ABSTRACT

Title of dissertation: A COMPREHENSIVE STUDY OF MULTIPLICATIVE ATTRIBUTE GRAPH MODEL

Sikai Qu

Dissertation directed by: Professors A.M. Makowski
Dept. of Electrical and Computer Engineering

Graphs are powerful tools to describe social, technological and biological networks, with nodes representing agents (people, websites, gene, etc.) and edges (or links) representing relations (or interactions) between agents. Examples of real-world networks include social networks, the World Wide Web, collaboration networks, protein networks, etc. Researchers often model these networks as random graphs.

In this dissertation, we study a recently introduced social network model, named the Multiplicative Attribute Graph model (MAG), which takes into account the randomness of nodal attributes in the process of link formation (i.e., the probability of a link existing between two nodes depends on their attributes). Kim and Lesckovec, who defined the model, have claimed that this model exhibit some of the properties a real world social network is expected to have. Focusing on a homogeneous version of this model, we investigate the existence of zero-one laws for graph properties, e.g., the absence of isolated nodes, graph connectivity and the emergence of triangles. We obtain conditions on the parameters of the model, so that these properties occur with high or vanishingly probability as the number of nodes becomes unboundedly large. In that regime, we also investigate the property of triadic closure and the nodal degree distribution.

A Comprehensive Study of Multiplicative Attribute Graph Model

by

Sikai Qu

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Advisory Committee:
Professor A.M. Makowski, Chair/Advisor
Professor R.J. La
Professor N. Martins
Professor A. Barg
Professor E. Slud

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Chapter 1

Introduction

An undirected and loopless graph is a collection of nodes and edges linking pairs of distinct end points from the node set. Such a graph is usually denoted by $G = (V, E)$ where V is a (non-empty) set of nodes and E is a set of edges. A graph is a powerful way to describe a set of agents (represented by nodes) and their interactions (represented by edges). These agents can be people forming social or collaborating networks; websites forming the World Wide Web (WWW); genes forming protein networks, etc. A link between two agents describes an interaction, such as a social relationship, a path for information diffusion or a physical connection.

Given the role graphs play in multiple disciplines and their complex structures, researchers have devoted intense efforts to their studies. Researchers have mainly focused on network modeling, network analysis and network inference. This thesis studies properties of a recently proposed social network model and contribute to the area of social network modeling.

1.1 Social network structures

To create a good model of social networks, we need to first understand some basic structures, at the root of modeling. Social networks have been studied for almost a century. As early as 1940, Brown [1] had already mentioned social structures embedded in human societies:

> The social phenomena which we observe in any human society are not the immediate result of the nature of individual human beings, but are the result of the social structure by which they are united.

There is a strong belief among researches that social phenomena are all connected tightly with social structures, either being implied in or resulting from them. These structures are of interest to researchers who study how humans self-organize, and to those who wish to study the implications of those structures in various (economic, political) interactions. They led to the study of mechanisms that generate different kinds of structures so that we can model these structures and make simple simulations of real world networks.

For the purpose of this dissertation, all graphs under considerations are simple (i.e., each pair of nodes can only be connected by one edge and no self-loop is allowed) and undirected. An extended introduction to on graph theory can be found in the reference [2].

1.2 Social network models

Generally speaking, there are two main modeling approaches. One relies on random graph theory while the other one is based on statistics models inferred from social network data. Here, we mainly focus on the random graph approach.

Given a probability triple $(\Omega, \mathcal{F}, \mathbb{P})$, a (undirected) random graph G on the vertex set $V_n = \{1, ..., n\}$ is a graph-valued random variable (rv) defined by

$$\mathbb{G}(n) : \Omega \to \mathcal{G}(V_n),$$

where $\mathcal{G}(V_n)$ is the set of all undirected simple graphs on V_n. Different graph-generating algorithms have been proposed in the literature and the properties emerging from the resulting graphs have been extensively studied. We are going to review some of the most classic models, starting with the standard Erdős-Rényi graphs [5, 6, 8].

Erdős-Rényi graphs Given a set of n nodes, the $\frac{n(n-1)}{2}$ distinct edges emerge independently with probability $p > 0$ which usually scales with n.

Several remarkable results were presented in [5], showing that sharp phases transitions arise under certain scalings $p : \mathbb{N}_0 \to (0, 1)$. For example, with $p_n = c\frac{\ln n}{n}$, if $c > 1$, the graph is connected (and therefore has no isolated nodes) with high probability. On the other hand, if $c < 1$, the graph contains isolated nodes (and therefore it is not connected) with high probability.

However this model fails to be a good social network model because its degree distribution converges to a Poisson distribution with parameter λ if $\lim_{n\to\infty} np_n = \lambda >$

0, which is not usually observed in the real data set. Moreover, Erdős-Rényi graphs does not have high clustering coefficients. However, Erdős-Rényi graphs frequently serve as benchmarks for other random graph models; its sharp phases transition thresholds shed light on similar properties in other models.

Next, we introduce an elegant model which has a power law degree distribution.

Barabási-Albert graphs A degree distribution is of power law if the frequency of degree d is asymptotically proportional to $d^{-\gamma}$ for some $\gamma > 0$. Data collected from real world networks reveal that a lot of social networks have power law degree distributions [10–14]. In order to construct a model with a power-law degree distribution, Barabási and Albert proposed a model [9] based on the idea of preferential attachment. This model reflects the notion that the rich get richer, as a "newborn" node will choose its neighbor from the existing nodes with a probability proportional to their current degrees. Barabási and Albert showed how this simple model can generate a power law degree distribution with $\gamma = 3$. However, its tree-like structure precludes it from being a good model for social networks.

To address this clustering issue, we bring the next model to readers' attention.

Watts-Strogatz graphs In order to address the combination of relative small diameters and high level of clustering, Watts and Strogatz [15] proposed a model which became known as the small world model. Having n nodes in a circle, each node initially connects to other nodes within k steps away on the circle. For each link, it is rewired with probability p in $(0, 1)$, i.e., with one end point fixed, the other one is chosen uniformly at random. For $p = 1$, the model reduce essentially to an Erdős-Rényi graph model, while for $p = 0$, the graph remains a lattice.

The small world model is so named since even for a small rewiring probability, the diameter of the graph will be significantly smaller than in the original lattice while the graph still maintains a high level of clustering.

However, it is not a growth model, as the number of nodes is fixed before links being rewired. Moreover, its degree distribution is somewhere between a Poisson distribution and a uniform distribution, a feature rarely observed in the real world.

We finish this brief introduction to random graph models by introducing the notion of zero-one laws, which are frequently explored in this dissertation. Fix $n = 2, 3, \ldots$, and assume that $\mathbb{G}(n)$ has vertex set $V_n = \{1, \ldots, n\}$. Often, the pmf of $\mathbb{G}(n)$ depends on a parameter (vector), say ν, in some subset $\mathcal{Y} \subseteq \mathbb{R}^d$ for some d in $\mathbb{N}_0$. This parameter is sometimes (partially) scaled with n so that the collection $\{\mathbb{G}(n; \nu_n),\ n = 2, 3, \ldots\}$ now defines a family of random graphs. One of the main goals of this dissertation is to obtain conditions on the scaling $\nu : \mathbb{N}_0 \to \mathcal{Y}$ such that either

$$\lim_{n\to\infty} \mathbb{P}\left[G(n; \nu_n) \text{ has property } \mathcal{A}\right] = 0 \ \textbf{(Zero-law)}$$

or

$$\lim_{n\to\infty} \mathbb{P}\left[G(n; \nu_n) \text{ has property } \mathcal{A}\right] = 1 \ \textbf{(One-law)}.$$

for a given graph property $\mathcal{A}$.

1.3 Modeling social networks with nodal attributes

In most networks, nodes (agents) themselves are associated with a rich set of properties. For example, a person has a profile in terms of gender, living locations, education background, working experiences, hobbies, etc.; web pages contain content, themes, domains; cities have population, geometric locations, economic performance indicators and so on. These features and properties should influence links formation and should affect network structures, both global and local. Discussions about the dependency between nodal attributes and a network can be found in [20] by Fosdick and Hoff. However, most of the existing random graph models, especially the three mentioned above, do not take nodal attributes into consideration.

A recent study by Kim and Leskovec [17], which used nodal attributes to govern the link establishment probabilities, attracts our attention. This model is claimed to exhibit complex behaviors while still being analytically tractable; it is referred as Multiplicative Attribute Graph (MAG). Each node u is associated with an E-valued random vector known as its attributes. These attributes are mutually independent. The probability of adjacency (or establishment of a link) is governed by the attributes of two end points through a symmetric mapping $Q : E \times E \to (0, 1)$. Some closely related models can also be found in [18, 19].

Kim and Leskovec [17] studied various properties under a homogeneity assumption. More precisely, we have $E = \{0, 1\}^L$, an L-length binary (random) vector, of which the component Bernoulli random variables are i.i.d.s. $Q = q^L$, where $q : \{0, 1\} \times \{0, 1\} \to (0, 1)$ is a symmetric mapping. The emergence of a link is governed by the attribute vectors of the two end points of this link, and link variables are mutually independent once attribute vectors of all the nodes are given. Formal definition will be given in a later chapter. Under a scaling of the form $L_n = \rho_n \ln n$ (where n is the number of the nodes) for some sequences $\{\rho_n, n = 1, 2, \dots\}$ where $\lim_{n\to\infty} \rho_n = \rho > 0$, the zero-one law for connectivity and approximations to the nodal degree distribution were given as the total number of nodes n grows unboundly large.

