

ABSTRACT

Title of dissertation: QUANTITATIVE DERIVATION OF
EFFECTIVE EVOLUTION EQUATIONS
FOR THE DYNAMICS OF
BOSE-EINSTEIN CONDENSATES

Elif Kuz, Doctor of Philosophy, 2016

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This thesis proves certain results concerning an important question in non-equilibrium quantum statistical mechanics which is the derivation of effective evolution equations approximating the dynamics of a system of large number of bosons initially at equilibrium (ground state at very low temperatures). The dynamics of such systems are governed by the time-dependent linear many-body Schrödinger equation from which it is typically difficult to extract useful information due to the number of particles being large. We will study quantitatively (i.e. with explicit bounds on the error) how a suitable one particle non-linear Schrödinger equation arises in the mean field limit as number of particles $N \rightarrow \infty$ and how the appropriate corrections to the mean field will provide better approximations of the exact dynamics.

In the first part of this thesis we consider the evolution of N bosons, where N is large, with two-body interactions of the form $N^{3\beta}v(N^\beta \cdot)$, $0 \leq \beta \leq 1$. The

parameter β measures the strength and the range of interactions. We compare the exact evolution with an approximation which considers the evolution of a mean field coupled with an appropriate description of pair excitations, see [18, 19] by Grillakis-Machedon-Margetis. We extend the results for $0 \leq \beta < 1/3$ in [19, 20] to the case of $\beta < 1/2$ and obtain an error bound of the form $p(t)/N^\alpha$, where $\alpha > 0$ and $p(t)$ is a polynomial, which implies a specific rate of convergence as $N \rightarrow \infty$.

In the second part, utilizing estimates of the type discussed in the first part, we compare the exact evolution with the mean field approximation in the sense of marginals. We prove that the exact evolution is close to the approximate in trace norm for times of the order $o(1)\sqrt{N}$ compared to $\log(o(1)N)$ as obtained in Chen-Lee-Schlein [6] for the Hartree evolution. Estimates of similar type are obtained for stronger interactions as well.

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EFFECTIVE EVOLUTION EQUATIONS FOR
THE DYNAMICS OF BOSE-EINSTEIN CONDENSATES

by

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Dissertation submitted to the Faculty of the Graduate School of the
University of Maryland, College Park in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
2016

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Acknowledgments

I am deeply grateful to my advisors Professor Manoussos Grillakis and Professor Matei Machedon for their invaluable guidance, support, encouragement and patience during my graduate study at University of Maryland.

I would like to thank Professor Dionisios Margetis for serving as a thesis committee member, reviewing the manuscript and for very useful seminars organized partly by him on modelling and analysis in atomic physics, particle systems and aspects of statistical mechanics from which I greatly benefited.

Many thanks are due to Professor Antoine Mellet and Professor Victor Galitski for agreeing to serve on my thesis committee and sparing their valuable time for reviewing the manuscript. I am also grateful to Professor Galitski for very useful lectures he gave on quantum mechanics on Coursera, which helped me develop a better understanding of certain phenomena early on in my research.

I would like to thank here Professor Alp Eden once again for all his support and encouragement while supervising my Masters thesis during my graduate studies at Boğaziçi University and applications to graduate school.

I owe my deepest thanks to my family - my deceased father, my mother, my sister and my brother for their support and huge sacrifices without which this work would be a distant goal to achieve.

I would also like to thank Chunting Lu, Ayşe and Mustafa Bilgic, Ayşe and Serdar Gürses for their friendship and support. I thank all those whose names I could not mention here but who helped in various ways during my studies.

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List of Abbreviations and Some Notation

$v_N(x)$	$= N^{3\beta}v(N^\beta x)$
$\phi^{\otimes N}(x_1, \dots, x_N)$	$= \prod_{j=1}^N \phi(x_j)$
$ \phi\rangle\langle\phi $	rank-one projection with kernel $(\phi \otimes \bar{\phi})(x, y) = \phi(x)\bar{\phi}(y)$
$\ \cdot\ _p$	norm on L^p spaces where $1 \leq p \leq \infty$
$\ \ \cdot\ _p\ _q$	norm on e.g. $L^q(\mathbb{R}^n; L^p(\mathbb{R}^m))$, space of measurable functions f from \mathbb{R}^n to $L^p(\mathbb{R}^m)$ such that $\ f(\cdot)\ _{L^p(\mathbb{R}^m)} \in L^q(\mathbb{R}^n)$
$\ \cdot\ _{L^q L^p}$	$= \ \ \cdot\ _p\ _q$
$\langle\cdot, \cdot\rangle_H$	inner product on Hilbert space H , linear in the first component
$L_s^2(\mathbb{R}^{3N})$	subspace of $L^2(\mathbb{R}^{3N})$ consisting of symmetric functions in x_1, x_2, \dots, x_N
$H^s(\mathbb{R}^n)$	Sobolev space of functions having s derivatives in $L^2(\mathbb{R}^n)$, equipped with the norm $\ f\ _{H^s} = \ (1 + \xi ^2)^{s/2} \hat{f}(\xi)\ _{L^2}$ where s is allowed to have non-integer values
$\text{Tr} \cdot $	trace norm on the space of trace class operators $\mathcal{L}_1(L^2(\mathbb{R}^3))$ on $L^2(\mathbb{R}^3)$ i.e. $\text{Tr} A = \text{Tr}((A^*A)^{1/2})$ for A satisfying $\sum_{f \in \mathcal{F}} \langle A f, f \rangle_{L^2(\mathbb{R}^3)} < \infty$ for any \mathcal{F} orthonormal basis of $L^2(\mathbb{R}^3)$
$\ \cdot\ _{\text{op}}$	operator norm e.g. for J an operator on $L^2(\mathbb{R}^3)$, $\ J\ _{\text{op}} = \sup_{\{f \in L^2(\mathbb{R}^3): \ f\ _2=1\}} \ Jf\ _2$
BEC	Bose-Einstein condensate (or condensation)
GMM	Grillakis-Machedon-Margetis
NLS	Non-linear Schrödinger equation
PDE	Partial differential equation
h.c.	hermitian conjugate of the preceding term
r.h.s. (or l.h.s.)	right (or left) hand side
w.r.t.	with respect to

Chapter 1: Introduction

1.1 Background

A Bose-Einstein condensate is a state of matter of a dilute gas of bosons at very low temperatures, in which particles macroscopically occupy the lowest energy state described by a single one particle wave function. This phenomenon was first predicted by Einstein in 1925 for non-interacting massive particles based on the ideas of Bose. The experimental realization of the first condensates was achieved in 1995 [1, 11] which has been followed by an increase in the experimental and theoretical activity on the study of the condensates.

In experiments, to obtain a condensate, weakly interacting atoms trapped by external potentials are cooled below a certain temperature depending on the density of the gas. Traps are then removed to observe the evolution of the condensate. The properties of interest are the macroscopic properties of the system describing the typical behavior of the particles resulting from averaging over a large number of particles. The limiting behavior as the number of particles goes to infinity is expected to be a good approximation for the macroscopic properties observed in the experiments for a system of large but finite number of particles. We can describe the corresponding mathematical model as follows. We consider a system of N

weakly interacting bosons, the dynamics of which is governed by the N -body linear Schrödinger equation

$$\frac{1}{i} \partial_t \psi_N = \left(\sum_{j=1}^N \Delta_{x_j} - \frac{1}{N} \sum_{j < k}^N \overbrace{N^{3\beta} v(N^\beta(x_j - x_k))}^{H_N} \right) \psi_N \quad (1.1)$$

\uparrow
 to treat
 H_N as
 a mean field
 Hamiltonian

$v_N(x_j - x_k) \xrightarrow{N \rightarrow \infty} \left(\int v(x) dx \right) \delta(x_j - x_k)$

where $0 \leq \beta \leq 1$. H_N acts on the wave functions

$$\psi_N \in L_s^2(\mathbb{R}^{3N}) \quad \text{with} \quad \|\psi_N\|_{L_s^2(\mathbb{R}^{3N})} = 1$$

where $L_s^2(\mathbb{R}^{3N})$ stands for the subspace of $L^2(\mathbb{R}^{3N})$ consisting of symmetric functions in x_1, x_2, \dots, x_N . The spherically symmetric non-negative potential $v \in L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ models two body interactions and the scaling parameter β describes the range and the strength of interactions. In a trap of fixed size the average inter-particle distance can be considered to scale with $N^{-1/3}$ compared to the range of the interaction of order $N^{-\beta}$. Hence for $\beta > 1/3$, each particle feels only the potential generated by itself, which is consistent with the Gross-Pitaevskii theory proposing to model the many-body effects by a strong on-site self interaction [23, 24, 34]. The problem becomes more difficult and interesting as $\beta \rightarrow 1$ since strong interactions in Gross-Pitaevskii regime ($\beta = 1$) produce short range particle correlations leading to the emergence of the scattering length of v in the limiting dynamics.

We consider the evolution in (1.1) with initial data coming from the ground

state of the following Hamiltonian describing the initially trapped gas:

$$H_N^{\text{trap}} := \sum_{j=1}^N \left(-\Delta_{x_j} + \underbrace{V_{\text{ext}}(x_j)}_{\substack{\sim |x_j|^2 \text{ typically} \\ \text{in available experiments}}} \right) + \frac{1}{2N} \sum_{j \neq k} N^{3\beta} v(N^\beta(x_j - x_k)).$$

The ground state of H_N^{trap} at zero temperature exhibits complete BEC as $N \rightarrow \infty$

in the sense that it looks like a factorized state:

$$\psi_N(0, x_1, \dots, x_N) \simeq \phi_0^{\otimes N} := \underbrace{\prod_{j=1}^N \phi_0(x_j)}_{\substack{\text{pure condensate in} \\ \text{case of no interaction}}} \quad (1.2)$$

as rigorously justified by the work of Lieb-Seiringer-Yngvason [30, 31] which showed

$$\gamma_N^{(1)}(0, x, x') := \int_{\mathbb{R}^{3(N-1)}} \psi_N(0, x, \mathbf{x}_{N-1}) \overline{\psi_N(0, x', \mathbf{x}_{N-1})} d\mathbf{x}_{N-1} \rightarrow \phi_0(x) \bar{\phi}_0(x') \quad (1.3)$$

in trace norm as $N \rightarrow \infty$ and for ϕ_0 minimizing the appropriate one-particle energy functional, subject to $\|\phi_0\|_{L^2(\mathbb{R}^3)} = 1$. The eigenvalues of $\gamma_N^{(1)}$ are interpreted as the probabilities of occupation of the corresponding eigenstates. Convergence to a rank-one projection as in (1.3) implies that in the limit of large N any randomly chosen particle occupies the one-particle state ϕ_0 (initial mean field) with probability one.

Concerning the time evolution of an initially factorized (or approximately factorized) state, preservation of condensation at later times in the sense of marginals as described above has been proved during the last 10-15 years mainly by Erdős-Schlein-Yau in a series of papers [13–16] under varying assumptions on the interac-

tion and the scaling parameter β . More precisely the solution ψ_N to (1.1) satisfies

$$\psi_N(t, x_1, \dots, x_N) \simeq \prod_{j=1}^N \phi(t, x_j) \quad (1.4)$$

in the sense that

$$\gamma_N^{(1)}(t, x, x') \rightarrow \phi(t, x) \bar{\phi}(t, x') \quad (1.5)$$

in trace norm as $N \rightarrow \infty$ where the limiting one-particle condensate wave function ϕ satisfies the Schrödinger equation

$$\frac{1}{i} \partial_t \phi = \Delta \phi - \begin{cases} (v * |\phi|^2) \phi, & \text{if } \beta = 0 \\ (\int v(x) dx) |\phi|^2 \phi, & \text{if } 0 < \beta < 1 \\ 8\pi a \downarrow |\phi|^2 \phi, & \text{if } \beta = 1 \\ \text{scattering length of } v \end{cases} \quad (1.6)$$

$$\phi(0, \cdot) = \phi_0.$$

Similarly for higher order marginals

$$\gamma_N^{(k)}(t, \mathbf{x}_k, \mathbf{x}'_k) := \int_{\mathbb{R}^{3(N-k)}} \psi_N(t, \mathbf{x}_k, \mathbf{x}_{N-k}) \bar{\psi}_N(t, \mathbf{x}'_k, \mathbf{x}_{N-k}) d\mathbf{x}_{N-k} \rightarrow \prod_{j=1}^k \phi(t, x_j) \bar{\phi}(t, x'_j)$$

in trace norm as $N \rightarrow \infty$. The strategy in the above mentioned papers was based on the work of Spohn [37] proving (1.4) for bounded potentials in case of $\beta = 0$ via BBGKY hierarchy. Recent simplifications and generalizations were given in [26], [7], [9, 10], [8].

One would also like to quantify the error in (1.5). Using the framework of the second quantization L. Chen-Lee-Schlein [6] (extending the techniques developed

by Rodnianski-Schlein [35]) and also Benedikter-Oliveira-Schlein [3] obtained the following results:

$$\mathrm{Tr} \left| \gamma_N^{(1)}(t, \cdot) - |\phi(t, \cdot)\rangle\langle\phi(t, \cdot)| \right| \lesssim \begin{cases} e^{Ct}/N & \text{by [6], for } \beta=0 \text{ and} \\ & \text{singular potentials} \\ & \text{including } v(x)=|x|^{-1}, \\ \frac{\exp(c_1 \exp(c_2 t))}{N^{1/4}} & \text{by [3], for } \beta=1 \text{ and} \\ & v \in L^1 \cap L^3(\mathbb{R}^3, (1+|x|^6)dx). \end{cases} \quad (1.7)$$

Convergence in the sense of marginals provides with partial information about the system since most of the variables are averaged out. Also, although the Hartree equation (corresp. to $\beta = 0$) or the cubic nonlinear Schrödinger equation provides a good description of the limiting behavior for the mean field represented by the condensate wave function, they fail to describe pair excitations i.e. the scattering of particles in pairs from the condensate to other states. Hence, Grillakis-Machedon-Margetis (GMM) [18, 19], inspired by but being different than that of Wu [39], introduced a Fock space approximation of the exact dynamics which considers pair excitations as a correction to the mean field. They obtained error bounds deteriorating more slowly in time compared to the exponential deterioration. Those results are valid for $\beta < 1/3$ with the assumptions in [18–20]. One of the issues we would like to address in this thesis is extending GMM-results to higher β values (i.e. stronger interactions with shorter range) under the same assumptions as in [20]. The other direction in this thesis is utilizing GMM-type Fock space estimates to improve the error bounds in (1.7). We will state and discuss our main results in subsections 1.3 and 1.4 after providing a brief overview of second quantization in the next section.

We will conclude this introductory chapter by an outline of the rest of this thesis.

1.2 Fock Space Formalism

1.2.1 Symmetric Fock Space

The *symmetric Fock space* is defined as

$$\mathbb{F} = \bigoplus_{n=0}^{\infty} \mathbb{F}_n; \quad \mathbb{F}_n = L_s^2(\mathbb{R}^{3n}) \text{ for } n > 0 \quad \text{and} \quad \mathbb{F}_0 = \mathbb{C}$$

containing vectors of the form

$$|\psi\rangle = (\psi_0, \psi_1, \psi_2, \dots)$$

and equipped with the inner product (linear in the first component)

$$\langle \psi | \phi \rangle = \psi_0 \bar{\phi}_0 + \sum_{n \geq 1} \int_{\mathbb{R}^{3n}} \psi_n(\mathbf{x}) \bar{\phi}_n(\mathbf{x}) d\mathbf{x}.$$

We will use the following notation for what is known to be the *vacuum state* with no particles:

$$|0\rangle = (1, 0, 0, \dots).$$

We define the *annihilation and creation operator-valued distributions* denoted by a_x and a_x^* and acting on Fock vectors of the form $(0, \dots, 0, \psi_{n+1}, 0, \dots)$ and

$(0, \dots, 0, \psi_{n-1}, 0, \dots)$ respectively as:

$$a_x(\psi_{n+1}) = \sqrt{n+1}\psi_{n+1}(x, x_1, \dots, x_n), \quad (1.8a)$$

$$a_x^*(\psi_{n-1}) = \frac{1}{\sqrt{n}} \sum_{j=1}^n \delta(x - x_j) \psi_{n-1}(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n). \quad (1.8b)$$

Note that they are adjoints of one another and they satisfy

$$[a_x, a_y] = [a_x^*, a_y^*] = 0, \quad [a_x, a_y^*] = \delta(x - y), \quad a_x|0\rangle = 0. \quad (1.9)$$

We also need to define the (unbounded) *number of particles operator*:

$$\begin{aligned} \mathcal{N} &:= \int dx a_x^* a_x \text{ acting as } \left(\mathcal{N}^\alpha |\psi\rangle \right)_n = n^\alpha \psi_n \\ &\text{for } \alpha > 0 \text{ and } |\psi\rangle \text{ satisfying } \sum_{n=1}^{\infty} n^{2\alpha} \|\psi_n\|_{L^2(\mathbb{R}^{3n})}^2 < \infty \end{aligned} \quad (1.10)$$

so that for $|\psi\rangle$ in Fock space with $\| |\psi\rangle \|_{\mathbb{F}} = 1$

$$\langle \psi | \mathcal{N} | \psi \rangle = \sum_{n=1}^{\infty} n \overbrace{\|\psi_n\|_{L^2(\mathbb{R}^{3n})}^2}^{\text{probability of having } n \text{ particles at } |\psi\rangle} \quad (1.11)$$

represents the *expectation of number of particles* in the state $|\psi\rangle$.

With the help of operator-valued distributions a_x, a_x^* defined in (1.8), we can introduce the unbounded, closed, densely-defined operators annihilating/creating particles at a state f :

$$a(\bar{f}) := \int dx \bar{f}(x) a_x \quad \text{and} \quad a^*(f) := \int dx f(x) a_x^* \quad \text{for } f \in L^2(\mathbb{R}^3). \quad (1.12)$$

We will need the following control of the annihilation $a(f)$ and the creation $a^*(f)$ operators in terms of the number of particles operator \mathcal{N} introduced in (1.10):

Lemma 1.1. *For $f \in L^2(\mathbb{R}^3)$, the following estimates hold:*

$$\|a(f)|\psi\rangle\| \leq \|f\|_2 \|\mathcal{N}^{1/2}|\psi\rangle\|, \quad (1.13a)$$

$$\|a^*(f)|\psi\rangle\| \leq \|f\|_2 \|(\mathcal{N} + 1)^{1/2}|\psi\rangle\|. \quad (1.13b)$$

Proof. (1.13a) follows by

$$\|a(f)|\psi\rangle\| = \left\| \int dx f(x) a_x |\psi\rangle \right\| \leq \int |f(x)| \|a_x |\psi\rangle\| dx \leq \|f\|_2 \left(\int \langle \psi | a_x^* a_x |\psi\rangle dx \right)^{1/2}$$

and recalling the definition of \mathcal{N} in (1.10). (1.13b) follows from

$$\begin{aligned} \|a^*(f)|\psi\rangle\|^2 &= \langle \psi | a(\bar{f}) a^*(f) |\psi\rangle \stackrel{\substack{\text{using the 2nd commutation} \\ \text{relation in (1.9)}}}{=} \langle \psi | \{ \|f\|_2^2 + a^*(f) a(\bar{f}) \} |\psi\rangle \\ &= \|f\|_2^2 \|\psi\rangle\|^2 + \|a(\bar{f})|\psi\rangle\|^2 \\ &\leq \|f\|_2^2 \|\psi\rangle\|^2 + \|f\|_2^2 \|\mathcal{N}^{1/2}|\psi\rangle\|^2 \end{aligned}$$

where the last inequality follows by (1.13a). □

1.2.2 Embedding the N -body Dynamics in Fock Space:

Fock Hamiltonian and Coherent States¹

Embedding the N -body dynamics in the Fock space representation provides with the advantage of considering all possible number of particles at the same time. One can then try to extract information for the relevant N -particle sector via projection. Also the algebra in Fock space is easier due to certain algebraic properties of the operators soon to be defined.

Let's first define the second quantization $d\Gamma(J)$ of a one-particle operator J acting on $L^2(\mathbb{R}^3)$:

$$(d\Gamma(J)|\psi\rangle)_n := \sum_{k=1}^n J_k \psi_n \quad (1.14)$$

where J_k denotes the action of J on ψ_n in the k th variable. If J has a corresponding kernel then we can write

$$d\Gamma(J) = \int J(x, y) a_x^* a_y dx dy.$$

Notice that $d\Gamma(\delta(x - y)) = \mathcal{N}$ recalling definition (1.10). We state a property of $d\Gamma(J)$ here for future reference:

Lemma 1.2. *For any $|\psi\rangle$ in the domain of \mathcal{N} and for any bounded operator J on*

¹Embedding the N -body system in Fock space and using coherent states as initial data was originally proposed by Hepp [25] to study the semi-classical limit of quantum many-body boson systems and then was extended by Ginibre-Velo [17] to singular potentials. This approach has been revived by [35] to obtain estimates on the rate of convergence to the limiting Hartree dynamics which further inspired [18, 19].

$L^2(\mathbb{R}^3)$, the following estimate holds:

$$\|\mathrm{d}\Gamma(J)|\psi\rangle\| \leq \|J\|_{\mathrm{op}}\|\mathcal{N}|\psi\rangle\|. \quad (1.15)$$

Proof. We can proceed as

$$\|\mathrm{d}\Gamma(J)|\psi\rangle\|^2 = \sum_{n=0}^{\infty} \left\langle \sum_{k=1}^n J_k \psi_n, \sum_{k=l}^n J_l \psi_n \right\rangle_{L^2} \leq \|J\|_{\mathrm{op}}^2 \sum_{n=0}^{\infty} n^2 \|\psi_n\|^2 = \|J\|_{\mathrm{op}}^2 \|\mathcal{N}|\psi\rangle\|^2$$

where $\|J\|_{\mathrm{op}} = \sup_{\{f \in L^2(\mathbb{R}^3): \|f\|_2=1\}} \|Jf\|_2$ i.e. the operator norm of J . \square

Similar to the definitions above, one can define the second quantization of a symmetric 2-particle kernel W as

$$\frac{1}{2} \int W(x, y; x', y') a_x^* a_y^* a_{x'} a_{y'} dx dy dx' dy'. \quad (1.16)$$

In order to embed the N -body system in the Fock space we need to define the *Fock Hamiltonian* acting on \mathbb{F} as

$$\mathcal{H} := \mathcal{H}_1 - N^{-1}\mathcal{V} \quad \text{where,} \quad (1.17a)$$

$$\mathcal{H}_1 := \mathrm{d}\Gamma(\Delta) = \int \Delta_x \delta(x - y) a_x^* a_y dx dy, \quad (1.17b)$$

$$\mathcal{V} := \frac{1}{2} \int v_N(x - y) a_x^* a_y^* a_x a_y dx dy \quad (1.17c)$$

$$\text{with } v_N(x) := N^{3\beta} v(N^\beta x) \quad (1.17d)$$

where we took $W(x, y; x', y') = v_N(x - y) \delta((x, y) - (x', y'))$ in (1.16) for the definition

of \mathcal{V} in (1.17c). \mathcal{H} is a diagonal operator on \mathbb{F} , acting on the n -particle sector as a regular n -body PDE Hamiltonian

$$H_{N,n} = \sum_{j=1}^n \Delta_{x_j} - \frac{1}{N} \sum_{j < k} v_N(x_j - x_k)$$

which is equal to H_N in (1.1) for $n = N$.

Before considering the initial value problem in the Fock space corresponding to the N -body equation (1.1) let's introduce the *coherent states* which we will use as our initial data. First define the skew-Hermitian operator:

$$\mathcal{A}(\phi_0) := \underbrace{\int dx \bar{\phi}_0(x) a_x}_{a(\bar{\phi}_0)} - \underbrace{\int dx \phi_0(x) a_x^*}_{a^*(\phi_0)} \quad \text{for } \phi_0 \in L^2(\mathbb{R}^3).$$

Setting $X = \sqrt{N} a^*(\phi_0)$ and $Y = -\sqrt{N} a(\bar{\phi}_0)$ for ϕ_0 with $\|\phi_0\|_2 = 1$ and using the Baker-Campbell-Hausdorff formula implying

$$e^{X+Y} = e^X e^Y e^{-\frac{1}{2}[X,Y]} \quad \text{for operators } X, Y \text{ with } [X, [X, Y]] = [Y, [X, Y]] = 0$$

together with $a|0\rangle = 0$, we can perform the following computation leading to an explicit formula for what is known to be the N -particle coherent state $e^{-\sqrt{N}\mathcal{A}(\phi_0)}|0\rangle$:

$$\begin{aligned} \overbrace{e^{-\sqrt{N}\mathcal{A}(\phi_0)}}^{\text{Weyl operator}} |0\rangle &= e^{-\frac{N}{2}} e^{\sqrt{N} a^*(\phi_0)} |0\rangle = e^{-\frac{N}{2}} \sum_{n=0}^{\infty} \frac{N^{n/2}}{n!} \overbrace{a^*(\phi_0)^n |0\rangle}^{(0, \dots, 0, \sqrt{n!} \phi_0^{\otimes n}, 0, \dots)} \\ &= \left(\dots, c_n \prod_{j=1}^n \phi_0(x_j), \dots \right) \quad \text{with } c_n = (e^{-N} N^n / n!)^{1/2} \end{aligned} \quad (1.18)$$

Coherent states, having a tensor product in each sector, are a generalization in Fock space of factorized states $\phi_0^{\otimes N}$ seen in (1.2). Some useful properties of them are stated in (iii)-(iv) of the following lemma:

Lemma 1.3. *Let $\phi \in L^2(\mathbb{R}^3)$.*

(i) *The Weyl operator $e^{\sqrt{N}\mathcal{A}(\phi)}$ is unitary.*

(ii) *We have*

$$e^{\sqrt{N}\mathcal{A}(\phi)} a_x e^{-\sqrt{N}\mathcal{A}(\phi)} = a_x + \sqrt{N}\phi(x), \quad (1.19)$$

$$e^{\sqrt{N}\mathcal{A}(\phi)} a_x^* e^{-\sqrt{N}\mathcal{A}(\phi)} = a_x^* + \sqrt{N}\bar{\phi}(x), \quad (1.20)$$

$$e^{\sqrt{N}\mathcal{A}(\phi)} \mathcal{N} e^{-\sqrt{N}\mathcal{A}(\phi)} = \mathcal{N} + \underbrace{\sqrt{N}\varphi(\phi)}_{a(\bar{\phi})+a^*(\phi)} + N\|\phi\|_2^2. \quad (1.21)$$

(iii) *Coherent states are eigenvectors of annihilation operators.*

(iv) *The expected number of particles in coherent state $e^{-\sqrt{N}\mathcal{A}(\phi_0)}|0\rangle$ with $\|\phi_0\|_2 = 1$ is $\langle 0|e^{\sqrt{N}\mathcal{A}(\phi_0)}\mathcal{N}e^{-\sqrt{N}\mathcal{A}(\phi_0)}|0\rangle = N$.*

Proof. (i) is implied by the fact that $\mathcal{A}(\phi) = a(\bar{\phi}) - a^*(\phi)$ is skew-Hermitian. (1.19) and (1.20) are adjoints of each other and either of them can be obtained by using the formula

$$e^{\mathcal{A}}\mathcal{H}e^{-\mathcal{A}} = \sum_{n=0}^{\infty} \frac{1}{n!} (\text{ad}_{\mathcal{A}})^n(\mathcal{H}) \text{ where } \text{ad}_{\mathcal{A}}(\mathcal{H}) := [\mathcal{A}, \mathcal{H}] \text{ for operators } \mathcal{A}, \mathcal{H} \quad (1.22)$$

and the commutation relation in (1.9). (1.21) can be obtained in the same way using

(1.22) or we can write

$$e^{\sqrt{N}\mathcal{A}(\phi)}\mathcal{N}e^{-\sqrt{N}\mathcal{A}(\phi)} = \int e^{\sqrt{N}\mathcal{A}(\phi)}\mathbf{a}_x^*e^{-\sqrt{N}\mathcal{A}(\phi)}e^{\sqrt{N}\mathcal{A}(\phi)}\mathbf{a}_xe^{-\sqrt{N}\mathcal{A}(\phi)}\mathbf{d}x$$

and then use (1.19)-(1.20). (iii) follows from (1.19) if we let both sides of (1.19) act on vacuum and use $a|0\rangle = 0$. (iv) is implied by (1.21) or we can recall (1.11) and use the explicit formula in (1.18) to compute the probability of having n particles at $e^{-\sqrt{N}\mathcal{A}(\phi_0)}|0\rangle$ as $c_n^2 = e^{-N}N^n/n!$ which implies a Poisson distribution with expected number of particles N . \square

Considering (1.17)-(1.18), the *initial value problem in the Fock space* can be written as:

$$\frac{1}{i}\partial_t|\psi\rangle = \mathcal{H}|\psi\rangle, \tag{1.23a}$$

$$|\psi(0)\rangle = e^{-\sqrt{N}\mathcal{A}(\phi_0)}|0\rangle \tag{1.23b}$$

so that on the N -particle sector we have the N -body equation (1.1) with the initial data $c_N\phi_0^{\otimes N}$ where $c_N = e^N N^N/N! \simeq (2\pi N)^{-1/4}$ has been estimated using Stirling's formula.

1.2.3 Mean Field Approximation and Second-order Corrections

The solution to the initial value problem (1.23a)-(1.23b) can be written formally as

$$|\psi_{\text{ex}}(t)\rangle = e^{it\mathcal{H}}e^{-\sqrt{N}\mathcal{A}(\phi_0)}|0\rangle \quad (\text{exact evolution}) \tag{1.24}$$

having $c_N e^{itH_N} \phi_0^{\otimes N}$ in the N -particle sector and the mean field evolution is described by

$$|\psi_{\text{MF}}(t)\rangle = e^{-\sqrt{N}\mathcal{A}(\phi(t))}|0\rangle \quad (1.25)$$

where ϕ satisfies

$$\frac{1}{i}\partial_t\phi - \Delta\phi + (v_N * |\phi|^2)\phi = 0 \text{ with } \phi(0, \cdot) = \phi_0. \quad (1.26)$$

However the mean field evolution does not track the exact dynamics in the Fock space norm. We can explain this briefly in the following way while referring to Section 3 in [18] for more details. We want to estimate

$$\| |\psi_{\text{ex}}\rangle - |\psi_{\text{MF}}\rangle \|_{\mathbb{F}} = \left\| \underbrace{e^{\sqrt{N}\mathcal{A}(\phi)} e^{it\mathcal{H}} e^{-\sqrt{N}\mathcal{A}(\phi_0)} |0\rangle}_{\text{define as } |\psi_1\rangle} - |0\rangle \right\|_{\mathbb{F}}. \quad (1.27)$$

To this aim we can compute the evolution for $|\psi_1\rangle$ in the above equation in a closed form as

$$\frac{1}{i}\partial_t|\psi_1\rangle = \left[\underbrace{\left(\frac{1}{i}\partial_t e^{\sqrt{N}\mathcal{A}(\phi)} \right) e^{-\sqrt{N}\mathcal{A}(\phi)} + e^{\sqrt{N}\mathcal{A}(\phi)} \mathcal{H} e^{-\sqrt{N}\mathcal{A}(\phi)}}_{\text{define as } L_1} \right] |\psi_1\rangle \quad (1.28)$$

which is equivalent to

$$\left(\frac{1}{i}\partial_t - L_1 \right) (|\psi_1\rangle - |0\rangle) = L_1|0\rangle.$$

Hence for the error in (1.27) to be small we need to prove that the forcing term $L_1|0\rangle$ in the last equation is small when measured in Fock space norm. L_1 in (1.28) has been explicitly computed in e.g. [18, 20] and shown to have the form

$$L_1 = N\mu_0 + N^{1/2}\mathcal{P}_1 + \mathcal{P}_2 + N^{-1/2}\mathcal{P}_3 + N^{-1}\mathcal{P}_4 \quad (1.29)$$

where \mathcal{P}_k stands for a polynomial of degree k in annihilation and creation operators (a, a^*) . Let's explain each term briefly:

- $N\mu_0 = (N/2)\langle v_N * |\phi|^2, |\phi|^2 \rangle_{L^2}$ in (1.29) is a zero order term w.r.t (a, a^*) . Despite being of $O(N)$ (i.e. not small), it can be absorbed as a phase factor.
- $\mathcal{P}_1 = \int dx \{h(t, x)a_x^* + \bar{h}(t, x)a_x\}$ with $h(t, x) = -(1/i)\partial_t\phi + \Delta\phi - (v_N * |\phi|^2)\phi$ and so drops out since ϕ satisfies the Hartree equation in (1.26).
- $\mathcal{P}_3 = - \int dx dy \left\{ v_N(x-y)(\phi(y)a_x^*a_y^*a_x + \text{h.c.}) \right\}$ has a^*a^*a - and a^*aa -terms so does not give any contribution when it acts on $|0\rangle$. The same is true for \mathcal{P}_4 which equals $-\mathcal{V}$, the potential part of the Fock Hamiltonian \mathcal{H} in (1.17).
- Finally

$$\begin{aligned} \mathcal{P}_2 = & d\Gamma \left(\left\{ \Delta_x - (v_N * |\phi|^2)(t, x) \right\} \delta(x-y) - v_N(x-y) \bar{\phi}(t, y) \phi(t, x) \right) \\ & - \frac{1}{2} \int dx dy \left\{ v_N(x-y) (\phi(t, x) \phi(t, y) a_x^* a_y^* + \text{h.c.}) \right\} \end{aligned}$$

first line of which is the second quantization of big-parenthesized terms and gives no contribution when acts on $|0\rangle$

Although several terms in $L_1|0\rangle$ drop out as explained, $\|L_1|0\rangle\|_{\mathbb{F}}$ will still not be small due to the presence of a^*a^* -term in \mathcal{P}_2 , which stands for two particles leaving the condensate and forming a pair $v_N(x-y)\phi(t,x)\phi(t,y)$ driving in turn the evolution of pair interactions. Hence the error (1.27) does not turn out to be small.

To circumvent the problem described above, in [18, 19], via the skew-Hermitian operator

$$\mathcal{B}(k) := \frac{1}{2} \int \{ \bar{k}(x, y) a_x a_y - k(x, y) a_x^* a_y^* \} dx dy, \quad (1.30)$$

a second order correction to the mean field $e^{-\sqrt{N}\mathcal{A}(\phi)}|0\rangle$ was introduced, namely, a state of the form

$$|\psi_{\text{ap}}\rangle := e^{iN\chi(t)} e^{-\sqrt{N}\mathcal{A}(\phi)} e^{-\mathcal{B}(k)} |0\rangle \quad (\text{approximate evolution}) \quad (1.31)$$

where $\chi(t)$ is an appropriately chosen phase function and $k(t, x, y)$ describes a pair of particles that scatter from condensate to other states. The dynamics of these pair excitations is computed in a way consistent with the N -body dynamics (1.1) which is explained more in section 2.1.

In order to write $e^{\mathcal{B}(k)}$ more explicitly we can consider the map \mathcal{I} from the space of complex L^2 symplectic matrices of the form

$$L := \begin{pmatrix} d(x, y) & k(x, y) \\ l(x, y) & -d(y, x) \end{pmatrix} \quad \text{where } d, k, l \in L^2(\mathbb{R}^6), \text{ and } k, l \text{ symmetric in } (x, y)$$

to quadratic expressions in (a, a^*) given by

$$\mathcal{I}(L) = \frac{1}{2} \int \left\{ (a_x, a_x^*) \begin{pmatrix} d(x, y) & k(x, y) \\ l(x, y) & -d(y, x) \end{pmatrix} \begin{pmatrix} -a_y^* \\ a_y \end{pmatrix} \right\} dx dy$$

which is the Lie algebra isomorphism considered in [18–20]. Then

$$K = \begin{pmatrix} 0 & \bar{k} \\ k & 0 \end{pmatrix} \xrightarrow{\mathcal{I}} \mathcal{B}(k) \quad \text{and} \quad \boxed{e^K = \begin{pmatrix} c & \bar{u} \\ u & \bar{c} \end{pmatrix} \mapsto e^{\mathcal{B}(k)}} \quad (1.32)$$

$$\text{with} \quad \begin{cases} u := \text{sh}(k) = k + \frac{1}{3!}k \circ \bar{k} \circ k + \dots, \\ c := \text{ch}(k) = \delta(x - y) + p = \delta(x - y) + \frac{1}{2!}\bar{k} \circ k + \dots \end{cases} \quad (1.33)$$

where $k \circ l$ needs to be understood in the following sense

$$(k \circ l)(x, y) := \int k(x, z)l(z, y)dz$$

for k and l symmetric Hilbert-Schmidt operators on $L^2(\mathbb{R}^3)$. Since $\mathcal{B}(k)$ is skew-Hermitian $e^{\mathcal{B}(k)}$ is unitary and therefore $e^K e^{-K} = I$ based on the correspondence in (1.32). This fact implies the following trigonometric identities which will play crucial role in our arguments:

$$c \circ c - \bar{u} \circ u = \delta(x - y) \quad \text{and} \quad u \circ c = \bar{c} \circ u. \quad (1.34)$$

Finally in this section we state for future reference the next lemma showing $e^{\mathcal{B}(k)}$ acts on annihilation and creation operators as a Bogoliubov transformation:

Lemma 1.4. $k \in L^2(\mathbb{R}^6)$ be symmetric in (x, y) . Also let $u := \text{sh}(k)$ and $c := \text{ch}(k)$ as in (1.33). Then

$$b_x := e^{\mathcal{B}(k)} a_x e^{-\mathcal{B}(k)} = \int (c(y, x) a_y + u(y, x) a_y^*) dy, \quad (1.35)$$

$$b_x^* := e^{\mathcal{B}(k)} a_x^* e^{-\mathcal{B}(k)} = \int (\bar{u}(y, x) a_y + \bar{c}(y, x) a_y^*) dy \quad (1.36)$$

and $[b_x, b_y^*] = \delta(x - y)$, $[b_x, b_y] = [b_x^*, b_y^*] = 0$.

