same after replacing $\overline{\theta}$ by θ. Hence, we may combine our preliminary martingale Lemma 3.2.8 with the strong convergence upgrades Corollary 3.2.15 and Lemma 3.2.13 to conclude that $(M_t)_{t=0}^T$ is a martingale. Moreover, our strong convergence results yield: $\tilde{\rho}_\delta\tilde{\theta}_\delta * \eta_\delta \to \rho\theta$ strongly in $L^p\big(\Omega; L_t^2(L_x^q)\big)$ for all $p \in [1,\infty)$ and $\frac{1}{q} = \frac{1}{\gamma} + \frac{1}{2} - \frac{1}{d}$. Using the Lipschitz continuity hypothesis 1.1.5 together with the arguments from [25], we may identify

$$M_t(\phi) = \sum_{k=1}^\infty \int_0^t \int_D \rho\sigma_k(\rho, \rho u, \rho\theta, x) \, dx d\beta_k(s).$$

This yields the momentum equation. $\square$

The proof of our main result is now complete. $\square$

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