1.4 Contribution

While giving Kim and Leskovec full credits for introducing the MAG model, we find that the proof of the zero-one law for connectivity is incorrect and the approximation to the degree distribution is inaccurate. Improving their modeling efforts, we re-investigate and give a correct proof to the zero-one law for connectivity. We also give a convergence result for the nodal degree distribution without any approximation. Our result relaxes the assumption of $\alpha > \beta > \gamma$ used by Kim and Leskovec and it is not limited to the tail of the degree distribution as suggested by these authors.

Moreover, we also study some properties that are not addressed by Kim and Leskovec. The zero-one law for the absence of isolated nodes is one of them. It is interesting to see whether the zero-one law for the absence of isolated nodes coincide with the zero-one law for connectivity, which is true for Erdös Rényi graphs [6], random geometric graphs [3] and random key graphs [27]. We establish a zero-one law for the absence of isolated nodes with an accurate proof. According to our results, the assumption $\alpha > \beta > \gamma$ can be relaxed. Yet the new conditions are too weak for the zero-one law for connectivity to hold. A counterexample is given.

Another property that has not been studied by Kim and Leskovec is the property of triadic closure. We show that MAG has the property of triadic closure, establish a zero-one law for the emergence of triangles and present a limiting result regarding the total clustering coefficient.

1.5 The road map

In Chapter 2, we first formally define the MAG model, and then discuss some of its basic properties which have been introduced in [17]. Some preliminary asymptotic laws and mathematical techniques are also included in this chapter. We establish the zero-one law for the absence of isolated nodes in Chapter 3, and give an alternative proof to the zero-one law for connectivity which bypass the technical mistake made by Kim and Leskovec in Chapter 4.

Both the property of triadic closure and the zero-one laws for the existence of triangles will be established in Chapter 5. Last but not least, we discuss the nodal degree distribution and its approximation in Chapter 6 and Chapter 7, respectively. Some additional proofs are given in Appendix.

1.6 Notation and conventions

Some frequently used notation in this dissertation:

All limiting statements, including asymptotic equivalence and convergence, are understood with the number of nodes n growing unboundedly large. All random variables (rvs) under considerations are defined on the same probability triple $(\Omega, \mathcal{F}, \mathbb{P})$; the corresponding expectation operator and variance operator are denoted by $\mathbb{E}$ and Var, respectively.

For a sequence of events $\{E_n,\ n = 1, 2, \ldots\}$, we say that E_n happens asymptotically almost surely (a.a.s.) if $\lim_{n\to\infty} \mathbb{P}[E_n] = 1$. For a sequence of $\mathbb{R}-$valued rvs $\{A, A_n,\ n = 1, 2, \ldots\}$, we write $A_n \xrightarrow{P}_n A$ to denote that A_n converges in probability to A. Similarly, $A_n \Rightarrow_n A$ denotes that A_n converges in distribution to A.

For sequences $a, b : \mathbb{N}_0 \to \mathbb{R}$, we write $a_n \sim b_n$ if $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$, and $a_n =_{\text{Approx}} b_n$ if $\lim_{n\to\infty}(a_n - b_n) = 0$. Additionally, we write $a_n = o(b_n)$ if $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, and we use the notation $a_n = \Omega(b_n)$ if there exist a constant $c > 0$ and a positive integer N such that $a_n \geqslant c \cdot b_n$ whenever $n \geqslant N$.

We also denote by $|S|$ the cardinality of the discrete set S. The indicator function of an event E is denoted by $\mathbf{1}[E]$.

Chapter 2

Introduction to Multiplicative Attribute Graph Model

2.1 General model

The multiplicative attribute graph (MAG) model is parameterized by a number of quantities, chief amongst them the number n of nodes and the number L of attributes associated with each node - Both n and L are positive integers. Nodes are labeled $u = 1, 2, \cdots, n$, while attributes are labeled $\ell = 1, 2, \cdots, L$.

As in the work of Kim and Leskovec [17], we assume that each of the L attributes associated with a node is binary in nature with 1 (resp. 0) signifying that the attribute is present (resp. absent). We conveniently organize these L attributes into a vector

$$\mathbf{a} = (a_1, \ldots, a_L), \qquad \mathbf{a} \in \{0, 1\}^L \tag{2.1}$$

with a_ℓ in $\{0, 1\}$ for $\ell = 1, \ldots, L$.

The propensity of nodes to attach to each other is governed by their attributes in a way to be clarified shortly. To formalize this notion, we follow the approach adopted by Kim and Leskovec [17], the construction used here being equivalent to the one found there.

2.1.1 The underlying random variables

On the probability triple $(\Omega, \mathcal{F}, \mathbb{P})$, we are given two set of rvs, namely the collection

$$\{A, A_\ell, A_\ell(u), \quad \ell = 1, \ldots, L;\ u = 1, \ldots, n\}$$

and the triangular array

$$\{U(u,v), \quad 1 \leqslant u < v \leqslant n\}.$$

The following assumptions are enforced throughout the dissertation:

(i) The collection $\{A, A_\ell, A_\ell(u),\ \ell = 1, \ldots, L;\ u = 1, \ldots, n\}$ and the triangular array

$\{U(u,v), \quad 1 \leqslant u < v \leqslant n\}$ are *mutually independent*;

(ii) The rvs $\{U(u,v),\ 1 \leqslant u < v \leqslant n\}$ are *i.i.d.* rvs, each of which is *uniformly* distributed on the interval $(0,1)$; and

(iii) The rvs $\{A, A_\ell, A_\ell(u),\ \ell = 1, \ldots, L; u = 1, \ldots, n\}$ form a collection of *i.i.d.* $\{0,1\}$-valued rvs with pmf $\boldsymbol{\mu} = (\mu(0), \mu(1))$ where

$$\mathbb{P}[A = 0] = \mu(0) \quad \text{and} \quad \mathbb{P}[A = 1] = \mu(1).$$

To avoid trivial situations of limited interest, we assume that both $\mu(0)$ and $\mu(1)$ are elements of the *open* interval $(0,1)$ such that $\mu(0)+\mu(1)=1$.

For each $L = 1, 2, \ldots$, we set

$$\boldsymbol{A}_L = (A_1, \ldots, A_L)$$

and

$$\boldsymbol{A}_L(u) = (A_1(u), \ldots, A_L(u)), \quad u = 1, 2, \ldots .$$

Under the enforced assumptions, the $\{0,1\}^L$-valued rvs $\{\mathbf{A}_L(u),\ u = 1, \ldots, n\}$ are mutually independent, each with i.i.d. components distributed like the generic random variable A. We also define

$$S_L(u) = \sum_{\ell=1}^{L} A_\ell(u), \quad u = 1, 2, \ldots, n, \tag{2.2}$$

and

$$S_L = \sum_{\ell=1}^{L} A_\ell. \tag{2.3}$$

Thus, $S_L(u)$ counts the number of attributes node u has. From the enforced assumptions, it is plain that the rvs $\{S_L(u),\ u = 1, 2, \ldots n\}$ form a sequence of i.i.d. rvs, each being distributed according to the rv S_L which is itself a Binomial rv $\mathrm{Bin}(L, \mu(1))$.

For notational reason we find it convenient augment the triangular array of uniform rvs into the larger collection $\{U(u,v),\ u,v = 1, 2, \ldots n\}$ through the definitions

$$U(u,u) = 1 \quad \text{and} \quad U(v,u) = U(u,v), \quad 1 \leqslant u < v \leqslant n.$$

2.1.2 Adjacency

On the way to defining MAGs, we introduce notions of *adjacency* between nodes based on their attributes. To do so, we start with 2×2 attribute score matrices $\mathcal{Q}_\ell$ given by

$$\mathcal{Q}_\ell = q_\ell(a,b) \equiv \begin{pmatrix} q_\ell(1,1) & q_\ell(1,0) \\ q_\ell(0,1) & q_\ell(0,0) \end{pmatrix}$$

for $\ell = 1, 2, \ldots, L$. Throughout we assume the *symmetry* conditions

$$q_\ell(1,0) = q_\ell(0,1), \quad \ell = 1, 2, \ldots, L \tag{2.4}$$

together with

$$q_\ell(a,b) \in [0,1], \quad \begin{array}{c} a, b \in \{0,1\} \\ \ell = 1, 2, \ldots, L. \end{array} \tag{2.5}$$

With these symmetric 2×2 matrices $\{\mathcal{Q}_\ell, \ \ell = 1, 2, \ldots, L\}$, we associate a mapping $Q_L : \{0,1\}^L \times \{0,1\}^L \to [0,1]$ given by

$$Q_L(\boldsymbol{a}_L, \boldsymbol{b}_L) = \prod_{\ell=1}^{L} q_\ell(a_\ell, b_\ell), \quad \boldsymbol{a}_L, \boldsymbol{b}_L \in \{0,1\}^L. \tag{2.6}$$

Interpretations for these quantities will be given shortly. The enforced assumptions (2.4)-(2.5) on the score matrices $\mathcal{Q}_1, \ldots, \mathcal{Q}_L$ readily imply

$$Q_L(\boldsymbol{b}_L, \boldsymbol{a}_L) = Q_L(\boldsymbol{a}_L, \boldsymbol{b}_L), \quad \boldsymbol{a}_L, \boldsymbol{b}_L \in \{0,1\}^L \tag{2.7}$$

with

$$0 \leqslant Q_L(\boldsymbol{a}_L, \boldsymbol{b}_L) \leqslant 1, \quad \boldsymbol{a}_L, \boldsymbol{b}_L \in \{0,1\}^L. \tag{2.8}$$

Pick two distinct nodes u, v in $\{1, 2, \ldots, n\}$. We say that node u is L-adjacent to node v, written $u \sim_L v$, if the condition

$$U(u,v) \leqslant Q_L(\boldsymbol{A}_L(u), \boldsymbol{A}_L(v)) \tag{2.9}$$

holds, in which case an (undirected) edge from node u to node v is said to exist. Obviously, L-adjacency is a binary relation on the set of all nodes. Since $U(u,v) = U(v,u)$, it is plain from (2.7) that u is L-adjacent to node v if and only if v is L-adjacent to node u — This will allow us to say that nodes u and v are *L*-adjacent without any risk of confusion.