Proof. For $f, g \in L^2(\mathbb{R}^3)$, we obtain the following

$$\begin{aligned} & e^{\mathcal{B}(k)} \left\{ \int (a_y, a_y^*) \begin{pmatrix} f(y) \\ g(y) \end{pmatrix} dy \right\} e^{-\mathcal{B}(k)} \\ &= e^{\mathcal{I}(K)} \left\{ \int (a_y, a_y^*) \begin{pmatrix} f(y) \\ g(y) \end{pmatrix} dy \right\} e^{-\mathcal{I}(K)} \\ & \quad \uparrow \\ & \quad \text{recalling (1.32)} \\ &= \int \left\{ (a_y, a_y^*) e^{K(y,z)} \begin{pmatrix} f(z) \\ g(z) \end{pmatrix} \right\} dy dz \end{aligned} \quad (1.37)$$

where the last step can be justified by the use of (1.22) (see e.g. Sect. 7 in [20]).

We obtain (1.35) if $f(z) = \delta(x - z)$ and $g(z) \equiv 0$ in (1.37). (1.36) is the adjoint of (1.35). Finally the commutation relations follow using trigonometric identities in (1.34). \square

1.3 Error Estimates in Fock Space for Stronger Interaction:

The case of $v_N = N^{3\beta}v(N^\beta \cdot)$ with $\beta < 1/2$

In this section we will state one of our main results. It is an improvement of the following result obtained via the GMM-approximation scheme introduced in (1.31).

Theorem 1.5. [19, 20] *Let ϕ and k seen in (1.31) satisfy*

$$\frac{1}{i}\partial_t\phi - \Delta\phi + (v_N * |\phi|^2)\phi = 0, \quad (1.38a)$$

$$\frac{1}{i}\partial_t\text{sh}(2k) + g^T \circ \text{sh}(2k) + \text{sh}(2k) \circ g = m \circ \text{ch}(2k) + \overline{\text{ch}(2k)} \circ m, \quad (1.38b)$$

$$\frac{1}{i}\partial_t\overline{\text{ch}(2k)} + [g^T, \overline{\text{ch}(2k)}] = m \circ \overline{\text{sh}(2k)} - \text{sh}(2k) \circ \bar{m}, \quad (1.38c)$$

where

$$g(t, x, y) := \left(-\Delta_x + (v_N * |\phi|^2)(t, x) \right) \delta(x - y) + v_N(x - y) \bar{\phi}(t, x) \phi(t, y), \quad (1.39)$$

$$m(t, x, y) := -v_N(x - y) \phi(t, x) \phi(t, y), \quad (1.40)$$

with prescribed initial conditions $\phi(0, \cdot) = \phi_0$, $k(0, \cdot, \cdot) = 0$. Then for some real phase function χ the following estimate holds:

$$\left\| \left| \psi_{\text{ex}}(t) \right\rangle - \left| \psi_{\text{ap}}(t) \right\rangle \right\|_{\mathbb{F}} \lesssim \begin{cases} \frac{\sqrt{1+t}}{\sqrt{N}} & \text{if } \beta=0 \text{ and } v(x)=\xi(x)|x|^{-1} \text{ for some } \xi \in C_0^\infty \text{ decreasing} \\ \frac{(1+t) \log^4(1+t)}{N^{(1-3\beta)/2}} & \text{if } 0 < \beta < 1/3 \text{ and } v(x) \text{ bounded, integrable.} \end{cases}$$

Our main result in this section extends the error estimates in Theorem 1.5 to

the case of $\beta < 1/2$:

Theorem 1.6. [28] *If ϕ and k seen in (1.31) satisfy the equations in (1.38) with prescribed initial data $\phi(0, \cdot) = \phi_0$ and $k(0, \cdot, \cdot) = 0$ and if v is bounded and integrable, then the following estimate holds:*

$$\begin{aligned} & \left\| |\psi_{\text{ex}}(t)\rangle - |\psi_{\text{ap}}(t)\rangle \right\|_{\mathbb{F}} \\ & \lesssim_{\epsilon, j} t^{\frac{j+3}{2}} \log^6(1+t) \cdot \begin{cases} N^{-\frac{1}{2} + \beta(1+\epsilon)} & \text{for } 0 < \beta \leq \frac{2j}{(1-2\epsilon+4j)}, \\ N^{-\frac{3+7\beta}{2} + (j-1)(-1+2\beta)} & \text{for } \frac{1+2j}{3+4j} > \beta > \frac{2j}{(1-2\epsilon+4j)}. \end{cases} \end{aligned} \quad (1.41)$$

The above estimate implies a decay as $N \rightarrow \infty$ for β as close as desired to $1/2$ if we choose first ϵ sufficiently small and then j sufficiently large depending on ϵ .

Remark 1.7.

(i) *Note that the bound in Theorem 1.6 gives a faster decay rate w.r.t. N for the case of $0 < \beta < 1/3$ compared to that of Theorem 1.5 but the error in Theorem 1.6 grows faster in time. Nevertheless it is still less than the exponential growth typical of previous works.*

(ii) *We also claim that with the uncoupled system given in (1.38) one can go only as far as $\beta < 1/2$ in terms of Fock space estimates of the type presented above. A heuristic argument supporting this claim will be provided in the next chapter. We note that [22] extended the estimates to the case of $\beta < 2/3$ (but locally in time) by considering a coupled system introduced in [21] instead of (1.38) we used for our results. Also, similar Fock space estimates have been obtained*

in [4] for $\beta \in (0, 1)$ using a certain class of initial data and an explicit choice of pair excitation function k . However the dependence of the error bounds on time in [4] is exponential.

(iii) Estimates of the type presented above have implications for N -particle wave function $\psi_N = e^{itH_N} \phi_0^{\otimes N}$ if we consider projection P_N onto the N -particle sector. Recalling that $P_N |\psi_{\text{ex}}(t)\rangle = c_N e^{itH_N} \phi_0^{\otimes N}$ with $c_N = O(N^{-1/4})$ (see (1.23)), we have

$$\left\| \psi_N - \frac{1}{c_N} P_N |\psi_{\text{ap}}\rangle \right\|_{L^2(\mathbb{R}^{3N})} \lesssim N^{1/4} \left\| |\psi_{\text{ex}}(t)\rangle - |\psi_{\text{ap}}(t)\rangle \right\|_{\mathbb{F}}.$$

Hence inserting estimate in Theorems 1.5 and 1.6 into the above inequality gives

$$\left\| \psi_N - \frac{1}{c_N} P_N |\psi_{\text{ap}}\rangle \right\|_{L^2(\mathbb{R}^{3N})} \sim O(N^{\beta-1/4}) \text{ for } 0 \leq \beta < 1/2$$

which implies a decay as long as $\beta < 1/4$. We do not know what $P_N |\psi_{\text{ap}}\rangle = e^{iN\chi(t)} P_N e^{-\sqrt{N}A(\phi)} e^{-\mathcal{B}(k)} |0\rangle$ exactly equals but the idea in considering such an approximation is that the N -particle sector should roughly look like

$$\phi(t, x_1) \dots \phi(t, x_N) \prod_{j,k} f(t, x_j, x_k)$$

where f is a function describing particle correlations.

1.4 Error Estimates in the Sense of Marginals:

Rate of Convergence to the Limiting Mean Field

For an N -particle wave function ψ_N we can express the one-particle marginal using (a, a^*) as

$$\begin{aligned}\gamma_N^{(1)}(x, y) &:= \int_{\mathbb{R}^{3(N-1)}} \psi_N(x, \mathbf{x}_{N-1}) \bar{\psi}_N(y, \mathbf{x}_{N-1}) d\mathbf{x}_{N-1} \\ &= \frac{1}{N} \langle \psi_N, a_x^* a_y \psi_N \rangle_{L^2(\mathbb{R}^{3(N-1)})}.\end{aligned}\tag{1.42}$$

For Fock space marginals, this generalizes to

$$\Gamma_{|\psi\rangle}^{(1)}(x, y) = \frac{1}{\langle \psi | \mathcal{N} | \psi \rangle} \langle \psi | a_x^* a_y | \psi \rangle \quad \text{for } |\psi\rangle \in \mathbb{F} \text{ with } \underbrace{\langle \psi | \mathcal{N} | \psi \rangle}_{\substack{\text{finite particle expectation} \\ \text{see (1.11)}}} < \infty \tag{1.43}$$

which agrees with (1.42) for an N -particle state $|\psi\rangle = (0, \dots, 0, \psi_N, 0, \dots)$.

Before stating our main results in this section let's also recall that $\text{Tr}|\cdot|$ denotes the trace norm on the space of trace class operators $\mathcal{L}_1(L^2(\mathbb{R}^3))$ on $L^2(\mathbb{R}^3)$ i.e. $\text{Tr}|A| = \text{Tr}((A^*A)^{1/2})$ for A satisfying $\sum_{f \in \mathcal{F}} \langle |A|f, f \rangle_{L^2(\mathbb{R}^3)} < \infty$ for any \mathcal{F} orthonormal basis of $L^2(\mathbb{R}^3)$.

Our main results in this section are the following:

Theorem 1.8. (partly from [27]) *Let $|\psi_{\text{ex}}(t)\rangle = e^{it\mathcal{H}} e^{-\sqrt{N}\mathcal{A}(\phi_0)} |0\rangle$ as in (1.24) where \mathcal{H} denotes the Fock Hamiltonian (1.17) and $\Gamma_{\text{ex}}^{(1)}(t, x, y) = \frac{\langle \psi_{\text{ex}} | a_x^* a_y | \psi_{\text{ex}} \rangle}{\langle \psi_{\text{ex}} | \mathcal{N} | \psi_{\text{ex}} \rangle}$ according to definition (1.43). Let $\sigma = 1/2$ for $0 < \beta \leq 1/6$ and $\sigma = 1 - 3\beta$ for $1/6 < \beta < 1/3$. Also for ϵ arbitrarily small let j be as large as to satisfy $2j/(1 - 2\epsilon + 4j) < 1/(2(1 + \epsilon))$.*

Then the following estimate holds:

$$\mathrm{Tr}|\Gamma_{\mathrm{ex}}^{(1)}(t) - |\phi(t)\rangle\langle\phi(t)|| \lesssim \begin{cases} \frac{1+t}{N} & \text{if } \beta=0 \text{ and } v(x)=\xi(|x|)/|x|, \xi \in C_0^\infty \text{ decreasing cutoff} \\ \frac{(1+t)^2 \log^{16}(1+t)}{N^\sigma} & \text{if } 0 < \beta < 1/3 \text{ and } v \text{ is bounded, integrable} \\ \frac{t^{j+3} \log^{20}(1+t)}{N^{1-2\beta(1+\epsilon)}} & \text{if } 1/3 \leq \beta \leq \frac{2j}{1-2\epsilon+4j} \text{ and } v \text{ bounded, integrable} \end{cases}$$

where ϕ solves

$$\frac{1}{i}\partial_t\phi = \Delta\phi - \begin{cases} (v * |\phi|^2)\phi, & \beta = 0 \\ (\int v(x)dx)|\phi|^2\phi, & 0 < \beta < 1/3 \end{cases} \quad (1.44)$$

with $\phi(0, \cdot) = \phi_0$ satisfying $\phi_0 \in H^1(\mathbb{R}^3)$ and also in $W^{l,1}(\mathbb{R}^3)$ for $l \geq 2$ in case $0 < \beta < 1/3$.

Theorem 1.9. (partly from [27]) Let $\psi_N(t) = e^{itH_N}\phi_0^{\otimes N}$ where H_N denotes the N -body Hamiltonian defined in (1.1) and

$$\gamma_N^{(1)}(t, x, y) = \int_{\mathbb{R}^{3(N-1)}} \psi_N(t, x, \mathbf{x}_{N-1}) \overline{\psi_N(t, y, \mathbf{x}_{N-1})} d\mathbf{x}_{N-1}.$$

Then we have

$$\mathrm{Tr}|\gamma_N^{(1)}(t) - |\phi(t)\rangle\langle\phi(t)|| \lesssim \begin{cases} \frac{\sqrt{1+t}}{N^{1/4}} & \text{if } \beta=0 \text{ and } v(x)=\xi(|x|)/|x|, \xi \in C_0^\infty \text{ decreasing cutoff} \\ \frac{(1+t) \log^8(1+t)}{N^{(1-6\beta)/4}} & \text{if } 0 < \beta < 1/6 \text{ and } v \text{ is bounded, integrable} \\ \frac{t^{(j+3)/2} \log^{10}(1+t)}{N^{1/4-\beta(1+\epsilon)}} & \text{if } 1/6 \leq \beta < \frac{1}{4(1+\epsilon)}, v \text{ bounded, integrable} \end{cases}$$

where $\epsilon > 0$ is as small as desired, j sufficiently large to satisfy $2j/(1-2\epsilon+4j) <$

$1/(2+2\epsilon)$ and ϕ solves (1.44) with ϕ_0 satisfying the same assumptions as in Theorem 1.8.

Remark 1.10.

- (i) Estimates similar to the ones in Theorem 1.8 have been obtained in [35] for $\beta = 0$ and $v(x) = |x|^{-1}$ and in [3] for $\beta = 1$ and more regular potentials. The error was of $O(N^{-1})$ in [35] which is known to be the optimal rate of convergence. We obtained the same rate and additionally our error estimate grows more slowly in time compared to the exponential growth of [35]. However we had to use the cut-offed Coulomb potential since our main tool in proving Theorems 1.8 and 1.9 was the Fock space estimates of the previous section which required faster decay at infinity than that of $|x|^{-1}$ in case of $\beta = 0$.
- (ii) We consider in Theorem 1.8 with initial data of the form $e^{-\sqrt{N}\mathcal{A}(\phi_0)}|0\rangle$ for suitable ϕ_0 . One might also think of replacing the vacuum $|0\rangle$ in $e^{-\sqrt{N}\mathcal{A}(\phi_0)}|0\rangle$ with a more general Fock space vector $|\psi\rangle$ (with only few particles) satisfying $\langle\psi|\mathcal{N}|\psi\rangle \leq C$ for some constant C . The main tool in the proof of Theorem 1.8 is Theorem 1.5 which also holds for initial data of the form $e^{-\sqrt{N}\mathcal{A}(\phi_0)}|\psi\rangle$ if we take $|\psi\rangle = e^{-\mathcal{B}(k_0)}|0\rangle$ with a symmetric $k_0 = k(0, \cdot) \in L^2(\mathbb{R}^6)$ to be prescribed such that $\langle\psi|\mathcal{N}|\psi\rangle = \langle 0|e^{\mathcal{B}(k_0)}\mathcal{N}e^{-\mathcal{B}(k_0)}|0\rangle = \|\text{sh}(k_0)\|_{L^2(\mathbb{R}^6)}^2$ (last equality follows by Lemma 1.4) is of $O(1)$ w.r.t. N . The case of initial data of the form $e^{-\sqrt{N}\mathcal{A}(\phi_0)}|\psi\rangle$ with a more general $|\psi\rangle$ remains to be investigated
- (iii) As can be seen in Theorem 1.9, in projecting onto the N -particle space there is some loss in the power of N which prevents obtaining a bound of $O(N^{-1})$

in case of $\beta = 0$. This was achieved in [6] but their bound was meaningful for times of order $\log(o(1)N)$ whereas our bound shows that the mean field approximation stays close to the exact dynamics at least for up to times of order $o(1)\sqrt{N}$.

(iv) The idea of the proof for the results of this section is to bound the trace norm of the difference between $\Gamma_{\text{ex}}^{(1)}$ (or $\gamma_N^{(1)}$) and their approximation by the number of particles at a reduced dynamics expected to be close to the vacuum. One then has to control the particle expectation in this reduced dynamics, which has been done so far via energy estimates being typical of [35], [6], [3]. The novelty of our work is in controlling particle expectation of the appropriately-defined reduced dynamics via the error in Fock space approximation on which we have a control from Theorems 1.5 and 1.6.

1.5 Outline of the Rest of the Thesis

The rest of this thesis is devoted to proving Theorems 1.6, 1.8 and 1.9. First in chapter 2 we prove the Fock space estimate of Theorem 1.6 and then in chapter 3 we will establish Theorems 1.8 and 1.9 on error estimates in the sense of marginals.

Chapter 2: Proof of Fock Space Estimate in Theorem 1.6

2.1 Preliminaries

The proof of Theorem 1.6 is based on estimating the deviation of the evolution from the vacuum state defined as

$$|\tilde{\psi}(t)\rangle := e^{-iN\chi(t)}|\psi_{\text{red}}(t)\rangle - |0\rangle \quad \text{where} \quad (2.1a)$$

$$|\psi_{\text{red}}(t)\rangle := e^{\mathcal{B}(k(t))} e^{\sqrt{N}\mathcal{A}(\phi(t))} \underbrace{e^{it\mathcal{H}} e^{-\sqrt{N}\mathcal{A}(\phi_0)} |0\rangle}_{|\psi_{\text{ex}}(t)\rangle} \quad (\text{reduced dynamics}) \quad (2.1b)$$

which satisfies

$$\| |\tilde{\psi}(t)\rangle \|_{\mathbb{F}} = \| |\psi_{\text{ex}}(t)\rangle - \overbrace{e^{iN\chi(t)} e^{-\sqrt{N}\mathcal{A}(\phi(t))} e^{-\mathcal{B}(k(t))} |0\rangle}^{|\psi_{\text{ap}}(t)\rangle} \|_{\mathbb{F}}$$

due to $e^{-\sqrt{N}\mathcal{A}}$ and $e^{-\mathcal{B}}$ being unitary. We can obtain the evolution for $|\tilde{\psi}\rangle$ as follows.

A straightforward computation gives the evolution of the reduced dynamics:

$$\frac{1}{i} \partial_t |\psi_{\text{red}}\rangle = \mathcal{H}_{\text{red}} |\psi_{\text{red}}\rangle \quad (2.2)$$

where

$$\mathcal{H}_{\text{red}} := \frac{1}{i}(\partial_t e^{\mathcal{B}})e^{-\mathcal{B}} + e^{\mathcal{B}} \left(\frac{1}{i}(\partial_t e^{\sqrt{N}\mathcal{A}})e^{-\sqrt{N}\mathcal{A}} + e^{\sqrt{N}\mathcal{A}}\mathcal{H}e^{-\sqrt{N}\mathcal{A}} \right) e^{-\mathcal{B}}. \quad (2.3)$$

As shown in section 2 of [20], if (1.38) holds, then

$$\mathcal{H}_{\text{red}} = N \underbrace{\mu(t)}_{\substack{\text{related} \\ \text{to } \chi \text{ via} \\ \chi = f \mu}} + \int dx dy \left\{ \underbrace{L(t, x, y)}_{\substack{\Delta_x \delta(x-y) \\ \text{+some self-adjoint} \\ \text{potential terms}}} a_x^* a_y \right\} - N^{-1/2} \underbrace{\mathcal{E}(t)}_{\substack{\text{a self-adjoint error} \\ \text{term being sum of} \\ \text{polynomials in } (a, a^*) \text{ of degree} \\ \text{one up to four}}}. \quad (2.4)$$

(2.4), (2.1a), (2.2) and the fact that $a|0\rangle = 0$ implies

$$\boxed{\left(\frac{1}{i} \partial_t - \mathcal{L} \right) |\tilde{\psi}\rangle = \underbrace{-N^{-1/2} \mathcal{E}(t) |0\rangle}_{=:(0, F_1, F_2, F_3, F_4, 0, \dots)} \text{ with } |\tilde{\psi}(0)\rangle = 0} \quad (2.5)$$

The integral term in (2.5) is the second quantization of the self-adjoint one-particle operator $L(t, x, y)$ which can be considered to be the sum of some kinetic and “potential” parts as follows:

$$\begin{aligned} L(t, x, y) := & \overbrace{\Delta_x \delta(x-y) - (v_N * |\phi|^2)(t, x) \delta(x-y) - v_N(x-y) \phi(t, x) \bar{\phi}(t, y)}^{-g(t, y, x) \text{ as defined in (1.39)}} \\ & + \frac{1}{2} \left(\underbrace{(\bar{c})^{-1}}_{\downarrow \text{ch}(k)} \circ \underbrace{m \circ \bar{u}}_{\downarrow -v_N(x-y)\phi(t,x)\phi(t,y) \text{ from (1.40)}} + \underbrace{u \circ \bar{m}}_{\downarrow \text{sh}(k)} \circ (\bar{c})^{-1} + \underbrace{[\mathbf{W}(\bar{c}), (\bar{c})^{-1}]}_{\downarrow \frac{1}{i} \partial_t \bar{c} + [g^T, \bar{c}] \text{ see top line for } g} \right). \end{aligned} \quad (2.6)$$

Before explaining the forcing term in (2.5), let's see in the next section how the error term in (2.4) looks like.

2.1.1 The Error Term $N^{-1/2}\mathcal{E}(t)$ in \mathcal{H}_{red} in (2.4)

$N^{-1/2}\mathcal{E}(t)$ in (2.4) is an error term containing polynomials in (a, a^*) up to degree four. It is a self-adjoint operator which can be written as:

$$N^{-1/2}\mathcal{E}(t) = \sum_{j=1}^4 \left\{ \mathcal{E}_j(t) + \overset{\substack{\text{denotes} \\ \text{adjoint}}}{\downarrow} \mathcal{E}_j^*(t) \right\} + \mathcal{E}_2^{\text{sa}}(t) + \mathcal{E}_4^{\text{sa}}(t) \quad (2.7)$$

where $\mathcal{E}_j(t)$ denotes a contribution consisting terms of degree j . We also have some self-adjoint contributions consisting terms of 2nd and 4th degrees and denoted by $\mathcal{E}_2^{\text{sa}}(t)$ and $\mathcal{E}_4^{\text{sa}}(t)$. The explicit forms of these terms needs to be given here for future reference.¹ Using the notation $\mathcal{D}_{xy} := a_x^* a_y$, $\mathcal{Q}_{xy}^* := a_x^* a_y^*$, $\mathcal{Q}_{xy} := a_x a_y$ and suppressing the time dependence of the functions ϕ , $c := \text{ch}(k) = \delta(x - y) + p$ and $u := \text{sh}(k)$ (recalling (1.33)) we have

$$\mathcal{E}_1(t) := N^{-1/2} \int dx_1 dx_2 dy_1 \left\{ (u \circ c)(x_1, x_2) v_N(x_1 - x_2) \bar{\phi}(x_2) \bar{u}(y_1, x_1) a_{y_1} \right. \quad (2.8a)$$

$$\left. + c(y_1, x_1) v_N(x_1 - x_2) \phi(x_2) (c \circ \bar{u})(x_1, x_2) a_{y_1} \right\} \quad (2.8b)$$

$$\mathcal{E}_2(t) := \frac{1}{2N} \int dx_1 dx_2 dy_1 dy_2 \left\{ (\bar{u} \circ \bar{c})(x_1, x_2) v_N(x_1 - x_2) c(y_1, x_1) u(x_2, y_2) \mathcal{D}_{y_2 y_1} \right. \quad (2.8c)$$

$$\left. + (\bar{u} \circ \bar{c})(x_1, x_2) v_N(x_1 - x_2) u(y_1, x_1) \bar{c}(x_2, y_2) \mathcal{D}_{y_1 y_2} \right. \quad (2.8d)$$

$$\left. + (\bar{u} \circ \bar{c})(x_1, x_2) v_N(x_1 - x_2) c(y_1, x_1) \bar{c}(x_2, y_2) \mathcal{Q}_{y_1 y_2} \right. \quad (2.8e)$$

$$\left. + (u \circ c)(x_1, x_2) v_N(x_1 - x_2) \bar{u}(y_1, x_1) \bar{u}(x_2, y_2) \mathcal{Q}_{y_1 y_2} \right\} \quad (2.8f)$$

¹See Section 5, [20] for the computations leading to this explicit form of $N^{-1/2}\mathcal{E}(t)$.

terms involved
are self-adjoint

$$\mathcal{E}_2^{\text{sa}}(t) := \frac{1}{2N} \int dx_1 dx_2 dy_1 dy_2$$

$$\left\{ (u \circ \bar{u})(x_1, x_2) v_N(x_1 - x_2) \bar{u}(y_1, x_1) u(x_2, y_2) \mathcal{D}_{y_2 y_1} \right. \quad (2.8g)$$

$$\left. + 2(u \circ \bar{u})(x_1, x_1) v_N(x_1 - x_2) \bar{u}(y_2, x_2) u(y_1, x_2) \mathcal{D}_{y_1 y_2} \right\} \quad (2.8h)$$

$$\mathcal{E}_3(t) := N^{-1/2} \int dx_1 dx_2 dy_1 dy_2 dy_3$$

$$\left\{ \bar{u}(y_1, x_1) v_N(x_1 - x_2) \phi(x_2) c(x_2, y_2) c(y_3, x_1) \mathcal{D}_{y_2 y_1} a_{y_3} \right. \quad (2.8i)$$

$$\left. + \bar{c}(y_1, x_1) v_N(x_1 - x_2) \phi(x_2) \bar{u}(x_2, y_2) c(y_3, x_1) \mathcal{D}_{y_1 y_2} a_{y_3} \right. \quad (2.8j)$$

$$\left. + \bar{c}(y_1, x_1) v_N(x_1 - x_2) \bar{\phi}(x_2) c(y_2, x_1) \bar{c}(x_2, y_3) a_{y_1}^* \mathcal{Q}_{y_2 y_3} \right. \quad (2.8k)$$

$$\left. + \bar{u}(y_1, x_1) v_N(x_1 - x_2) \phi(x_2) \bar{u}(x_2, y_2) c(y_3, x_1) \mathcal{Q}_{y_1 y_2} a_{y_3} \right. \quad (2.8l)$$

$$\left. + \bar{u}(y_1, x_1) v_N(x_1 - x_2) \bar{\phi}(x_2) u(y_2, x_1) \bar{c}(x_2, y_3) a_{y_1} \mathcal{D}_{y_2 y_3} \right. \quad (2.8m)$$

$$\left. + \bar{u}(y_1, x_1) v_N(x_1 - x_2) \bar{\phi}(x_2) c(y_2, x_1) u(x_2, y_3) a_{y_1} \mathcal{D}_{y_3 y_2} \right. \quad (2.8n)$$

$$\left. + \bar{u}(y_1, x_1) v_N(x_1 - x_2) \bar{\phi}(x_2) c(y_2, x_1) \bar{c}(x_2, y_3) a_{y_1} \mathcal{Q}_{y_2 y_3} \right. \quad (2.8o)$$

$$\left. + \bar{u}(y_1, x_1) v_N(x_1 - x_2) \phi(x_2) \bar{u}(x_2, y_2) u(y_3, x_1) \mathcal{Q}_{y_1 y_2} a_{y_3}^* \right\} \quad (2.8p)$$

$$\mathcal{E}_4(t) := \frac{1}{2N} \int dx_1 dx_2 dy_1 dy_2 dy_3 dy_4$$

$$\left\{ \bar{u}(y_1, x_1) c(x_2, y_2) v_N(x_1 - x_2) c(y_3, x_1) u(x_2, y_4) \mathcal{D}_{y_2 y_1} \mathcal{D}_{y_4 y_3} \right. \quad (2.8q)$$

$$\left. + \bar{c}(y_1, x_1) \bar{u}(x_2, y_2) v_N(x_1 - x_2) c(y_3, x_1) \bar{c}(x_2, y_4) \mathcal{D}_{y_1 y_2} \mathcal{Q}_{y_3 y_4} \right. \quad (2.8r)$$

$$\left. + \bar{u}(y_1, x_1) \bar{u}(x_2, y_2) v_N(x_1 - x_2) c(y_3, x_1) u(x_2, y_4) \mathcal{Q}_{y_1 y_2} \mathcal{D}_{y_4 y_3} \right. \quad (2.8s)$$

$$\left. + \bar{u}(y_1, x_1) c(x_2, y_2) v_N(x_1 - x_2) c(y_3, x_1) \bar{c}(x_2, y_4) \mathcal{D}_{y_2 y_1} \mathcal{Q}_{y_3 y_4} \right. \quad (2.8t)$$

$$\left. + \bar{u}(y_1, x_1) \bar{u}(x_2, y_2) v_N(x_1 - x_2) u(y_3, x_1) \bar{c}(x_2, y_4) \mathcal{Q}_{y_1 y_2} \mathcal{D}_{y_3 y_4} \right. \quad (2.8u)$$

$$\left. + \bar{u}(y_1, x_1) \bar{u}(x_2, y_2) v_N(x_1 - x_2) c(y_3, x_1) \bar{c}(x_2, y_4) \mathcal{Q}_{y_1 y_2} \mathcal{Q}_{y_3 y_4} \right\} \quad (2.8v)$$

$$\mathcal{E}_4^{\text{sa}}(t) := \frac{1}{2N} \int dx_1 dx_2 dy_1 dy_2 dy_3 dy_4$$

$$\left\{ \bar{c}(y_1, x_1) \bar{u}(x_2, y_2) v_N(x_1 - x_2) c(y_3, x_1) u(x_2, y_4) \mathcal{D}_{y_1 y_2} \mathcal{D}_{y_4 y_3} \right. \quad (2.8w)$$

$$+ \bar{u}(y_1, x_1) c(x_2, y_2) v_N(x_1 - x_2) u(y_3, x_1) \bar{c}(x_2, y_4) \mathcal{D}_{y_2 y_1} \mathcal{D}_{y_3 y_4} \quad (2.8x)$$

$$+ \bar{c}(y_1, x_1) c(x_2, y_2) v_N(x_1 - x_2) c(y_3, x_1) \bar{c}(x_2, y_4) \mathcal{Q}_{y_1 y_2}^* \mathcal{Q}_{y_3 y_4} \quad (2.8y)$$

$$+ \bar{u}(y_1, x_1) \bar{u}(x_2, y_2) v_N(x_1 - x_2) u(y_3, x_1) u(x_2, y_4) \mathcal{D}_{y_3 y_1} \mathcal{D}_{y_4 y_2} \left. \right\}. \quad (2.8z)$$

We will estimate the above terms in various ways to be explained later.

2.1.2 The Forcing Term $N^{-1/2} \mathcal{E}(t) |0\rangle$ in (2.5)

Based on the explicit form of $N^{-1/2} \mathcal{E}(t)$ given by (2.7)-(2.8) and recalling $c := ch(k) = \delta(x - y) + p$, the sectors of the forcing term $-N^{-1/2} \mathcal{E}(t) |0\rangle$ in (2.5) can be computed (up to symmetrization in the 2nd, 3rd and 4th sectors) as²:

- Sector \mathbb{F}_1 :

$$F_1(t, y_1) := -N^{-1/2} \left(\int dx_1 dx_2 v_N(x_1 - x_2) \left\{ u(y_1, x_2) (\bar{u} \circ u)(x_1, x_1) \bar{\phi}(x_2) \right. \right. \quad (2.9a)$$

$$+ \bar{p}(y_1, x_2) (u \circ \bar{u})(x_1, x_1) \phi(x_2) \quad (2.9b)$$

$$+ u(y_1, x_1) (\bar{u} \circ u)(x_1, x_2) \bar{\phi}(x_2) \quad (2.9c)$$

$$+ \bar{p}(y_1, x_1) (\bar{p} \circ u)(x_1, x_2) \bar{\phi}(x_2) \quad (2.9d)$$

$$+ \bar{p}(y_1, x_1) (u \circ \bar{u})(x_1, x_2) \phi(x_2) \quad (2.9e)$$

$$+ u(y_1, x_1) (\bar{u} \circ \bar{p})(x_1, x_2) \phi(x_2) \quad (2.9f)$$

²The main idea of this computation is to commute a (if there is any), to the right hand side, with a^* operators in those terms in (2.7) which do not annihilate the vacuum. This produces some lower order terms (contributions of which we see in (2.9a)-(2.12d)) and terms which annihilate $|0\rangle$ since $a|0\rangle = 0$. See Section 5, [20] for the details.

$$+ \bar{p}(y_1, x_1)u(x_1, x_2)\bar{\phi}(x_2) \quad (2.9g)$$

$$+ u(y_1, x_1)\bar{u}(x_1, x_2)\phi(x_2) \quad (2.9h)$$

$$+ \int dx_1 v_N(y_1 - x_1) \left\{ u(y_1, x_1)\bar{\phi}(x_1) \quad (2.9i)$$

$$+ (u \circ \bar{u})(y_1, x_1)\phi(x_1) \quad (2.9j)$$

$$+ (\bar{p} \circ u)(y_1, x_1)\bar{\phi}(x_1) \quad (2.9k)$$

$$+ (u \circ \bar{u})(x_1, x_1)\phi(y_1) \left. \right\} \quad (2.9l)$$

• Sector \mathbb{F}_2 :

$$F_2(t, y_1, y_2) :=$$

$$- \frac{1}{2N} \left(v_N(y_1 - y_2) \left\{ u(y_1, y_2) + (\bar{p} \circ u)(y_1, y_2) \right\} \quad (2.10a)$$

$$+ \int dx_1 dx_2 v_N(x_1 - x_2) \left\{ 2\bar{p}(y_1, x_2)u(x_2, y_2)(\bar{u} \circ u)(x_1, x_1) \quad (2.10b)$$

$$+ 2\bar{p}(y_1, x_2)u(x_1, y_2)(\bar{u} \circ u)(x_1, x_2) \quad (2.10c)$$

$$+ u(y_1, x_1)u(x_2, y_2)(\bar{u} \circ \bar{p})(x_1, x_2) \quad (2.10d)$$

$$+ \bar{p}(y_1, x_1)p(x_2, y_2)(\bar{p} \circ u)(x_1, x_2) \quad (2.10e)$$

$$+ u(y_1, x_1)u(x_2, y_2)\bar{u}(x_1, x_2) \quad (2.10f)$$

$$+ \bar{p}(y_1, x_1)p(x_2, y_2)u(x_1, x_2) \left. \right\} \quad (2.10g)$$

$$+ \int dx_1 v_N(y_1 - x_1) \left\{ 2u(y_1, y_2)(\bar{u} \circ u)(x_1, x_1) \quad (2.10h)$$

$$+ \bar{p}(y_2, x_1)u(x_1, y_1) \quad (2.10i)$$

$$+ 2u(x_1, y_2)(\bar{u} \circ u)(x_1, y_1) \quad (2.10j)$$

$$+ \bar{p}(y_2, x_1)(\bar{p} \circ u)(y_1, x_1) \left. \right\} \quad (2.10k)$$

$$+ \int dx_1 v_N(x_1 - y_2) \bar{p}(y_1, x_1) (\bar{c} \circ u)(x_1, y_2) \Big) \quad (2.10l)$$

• Sector \mathbb{F}_3 :

$$F_3(t, y_1, y_2, y_3) :=$$

$$- N^{-1/2} \left\{ v_N(y_1 - y_2) \phi(y_2) u(y_3, y_1) \right. \quad (2.11a)$$

$$+ \int dx \{ v_N(y_1 - x) \bar{\phi}(x) u(x, y_3) \} u(y_2, y_1) \quad (2.11b)$$

$$+ \int dx \{ \bar{p}(y_1, x) v_N(x - y_2) u(y_3, x) \} \phi(y_2) \quad (2.11c)$$

$$+ \int dx \{ \bar{p}(y_2, x) v_N(y_1 - x) \phi(x) \} u(y_3, y_1) \quad (2.11d)$$

$$+ \int dx_1 dx_2 \{ \bar{p}(y_1, x_1) v_N(x_1 - x_2) \bar{\phi}(x_2) u(y_2, x_1) u(x_2, y_3) \} \quad (2.11e)$$

$$+ \int dx_1 dx_2 \{ \bar{p}(y_1, x_1) p(x_2, y_2) v_N(x_1 - x_2) \phi(x_2) u(y_3, x_1) \} \Big\} \quad (2.11f)$$

• Sector \mathbb{F}_4 :

$$F_4(t, y_1, y_2, y_3, y_4) :=$$

$$- (1/2N) \left\{ v_N(y_1 - y_2) u(y_3, y_1) u(y_2, y_4) \right. \quad (2.12a)$$

$$+ \int dx \{ \bar{p}(y_2, x) v_N(y_1 - x) u(x, y_4) \} u(y_3, y_1) \quad (2.12b)$$

$$+ \int dx \{ \bar{p}(y_1, x) v_N(x - y_2) u(y_3, x) \} u(y_2, y_4) \quad (2.12c)$$

$$+ \int dx_1 dx_2 \{ \bar{p}(y_1, x_1) p(x_2, y_2) v_N(x_1 - x_2) u(y_3, x_1) u(x_2, y_4) \} \Big\}. \quad (2.12d)$$

A standard energy estimate applied to (2.5) using self-adjointness of \mathcal{L} implies

$$\| |\psi_{\text{ex}}(t)\rangle - |\psi_{\text{ap}}(t)\rangle \|_{\mathbb{F}} = \| |\tilde{\psi}(t)\rangle \|_{\mathbb{F}} \leq N^{-1/2} \int_0^t \| \mathcal{E}(t_1)|0\rangle \|_{\mathbb{F}} dt_1.$$

For an estimate of the right hand side of the above inequality, we need L^2 -norm estimates of the terms in (2.9a)-(2.12d). This was done in [20] using the decay estimate $\| \phi(t, \cdot) \|_{L^\infty(\mathbb{R}^3)} \lesssim 1/(1 + t^{3/2})$ and the estimate $\| u(t, \cdot) \|_{L^2(\mathbb{R}^6)} \lesssim \log(1 + t)$. However $\beta < 1/3$ had to be assumed there for the final estimate in Theorem 1.5 to be meaningful. We consider a modified approach in which we treat the singular terms (i.e. terms not having sufficient integrability properties) in $N^{-1/2}\mathcal{E}(t)|0\rangle$ separately to be explained in the next section.