We can readily encode L-adjacency through the $\{0,1\}$-valued rvs

$$\left\{ \chi_L(u,v), \quad \begin{array}{c} u, v = 1, 2, \ldots, n \\ u \neq v \end{array} \right\}$$

given by

$$\chi_L(u,v) = \mathbf{1}[U(u,v) \leqslant Q_L(\boldsymbol{A}_L(u), \boldsymbol{A}_L(v))], \qquad \begin{array}{c} u, v = 1, 2, \ldots, n \\ u \neq v \end{array} \tag{2.10}$$

with $\chi_L(u,v) = 1$ (resp. $\chi_L(u,v) = 0$) corresponding to the existence (absence) of an edge between u and v. In view of earlier remarks, we observe that the conditions

$$\chi_L(u,v) = \chi_L(v,u), \qquad \begin{array}{c} u, v = 1, 2, \ldots, n \\ u \neq v \end{array} \tag{2.11}$$

and

$$\chi_L(u,u) = 0, \ \ u = 1, 2, \ldots, n. \tag{2.12}$$

are all satisfied.

2.1.3 Defining MAGs

Fix $n = 1, 2, \ldots,$ and $L = 1, 2, \ldots$. A MAG over a set of n nodes, labeled $1, 2, \ldots, n$, with each node having L attributes, labeled $1, 2, \ldots, L$, is the random graph $\mathbb{M}(n; L)$, whose edge set is determined through the rvs in (2.10). From (2.11)-(2.12) it follows that edges in $\mathbb{M}(n; L)$ are undirected and that there are no self-loops.

For the sake of simplicity, for each $n = 1, 2, \ldots,$ we denote the node set of $\mathbb{M}(n; L)$ by $V_n = \{1, \ldots, n\}$. This will allow us to say u in V_n is equivalent to $u = 1, \ldots, n$, and we will use the notion u in V_n in the rest of this dissertation.

This definition is equivalent to the one given by Kim and Leskovec [17]. Indeed, with the help of Assumptions (i) and (ii), the rvs forming the triangular array

$$\chi_L(u,v) = \mathbf{1}[U(u,v) \leqslant Q_L(\boldsymbol{A}_L(u), \boldsymbol{A}_L(v))], \qquad \begin{array}{c} u, v \in V_n \\ \\ u < v \end{array}$$

are *conditionally* mutually independent given the i.i.d. attribute rvs $\{\mathbf{A}_L(u),\ u \in V_n\}$. Indeed, we have

$$\begin{aligned}
& \mathbb{P}[\chi_L(u,v) = 1,\ 1 \leqslant u < v \leqslant n | \mathbf{A}_L(w) = \mathbf{a}_L(w), w \in V_n] \\
= & \prod_{1 \leqslant u < v \leqslant n} \mathbb{P}[\chi_L(u,v) = 1 | \mathbf{A}_L(w) = \mathbf{a}_L(w), w \in V_n] \\
= & \prod_{1 \leqslant u < v \leqslant n} \mathbb{P}[\chi_L(u,v) = 1 | \mathbf{A}_L(u) = \mathbf{a}_L(u), \mathbf{A}_L(v) = \mathbf{a}_L(v)] \\
= & \prod_{1 \leqslant u < v \leqslant n} Q_L(\mathbf{a}_L(u), \mathbf{a}_L(v)),
\end{aligned} \tag{2.13}$$

with

$$\begin{aligned}
&\mathbb{P}[\chi_L(u,v) = 1|\mathbf{A}_L(u), \mathbf{A}_L(v)] \\
&= Q_L(\mathbf{A}_L(u), \mathbf{A}_L(v)) \\
&= \prod_{\ell=1}^{L} q_\ell(A_\ell(u), A_\ell(v)), \qquad \begin{array}{c} u, v \in V_n \\ \\ u < v. \end{array}
\end{aligned} \tag{2.14}$$

2.1.4 A homogeneous version

For the MAGs defined in the previous section, there are $3L + 3$ parameters, namely n, L, $\mu(1)$ and $\{q_\ell(1,1), \quad q_\ell(1,0), \quad q_\ell(0,0), \ \ell = 1, 2, \ldots, L\}$. But it is hard to obtain close form results when the parameter set is large. Following Kim and Leskovec [17], we shall consider a homogeneous version of MAGs whose score matrices are identical, namely

$$\mathcal{Q}_\ell \equiv \mathcal{Q}, \quad \ell = 1, 2, \ldots, L.$$

In the rest of this dissertation, unless explicitly stated otherwise, this homogeneity assumption will be enforced throughout. To further simplify the notation, we write

$$q_\ell(1,1) = q(1,1) = \alpha,$$

$$q_\ell(1,0) = q_\ell(0,1) = q(1,0) = q(0,1) = \beta$$

and

$$q_\ell(0,0) = q(0,0) = \gamma$$

for $\ell = 1, 2, \ldots, L$. We also define

$$\Gamma(1) = \mathbb{E}[q(1, A)] = \mu(1)\alpha + \mu(0)\beta \tag{2.15}$$

and

$$\Gamma(0) = \mathbb{E}[q(0, A)] = \mu(1)\beta + \mu(0)\gamma. \tag{2.16}$$

These building blocks are assumed given and held *fixed* during the discussions — They will not be explicitly displayed in the notation.

2.2 Properties of MAG

Before discussing various structural properties of MAGs, we present some fundamental properties which build the ground of MAGs. We first review link establishment probabilities given by Kim and Leskovec [17]. This is then followed by a discussion of the independence of the edge assignment rvs.

2.2.1 Link establishment

Fix $n = 2, 3, \ldots$ and $L = 1, 2, \ldots$. For distinct nodes u, v in V_n, easy calculations [17] yield

$$\mathbb{P}[u \sim_L v | S_L(u) = \ell] \quad = \quad \Gamma(1)^\ell \Gamma(0)^{L-\ell}, \quad \ell = 0, 1, \ldots, L. \tag{2.17}$$

Taking expectation we get

$$\begin{aligned} \mathbb{P}[\chi_L(u, v) = 1] &= \mathbb{P}[u \sim_L v] \\ &= \mathbb{E}[\mathbb{P}[u \sim_L v | S_L(u)]] \\ &= (\mu(1)\Gamma(1) + \mu(0)\Gamma(0))^L \end{aligned} \tag{2.18}$$

as we used the fact that $S_L(u)$ is a Binomial rv with parameters $(L, \mu(1))$.

2.2.2 Independence

Many probabilistic bounds and properties rely on the mutual independence of the rvs involved. We discuss two such instances.

Lemma 2.1. *The link rvs in the triangular array* $\{\chi_L(u,v),\ u,v \in V_n,\ u < v\}$ *are not mutually independent.*

To establish Lemma 2.1, we only need to show that, for three distinct nodes u, v and w in V_n,

$$\mathbb{P}[\chi_L(u,v) = 1, \chi_L(u,w) = 1] \neq \mathbb{P}[\chi_L(u,v) = 1]\mathbb{P}\left[\chi_L(u,w) = 1\right].$$

Proof. Fix $n = 2, 3, \ldots$ and $L = 1, 2, \ldots$. For three distinct nodes u, v and w in V_n, by iterated expectations, we have

$$\begin{aligned} \mathbb{P}[\chi_L(u,v) = 1, \chi_L(u,w) = 1] &= \mathbb{E}[\mathbf{1}[\chi_L(u,v) = 1, \chi_L(u,w) = 1]] \\ &= \mathbb{E}[\mathbb{E}[\mathbf{1}[\chi_L(u,v) = 1, \chi_L(u,w) = 1]|\mathbf{A}_L(u)]]. \end{aligned} \tag{2.19}$$

From the definition (2.10) of link variables, we can rewrite (2.19) as

$$\begin{aligned} &\mathbb{E}[\mathbf{1}[\chi_L(u,v) = 1, \chi_L(u,w) = 1]|\mathbf{A}_L(u)] \\ &\quad = \mathbb{E}\left[\mathbf{1}[U(u,v) \leqslant Q_L(\mathbf{a}_L(u), \mathbf{A}_L(v))]\mathbf{1}[U(u,w) \leqslant Q_L(\mathbf{a}_L(u), \mathbf{A}_L(w))]\right]_{\mathbf{a}_L(u)=\mathbf{A}_L(u)} \\ &. \end{aligned} \tag{2.20}$$