2.2 General Strategy and Outline of the Proof of Theorem 1.6

As mentioned at the beginning of this chapter we need to estimate the error $|\tilde{\psi}\rangle$ defined in (2.1a) and satisfying the equation (2.5) with the forcing $-N^{-1/2}\mathcal{E}(t)|0\rangle$ which contains the terms in (2.9)-(2.12). In all of them except (2.10a), (2.11a) and (2.12a), the singularity associated with the interaction $v_N(x) = N^{3\beta}v(N^\beta x)$, which converges to $(\int v)\delta(x)$ as $N \rightarrow \infty$, is smoothed out due to the integration against functions with sufficient integrability properties. Hence we separate $F_l(t, \cdot)$ defined in (2.9)-(2.12) into their regular and singular parts as follows, where super-scripts “r” and “s” stand for “regular” and “singular” respectively:

$$F_2^s(t, y_1, y_2) := -(1/2N)v_N(y_1 - y_2) \left\{ u(t, y_1, y_2) + (\bar{p} \circ u)(t, y_1, y_2) \right\}, \quad (2.13a)$$

$$F_3^s(t, y_1, y_2, y_3) := -N^{-1/2}v_N(y_1 - y_2)\phi(t, y_2)u(t, y_3, y_1), \quad (2.13b)$$

$$F_4^s(t, y_1, y_2, y_3, y_4) := -(1/2N)v_N(y_1 - y_2)u(t, y_3, y_1)u(t, y_2, y_4) \text{ and} \quad (2.13c)$$

$$F_l^r := F_l - F_l^s \text{ for } l = 2, 3, 4. \quad (2.13d)$$

Using this, we split the error $|\tilde{\psi}\rangle$ in (2.1a) first into its singular and regular parts as

$$|\tilde{\psi}\rangle = |\tilde{\psi}^r\rangle + |\tilde{\psi}^s\rangle \text{ where}$$

$$\left(\frac{1}{i}\partial_t - \mathcal{L}\right)|\tilde{\psi}^r\rangle = (0, F_1, F_2^r, F_3^r, F_4^r, 0, \dots), \quad (2.14a)$$

$$\left(\frac{1}{i}\partial_t - \mathcal{L}\right)|\tilde{\psi}^s\rangle = (0, 0, F_2^s, F_3^s, F_4^s, 0, \dots), \quad (2.14b)$$

$$|\tilde{\psi}^r(0)\rangle = |\tilde{\psi}^s(0)\rangle = 0$$

which follows from (2.5). Energy estimate applied to (2.14a) implies

$$\| |\tilde{\psi}^r(t)\rangle \|_{\mathbb{F}} \lesssim \int_0^t \left(\|F_1(t_1)\|_{L^2(\mathbb{R}^3)} + \sum_{l=2}^4 \|F_l^r(t_1)\|_{L^2(\mathbb{R}^{3l})} \right) dt_1. \quad (2.15)$$

Hence we need to obtain L^2 -norm estimates of F_1 and $F_l^r, l = 2, 3, 4$, which we do in section 2.4 after obtaining a priori estimates on the pair excitations in section 2.3.

We will start dealing with the singular part of $|\tilde{\psi}\rangle$ in section 2.5 in which we will split $|\tilde{\psi}^s\rangle$ in (2.14b) into its approximate and error parts as follows

$$|\tilde{\psi}^s\rangle = |\tilde{\psi}_1^a\rangle + |\tilde{\psi}_1^e\rangle \text{ where}$$

$$\left(\frac{1}{i}\partial_t - \int L(t, x, y) a_x^* a_y \, dx dy\right) |\tilde{\psi}_1^a\rangle = (0, 0, F_2^s, F_3^s, F_4^s, 0, \dots), \quad (2.16a)$$

$$\left(\frac{1}{i}\partial_t - \mathcal{L}\right) |\tilde{\psi}_1^e\rangle = -N^{-1/2} \mathcal{E}(t) |\tilde{\psi}_1^a\rangle, \quad (2.16b)$$

$$|\tilde{\psi}_1^a(0)\rangle = |\tilde{\psi}_1^e(0)\rangle = 0.$$

First we will obtain estimates on $|\tilde{\psi}_1^a\rangle$ using an elliptic estimate and also Strichartz estimates along with Christ-Kiselev Lemma after a suitable change of variables. Those will not provide us with sufficient integrability properties for the forcing term in (2.16b). Hence we will also discuss the necessity to iterate the splitting procedure for some finitely many times before applying a final energy estimate to the error part of the solution at the final step of iteration. We will prove the inductive step of the iteration and discuss its implications in section 2.6. Theorem 2.8 in section 2.4, Theorem 2.14 and Corollary 2.15 in section 2.6 will lead to our main result Theorem 1.6 as proved in section 2.7.

2.3 A priori Estimates for the Pair Excitations

In this section we will prove estimates on mixed L^p and Sobolev norms of the pair excitations which will be needed in estimating the terms in (2.9)-(2.12). To keep the notation simple in what follows let's define

$$s_2 := \text{sh}(2k) = 2\text{sh}(k) \circ \text{ch}(k) \quad (2.17a)$$

$$p_2 := \text{ch}(2k) - \delta(x - y) \quad (2.17b)$$

and also the operators

$$\mathbf{S}(s) := \frac{1}{i} \partial_t s + g^T \circ s + s \circ g \underbrace{\left(-\Delta_x + (v_N * |\phi|^2)(t, x) \right) \delta(x-y) + v_N(x-y) \bar{\phi}(t, x) \phi(t, y)}_{\text{Schrödinger-type}}, \quad (2.18a)$$

$$\mathbf{W}(p) := \frac{1}{i} \partial_t p + [g^T, p] \quad (\text{Wigner-type}). \quad (2.18b)$$

Then (1.38b)-(1.38c) becomes

$$\mathbf{S}(s_2) = 2m + m \circ p_2 + \bar{p}_2 \circ m, \quad (2.19a)$$

$$\mathbf{W}(\bar{p}_2) = m \circ \bar{s}_2 - s_2 \circ \bar{m}, \quad (2.19b)$$

$$s_2(0, \cdot) = p_2(0, \cdot) = 0.$$

Let's also recall our notation (see (1.33)) $u := \text{sh}(k)$, $c := \text{ch}(k) = \delta(x-y) + p$ from the previous section. Our main result in this section is the following:

Theorem 2.1. *Let the initial data ϕ_0 for (1.38a) be in $W^{m,1}(\mathbb{R}^3)$ (m derivatives in L^1) for $m \geq 6$ and let $(\partial_t s_2)(0, \cdot)$ be sufficiently regular (to be specified later in the proof). Then the following estimates hold:*

$$\|\partial_t^j s_2(t, \cdot)\|_{H^{3/2}} \lesssim_\epsilon N^{\beta(1+\epsilon)} \log(1+t) \quad \text{for } j = 0, 1 \quad (2.20)$$

$$\|u(t, \cdot)\|_{H^{3/2}} \lesssim_\epsilon N^{\beta(1+\epsilon)} \log(1+t) \quad (2.21)$$

$$\|u(t, x, y)\|_{L^\infty(\text{d}y; L^2(\text{d}x))} := \left\| \|u(t, x, y)\|_{L^2(\text{d}x)} \right\|_{L^\infty(\text{d}y)} \lesssim_\epsilon N^{\beta(1+\epsilon)} \log(1+t) \quad (2.22)$$

for any $\epsilon > 0$ and $0 < \beta \leq 1$.

We will need the following lemmas for the proof Theorem 2.1:

Lemma 2.2. (Proposition 3.3, Corollary 3.4, Corollary 3.5 in [20]) *Let ϕ be a solution of (1.38a) with initial data ϕ_0 .*

(i) *There exists C_s depending only on $\|\phi_0\|_{H^s(\mathbb{R}^3)}$ such that*

$$\|\phi(t, \cdot)\|_{H^s(\mathbb{R}^3)} \leq C_s \text{ uniformly in time.} \quad (2.23)$$

(ii) *Assuming $\phi_0 \in W^{m,1}$ for $m \geq 2$,*

$$\|\partial_t^j \phi(t, \cdot)\|_{L^\infty(\mathbb{R}^3)} \lesssim (1 + t^{3/2})^{-1} \text{ and} \quad (2.24)$$

$$\|\partial_t^j \phi(t, \cdot)\|_{L^3(\mathbb{R}^3)} \lesssim (1 + t^{1/2})^{-1} \text{ for } j = 0, 1. \quad (2.25)$$

Remark 2.3. *Note that in case of $j = 0$, (2.25) follows by interpolating (2.24) with mass conservation and in case of $j = 1$, by interpolating (2.24) with*

$$\|\partial_t \phi(t, \cdot)\|_{L^2(\mathbb{R}^3)} \lesssim \|\phi(t, \cdot)\|_{H^2(\mathbb{R}^3)} + \underbrace{\|(v_N * |\phi(t, \cdot)|^2)\phi(t, \cdot)\|_{L^2(\mathbb{R}^3)}}_{\leq \|v\|_1 \|\phi\|_4^2 \|\phi\|_\infty} \leq \text{const.} \quad (2.26)$$

We will also frequently use

$$\|\partial_t^j \phi(t, \cdot)\|_{L^4(\mathbb{R}^3)} \lesssim (1 + t^{3/4})^{-1} \text{ for } j = 0, 1 \quad (2.27)$$

which follows again by interpolation.

Corollary 2.4. (2.24)-(2.25) *hold for $j \geq 2$ if $\phi_0 \in W^{m,1}$ for m sufficiently large.*

Proof Sketch. Since we will only need the estimates on the second and third order time derivatives, let's provide here with an outline of the proof in case of the second order time derivative, which can be modified to obtain estimates on higher time derivatives. We claim that if ϕ solves (1.38a) with initial data $\phi_0 \in W^{m,1}$ for $m \geq 4$ ($m \geq 6$ in case of third order time derivative) then we have

$$\|\partial_t^2 \phi(t)\|_{L^\infty(\mathbb{R}^3)} \lesssim \frac{1}{1+t^{3/2}}. \quad (2.28)$$

To prove this estimate let's differentiate (1.38a) with respect to time twice and solve the resulting equation for $\partial_t^2 \phi$ by Duhamel's formula. Then we have

$$\|\partial_t^2 \phi(t)\|_\infty \leq \|e^{it\Delta}(\partial_t^2 \phi)_0\|_\infty + \int_0^t \left\| e^{i(t-s)\Delta} \partial_s^2 [(v_N * |\phi|^2)\phi(s)] \right\|_\infty ds. \quad (2.29)$$

Assuming $t > 1$, we split the above integral and, to estimate the integrand, we use the standard $L^\infty L^1$ decay estimate for the linear equation when we integrate over $(0, t-1)$. For the part of the same integral on $(t-1, t)$, we first use the Sobolev embedding $W^{3+\epsilon,1}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$ and then the $L^3 L^{3/2}$ decay estimate for the linear equation, up to modifying the exponents by a small amount. Hence we obtain

$$\begin{aligned} \|\partial_t^2 \phi(t)\|_\infty &\lesssim \frac{\|(\partial_t^2 \phi)_0\|_1}{1+t^{3/2}} + \int_0^{t-1} \frac{1}{1+|t-s|^{3/2}} \|\partial_s^2 [(v_N * |\phi|^2)\phi(s)]\|_1 ds \\ &\quad + \int_{t-1}^t \frac{1}{1+|t-s|^{1/2+\epsilon}} \|\nabla \partial_s^2 [(v_N * |\phi|^2)\phi(s)]\|_{3/2-\epsilon} ds. \end{aligned} \quad (2.30)$$

Now we have $\|\partial^\alpha \phi\|_2 \lesssim \|\phi\|_{H^{|\alpha|}} \leq C_{|\alpha|}$ and also $\|\partial^\alpha \phi\|_\infty \lesssim \|\partial^\alpha \phi\|_{H^2} \leq C_{2+|\alpha|}$ which

follow from (2.23). Interpolating, we obtain

- $\|\partial^\alpha \phi\|_p \leq C_{\alpha,p}$ for $p \geq 2$ and for spatial derivatives ∂^α of all orders.
- We can extend this to the case of derivatives including the time variable if we take derivatives in (1.38a) as needed and use the estimates obtained so far.

(2.31)

These regularity and integrability properties together with (2.24) imply

$$\begin{aligned} \|\partial_s^2[(v_N * |\phi|^2)\phi(s)]\|_1 &\lesssim \|\phi(s)\|_\infty \lesssim \frac{1}{1+s^{3/2}}, \\ \|\nabla \partial_s^2[(v_N * |\phi|^2)\phi(s)]\|_{3/2-\epsilon'} &\lesssim \|\phi(s)\|_\infty + \|\partial_s \phi(s)\|_\infty \lesssim \frac{1}{1+s^{3/2}}. \end{aligned}$$

Inserting these in (2.30) implies our claim in (2.28). We also have

$$\|\partial_t^2 \phi(t)\|_3 \lesssim \frac{1}{1+t^{1/2}} \quad (2.32)$$

by interpolation between (2.28) and L^2 -norm which is uniformly bounded. \square

Before stating the next lemma, let's write the kinetic and the potential parts of g (see (1.39)) separately as

$$g = -\Delta_x \delta(x-y) + g_{\text{pot}}. \quad (2.33)$$

then we can define V as follows

$$V(u) := g_{\text{pot}}^T \circ u + u \circ g_{\text{pot}}. \quad (2.34)$$

Explicitly,

$$\begin{aligned}
V(u)(t, x, y) = & \\
& \left((v_N * |\phi|^2)(t, x) + (v_N * |\phi|^2)(t, y) \right) u(x, y) \\
& + \int v_N(x - z) \phi(t, x) \bar{\phi}(t, z) u(z, y) dz + \int u(x, z) v_N(z - y) \bar{\phi}(t, z) \phi(t, y) dz.
\end{aligned} \tag{2.35}$$

This allows us to write the potential part of \mathbf{S} (see (2.18a)) separately:

$$\mathbf{S}(\cdot) = \left(\frac{1}{i} \partial_t - \Delta \right) (\cdot) + V(\cdot). \tag{2.36}$$

We will split s_2 satisfying (2.19a) as

$$s_2 = s_a + s_e \tag{2.37}$$

where s_a satisfies the equation $\mathbf{S}(s_a) = 2m = -2v_N(x - y)\phi(t, x)\phi(t, y)$ and it represents the singular part of s_2 since

$$\begin{aligned}
\|m(t, \cdot)\|_{L^2(\mathbb{R}^6)} &= \left(v_N^2 * |\phi(t, \cdot)|^2, |\phi(t, \cdot)|^2 \right)^{\frac{1}{2}} \lesssim \|v_N\|_{L^2(\mathbb{R}^3)} \|\phi(t, \cdot)\|_{L^4(\mathbb{R}^3)}^2 \\
&\stackrel{\text{by (2.27)}}{\lesssim} N^{3\beta/2} (1 + t^{3/2})^{-1}
\end{aligned} \tag{2.38}$$

blows up as $N \rightarrow \infty$. We further split s_a into its approximate and error parts as

$$s_a = s_a^0 + s_a^1 \tag{2.39}$$

and we have the following set of equations being equivalent to (2.19a):

$$\left(\frac{1}{i}\partial_t - \Delta\right)s_a^0 = 2m \quad (2.40a)$$

$$\mathbf{S}(s_a^1) = -V(s_a^0) \quad (2.40b)$$

$$\mathbf{S}(s_e) = m \circ p_2 + \bar{p}_2 \circ m \quad (2.40c)$$

$$s_a^0(0) = s_a^1(0) = s_e(0) = 0.$$

We are ready to state the next lemma:

Lemma 2.5. *Assuming $\phi_0 \in W^{m,1}$ for $m \geq 2$ as initial data for (1.38a), the following estimates hold:*

$$\|s_a^0(t, \cdot)\|_{L^2(\mathbb{R}^6)} \lesssim \log(1+t), \quad \|s_a^1(t, \cdot)\|_{L^2(\mathbb{R}^6)} \lesssim 1 \quad (2.41)$$

which imply

$$\|s_e(t, \cdot)\|_{L^2(\mathbb{R}^6)} \lesssim 1, \quad \|p_2(t, \cdot)\|_{L^2(\mathbb{R}^6)} \lesssim 1. \quad (2.42)$$

Since $s_2 = s_a^0 + s_a^1 + s_e$, we also have

$$\|s_2(t, \cdot)\|_{L^2(\mathbb{R}^6)} \lesssim \log(1+t). \quad (2.43)$$

Finally since $s_2 = \text{sh}(2k) = 2\text{sh}(k) \circ \text{ch}(k)$ and $\|\text{ch}(k)^{-1}\|_{\text{operator}}$ is uniformly bounded, recalling the notation $u = \text{sh}(k)$ and $p = \text{ch}(k) - \delta(x-y)$, we have

$$\|p(t, \cdot)\|_{L^2(\mathbb{R}^6)} \leq \|u(t, \cdot)\|_{L^2(\mathbb{R}^6)} \lesssim \log(1+t) \quad (2.44)$$

where the first inequality follows from taking traces in the relation $p \circ p + 2p = \bar{u} \circ u$ (see (1.34)) and using $\overline{p(x, x)} \geq 0$. Constants involved in the above estimates depend only on $\|\phi_0\|_{W^{m,1}}$.

Remark 2.6. For the proof of the first inequality in (2.41), one solves equation (2.40a) by Duhamel's formula and, after an integration by parts, uses the elliptic estimates below (Lemma 4.3 in [20]) along with (2.25):

$$\int \frac{|\hat{m}(t, \xi, \eta)|^2}{(|\xi|^2 + |\eta|^2)^2} d\xi d\eta \lesssim \|\phi(t, \cdot)\|_3^4, \quad (2.45a)$$

$$\int \frac{|\partial_t \hat{m}(t, \xi, \eta)|^2}{(|\xi|^2 + |\eta|^2)^2} d\xi d\eta \lesssim \|\phi(t, \cdot)\|_3^2 \|\partial_t \phi(t, \cdot)\|_3^2 \text{ and} \quad (2.45b)$$

similar estimates hold for higher time derivatives.

The proof of the second inequality in (2.41) is achieved by applying an energy estimate to the equation (2.40b) and using the first inequality in (2.41). A final application of energy estimates to the equations (2.40c) and (2.19b) together with the estimates in (2.41) implies the estimates in (2.42). We refer for more details to the proofs of Lemma 4.4 and Lemma 4.5 in [20].

Proof of Theorem 2.1. Proof of (2.20). Recalling that $s_2 = s_a^0 + s_a^1 + s_e$ from (2.37) and (2.39), we will prove (2.20) in two steps.

Step 1 Estimates on $\|\partial_t^j s_a^0\|_{H^{3/2}}$ for $j = 0, 1$: We will first estimate H^2 and $H^{1/2-\epsilon}$ -norms and then interpolate.

Differentiating (2.40a) as needed, solving the corresponding equations by Duha-

mel's formula and using integration by parts give

$$\begin{aligned}
& (\partial_t^j \widehat{s}_a^0)(t, \xi, \eta) = \\
& \underbrace{(\partial_t^j \widehat{s}_a^0)(0, \xi, \eta)}_{\text{equals 0 if } j=0} + \frac{2e^{-it(|\xi|^2+|\eta|^2)}}{|\xi|^2 + |\eta|^2} \left(\partial_t^j \widehat{m}(t, \xi, \eta) e^{it(|\xi|^2+|\eta|^2)} - (\partial_t^j \widehat{m})(0, \xi, \eta) \right. \\
& \qquad \qquad \qquad \left. - \int_0^t e^{is(|\xi|^2+|\eta|^2)} \partial_s^{j+1} \widehat{m}(s, \xi, \eta) ds \right) \quad (2.46)
\end{aligned}$$

which implies

$$\begin{aligned}
\|\Delta \partial_t^j s_a^0(t, \cdot)\|_2 & \lesssim \overbrace{\|\Delta(\partial_t^j s_a^0)(0, \cdot)\|_2}^{\text{equals 0 if } j=0 \text{ and} \\ & \text{assumed to be finite for } j=1} + \|(\partial_t^j m)(0, \cdot)\|_2 + \|\partial_t^j m(t, \cdot)\|_2 \\
& \quad + \int_0^t \|\partial_s^{j+1} m(s, \cdot)\|_2 ds \quad (2.47)
\end{aligned}$$

Applying estimate (2.38) and the following estimates

$$\begin{aligned}
\|\partial_s m(s, \cdot)\|_2 & \lesssim \langle v_N^2 * |\partial_s \phi(s, \cdot)|^2, |\phi(s, \cdot)|^2 \rangle^{1/2} \leq \|\phi(s, \cdot)\|_\infty \|v_N\|_2 \|\partial_s \phi(s, \cdot)\|_2 \\
& \lesssim N^{3\beta/2} (1 + s^{3/2})^{-1} \\
& \quad \text{(2.24) and (2.26)}
\end{aligned}$$

$$\begin{aligned}
\|\partial_s^2 m(s, \cdot)\|_2 & \lesssim \langle v_N^2 * |\partial_s^2 \phi(s, \cdot)|^2, |\phi(s, \cdot)|^2 \rangle^{1/2} + \langle v_N^2 * |\partial_s \phi(s, \cdot)|^2, |\partial_s \phi(s, \cdot)|^2 \rangle^{1/2} \\
& \leq \|\phi(s, \cdot)\|_\infty \|v_N\|_2 \|\partial_s^2 \phi(s, \cdot)\|_2 + \|\partial_s \phi(s, \cdot)\|_\infty \|v_N\|_2 \|\partial_s \phi(s, \cdot)\|_2 \\
& \lesssim N^{3\beta/2} (1 + s^{3/2})^{-1} \\
& \quad \text{(2.24) and (2.31)}
\end{aligned}$$

to (2.47) and considering (2.44), we obtain

$$\boxed{\|\partial_t^j s_a^0(t, \cdot)\|_{H^2} \lesssim N^{3\beta/2} \log(1+t) \quad \text{for } j = 0, 1.} \quad (2.48)$$

We will next estimate $\|\partial_t^j s_a^0(t, \cdot)\|_{H^{1/2-\epsilon'}}$, $j = 0, 1$ for $\epsilon' > 0$ small and to be determined later. Again by using (2.46)

$$\begin{aligned}
& \|D^{1/2-\epsilon'} \partial_t^j s_a^0(t, \cdot)\|_2 \simeq \|(|\xi| + |\eta|)^{1/2-\epsilon'} \partial_t^j \widehat{s}_a^0(t, \xi, \eta)\|_2 \\
& \lesssim \overbrace{\|D^{1/2-\epsilon'} (\partial_t^j s_a^0)(0, \cdot)\|_2}^{\text{equals 0 if } j=0} + \left\| \frac{(\partial_t^j \widehat{m})(0, \xi, \eta)}{(|\xi| + |\eta|)^{3/2+\epsilon'}} \right\|_2 + \left\| \frac{\partial_t^j \widehat{m}(t, \xi, \eta)}{(|\xi| + |\eta|)^{3/2+\epsilon'}} \right\|_2 \\
& \quad + \int_0^t \left\| \frac{\partial_s^{j+1} \widehat{m}(s, \xi, \eta)}{(|\xi| + |\eta|)^{3/2+\epsilon'}} \right\|_2 ds. \tag{2.49}
\end{aligned}$$

Now we need estimates of

$$\left\| \frac{\partial_t^j \widehat{m}(t, \xi, \eta)}{(|\xi| + |\eta|)^{3/2+\epsilon'}} \right\|_2 \quad \text{for } j = 0, 1, 2.$$

We will prove the estimates on the above terms similarly to the proof of (2.45).

Let's do it first for the case $j = 0$. Writing

$$-m(t, x, y) = v_N(x - y) \phi(t, x) \phi(t, y) = \int \delta(x - y - z) v_N(z) \phi(t, x) \phi(t, y) dz$$

and considering the Fourier transform of $\delta(x - y - z) \phi(t, x) \phi(t, y)$ in the variables

x, y :

$$e^{iz \cdot \eta} \widehat{\phi \phi_z}(t, \xi + \eta) \quad \text{where } \phi_z(x) = \phi(x - z)$$

we can write

$$|\widehat{m}(t, \xi, \eta)|^2 = \left| \int v_N(z) e^{iz \cdot \eta} \widehat{\phi \phi_z}(t, \xi + \eta) dz \right|^2 \leq \|v\|_1 \int |v_N(z)| |\widehat{\phi \phi_z}(t, \xi + \eta)|^2 dz.$$

Hence after a change of variables

$$\begin{aligned} \left\| \frac{\hat{m}(t, \xi, \eta)}{(|\xi| + |\eta|)^{3/2+\epsilon'}} \right\|_2^2 &\lesssim \int |v_N(z)| \frac{|\widehat{\phi\phi_z}(t, \xi)|^2}{(|\xi| + |\eta|)^{3+2\epsilon'}} d\xi d\eta dz \\ &\lesssim \frac{1}{\epsilon'} \int |v_N(z)| \frac{|\widehat{\phi\phi_z}(t, \xi)|^2}{|\xi|^{2\epsilon'}} d\xi dz. \end{aligned}$$

Combining this last estimate with

$$\int \frac{|\widehat{\phi\phi_z}(t, \xi)|^2}{|\xi|^{2\epsilon'}} d\xi \lesssim \|D^{-\epsilon'}(\phi\phi_z)\|_2^2 \underset{\substack{\text{Hardy-} \\ \text{Littlewood-} \\ \text{Sobolev}}}{\lesssim_{\epsilon'}} \|\phi\phi_z\|_{2-\epsilon''}^2 \leq \|\phi\|_{4-2\epsilon''}^4 \text{ where } \epsilon'' = \frac{4\epsilon'}{3+2\epsilon'}$$

gives

$$\left\| \frac{\hat{m}(t, \xi, \eta)}{(|\xi| + |\eta|)^{3/2+\epsilon'}} \right\|_2 \lesssim_{\epsilon'} \|\phi\|_{4-2\epsilon''}^2. \quad (2.50)$$

We can prove similarly in general the following estimate:

$$\left\| \frac{\partial_t^j \hat{m}(t, \xi, \eta)}{(|\xi| + |\eta|)^{3/2+\epsilon'}} \right\|_2 \lesssim_{\epsilon'} \sum_{l=0}^j \|\partial_t^l \phi\|_{4-2\epsilon''} \|\partial_t^{j-l} \phi\|_{4-2\epsilon''}. \quad (2.51)$$

Inserting estimates (2.50)-(2.51) into (2.49) gives

$$\begin{aligned} \|D^{1/2-\epsilon'} s_a^0(t, \cdot)\|_2 &\lesssim_{\epsilon'} \|\phi(0, \cdot)\|_{4-2\epsilon''}^2 + \|\phi(t, \cdot)\|_{4-2\epsilon''}^2 \\ &\quad + \int_0^t \|\phi(s, \cdot)\|_{4-2\epsilon''} \|\partial_s \phi(s, \cdot)\|_{4-2\epsilon''} ds \\ \|D^{1/2-\epsilon'} \partial_t s_a^0(t, \cdot)\|_2 &\lesssim_{\epsilon'} \|(\partial_t s_a^0)(0, \cdot)\|_{H^{1/2-\epsilon'}} + \|\phi(0, \cdot)\|_{4-2\epsilon''} \|(\partial_t \phi)(0, \cdot)\|_{4-2\epsilon''} \\ &\quad + \|\phi(t, \cdot)\|_{4-2\epsilon''} \|\partial_t \phi(t, \cdot)\|_{4-2\epsilon''} \\ &\quad + \int_0^t \left(\|\phi(s, \cdot)\|_{4-2\epsilon''} \|\partial_s^2 \phi(s, \cdot)\|_{4-2\epsilon''} + \|\partial_s \phi(s, \cdot)\|_{4-2\epsilon''}^2 \right) ds. \end{aligned}$$

Using $\|\partial_t^j \phi(t, \cdot)\|_{4-2\epsilon''} \lesssim (1+t^{3/4-\epsilon'/2})^{-1}$ for $j = 0, 1, 2$, which follow by interpolating L^2 -norm with L^∞ estimates (see Corollary 2.4 and (2.31)) and recalling (2.44), we obtain

$$\boxed{\|\partial_t^j s_a^0(t, \cdot)\|_{H^{1/2-\epsilon'}} \lesssim_{\epsilon'} \log(1+t) \quad \text{for } j = 0, 1.} \quad (2.52)$$

Interpolating this with (2.48) gives

$$\|\partial_t^j s_a^0\|_{H^{3/2}} \leq \|\partial_t^j s_a^0\|_{H^2}^{\frac{2+2\epsilon'}{3+2\epsilon'}} \|\partial_t^j s_a^0\|_{H^{1/2-\epsilon'}}^{\frac{1}{3+2\epsilon'}} \lesssim_{\epsilon'} (N^{\frac{3\beta}{2}})^{\frac{2+2\epsilon'}{3+2\epsilon'}} \log(1+t) \quad \text{for } j = 0, 1.$$

Hence finally we obtain

$$\boxed{\|\partial_t^j s_a^0(t, \cdot)\|_{H^{3/2}} \lesssim_{\epsilon} N^{\beta(1+\epsilon)} \log(1+t) \quad \text{for } j = 0, 1} \quad \text{where } \epsilon = \frac{\epsilon'}{3+2\epsilon'} \quad (2.53)$$

So for $\epsilon > 0$ arbitrarily small, we can choose $\epsilon' = 3\epsilon/(1-2\epsilon)$ in the above estimates leading to (2.53).

Step 2 Estimates on $\|\partial_t^j s_a^1\|_{H^{3/2}}$ and $\|\partial_t^j s_e\|_{H^{3/2}}$ for $j = 0, 1$: We will first estimate H^2 -norms then we will use the Sobolev embedding $H^2 \hookrightarrow H^{3/2}$. We will obtain H^2 -estimates of $\partial_t^j s_a^1$ and $\partial_t^j s_e$ by estimating $\partial_t^{j+1} s_a^1$ and $\partial_t^{j+1} s_e$ in L^2 first and then using the equations satisfied by $\partial_t^j s_a^1$ and $\partial_t^j s_e$. If we take derivative on both sides in (2.40b) and recall that $s_a = s_a^0 + s_a^1$ from (2.39), we can write

$$\mathbf{S}(\partial_t s_a^1) = -V(\partial_t s_a^0) - \left((\partial_t g_{\text{pot}}^{\text{T}}) \circ s_a + s_a \circ (\partial_t g_{\text{pot}}) \right), \quad (2.54)$$

$$\begin{aligned} \mathbf{S}(\partial_t^2 s_a^1) &= -V(\partial_t^2 s_a^0) - 2 \left((\partial_t g_{\text{pot}}^{\text{T}}) \circ \partial_t s_a + (\partial_t s_a) \circ (\partial_t g_{\text{pot}}) \right) \\ &\quad - \left((\partial_t^2 g_{\text{pot}}^{\text{T}}) \circ s_a + s_a \circ (\partial_t^2 g_{\text{pot}}) \right) \end{aligned} \quad (2.55)$$

where

$$\begin{aligned} \partial_t g_{\text{pot}}(t, x, y) &= \left(v_N * \left(2\text{Re}(\bar{\phi} \partial_t \phi) \right) \right) (t, x) \delta(x - y) \\ &\quad + v_N(x - y) \partial_t \bar{\phi}(t, x) \phi(t, y) + v_N(x - y) \bar{\phi}(t, x) \partial_t \phi(t, y) \end{aligned}$$

and we can compute $\partial_t^2 g_{\text{pot}}$ likewise. We will apply an energy estimates to (2.54)-(2.55). Let's define

$$(\partial_t^j V)(u) := \left(\partial_t^j g_{\text{pot}}^{\text{T}} \right) \circ u + u \circ \left(\partial_t^j g_{\text{pot}} \right) \quad \text{for } j = 1, 2. \quad (2.56)$$

Using this definition and (2.54)-(2.55) and also recalling (2.18), we have

$$\begin{aligned} \mathbf{W}((\partial_t s_a^1) \circ \partial_t \bar{s}_a^1) &= \mathbf{S}(\partial_t s_a^1) \circ \partial_t \bar{s}_a^1 - (\partial_t s_a^1) \circ \overline{\mathbf{S}(\partial_t s_a^1)} \\ &= -V(\partial_t s_a^0) \circ \partial_t \bar{s}_a^1 + (\partial_t s_a^1) \circ \overline{V(\partial_t s_a^0)} - ((\partial_t V)(s_a)) \circ \partial_t \bar{s}_a^1 + (\partial_t s_a^1) \circ \overline{(\partial_t V)(s_a)}, \\ \mathbf{W}((\partial_t^2 s_a^1) \circ \partial_t^2 \bar{s}_a^1) &= \mathbf{S}(\partial_t^2 s_a^1) \circ \partial_t^2 \bar{s}_a^1 - (\partial_t^2 s_a^1) \circ \overline{\mathbf{S}(\partial_t^2 s_a^1)} \\ &= -V(\partial_t^2 s_a^0) \circ \partial_t^2 \bar{s}_a^1 + (\partial_t^2 s_a^1) \circ \overline{V(\partial_t^2 s_a^0)} \\ &\quad - 2 \left[((\partial_t V)(\partial_t s_a)) \circ \partial_t^2 \bar{s}_a^1 - (\partial_t^2 s_a^1) \circ \overline{(\partial_t V)(\partial_t s_a)} \right] \\ &\quad - (\partial_t^2 V)(s_a) \circ \partial_t^2 \bar{s}_a^1 + (\partial_t^2 s_a^1) \circ \overline{(\partial_t^2 V)(s_a)}. \end{aligned}$$

To obtain L^2 -norm estimates, we take traces on both sides of the above equations and make the following estimates:

$$\partial_t \|\partial_t s_a^1\|_2^2 \leq 2 \left(\|V(\partial_t s_a^0)\|_2 + \|(\partial_t V)(s_a)\|_2 \right) \|\partial_t s_a^1\|_2 \quad (2.57)$$

$$\partial_t \|\partial_t^2 s_a^1\|_2^2 \leq 2 \left(\|V(\partial_t^2 s_a^0)\|_2 + 2 \|(\partial_t V)(\partial_t s_a)\|_2 + \|(\partial_t^2 V)(s_a)\|_2 \right) \|\partial_t^2 s_a^1\|_2 \quad (2.58)$$

Both V (see (2.35)) and $\partial_t^j V$ for $j = 1, 2$ (see (2.56)) are bounded from L^2 to L^2 since the inequalities

$$\left. \begin{aligned} & \| (v_N * \partial_t^j |\phi|^2)(t, x) u(x, y) \|_{L^2_{x,y}} \\ & \lesssim \|v_N\|_{L^1(\mathbb{R}^3)} \left(\sum_{k=0}^j \|\partial_t^k \phi(t, \cdot)\|_{L^\infty(\mathbb{R}^3)} \|\partial_t^{j-k} \phi(t, \cdot)\|_{L^\infty(\mathbb{R}^3)} \right) \|u\|_{L^2(\mathbb{R}^6)}, \\ & \left\| \int v_N(x-z) \partial_t^j (\bar{\phi}(t, x) \phi(t, z)) u(z, y) dz \right\|_{L^2_{x,y}} \\ & \lesssim \left(\sum_{k=0}^j \|\partial_t^k \phi(t, \cdot)\|_{L^\infty(\mathbb{R}^3)} \|\partial_t^{j-k} \phi(t, \cdot)\|_{L^\infty(\mathbb{R}^3)} \right) \underbrace{\| (v_N * \|u(\cdot, y)\|_{L^2_y})(x) \|_{L^2_x}}_{\lesssim \|v_N\|_{L^1(\mathbb{R}^3)} \|u\|_{L^2(\mathbb{R}^6)}} \end{aligned} \right\} \quad (2.59)$$

for $j = 0, 1, 2$ and $\|\partial_t^j \phi(t, \cdot)\|_{L^\infty(\mathbb{R}^3)} \lesssim 1/(1+t^{3/2})$ (see Corollary 2.4) imply

$$\left. \begin{aligned} & \|V\|_{\text{op}} \lesssim \|\phi(t, \cdot)\|_\infty^2 \lesssim (1+t^3)^{-1}, \\ & \|\partial_t V\|_{\text{op}} \lesssim \|\phi(t, \cdot)\|_\infty \|\partial_t \phi(t, \cdot)\|_\infty \lesssim (1+t^3)^{-1}, \\ & \|\partial_t^2 V\|_{\text{op}} \lesssim \|\phi(t, \cdot)\|_\infty \|\partial_t^2 \phi(t, \cdot)\|_\infty + \|\partial_t \phi(t, \cdot)\|_\infty^2 \lesssim (1+t^3)^{-1}. \end{aligned} \right\} \quad (2.60)$$

Hence (2.57)-(2.58) take the form

$$\partial_t \|\partial_t s_a^1(t, \cdot)\|_2 \lesssim \frac{1}{1+t^3} \left(\|\partial_t s_a^0(t, \cdot)\|_2 + \underbrace{\|s_a(t, \cdot)\|_2}_{\substack{\lesssim \log(1+t) \text{ by (2.41)} \\ \text{since } s_a = s_a^0 + s_a^1}} \right), \quad (2.61)$$

$$\partial_t \|\partial_t^2 s_a^1(t, \cdot)\|_2 \lesssim \frac{1}{1+t^3} \left(\|\partial_t^2 s_a^0(t, \cdot)\|_2 + \|\partial_t s_a(t, \cdot)\|_2 + \underbrace{\|s_a(t, \cdot)\|_2}_{\lesssim \log(1+t)} \right). \quad (2.62)$$

Now we need estimates of $\|\partial_t^j s_a^0\|_2$, $j = 1, 2$. Taking L^2 -norms in (2.46) and using