The two pair of rvs $(U(u,v), \mathbf{A}_L(v))$ and $(U(u,w), \mathbf{A}_L(w))$ are independent. As a result, for each $\mathbf{a}_L(u)$ in $\{0,1\}^L$, it holds that

$$\begin{aligned} & \mathbb{E}[\mathbf{1}[U(u,v) \leqslant Q_L(\mathbf{a}_L(u), \mathbf{A}_L(v))]\mathbf{1}[U(u,w) \leqslant Q_L(\mathbf{a}_L(u), \mathbf{A}_L(w))]] \\ = \ & \mathbb{E}[\mathbf{1}[U(u,v) \leqslant Q_L(\mathbf{a}_L(u), \mathbf{A}_L(v))]]\mathbb{E}[\mathbf{1}[U(u,w) \leqslant Q_L(\mathbf{a}_L(u), \mathbf{A}_L(w))]] \\ = \ & \mathbb{E}[\mathbf{1}[U(u,v) \leqslant Q_L(\mathbf{a}_L(u), \mathbf{A}_L(v))]]^2 \end{aligned} \tag{2.21}$$

where the last step is based on the fact that the collections $\{\mathbf{A}_L(t),\ t \in V_n\}$ and $\{U(t,s),\ t,s \in V_n,\ t < s\}$ are both collections of i.i.d. rvs. It is also plain that

$$\begin{aligned} & \mathbb{E}[\mathbf{1}[U(u,v) \leqslant Q_L(\mathbf{a}_L(u), \mathbf{A}_L(v))]] \\ = \ & \mathbb{E}\left[\mathbb{E}\left[\mathbf{1}[U(u,v) \leqslant Q_L(\mathbf{a}_L(u), \mathbf{A}_L(v))]|\mathbf{A}_L(v)\right]\right] \\ = \ & \mathbb{E}[Q_L(\mathbf{a}_L(u), \mathbf{A}_L(v))]. \end{aligned} \tag{2.22}$$

Upon setting

$$Q_L^\star(\boldsymbol{a}_L) = \mathbb{E}\left[Q_L\left(\boldsymbol{a}_L, \boldsymbol{A}_L\right)\right], \quad \boldsymbol{a}_L \in \{0,1\}^L, \tag{2.23}$$

we see that

$$\begin{aligned} Q_L^\star(\boldsymbol{a}_L) &= \mathbb{E}\left[\prod_{\ell=1}^L q(a_\ell, A_\ell)\right] \\ &= \prod_{\ell=1}^L \mathbb{E}\left[q(a_\ell, A_\ell)\right] \\ &= \prod_{\ell=1}^L \mathbb{E}\left[q(1, A_\ell)\right]^{a_\ell} \cdot \mathbb{E}\left[q(0, A_\ell)\right]^{1-a_\ell} \\ &= \Gamma(1)^{\sum_{\ell=1}^L a_\ell} \cdot \Gamma(0)^{\sum_{\ell=1}^L (1-a_\ell)} \end{aligned} \tag{2.24}$$

with the help of the notation (2.15)-(2.16). In particular it follows that

$$Q_L^\star(\boldsymbol{A}_L(u)) = \Gamma(1)^{S_L(u)}\Gamma(0)^{L-S_L(u)}, \quad u \in V_n. \tag{2.25}$$

Therefore, by virtue of (2.20)-(2.24), (2.19) now becomes

$$\begin{aligned}
\mathbb{P}[\chi_L(u,v)=1,\chi_L(u,w)=1] &= \mathbb{E}\left[\mathbb{E}[Q_L(\mathbf{a}_L(u),\mathbf{A}_L(u))]^2_{\mathbf{a}_L(u)=\mathbf{A}_L(u)}\right] \\
&= \mathbb{E}\left[Q^\star_L(\boldsymbol{a}_L(u))^2_{\mathbf{a}_L(u)=\mathbf{A}_L(u)}\right] \\
&= \mathbb{E}\left[Q^\star_L(\boldsymbol{A}_L(u))^2\right] \\
&= \mathbb{E}\left[\left(\Gamma(1)^{S_L(u)}\Gamma(0)^{L-S_L(u)}\right)^2\right] \\
&= \left(\mu(1)\Gamma(1)^2+\mu(0)\Gamma(0)^2\right)^L \qquad (2.26)
\end{aligned}$$

upon using the fact that the rv $S_L(u)$ is a binomial rv with parameters $(L,\mu(1))$.

However, using (2.18) we have

$$\mathbb{P}[\chi_L(u,v)=1]\mathbb{P}[\chi_L(u,w)=1]=(\mu(1)\Gamma(1)+\mu(0)\Gamma(0))^{2L}. \qquad (2.27)$$

This implies

$$\mathbb{P}[\chi_L(u,v)=1,\chi_L(u,w)=1]\neq\mathbb{P}[\chi_L(u,v)=1]\mathbb{P}[\chi_L(u,w)=1]$$

and the link variables $\chi_L(u,v)$ and $\chi_L(u,w)$ are not mutually independent. ■

On the other hand, the following fact holds.

Lemma 2.2. *For $n=2,3,\ldots,$ and any node u in V_n, the rvs $\{\chi_L(u,v),\ v\in V_n,\ v\neq u\}$ are conditionally mutually independent given $S_L(u)$.*

Proof. Fix $n=2,3,\ldots$ and a node u in V_n. For $\ell=1,2,\ldots,L$, we have

$$\begin{aligned}
&\mathbb{P}\left[\bigcap_{v\in V_n,\ v\neq u}[\chi_L(u,v)=b_v]\middle| S_L(u)=\ell\right] \\
&= \mathbb{E}\left[\prod_{v\in V_n,\ v\neq u}(b_v\chi_L(u,v)+(1-b_v)(1-\chi_L(u,v)))\middle| S_L(u)=\ell\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\prod_{v\in V_n,\ v\neq u}(b_v\chi_L(u,v)+(1-b_v)(1-\chi_L(u,v)))\middle|\mathbf{A}_L(u)\right]\middle| S_L(u)=\ell\right]
\end{aligned}$$

where b_v in $\{0,1\}$ for v in V_n and $v \neq u$. From (2.21)-(2.25), we get

$$\begin{aligned}
&\mathbb{E}\left[\prod_{v\in V_n,\ v\neq u}(b_v\chi_L(u,v)+(1-b_v)(1-\chi_L(u,v)))\middle|\mathbf{A}_L(u)\right] \\
&= \mathbb{E}\left[\prod_{v\in V_n,\ v\neq u}\left(\begin{array}{l} b_v\mathbf{1}[U(u,v)\leqslant Q(\mathbf{a}_L(u),\mathbf{A}_L(v))] \\ +(1-b_v)\mathbf{1}[U(u,v)>Q(\mathbf{a}_L(u),\mathbf{A}_L(v))]\end{array}\right)\right]_{\mathbf{a}_L(u)=\mathbf{A}_L(u)} \\
&= \left(\prod_{v\in V_n,\ v\neq u}\Big(b_vQ_L^*(\mathbf{a}_L(u))+(1-b_v)(1-Q_L^*(\mathbf{a}_L(u)))\Big)\right)_{\mathbf{a}_L(u)=\mathbf{A}_L(u)} \\
&= \left(\Gamma(1)^{S_L(u)}\Gamma(0)^{L-S_L(u)}\right)^{\sum_{v\in V_n,\ v\neq u}b_v}\left(1-\Gamma(1)^{S_L(u)}\Gamma(0)^{L-S_L(u)}\right)^{\sum_{v\in V_n,\ v\neq u}1-b_v}
\end{aligned}$$

since the rvs $\{(U(u,v),\mathbf{A}_L(v)),\ v\in V_n,\ v\neq u\}$ form a collection of i.i.d. rvs. Consequently, it is plain that

$$\begin{aligned}
&\mathbb{P}\left[\bigcap_{v\in V_n,\ v\neq u}[\chi_L(u,v)=b_v]\middle|S_L(u)=\ell\right] \\
&= \mathbb{E}\left[\begin{array}{l}\left(\Gamma(1)^{S_L(u)}\Gamma(0)^{L-S_L(u)}\right)^{\sum_{v\in V_n,\ v\neq u}b_v} \\ \times\left(1-\Gamma(1)^{S_L(u)}\Gamma(0)^{L-S_L(u)}\right)^{\sum_{v\in V_n,\ v\neq u}1-b_v}\end{array}\middle|S_L(u)=\ell\right] \\
&= \left(\Gamma(1)^{\ell}\Gamma(0)^{L-\ell}\right)^{\sum_{v\in V_n,\ v\neq u}b_v}\left(1-\Gamma(1)^{\ell}\Gamma(0)^{L-\ell}\right)^{\sum_{v\in V_n,\ v\neq u}1-b_v} \\
&= \prod_{v:b_v=1}\mathbb{P}[\chi_L(u,v)=1|S_L(u)=\ell]\prod_{v:b_v=0}(1-\mathbb{P}[\chi_L(u,v)=1|S_L(u)=\ell]) \\
&= \prod_{v=1;\ v\neq u}^{n}\mathbb{P}[\chi_L(u,v)=b_v|S_L(u)=\ell],\qquad \begin{array}{c}b_v\in\{0,1\}, \\ v\in V_n \\ v\neq u\end{array}
\end{aligned} \tag{2.28}$$

as we made use of (2.17) and the desired mutual independence follows. ■

2.3 The asymptotic theory

Recall from [5, 6] that an Erdős-Rényi graph $G(n,p)$ is an undirected graph with n nodes, labeled $1,\ldots,n$, where adjacency is defined through the i.i.d. $\{0,1\}$-valued rvs $\{\tau(u,v),\ 1 \leqslant u < v \leqslant n\}$ with

$$\mathbb{P}[\tau(u,v)=1] = \mathbb{P}[u \sim v] = p, \qquad 1 \leqslant u < v \leqslant n. \tag{2.29}$$

The asymptotic properties of $G(n,p)$ when n grows unboundedly large have been extensively studied. Under scalings $p : \mathbb{N}_0 \to (0,1)$ such that $\lim_{n\to\infty} p_n = 0$, critical conditions for various zero-one laws [8] have been found for several properties, including the emergence of giant components, the absence of isolated nodes, connectivity and the emergence of triangles.

For the MAG model, we are interested in establishing such zero-one laws. For any two distinct nodes u, v in V_n, in order to have a scaling such that $\lim_{n\to\infty} \mathbb{P}[\chi_L(u,v) = 1] = 0$, we consider a scaling $\mathtt{L} : \mathbb{N}_0 \to \mathbb{N}_0 : n \to L_n$ with $\lim_{n\to\infty} L_n = \infty$. With $\alpha, \beta, \gamma, \mu(1)$ in $(0,1)$ fixed, it holds that

$$\lim_{n\to\infty} \mathbb{P}[u \sim_{L_n} v] = \lim_{n\to\infty} \left(\mu(1)\Gamma(1) + \mu(0)\Gamma(0)\right)^{L_n} = 0. \tag{2.30}$$

With $\rho > 0$, the scaling $\mathtt{L} : \mathbb{N}_0 \to \mathbb{N}_0$ is said to be ρ-admissible if

$$L_n \sim \rho \ln n \tag{2.31}$$

in which case we can write

$$L_n = \rho_n \ln n, \quad n = 1, 2, \ldots$$

for some sequence $\varrho : \mathbb{N}_0 \to \mathbb{R}_+$ such that $\lim_{n\to\infty} \rho_n = \rho > 0$. For reasons behind selecting (2.31), please refer to [17]. In this dissertation, unless explicitly specified, all scalings are assumed to be ρ-admissible. An asymptotic MAG with parameters $(n, \mu(1), \rho_n, \alpha, \beta, \gamma)$ is denoted by $\mathbb{M}(n; L_n)$, where $(\mu(1), \alpha, \beta, \gamma)$ are assumed fixed and will not be explicitly displayed in the notation.

2.4 Four useful techniques

While establishing various properties for MAGs, some mathematical techniques will be repeatedly used. In this section we present four of these techniques for future reference.

2.4.1 Behaviors of $(1 + x_n)^n$

Consider a scaling: $\mathbb{N}_0 \to (-1, 1) : n \to x_n$. We are interested in bounds and limits of the sequence $n \to (1 + x_n)^n$ where n becomes unboundedly large. The following fact is crucial:

Proposition 2.3. *For x in $(-1, 1)$ and $p > 0$ fixed, we have*

$$(1 + x)^p \leqslant e^{px}. \tag{2.32}$$

Proof. Fix x in $(-1, 1)$ and note that

$$(1 + x)^p = e^{p(\ln(1+x))}. \tag{2.33}$$

The result is now a simple consequence of the fact that

$$\ln(1+x) \quad = \quad \sum_{i=1}^{+\infty}(-1)^{i+1}\frac{x^i}{i} \leqslant x \tag{2.34}$$

by Taylor series expansion. ■

For any sequence $\mathbb{N}_0 \to \mathbb{R} : n \to x_n$, if $\lim_{n\to\infty} x_n = 0$, it is then plain from (2.34) that

$$\ln(1+x_n) = x_n(1+o(1)),$$

and we readily obtain [25, Prop. 3.1.1, p. 116], the following useful fact:

Lemma 2.4. *For any sequence $x : \mathbb{N}_0 \to (-1,1)$, there exists c in $[-\infty,+\infty]$ such that if*

$$\lim_{n\to\infty} nx_n = c,$$

then

$$\lim_{n\to\infty}(1+x_n)^n = e^c. \tag{2.35}$$

2.4.2 The limit of $nC_n^{L_n}$

Lemma 2.5. *Consider a ρ-admissible scaling $\mathtt{L} : \mathbb{N}_0 \to \mathbb{N}_0$ for some $\rho > 0$. For any sequence $\mathtt{C} : \mathbb{N}_0 \to (0,\infty)$ such that $\lim_{n\to\infty} C_n = C$ for some $C > 0$, it holds that*

$$\lim_{n\to\infty} nC_n^{L_n} = \begin{cases} \infty & \text{if } 1+\rho\ln C > 0 \\ \\ 0 & \text{if } 1+\rho\ln C < 0. \end{cases} \tag{2.36}$$

Proof. The ρ-admissibility of the scaling $\mathtt{L} : \mathbb{N}_0 \to \mathbb{N}_0$ yields

$$nC_n^{L_n} = ne^{L_n \log C_n} = ne^{\rho_n \ln C_n \cdot \ln n} = n^{1+\rho_n \ln C_n}, \quad n = 2, 3, \ldots \tag{2.37}$$

for some sequence $\varrho : \mathbb{N}_0 \to \mathbb{N}_0$ such that $\lim_{n\to\infty} \rho_n = \rho$. Letting n go to infinity readily yields the desired conclusion (2.36). ■

2.4.3 Stirling's approximation for binomial coefficient

In the later chapters, we will have the opportunity to use Stirling's approximation for factorials given by

$$p! \sim \sqrt{2\pi p} \left(\frac{p}{e}\right)^p \quad (p \to \infty). \tag{2.38}$$

The following lemma is a direct consequence of (2.38).

Lemma 2.6. *Consider a scaling $t : \mathbb{N}_0 \to \mathbb{N}_0 : n \to t_n$ with $\lim_{n\to\infty} t_n = \infty$ and another scaling $\tau : \mathbb{N}_0 \to \mathbb{N}_0 : n \to \tau_n$ such that $\lim_{n\to\infty} \tau_n = \infty$ and $\tau_n < t_n$ for $n = 1, 2, \ldots$. It holds that*

$$\binom{t_n}{\tau_n} \sim \frac{\sqrt{t_n}}{\sqrt{2\pi\tau_n(t_n - \tau_n)}} \left(\frac{t_n}{\tau_n}\right)^{\tau_n} \left(\frac{t_n}{t_n - \tau_n}\right)^{t_n - \tau_n}.$$

Proof. Lemma 2.6 is established once we notice that

$$\binom{t_n}{\tau_n} = \frac{t_n!}{\tau_n!(t_n - \tau_n)!}$$

for $n = 1, 2, \ldots$. ■

2.4.4 The method of first and second moments

A conventional way to obtain zero-one laws is through the method of first and second moments. In this subsection we provide the main ingredients of this approach as we will need it in its various applications.

Let $\{Z_n,\ n=1,2,\ldots\}$ denote a collection of $\mathbb{N}$-valued rvs such that $\mathbb{E}\left[Z_n^2\right] < \infty$ for each $n = 1, 2, \ldots$. The method of first moment [26, Eqn. (3.10), p. 55] relies on the well-known bound

$$1 - \mathbb{E}\left[Z_n\right] \leqslant \mathbb{P}\left[Z_n = 0\right], \quad n = 1, 2, \ldots \tag{2.39}$$

while the method of second moment [26, Remark 3.1, p. 55] has its starting point in the inequality

$$\mathbb{P}\left[Z_n = 0\right] \leqslant 1 - \frac{\left(\mathbb{E}\left[Z_n\right]\right)^2}{\mathbb{E}\left[Z_n^2\right]}, \quad n = 1, 2, \ldots. \tag{2.40}$$

Letting n go to infinity in the resulting inequalities, we conclude from (2.39) that

$$\lim_{n\to\infty} \mathbb{P}\left[Z_n = 0\right] = 1 \tag{2.41}$$

if

$$\lim_{n\to\infty} \mathbb{E}\left[Z_n\right] = 0, \tag{2.42}$$

while the bound (2.40) implies

$$\lim_{n\to\infty} \mathbb{P}\left[Z_n = 0\right] = 0 \tag{2.43}$$

whenever

$$\limsup_{n\to\infty} \frac{\mathbb{E}\left[Z_n^2\right]}{\left(\mathbb{E}\left[Z_n\right]\right)^2} \leqslant 1. \tag{2.44}$$

This strategy is often used when the rvs $\{Z_n,\ n=1,2,\ldots\}$ are count variables with the following structure (as will be the cases we are going to handle): For $n=1,2,\ldots$, assume the rv Z_n has the form

$$Z_n = \sum_{u=1}^{n} \zeta_{n,u}$$

where the rvs $\zeta_{n,1},\ldots,\zeta_{n,n}$ are $\{0,1\}$-valued rvs. If in addition, the rvs $\zeta_{n,1},\ldots,\zeta_{n,n}$ are *exchangeable* (i.e. they are identically distributed), then we easily arrive at the expressions

$$\mathbb{E}\left[Z_n\right] = \mathbb{E}\left[\sum_{u=1}^{n} \zeta_{n,u}\right] = n\mathbb{E}\left[\zeta_{n,1}\right] \tag{2.45}$$

and

$$\mathbb{E}\left[Z_n^2\right] = \mathbb{E}\left[\left(\sum_{u=1} \zeta_{n,u}\right)^2\right] = n\mathbb{E}\left[\zeta_{n,1}\right] + n(n-1)\mathbb{E}\left[\zeta_{n,1}\cdot\zeta_{n,2}\right] \tag{2.46}$$

by virtue of the binary nature of the rvs involved. Therefore, using (2.45) we find

$$\frac{\mathbb{E}\left[Z_n^2\right]}{\left(\mathbb{E}\left[Z_n\right]\right)^2} = \frac{1}{\mathbb{E}\left[Z_n\right]} + \frac{n-1}{n}\cdot\frac{\mathbb{E}\left[\zeta_{n,1}\cdot\zeta_{n,2}\right]}{\left(\mathbb{E}\left[\zeta_{n,1}\right]\right)^2}. \tag{2.47}$$

It is now plain that (2.44) can be achieved if we show the two convergence statements

$$\lim_{n\to\infty} \mathbb{E}\left[Z_n\right] = \infty, \tag{2.48}$$

and

$$\limsup_{n\to\infty} \frac{\mathbb{E}\left[\zeta_{n,1}\cdot\zeta_{n,2}\right]}{\left(\mathbb{E}\left[\zeta_{n,1}\right]\right)^2} \leqslant 1. \tag{2.49}$$

Chapter 3

Absence of Isolated Nodes

In the previous chapter, we introduced the MAG model and gave the homogeneous version that will be considered in this dissertation. In this present chapter, we are interested in establishing a zero-one law for the absence of isolated nodes in MAGs when the number n of nodes and the number L of nodal attributes grow unboundedly large, the latter quantity scaling with the former.