(2.45), we can obtain the following estimate:

$$\begin{aligned}
& \|\partial_t^j s_a^0(t, \cdot)\|_2 \\
& \leq \|(\partial_t^j s_a^0)(0, \cdot)\|_2 + \left\| \frac{(\partial_t^j \hat{m})(0, \xi, \eta)}{|\xi|^2 + |\eta|^2} \right\|_2 + \left\| \frac{\partial_t^j \hat{m}(t, \xi, \eta)}{|\xi|^2 + |\eta|^2} \right\|_2 \\
& \quad + \left\| \int_0^t e^{is(|\xi|^2 + |\eta|^2)} \frac{\partial_s^{j+1} \hat{m}(s, \xi, \eta)}{|\xi|^2 + |\eta|^2} ds \right\|_2 \\
& \lesssim \|(\partial_t^j s_a^0)(0, \cdot)\|_2 + \sum_{l=0}^j \left(\|(\partial_t^l \phi)(0, \cdot)\|_3 \|(\partial_t^{j-l} \phi)(0, \cdot)\|_3 + \|\partial_t^l \phi(t, \cdot)\|_3 \|\partial_t^{j-l} \phi(t, \cdot)\|_3 \right) \\
& \quad + \int_0^t \left\{ \sum_{l=0}^{j+1} \|\partial_t^l \phi(t, \cdot)\|_3 \|\partial_t^{j+1-l} \phi(t, \cdot)\|_3 \right\} ds.
\end{aligned}$$

This last estimate considered with (2.25) and Corollary 2.4 imply

$$\|\partial_t^j s_a^0(t, \cdot)\|_2 \lesssim \log(1+t) \quad \text{for } j = 1, 2. \quad (2.63)$$

Inserting this in (2.61) gives

$$\partial_t \|\partial_t s_a^1\|_2 \lesssim \frac{\log(1+t)}{1+t^3} \quad (2.64)$$

which implies uniform-in-time boundedness of $\|\partial_t s_a^1\|_2$. This together with (2.63)

implies

$$\|\partial_t s_a\|_2 \lesssim \log(1+t) \quad (2.65)$$

since $s_a = s_a^0 + s_a^1$. Inserting this last estimate and estimate (2.63) in (2.62) implies

$$\partial_t \|\partial_t^2 s_a^1\|_2 \lesssim \frac{\log(1+t)}{1+t^3} \quad (2.66)$$

yielding uniform-in-time boundedness of $\|\partial_t^2 s_a^1\|_2$. With the help of the uniform bounds on $\|\partial_t^j s_a^1\|_2$, $j = 1, 2$, we can control Δs_a^1 and $\Delta \partial_t s_a^1$ using equations (2.40b) and (2.54) satisfied by s_a^1 and $\partial_t s_a^1$ respectively:

$$\|\Delta s_a^1\|_2 \leq \underbrace{\|\partial_t s_a^1\|_2}_{\text{unif. bounded}} + \underbrace{\|V(s_a)\|_2}_{\lesssim \frac{\log(1+t)}{(1+t^3)} \text{ by (2.60), (2.41)}} \lesssim \log(1+t), \quad (2.67)$$

$$\|\Delta \partial_t s_a^1\|_2 \leq \underbrace{\|\partial_t^2 s_a^1\|_2}_{\text{unif. bounded}} + \underbrace{\|V(\partial_t s_a)\|_2 + \|(\partial_t V)(s_a)\|_2}_{\lesssim \frac{\log(1+t)}{(1+t^3)} \text{ by (2.60), (2.41), (2.65)}} \lesssim \log(1+t). \quad (2.68)$$

Since we have $H^2 \hookrightarrow H^{3/2}$, we obtain

$$\boxed{\|\partial_t^j s_a^1(t, \cdot)\|_{H^{3/2}} \lesssim \log(1+t) \quad \text{for } j = 0, 1.} \quad (2.69)$$

Finally for estimating $\|\partial_t^j s_e\|_{H^{3/2}}$ for $j = 0, 1$, again we will estimate $\partial_t^{j+1} s_e$ in L^2 and use the equations satisfied by $\partial_t^j s_e$ to estimate $\Delta \partial_t^j s_e$ and then the embedding $H^2 \hookrightarrow H^{3/2}$. If we take derivatives of equations (2.40c) and (2.19b), we obtain the following equations to which we will apply energy estimates:

$$\left. \begin{aligned} & \mathbf{S}(\partial_t s_e) \\ & = -(\partial_t V)(s_e) + (\partial_t m) \circ p_2 + m \circ \partial_t p_2 + (\partial_t \bar{p}_2) \circ m + \bar{p}_2 \circ (\partial_t m) \\ & \mathbf{W}(\partial_t \bar{p}_2) \\ & = -[\partial_t g_{\text{pot}}^T, \bar{p}_2] + \partial_t M + \underbrace{(\partial_t m) \circ \bar{s}_e + m \circ \partial_t \bar{s}_e}_{m \circ \bar{s}_a - s_a \circ \bar{m}} - (\partial_t s_e) \circ \bar{m} - s_e \circ \partial_t \bar{m} \end{aligned} \right\} \quad (2.70)$$

$$\begin{aligned}
& \mathbf{S}(\partial_t^2 s_e) \\
&= -2(\partial_t V)(\partial_t s_e) - (\partial_t^2 V)(s_e) + (\partial_t^2 m) \circ p_2 + \bar{p}_2 \circ \partial_t^2 m \\
&\quad + 2\left[(\partial_t m) \circ \partial_t p_2 + (\partial_t \bar{p}_2) \circ \partial_t m\right] + m \circ \partial_t^2 p_2 + (\partial_t^2 \bar{p}_2) \circ m \\
& \mathbf{W}(\partial_t^2 \bar{p}_2) \\
&= -\left[\partial_t^2 g_{\text{pot}}^T, \bar{p}_2\right] - 2\left[\partial_t g_{\text{pot}}^T, \partial_t \bar{p}_2\right] + \partial_t^2 M + (\partial_t^2 m) \circ \bar{s}_e - s_e \circ \partial_t^2 \bar{m} \\
&\quad + 2\left[(\partial_t m) \circ \partial_t \bar{s}_e - (\partial_t s_e) \circ \partial_t \bar{m}\right] + m \circ \partial_t^2 \bar{s}_e - (\partial_t^2 s_e) \circ \bar{m}
\end{aligned} \tag{2.71}$$

where $M := m \circ \bar{s}_a - s_a \circ \bar{m}$. Now we add the equations

$$\begin{aligned}
\mathbf{W}((\partial_t^j s_e) \circ \partial_t^j \bar{s}_e) &= \mathbf{S}(\partial_t^j s_e) \circ \partial_t^j \bar{s}_e - (\partial_t^j s_e) \circ \overline{\mathbf{S}(\partial_t^j s_e)} \\
\mathbf{W}((\partial_t^j \bar{p}_2) \circ \partial_t^j \bar{p}_2) &= \mathbf{W}(\partial_t^j \bar{p}_2) \circ \partial_t^j \bar{p}_2 + (\partial_t^j \bar{p}_2) \circ \mathbf{W}(\partial_t^j \bar{p}_2)
\end{aligned}$$

side by side and then take traces to make the following estimate:

$$\begin{aligned}
& \partial_t \left(\overbrace{\|\partial_t^j s_e\|_2^2 + \|\partial_t^j p_2\|_2^2}^{=: E_j^2(t)} \right) \\
& \lesssim \|\mathbf{S}(\partial_t^j s_e)\|_2 \|\partial_t^j s_e\|_2 + \|\mathbf{W}(\partial_t^j \bar{p}_2)\|_2 \|\partial_t^j p_2\|_2 \quad \text{for } j = 1, 2.
\end{aligned} \tag{2.73}$$

We already know from (2.60) that $\|\partial_t^j V\|_{\text{op}} \lesssim (1+t^3)^{-1}$ for $j = 0, 1, 2$. Similarly

$$\|[\partial_t^j g_{\text{pot}}^T, (\cdot)]\|_{\text{op}} \lesssim (1+t^3)^{-1}. \tag{2.74}$$

Recalling $m(t, x, y) = -v_N(x-y)\phi(t, x)\phi(t, y)$, the definition of M from (2.70) and

using estimates similar to the second one in (2.59) we obtain

$$\left. \begin{aligned} \|(\partial_t^j m) \circ u\|_2 &\leq (1+t^3)^{-1} \|u\|_2, \\ \|\partial_t^j M\|_2 &\lesssim (1+t^3)^{-1} \sum_{k=0}^j \|\partial_t^k s_a\|_2 \end{aligned} \right\} \text{ for } j = 1, 2. \quad (2.75)$$

Considering all these estimates together with (2.70) implies

$$\begin{aligned} \|\mathbf{S}(\partial_t s_e)\|_2 &\lesssim \frac{1}{1+t^3} \left(\overbrace{\|s_e\|_2 + \|p_2\|_2 + \|\partial_t p_2\|_2}^{O(1) \text{ by (2.42)}} \right), \\ \|\mathbf{W}(\partial_t \bar{p}_2)\|_2 &\lesssim \frac{1}{1+t^3} \left(\|p_2\|_2 + \underbrace{\|s_a\|_2 + \|\partial_t s_a\|_2}_{\lesssim \log(1+t) \text{ by (2.41), (2.65)}} + \|s_e\|_2 + \|\partial_t s_e\|_2 \right). \end{aligned}$$

Inserting the above estimates in (2.73) for $j = 1$, we obtain

$$\partial_t \left(\overbrace{\|\partial_t s_e\|_2^2 + \|\partial_t p_2\|_2^2}^{=: E_1^2(t)} \right) \lesssim \frac{1}{1+t^3} \left(\overbrace{\|\partial_t s_e\|_2}^{\leq E_1(t)} + \overbrace{\|\partial_t s_e\|_2 \|\partial_t p_2\|_2}^{\lesssim E_1^2(t)} + (1 + \log(1+t)) \overbrace{\|\partial_t p_2\|_2}^{\leq E_1(t)} \right)$$

from which it follows that

$$\partial_t E_1(t) \lesssim \frac{1}{1+t^3} E_1(t) + \frac{1 + \log(1+t)}{1+t^3}.$$

This in turn implies that $E_1(t)$ is uniformly bounded in time. Using this, we can deduce

$$\|\partial_t s_e(t, x, y)\|_{L_{x,y}^2} \lesssim 1 \quad \text{and} \quad \|\partial_t p_2(t, x, y)\|_{L_{x,y}^2} \lesssim 1. \quad (2.76)$$

Now considering estimates (2.60), (2.74), (2.75) together with (2.71) implies

$$\begin{aligned} \|\mathbf{S}(\partial_t^2 s_e)\|_2 &\lesssim \frac{1}{1+t^3} \left(\overbrace{\|s_e\|_2 + \|p_2\|_2}^{O(1) \text{ by (2.42)}} + \overbrace{\|\partial_t s_e\|_2 + \|\partial_t p_2\|_2 + \|\partial_t^2 p_2\|_2}^{O(1) \text{ by (2.76)}} \right), \\ \|\mathbf{W}(\partial_t^2 \bar{p}_2)\|_2 &\lesssim \frac{1}{1+t^3} \left(\underbrace{\sum_{j=0}^1 (\|\partial_t^j s_e\|_2 + \|\partial_t^j p_2\|_2)}_{O(1) \text{ as above}} + \underbrace{\sum_{j=0}^2 \|\partial_t^j s_a\|_2 + \|\partial_t^2 s_e\|_2}_{\substack{\lesssim \log(1+t) \\ \text{by (2.41), (2.65), (2.63), (2.66)} \\ \text{and recalling } s_a = s_a^0 + s_a^1}} \right). \end{aligned}$$

Inserting the above estimates in (2.73) for $j = 2$, we obtain

$$\partial_t \left(\overbrace{\|\partial_t^2 s_e\|_2^2 + \|\partial_t^2 p_2\|_2^2}^{=: E_2^2(t)} \right) \lesssim \frac{1}{1+t^3} \left(\overbrace{\|\partial_t^2 s_e\|_2}^{\leq E_2(t)} + \overbrace{\|\partial_t^2 s_e\|_2 \|\partial_t^2 p_2\|_2}^{\lesssim E_2^2(t)} + (1 + \log(1+t)) \overbrace{\|\partial_t^2 p_2\|_2}^{\leq E_2(t)} \right)$$

which yields

$$\partial_t E_2(t) \lesssim \frac{1}{1+t^3} E_2(t) + \frac{1 + \log(1+t)}{1+t^3}.$$

This implies that $E_2(t)$ is uniformly bounded in time, which helps us conclude

$$\|\partial_t^2 s_e(t, x, y)\|_{L_{x,y}^2} \lesssim 1 \quad \text{and} \quad \|\partial_t^2 p_2(t, x, y)\|_{L_{x,y}^2} \lesssim 1. \quad (2.77)$$

Now we can estimate $\|\Delta \partial_t^j s_e\|_2$, $j = 0, 1$ using (2.40c) and the first equation in (2.70) as follows:

$$\|\Delta s_e\|_2 \leq \|\partial_t s_e\|_2 + \|V(s_e)\|_2 + \|m \circ p_2\|_2 + \|\bar{p}_2 \circ m\|_2 \lesssim 1 + \frac{1}{1+t^3} \quad (2.78)$$

$$\begin{aligned} \|\Delta \partial_t s_e\|_2 &\leq \|\partial_t^2 s_e\|_2 + \|(\partial_t m) \circ p_2\|_2 + \|\bar{p}_2 \circ \partial_t m\|_2 + \|m \circ \partial_t p_2\|_2 + \|(\partial_t \bar{p}_2) \circ m\|_2 \\ &\lesssim 1 + \frac{1}{1+t^3} \end{aligned} \quad (2.79)$$

where we used (2.60), (2.75), (2.42), (2.76) and (2.77). The estimates above imply

$$\boxed{\|\partial_t^j s_e(t, \cdot)\|_{H^{3/2}} \lesssim 1 \quad \text{for } j = 0, 1.} \quad (2.80)$$

due to the Sobolev embedding $H^2 \hookrightarrow H^{3/2}$. Recalling $s_2 = s_a^0 + s_a^1 + s_e$ and combining (2.53), (2.69) and (2.80) imply

$$\|\partial_t^j s_2\|_{H^{3/2}} \lesssim_\epsilon N^{\beta(1+\epsilon)} \log(1+t) \quad \text{for } j = 0, 1$$

which proves (2.20).

Proof of (2.21). This is based on the identity $s_2 = 2u \circ c = 2\bar{c} \circ u$. We have

$$D_x^\sigma u(t, x, y) = \frac{1}{2}(D_x^\sigma s_2) \circ c^{-1} \quad \text{and} \quad D_y^\sigma u(t, x, y) = \frac{1}{2}\bar{c}^{-1} \circ D_y^\sigma s_2 \quad (2.81)$$

where $\sigma \in \mathbb{R}$ denotes the order of the derivative. (2.81) implies

$$\begin{aligned} \|D^\sigma u(t, \cdot)\|_2^2 &= \int (|\xi|^2 + |\eta|^2)^\sigma |\hat{u}(t, \xi, \eta)|^2 d\xi d\eta \\ &\lesssim \int |\xi|^{2\sigma} |\hat{u}(t, \xi, \eta)|^2 d\xi d\eta + \int |\eta|^{2\sigma} |\hat{u}(t, \xi, \eta)|^2 d\xi d\eta \\ &= \|D_x^\sigma u(t, \cdot)\|_2^2 + \|D_y^\sigma u(t, \cdot)\|_2^2 \lesssim \|s_2\|_{H^\sigma}^2 \end{aligned} \quad (2.82)$$

where the last inequality follows from (2.81) since $\|c^{-1}\|_{\text{op}}$ is uniformly bounded.

Taking $\sigma = 3/2$ in (2.81)-(2.82) proves (2.21).

Proof of (2.22). For $\tilde{\epsilon} > 0$ small, we can make the following estimate:

$$\left\| \|u(t, x, y)\|_{L^2(dx)} \right\|_{L^\infty(dy)} \leq \left\| \|u(t, x, y)\|_{L^\infty(dy)} \right\|_{L^2(dx)} \lesssim_{\tilde{\epsilon}} \|D_y^{\frac{3}{2}+\tilde{\epsilon}} u(t, \cdot)\|_2 \lesssim \|u\|_{H^{\frac{3}{2}+\tilde{\epsilon}}} \quad (2.83)$$

where, for the second last inequality, we have used $H^s(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$ for $s > n/2$ with $n = 3$ (see e.g. Remark 1.4.1 (v) in [5]). Considering $\sigma = 2$ in (2.81)-(2.82), one can prove $\|u(t, \cdot)\|_{H^2} \lesssim \|s_2(t, \cdot)\|_{H^2} \lesssim N^{3\beta/2} \log(1+t)$ where the last inequality follows from (2.48), (2.67) and (2.78). Interpolating between this H^2 -norm estimate and the previously obtained $H^{3/2}$ -norm estimate (see (2.21)) gives

$$\|u(t, \cdot)\|_{H^{3/2+\tilde{\epsilon}}} \lesssim \underbrace{(N^{\beta(1+\epsilon)})^{1-2\tilde{\epsilon}} (N^{3\beta/2})^{2\tilde{\epsilon}}}_{N^{\beta[1+\epsilon+\tilde{\epsilon}(1-2\epsilon)]}} \log(1+t). \quad (2.84)$$

This last estimate considered with (2.83) proves (2.22). \square

Remark 2.7.

(i) In the following section we will frequently use an estimate of $\|\|u(t, \cdot)\|_2\|_4 := \|\|u(t, x, y)\|_{L_x^2}\|_{L_y^4}$ to control most of the contributions in (2.9)-(2.12). This follows by interpolation between $\|\|u\|_2\|_\infty$ and $\|u\|_2 = \|\|u\|_2\|_2$ i.e. we have

$$\|\|u(t, \cdot)\|_2\|_4 \leq \|\|u(t, \cdot)\|_2\|_\infty^{1/2} \|u(t, \cdot)\|_2^{1/2} \lesssim N^{(\beta/2)(1+\epsilon)} \log(1+t), \quad \epsilon > 0 \quad (2.85)$$

where for the last inequality we used (2.44) and (2.22).

(ii) Recalling the relation $p \circ p + 2p = \bar{u} \circ u$ and the fact that $(p \circ p)(t, x, x) \geq 0$

and also $p(t, x, x) \geq 0$, we have

$$\|p(t, x, y)\|_{L^2(dx)}^2 = (p \circ p)(t, y, y) \leq (\bar{u} \circ u)(t, y, y) = \|u(t, x, y)\|_{L^2(dx)}^2$$

which implies (for any $\epsilon > 0$)

$$\| \|p(t, \cdot)\|_2 \|_\infty \leq \| \|u(t, \cdot)\|_2 \|_\infty \lesssim N^{\beta(1+\epsilon)} \log(1+t) \quad (2.86)$$

$$\| \|p(t, \cdot)\|_2 \|_4 \leq \| \|u(t, \cdot)\|_2 \|_4 \lesssim N^{(\beta/2)(1+\epsilon)} \log(1+t) \quad (2.87)$$

using (2.22) and (2.85).

2.4 The Regular Part of the Error $|\tilde{\psi}\rangle$

Our main result in this section is the following:

Theorem 2.8. *We have the following estimate for $|\tilde{\psi}^f\rangle$ solving equation (2.14a):*

$$\| |\tilde{\psi}^f(t)\rangle \|_{\mathbb{F}} \lesssim_\epsilon N^{-1/2+\beta(1+\epsilon)} t \log^4(1+t) \quad (2.88)$$

for any $\epsilon > 0$.

We will need the following lemma for the proof of Theorem 2.8:

Lemma 2.9. *Given the definitions in (2.9)-(2.12) and (2.13), the following esti-*

mates hold:

$$\|F_l^\downarrow(t)\|_{L^2(\mathbb{R}^{3l})} \lesssim_\epsilon \begin{cases} N^{-1/2+\beta(1+\epsilon)} \log^3(1+t)/(1+t^{3/2}), & l = 1, 3 \\ N^{-1+2\beta(1+\epsilon)} \log^4(1+t), & l = 2, 4. \end{cases} \quad (2.89)$$

for any $\epsilon > 0$.

Proof. Let's prove (2.89) for $l = 1, 2$ first. We need to estimate the L^2 -norms of the contributions in (2.9a)-(2.9i) and the ones in (2.10b)-(2.10l). Estimate for the term in (2.10b) can be made as follows:

$$\begin{aligned} & N^{-1} \left\| \int dx_1 dx_2 v_N(x_1 - x_2) p(x_2, y_1) u(x_2, y_2) (\bar{u} \circ u)(x_1, x_1) \right\|_{L^2(dy_1 dy_2)} \quad (2.90) \\ & \leq N^{-1} \int dx_1 dx_2 v_N(x_1 - x_2) \|p(x_2, y_1)\|_{L^2(dy_1)} \|u(x_2, y_2)\|_{L^2(dy_2)} (\bar{u} \circ u)(x_1, x_1) \\ & \leq N^{-1} \| \|p\|_2 \|u\|_2 \|v_N\|_1 \| \underbrace{(\bar{u} \circ u)(x_2, x_2)}_{\|u\|_2^2} \|_{L^1(dx_2)} \underset{\substack{\text{by} \\ (2.44), \\ (2.22) \text{ and } (2.86)}}{\lesssim_\epsilon} N^{-1+2\beta(1+\epsilon)} \log^4(1+t). \end{aligned}$$

Estimates of the terms in (2.10c)-(2.10e) are similar and differ slightly from (2.90). We estimate only for (2.10c):

$$\begin{aligned} & N^{-1} \left\| \int dx_1 dx_2 v_N(x_1 - x_2) p(x_2, y_1) u(x_1, y_2) (\bar{u} \circ u)(x_1, x_2) \right\|_{L^2(dy_1 dy_2)} \\ & \leq N^{-1} \int dx_1 dx_2 v_N(x_1 - x_2) \|p(x_2, y_1)\|_{L^2(dy_1)} \|u(x_1, y_2)\|_{L^2(dy_2)} |(\bar{u} \circ u)(x_1, x_2)| \\ & \leq N^{-1} \| \|p\|_2 \|u\|_2 \| \int dx_2 v_N(x_2) \underbrace{\left(\int dx_1 |(\bar{u} \circ u)(x_1, x_1 - x_2)| \right)}_{\leq \|u\|_2^2 \text{ uniformly in } x_2} \| \\ & \leq N^{-1} \| \|p\|_2 \|u\|_2 \|v_N\|_1 \|u\|_2^2 \underset{\substack{\text{by} \\ (2.44), \\ (2.22) \text{ and } (2.86)}}{\lesssim_\epsilon} N^{-1+2\beta(1+\epsilon)} \log^4(1+t). \quad (2.91) \end{aligned}$$

Estimates of (2.9a)-(2.9b) are similar to (2.90) and the estimates of (2.9c)-(2.9f) are similar to (2.91); the only difference being that, in (2.90)-(2.91), we were able to pull two factors out of the integral in L^∞ -norm, each of which is either a p -term or a u -term whereas in estimates of (2.9a)-(2.9c) there is only one u (or p)-term available for us to pull out in the same manner and we also need to pull ϕ out of the integral in L^∞ -norm. This explains the the difference between the powers of N and the time dependence of the bounds in the estimates in (2.89), in cases of $l = 1$ and $l = 2$.

Estimates of (2.10f)-(2.10g) are similar so let's just look at the estimate of (2.10f):

$$\begin{aligned}
& (1/2N) \left\| \int dx_1 dx_2 v_N(x_1 - x_2) u(y_1, x_1) u(x_2, y_2) \bar{u}(x_1, x_2) \right\|_{L^2(dy_1 dy_2)} \\
& \leq (1/2N) \int dx_1 dx_2 v_N(x_1 - x_2) \|u(y_1, x_1)\|_{L^2(dy_1)} \|u(x_2, y_2)\|_{L^2(dy_2)} |u(x_1, x_2)| \\
& \leq (1/2N) \| \|u\|_2 \|_\infty \int dx_2 v_N(x_2) \underbrace{\left(\int dx_1 |u(x_1, x_1 - x_2)| \|u(y_1, x_1)\|_{L^2(dy_1)} \right)}_{\leq \|u\|_{H^{3/2+\epsilon}} \|u\|_2 \text{ unif. in } x_2} \\
& \leq (1/2N) \| \|u\|_2 \|_\infty \|v_N\|_1 \|u\|_{H^{3/2+\epsilon}} \|u\|_2 \underset{\substack{\text{by} \\ (2.44), \\ (2.84) \text{ and } (2.22)}}{\lesssim_\epsilon} N^{-1+2\beta(1+\epsilon)} \log^3(1+t). \tag{2.92}
\end{aligned}$$

Estimate of the more singular term (2.10i) differs slightly from the above estimate:

$$\begin{aligned}
& (1/2N) \left\| \int dx_1 v_N(y_1 - x_1) p(x_1, y_2) u(y_1, x_1) \right\|_{L^2(dy_1 dy_2)} \\
& \leq (1/2N) \left\| \int dx_1 v_N(y_1 - x_1) \|p(x_1, y_2)\|_{L^2(dy_2)} u(y_1, x_1) \right\|_{L^2(dy_1)}
\end{aligned}$$

$$\begin{aligned}
&\leq (1/2N) \left\| \|p\|_2 \right\|_\infty \int dx_1 v_N(x_1) \underbrace{\|u(y_1, y_1 - x_1)\|_{L^2(dy_1)}}_{\leq \|u\|_{H^{3/2+\epsilon}} \text{ unif. in } x_1} \\
&\leq (1/2N) \left\| \|p\|_2 \right\|_\infty \|v_N\|_1 \|u\|_{H^{3/2+\epsilon}} \stackrel{\text{by}}{\lesssim_\epsilon} N^{-1+2\beta(1+\epsilon)} \log^2(1+t). \tag{2.93} \\
&\hspace{10em} \text{(2.84) and (2.86)}
\end{aligned}$$

Now let's consider the estimate of (2.10h):

$$\begin{aligned}
&N^{-1} \left\| \int dx_1 v_N(x_1 - y_1) u(y_1, y_2) (\bar{u} \circ u)(x_1, x_1) \right\|_{L^2(dy_1 dy_2)} \\
&\leq N^{-1} \left\| \int dx_1 v_N(x_1 - y_1) \|u(y_1, y_2)\|_{L^2(dy_2)} (\bar{u} \circ u)(x_1, x_1) \right\|_{L^2(dy_1)} \\
&\leq N^{-1} \left\| \left(v_N * ((\bar{u} \circ u)(\cdot, \cdot)) \right) (y_1) \|u(y_1, y_2)\|_{L^2(dy_2)} \right\|_{L^2(dy_1)} \\
&\leq N^{-1} \left\| \|u\|_2 \right\|_\infty \|v_N\|_1 \underbrace{\|(\bar{u} \circ u)(y_1, y_1)\|_{L^2(dy_1)}}_{\left\| \|u\|_2 \right\|_4^2} \stackrel{\text{by}}{\lesssim_\epsilon} N^{-1+2\beta(1+\epsilon)} \log^3(1+t). \tag{2.94} \\
&\hspace{10em} \text{(2.22) and (2.85)}
\end{aligned}$$

Estimates of (2.10j)-(2.10k) are similar and differ slightly from (2.94). We will estimate for (2.10j) in the following way:

$$\begin{aligned}
&N^{-1} \left\| \int dx_1 v_N(x_1 - y_1) u(x_1, y_2) (\bar{u} \circ u)(x_1, y_1) \right\|_{L^2(dy_1 dy_2)} \\
&\leq N^{-1} \left\| \int dx_1 v_N(y_1 - x_1) (\bar{u} \circ u)(x_1, y_1) \|u(x_1, y_2)\|_{L^2(dy_2)} \right\|_{L^2(dy_1)} \\
&\leq N^{-1} \left\| \|u\|_2 \right\|_\infty \int dx_1 v_N(x_1) \underbrace{\|(\bar{u} \circ u)(y_1 - x_1, y_1)\|_{L^2(dy_1)}}_{\leq \left\| \|u\|_2 \right\|_4^2 \text{ uniformly in } x_1} \\
&\leq N^{-1} \left\| \|u\|_2 \right\|_\infty \|v_N\|_1 \left\| \|u\|_2 \right\|_4^2 \stackrel{\text{by}}{\lesssim_\epsilon} N^{-1+2\beta(1+\epsilon)} \log^3(1+t). \tag{2.95} \\
&\hspace{10em} \text{(2.22) and (2.85)}
\end{aligned}$$

(2.10l) is similar to the sum of the terms in (2.10i) and (2.10k) whose estimates have already been discussed.

Estimates of (2.9g)-(2.9h) are similar to (2.92). The estimate of (2.9i) resembles (2.93). Estimate of (2.9l) is similar to (2.94) and estimates of (2.9j)-(2.9k) resemble (2.95). However, similar to the remarks coming right after (2.91), in (2.9g)-(2.9l), there is no u (or p)-term available for us to pull out of the integral in the way we did in (2.92)-(2.95). Instead, we can pull ϕ out in L^∞ -norm, which explains the difference in the powers of N and the time dependence of the bounds in (2.89), in cases $l = 1, l = 2$.

In order to prove (2.89) for $l = 3, 4$, we need to consider L^2 -norms of the terms in (2.11b)-(2.11f) and the terms in (2.12b)-(2.12d). Estimates of (2.12b) and (2.12c) are similar so let's make it for (2.12b):

$$\begin{aligned}
& (1/2N) \left\| \int dx \bar{p}(y_2, x) v_N(y_1 - x) u(x, y_4) u(y_3, y_1) \right\|_{L^2(dy_1 dy_2 dy_3 dy_4)} \\
& \leq (1/2N) \left\| \int dx v_N(y_1 - x) \|p(x, y_2)\|_{L^2(dy_2)} \|u(x, y_4)\|_{L^2(dy_4)} \|u(y_3, y_1)\|_{L^2(dy_3)} \right\|_{L^2(dy_1)} \\
& \leq (1/2N) \left\| \|u\|_2 \right\|_\infty^2 \left\| (v_N * \|p(\cdot, y_2)\|_{L^2(dy_2)}) \right\|_{L^2(dy_1)} \\
& \leq (1/2N) \left\| \|u\|_2 \right\|_\infty^2 \|v_N\|_1 \|p\|_2 \underset{\substack{\lesssim_\epsilon \\ \text{by} \\ (2.44) \text{ and } (2.22)}}{N^{-1+2\beta(1+\epsilon)} \log^3(1+t)}.
\end{aligned}$$

Estimates of (2.11b)-(2.11d) are similar but we need to pull out $\| \|u(y_2, y_1)\|_{L^2_{y_2}} \|_{L^\infty_{y_1}}$ in (2.11b), $\| \|u(y_3, x)\|_{L^2_{y_3}} \|_{L^\infty_x}$ in (2.11c), $\| \|u(y_3, y_1)\|_{L^2_{y_3}} \|_{L^\infty_{y_1}}$ in (2.11d) and also $\|\phi\|_\infty$ in all three of them, instead of the $\| \|u\|_2 \|_\infty^2$ factor in the above estimate. That again causes the difference in the powers of N and the time dependence of the bounds in (2.89) in cases $l = 3, l = 4$.

And our last estimate is for (2.12d):

$$\begin{aligned}
& (1/2N) \left\| \int dx_1 dx_2 \bar{p}(y_1, x_1) p(x_2, y_2) v_N(x_1 - x_2) u(y_3, x_1) u(x_2, y_4) \right\|_{L^2(dy_1 dy_2 dy_3 dy_4)} \\
& \leq (1/2N) \int dx_1 dx_2 \left\{ v_N(x_1 - x_2) \right. \\
& \quad \left. \times \|p(x_1, y_1)\|_{L^2(dy_1)} \|p(x_2, y_2)\|_{L^2(dy_2)} \|u(x_1, y_3)\|_{L^2(dy_3)} \|u(x_2, y_4)\|_{L^2(dy_4)} \right\} \\
& \leq (1/2N) \left\| \|p\|_2 \right\|_\infty \left\| \|u\|_2 \right\|_\infty \int dx_1 \left(v_N * (\|p(\cdot, y_2)\|_{L^2(dy_2)} \|u(\cdot, y_4)\|_{L^2(dy_4)}) \right) (x_1) \\
& \leq (1/2N) \left\| \|p\|_2 \right\|_\infty \left\| \|u\|_2 \right\|_\infty \|v_N\|_1 \|p\|_2 \|u\|_2 \stackrel{\substack{\lesssim_\epsilon \\ \text{by} \\ (2.44), \\ (2.22) \text{ and } (2.86)}}}{\leq} N^{-1+2\beta(1+\epsilon)} \log^4(1+t).
\end{aligned}$$

Estimates of (2.11e)-(2.11f) are similar but we need to pull out $\| \|u(x_2, y_3)\|_{L^2_{y_3}} \|_{L^\infty_{x_2}}$ in (2.11e), $\| \|p(x_2, y_2)\|_{L^2_{y_2}} \|_{L^\infty_{x_2}}$ in (2.11f) and also $\|\phi\|_\infty$ in both of them. \square

Proof of Theorem 2.8. Recalling the equation (2.14a) satisfied by $|\tilde{\psi}^r\rangle$ and the energy estimate (2.15) obtained from it, one can insert estimates in Lemma 2.9 into the energy estimate (2.15) and this implies our claim in Theorem 2.8. \square

2.5 The Singular Part of $|\tilde{\psi}\rangle$

The singular part of $|\tilde{\psi}\rangle$, denoted by $|\tilde{\psi}^s\rangle$, satisfies equation (2.14b). Let's recall from (2.16a)-(2.16b) how we split $|\tilde{\psi}^s\rangle$:

$$|\tilde{\psi}^s\rangle = |\tilde{\psi}_1^a\rangle + |\tilde{\psi}_1^e\rangle \text{ where}$$

$$\left(\frac{1}{i}\partial_t - \int L(t, x, y) a_x^* a_y dx dy\right) |\tilde{\psi}_1^a\rangle = (0, 0, F_2^s, F_3^s, F_4^s, 0, \dots), \quad (2.96a)$$

$$\left(\frac{1}{i}\partial_t - \mathcal{L}\right) |\tilde{\psi}_1^e\rangle = -N^{-1/2} \mathcal{E}(t) |\tilde{\psi}_1^a\rangle, \quad (2.96b)$$

$$|\tilde{\psi}_1^a(0)\rangle = |\tilde{\psi}_1^e(0)\rangle = 0.$$

First we want to obtain estimates on the components of $|\tilde{\psi}_1^a\rangle$ and then use them to estimate the error part $|\tilde{\psi}_1^e\rangle$. We would like to do the latter by applying an energy estimate to (2.96b). But the estimates we obtain for $|\tilde{\psi}_1^a\rangle$ will still not ensure sufficient L^2 -integrability for the components of the forcing term in (2.96b) as $N \rightarrow \infty$ and for β close to $1/2$. Hence we will need to split $|\tilde{\psi}_1^e\rangle$ further into its regular and singular parts and we will repeat similar splitting procedure for some finitely many times before a final application of an energy estimate.

Recalling the explicit formula for $L(t, x, y)$ from (2.6), let's define $\tilde{V}(t, x, y)$ via the equation

$$L(t, x, y) = \Delta_x \delta(x - y) - \tilde{V}(t, x, y). \quad (2.97)$$

Let's also define the operator

$$\mathbf{S}_j = \frac{1}{i} \partial_t - \Delta_{\mathbb{R}^{3j}} + \underbrace{\sum_{k=1}^j \overbrace{\left(\tilde{V}(t) \right)_k}^{\substack{\text{action of} \\ \tilde{V}(t) \text{ on a function} \\ \text{in the} \\ k\text{th variable}}}}_{\mathbf{V}_j} \quad (2.98)$$

Hence we have the following set of equations being equivalent to (2.96a):

$$\begin{aligned} \mathbf{S}_j \psi_1^{(j)} &= F_j^s \text{ with } \psi_1^{(j)}(0) = 0 \text{ for } j = 2, 3, 4 \text{ and} \\ |\tilde{\psi}_1^a\rangle &= (0, 0, \psi_1^{(2)}, \psi_1^{(3)}, \psi_1^{(4)}, 0, \dots). \end{aligned} \quad (2.99)$$

Our main result in this section is the following:

Theorem 2.10. *We have the following estimates for $\psi_1^{(j)}$ satisfying (2.99):*

$$\|\psi_1^{(2)}\|_{L^2(\mathbb{R}^6)} \lesssim_\epsilon N^{-1+\beta+\beta\epsilon} t \log(1+t) \text{ for any } \epsilon > 0, \quad (2.100a)$$

$$\|\psi_1^{(3)}\|_{L^2(\mathbb{R}^9)} \lesssim N^{\frac{-1+\beta}{2}}, \quad (2.100b)$$

$$\|\psi_1^{(4)}\|_{L^2(\mathbb{R}^{12})} \lesssim_\epsilon N^{-1+\frac{3\beta}{2}+\beta\epsilon} \sqrt{t} \log^2(1+t) \text{ for any } \epsilon > 0 \quad (2.100c)$$

which imply the following estimate for $|\tilde{\psi}_1^a\rangle$ satisfying (2.96a):

$$\| |\tilde{\psi}_1^a\rangle \|_{\mathbb{F}} \lesssim N^{\frac{-1+\beta}{2}} t \log^2(1+t) \text{ for } \beta < 1/2. \quad (2.101)$$

We will need the following lemmas to prove Theorem 2.10:

Lemma 2.11. (Christ-Kiselev Lemma, see e.g. Lemma 2.4 in [38]) *Let X, Y be Banach spaces, let I be a time interval, and let $K \in C^0(I \times I; B(X, Y))$ be a kernel taking values in the space of bounded operators from X to Y . Suppose that $1 \leq p < q \leq \infty$ is such that*

$$\left\| \int_I K(t, s) f(s) \, ds \right\|_{L^q(I, Y)} \lesssim \|f\|_{L^p(I, X)}$$

for all $f \in L^p(I, X)$. Then one also has

$$\left\| \int_{s \in I: s < t} K(t, s) f(s) \, ds \right\|_{L^q(I, Y)} \lesssim_{p, q} \|f\|_{L^p(I, X)}.$$

Lemma 2.12. *For the operator norm of \mathbf{V}_j defined in (2.98), we have the following*

estimate:

$$\begin{aligned} \|\mathbf{V}_j\|_{L^2(\mathbb{R}^{3j}) \rightarrow L^2(\mathbb{R}^{3j})} &\leq j \|\tilde{V}(t)\|_{L^2(\mathbb{R}^3) \rightarrow L^2(\mathbb{R}^3)} \\ &\lesssim \frac{j}{1+t^3} \|u(t)\|_{L^2(\mathbb{R}^6)}^4 \lesssim \frac{j \log^4(1+t)}{1+t^3}. \end{aligned} \quad (2.102)$$

Proof. The first inequality follows from the definition of \mathbf{V}_j in (2.98). For the second inequality let's write $\tilde{V}(t)$ explicitly recalling (2.6) and (2.97):

$$\begin{aligned} (\tilde{V}(t)f)(x) &= \int \tilde{V}(t, x, y) f(y) dy \\ &= (v_N * |\phi|^2)(t, x) f(x) + \int v_N(x-y) \phi(t, x) \bar{\phi}(t, y) f(y) dy \end{aligned} \quad (2.103a)$$

$$- \frac{1}{2} \left((\bar{c})^{-1} \circ m \circ \bar{u} + u \circ \bar{m} \circ (\bar{c})^{-1} + [\mathbf{W}(\bar{c}), (\bar{c})^{-1}] \right) \circ f. \quad (2.103b)$$

We can estimate L^2 -norms of the terms in (2.103a) as:

$$\|(v_N * |\phi|^2)f\|_2 \leq \|v_N\|_1 \|\phi\|_\infty^2 \|f\|_2 \lesssim \frac{1}{1+t^3} \|f\|_2, \quad (2.104)$$

$$\left\| \int v_N(x-y) \phi(t, x) \bar{\phi}(t, y) f(y) dy \right\|_2 \leq \|\phi\|_\infty^2 \|v_N * f\|_2 \lesssim \frac{1}{1+t^3} \|f\|_2. \quad (2.105)$$

where we used $\|\phi(t)\|_{L^\infty(\mathbb{R}^3)} \lesssim 1/(1+t^{3/2})$ from (2.24). Similarly to (2.105), one can prove for $m(t, x, y) = -v_N(x-y)\phi(t, x)\phi(t, y)$ that

$$\|m \circ l\|_{L^2(\mathbb{R}^6)} \lesssim \frac{1}{1+t^3} \|l\|_{L^2(\mathbb{R}^6)} \quad (2.106)$$

for any $l \in L^2(\mathbb{R}^6)$.