We remind readers that we have the quantities $\Gamma(0)$ and $\Gamma(1)$ were defined by

$$\Gamma(1) = \mathbb{E}[q(1, A)] = \mu(1)\alpha + \mu(0)\beta$$

and

$$\Gamma(0) = \mathbb{E}[q(0, A)] = \mu(1)\beta + \mu(0)\gamma$$

and that the results are all given under the condition $\Gamma(0) \leqslant \Gamma(1)$. When $\Gamma(1) < \Gamma(0)$, the results can be obtained *mutatis mutandis* by exchanging the roles of the attributes 0 and 1, i.e., the roles of $\mu(0)$ (resp. $\Gamma(0)$) and $\mu(1)$ (resp. $\Gamma(1)$) need to be interchanged in various statements. Details are left to the interested reader.

3.1 The zero-one laws

The zero-one law for the absence of isolated nodes is given in two parts determined by the sign of $1 + \rho \ln \mu(0)$.

Theorem 3.1. *Assume* $\Gamma(0) < \Gamma(1)$. *With* $\rho > 0$, *we further assume that*

$$1 + \rho \ln \mu(0) > 0. \tag{3.1}$$

Then, for any ρ*-admissible scaling* $\mathtt{L} : \mathbb{N}_0 \to \mathbb{N}_0$, *we have*

$$\begin{aligned} &\lim_{n\to\infty} \mathbb{P}\left[\ \mathbb{M}(n; L_n) \text{ contains no isolated nodes } \right] \\ &= \begin{cases} 0 & \text{if } 1 + \rho \ln \Gamma(0) < 0 \\ \\ 1 & \text{if } 1 + \rho \ln \Gamma(0) > 0. \end{cases} \end{aligned} \tag{3.2}$$

Theorem 3.1 takes a very different form when (3.1) does not hold. To state the results, we introduce the quantity

$$G(\nu, \mu) = \left(\frac{\mu}{\nu}\right)^{\nu} \left(\frac{1-\mu}{1-\nu}\right)^{1-\nu}, \quad 0 < \nu, \mu < 1. \tag{3.3}$$

For each μ in $(0,1)$, the mapping $(0,1) \to \mathbb{R}_+ : \nu \to G(\nu, \mu)$ is well defined and continuous. By continuity we can extend it into a continuous mapping defined on the *closed* interval $[0,1]$ with

$$G(0, \mu) = \lim_{\nu \downarrow 0} G(\nu, \mu) = 1 - \mu$$

and

$$G(1, \mu) = \lim_{\nu \uparrow 1} G(\nu, \mu) = \mu.$$

These limits are using the convention $0^0 = 1$ in the expression (3.3). In a similar way, for each μ in $(0,1)$ the mapping $(0,1) \to \mathbb{R} : \nu \to \ln G(\nu,\mu)$ is well defined and continuous with

$$\ln G(\nu,\mu) = -\nu \ln \left(\frac{\nu}{\mu}\right) - (1-\nu)\ln\left(\frac{1-\nu}{1-\mu}\right), \quad 0 < \nu < 1. \tag{3.4}$$

We can also extend this second mapping into a continuous mapping defined on the closed interval $[0,1]$ with

$$\ln G(0,\mu) = \lim_{\nu \downarrow 0} \ln G(\nu,\mu) = \ln(1-\mu)$$

and

$$\ln G(1,\mu) = \lim_{\nu \uparrow 1} \ln G(\nu,\mu) = \ln \mu.$$

This is consistent with applying the usual convention $0 \ln 0 = 0$ in the expression (3.4). Elementary calculus shows that the mapping $[0,1] \to \mathbb{R} : \nu \to \ln G(\nu,\mu)$ is concave, and that its maximum is achieved at $\nu = \mu$ with $\ln G(\mu,\mu) = 0$. Thus, the mapping $[0,1] \to \mathbb{R} : \nu \to \ln G(\nu,\mu)$ increases on $(0,\mu)$, reaches its maximum at $\nu = \mu$ and then decreases on $(\mu,1)$.

With these preliminaries in place, for each μ in $(0,1)$ and $\rho > 0$, consider the non-linear equation

$$1 + \rho \ln G(\nu,\mu) = 0, \quad \nu \in [0,1]. \tag{3.5}$$

If the condition

$$1 + \rho \ln(1-\mu) < 0$$

holds, then the equation (3.5) has a *non*-empty set of solutions. More precisely, there always exists a root, denoted $\nu_\star(\rho)$, in the interval $(0,\mu)$ since $1+\rho \ln G(0,\mu) =$

$1 + \rho \ln(1-\mu) < 0$ while $1 + \rho \ln G(\mu, \mu) = 1$. Additionally, only when

$$1 + \rho \ln G(1, \mu) = 1 + \rho \ln \mu \leqslant 0,$$

does there exist a second root located in the interval $(\mu, 1]$.

Theorem 3.2. *Assume* $\Gamma(0) < \Gamma(1)$. *With* $\rho > 0$, *we further assume that*

$$1 + \rho \ln \mu(0) < 0. \tag{3.6}$$

Then, for any ρ*-admissble scaling* $\mathtt{L} : \mathbb{N}_0 \to \mathbb{N}_0$, *we have*

$$\lim_{n\to\infty} \mathbb{P}\left[\ \mathbb{M}(n; L_n) \text{ contains no isolated nodes } \right] = \begin{cases} 0 & \text{if } 1 + \rho \ln\left(\Gamma(1)^{\nu_\star(\rho)}\Gamma(0)^{1-\nu_\star(\rho)}\right) < 0 \\ & \\ 1 & \text{if } 1 + \rho \ln\left(\Gamma(1)^{\nu_\star(\rho)}\Gamma(0)^{1-\nu_\star(\rho)}\right) > 0 \end{cases} \tag{3.7}$$

where $\nu_\star(\rho)$ *is the unique solution in the interval* $(0, \mu(1))$ *to the equation*

$$1 + \rho \ln G(\nu, \mu(1)) = 0, \quad \nu \in [0, 1]. \tag{3.8}$$

For future reference, in order to avoid repetitions, we close with a discussion of the constraint on the sign of $1 + \rho \ln\left(\Gamma(1)^{\nu_\star(\rho)}\Gamma(0)^{1-\nu_\star(\rho)}\right)$ which appears in the statement of Theorem 3.2. As we will discover shortly in subsequent sections, forthcoming arguments will require asserting either the existence of a value ν in the interval $(0, \nu_\star(\rho))$ such that

$$1 + \rho \ln\left(\Gamma(1)^{\nu}\Gamma(0)^{1-\nu}\right) < 0, \tag{3.9}$$

or the existence of a value ν in the interval $(\nu_\star(\rho), \mu(1))$ such that

$$1 + \rho \ln \left(\Gamma(1)^{\nu}\Gamma(0)^{1-\nu}\right) > 0. \tag{3.10}$$

We now argue that the existence of a value ν in the requisite intervals is indeed guaranteed by the conditions

$$1 + \rho \ln \left(\Gamma(1)^{\nu_\star(\rho)}\Gamma(0)^{1-\nu_\star(\rho)}\right) < 0 \tag{3.11}$$

and

$$1 + \rho \ln \left(\Gamma(1)^{\nu_\star(\rho)}\Gamma(0)^{1-\nu_\star(\rho)}\right) > 0, \tag{3.12}$$

respectively. In fact a little bit more holds:

Indeed, using the fact that

$$1 + \rho \ln \left(\Gamma(1)^{\nu}\Gamma(0)^{1-\nu}\right) = 1 + \rho \left(\nu \ln \Gamma(1) + (1-\nu) \ln \Gamma(0)\right), \quad \nu \in [0, 1],$$

we note that the mapping $\nu \to 1 + \rho \ln \left(\Gamma(1)^{\nu}\Gamma(0)^{1-\nu}\right)$ is *affine* (thus continuous) on $[0, 1]$ and strictly increasing (since $\Gamma(0) < \Gamma(1)$) with intercepts at $\nu = 0$ and $\nu = 1$ given by $1 + \rho \ln \Gamma(0)$ and $1 + \rho \ln \Gamma(1)$, respectively. This elementary observation has the following implications: If (3.11) holds, then by continuity and monotonicity there exists a non-trivial interval $I_-(\rho) = (\alpha_-(\rho), \beta_-(\rho))$ contained in $(0, \mu(1))$ with the following properties: The interval $I_-(\rho)$ contains $\nu_\star(\rho)$ and (3.9) holds on it. When $\Gamma(0) = \Gamma(1)$, it is easy to check that we can take $I_-(\rho) = (0, \mu(1))$.

On the other hand, if (3.12) holds, then by continuity and monotonicity there now exists a non-trivial interval $I_+(\rho) = (\alpha_+(\rho), \beta_+(\rho))$ contained in $(0, \mu(1))$ with the following properties: The interval $I_+(\rho)$ contains $\nu_\star(\rho)$ and (3.10) holds on it. When $\Gamma(0) = \Gamma(1)$, it is easy to check that we can take $I_+(\rho) = (0, \mu(1))$.