Recalling the relation $\bar{c}^2 = \bar{c} \circ \bar{c} = \delta(x - y) + u \circ \bar{u}$ and considering a contour Γ enclosing the spectrum of the non-negative Hilbert-Schmidt operator $q := u \circ \bar{u}$ one can write

$$\mathbf{W}(\bar{c}) = \mathbf{W}(\sqrt{1+q}) = \frac{1}{2\pi i} \int_{\Gamma} (q-z)^{-1} \underbrace{\mathbf{W}(q)}_{\substack{m\circ\bar{u}\circ\bar{c}-u\circ c\circ\bar{m} \\ \text{by (92) in [20]}}}(q-z)^{-1} \sqrt{1+z} dz \quad (2.107)$$

Since $(\bar{c})^{-1}$ and $(q-z)^{-1}$ have uniformly bounded operator norms and $|z| \lesssim \|u\|_2^2$, (2.106) and (2.107) help us dominate L^2 -norm of (2.103b) with

$$\frac{1}{1+t^3} \|u(t)\|_{L^2(\mathbb{R}^6)}^4 \|f\|_{L^2(\mathbb{R}^3)}.$$

This last bound considered together with the estimates in (2.104)-(2.105) proves the second inequality in (2.102). The last inequality in (2.102) follows from the estimate $\|u(t)\|_{L^2(\mathbb{R}^6)} \lesssim \log(1+t)$ as we recall from (2.44). \square

Proof of Theorem 2.10. (2.99) is equivalent to the following set of equations:

$$\psi_1^{(j)} = \psi_{1,a}^{(j)} + \psi_{1,e}^{(j)} \text{ where} \quad (2.108a)$$

$$\left(\frac{1}{i}\partial_t - \Delta_{\mathbb{R}^{3j}}\right)\psi_{1,a}^{(j)} = F_j^s, \quad (2.108b)$$

$$\mathbf{S}_j \psi_{1,e}^{(j)} = -\mathbf{V}_j \psi_{1,a}^{(j)}, \quad (2.108c)$$

$$\psi_{1,a}^{(j)}(0) = \psi_{1,e}^{(j)}(0) = 0 \text{ for } j = 2, 3, 4.$$

We will try to obtain estimates on $\|\psi_{1,a}^{(j)}\|_{L^2(\mathbb{R}^{3j})}$ using an elliptic estimate in case of

$j = 2$ and for the cases $j = 3, 4$ we will make use of the end-point Strichartz estimates along with TT^* -method (to be explained shortly) and Christ-Kiselev Lemma (see Lemma 2.11). Then we will use the following energy estimate to control $\psi_{1,e}^{(j)}$:

$$\begin{aligned}
\partial_t \|\psi_{1,e}^{(j)}\|_{L^2(\mathbb{R}^{3j})}^2 &\leq -2\text{Im} \left((\Delta_{\mathbb{R}^{3j}} - \mathbf{V}_j) \psi_{1,e}^{(j)} - \mathbf{V}_j \psi_{1,a}^{(j)}, \psi_{1,e}^{(j)} \right) \\
&\lesssim \|\mathbf{V}_j \psi_{1,a}^{(j)}\|_{L^2(\mathbb{R}^{3j})} \|\psi_{1,e}^{(j)}\|_{L^2(\mathbb{R}^{3j})} \quad \text{since } \mathbf{V}_j \text{ is self-adjoint} \\
&\lesssim \frac{j \log^4(1+t)}{1+t^3} \|\psi_{1,a}^{(j)}\|_{L^2(\mathbb{R}^{3j})} \|\psi_{1,e}^{(j)}\|_{L^2(\mathbb{R}^{3j})} \quad \text{by Lemma 2.12}
\end{aligned}$$

which implies

$$\|\psi_{1,e}^{(j)}(t)\|_{L^2(\mathbb{R}^{3j})} \lesssim \int_0^t \frac{j \log^4(1+t_1)}{1+t_1^3} \|\psi_{1,a}^{(j)}(t_1)\|_{L^2(\mathbb{R}^{3j})} dt_1. \quad (2.109)$$

Case 1: $j = 2$. For $j = 2$, recalling (2.13a), (2.108b) becomes:

$$\left(\frac{1}{i} \partial_t - \Delta_{\mathbb{R}^6} \right) \psi_{1,a}^{(2)} = -\frac{1}{2N} v_N(y_1 - y_2) \left\{ \overbrace{u(t, y_1, y_2) + (\bar{p} \circ u)(t, y_1, y_2)}^{\bar{c} \circ u = u \circ c = \frac{1}{2} s_2} \right\}. \quad (2.110)$$

Solving (2.110) by Duhamel's formula and using integration by parts we get

$$\begin{aligned}
\|\psi_{1,a}^{(2)}(t, \cdot)\|_{L^2(\mathbb{R}^6)} &\lesssim \left\| \int_0^t e^{it_1(|\xi|^2 + |\eta|^2)} \widehat{F}_2^s(t_1, \xi, \eta) dt_1 \right\|_{L^2(\mathbb{R}^6)} \\
&\lesssim \left\| \frac{\widehat{F}_2^s(0, \xi, \eta)}{|\xi|^2 + |\eta|^2} \right\|_{L^2(\mathbb{R}^6)} + \left\| \frac{\widehat{F}_2^s(t, \xi, \eta)}{|\xi|^2 + |\eta|^2} \right\|_{L^2(\mathbb{R}^6)} \\
&\quad + \left\| \int_0^t e^{it_1(|\xi|^2 + |\eta|^2)} \frac{\partial_{t_1} \widehat{F}_2^s(t_1, \xi, \eta)}{|\xi|^2 + |\eta|^2} dt_1 \right\|_{L^2(\mathbb{R}^6)}. \quad (2.111)
\end{aligned}$$

Now we need estimates of

$$\left\| \frac{\partial_t^j \widehat{F}_2^s(t, \xi, \eta)}{|\xi|^2 + |\eta|^2} \right\|_{L^2(\mathbb{R}^6)} \quad \text{for } j = 0, 1.$$

Writing

$$\partial_t^j F_2^s(t, x, y) = -\frac{1}{4N} v_N(x-y) \partial_t^j s_2(t, x, y) = -\frac{1}{4N} \int \delta(x-y-z) v_N(z) \partial_t^j s_2(t, x, y) dz$$

and considering the Fourier transform of $\delta(x-y-z) \partial_t^j s_2(t, x, y)$ in variables x, y :

$$e^{iz \cdot \eta} \partial_t^j \widehat{s}_2^z(t, \xi + \eta) \quad \text{where} \quad s_2^z(t, x) = s_2(t, x, x-z)$$

we can write

$$\begin{aligned} |\partial_t^j \widehat{F}_2^s(t, \xi, \eta)|^2 &= \frac{1}{16N^2} \left| \int v_N(z) e^{iz \cdot \eta} \partial_t^j \widehat{s}_2^z(t, \xi + \eta) dz \right|^2 \\ &\lesssim \frac{\|v\|_1}{N^2} \int |v_N(z)| |\partial_t^j \widehat{s}_2^z(t, \xi + \eta)|^2 dz. \end{aligned}$$

Hence after a change of variables

$$\begin{aligned} \left\| \frac{\partial_t^j \widehat{F}_2^s(t, \xi, \eta)}{|\xi|^2 + |\eta|^2} \right\|_{L^2(\mathbb{R}^6)}^2 &\lesssim \frac{1}{N^2} \int |v_N(z)| \frac{|\partial_t^j \widehat{s}_2^z(t, \xi)|^2}{(|\xi|^2 + |\eta|^2)^2} d\xi d\eta dz \\ &\lesssim \frac{1}{N^2} \int |v_N(z)| \underbrace{\left(\int \frac{|\partial_t^j \widehat{s}_2^z(t, \xi)|^2}{|\xi|} d\xi \right)}_{\lesssim \|D^{-1/2} \partial_t^j s_2^z\|_{L^2(\mathbb{R}^3)}^2} dz \\ &\stackrel{\text{by Trace theorem}}{\lesssim} \frac{1}{N^2} \|\partial_t^j s_2\|_{H^{\frac{3}{2}+}(\mathbb{R}^6)}^2. \end{aligned} \tag{2.112}$$

Now since $s_2 = s_a^0 + s_a^1 + s_e$, $\|\partial_t^j s_2\|_{H^2} \lesssim N^{3\beta/2} \log(1+t)$ by (2.48), (2.67)-(2.68) and (2.78)-(2.79). Interpolating this H^2 -norm estimate with $\|\partial_t^j s_2\|_{H^{3/2}} \lesssim N^{\beta(1+\epsilon)} \log(1+t)$ (see (2.20)) and applying the resulting estimate in (2.112) imply

$$\left\| \frac{\partial_t^j \widehat{F}_2^s(t, \xi, \eta)}{|\xi|^2 + |\eta|^2} \right\|_{L^2(\mathbb{R}^6)} \lesssim_\epsilon N^{-1+\beta+\beta\epsilon} \log(1+t) \quad \text{for any } \epsilon > 0 \text{ and for } j = 0, 1. \quad (2.113)$$

This inserted in (2.111) implies

$$\|\psi_{1,a}^{(2)}(t)\|_{L^\infty((0,t);L^2(\mathbb{R}^6))} \lesssim_\epsilon N^{-1+\beta+\beta\epsilon} t \log(1+t) \quad \text{for any } \epsilon > 0. \quad (2.114)$$

(2.114) considered with (2.109) for $j = 2$ gives

$$\|\psi_{1,e}^{(2)}(t)\|_{L^2(\mathbb{R}^6)} \lesssim_\epsilon N^{-1+\beta+\beta\epsilon} \int_0^t \frac{t_1 \log^5(1+t_1)}{1+t_1^3} dt_1 \lesssim N^{-1+\beta+\beta\epsilon} t \log(1+t).$$

Since $\psi_1^{(2)} = \psi_{1,a}^{(2)} + \psi_{1,e}^{(2)}$ as we recall from (2.108a), we can combine our last estimate with (2.114) to obtain

$$\boxed{\|\psi_1^{(2)}(t)\|_{L^2(\mathbb{R}^6)} \lesssim_\epsilon N^{-1+\beta+\beta\epsilon} t \log(1+t)} \quad \text{for any } \epsilon > 0. \quad (2.115)$$

Case 2: $j = 3$. For $j = 3$, recalling (2.13b), (2.108b) becomes:

$$\left(\frac{1}{i}\partial_t - \Delta_{\mathbb{R}^9}\right)\psi_{1,a}^{(3)} = -N^{-1/2}v_N(y_1 - y_2)\phi(t, y_2)u(t, y_1, y_3). \quad (2.116)$$

To put the forcing term in a more suitable form for the mixed space-time norm

estimates so that the power of N will depend on β in the desired way, we need the change of variables $x_1 = y_1 - y_2$ and $x_2 = y_1 + y_2$ which is inspired by the technique introduced in Lemma 4.6, [9] (see also the remark following Lemma 5.3 in [10]). So (2.116) takes the form

$$\begin{aligned} & \left(\frac{1}{i} \partial_t - 2(\Delta_{x_1} + \Delta_{x_2}) - \Delta_{y_3} \right) \psi_{1,a}^{(3)} \left(t, \frac{x_1+x_2}{2}, \frac{x_2-x_1}{2}, y_3 \right) \\ &= -N^{-1/2} v_N(x_1) \phi \left(t, \frac{x_2-x_1}{2} \right) u \left(t, \frac{x_1+x_2}{2}, y_3 \right). \end{aligned} \quad (2.117)$$

Now if we consider the solution operator $T := e^{it\{2(\Delta_{x_1}+\Delta_{x_2})+\Delta_{y_3}\}}$ for the corresponding free Schrödinger equation, we have the following estimate:

$$\begin{aligned} \|Tf_0\|_{L_t^2 L_{x_1}^6 L_{x_2 y_3}^2} &= \underbrace{\|Tf_0\|_{L_{x_2 y_3}^2}}_{=\|e^{2it\Delta_{x_1}} f_0\|_{L_{x_2 y_3}^2}} \Big\|_{L_t^2 L_{x_1}^6} \leq \underbrace{\|e^{2it\Delta_{x_1}} f_0\|_{L_t^2 L_{x_1}^6}}_{\lesssim \|f_0(x_1, x_2, y_3)\|_{L_{x_1}^2}} \Big\|_{L_{x_2 y_3}^2} \lesssim \|f_0\|_{L^2(\mathbb{R}^9)} \\ &\quad \text{by the end-point Strichartz estimates in dimension 3} \end{aligned}$$

which proves

$$T : L^2(\mathbb{R}^9) \rightarrow L_t^2 L_{x_1}^6 L_{x_2 y_3}^2. \quad (2.118)$$

Similarly we also have

$$T : L^2(\mathbb{R}^9) \rightarrow L_t^\infty L_{x_1 x_2 y_3}^2. \quad (2.119)$$

If we consider

$$(T^* f)(x_1, x_2, y_3) = \int_{\mathbb{R}} e^{-is\{2(\Delta_{x_1}+\Delta_{x_2})+\Delta_{y_3}\}} f(s, x_1, x_2, y_3) ds, \quad (2.120)$$

then (2.118) is equivalent to

$$T^* : L_t^2 L_{x_1}^{6/5} L_{x_2 y_3}^2 \rightarrow L^2(\mathbb{R}^9). \quad (2.121)$$

(2.119) and (2.121) imply

$$TT^* : L_t^2 L_{x_1}^{6/5} L_{x_2, y_3}^2 \rightarrow L_t^\infty L_{x_1, x_2, y_3}^2. \quad (2.122)$$

Using (2.122) and Christ Kiselev Lemma 2.11 with $K(t, s) = e^{i(t-s)\{2(\Delta_{x_1} + \Delta_{x_2}) + \Delta_{y_3}\}}$, f being the right hand side of (2.117), $X = L^{6/5}(\mathbb{R}^3; L^2(\mathbb{R}^6))$, $Y = L^2(\mathbb{R}^9)$ and $p = 2, q = \infty$, we obtain the first inequality in the following estimate:

$$\begin{aligned} \|\psi_{1,a}^{(3)}\|_{L^\infty((0,t); L^2(\mathbb{R}^9))} &\lesssim N^{-1/2} \|v_N(x_1) \phi(t, \frac{x_2-x_1}{2}) u(t, \frac{x_1+x_2}{2}, y_3)\|_{L_t^2 L_{x_1}^{6/5} L_{x_2, y_3}^2} \\ &\leq N^{-1/2} \left(\int_0^t \|\phi(t_1)\|_{L^\infty(\mathbb{R}^3)}^2 \right. \\ &\quad \left. \times \left(\int v_N^{6/5}(x_1) \left(\int |u(t_1, \frac{x_1+x_2}{2}, y_3)|^2 dx_2 dy_3 \right)^{\frac{1}{2}} \cdot \frac{6}{5} dx_1 \right)^{\frac{5}{6} \cdot 2} dt_1 \right)^{\frac{1}{2}} \\ &\lesssim N^{-1/2} \|v_N\|_{L^{6/5}(\mathbb{R}^3)} \left(\int_0^t \|\phi(t_1)\|_{L^\infty(\mathbb{R}^3)}^2 \|u(t_1)\|_{L^2(\mathbb{R}^6)}^2 dt_1 \right)^{1/2} \\ &\stackrel{\text{by (2.24) and (2.44)}}{\lesssim} N^{(-1+\beta)/2} \left(\int_0^t \frac{\log^2(1+t_1)}{1+t_1^3} dt_1 \right)^{1/2} \lesssim N^{(-1+\beta)/2}. \end{aligned} \quad (2.123)$$

This inserted in (2.109) for $j = 3$ implies:

$$\|\psi_{1,e}^{(3)}(t)\|_{L^2(\mathbb{R}^9)} \lesssim N^{(-1+\beta)/2} \int_0^t \frac{\log^4(1+t_1)}{1+t_1^3} dt_1 \lesssim N^{(-1+\beta)/2}.$$

Combining the last estimate with (2.123) gives

$$\boxed{\|\psi_1^{(3)}\|_{L^2(\mathbb{R}^9)} \lesssim N^{(-1+\beta)/2}} \quad (2.124)$$

since $\psi_1^{(3)} = \psi_{1,a}^{(3)} + \psi_{1,e}^{(3)}$ by (2.108a).

Case 3: $j = 4$. Finally for $j = 4$, recalling (2.13c), (2.108b) becomes:

$$\left(\frac{1}{i}\partial_t - \Delta_{\mathbb{R}^{12}}\right)\psi_{1,a}^{(4)} = -\frac{1}{2N}v_N(y_1 - y_2)u(t, y_3, y_1)u(t, y_2, y_4). \quad (2.125)$$

Doing the same change of variables as before, i.e. $x_1 = y_1 - y_2$ and $x_2 = y_1 + y_2$ in (2.125) and letting T denote the corresponding free propagator, this time we have $TT^* : L_t^2 L_{x_1}^{6/5} L_{x_2, y_3, y_4}^2 \rightarrow L_t^\infty L_{x_1, x_2, y_3, y_4}^2$. We again use Lemma 2.11 to obtain the first inequality in the following:

$$\begin{aligned} \|\psi_{1,a}^{(4)}\|_{L^\infty((0,t);L^2(\mathbb{R}^{12}))} &\lesssim N^{-1} \left\| v_N(x_1)u(t, y_3, \frac{x_1+x_2}{2})u(t, \frac{x_2-x_1}{2}, y_4) \right\|_{L_t^2 L_{x_1}^{6/5} L_{x_2, y_3, y_4}^2} \\ &= N^{-1} \left(\int_0^t \left(\int v_N^{6/5}(x_1) \right. \right. \\ &\quad \times \underbrace{\left(\int \|u(t_1, y_3, \frac{x_1+x_2}{2})\|_{L_{y_3}^2}^2 \|u(t_1, \frac{x_2-x_1}{2}, y_4)\|_{L_{y_4}^2}^2 dx_2 \right)^{\frac{1}{2} \cdot \frac{6}{5}} dx_1}_{\lesssim \|u(t_1, x, y)\|_{L_x^2}^4 \|u\|_{L_y^4}^4 \lesssim_\epsilon N^{2\beta(1+\epsilon)} \log^4(1+t_1) \text{ by (2.85)}} \left. dt_1 \right)^{\frac{1}{2}} \\ &\lesssim_\epsilon N^{-1+\beta(1+\epsilon)} \|v_N\|_{L^{6/5}} \left(\int_0^t \log^4(1+t_1) dt_1 \right)^{1/2} \\ &\lesssim N^{-1+\frac{3\beta}{2}+\beta\epsilon} t^{1/2} \log^2(1+t) \end{aligned} \quad (2.126)$$

for any $\epsilon < 0$. Inserting this in (2.109) for $j = 4$ implies

$$\|\psi_{1,e}^{(4)}(t)\|_{L^2(\mathbb{R}^{12})} \lesssim_\epsilon N^{-1+\frac{3\beta}{2}+\beta\epsilon} \int_0^t \frac{t_1^{1/2} \log^6(1+t_1)}{1+t_1^3} dt_1 \lesssim N^{-1+\frac{3\beta}{2}+\beta\epsilon} \log^2(1+t)$$

which, when combined with (2.126), gives

$$\boxed{\|\psi_1^{(4)}(t)\|_{L^2(\mathbb{R}^{12})} \lesssim_\epsilon N^{-1+\frac{3\beta}{2}+\beta\epsilon} t^{1/2} \log^2(1+t) \text{ for any } \epsilon > 0} \quad (2.127)$$

since $\psi_1^{(4)} = \psi_{1,a}^{(4)} + \psi_{1,e}^{(4)}$ by (2.108a). \square

Theorem 2.10, estimate (2.101) provides us with an estimate of $\|\tilde{\psi}_1^a\|_{\mathbb{F}}$ in case of $\beta < 1/2$, which decays as $N \rightarrow \infty$. We still need to estimate the error part $|\tilde{\psi}_1^e\rangle$. Recalling (2.96a)-(2.96b), at this point, one might think of applying the standard L^2 -energy estimate to (2.96b) to obtain

$$\|\tilde{\psi}_1^e(t)\|_{\mathbb{F}} \lesssim N^{-1/2} \int_0^t \|\mathcal{E}(t_1)|\tilde{\psi}_1^a(t_1)\|_{\mathbb{F}} dt_1 \quad (2.128)$$

in which we want to estimate the right hand side by using the estimates in Theorem 2.10. However, as we will explain shortly, we will not be able to pick up the desired powers of N from the estimate of $\|\mathcal{E}(t)|\tilde{\psi}_1^a(t)\|_{\mathbb{F}}$ to ensure a decay as $N \rightarrow \infty$ for $\beta < 1/2$. This problem is due to the contribution to $N^{-1/2}\mathcal{E}(t)$ coming from the term (2.8y), considered only with δ -parts of $c(x, y) = \text{ch}(k)(x, y) = \delta(x - y) + p(x, y)$ factors in it, namely

$$\frac{1}{2N} \int dy_1 dy_2 v_N(y_1 - y_2) \mathcal{Q}_{y_1 y_2}^* \mathcal{Q}_{y_1 y_2}. \quad (2.129)$$

Notice that this corresponds to the potential part of the original Hamiltonian (see (1.17)-(1.17c)). So let's define the Fock space operators

$$\tilde{\mathbb{H}} := \frac{1}{2N} \int dy_1 dy_2 v_N(y_1 - y_2) \mathcal{Q}_{y_1 y_2}^* \mathcal{Q}_{y_1 y_2}, \quad (2.130)$$

$$\mathbb{H} := N^{-1/2} \mathcal{E}(t) - \tilde{\mathbb{H}}. \quad (2.131)$$

Then we can rewrite (2.128) as

$$\begin{aligned} \|\tilde{\psi}_1^e(t)\|_{\mathbb{F}} &\lesssim \int_0^t \left\{ \underbrace{\|\mathbb{H} |\tilde{\psi}_1^a(t_1)\rangle\|_{\mathbb{F}} + \|\tilde{\mathbb{H}} |\tilde{\psi}_1^a(t_1)\rangle\|_{\mathbb{F}}}_{\substack{(0, 0, \psi_1^{(2)}(t_1), \psi_1^{(3)}(t_1), \psi_1^{(4)}(t_1), 0 \dots) \\ \text{from (2.99)}}} \right\} dt_1. \end{aligned} \quad (2.132)$$

We need the following operator norm estimates on \mathbb{H} and $\tilde{\mathbb{H}}$:

Lemma 2.13. *Based on the definitions (2.130)-(2.131), we have the following estimates for the actions of \mathbb{H} and $\tilde{\mathbb{H}}$ on the j th sector of Fock space:*

$$\begin{aligned} \|\mathbb{H}\psi^{(j)}\|_{\mathbb{F}} &\lesssim_{\epsilon, j} \left(N^{-1/2+\beta(1+\epsilon)} \log^4(1+t) + N^{-1+5\beta/2+\beta\epsilon} \log^2(1+t) \right. \\ &\quad \left. + N^{(-1+3\beta)/2} \frac{\log(1+t)}{1+t^{3/2}} \right) \|\psi^{(j)}\|_{L^2(\mathbb{R}^{3j})} \end{aligned} \quad (2.133a)$$

$$\|\tilde{\mathbb{H}}\psi^{(j)}\|_{\mathbb{F}} \lesssim N^{-1+3\beta} \|\psi^{(j)}\|_{L^2(\mathbb{R}^{3j})} \quad (2.133b)$$

for any $\psi^{(j)} \in L_s^2(\mathbb{R}^{3j})$ and $\epsilon > 0$.

We prove Lemma 2.13 in Appendix A.

Now turning back to the energy estimate (2.132), the inequalities given by (2.100a)-(2.100c) and (2.133a) imply that the first term inside the integral on the

left hand side of (2.132) i.e. $\|\mathbb{H}|\tilde{\psi}_1^a\rangle\|_{\mathbb{F}}$ is of order $N^{(-1+\beta)/2}N^{(-1+3\beta)/2}$ for $\beta < 1/2$ implying a decay as $N \rightarrow \infty$. However, the second term $\|\tilde{\mathbb{H}}|\tilde{\psi}_1^a\rangle\|_{\mathbb{F}}$ is of order $N^{-1+3\beta}N^{(-1+\beta)/2}$ using (2.100b) and (2.133b). In that case, we have a decay as $N \rightarrow \infty$ as long as we choose $\beta < 3/7$ which is not good enough but we can improve it as we will describe in the next section.

2.6 Iterating the Splitting Method

Let's recall how we split $|\tilde{\psi}\rangle$ which is defined by (2.1a) and satisfies equation (2.5). We first split $|\tilde{\psi}\rangle$ into its regular and singular parts as $|\tilde{\psi}^r\rangle + |\tilde{\psi}^s\rangle$ where $|\tilde{\psi}^r\rangle, |\tilde{\psi}^s\rangle$ satisfy equations (2.14a)-(2.14b) respectively. We obtained an estimate on $\| |\tilde{\psi}^r\rangle \|_{\mathbb{F}}$ in Theorem 2.8. We then split $|\tilde{\psi}^s\rangle$ into its approximate and error parts as $|\tilde{\psi}_1^a\rangle + |\tilde{\psi}_1^e\rangle$ where $|\tilde{\psi}_1^a\rangle, |\tilde{\psi}_1^e\rangle$ satisfy (2.96a)-(2.96b) respectively. We obtained an estimate on $\| |\tilde{\psi}_1^a\rangle \|_{\mathbb{F}}$ in Theorem 2.10. Theorems 2.8 and 2.10 not only provide with bounds that are slowly deteriorating in time but also imply a decay as $N \rightarrow \infty$ for $\beta < 1/2$. We then considered analyzing $|\tilde{\psi}_1^e\rangle$ to see if we can extend these observations to the case of the full error $\| |\psi_{\text{ex}}\rangle - |\psi_{\text{ap}}\rangle \|_{\mathbb{F}} = \| |\tilde{\psi}\rangle \|_{\mathbb{F}}$ since $|\tilde{\psi}\rangle = |\tilde{\psi}^r\rangle + |\tilde{\psi}_1^a\rangle + |\tilde{\psi}_1^e\rangle$. As we discussed at the end of the previous section, an approach based solely on the energy estimate (2.128), which is rewritten in (2.132), only provides with a bound which is meaningful as long as $\beta < 3/7$. The problem is due

to the term $\tilde{\mathbb{H}}|\tilde{\psi}_1^a\rangle$ on the right hand side of the equation for $|\tilde{\psi}_1^e\rangle$:

$$\begin{aligned} \left(\frac{1}{i}\partial_t - \mathcal{L}\right)|\tilde{\psi}_1^e\rangle &= \overbrace{-\mathbb{H}|\tilde{\psi}_1^a\rangle - \tilde{\mathbb{H}}|\tilde{\psi}_1^a\rangle}^{-N^{-1/2}\mathcal{E}(t)|\tilde{\psi}_1^a\rangle \text{ by (2.131)}} \\ &\quad \uparrow \\ &\quad \underbrace{(0, 0, \psi_1^{(2)}, \psi_1^{(3)}, \psi_1^{(4)}, 0, \dots)}_{\psi_1^{(j)} \text{ satisfy (2.99) which is equivalent to (2.96a)}} \end{aligned}$$

For an improvement, we now consider splitting $|\tilde{\psi}_1^e\rangle$ into its regular and singular parts as $|\tilde{\psi}_1^r\rangle + |\tilde{\psi}_1^s\rangle$ where

$$\begin{aligned} \left(\frac{1}{i}\partial_t - \mathcal{L}\right)|\tilde{\psi}_1^r\rangle &= -\mathbb{H}|\tilde{\psi}_1^a\rangle \text{ with } |\tilde{\psi}_1^r(0)\rangle = 0, \\ \left(\frac{1}{i}\partial_t - \mathcal{L}\right)|\tilde{\psi}_1^s\rangle &= -\tilde{\mathbb{H}}|\tilde{\psi}_1^a\rangle \text{ with } |\tilde{\psi}_1^s(0)\rangle = 0 \end{aligned}$$

and then we again split $|\tilde{\psi}_1^s\rangle$ into its approximate and error parts as $|\tilde{\psi}_2^a\rangle + |\tilde{\psi}_2^e\rangle$ where

$$\begin{aligned} \left(\frac{1}{i}\partial_t - \int L(t, x, y) a_x^* a_y dx dy\right)|\tilde{\psi}_2^a\rangle &= -\tilde{\mathbb{H}}|\tilde{\psi}_1^a\rangle \text{ with } |\tilde{\psi}_2^a(0)\rangle = 0, \\ \left(\frac{1}{i}\partial_t - \mathcal{L}\right)|\tilde{\psi}_2^e\rangle &= -N^{-1/2}\mathcal{E}(t)|\tilde{\psi}_2^a\rangle = -\mathbb{H}|\tilde{\psi}_2^a\rangle - \tilde{\mathbb{H}}|\tilde{\psi}_2^a\rangle \text{ with } |\tilde{\psi}_2^e(0)\rangle = 0 \end{aligned}$$

where $|\tilde{\psi}_2^a\rangle = (0, 0, \psi_2^{(2)}, \psi_2^{(3)}, \psi_2^{(4)}, 0, \dots)$ and

$$\mathbf{S}_l \psi_2^{(l)} \underset{\substack{\downarrow \\ \text{up to} \\ \text{symmetrizations}}}{=} -\frac{1}{2N} v_N (y_1 - y_2) \psi_1^{(l)}(t, y_1, y_2, \dots, y_l), \quad l = 2, 3, 4 \quad (\text{recalling (2.98), (2.130)}).$$

We will iterate splitting in this manner for $j - 1$ times and at j th step we will only split into approximate and error parts as $|\tilde{\psi}_{j-1}^s\rangle = |\tilde{\psi}_j^a\rangle + |\tilde{\psi}_j^e\rangle$ where j is to be

determined later. We can summarize our iteration scheme by the following set of equations:

$$|\tilde{\psi}\rangle = |\tilde{\psi}^r\rangle + |\tilde{\psi}_1^a\rangle + |\tilde{\psi}_1^r\rangle + \underbrace{|\tilde{\psi}_2^a\rangle + \cdots + |\tilde{\psi}_{j-1}^r\rangle}_{|\tilde{\psi}_{j-1}^s\rangle} + \underbrace{|\tilde{\psi}_j^a\rangle + |\tilde{\psi}_j^e\rangle}_{|\tilde{\psi}_{j-1}^e\rangle} \quad \text{where} \quad (2.134a)$$

$$\left(\frac{1}{i}\partial_t - \mathcal{L}\right)|\tilde{\psi}^r\rangle = (0, F_1, F_2^r, F_3^r, F_4^r, 0, \dots) \quad \text{with} \quad |\tilde{\psi}^r(0)\rangle = 0, \quad (2.134b)$$

$$\left(\frac{1}{i}\partial_t - \int L(t, x, y) a_x^* a_y dx dy\right)|\tilde{\psi}_1^a\rangle = (0, 0, F_2^s, F_3^s, F_4^s, 0, \dots) \quad \text{with} \quad |\tilde{\psi}_1^a(0)\rangle = 0, \quad (2.134c)$$

$$\left(\frac{1}{i}\partial_t - \mathcal{L}\right)|\tilde{\psi}_1^r\rangle = -\mathbb{H}|\tilde{\psi}_1^a\rangle \quad \text{with} \quad |\tilde{\psi}_1^r(0)\rangle = 0, \quad (2.134d)$$

$$\left(\frac{1}{i}\partial_t - \int L(t, x, y) a_x^* a_y dx dy\right)|\tilde{\psi}_2^a\rangle = -\tilde{\mathbb{H}}|\tilde{\psi}_1^a\rangle \quad \text{with} \quad |\tilde{\psi}_2^a(0)\rangle = 0, \quad (2.134e)$$

⋮

$$\left(\frac{1}{i}\partial_t - \mathcal{L}\right)|\tilde{\psi}_{j-1}^r\rangle = -\mathbb{H}|\tilde{\psi}_{j-1}^a\rangle \quad \text{with} \quad |\tilde{\psi}_{j-1}^r(0)\rangle = 0, \quad (2.134f)$$

$$\left(\frac{1}{i}\partial_t - \int L(t, x, y) a_x^* a_y dx dy\right)|\tilde{\psi}_j^a\rangle = -\tilde{\mathbb{H}}|\tilde{\psi}_{j-1}^a\rangle \quad \text{with} \quad |\tilde{\psi}_j^a(0)\rangle = 0, \quad (2.134g)$$

$$\left(\frac{1}{i}\partial_t - \mathcal{L}\right)|\tilde{\psi}_j^e\rangle = -N^{-1/2}\mathcal{E}(t)|\tilde{\psi}_j^a\rangle \quad \text{with} \quad |\tilde{\psi}_j^e(0)\rangle = 0 \quad \text{where} \quad (2.134h)$$

$$|\tilde{\psi}_j^a\rangle := (0, 0, \psi_j^{(2)}, \psi_j^{(3)}, \psi_j^{(4)}, 0, \dots) \quad \text{and} \quad (2.134i)$$

$$\mathbf{S}_t \psi_j^{(l)} \underset{\substack{\uparrow \\ \text{means} \\ \text{"equal up to} \\ \text{symmetrizations"}}}{\simeq}_l -\frac{1}{2N} v_N(y_1 - y_2) \psi_{j-1}^{(l)}(t, y_1, y_2, \dots, y_l), \quad l = 2, 3, 4. \quad (\text{recalling (2.98), (2.130)}).$$

We have the following result on the inductive step of the iteration:

Theorem 2.14. *Under the above setting and based on the estimates in Theorem 2.10 and Lemma 2.13, we have the following estimates:*

$$\|\tilde{\psi}_j^{(1)}(t)\|_{\mathbb{F}} \lesssim N^{j(-1+2\beta)} t^{(j+3)/2} \log^6(1+t), \quad (2.135a)$$

$$\|\psi_j^{(2)}(t)\|_{L^2(\mathbb{R}^6)} \lesssim_\epsilon N^{(j-1)(-1+2\beta)} N^{-1+\beta+\beta\epsilon} t^{(j+1)/2} \log(1+t), \quad (2.135b)$$

$$\|\psi_j^{(3)}(t)\|_{L^2(\mathbb{R}^9)} \lesssim N^{(j-1)(-1+2\beta)} N^{(-1+\beta)/2} t^{(j-1)/2}, \quad (2.135c)$$

$$\|\psi_j^{(4)}(t)\|_{L^2(\mathbb{R}^{12})} \lesssim_\epsilon N^{(j-1)(-1+2\beta)} N^{-1+\beta(3/2+\epsilon)} t^{j/2} \log^2(1+t) \quad (2.135d)$$

for all $j \geq 1$ and for every $\epsilon > 0$.

Proof. Let's prove first prove (2.135b)-(2.135d). The case $j = 1$ for (2.135b)-(2.135b) was handled in Theorem 2.10. Hence, for the inductive step, assuming (2.135b)-(2.135d), we will provide with a proof of the case $j + 1$.

Now let's consider the equation (2.134g) by replacing j with $j + 1$. It will be equivalent to the following set of equations where we need to recall (2.98), (2.130):

$$\begin{aligned} \mathbf{S}_l \psi_{j+1}^{(l)} &\simeq_l -\frac{1}{2N} v_N(y_1 - y_2) \psi_j^{(l)}(t, y_1, y_2, \dots, y_l) \text{ with } \psi_{j+1}^{(l)}(0) = 0 \text{ for } l = 2, 3, 4 \text{ and} \\ |\tilde{\psi}_{j+1}^a\rangle &= (0, 0, \psi_{j+1}^{(2)}, \psi_{j+1}^{(3)}, \psi_{j+1}^{(4)}, 0, \dots). \end{aligned} \quad (2.136a)$$

We can split $\psi_{j+1}^{(l)}$ similar to what we did in Theorem 2.10 as follows:

$$\psi_{j+1}^{(l)} = \psi_{j+1,a}^{(l)} + \psi_{j+1,e}^{(l)} \text{ where} \quad (2.137a)$$

$$\left(\frac{1}{i}\partial_t - \Delta_{\mathbb{R}^{3l}}\right)\psi_{j+1,a}^{(l)} \simeq_l -\frac{1}{2N}v_N(y_1 - y_2)\psi_j^{(l)}(t, y_1, y_2, \dots, y_l), \quad (2.137b)$$

$$\mathbf{S}l\psi_{j+1,e}^{(l)} = -\mathbf{V}l\psi_{j+1,a}^{(l)}, \quad (2.137c)$$

$$\psi_{j+1,a}^{(l)}(0) = \psi_{j+1,e}^{(l)}(0) = 0 \text{ for } l = 2, 3, 4$$

and again after estimating $\|\psi_{j+1,a}^{(l)}\|_{L^2(\mathbb{R}^{3l})}$, we can use the energy estimate

$$\begin{aligned} \|\psi_{j+1,e}^{(l)}(t)\|_{L^2(\mathbb{R}^{3l})} &\leq \int_0^t \|\mathbf{V}l(t_1)\psi_{j+1,a}^{(l)}(t_1)\|_{L^2(\mathbb{R}^{3l})} dt_1 \\ &\lesssim_l \int_0^t \frac{\log^4(1+t_1)}{1+t_1^3} \|\psi_{j+1,a}^{(l)}(t_1)\|_{L^2(\mathbb{R}^{3l})} dt_1. \end{aligned} \quad (2.138)$$

by Lemma 2.12

Hence let's prove the estimate on $\|\psi_{j+1,a}^{(l)}\|_{L^2(\mathbb{R}^{3l})}$ first. Similar to Case 2 in the proof of Theorem 2.10, after a change of variables in equation (2.137b) and using Strichartz estimates, TT^* -method and Christ-Kiselev Lemma we can make the following estimate:

$$\begin{aligned} &\|\psi_{j+1,a}^{(l)}(t)\|_{L^\infty((0,t);L^2(\mathbb{R}^{3l}))} \\ &\lesssim_l N^{-1} \left(\int_0^t \left\| v_N(x_1)\psi_j^{(l)}\left(t_1, \frac{x_1+x_2}{2}, \frac{x_2-x_1}{2}, y_3, \dots, y_l\right) \right\|_{L_{x_1}^{6/5} L_{x_2 y_3 \dots y_l}^2}^2 dt_1 \right)^{1/2} \\ &= N^{-1} \left(\int_0^t \left(\int \underbrace{v_N^{6/5}(x_1)}_{5/2} \right. \right. \\ &\quad \left. \left. \times \underbrace{\left(\int |\psi_j^{(l)}(t_1, \frac{x_1+x_2}{2}, \frac{x_2-x_1}{2}, y_3, \dots, y_l)|^2 dx_2 dy_3 \dots dy_l \right)^{\frac{1}{2} \cdot \frac{6}{5}} dx_1}_{5/3} \right)^{\frac{5}{6} \cdot 2} dt_1 \right)^{1/2} \end{aligned}$$

$$\underset{\substack{\text{H\"older} \\ \text{in } x_1 \\ \text{with} \\ (5/2, 5/3)}}{\lesssim} N^{-1} \underbrace{\|v_N\|_{L^3(\mathbb{R}^3)}}_{\lesssim N^{2\beta}} \left(\int_0^t \|\psi_j^{(l)}(t_1)\|_{L^2(\mathbb{R}^{3l})}^2 dt_1 \right)^{1/2} \quad \text{for } l = 2, 3, 4.$$

Now inserting the bounds in (2.135b)-(2.135d) in the last line of the above estimate implies

$$\|\psi_{j+1,a}^{(l)}\|_{L^\infty((0,t);L^2(\mathbb{R}^{3l}))} \lesssim N^{j(-1+2\beta)} \cdot \begin{cases} N^{-1+\beta(1+\epsilon)} t^{(j+2)/2} \log(1+t) & \text{for } l = 2 \\ N^{(-1+\beta)/2} t^{j/2} & \text{for } l = 3 \\ N^{-1+\beta(3/2+\epsilon)} t^{(j+1)/2} \log^2(1+t) & \text{for } l = 4 \end{cases}$$

\uparrow
 constant
 involved
 depends on ϵ for $l=2,4$

Finally inserting this in (2.138) yields the same bounds for $\|\psi_{j+1,e}^{(l)}(t)\|_{L^\infty((0,t);L^2(\mathbb{R}^{3l}))}$ because $\log^4(1+t)/(1+t^3)$ inside the integral in line (2.138) is integrable. Since $\psi_{j+1}^{(l)} = \psi_{j+1,a}^{(l)} + \psi_{j+1,e}^{(l)}$, we completed the inductive step of proving (2.135b)-(2.135d).

Now let's move on to proving (2.135a). Replacing $j-1$ with j in (2.134f), applying the L^2 -energy estimate to the resulting equation and using (2.135b)-(2.135d) we can make the following estimate for any $j \geq 1$:

$$\begin{aligned} & \underbrace{(0, 0, \psi_j^{(2)}(t_1), \psi_j^{(3)}(t_1), \psi_j^{(4)}(t_1), 0, \dots)}_{\uparrow} \\ \|\tilde{\psi}_j^r(t)\|_{\mathbb{F}} & \lesssim \int_0^t \|\mathbb{H}[\tilde{\psi}_j^a(t_1)]\|_{\mathbb{F}} dt_1 \\ & \lesssim_{\substack{\epsilon \\ \text{by} \\ (2.133a), \\ (2.135b)-(2.135d)}} \int_0^t N^{(-1+3\beta)/2} N^{(-1+\beta)/2+(j-1)(-1+2\beta)} t_1^{(j+1)/2} \log^6(1+t_1) dt_1 \end{aligned} \quad (2.139)$$

which implies (2.135a). □

Now let's see what the energy estimate applied to (2.134h) would imply if we

were to stop the iteration at the j th step:

Corollary 2.15. For $|\tilde{\psi}_j^e\rangle$ satisfying equation (2.134h) which is

$$\left(\frac{1}{i}\partial_t - \mathcal{L}\right)|\tilde{\psi}_j^e\rangle = -N^{-1/2}\mathcal{E}(t)|\tilde{\psi}_j^a\rangle \text{ with } |\tilde{\psi}_j^e(0)\rangle = 0$$

we have the following estimate

$$\begin{aligned} & \left\| |\tilde{\psi}_j^e(t)\rangle \right\|_{\mathbb{F}} \\ & \lesssim_j \left(N^{j(-1+2\beta)} + N^{-1+3\beta+(j-1)(-1+2\beta)+(-1+\beta)/2} \right) t^{(j+3)/2} \log^6(1+t). \end{aligned} \quad (2.140)$$

In particular, $\left\| |\tilde{\psi}_j^e(t)\rangle \right\|_{\mathbb{F}} = O(N^{(-3+7\beta)/2+(j-1)(-1+2\beta)})$ for $1/3 \leq \beta < 1/2$. To ensure a decay we also need to choose

$$\beta < \frac{1+2j}{3+4j}.$$

Hence, if j is sufficiently large, β will be as close as desired to $1/2$ in which case we will also have $\left\| |\tilde{\psi}_j^e(t)\rangle \right\|_{\mathbb{F}}$ decaying as $N \rightarrow \infty$.

Proof. Applying the standard energy estimate to (2.134h) gives

$$\begin{aligned} \left\| |\tilde{\psi}_j^e(t)\rangle \right\|_{\mathbb{F}} & \lesssim \int_0^t \left\| N^{-1/2}\mathcal{E}(t_1)|\tilde{\psi}_j^a(t_1)\rangle \right\|_{\mathbb{F}} dt_1 \\ & \lesssim \int_0^t \left\{ \underbrace{\left\| \mathbb{H}|\tilde{\psi}_j^a(t_1)\rangle \right\|_{\mathbb{F}}}_{\lesssim N^{j(-1+2\beta)}t_1^{(j+1)/2}\log^6(1+t_1) \text{ as in line (2.139)}} + \underbrace{\left\| \tilde{\mathbb{H}}|\tilde{\psi}_j^a(t_1)\rangle \right\|_{\mathbb{F}}}_{\lesssim N^{-1+3\beta}N^{(j-1)(-1+2\beta)}N^{(-1+\beta)/2}t_1^{(j+1)/2}\log^2(1+t_1) \text{ by (2.133b) and (2.135b)-(2.135d)}} \right\} dt_1 \end{aligned}$$

which implies estimate (2.140). □

2.7 Final Step

Proof of Theorem 1.6 (Main result). Considering (2.134a), Theorem 2.8, Theorem 2.14 and Corollary 2.15, we have

$$\begin{aligned}
& \|\tilde{\psi}(t)\|_{\mathbb{F}} \\
& \leq \underbrace{\|\tilde{\psi}^r(t)\|_{\mathbb{F}}}_{\substack{\text{by Theorem 2.8} \\ \lesssim_{\epsilon} N^{-1/2+\beta(1+\epsilon)} t \log^4(1+t)}} + \underbrace{\sum_{m=1}^j \|\tilde{\psi}_m^a(t)\|_{\mathbb{F}}}_{\substack{\text{by Theorem 2.14 for } \beta < 1/2, \\ \lesssim N^{(-1+\beta)/2} t^{(j+1)/2} \log^2(1+t)}} \\
& \quad + \underbrace{\sum_{m=1}^{j-1} \|\tilde{\psi}_m^r(t)\|_{\mathbb{F}}}_{\substack{\lesssim N^{-1+2\beta} t^{(j+3)/2} \log^6(1+t) \\ \text{by Theorem 2.14}}} + \underbrace{\|\tilde{\psi}_j^e(t)\|_{\mathbb{F}}}_{\substack{\lesssim \begin{cases} N^{j(-1+2\beta)} t^{(j+3)/2} & \text{if } 0 < \beta < 1/3 \\ N^{(-3+7\beta)/2+(j-1)(-1+2\beta)} t^{(j+3)/2} \log^6(1+t) & \text{if } 1/3 \leq \beta < 1/2 \end{cases} \\ \text{by Corollary 2.15}}} \\
& \lesssim_{\epsilon} \left(N^{-1/2+\beta(1+\epsilon)} + N^{(-3+7\beta)/2+(j-1)(-1+2\beta)} \right) t^{(j+3)/2} \log^6(1+t). \tag{2.141}
\end{aligned}$$

For $0 < \beta \leq 2j/(1 - 2\epsilon + 4j)$, (2.141) will decay as $N^{-1/2+\beta(1+\epsilon)}$ as $N \rightarrow \infty$ and for $2j/(1 - 2\epsilon + 4j) < \beta < (1+2j)/(3+4j)$, (2.141) will decay as $N^{(-3+7\beta)/2+(j-1)(-1+2\beta)}$, which implies estimate (1.41). \square

2.8 Uncoupled System: Error Estimates only up to $\beta < 1/2$

While, in the current work, we extend the estimates on the error to the case of $\beta < 1/2$ as stated in our main result (Theorem 1.6), we can also provide here with the following heuristic argument suggesting that the uncoupled system consisting of (1.38) does not provide an approximation for $\beta \geq 1/2$. Indeed, we can write $|\tilde{\psi}\rangle$

e.g. as

$$|\tilde{\psi}\rangle = (0, \psi_{(2.9i)}, 0, \dots) + \text{other contributions}$$

where $\psi_{(2.9i)}$ satisfies

$$\begin{aligned} & \left(\frac{1}{i} \partial_t - \Delta_{\mathbb{R}^3} \right) \psi_{(2.9i)}(t, y) \\ &= N^{-1/2} \int v_N(y-x) u(t, y, x) \bar{\phi}^{(N)}(t, x) dx \quad \text{with} \quad \psi_{(2.9i)}(0) = 0 \end{aligned} \quad (2.142)$$

in which the integral term on the right hand side comes from (2.9i). We added the superscript (N) to ϕ for recalling that it solves (1.38a) and hence it is N -dependent. We could have checked other similar contributions coming from (2.9a)-(2.12d) but we will consider (2.9i) as a typical example. At this point using (2.19a) (i.e. (1.38b)) we can consider an approximate equation for $u = \text{sh}(k)$. Recalling $s_2 = \text{sh}(2k) = 2u \circ c = 2u + 2u \circ p$, let's just look at

$$\frac{1}{i} \partial_t u - \Delta u + v_N(y_1 - y_2) \phi^{(N)}(t, y_1) \phi^{(N)}(t, y_2) = 0.$$

If we make the change of variables $x_1 := y_1 - y_2$ and $x_2 := y_1 + y_2$ then we have

$$\left(\frac{1}{i} \partial_t - 2(\Delta_{x_1} + \Delta_{x_2}) \right) u(t, \frac{x_1+x_2}{2}, \frac{x_2-x_1}{2}) = -v_N(x_1) \phi^{(N)}(t, \frac{x_1+x_2}{2}) \phi^{(N)}(t, \frac{x_2-x_1}{2}).$$

Hence one can consider an ‘‘approximate’’ solution

$$u(t, y_1, y_2) = -N^\beta w(N^\beta(y_1 - y_2)) \phi^{(N)}(t, y_1) \phi^{(N)}(t, y_2) \quad \text{where} \quad \Delta w = -\frac{1}{2}v.$$

Inserting the above ansatz in (2.142) gives

$$\left(\frac{1}{i}\partial_t - \Delta_{\mathbb{R}^3}\right)\psi_{(2.9i)} = -N^{\beta-1/2} \underbrace{\left\{ \left(N^{3\beta} v(N^\beta \cdot) w(N^\beta \cdot) \right) * |\phi^{(N)}|^2 \right\}}_{\substack{\text{converges to } (\int vw)|\phi(t,y)|^2\phi(t,y) \text{ as } N \rightarrow \infty \\ \text{since } \phi^{(N)} \rightarrow \phi \text{ in } L^2 \text{ as in Appendix B}}}(t, y)\phi^{(N)}(t, y)$$

where $\phi^{(N)}$ and ϕ solve equations (1.38a) and (1.6) for $0 < \beta < 1$ respectively.

Hence, to ensure a decay for $\psi_{(2.9i)}$ as $N \rightarrow \infty$, we have to consider $\beta < 1/2$.

2.9 Conclusions

In this chapter we provided a quantitative derivation of some effective evolution equations for the dynamics of a bosonic system of N -particles interacting via two-body potential $v_N(x) = N^{3\beta}v(N^\beta x)$, $x \in \mathbb{R}^3$, $1/3 \leq \beta < 1/2$. This together with previous results gives explicit rates of convergence for a Fock space approximation of the exact dynamics in case of short-range strong interactions described by $\beta < 1/2$. The approximation scheme employed here considers an appropriate description of pair excitations as a correction to mean field. We also provided with an argument showing that, with the uncoupled system of evolution equations at hand, the same approximation works well only up to $\beta < 1/2$. Our rates of convergence deteriorates more slowly in time compared to the exponential deterioration typical of previous works.

Chapter 3: Proof of Error Estimates for Marginals:

Theorems 1.8 and 1.9

3.1 Main Ideas and the General Strategy

Throughout this chapter *unless stated otherwise* let $\phi^{(N)}$ denote the solution of (1.38a)

$$\frac{1}{i} \partial_t \phi^{(N)} = \Delta \phi^{(N)} - (v_N * |\phi^{(N)}|^2) \phi^{(N)}$$

where $v_N = N^{3\beta} v(N^\beta \cdot)$ and $\phi^{(N)}(0, \cdot) = \phi_0$

and ϕ denote the solution to the equation (1.44)

$$\frac{1}{i} \partial_t \phi = \Delta \phi - \begin{cases} (v * |\phi|^2) \phi & \text{if } \beta = 0 \\ (\int v(x) dx) |\phi|^2 \phi & \text{if } 0 < \beta < 1 \end{cases} \quad \text{with } \phi(0, \cdot) = \phi_0$$

with initial data ϕ_0 as described in Theorem 1.8. Note that in case of $\beta = 0$, $\phi^{(N)} = \phi$ and for $0 < \beta < 1$, $\phi^{(N)} \rightarrow \phi$ in $L^2(\mathbb{R}^3)$ as proved in Appendix B.

Based on this let's recall the exact and approximate evolutions of previous

chapters by rewriting them here:

$$|\psi_{\text{ex}}(t)\rangle = e^{it\mathcal{H}}e^{-\sqrt{N}\mathcal{A}(\phi_0)}|0\rangle \quad (3.1)$$

$$|\psi_{\text{ap}}(t)\rangle = e^{iN\chi(t)}e^{-\sqrt{N}\mathcal{A}(\phi^{(N)}(t))}e^{-\mathcal{B}(k(t))}|0\rangle \quad (3.2)$$

along with the reduced dynamics

$$|\psi_{\text{red}}(t)\rangle = e^{\mathcal{B}(k(t))}e^{\sqrt{N}\mathcal{A}(\phi^{(N)}(t))}\underbrace{e^{it\mathcal{H}}e^{-\sqrt{N}\mathcal{A}(\phi_0)}|0\rangle}_{|\psi_{\text{ex}}(t)\rangle} \quad (3.3)$$

i.e. propagate forward using the exact dynamics until time t and come back following the approximate evolution. As noted earlier, due to $e^{\sqrt{N}\mathcal{A}}$ and $e^{\mathcal{B}}$ being unitary, we have

$$\| \underbrace{e^{-iN\chi(t)}|\psi_{\text{red}}(t)\rangle - |0\rangle}_{|\tilde{\psi}(t)\rangle} \|_{\mathbb{F}} = \| |\psi_{\text{ex}}(t)\rangle - |\psi_{\text{ap}}(t)\rangle \|_{\mathbb{F}} \quad (3.4)$$

which is supposed to be small in the limit of large N due to error estimates of section 1.3, provided $\phi^{(N)}$ and k satisfy suitable equations and the phase factor $\chi(t)$ is chosen accordingly. This in turn implies that $|\psi_{\text{red}}(t)\rangle$ stays close to the vacuum and hence the expected number of particles $\langle \psi_{\text{red}} | \mathcal{N} | \psi_{\text{red}} \rangle$ at the reduced dynamics should not grow fast.

The above observation will help us summarize our general strategy in proving Theorems 1.8 and 1.9:

Step 1: Estimate $\langle \psi_{\text{red}} | \mathcal{N} | \psi_{\text{red}} \rangle$ in terms of the error $\| |\tilde{\psi}\rangle \|_{\mathbb{F}} = \| |\psi_{\text{ex}}\rangle - |\psi_{\text{ap}}\rangle \|_{\mathbb{F}}$.

This is common to both proofs hence we prefer to do it first. When doing so, we will benefit from the conservation of number of particles by the exact dynamics.

Step 2: Estimate $\text{Tr}|\Gamma_{\text{ex}}^{(1)} - |\phi\rangle\langle\phi||$ and $\text{Tr}|\gamma_N^{(1)} - |\phi\rangle\langle\phi||$ in terms of $\langle\psi_{\text{red}}|\mathcal{N}|\psi_{\text{red}}\rangle$ and obtain bounds using step 1.

Step 3: Use the Fock space estimates on $\|\tilde{\psi}\|_{\mathbb{F}} = \|\psi_{\text{ex}} - \psi_{\text{ap}}\|_{\mathbb{F}}$ from section 1.3 to obtain final bounds.

We will complete step 1 in the next section. Then we will present proofs for Theorems 1.8 and 1.9 in sections 3.3 and 3.4 respectively, where we will follow steps 2 and 3 in each case.

3.2 Estimating $\langle\psi_{\text{red}}|\mathcal{N}|\psi_{\text{red}}\rangle$ in Terms of the Error $\|\tilde{\psi}\|_{\mathbb{F}}$

Our main result in this section is the following:

Proposition 3.1. *Let*

$$|\psi_{\text{red}}(t)\rangle = e^{\mathcal{B}(k(t))} e^{\sqrt{N}\mathcal{A}(\phi^{(N)}(t))} e^{it\mathcal{H}} e^{-\sqrt{N}\mathcal{A}(\phi_0)} |0\rangle$$

$$|\tilde{\psi}(t)\rangle = e^{-iN\chi(t)} |\psi_{\text{red}}(t)\rangle - |0\rangle$$

$$\text{and } u = \text{sh}(k)$$

as before. Let $\phi^{(N)}$ with $\|\phi^{(N)}(t)\|_{L^2(\mathbb{R}^3)} = 1$ and $k(t, x, y) \in L^2(\mathbb{R}^6)$ symmetric in (x, y) satisfy suitable equations with prescribed initial data $\phi(0, \cdot) = \phi_0$ and $k(0, \cdot, \cdot) = 0$ so that the error $\|\tilde{\psi}\|_{\mathbb{F}}$ is small (in the context of the current work

these are the equations in (1.38)). Then we have the following estimate:

$$\langle \psi_{\text{red}} | \mathcal{N} | \psi_{\text{red}} \rangle \lesssim N \| \tilde{\psi} \|_{\mathbb{F}}^2 (1 + \| u \|_{L^2(\mathbb{R}^6)}^6). \quad (3.5)$$

The proof uses the following lemma:

Lemma 3.2. *Let $k \in L^2(\mathbb{R}^6)$ be symmetric in (x, y) , $x, y \in \mathbb{R}^3$ and $u = \text{sh}(k)$.*

Then the following operator inequality holds:

$$e^{\mathcal{B}(k)} \mathcal{N} e^{-\mathcal{B}(k)} \leq C(1 + \| u \|_2^2)(\mathcal{N} + 1) \quad (3.6)$$

for some constant C independent of t .

Proof. We will use the notation

$$a^\sharp(f_x) := \int dz f(z, x) a_z^\sharp \text{ where } a^\sharp = a \text{ or } a^* \quad (3.7)$$

and estimates

$$\| a(f) | \psi \rangle \| \leq \| f \|_2 \| \mathcal{N}^{1/2} | \psi \rangle \| \quad \text{and} \quad \| a^*(f) | \psi \rangle \| \leq \| f \|_2 \| (\mathcal{N} + 1)^{1/2} | \psi \rangle \| \quad (3.8)$$

from Lemma 1.1. Using the shorthand notation $e^{\mathcal{B}}$ for $e^{\mathcal{B}(k)}$, we will also make use of (1.35)-(1.36) which takes the form

$$e^{\mathcal{B}} a_x e^{-\mathcal{B}} = a(c_x) + a^*(u_x) \text{ and } e^{\mathcal{B}} a_x^* e^{-\mathcal{B}} = a^*(\bar{c}_x) + a(\bar{u}_x) \quad (3.9)$$

due to the choice of notation in (3.7). We prove the following estimate:

$$\begin{aligned}
\langle \psi | e^{\mathcal{B}} \mathcal{N} e^{-\mathcal{B}} | \psi \rangle &= \int \langle \psi | e^{\mathcal{B}} a_x^* e^{-\mathcal{B}} e^{\mathcal{B}} a_x e^{-\mathcal{B}} | \psi \rangle dx = \int \| e^{\mathcal{B}} a_x e^{-\mathcal{B}} | \psi \rangle \|^2 dx \\
&= \int \| (a_x + a(p_x) + a^*(u_x)) | \psi \rangle \|^2 dx \quad (\text{using (3.9) and } c:=\text{ch}(k)=\delta+p \text{ from (1.33)}) \\
&\leq C((1 + \|p\|_2^2) \langle \psi | \mathcal{N} | \psi \rangle + \|u\|_2^2 \langle \psi | (\mathcal{N} + 1) | \psi \rangle) \quad (\text{using (3.8)}) \\
&\lesssim (1 + \|u\|_2^2) \langle \psi | (\mathcal{N} + 1) | \psi \rangle
\end{aligned}$$

where in the last step we used $\|p\|_2 \leq \|u\|_2$ from (2.44). \square

Proof of Proposition 3.1. First let's note that the exact dynamics conserves the number of particles since

$$\begin{aligned}
\langle \psi_{\text{ex}} | \mathcal{N} | \psi_{\text{ex}} \rangle &= \langle 0 | e^{\sqrt{N}\mathcal{A}(\phi_0)} e^{-it\mathcal{H}} \mathcal{N} e^{it\mathcal{H}} e^{-\sqrt{N}\mathcal{A}(\phi_0)} | 0 \rangle \\
&\stackrel{\substack{\text{by} \\ [\mathcal{H}, \mathcal{N}] = 0}}{=} \langle 0 | e^{\sqrt{N}\mathcal{A}(\phi_0)} \mathcal{N} e^{-\sqrt{N}\mathcal{A}(\phi_0)} | 0 \rangle \stackrel{\substack{\text{by} \\ \text{Lemma 1.3, (iv)}}{=}}{=} N.
\end{aligned} \tag{3.10}$$

Using (3.10) and the shorthand notation $e^{\sqrt{N}\mathcal{A}}$ for $e^{\sqrt{N}\mathcal{A}(\phi^{(N)})}$ we obtain:

$$\begin{aligned}
N &= \langle \psi_{\text{ex}} | \mathcal{N} | \psi_{\text{ex}} \rangle = \langle \psi_{\text{red}} | e^{\mathcal{B}} \overbrace{e^{\sqrt{N}\mathcal{A}} \mathcal{N} e^{-\sqrt{N}\mathcal{A}}}^{\mathcal{N} + \sqrt{N}(a(\overline{\phi^{(N)}}) + a^*(\phi^{(N)})) + N} e^{-\mathcal{B}} | \psi_{\text{red}} \rangle \\
&= \langle \psi_{\text{red}} | \{ \mathcal{P}_2 + \sqrt{N}\mathcal{P}_1 \} | \psi_{\text{red}} \rangle + N
\end{aligned} \tag{3.11}$$

where we also used (1.21) and the definitions

$$\left. \begin{aligned}
\mathcal{P}_2 &:= e^{\mathcal{B}} \mathcal{N} e^{-\mathcal{B}} \quad \text{and} \\
\mathcal{P}_1 &:= \int \mathcal{P}_{x,x}^{(1)} dx = \int \left\{ \overline{\phi^{(N)}}(t, x) \overbrace{e^{\mathcal{B}} a_x e^{-\mathcal{B}}}^{a(c_x) + a^*(u_x)} + \phi^{(N)}(t, x) \overbrace{e^{\mathcal{B}} a_x^* e^{-\mathcal{B}}}^{a(\bar{u}_x) + a^*(\bar{c}_x)} \right\} dx \\
&= a(\bar{l}) + a^*(l) \\
\text{with } l &:= u \circ \overline{\phi^{(N)}} + \bar{c} \circ \phi^{(N)} = u \circ \overline{\phi^{(N)}} + \phi^{(N)} + \bar{p} \circ \phi^{(N)} \quad (\text{see (1.33)}).
\end{aligned} \right\} \quad (3.12)$$

Equation (3.11) implies

$$\langle \psi_{\text{red}} | \mathcal{P}_2 | \psi_{\text{red}} \rangle = -\sqrt{N} \langle \psi_{\text{red}} | \mathcal{P}_1 | \psi_{\text{red}} \rangle. \quad (3.13)$$

We will estimate separately the left- and the right-hand sides of this last equation in terms of number of particles. For the term on the right-hand side, we have

$$\langle \psi_{\text{red}} | \mathcal{P}_1 | \psi_{\text{red}} \rangle = e^{-iN\chi(t)} \langle \psi_{\text{red}} | a(\bar{l}) | \tilde{\psi} \rangle + e^{iN\chi(t)} \langle \tilde{\psi} | a^*(l) | \psi_{\text{red}} \rangle \quad (3.14)$$

where we used again $a|0\rangle = 0$ recalling $|\tilde{\psi}\rangle = e^{-iN\chi(t)} |\psi_{\text{red}}\rangle - |0\rangle$ from (3.4). Hence taking absolute values and using Cauchy-Schwarz inequality in (3.14) gives:

$$\begin{aligned}
|\langle \psi_{\text{red}} | \mathcal{P}_1 | \psi_{\text{red}} \rangle| &\leq 2 \| |\tilde{\psi}\rangle \| \| |a^*(l) | \psi_{\text{red}} \rangle \| \\
&\leq C_1 \| |\tilde{\psi}\rangle \| (1 + \|u\|_2) \langle \psi_{\text{red}} | (\mathcal{N} + 1) | \psi_{\text{red}} \rangle^{1/2}
\end{aligned} \quad (3.15)$$

where we used (3.8) and $\|l\|_2 \lesssim 1 + \|u\|_2$ (since $\|p\|_2 \leq \|u\|_2$). The only other thing we need in order to obtain the bound for the number of particles is a rewrite of (3.6)

which is true for any $|\psi\rangle$:

$$\begin{aligned} \langle \psi | e^{\mathcal{B}}(\mathcal{N} + 1)e^{-\mathcal{B}}|\psi\rangle &= ((\mathcal{N} + 1)e^{-\mathcal{B}}|\psi\rangle, e^{-\mathcal{B}}|\psi\rangle) \\ &\geq \frac{C_2}{1 + \|u\|_2^2} \overbrace{\langle \psi | e^{\mathcal{B}}(e^{-\mathcal{B}}\mathcal{N}e^{\mathcal{B}})e^{-\mathcal{B}}|\psi\rangle}^{\langle \psi | \mathcal{N} | \psi \rangle} \end{aligned} \quad (3.16)$$

where we used (3.6) by replacing k with $-k$ for the last inequality. Constants showing up in the last two inequalities are independent of t as before. Combining what we know from (3.13), (3.15) and (3.16) gives

$$\frac{C_2}{1 + \|u\|_2^2} \langle \psi_{\text{red}} | \mathcal{N} | \psi_{\text{red}} \rangle - 1 \leq C_1 \sqrt{N} \|\tilde{\psi}\| (1 + \|u\|_2) \langle \psi_{\text{red}} | (\mathcal{N} + 1) | \psi_{\text{red}} \rangle^{1/2}.$$

Collecting all terms on the left hand side provides with a quadratic expression in $\langle \psi_{\text{red}} | \mathcal{N} | \psi_{\text{red}} \rangle^{1/2}$ which is non-positive:

$$\begin{aligned} 0 &\geq \frac{C_2}{1 + \|u\|_2^2} \langle \psi_{\text{red}} | \mathcal{N} | \psi_{\text{red}} \rangle \\ &\quad - C_1 \sqrt{N} \|\tilde{\psi}\| (1 + \|u\|_2) \langle \psi_{\text{red}} | \mathcal{N} | \psi_{\text{red}} \rangle^{1/2} \\ &\quad - (1 + C_1 \sqrt{N} \|\tilde{\psi}\| (1 + \|u\|_2)) \end{aligned}$$

The positive root of the corresponding quadratic function, namely,

$$\begin{aligned} &\frac{1}{2C_2/(1 + \|u\|_2^2)} \left(C_1 \sqrt{N} \|\tilde{\psi}\| (1 + \|u\|_2) \right. \\ &\quad \left. + \sqrt{C_1^2 N \|\tilde{\psi}\|^2 (1 + \|u\|_2)^2 + 4 \frac{C_2}{1 + \|u\|_2^2} (1 + C_1 \sqrt{N} \|\tilde{\psi}\| (1 + \|u\|_2))} \right) \end{aligned}$$

can be bounded in the limit of large N with a constant multiple of $\sqrt{N} \|\tilde{\psi}\| (1 + \|u\|_2^3)$ since the second term inside the square root in the above expression is lower order provided $\|\tilde{\psi}\|_{\mathbb{F}}$ is small. This gives us an upper bound of the form seen in (3.5). \square

3.3 Proof of Theorem 1.8

We will first estimate $\text{Tr}|\Gamma_{\text{ex}}^{(1)} - |\phi\rangle\langle\phi||$ in terms of $\langle\psi_{\text{red}}|\mathcal{N}|\psi_{\text{red}}\rangle$ and then will use Proposition 3.1 and Fock space estimates of 1.3.

3.3.1 Splitting the Error via $\Gamma_{\text{ap}}^{(1)}$ and

Marginals as Mean Field + Fluctuations

We can split $\text{Tr}|\Gamma_{\text{ex}}^{(1)} - |\phi\rangle\langle\phi||$ as

$$\text{Tr}|\Gamma_{\text{ex}}^{(1)} - |\phi\rangle\langle\phi|| \leq \text{Tr}|\Gamma_{\text{ex}}^{(1)} - \Gamma_{\text{ap}}^{(1)}| + \text{Tr}|\Gamma_{\text{ap}}^{(1)} - |\phi\rangle\langle\phi|| \quad (3.17)$$

where

$$\Gamma_{\text{ap}}^{(1)}(t, x, y) := \Gamma_{|\psi_{\text{ap}}\rangle}^{(1)}(t, x, y) = \frac{\langle\psi_{\text{ap}}|a_x^* a_y|\psi_{\text{ap}}\rangle}{\langle\psi_{\text{ap}}|\mathcal{N}|\psi_{\text{ap}}\rangle} \quad (3.18)$$

$$\text{with } |\psi_{\text{ap}}\rangle = e^{iN\chi(t)} e^{-\sqrt{N}\mathcal{A}(\phi^{(N)}(t))} e^{-\mathcal{B}(k(t))} |0\rangle.$$

We would like to write both $\Gamma_{\text{ex}}^{(1)}$ and $\Gamma_{\text{ap}}^{(1)}$ as the sum of N -particle mean field and fluctuations around it which will be useful in our proof of Theorem 1.8:

Lemma 3.3. *We have the following formulas for the one-particle marginal densities*

of $|\psi_{\text{ex}}(t)\rangle = e^{it\mathcal{H}}e^{-\sqrt{N}\mathcal{A}(\phi_0)}|0\rangle$ and $|\psi_{\text{ap}}(t)\rangle = e^{iN\chi(t)}e^{-\sqrt{N}\mathcal{A}(\phi^{(N)}(t))}e^{-\mathcal{B}(k(t))}|0\rangle$:

$$\begin{aligned}\Gamma_{\text{ex}}^{(1)}(t, x, y) &= N^{-1}\langle\psi_{\text{red}}|\overbrace{e^{\mathcal{B}}a_x^*a_ye^{-\mathcal{B}}}^{\mathcal{P}_{x,y}^{(2)}:=}|\psi_{\text{red}}\rangle \\ &\quad + N^{-1/2}\langle\psi_{\text{red}}|\overbrace{e^{\mathcal{B}}\{\phi^{(N)}(t, y)a_x^* + \overline{\phi^{(N)}}(t, x)a_y\}e^{-\mathcal{B}}}^{\mathcal{P}_{x,y}^{(1)}:=}|\psi_{\text{red}}\rangle \\ &\quad + |\phi^{(N)}\rangle\langle\phi^{(N)}|,\end{aligned}\tag{3.19}$$

$$\Gamma_{\text{ap}}^{(1)}(t, x, y) = \frac{N|\phi^{(N)}\rangle\langle\phi^{(N)}| + u \circ \bar{u}}{N + \|u\|_2^2} \simeq |\phi^{(N)}\rangle\langle\phi^{(N)}| + N^{-1}u \circ \bar{u}\tag{3.20}$$

where $u = \text{sh}(k)$. The equalities above are true for any $\phi_0, \phi^{(N)}$ with $\|\phi_0\|_{L^2(\mathbb{R}^3)} = \|\phi^{(N)}\|_{L^2(\mathbb{R}^3)} = 1$ and $k \in L^2(\mathbb{R}^6)$ symmetric in (x, y) . The approximation in (3.20) is true in the limit of large N if $\|u\|_{L^2(\mathbb{R}^6)} = O(1)$ w.r.t N , which holds if $\phi^{(N)}$ and k satisfy (1.38) (see (2.44) for $\beta > 0$ and Corollary 3.3 in [19] for $\beta = 0$).

Proof. We will write $e^{\sqrt{N}\mathcal{A}}$ for $e^{\sqrt{N}\mathcal{A}(\phi^{(N)})}$, $e^{\sqrt{N}\mathcal{A}_0}$ for $e^{\sqrt{N}\mathcal{A}(\phi_0)}$ and $e^{\mathcal{B}}$ for $e^{\mathcal{B}(k)}$ shortly.

Let's compute first $\Gamma_{\text{ex}}^{(N)}$:

$$\begin{aligned}\Gamma_{\text{ex}}^{(1)}(t, x, y) &= \frac{\langle\psi_{\text{ex}}|a_x^*a_y|\psi_{\text{ex}}\rangle}{\langle\psi_{\text{ex}}|\mathcal{N}|\psi_{\text{ex}}\rangle} \stackrel{\text{using (3.10) for the denominator}}{\downarrow} \frac{\langle\psi_{\text{red}}|e^{\mathcal{B}}\overbrace{e^{\sqrt{N}\mathcal{A}}a_x^*e^{-\sqrt{N}\mathcal{A}}}^{a_x^* + \sqrt{N}\overline{\phi^{(N)}}(t, x)}\overbrace{e^{\sqrt{N}\mathcal{A}}a_ye^{-\sqrt{N}\mathcal{A}}}^{|\psi_{\text{ex}}\rangle}e^{-\mathcal{B}}|\psi_{\text{red}}\rangle}{N} \\ &= \frac{\langle\psi_{\text{red}}|e^{\mathcal{B}}\{a_x^*a_y + \sqrt{N}(\phi^{(N)}(t, y)a_x^* + \overline{\phi^{(N)}}(t, x)a_y) + N|\phi^{(N)}\rangle\langle\phi^{(N)}|\}e^{-\mathcal{B}}|\psi_{\text{red}}\rangle}{N} \\ &= N^{-1}\langle\psi_{\text{red}}|e^{\mathcal{B}}a_x^*a_ye^{-\mathcal{B}}|\psi_{\text{red}}\rangle + N^{-1/2}\langle\psi_{\text{red}}|e^{\mathcal{B}}\{\phi^{(N)}(t, y)a_x^* + \overline{\phi^{(N)}}(t, x)a_y\}e^{-\mathcal{B}}|\psi_{\text{red}}\rangle \\ &\quad + |\phi^{(N)}\rangle\langle\phi^{(N)}|\end{aligned}$$

where we used (1.19)-(1.20) to write the numerator as it appears in the second line of the above computation.