3.2 Counting isolated nodes

Fix $n = 2, 3, \ldots$ and $L = 1, 2, \ldots$. For each $u = 1, \ldots, n$, node u is *isolated in* $\mathbb{M}(n; L)$ if no other node (in $V_n \backslash \{u\}$) is L-adjacent to node u. The $\{0, 1\}$-valued rv $\xi_{n,L}(u)$ given by

$$\xi_{n,L}(u) = \prod_{v \in V_n,\ v \neq u}^{n} (1 - \chi_L(u, v)) \tag{3.13}$$

encodes the fact that node u is isolated in $\mathbb{M}(n; L)$. To count the number of isolated nodes in $\mathbb{M}(n; L)$ we introduce the random variable $I_n(L)$ given by

$$I_n(L) = \sum_{u=1}^{n} \xi_{n,L}(u). \tag{3.14}$$

Interest in these count variables stems from the observation that $\mathbb{M}(n; L)$ contains no isolated nodes if and only if $I_n(L) = 0$, leading to the key relation

$$\mathbb{P}\left[\ \mathbb{M}(n; L) \text{ contains no isolated nodes }\right] = \mathbb{P}\left[I_n(L) = 0\right]. \tag{3.15}$$

This fact will be used to establish Theorems 3.1 and 3.2 by leveraging easy bounds on the probability $\mathbb{P}\left[I_n(L) = 0\right]$ in terms of the first and second moments of the random variable $I_n(L)$ (as discussed in Section 2.4.4).

However, some of the forthcoming arguments will require a finer accounting which we now introduce. Recall that for each node u in V_n, the number of attributes exhibited by node u amongst its L attributes is captured by the rv $S_L(u)$ introduced at (2.2). For each $\ell = 0, 1, \ldots, L$, the $\{0, 1\}$-valued random variable $\xi_{n,L}^{(\ell)}(u)$ indicates whether node u is isolated in $\mathbb{M}(n; L)$ while exhibiting ℓ attributes amongst its L attributes, thus

$$\xi_{n,L}^{(\ell)}(u) = \xi_{n,L}(u) \mathbf{1}\left[S_L(u) = \ell\right]. \tag{3.16}$$

The total number of nodes which are isolated and have ℓ attributes amongst the first L attributes is then given by

$$I_n^{(\ell)}(L) = \sum_{u=1}^{n} \xi_{n,L}^{(\ell)}(u) = \sum_{u=1}^{n} \xi_{n,L}(u) \mathbf{1}\left[S_L(u) = \ell\right]. \tag{3.17}$$

Simple accounting now shows that

$$\xi_{n,L}(u) = \sum_{\ell=0}^{L} \xi_{n,L}^{(\ell)}(u) \tag{3.18}$$

and

$$I_n(L) = \sum_{\ell=0}^{L} I_n^{(\ell)}(L), \tag{3.19}$$

whence the elementary bounds

$$I_n^{(\ell)}(L) \leqslant I_n(L), \quad \ell = 0, 1, \ldots, L. \tag{3.20}$$

3.3 Useful lemmas for Theorem 3.1

We begin with an easy calculation of the first moments.

Lemma 3.3. *Consider arbitrary $n = 2, 3, \ldots$ and $L = 1, 2, \ldots$. For each u in V_n, with $S_L(u)$ given by (2.2), it holds that*

$$\mathbb{E}\left[\xi_{n,L}^{(\ell)}(u)\right] = \left(1 - \Gamma(1)^{\ell}\Gamma(0)^{L-\ell}\right)^{n-1} \cdot \mathbb{P}\left[S_L(u) = \ell\right], \quad \ell = 0, 1, \ldots, L \tag{3.21}$$

and

$$\mathbb{E}\left[\xi_{n,L}(u)\right] = \mathbb{E}\left[\left(1 - \Gamma(1)^{S_L(u)}\Gamma(0)^{L-S_L(u)}\right)^{n-1}\right]. \tag{3.22}$$

Recall that the rvs $\{A, A_\ell,\ \ell = 1, 2, \ldots\}$ are i.i.d. $\{0,1\}$-valued rvs with pmf $\boldsymbol{\mu}$, and corresponding sequence of partial sums $\{S_L,\ L = 1, 2, \ldots\}$ given by (2.3). Under the enforced Assumptions (i)-(iii) it is plain that for each $L = 1, 2, \ldots$, the rvs $S_L(1), S_L(2), \ldots, S_L(n)$ are i.i.d., each distributed according to the random variable S_L. The two relations

$$\mathbb{E}\left[I_n^{(\ell)}(L)\right] = n\left(1 - \Gamma(1)^\ell \Gamma(0)^{L-\ell}\right)^{n-1} \cdot \mathbb{P}\left[S_L = \ell\right], \quad \ell = 0, 1, \ldots, L \tag{3.23}$$

and

$$\mathbb{E}\left[I_n(L)\right] = n\mathbb{E}\left[\left(1 - \Gamma(1)^{S_L}\Gamma(0)^{L-S_L}\right)^{n-1}\right] \tag{3.24}$$

are now immediate consequences of the relations (3.17) and (3.19), respectively. Also recall (2.23), namely

$$Q_L^\star(\boldsymbol{a}_L) = \mathbb{E}\left[Q_L\left(\boldsymbol{a}_L, \boldsymbol{A}_L\right)\right], \quad \boldsymbol{a}_L \in \{0,1\}^L,$$

so that

$$Q_L^\star(\boldsymbol{A}_L(u)) = \Gamma(1)^{S_L(u)}\Gamma(0)^{L-S_L(u)}, \quad u \in V_n.$$

Proof. It suffices to show that (3.21) holds since (3.22) follows as an easy consequence of the expression (3.18). Pick positive $n = 2, 3, \ldots$ and $L = 1, 2, \ldots$, and consider node u V_n. For each $\ell = 0, 1, \ldots, L$, with the relation (3.16) holding, a standard preconditioning argument yields

$$\mathbb{E}\left[\xi_{n,L}^{(\ell)}(u)\right] = \mathbb{E}\left[\mathbf{1}\left[S_L(u) = \ell\right] \cdot \mathbb{E}\left[\xi_{n,L}(u) \middle| \boldsymbol{A}_L(u)\right]\right] \tag{3.25}$$

as we note that the rv $S_L(u)$ is determined by the attribute vector $\boldsymbol{A}_L(u)$.

With (3.13) as a point of departure, we have

$$\xi_{n,L}(u) = \prod_{v \in V_n,\ v \neq u} (1 - \chi_L(u,v)) \tag{3.26}$$

Because of the conditional independence established in Lemma 2.2, we have

$$\begin{aligned}
\mathbb{E}\left[\xi_{n,L}(u) \middle| \boldsymbol{A}_L(u)\right] &= \mathbb{E}\left[\prod_{v \in V_n,\ v \neq u} (1 - \chi_L(u,v)) \middle| \boldsymbol{A}_L(u)\right] \\
&= \prod_{v \in V_n,\ v \neq u} \mathbb{E}\left[(1 - \chi_L(u,v)) \middle| \boldsymbol{A}_L(u)\right] \\
&= \left(1 - \mathbb{E}\left[Q_L(\boldsymbol{A}_L(u), \boldsymbol{A}_L(v)) | \boldsymbol{A}_L(u)\right]\right)^{n-1} \\
&= \left(1 - Q^\star_L(\boldsymbol{A}_{\boldsymbol{L}}(u))\right)^{n-1} \qquad (3.27)
\end{aligned}$$

where the last two steps made use of the fact that the rvs $\{\chi_L(u,v),\ v \in V_n,\ v \neq u\}$ are i.i.d. rvs conditioning on $\boldsymbol{A}_L(u)$. Using (3.25) yields

$$\begin{aligned}
\mathbb{E}\left[\xi^{(\ell)}_{n,L}(u)\right] &= \mathbb{E}\left[\mathbf{1}\left[S_L(u) = \ell\right] \cdot \left(1 - Q^\star_L(\boldsymbol{A}_{\boldsymbol{L}}(u))\right)^{n-1}\right] \\
&= \mathbb{E}\left[\mathbf{1}\left[S_L(u) = \ell\right] \cdot \left(1 - \Gamma(1)^{S_L(u)} \Gamma(0)^{L - S_L(u)}\right)^{n-1}\right] \qquad (3.28)
\end{aligned}$$

by virtue of (2.25), and the desired conclusion (3.21) follows in a straightforward manner. ■

The expressions for the second order quantities are much more involved as the next intermediary result already shows.