Next we will compute the marginal for $|\psi_{\text{ap}}\rangle$

$$\Gamma_{\text{ap}}^{(1)}(t, x, y) = \frac{\langle \psi_{\text{ap}} | a_x^* a_y | \psi_{\text{ap}} \rangle}{\langle \psi_{\text{ap}} | \mathcal{N} | \psi_{\text{ap}} \rangle}.$$

Note that the denominator is the trace of the numerator hence it is sufficient to compute the numerator. Similar to the above computations, we can proceed as

$$\begin{aligned} \langle \psi_{\text{ap}} | a_x^* a_y | \psi_{\text{ap}} \rangle &= \langle 0 | e^{\mathcal{B}} e^{\sqrt{N}\mathcal{A}} a_x^* a_y e^{-\sqrt{N}\mathcal{A}} e^{-\mathcal{B}} | 0 \rangle \\ &= \langle 0 | e^{\mathcal{B}} \{ a_x^* a_y + \sqrt{N}(\phi^{(N)}(t, y) a_x^* + \overline{\phi^{(N)}}(t, x) a_y) + N |\phi^{(N)}\rangle \langle \phi^{(N)}| \} e^{-\mathcal{B}} | 0 \rangle \\ &= \langle 0 | e^{\mathcal{B}} a_x^* a_y e^{-\mathcal{B}} | 0 \rangle + \sqrt{N} \langle 0 | \mathcal{P}_{x,y}^{(1)} | 0 \rangle + N |\phi^{(N)}\rangle \langle \phi^{(N)}| \quad \text{recalling } \mathcal{P}_{x,y}^{(1)} \text{ from (3.19)} \end{aligned} \quad (3.21)$$

The middle term in (3.21) vanishes because of the definition of $\mathcal{P}_{x,y}^{(1)}$ from (3.19), the identities (1.35)-(1.36) and the property $a|0\rangle = 0$. The first term in (3.21) can be computed using the same properties as

$$\begin{aligned} \langle 0 | e^{\mathcal{B}} a_x^* a_y e^{-\mathcal{B}} | 0 \rangle &= \langle 0 | e^{\mathcal{B}} a_x^* e^{-\mathcal{B}} e^{\mathcal{B}} a_y e^{-\mathcal{B}} | 0 \rangle \\ &= \langle 0 | \int \bar{u}(z, x) u(w, y) a_z a_w^* dz dw | 0 \rangle = u \circ \bar{u}. \end{aligned}$$

Inserting this into (3.21) gives

$$\langle \psi_{\text{ap}} | a_x^* a_y | \psi_{\text{ap}} \rangle = N |\phi^{(N)}\rangle \langle \phi^{(N)}| + u \circ \bar{u}.$$

Finally

$$\langle \psi_{\text{ap}} | \mathcal{N} | \psi_{\text{ap}} \rangle = \text{Tr}(\langle \psi_{\text{ap}} | a_x^* a_y | \psi_{\text{ap}} \rangle) = N + \|u\|_2^2 \quad \text{since } \|\phi^{(N)}\|_2 = 1.$$

Hence we have

$$\Gamma_{\text{ap}}^{(1)}(t, x, y) = \frac{\langle \psi_{\text{ap}} | a_x^* a_y | \psi_{\text{ap}} \rangle}{\langle \psi_{\text{ap}} | \mathcal{N} | \psi_{\text{ap}} \rangle} = \frac{N |\phi^{(N)}\rangle \langle \phi^{(N)}| + u \circ \bar{u}}{N + \|u\|_2^2}.$$

□

3.3.2 Estimate on $\text{Tr}|\Gamma_{\text{ex}}^{(1)} - \Gamma_{\text{ap}}^{(1)}|$

Recalling (3.17), we will first estimate $\text{Tr}|\Gamma_{\text{ex}}^{(1)} - \Gamma_{\text{ap}}^{(1)}|$.

Proposition 3.4. *Let $\phi^{(N)}$ and k satisfy suitable equations (the uncoupled system (1.38) in the current work) so that the error $\|\tilde{\psi}\|_{\mathbb{F}} = \|\psi_{\text{ex}} - \psi_{\text{ap}}\|_{\mathbb{F}}$ is small and $\|u(t, \cdot)\|_{L^2} = O(1)$ w.r.t. N where $u = \text{sh}(k)$. Then*

$$\text{Tr}|\Gamma_{\text{ex}}^{(1)} - \Gamma_{\text{ap}}^{(1)}| \lesssim (1 + \|u\|_2^8) \|\tilde{\psi}\|_{\mathbb{F}}^2 + \frac{\|u\|_2^2}{N}. \quad (3.22)$$

where $\Gamma_{\text{ex}}^{(1)}, \Gamma_{\text{ap}}^{(1)}$ are as they have been computed in Lemma 3.3.

A note on notation. We will use $\text{Tr}|\cdot|$ to denote the trace norm in the space of the trace class operators $\mathcal{L}_1(L^2(\mathbb{R}^3))$ as explained just before Theorem 1.8. In what follows, by an abuse of notation, we will identify an operator with its kernel if there exists any. In that case, $\text{Tr}|\gamma(x, y)|$ denotes the trace of the absolute value of the

operator with the kernel $\gamma(x, y)$, not the trace of the absolute value of the kernel. However, if γ is a positive trace class operator with continuous kernel $\gamma(x, y)$, then we indeed have $\text{Tr}|\gamma| = \int \gamma(x, x) dx$ (see, for instance, Theorem 2.12 in [36]).

Proof of Proposition 3.4. Let's first obtain a bound on $\text{Tr}|\Gamma_{\text{ex}}^{(1)} - \Gamma_{\text{ap}}^{(1)}|$ in terms of $\langle \psi_{\text{red}} | \mathcal{N} | \psi_{\text{red}} \rangle$. Recalling (3.19) and (3.20), we need to estimate the terms on the right hand side of the following inequality (see the note on the notation preceding this proof):

$$\text{Tr}|\Gamma_{\text{ex}}^{(1)} - \Gamma_{\text{ap}}^{(1)}| \leq \frac{1}{N} \text{Tr}|\langle \psi_{\text{red}} | \mathcal{P}_{x,y}^{(2)} | \psi_{\text{red}} \rangle| + \frac{1}{\sqrt{N}} \text{Tr}|\langle \psi_{\text{red}} | \mathcal{P}_{x,y}^{(1)} | \psi_{\text{red}} \rangle| + \frac{\|u\|_2^2}{N}. \quad (3.23)$$

As for the first term on the right hand side of (3.23), the one particle operator $\langle \psi_{\text{red}} | \mathcal{P}_{x,y}^{(2)} | \psi_{\text{red}} \rangle$ is positive-semidefinite since

$$\begin{aligned} \int \langle \psi_{\text{red}} | \mathcal{P}_{x,y}^{(2)} | \psi_{\text{red}} \rangle f(y) \bar{f}(x) dx dy &= \langle \psi_{\text{red}} | e^{\mathcal{B}} a^*(f) a(\bar{f}) e^{-\mathcal{B}} | \psi_{\text{red}} \rangle \\ &= \|a(\bar{f}) e^{-\mathcal{B}} | \psi_{\text{red}} \rangle\|^2 \geq 0. \end{aligned}$$

Hence we have

$$\begin{aligned} \text{Tr}|\langle \psi_{\text{red}} | \mathcal{P}_{x,y}^{(2)} | \psi_{\text{red}} \rangle| &= \int \langle \psi_{\text{red}} | \mathcal{P}_{x,x}^{(2)} | \psi_{\text{red}} \rangle dx = \langle \psi_{\text{red}} | e^{\mathcal{B}} \mathcal{N} e^{-\mathcal{B}} | \psi_{\text{red}} \rangle \\ &\lesssim (1 + \|u\|_2^2) \langle \psi_{\text{red}} | (\mathcal{N} + 1) | \psi_{\text{red}} \rangle \quad \text{using (3.6)}. \end{aligned} \quad (3.24)$$

For estimating the second term on the right hand side of (3.23), first we can use

(3.9) to compute $\mathcal{P}_{x,y}^{(1)}$ defined in (3.19) explicitly as

$$\mathcal{P}_{x,y}^{(1)} = \phi^{(N)}(t, y) \{a(\bar{u}_x) + a^*(\bar{c}_x)\} + \overline{\phi^{(N)}}(t, x) \{a(c_y) + a^*(u_y)\}.$$

Using $a|0\rangle = 0$ and also the definition of $|\tilde{\psi}\rangle := e^{-iN\chi(t)}|\psi_{\text{red}}\rangle - |0\rangle$ from (3.4), we obtain the following equation:

$$\begin{aligned} \langle\psi_{\text{red}}|\mathcal{P}_{x,y}^{(1)}|\psi_{\text{red}}\rangle &= e^{-iN\chi(t)}\langle\psi_{\text{red}}|\{\phi^{(N)}(t, y)a(\bar{u}_x) + \overline{\phi^{(N)}}(t, x)a(c_y)\}|\tilde{\psi}\rangle \\ &\quad + e^{iN\chi(t)}\langle\tilde{\psi}|\{\phi^{(N)}(t, y)a^*(\bar{c}_x) + \overline{\phi^{(N)}}(t, x)a^*(u_y)\}|\psi_{\text{red}}\rangle. \end{aligned} \quad (3.25)$$

We will estimate the trace norm of (3.25) by using a duality argument. Let \mathcal{L}_1 and \mathcal{K} denote the spaces of trace class and compact operators on $L^2(\mathbb{R}^3)$ respectively. Since $\mathcal{L}_1 \cong \mathcal{K}^*$ under $\gamma \mapsto \text{Tr}(\cdot \gamma)$ where $\text{Tr}(\cdot \gamma) : J \mapsto \text{Tr}(J\gamma)$, we have

$$\text{Tr}|\langle\psi_{\text{red}}|\mathcal{P}_{x,y}^{(1)}|\psi_{\text{red}}\rangle| = \sup_{\substack{J \in \mathcal{K} \\ \text{with } \|J\|_{\text{op}}=1}} |\text{Tr}(J\langle\psi_{\text{red}}|\mathcal{P}_{x,y}^{(1)}|\psi_{\text{red}}\rangle)|. \quad (3.26)$$

Without loss of generality we can consider J satisfying $\bar{J}^T = J$. Thus we will estimate $|\text{Tr}(J\langle\psi_{\text{red}}|\mathcal{P}_{x,y}^{(1)}|\psi_{\text{red}}\rangle)|$ using the formula (3.25) as:

$$\begin{aligned} &|\text{Tr}(J\langle\psi_{\text{red}}|\mathcal{P}_{x,y}^{(1)}|\psi_{\text{red}}\rangle)| \\ &= \left| \int J(x, y) e^{-iN\chi(t)} \left(\langle\psi_{\text{red}}|\{\phi^{(N)}(t, x)a(\bar{u}_y) + \overline{\phi^{(N)}}(t, y)a(c_x)\}|\tilde{\psi}\rangle \right. \right. \\ &\quad \left. \left. + \langle\tilde{\psi}|\{\phi^{(N)}(t, x)a^*(\bar{c}_y) + \overline{\phi^{(N)}}(t, y)a^*(u_x)\}|\psi_{\text{red}}\rangle \right) dx dy \right| \end{aligned}$$

$$\begin{aligned}
&\leq \left| \left\langle \underbrace{\int J(x, y) \overline{\phi^{(N)}}(t, x) u(z, y) a_z^* dx dy dz}_{a^*(u \circ \bar{J} \circ \overline{\phi^{(N)}})} |\psi_{\text{red}}\rangle, |\tilde{\psi}\rangle \right\rangle_{\mathbb{F}} \right| \\
&\quad + \left| \left\langle \underbrace{\int J(x, y) \phi^{(N)}(t, y) \bar{c}(z, x) a_z^* dx dy dz}_{a^*(\bar{c} \circ J \circ \phi^{(N)})} |\psi_{\text{red}}\rangle, |\tilde{\psi}\rangle \right\rangle_{\mathbb{F}} \right| \\
&\quad + \left| \langle \tilde{\psi} | \underbrace{\int \bar{J}(x, y) \phi^{(N)}(t, x) \bar{c}(z, y) a_z^* dx dy dz}_{a^*(\bar{c} \circ J \circ \phi^{(N)})} |\psi_{\text{red}}\rangle \right| \\
&\quad + \left| \langle \tilde{\psi} | \underbrace{\int \bar{J}(x, y) \overline{\phi^{(N)}}(t, y) u(z, x) a_z^* dx dy dz}_{a^*(u \circ \bar{J} \circ \overline{\phi^{(N)}})} |\psi_{\text{red}}\rangle \right| \\
&\lesssim \| |\tilde{\psi}\rangle \| \left(\| a^*(u \circ \bar{J} \circ \overline{\phi^{(N)}}) |\psi_{\text{red}}\rangle \| + \| a^*(\bar{c} \circ J \circ \phi^{(N)}) |\psi_{\text{red}}\rangle \| \right) \\
&\leq \| |\tilde{\psi}\rangle \| \left(\| u \circ \bar{J} \circ \overline{\phi^{(N)}} \|_2 + \| \bar{c} \circ J \circ \phi^{(N)} \|_2 \right) \| (\mathcal{N} + 1)^{1/2} |\psi_{\text{red}}\rangle \| \quad \text{using (1.13)} \\
&\leq \| J \|_{\text{op}} (1 + \| u \|_2) \| |\tilde{\psi}\rangle \| \langle \psi_{\text{red}} | (\mathcal{N} + 1) | \psi_{\text{red}} \rangle^{1/2} \tag{3.27}
\end{aligned}$$

where for the last inequality we used $c = \text{ch}(k) = \delta(x - y) + p$ from (1.33) and then $\| p \|_2 \leq \| u \|_2$ from (2.44). (3.26) and (3.27) imply

$$\text{Tr} \langle \psi_{\text{red}} | \mathcal{P}_{x,y}^{(1)} | \psi_{\text{red}} \rangle \lesssim (1 + \| u \|_2) \| |\tilde{\psi}\rangle \| \langle \psi_{\text{red}} | (\mathcal{N} + 1) | \psi_{\text{red}} \rangle^{1/2} \tag{3.28}$$

Inserting (3.28) and (3.24) in (3.23) yields the following estimate

$$\begin{aligned}
\text{Tr} |\Gamma_{\text{ex}}^{(1)} - \Gamma_{\text{ap}}^{(1)}| &\lesssim \frac{(1 + \| u \|_2^2)}{N} \langle \psi_{\text{red}} | (\mathcal{N} + 1) | \psi_{\text{red}} \rangle \\
&\quad + \frac{(1 + \| u \|_2)}{\sqrt{N}} \| |\tilde{\psi}\rangle \| \langle \psi_{\text{red}} | (\mathcal{N} + 1) | \psi_{\text{red}} \rangle^{\frac{1}{2}} + \frac{\| u \|_2^2}{N} \tag{3.29}
\end{aligned}$$

in which when we insert $\langle \psi_{\text{red}} | \mathcal{N} | \psi_{\text{red}} \rangle \lesssim N(1 + \|u\|_2^6) \|\tilde{\psi}\|^2$ from Proposition 3.1 we obtain (3.22). \square

Remark 3.5. Proposition 3.4 shows in particular that the mean field with second order corrections approximates the exact dynamics well also in the sense of marginals.

3.3.3 Estimate on $\text{Tr}|\Gamma_{\text{ap}}^{(1)} - |\phi\rangle\langle\phi||$

Here we compare the approximate evolution with the mean field evolution:

Proposition 3.6. Let $\phi^{(N)}$ and ϕ satisfy

$$\frac{1}{i}\partial_t\phi^{(N)} = \Delta\phi^{(N)} - (v_N * |\phi^{(N)}|^2)\phi^{(N)}$$

where $v_N = N^{3\beta}v(N^\beta\cdot)$ and $\phi^{(N)}(0, \cdot) = \phi_0$

and

$$\frac{1}{i}\partial_t\phi = \Delta\phi - \begin{cases} (v * |\phi|^2)\phi & \text{if } \beta = 0 \\ (\int v(x)dx) |\phi|^2\phi & \text{if } 0 < \beta < 1 \end{cases} \quad \text{with } \phi(0, \cdot) = \phi_0$$

respectively where ϕ_0 is as stated in Theorem 1.8 and let k satisfy (1.38b)-(1.38c).

Then

$$\text{Tr}|\Gamma_{\text{ap}}^{(1)} - |\phi\rangle\langle\phi|| \lesssim \begin{cases} \frac{1}{N}, & \text{if } \beta=0 \text{ and } v(x)=\xi(|x|)/|x|, \xi \in C_0^\infty \text{ decreasing cutoff} \\ \frac{\log^2(1+t)}{\sqrt{N}}, & \text{if } 0 < \beta < 1 \text{ and } v \text{ is bounded, integrable.} \end{cases}$$

Proof. In case of $\beta = 0$ we have $\phi^{(N)} = \phi$ and for N large, $\text{Tr}|\Gamma_{\text{ap}}^{(1)} - |\phi\rangle\langle\phi|| \simeq \|u\|_2^2/N = O(N^{-1})$ where we used $\|u\|_2 = O(1)$ w.r.t. N and t for $u = \text{sh}(k)$ as

proved in Corollary 3.3, [19].

In case of $0 < \beta < 1$, we can make the following estimate:

$$\begin{aligned} \mathrm{Tr}|\Gamma_{\mathrm{ap}}^{(1)} - |\phi\rangle\langle\phi|| &\leq \mathrm{Tr}||\phi^{(N)}\rangle\langle\phi^{(N)}| - |\phi\rangle\langle\phi|| + \frac{\|u\|_2^2}{N} \\ &\lesssim 2\|\phi^{(N)} - \phi\|_2 + \frac{\log^2(1+t)}{N} \lesssim \frac{\log^2(1+t)}{\sqrt{N}} \end{aligned}$$

where we used (2.44) and the fact $\mathrm{Tr}||\phi^{(N)}\rangle\langle\phi^{(N)}| - |\phi\rangle\langle\phi|| \leq 2\|\phi^{(N)} - \phi\|_2 \lesssim N^{-1/2}$ as proved in Appendix B which compares the N -particle (N finite) mean field $\phi^{(N)}$ to the limiting mean field ϕ in case of $0 < \beta < 1$. \square

3.3.4 Conclusion

We will use the following corollary to obtain the final estimate:

Corollary 3.7. *Let $\phi^{(N)}$ and k satisfy the uncoupled system (1.38) of Theorem 1.5 with initial data satisfying the same assumptions there. Then*

$$\mathrm{Tr}|\Gamma_{\mathrm{ex}}^{(1)}(t) - \Gamma_{\mathrm{ap}}^{(1)}(t)| \lesssim \begin{cases} \frac{1+t}{N} & \text{for } \beta = 0 \text{ and } v \text{ cut-offed Coulomb} \\ \frac{(1+t)^2 \log^{16}(1+t)}{N^{1-3\beta}} & \text{for } 0 < \beta < 1/3 \text{ and } v \text{ bounded, integrable} \\ \frac{t^{j+3} \log^{20}(1+t)}{N^{1-2\beta(1+\epsilon)}} & \text{for } 1/3 \leq \beta \leq \frac{2j}{1-2\epsilon+4j}, \epsilon \text{ small, } j \text{ sufficiently large, } v \text{ bounded, integrable} \end{cases} \quad (3.30)$$

Proof. Recalling that $\|\tilde{\psi}\|_{\mathbb{F}} = \|\psi_{\mathrm{ex}} - \psi_{\mathrm{ap}}\|_{\mathbb{F}}$ from (3.4) and inserting the estimates of Theorem 1.5 for $0 \leq \beta < 1/3$ and Theorem 1.6 for $\beta \geq 1/3$ into estimate (3.22) in Proposition 3.4 implies the above corollary. \square

Now to get the final estimate in Theorem 1.8 we can insert (3.30) in (3.17),

namely

$$\mathrm{Tr}|\Gamma_{\mathrm{ex}}^{(1)} - |\phi\rangle\langle\phi|| \leq \mathrm{Tr}|\Gamma_{\mathrm{ex}}^{(1)} - \Gamma_{\mathrm{ap}}^{(1)}| + \mathrm{Tr}|\Gamma_{\mathrm{ap}}^{(1)} - |\phi\rangle\langle\phi||$$

and using Proposition 3.6 for the second term on the r.h.s. proves Theorem 1.8.

3.4 Proof of Theorem 1.9

We will first estimate $\mathrm{Tr}|\gamma_N^{(1)} - |\phi\rangle\langle\phi||$ in terms of $\langle\psi_{\mathrm{red}}|\mathcal{N}|\psi_{\mathrm{red}}\rangle$ and then will use Proposition 3.1 and Fock space estimates of 1.3.

3.4.1 Projecting onto N -particle Sector and

Expanding $\gamma_N^{(1)}$ around N -particle Mean Field

Let's recall the following:

$$\gamma_N^{(1)}(t, x, y) = \frac{1}{N} \langle \psi_N, a_x^* a_y \psi_N \rangle_{L^2(\mathbb{R}^{3N})} \quad \text{where } \psi_N(t) = \overset{\substack{N\text{-body Hamiltonian} \\ \text{in (1.1)}}}{\downarrow} e^{itH_N} \phi_0^{\otimes N} \quad (3.31)$$

and

$$|\psi_{\mathrm{ex}}(t)\rangle = \overset{\substack{\text{Fock Hamiltonian} \\ \text{in (1.17)}}}{\downarrow} e^{it\mathcal{H}} e^{-\sqrt{N}\mathcal{A}(\phi_0)} |0\rangle = (\dots, \underbrace{c_N e^{itH_N} \phi_0^{\otimes N}}_{\substack{N\text{-particle sector} \\ \downarrow}}, \dots), \quad c_N = O(N^{-1/4}). \quad (3.32)$$

If P_N denotes projection onto the N -particle sector, considering (3.31)-(3.32) we can rewrite $\gamma_N^{(1)}$ as in the following line and then expand it around N -particle mean field, where we write shortly $e^{\sqrt{N}\mathcal{A}}$ for $e^{\sqrt{N}\mathcal{A}(\phi^{(N)})}$ and use (1.19)-(1.20) in the second

line:

$$\begin{aligned}
\gamma_N^{(1)}(t, x, y) &= \frac{1}{c_N^2 N} \left\langle e^{it\mathcal{H}} P_N e^{-\sqrt{N}\mathcal{A}(\phi_0)} |0\rangle, a_x^* a_y e^{it\mathcal{H}} e^{-\sqrt{N}\mathcal{A}(\phi_0)} |0\rangle \right\rangle_{\mathbb{F}} \\
&= \frac{1}{c_N^2 N} \left\langle e^{it\mathcal{H}} P_N e^{-\sqrt{N}\mathcal{A}(\phi_0)} |0\rangle, \right. \\
&\quad \left. e^{-\sqrt{N}\mathcal{A}} \underbrace{e^{\sqrt{N}\mathcal{A}} a_x^* e^{-\sqrt{N}\mathcal{A}}}_{a_x^* + \sqrt{N}\bar{\phi}^{(N)}(t,x)} \underbrace{e^{\sqrt{N}\mathcal{A}} a_y e^{-\sqrt{N}\mathcal{A}}}_{a_y + \sqrt{N}\phi^{(N)}(t,y)} e^{\sqrt{N}\mathcal{A}} e^{it\mathcal{H}} e^{-\sqrt{N}\mathcal{A}(\phi_0)} |0\rangle \right\rangle_{\mathbb{F}} \\
&= \frac{1}{c_N^2 N} \left\langle e^{it\mathcal{H}} P_N e^{-\sqrt{N}\mathcal{A}(\phi_0)} |0\rangle, e^{-\sqrt{N}\mathcal{A}} a_x^* a_y e^{\sqrt{N}\mathcal{A}} e^{it\mathcal{H}} e^{-\sqrt{N}\mathcal{A}(\phi_0)} |0\rangle \right\rangle_{\mathbb{F}} \\
&\quad + \frac{\phi^{(N)}(t, x)}{c_N^2 \sqrt{N}} \left\langle e^{it\mathcal{H}} P_N e^{-\sqrt{N}\mathcal{A}(\phi_0)} |0\rangle, e^{-\sqrt{N}\mathcal{A}} a_y e^{\sqrt{N}\mathcal{A}} e^{it\mathcal{H}} e^{-\sqrt{N}\mathcal{A}(\phi_0)} |0\rangle \right\rangle_{\mathbb{F}} \\
&\quad + \frac{\bar{\phi}^{(N)}(t, y)}{c_N^2 \sqrt{N}} \left\langle e^{it\mathcal{H}} P_N e^{-\sqrt{N}\mathcal{A}(\phi_0)} |0\rangle, e^{-\sqrt{N}\mathcal{A}} a_x^* e^{\sqrt{N}\mathcal{A}} e^{it\mathcal{H}} e^{-\sqrt{N}\mathcal{A}(\phi_0)} |0\rangle \right\rangle_{\mathbb{F}} \\
&\quad + \phi^{(N)}(t, x) \bar{\phi}^{(N)}(t, y). \tag{3.33}
\end{aligned}$$

3.4.2 Duality Argument for Estimating Fluctuations

We will prove the following proposition:

Proposition 3.8. *Let $\phi^{(N)}$ and k satisfy suitable equations so that the error*

$$\| \overbrace{e^{-itN\chi(t)} |\psi_{\text{red}}\rangle - |0\rangle}^{|\bar{\psi}\rangle} \|_{\mathbb{F}} = \| \overbrace{e^{it\mathcal{H}} e^{-\sqrt{N}\mathcal{A}(\phi_0)} |0\rangle}^{|\psi_{\text{ex}}\rangle} - \overbrace{e^{\mathcal{B}(k)} e^{\sqrt{N}\mathcal{A}(\phi^{(N)})} e^{it\mathcal{H}} e^{-\sqrt{N}\mathcal{A}(\phi_0)} |0\rangle}^{|\psi_{\text{ap}}\rangle} \|_{\mathbb{F}}$$

is small (again in the current work we can take $\phi^{(N)}$ and k as the solutions of (1.38))

with prescribed initial data). Then

$$\mathrm{Tr}|\gamma_N^{(1)} - |\phi^{(N)}\rangle\langle\phi^{(N)}|| \lesssim (1 + \|u\|_2^4)N^{1/4}\left(\|\tilde{\psi}\|_{\mathbb{F}} + N^{-1/2}\right). \quad (3.34)$$

Proof. The proof is based on a duality argument as in Appendix C of [3] where they considered a more general N -particle state as the initial data. We will continue with the notations introduced in the previous sections.

Because of $\mathcal{L}_1 \cong \mathcal{K}^*$ where \mathcal{L}_1 and \mathcal{K} stand for the spaces of trace class and compact operators on $L^2(\mathbb{R}^3)$ respectively as before (see the lines leading to (3.26)), we have

$$\mathrm{Tr}|\gamma_N^{(1)} - |\phi^{(N)}\rangle\langle\phi^{(N)}|| = \sup_{\substack{J \in \mathcal{K} \\ \text{with } \|J\|_{\mathrm{op}}=1}} |\mathrm{Tr}(J(\gamma_N^{(1)} - |\phi^{(N)}\rangle\langle\phi^{(N)}|))|. \quad (3.35)$$

Again we consider J satisfying $\bar{J}^T = J$. Hence considering the expansion in (3.33), we will estimate the difference

$$\begin{aligned} \mathrm{Tr}(J(\gamma_N^{(1)} - |\phi^{(N)}\rangle\langle\phi^{(N)}|)) &= \int J(x, y)(\gamma_N^{(1)}(t, y, x) - \phi^{(N)}(t, y)\bar{\phi}^{(N)}(t, x))dx dy \\ &= \frac{1}{c_N^2 N} \left\langle e^{it\mathcal{H}} P_N e^{-\sqrt{N}\mathcal{A}(\phi_0)} |0\rangle, e^{-\sqrt{N}\mathcal{A}} d\Gamma(J) e^{-\mathcal{B}} \overbrace{e^{\mathcal{B}} e^{\sqrt{N}\mathcal{A}} |\psi_{\mathrm{ex}}\rangle}^{|\psi_{\mathrm{red}}\rangle \text{ by (3.3)}} \right\rangle_{\mathbb{F}} \\ &\quad + \frac{1}{c_N^2 \sqrt{N}} \left\langle e^{it\mathcal{H}} P_N e^{-\sqrt{N}\mathcal{A}(\phi_0)} |0\rangle, e^{-\sqrt{N}\mathcal{A}} (a(\overline{J \circ \phi^{(N)}}) + a^*(J \circ \phi^{(N)})) e^{-\mathcal{B}} |\psi_{\mathrm{red}}\rangle \right\rangle_{\mathbb{F}} \end{aligned}$$

using (i) Cauchy-Schwarz inequality, (ii) $\|e^{it\mathcal{H}} P_N e^{-\sqrt{N}\mathcal{A}(\phi_0)} |0\rangle\| = c_N$, (iii) the fact that $\|d\Gamma(J)|\psi\rangle\| \leq \|J\| \|\mathcal{N}|\psi\rangle\|$ from Lemma 1.2 and (iv) the estimates (1.13) as

follows:

$$\begin{aligned}
& |\mathrm{Tr}(J(\gamma_N^{(1)} - |\phi^{(N)}\rangle\langle\phi^{(N)}|))| \\
& \lesssim \frac{\|J\|}{c_N N} \|\mathcal{N}e^{-\mathcal{B}}|\psi_{\mathrm{red}}\rangle\| + \frac{\|J\|}{c_N \sqrt{N}} \|(\mathcal{N} + 1)^{1/2}e^{-\mathcal{B}}|\psi_{\mathrm{red}}\rangle\| \\
& \lesssim \|J\| \left(\frac{1}{N^{3/4}} \|\mathcal{N}e^{-\mathcal{B}}|\psi_{\mathrm{red}}\rangle\| + \frac{(1 + \|u\|_2)}{N^{1/4}} \|(\mathcal{N} + 1)^{1/2}|\psi_{\mathrm{red}}\rangle\| \right) \quad (3.36)
\end{aligned}$$

where we used (3.6) and $c_N = O(N^{-1/4})$ for the second inequality. Hence by (3.35)

$$\begin{aligned}
& \mathrm{Tr}|\gamma_N^{(1)} - |\phi^{(N)}\rangle\langle\phi^{(N)}|| \\
& \lesssim \left(\frac{1}{N^{3/4}} \|\mathcal{N}e^{-\mathcal{B}}|\psi_{\mathrm{red}}\rangle\| + \frac{(1 + \|u\|_2)}{N^{1/4}} \|(\mathcal{N} + 1)^{1/2}|\psi_{\mathrm{red}}\rangle\| \right) \quad (3.37)
\end{aligned}$$

Based on the last inequality, it remains to estimate the expression

$$\|\mathcal{N}e^{-\mathcal{B}}|\psi_{\mathrm{red}}\rangle\| = \langle\psi_{\mathrm{red}}|e^{\mathcal{B}}\mathcal{N}^2e^{-\mathcal{B}}|\psi_{\mathrm{red}}\rangle^{1/2}.$$

Using (1.19)-(1.21) and the shorthand notations $\varphi(\phi) := a(\bar{\phi}) + a^*(\phi)$ and $e^{\sqrt{N}\mathcal{A}_0} := e^{\sqrt{N}\mathcal{A}(\phi_0)}$, we will proceed as follows (a simplified version of Proposition 4.2, [3]):

$$\begin{aligned}
\langle\psi_{\mathrm{red}}|e^{\mathcal{B}}\mathcal{N}^2e^{-\mathcal{B}}|\psi_{\mathrm{red}}\rangle & = \left\langle \mathcal{N}e^{-\mathcal{B}}|\psi_{\mathrm{red}}\rangle, e^{\sqrt{N}\mathcal{A}} \underbrace{e^{-\sqrt{N}\mathcal{A}}\mathcal{N}e^{\sqrt{N}\mathcal{A}}}_{\mathcal{N} - \sqrt{N}\varphi(\phi^{(N)}) + N} e^{it\mathcal{H}} e^{-\sqrt{N}\mathcal{A}_0} |0\rangle \right\rangle_{\mathbb{F}} \\
& = \left\langle \mathcal{N}e^{-\mathcal{B}}|\psi_{\mathrm{red}}\rangle, e^{\sqrt{N}\mathcal{A}} e^{it\mathcal{H}} e^{-\sqrt{N}\mathcal{A}_0} \underbrace{e^{\sqrt{N}\mathcal{A}_0}\mathcal{N}e^{-\sqrt{N}\mathcal{A}_0}}_{\mathcal{N} + \sqrt{N}\varphi(\phi_0) + N} |0\rangle \right\rangle_{\mathbb{F}} \\
& \quad - \sqrt{N} \left\langle \mathcal{N}e^{-\mathcal{B}}|\psi_{\mathrm{red}}\rangle, \underbrace{e^{\sqrt{N}\mathcal{A}}\varphi(\phi^{(N)})e^{-\sqrt{N}\mathcal{A}}}_{\varphi(\phi^{(N)}) + 2\sqrt{N}} e^{\sqrt{N}\mathcal{A}} e^{it\mathcal{H}} e^{-\sqrt{N}\mathcal{A}_0} |0\rangle \right\rangle_{\mathbb{F}}
\end{aligned}$$

$$\begin{aligned}
& + N \langle \psi_{\text{red}} | e^{\mathcal{B}} \mathcal{N} e^{-\mathcal{B}} | \psi_{\text{red}} \rangle \\
& = \sqrt{N} \left\{ \left\langle \mathcal{N} e^{-\mathcal{B}} | \psi_{\text{red}} \right\rangle, e^{\sqrt{N}\mathcal{A}} e^{it\mathcal{H}} e^{-\sqrt{N}\mathcal{A}_0} \varphi(\phi_0) | 0 \right\rangle_{\mathbb{F}} \right. \\
& \quad \left. - \left\langle \mathcal{N} e^{-\mathcal{B}} | \psi_{\text{red}} \right\rangle, \varphi(\phi^{(N)}) e^{\sqrt{N}\mathcal{A}} e^{it\mathcal{H}} e^{-\sqrt{N}\mathcal{A}_0} | 0 \right\rangle_{\mathbb{F}} \right\} \\
& \leq \sqrt{N} \|\mathcal{N} e^{-\mathcal{B}} | \psi_{\text{red}} \rangle\| \left(\|\varphi(\phi_0) | 0 \rangle\| + \|\varphi(\phi^{(N)}) e^{-\mathcal{B}} | \psi_{\text{red}} \rangle\| \right) \\
& \leq \epsilon \langle \psi_{\text{red}} | e^{\mathcal{B}} \mathcal{N}^2 e^{-\mathcal{B}} | \psi_{\text{red}} \rangle + CN \left(\|\varphi(\phi_0) | 0 \rangle\|^2 + \|\varphi(\phi^{(N)}) e^{-\mathcal{B}} | \psi_{\text{red}} \rangle\|^2 \right) \text{ for some } \epsilon < 1.
\end{aligned}$$

In the last line it is enough to consider $\epsilon = 1/2$ and $C = 1$. This last estimate implies

$$\begin{aligned}
\langle \psi_{\text{red}} | e^{\mathcal{B}} \mathcal{N}^2 e^{-\mathcal{B}} | \psi_{\text{red}} \rangle & \lesssim N \left(\|\varphi(\phi_0) | 0 \rangle\|^2 + \|\varphi(\phi) e^{-\mathcal{B}} | \psi_{\text{red}} \rangle\|^2 \right) \\
& \lesssim N(1 + \|u\|_2^2) \langle \psi_{\text{red}} | (\mathcal{N} + 1) | \psi_{\text{red}} \rangle \quad \text{using (1.13), (3.6)}
\end{aligned}$$

which implies $\|\mathcal{N} e^{-\mathcal{B}} | \psi_{\text{red}} \rangle\| \lesssim \sqrt{N}(1 + \|u\|_2) \langle \psi_{\text{red}} | (\mathcal{N} + 1) | \psi_{\text{red}} \rangle^{1/2}$. This inserted in (3.37) gives

$$\text{Tr} |\gamma_N^{(1)} - |\phi^{(N)}\rangle\langle\phi^{(N)}|| \lesssim \frac{1 + \|u\|_2}{N^{1/4}} \langle \psi_{\text{red}} | (\mathcal{N} + 1) | \psi_{\text{red}} \rangle^{1/2}. \quad (3.38)$$

(3.34) follows by inserting the bound on particle expectation of reduced dynamics from Proposition 3.1 in the above estimate. \square

3.4.3 Conclusion

Note that in case of $\beta = 0$, the N -particle mean field $\phi^{(N)}$ solving (1.38a) equals the limiting mean field ϕ solving (1.44). Therefore inserting *the estimate on the error for Fock space approximation from Theorem 1.5 in case of $\beta = 0$* in (3.34) proves Theorem 1.9 for the case $\beta = 0$.

For $0 < \beta < 1$, we can write

$$\mathrm{Tr}|\gamma_N^{(1)} - |\phi\rangle\langle\phi|| \leq \underbrace{\mathrm{Tr}|\gamma_N^{(1)} - |\phi^{(N)}\rangle\langle\phi^{(N)}||}_{\substack{\lesssim (1+\|u\|_2^4)N^{1/4}(\|\tilde{\psi}\|_{\mathbb{F}+N^{-1/2}}) \\ \text{from (3.34)}}} + \underbrace{\mathrm{Tr}||\phi^{(N)}\rangle\langle\phi^{(N)}| - |\phi\rangle\langle\phi||}_{\substack{O(N^{-1/2}) \\ \text{as proved in Appendix B}}}$$

Hence inserting $\|u(t)\|_2 \lesssim \log(1+t)$ from (2.44) (which holds for $\beta > 0$) and $\|\tilde{\psi}\| \lesssim N^{(-1+3\beta)/2}(1+t)\log^4(1+t)$ for $0 < \beta < 1/3$ from Theorem 1.5 into the above estimate implies Theorem 1.9 for $0 < \beta < 1/6$. To get the estimate for $\beta \geq 1/6$ we can use $\|\tilde{\psi}\| \lesssim t^{(j+3)/2}\log^6(1+t)N^{-1/2+\beta(1+\epsilon)}$ (from Theorem 1.6) which holds for $0 < \beta \leq 2j/(1-2\epsilon+4j)$ for ϵ small and j sufficiently large depending on ϵ as explained in Theorem 1.9.

3.5 Concluding Remarks for Chapter 3

In this chapter we established the following general result:

If the N -particle mean field $\phi^{(N)}(t, x)$ with $\|\phi^{(N)}\|_{L^2(\mathbb{R}^3)} = 1$ and the pair excitations function $k(t, x, y) \in L^2(\mathbb{R}^6)$ symmetric w.r.t (x, y) satisfy suitable equations with appropriate initial data so that

(i) $\|\text{sh}(k(t))\|_{L^2(\mathbb{R}^6)} = O(1)$ w.r.t. N and

(ii) The error $\left\| \underbrace{e^{it\mathcal{H}} e^{-\sqrt{N}\mathcal{A}(\phi_0)}|0\rangle}_{|\psi_{\text{ex}}(t)\rangle} - \underbrace{e^{iN\chi(t)} e^{-\sqrt{N}\mathcal{A}(\phi^{(N)}(t))} e^{-\mathcal{B}(k(t))}|0\rangle}_{|\psi_{\text{ap}}(t)\rangle} \right\|_{\mathbb{F}}$ is small

subitabaly-chosen real phase
depending on $\phi^{(N)}$ and k

then

$$\begin{aligned} \text{Tr}|\Gamma_{\text{ex}}^{(1)} - \Gamma_{\text{ap}}^{(1)}| &\leq C(\|\text{sh}(k)\|_2) \left(\left\| |\psi_{\text{ex}}\rangle - |\psi_{\text{ap}}\rangle \right\|_{\mathbb{F}}^2 + N^{-1} \right) \\ \text{Tr}|\Gamma_{\text{ex}}^{(1)} - |\phi^{(N)}\rangle\langle\phi^{(N)}|| &\leq C(\|\text{sh}(k)\|_2) \left(\left\| |\psi_{\text{ex}}\rangle - |\psi_{\text{ap}}\rangle \right\|_{\mathbb{F}}^2 + N^{-1} \right) \end{aligned} \quad (3.39)$$

$$\text{Tr}|\gamma_N^{(1)} - |\phi^{(N)}\rangle\langle\phi^{(N)}|| \leq C(\|\text{sh}(k)\|_2) N^{1/4} \left(\left\| |\psi_{\text{ex}}\rangle - |\psi_{\text{ap}}\rangle \right\|_{\mathbb{F}} + N^{-1/2} \right) \quad (3.40)$$

where $C(\|\text{sh}(k)\|_2)$ denotes different constants depending on $\|\text{sh}(k)\|_2$.

We obtained explicit rates of convergence as $N \rightarrow \infty$ inserting the ones we have for $\|\text{sh}(k)\|_2$ and $\left\| |\psi_{\text{ex}}\rangle - |\psi_{\text{ap}}\rangle \right\|_{\mathbb{F}}$ when $\phi^{(N)}$ and k satisfy (1.38) into the above estimates.

If we also know $\text{Tr}|\phi^{(N)}\rangle\langle\phi^{(N)}| - |\phi\rangle\langle\phi| = O(N^\sigma)$ with some $\sigma < 0$ in case of $0 < \beta < 1$ for ϕ solving

$$\frac{1}{i}\partial_t\phi = \Delta\phi - \begin{cases} (v * |\phi|^2)\phi & \text{if } \beta = 0 \\ (\int v(x)dx) |\phi|^2\phi & \text{if } 0 < \beta < 1 \end{cases}$$

then we can obtain estimates for convergence to the limiting mean field ϕ replacing the $\phi^{(N)}$'s in (3.39)-(3.40) (for $\phi^{(N)}$ satisfying (1.38a) we obtained $\sigma = 1/2$).

If $\phi^{(N)}$ and k satisfy the uncoupled system (1.38) then the condition (i) above

is true for any $\beta > 0$ and the condition (ii) was shown to be true for $\beta < 1/2$ at most.

Appendix A: Proof of Lemma 2.13:

Operator Norm Estimates on $N^{-1/2}\mathcal{E}(t)$

Proof of Lemma 2.13. Recalling

$$\begin{aligned} N^{-1/2}\mathcal{E}(t) &= \sum_{j=1}^4 \left(\mathcal{E}_j(t) + \mathcal{E}_j^*(t) \right) + \mathcal{E}_2^{\text{sa}}(t) + \mathcal{E}_4^{\text{sa}}(t) \text{ from (2.7)} \\ &= \mathbb{H} + \underbrace{\frac{1}{2N} \int dy_1 dy_2 v_N(y_1 - y_2) \mathcal{Q}_{y_1 y_2}^* \mathcal{Q}_{y_1 y_2}}_{\mathbb{H}} \text{ from (2.131) - (2.130),} \end{aligned}$$

it is sufficient to obtain operator norm estimates for the terms listed in (2.8) since from the general theory of bounded linear operators on Hilbert spaces, the adjoint of an operator will have the same operator norm as the operator itself.

A typical contribution to \mathbb{H} coming from the contributions involved in the terms in (2.7) is of the form

$$\int dy_1 \dots dy_l f(y_1, \dots, y_l) \underbrace{(a, a^*)_l}_{\substack{l\text{th order} \\ \text{term in } a, a^*}} \text{ where } l = 1, 2, 3, 4.$$

Let's first consider estimating the second and the fourth order terms.

(2.8s) and (2.8u) are similar terms. If we consider (2.8s) in which we have

$l = 4$, $(a, a^*)_4 = \mathcal{Q}_{y_1 y_2} \mathcal{D}_{y_4 y_3}$ and f being equal to

$$f_{(2.8s)}(y_1, y_2, y_3, y_4) = \frac{1}{2N} \int dx_1 dx_2 \{ \bar{u}(y_1, x_1) \bar{u}(x_2, y_2) v_N(x_1 - x_2) c(y_3, x_1) u(x_2, y_4) \},$$

we can write the contribution to $\mathbb{H}\psi^{(j)}$ coming from (2.8s) as

$$\int dy_1 dy_2 dy_3 dy_4 \{ f_{(2.8s)}(y_1, y_2, y_3, y_4) \mathcal{Q}_{y_1 y_2} \mathcal{D}_{y_4 y_3} \} (\psi^{(j)}) \quad (\text{A.1})$$

producing a function in sector $j-2$ for $j \geq 2$, L^2 -norm of which we want to estimate.

We have the following typical estimates among others arising from symmetrizations involved in the definition of the creation operators:

$$\begin{aligned} \text{Type 1: } & \left\| \int \left(\int f_{(2.8s)}(y_1, y_2, y_3, y_2) dy_2 \right) \psi^{(j)}(y_3, y_1, z_1, \dots, z_{j-2}) dy_1 dy_3 \right\|_{L^2(\mathbb{R}^{3(j-2)})} \\ & \leq \underbrace{\left\| \int f_{(2.8s)}(y_1, y_2, y_3, y_2) dy_2 \right\|_{L^2_{y_1 y_3}}}_{\substack{\leq \text{sum of } L^2\text{-norms of} \\ \text{the terms like (2.10b), (2.10h)}}} \|\psi^{(j)}\|_{L^2(\mathbb{R}^{3j})} \\ & \stackrel{\substack{\lesssim_\epsilon \\ \text{by} \\ (2.89), l=2}}{\leq} N^{-1+2\beta(1+\epsilon)} \log^4(1+t) \|\psi^{(j)}\|_{L^2(\mathbb{R}^{3j})} \end{aligned}$$

$$\begin{aligned} \text{Type 2: } & \left\| \int \left(\int f_{(2.8s)}(y_1, y_2, y_3, y_1) dy_1 \right) \psi^{(j)}(y_3, y_2, z_1, \dots, z_{j-2}) dy_1 dy_3 \right\|_{L^2(\mathbb{R}^{3(j-2)})} \\ & \leq \underbrace{\left\| \int f_{(2.8s)}(y_1, y_2, y_3, y_1) dy_1 \right\|_{L^2_{y_1 y_3}}}_{\substack{\leq \text{sum of } L^2\text{-norms of} \\ \text{the terms like (2.10c), (2.10j)}}} \|\psi^{(j)}\|_{L^2(\mathbb{R}^{3j})} \\ & \stackrel{\substack{\lesssim_\epsilon \\ \text{by} \\ (2.89), l=2}}{\leq} N^{-1+2\beta(1+\epsilon)} \log^4(1+t) \|\psi^{(j)}\|_{L^2(\mathbb{R}^{3j})} \end{aligned}$$

$$\text{Type 3: } \left\| \int dy_1 dy_2 dy_3 f_{(2.8s)}(y_1, y_2, y_3, z_1) \psi^{(j)}(y_3, y_2, y_1, z_2, \dots, z_{j-2}) \right\|_{L^2(\mathbb{R}^{3(j-2)})}$$

$$\leq \underbrace{\|f_{(2.8s)}\|}_{\substack{\text{sum of} \\ (2.12c)-(2.12d) \text{ like terms}}} \|_{L^2(\mathbb{R}^{12})} \|\psi^{(j)}\|_{L^2(\mathbb{R}^{3j})} \underset{\substack{\text{by} \\ (2.89), l=4}}{\lesssim_\epsilon} N^{-1+2\beta(1+\epsilon)} \log^4(1+t) \|\psi^{(j)}\|_{L^2(\mathbb{R}^{3j})}.$$

With the above estimates we can estimate the contribution (A.1) as:

$$\begin{aligned} & \left\| \int dy_1 dy_2 dy_3 dy_4 \{f_{(2.8s)}(y_1, y_2, y_3, y_4) \mathcal{Q}_{y_1 y_1} \mathcal{D}_{y_4 y_3}\}(\psi^{(j)}) \right\|_{L^2(\mathbb{R}^{3(j-2)})} \\ & \lesssim_{j,\epsilon} N^{-1+2\beta(1+\epsilon)} \log^4(1+t) \|\psi^{(j)}\|_{L^2(\mathbb{R}^{3j})}. \end{aligned}$$

If we consider the contribution to $\mathbb{H}\psi^{(j)}$ coming from (2.8v) and its adjoint, we have

$$\begin{aligned} & \int dy_1 dy_2 dy_3 dy_4 \left\{ f_{(2.8v)}(y_1, y_2, y_3, y_4) \overbrace{\mathcal{Q}_{y_1 y_2} \mathcal{Q}_{y_3 y_4}}^{j \geq 4 \text{ should hold for non-trivial contribution}} \right. \\ & \quad \left. + \overline{f_{(2.8v)}}(y_1, y_2, y_3, y_4) \mathcal{Q}_{y_1 y_2}^* \mathcal{Q}_{y_3 y_4}^* \right\} (\psi^{(j)}) \end{aligned}$$

with $f_{(2.8v)}(y_1, y_2, y_3, y_4)$

$$= \frac{1}{2N} \int dx_1 dx_2 \{ \bar{u}(y_1, x_1) \bar{u}(x_2, y_2) v_N(x_1 - x_2) c(y_3, x_1) \bar{c}(x_2, y_4) \} \quad (\text{A.2})$$

which will produce a contribution to $\mathbb{H}\psi^{(j)}$ of the following type:

$$\begin{aligned} & \left(0, \dots, 0, \int_{\mathbb{R}^{12}} d\mathbf{y} \{ f_{(2.8v)}(\mathbf{y}) \psi^{(j)}(\mathbf{y}, z_1, \dots, z_{j-4}) \}, 0, \dots \right. \\ & \quad \left. \dots, 0, (\overline{f_{(2.8v)}} \otimes \psi^{(j)})(z_1, \dots, z_{j+4}), 0, \dots \right) \quad (\text{A.3}) \end{aligned}$$

Fock space norm of which is

$$\begin{aligned}
& \lesssim_j \underbrace{\|f(2.8v)\|_{L^2(\mathbb{R}^{12})} \|\psi^{(j)}\|_{L^2(\mathbb{R}^{3j})}}_{\leq \text{sum of } L^2\text{-norms of terms like (2.12a)-(2.12d)}} \\
& \lesssim_\epsilon \left(N^{-1+2\beta(1+\epsilon)} \log^4(1+t) + N^{-1+5\beta/2+\beta\epsilon} \log^2(1+t) \right) \|\psi^{(j)}\|_{L^2(\mathbb{R}^{3j})}
\end{aligned}$$

where the last inequality follows by (2.89), $l = 4$ and also by the following estimate (see (2.13c) for F_4^s):

$$\begin{aligned}
\|F_4^s\|_{L^2(\mathbb{R}^{12})} & \lesssim N^{-1} \left(\int v_N^2(y_1 - y_2) \|u(y_3, y_1)\|_{L_{y_3}^2}^2 \|u(y_2, y_4)\|_{L_{y_4}^2}^2 dy_1 dy_2 \right)^{1/2} \\
& \lesssim N^{-1} \left\| \|u(y_2, y_4)\|_{L_{y_4}^2} \right\|_{L_{y_2}^\infty} \left\| (v_N^2 * \|u(y_3, \cdot)\|_{L_{y_3}^2}^2)(y_2) \right\|_{L_{y_2}^1}^{1/2} \\
& \lesssim_\epsilon N^{-1+\beta(1+\epsilon)} \log(1+t) \|v_N\|_{L^2(\mathbb{R}^3)} \|u\|_{L^2(\mathbb{R}^6)} \text{ by (2.22)} \\
& \lesssim N^{-1+5\beta/2+\beta\epsilon} \log^2(1+t). \tag{A.4}
\end{aligned}$$

Now let's look at the contribution coming from (2.8w) only in the most singular case which corresponds to keeping only the δ -parts of c -terms recalling $c(x, y) = \delta(x - y) + p(x, y)$:

$$\int dy_1 dy_2 dy_4 \left\{ \underbrace{\left(\frac{1}{2N} \int dx_2 \bar{u}(x_2, y_2) v_N(y_1 - x_2) u(x_2, y_4) \right)}_{=: f(y_1, y_2, y_4)} \mathcal{D}_{y_1 y_2} \mathcal{D}_{y_4 y_1} \right\} (\psi^{(j)}). \tag{A.5}$$

This will not cause any sector shifts. We have the following typical estimates among

others arising from symmetrization:

$$\begin{aligned}
\text{Type 1: } & \left\| \left(\int dy_2 f(z_1, y_2, y_2) \right) \psi^{(j)}(z_1, \dots, z_j) \right\|_{L^2(\mathbb{R}^{3j})} \\
&= \frac{1}{2N} \left\| \left(\int dx_2 (u \circ \bar{u})(x_2, x_2) v_N(z_1 - x_2) \right) \psi^{(j)}(z_1, \dots, z_j) \right\|_{L^2(\mathbb{R}^{3j})} \\
&\lesssim N^{-1} \left\| \left(v_N * (u \circ \bar{u})(\cdot, \cdot) \right) (z_1) \right\|_{L_{z_1}^\infty} \|\psi^{(j)}\|_{L^2(\mathbb{R}^{3j})} \\
&\leq N^{-1} \|v_N\|_{L^1(\mathbb{R}^3)} \left\| \|u(x, z_1)\|_{L_x^2} \right\|_{L_{z_1}^\infty}^2 \|\psi^{(j)}\|_{L^2(\mathbb{R}^{3j})} \\
&\lesssim_\epsilon N^{-1+2\beta(1+\epsilon)} \log^2(1+t) \|\psi^{(j)}\|_{L^2(\mathbb{R}^{3j})} \quad \text{by (2.22)}
\end{aligned}$$

$$\begin{aligned}
\text{Type 2: } & \left\| \int f(z_1, y_2, z_2) \psi^{(j)}(z_1, y_2, z_3, \dots, z_j) dy_2 \right\|_{L^2(\mathbb{R}^{3j})} \\
&= \frac{1}{2N} \left\| \int \left(\int \bar{u}(x_2, y_2) v_N(z_1 - x_2) u(x_2, z_2) dx_2 \right) \right. \\
&\quad \left. \times \psi^{(j)}(z_1, y_2, z_3, \dots, z_j) dy_2 \right\|_{L^2(\mathbb{R}^{3j})} \\
&\lesssim N^{-1} \left\| \|u(x_2, z_2)\|_{L_{z_2}^2} \right\|_{L_{x_2}^\infty} \\
&\quad \times \left\| \int v_N(x_2) \underbrace{\left(\int \bar{u}(z_1 - x_2, y_2) \psi^{(j)}(z_1, y_2, z_3, \dots, z_j) dy_2 \right)}_{\leq \|u(z_1 - x_2, y_2)\|_{L_{y_2}^2} \|\psi^{(j)}(z_1, y_2, z_3, \dots, z_j)\|_{L_{y_2}^2}} dx_2 \right\|_{L^2(\mathbb{R}^{3(j-1)})} \\
&\leq N^{-1} \|v_N\|_{L^1(\mathbb{R}^3)} \left\| \|u(x_2, z)\|_{L_z^2} \right\|_{L_{x_2}^\infty}^2 \|\psi^{(j)}\|_{L^2(\mathbb{R}^{3j})} \\
&\lesssim_\epsilon N^{-1+2\beta(1+\epsilon)} \log^2(1+t) \|\psi^{(j)}\|_{L^2(\mathbb{R}^{3j})} \quad \text{by (2.22)}.
\end{aligned}$$

Estimate for the contribution coming from (2.8x) is almost the same with the above and (2.8q) can be estimated similarly. The other \mathcal{DD} -contribution comes from (2.8z) but this term is even less singular due to not having any c -factors.

Contributions to $\mathbb{H}\psi^{(j)}$ coming from (2.8t) and (2.8r) are similar hence if we look at the contribution from (2.8r), considered only in the most singular case cor-

responding to keeping only the δ -parts of c -terms, it has the form

$$\frac{1}{2N} \int dy_1 dy_2 dy_4 \{ \bar{u}(y_4, y_2) v_N(y_1 - y_4) \mathcal{D}_{y_1 y_2} \mathcal{Q}_{y_1 y_4} \} (\psi^{(j)}) \quad (\text{A.6})$$

lowering the sector by two. We can make the following typical estimate for this contribution up to symmetrizations:

$$\begin{aligned} & \frac{1}{2N} \left\| \int v_N(z_1 - y_4) \bar{u}(y_4, y_2) \psi^{(j)}(y_2, y_4, z_1, \dots, z_{j-2}) dy_2 dy_4 \right\|_{L^2(\mathbb{R}^{3(j-2)})} \\ & \lesssim N^{-1} \left\| \left\| v_N(z_1 - y_4) \right\|_{L^2_{y_2}} \left\| u(y_4, y_2) \right\|_{L^2_{y_4}} \left\| \psi^{(j)}(y_2, y_4, z_1, \dots, z_{j-2}) \right\|_{L^2_{y_2 y_4}} \right\|_{L^2_{z_1 \dots z_{j-2}}} \\ & \leq N^{-1} \|v_N\|_{L^2(\mathbb{R}^3)} \left\| \left\| u(y_2, y_4) \right\|_{L^2_{y_2}} \right\|_{L^\infty_{y_4}} \left\| \psi^{(j)} \right\|_{L^2(\mathbb{R}^{3j})} \\ & \lesssim_\epsilon N^{-1+5\beta/2+\beta\epsilon} \log(1+t) \left\| \psi^{(j)} \right\|_{L^2(\mathbb{R}^{3j})} \text{ by (2.22)} \end{aligned}$$

Similar estimates can be made for the contributions to $\mathbb{H}\psi^{(j)}$ coming from the term in (2.8y) provided we keep the p -part of $\bar{c}(y_1, x_1)$ (or of $c(y_3, x_1)$) and replace the remaining three c -factors with their corresponding δ -parts.

We move on to checking the second order contributions to $\mathbb{H}\psi^{(j)}$.

(2.8c) and (2.8d) are similar terms. (2.8h) seems to be more singular compared to (2.8g). So let's estimate the contributions to $\mathbb{H}\psi^{(j)}$ coming from (2.8d) and (2.8h) which can be considered together in the form:

$$\begin{aligned} & \int dy_1 dy_2 \{ f(y_1, y_2) \mathcal{D}_{y_1 y_2} \} (\psi^{(j)}) \quad \text{where} \quad (\text{A.7}) \\ & f(y_1, y_2) = \frac{1}{2N} \int dx_1 dx_2 \{ v_N(x_1 - x_2) [(\bar{u} \circ \bar{c})(x_1, x_2) u(y_1, x_1) \bar{c}(x_2, y_2) \\ & \quad + 2(u \circ \bar{u})(x_1, x_1) \bar{u}(y_2, x_2) u(y_1, x_2)] \} \end{aligned}$$

and we can estimate it as follows:

$$\begin{aligned}
& \left\| \int dy_1 dy_2 \{ f(y_1, y_2) \mathcal{D}_{y_1 y_2} \} (\psi^{(j)}) \right\|_{L^2(\mathbb{R}^{3j})} \\
& \leq \sum_{k=1}^j \left\| \int f(z_k, y_2) \psi^{(j)}(y_2, \overbrace{z_1, \dots, z_j}^{z_k \text{ missing}}) dy_2 \right\|_{L^2(\mathbb{R}^{3j})} \\
& \lesssim_j \underbrace{\|f\|_{L^2(\mathbb{R}^6)}}_{\substack{\leq \text{sum of } L^2\text{-norms} \\ \text{of terms similar to (2.10b),} \\ \text{(2.10d), (2.10f), (2.10g)}}} \|\psi^{(j)}\|_{L^2(\mathbb{R}^{3j})} \underset{\substack{\text{by} \\ \text{(2.89), } l=2}}{\lesssim_{j,\epsilon}} N^{-1+2\beta(1+\epsilon)} \log^4(1+t) \|\psi^{(j)}\|_{L^2(\mathbb{R}^{3j})}. \tag{A.8}
\end{aligned}$$

If we consider the contribution to $\mathbb{H}\psi^{(j)}$ coming from (2.8e)-(2.8f) and their adjoints, we have

$$\begin{aligned}
& \int dy_1 dy_2 \{ f(y_1, y_2) \mathcal{Q}_{y_1 y_2} + \bar{f}(y_1, y_2) \mathcal{Q}_{y_1 y_2}^* \} (\psi^{(j)}) \quad \text{where} \tag{A.9} \\
& f(y_1, y_2) = f_{(2.8e)}(y_1, y_2) + f_{(2.8f)}(y_1, y_2) \quad \text{and} \\
& f_{(2.8e)}(y_1, y_2) = \frac{1}{2N} \int dx_1 dx_2 (\bar{u} \circ \bar{c})(x_1, x_2) v_N(x_1 - x_2) c(y_1, x_1) \bar{c}(x_2, y_2), \\
& f_{(2.8f)}(y_1, y_2) = \frac{1}{2N} \int dx_1 dx_2 (u \circ c)(x_1, x_2) v_N(x_1 - x_2) \bar{u}(y_1, x_1) \bar{u}(x_2, y_2)
\end{aligned}$$

which will produce a contribution to $\mathbb{H}\psi^{(j)}$ of the following form

$$\left(0, \dots, 0, \int_{\mathbb{R}^6} dy \{ f(y) \psi^{(j)}(y, z_1, \dots, z_{j-2}) \}, 0, 0, 0, (\bar{f} \otimes \psi^{(j)})(z_1, \dots, z_{j+2}), 0, \dots \right)$$

Fock space norm of which is

$$\lesssim_j \left(\underbrace{\|f_{(2.8e)}\|_{L^2(\mathbb{R}^6)} + \|f_{(2.8f)}\|_{L^2(\mathbb{R}^6)}}_{\substack{\leq \text{sum of } L^2\text{-norms of terms like} \\ \text{(2.10a), (2.10e), (2.10g), (2.10l)}}} \right) \|\psi^{(j)}\|_{L^2(\mathbb{R}^{3j})}$$

$$\lesssim_\epsilon (N^{-1+2\beta(1+\epsilon)} \log^4(1+t) + N^{-1+5\beta/2+\beta\epsilon} \log^2(1+t)) \|\psi^{(j)}\|_{L^2(\mathbb{R}^{3j})}$$

where the last inequality follows by (2.89), $l = 2$ and the following estimate (see (2.13a) for F_2^s):

$$\begin{aligned} \|F_2^s\|_{L^2(\mathbb{R}^6)} &\lesssim N^{-1} \left(\int v_N^2(y_2) \right. \\ &\quad \left. \times \left(\int \{|u(y_1, y_1 - y_2)|^2 + |(\bar{p} \circ u)(y_1, y_1 - y_2)|^2\} dy_1 \right) dy_2 \right)^{\frac{1}{2}} \\ &\lesssim N^{-1} \underbrace{\|v_N\|_{L^2(\mathbb{R}^3)}}_{O(N^{3\beta/2})} (\|u\|_{H^{\frac{3}{2}+(\mathbb{R}^6)}} + \| \|u(x, y)\|_{L_x^2 L_y^4}^2) \\ &\lesssim_\epsilon N^{-1+5\beta/2+\beta\epsilon} \log^2(1+t) \text{ by (2.84), (2.85), (2.87)}. \end{aligned} \tag{A.10}$$

Next let's deal with the third order terms. (2.8i), (2.8j), (2.8k) are providing $\mathcal{D}a$ (or $a^* \mathcal{Q}$)-terms which lower the sector by one. The most singular contribution comes from (2.8k). Let's consider its estimate in the most singular case by keeping the δ -parts of c -factors. The corresponding contribution to $\mathbb{H}\psi^{(j)}$ will have the following form:

$$\frac{1}{\sqrt{N}} \int dy_1 dy_3 \{v_N(y_1 - y_3) \bar{\phi}(y_3) a_{y_1}^* a_{y_1} a_{y_3}\} (\psi^{(j)}) \tag{A.11}$$

whose L^2 -norm is

$$\simeq_j \frac{1}{\sqrt{N}} \left\| \int dy_3 \{v_N(z_1 - y_3) \bar{\phi}(y_3) \psi^{(j)}(y_3, z_1, \dots, z_{j-1})\} \right\|_{L^2(\mathbb{R}^{3(j-1)})}$$

$$\begin{aligned}
&\leq \frac{\|\phi(t, \cdot)\|_{L^\infty(\mathbb{R}^3)}}{\sqrt{N}} \|v_N\|_{L^2(\mathbb{R}^3)} \|\psi^{(j)}(y_3, z_1, \dots, z_{j-1})\|_{L^2_{y_3}} \|\psi^{(j)}\|_{L^2_{z_1 \dots z_{j-1}}} \\
&\lesssim \frac{N^{(-1+3\beta)/2}}{1+t^{3/2}} \|\psi^{(j)}\|_{L^2(\mathbb{R}^{3j})}.
\end{aligned}$$

We can write the contributions to $\mathbb{H}\psi^{(j)}$ coming from (2.81) and (2.80) together with their adjoints in the form:

$$\begin{aligned}
&\int dy_1 dy_2 dy_3 \{ f(y_1, y_2, y_3) \mathcal{Q}_{y_1 y_2} a_{y_3} + \bar{f}(y_1, y_2, y_3) a_{y_1}^* \mathcal{Q}_{y_2 y_3}^* \} (\psi^{(j)}) \quad \text{where (A.12)} \\
&f(y_1, y_2, y_3) = N^{-1/2} \int dx_1 dx_2 v_N(x_1 - x_2) \{ \bar{u}(y_1, x_1) \phi(x_2) \bar{u}(x_2, y_2) c(y_3, x_1) \\
&\quad + \bar{u}(y_1, x_1) \bar{\phi}(x_2) c(y_2, x_1) \bar{c}(x_2, y_3) \}
\end{aligned}$$

which will produce a contribution of the following form:

$$\left(0, \dots, 0, \int_{\mathbb{R}^9} d\mathbf{y} \{ f(\mathbf{y}) \psi^{(j)}(\mathbf{y}, z_1, \dots, z_{j-3}) \}, 0, \dots, 0, (\bar{f} \otimes \psi^{(j)})(z_1, \dots, z_{j+3}), 0, \dots \right)$$

Fock space norm of which is

$$\begin{aligned}
&\lesssim_j \underbrace{\|f\|_{L^2(\mathbb{R}^9)}}_{\substack{\lesssim \text{sum of } L^2\text{-norms of} \\ \text{tems like (2.11a)-(2.11f)}}} \|\psi^{(j)}\|_{L^2(\mathbb{R}^{3j})} \\
&\lesssim_\epsilon \left(N^{-1/2+\beta(1+\epsilon)} \log^3(1+t)/(1+t^{3/2}) \right. \\
&\quad \left. + N^{(-1+3\beta)/2} \log(1+t)/(1+t^{3/2}) \right) \|\psi^{(j)}\|_{L^2(\mathbb{R}^{3j})}
\end{aligned}$$

by (2.89), $l = 3$ and the following estimate (see (2.13b) for F_3^s):

$$\begin{aligned}
\|F_3^s\|_{L^2(\mathbb{R}^9)} &\lesssim N^{-1/2}\|\phi\|_{L^\infty(\mathbb{R}^3)}\left(\int v_N^2(y_1 - y_2)\|u(y_3, y_1)\|_{L_{y_3}^2}^2 dy_1 dy_2\right)^{1/2} \\
&\lesssim \frac{N^{-1/2}}{1 + t^{3/2}}\|v_N\|_{L^2(\mathbb{R}^3)}\|u\|_{L^2(\mathbb{R}^6)} \\
&\lesssim N^{(-1+3\beta)/2}\log(1 + t)/(1 + t^{3/2}) \quad \text{by (2.44)}. \tag{A.13}
\end{aligned}$$

Other third order contributions to $\mathbb{H}\psi^{(j)}$ are less singular and can be estimated similarly. The first order contributions in (2.8a)-(2.8b) are providing with similar bounds and the estimates for them are similar to the previous estimates. The estimates so far prove (2.133a).

Finally let's prove the estimate (2.133b) on $\tilde{\mathbb{H}}\psi^{(j)}$. This is the contribution coming from (2.8y) when all c -factors are replaced with their corresponding δ -parts as we can recall from the definition (2.130). We have the following estimate:

$$\begin{aligned}
&\frac{1}{2N}\left\|\int dy_1 dy_2\{v_N(y_1 - y_2)\mathcal{Q}_{y_1 y_2}^*\mathcal{Q}_{y_1 y_2}\}(\psi^{(j)})\right\|_{L^2(\mathbb{R}^{3j})} \\
&\simeq_j N^{-1}\|v_N(z_1 - z_2)\psi^{(j)}(z_1, z_2, \dots, z_j)\|_{L^2(\mathbb{R}^{3j})} \\
&\lesssim N^{-1}\|v_N\|_{L^\infty(\mathbb{R}^3)}\|\psi^{(j)}\|_{L^2(\mathbb{R}^{3j})} \lesssim N^{-1+3\beta}\|\psi^{(j)}\|_{L^2(\mathbb{R}^{3j})}
\end{aligned}$$

with which we completed proving Lemma 2.13. □

Appendix B: Comparison of N -particle Mean Field to the Limiting Mean Field

The purpose here is to compare the N -particle (N finite) mean field $\phi^{(N)}$ satisfying (1.38a) i.e.

$$\frac{1}{i}\partial_t\phi^{(N)} - \Delta\phi^{(N)} + (v_N * |\phi^{(N)}|^2)\phi^{(N)} = 0 \text{ where } v_N = N^{3\beta}v(N^\beta\cdot), 0 < \beta < 1$$

with the mean field ϕ (in the limit as $N \rightarrow \infty$) which is the solution of the formal limit of the above equation, namely,

$$\frac{1}{i}\partial_t\phi - \Delta\phi + \left(\int v(x)dx\right)|\phi|^2\phi = 0. \tag{B.1}$$

Proposition B.1. *Let $0 < \beta < 1$ and $\phi^{(N)}(t, \cdot)$ and $\phi(t, \cdot)$ denote the solutions of (1.38a) and (B.1) respectively, with initial data $\phi_0 \in H^1 \cap W^{l,1}$ for $l \geq 2$. Then for every t*

$$\text{Tr}||\phi^{(N)}(t, \cdot)\rangle\langle\phi^{(N)}(t, \cdot)| - |\phi(t, \cdot)\rangle\langle\phi(t, \cdot)|| \leq 2\|\phi^{(N)}(t, \cdot) - \phi(t, \cdot)\|_2 \leq \frac{C}{\sqrt{N}}$$

for some constant C independent of t and N .

Proof. The first inequality follows from the duality of trace class operators with the compact operators on $L^2(\mathbb{R}^3)$ as discussed earlier in chapter 3. The proof of the second inequality follows as in Appendix A of [3] with some alterations where we make use of the L^∞ -decay estimates ((B.4a)-(B.4b)) for the solutions of the equations (1.38a) and (B.1). In the following, we will suppress the time dependence of $\phi^{(N)}$ and ϕ notation-wise. The idea of the proof is applying energy estimates to the equation

$$((1/i)\partial_t - \Delta)(\phi^{(N)} - \phi) = \left(\int v(x)dx \right) |\phi|^2 \phi - (v_N * |\phi^{(N)}|^2) \phi^{(N)}.$$

Let $c := \int v(x)dx$. Then we have:

$$\begin{aligned} \partial_t \|\phi^{(N)} - \phi\|_2^2 &= 2 \operatorname{Im} \langle i\partial_t \phi^{(N)} - i\partial_t \phi, \phi^{(N)} - \phi \rangle & (B.2) \\ &= 2 \operatorname{Im} \left\langle -\Delta \phi^{(N)} + (v_N * |\phi^{(N)}|^2) \phi^{(N)} + \Delta \phi - c|\phi|^2 \phi, \phi^{(N)} - \phi \right\rangle \\ &= -2 \operatorname{Im} \left\langle (v_N * |\phi^{(N)}|^2) \phi^{(N)}, \phi \right\rangle - 2 \underbrace{\operatorname{Im} \langle c|\phi|^2 \phi, \phi^{(N)} \rangle}_{-\operatorname{Im} \langle c|\phi|^2 \phi^{(N)}, \phi \rangle} \\ &= -2 \operatorname{Im} \left\langle (v_N * |\phi^{(N)}|^2 - c|\phi|^2) \phi^{(N)}, \phi \right\rangle \quad (\text{add \& subtract } v_N * |\phi|^2 \text{ to the terms in } (\cdot)) \\ &= -2 \underbrace{\operatorname{Im} \left\langle (v_N * (|\phi^{(N)}|^2 - |\phi|^2)) \phi^{(N)}, \phi \right\rangle}_{\operatorname{Im} \left\langle (v_N * (|\phi^{(N)}|^2 - |\phi|^2)) (\phi^{(N)} - \phi), \phi \right\rangle = (I)} - 2 \underbrace{\operatorname{Im} \left\langle (v_N * |\phi|^2 - c|\phi|^2) \phi^{(N)}, \phi \right\rangle}_{(II)} \end{aligned}$$

We estimate (I) in the last line of (B.2) as follows:

$$\begin{aligned} |(I)| &\leq \left\| (v_N * (|\phi^{(N)}|^2 - |\phi|^2)) (\phi^{(N)} - \phi) \right\|_1 \|\phi\|_\infty \\ &\leq \|v_N * (|\phi^{(N)}|^2 - |\phi|^2)\|_2 \|\phi^{(N)} - \phi\|_2 \|\phi\|_\infty \end{aligned}$$

$$\begin{aligned}
&\leq \|v_N\|_1 \left\| |\phi^{(N)}|^2 - |\phi|^2 \right\|_2 \|\phi^{(N)} - \phi\|_2 \|\phi\|_\infty \\
&\leq \|v_N\|_1 (\|\phi^{(N)}\|_\infty + \|\phi\|_\infty) \|\phi^{(N)} - \phi\|_2^2 \|\phi\|_\infty \\
&\leq C(1+t^3)^{-1} \|\phi^{(N)} - \phi\|_2^2
\end{aligned} \tag{B.3}$$

where we used the facts

$$\|\phi^{(N)}(t, \cdot)\|_\infty \leq C(1+t^{3/2})^{-1} \text{ by Corollary 3.4 in [20]}, \tag{B.4a}$$

$$\|\phi(t, \cdot)\|_\infty \leq C(1+t^{3/2})^{-1} \text{ by Theorem 2 in [32]} \tag{B.4b}$$

for the last inequality in (B.3) (constants in (B.4a), (B.4b) depend only on $\|\phi_0\|_{W^{l,1}}$).

This is the point which makes the argument different than that of [3].

We move on to estimating (II) in the last line of (B.2) recalling that the time dependence of $\phi^{(N)}$, ϕ is notationally omitted for ease of notation. Again we proceed as in [3]:

$$\begin{aligned}
|(\text{II})| &\leq \int v_N(y) \left| |\phi(x-y)|^2 - |\phi(x)|^2 \right| |\phi^{(N)}(x)| |\phi(x)| \, dx \, dy \quad (\text{using } \int v_N(x) dx = c) \\
&= \int v(y) \left| \underbrace{|\phi(x-y/N)|^2 - |\phi(x)|^2}_{\int_0^1 \frac{d}{ds} |\phi(x-sy/N)|^2 ds} \right| |\phi^{(N)}(x)| |\phi(x)| \, dx \, dy \\
&\leq \int v(y) \left(2|y|N^{-1} \int_0^1 |\nabla \phi(x-sy/N)| |\phi(x-sy/N)| \, ds \right) |\phi^{(N)}(x)| |\phi(x)| \, dx \, dy \\
&\leq N^{-1} \|\phi\|_\infty^2 (\|\nabla \phi\|_2^2 + \|\phi^{(N)}\|_2^2) \int v(y) |y| \, dy \\
&\leq C(1+t^3)^{-1} N^{-1} (1 + \|\phi\|_{H^1}^2) \int v(y) |y| \, dy \quad (\text{using (B.4b) being different than [3]}) \\
&\leq C(1+t^3)^{-1} N^{-1}
\end{aligned} \tag{B.5}$$

Inserting (B.3) and (B.5) in the last line of (B.2):

$$\partial_t \|\phi^{(N)} - \phi\|_2^2 \leq C(1+t^3)^{-1} \|\phi^{(N)} - \phi\|_2^2 + \frac{C(1+t^3)^{-1}}{N}$$

Gronwall's inequality implies:

$$\|\phi^{(N)} - \phi\|_2^2 \leq CN^{-1} e^{C(\int_0^t (1+s^3)^{-1} ds)} \int_0^t (1+s^3)^{-1} ds$$

which gives the desired rate of convergence. □

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