Lemma 3.4. *Consider arbitrary $n = 2, 3, \ldots$ and $L = 1, 2, \ldots$. For distinct u, v in V_n, it holds that*

$$\begin{aligned}
&\mathbb{E}\left[\xi_{n,L}(u)\xi_{n,L}(v) \middle| \boldsymbol{A}_L(u), \boldsymbol{A}_L(v)\right] \qquad (3.29) \\
= \ &\left(1 - Q_L(\boldsymbol{A}_L(u), \boldsymbol{A}_L(v))\right) \cdot \left(1 - Q^\star_L(\boldsymbol{A}_L(u)) - Q^\star_L(\boldsymbol{A}_L(v)) + Q^{\star\star}_L(\boldsymbol{A}_L(u), \boldsymbol{A}_L(v))\right)^{n-2}
\end{aligned}$$

where

$$Q_L^{\star\star}(\boldsymbol{a_L}, \boldsymbol{b}_L) = \mathbb{E}\left[Q_L(\boldsymbol{a}_L, \boldsymbol{A}_L)Q_L(\boldsymbol{b}_L, \boldsymbol{A}_L)\right], \quad \boldsymbol{a_L}, \boldsymbol{b}_L \in \{0,1\}^L. \tag{3.30}$$

The proof of this result can be found in Appendix A. In principle, it is now possible to evaluate the expressions

$$\mathbb{E}\left[\xi_{n,L}^{(k)}(u)\xi_{n,L}^{(\ell)}(v)\right], \quad k, \ell = 0, \ldots, L$$

Indeed, for $k, \ell = 0, 1, \ldots, L$, not necessarily distinct, the relation (3.16) yields

$$\xi_{n,L}^{(k)}(u)\xi_{n,L}^{(\ell)}(v) = \mathbf{1}\left[S_L(u) = k\right]\mathbf{1}\left[S_L(v) = \ell\right]\xi_{n,L}(u)\xi_{n,L}(v) \tag{3.31}$$

and an easy preconditioning argument leads to

$$\begin{aligned} &\mathbb{E}\left[\xi_{n,L}^{(k)}(u) \cdot \xi_{n,L}^{(\ell)}(v)\right] \\ &= \mathbb{E}\left[\mathbf{1}\left[S_L(u) = k\right]\mathbf{1}\left[S_L(v) = \ell\right] \cdot \mathbb{E}\left[\xi_{n,L}(u)\xi_{n,L}(v)\middle|\boldsymbol{A}_L(u), \boldsymbol{A}_L(v)\right]\right] \end{aligned} \tag{3.32}$$

because the rvs $S_L(u)$ and $S_L(v)$ are determined by the attribute vectors $\boldsymbol{A}_L(u)$ and $\boldsymbol{A}_L(v)$, respectively. Using (3.18) we readily obtain

$$\mathbb{E}\left[\xi_{n,L}(u)\xi_{n,L}(v)\right] = \sum_{k=0}^{L}\sum_{\ell=0}^{L}\mathbb{E}\left[\xi_{n,L}^{(k)}(u)\xi_{n,L}^{(\ell)}(v)\right].$$

It is plain from (3.29) that these expressions are becoming quite unwieldy. To see why this is so, with arbitrary $\boldsymbol{a_L}$ and $\boldsymbol{b}_L$ in $\{0,1\}^L$, we note that

$$
\begin{aligned}
&Q_L^{\star\star}(\boldsymbol{a_L}, \boldsymbol{b}_L) \\
&\quad = \mathbb{E}\left[Q_L(\boldsymbol{a}_L, \boldsymbol{A}_L) Q_L(\boldsymbol{b}_L, \boldsymbol{A}_L)\right] \\
&\quad = \mathbb{E}\left[\prod_{\ell=1}^{L} q(a_\ell, A_\ell) q(b_\ell, A_\ell)\right] \\
&\quad = \prod_{\ell=1}^{L} \mathbb{E}\left[q(a_\ell, A_\ell) q(b_\ell, A_\ell)\right] \\
&\quad = \prod_{\ell=1}^{L} \mathbb{E}\left[q(a_\ell, A) q(b_\ell, A)\right] \\
&\quad = \prod_{\ell=1}^{L} \mathbb{E}\left[q(1,A)^2\right]^{a_\ell b_\ell} \mathbb{E}\left[q(1,A)q(0,A)\right]^{a_\ell(1-b_\ell)+b_\ell(1-a_\ell)} \mathbb{E}\left[q(0,A)^2\right]^{(1-a_\ell)(1-b_\ell)} \\
&\quad = \mathbb{E}\left[q(1,A)^2\right]^{\sum_{\ell=1}^L a_\ell b_\ell} \mathbb{E}\left[q(1,A)q(0,A)\right]^{\sum_{\ell=1}^L a_\ell(1-b_\ell)+b_\ell(1-a_\ell)} \mathbb{E}\left[q(0,A)^2\right]^{\sum_{\ell=1}^L (1-a_\ell)(1-b_\ell)}
\end{aligned}
$$

by arguments similar to the ones used for reaching the expression (2.24). Here lies the rub: The quantities $Q_L^\star(\boldsymbol{A_L}(u))$ and $Q_L^\star(\boldsymbol{A_L}(v))$ depend on $\boldsymbol{A_L}(u)$ and $\boldsymbol{A_L}(v)$ *only* through the sums $S_L(u)$ and $S_L(v)$, respectively. On the other hand, the rv $Q_L^{\star\star}(\boldsymbol{A_L}(u), \boldsymbol{A}_L(v))$ does *not* depend on the sums $S_L(u)$ and $S_L(v)$, but instead on the three sums

$$\sum_{\ell=1}^{L} A_\ell(u) A_\ell(v),$$

$$\sum_{\ell=1}^{L} \left(A_\ell(u)\left(1 - A_\ell(v)\right) + A_\ell(v)\left(1 - A_\ell(u)\right)\right)$$

and

$$\sum_{\ell=1}^{L} \left(1 - A_\ell(u)\right)\left(1 - A_\ell(v)\right).$$

Fortunately, the *exact* expression (3.29) (and its consequences) will not be needed as only the following crude bounds will suffice: For $k, \ell = 0, 1, \ldots, L$, not necessarily distinct, the expression (3.32) yields the bound

$$\begin{aligned}\mathbb{E}\left[\xi_{n,L}^{(k)}(u)\cdot\xi_{n,L}^{(\ell)}(v)\right] &\leqslant \mathbb{P}\left[S_L(u)=k, S_L(v)=\ell\right] \\ &= \mathbb{P}\left[S_L(u)=k\right]\mathbb{P}\left[S_L(v)=\ell\right] \end{aligned} \tag{3.33}$$

since

$$\mathbb{E}\left[\xi_{n,L}(u)\xi_{n,L}(v)\Big|\boldsymbol{A}_L(u),\boldsymbol{A}_L(v)\right]\leqslant 1 \quad a.s.$$

3.4 A zero-infinity law when $1+\rho\ln\mu(0)>0$

The proof of Theorem 3.1 proceeds in two steps which are presented in this and the next sections. Throughout condition (3.1) is assumed to hold.

The first step deals with the first moment conditions (2.42) and (2.48), and is contained in the following "zero-infinity" law for the first moment. Note the analogy with Theorem 3.1.

Proposition 3.5. *Assume $\Gamma(0)<\Gamma(1)$. With $\rho>0$, assume that (3.1) holds. For any ρ-admissble scaling $\mathtt{L}:\mathbb{N}_0\to\mathbb{N}_0$, we have*

$$\lim_{n\to\infty}\mathbb{E}\left[I_n(L_n)\right]=\begin{cases}\infty & \text{if } 1+\rho\ln\Gamma(0)<0\\ \\ 0 & \text{if } 1+\rho\ln\Gamma(0)>0.\end{cases} \tag{3.34}$$

Proof. Fix $n=2,3,\ldots$. Under the assumed inequality $\Gamma(0)<\Gamma(1)$, the expression

(3.24) implies

$$\begin{aligned}\mathbb{E}\left[I_n(L)\right] &\leqslant n\left(1-\Gamma(0)^L\right)^{n-1} \\ &\leqslant ne^{-(n-1)\Gamma(0)^L} \\ &= e^{\ln n-(n-1)\Gamma(0)^L}, \quad L=1,2,\ldots \end{aligned} \tag{3.35}$$

Now, for any ρ-admissible scaling $\mathtt{L}:\mathbb{N}_0\to\mathbb{N}_0$ we have

$$\mathbb{E}\left[I_n(L_n)\right] \leqslant e^{\ln n-(n-1)\Gamma(0)^{L_n}} \tag{3.36}$$

with

$$\ln n-(n-1)\Gamma(0)^{L_n} = \ln n-(n-1)\Gamma(0)^{\rho_n \ln n} = \ln n - \frac{n-1}{n}n^{1+\rho_n \ln\Gamma(0)} \tag{3.37}$$

for some sequence $\varrho:\mathbb{N}_0\to\mathbb{N}_0$ satisfying $\lim_{n\to\infty}\rho_n=\rho$. Under the condition $1+\rho\ln\Gamma(0)>0$, we have

$$\lim_{n\to\infty}\left(\ln n-(n-1)\Gamma(0)^{L_n}\right)=-\infty$$

and the conclusion $\lim_{n\to\infty}\mathbb{E}\left[I_n(L_n)\right]=0$ follows upon letting n go to infinity in (3.36)-(3.37).

We now turn to the case $1+\rho\ln\Gamma(0)<0$: Fix $n=2,3,\ldots$. For each $L=1,2,\ldots$, the bound (3.20) (with $\ell=0$) yields

$$\mathbb{E}\left[I_n^{(0)}(L)\right]=n\left(1-\Gamma(0)^L\right)^{n-1}\cdot\mathbb{P}\left[S_L=0\right]\leqslant\mathbb{E}\left[I_n(L)\right] \tag{3.38}$$

as we make use of (3.23) (with $\ell=0$). Recall that $\mathbb{P}\left[S_L=0\right]=\mu(0)^L$ since S_L is a binomial rv $\mathrm{Bin}(L,\mu(1))$. Now, for any ρ-admissible scaling $\mathtt{L}:\mathbb{N}_0\to\mathbb{N}_0$ we can write

$$\mathbb{E}\left[I_n^{(0)}(L_n)\right]=n\mu(0)^{L_n}\left(1-\Gamma(0)^{L_n}\right)^{n-1}\leqslant\mathbb{E}\left[I_n(L_n)\right] \tag{3.39}$$

for some sequence $\varrho : \mathbb{N}_0 \to \mathbb{N}_0$ such that $\lim_{n\to\infty} \rho_n = \rho$. Let n go to infinity in (3.39): Lemma 2.4 (with $x_n = \Gamma(0)^{L_n}$) gives $\lim_{n\to\infty} \left(1 - \Gamma(0)^{L_n}\right)^{n-1} = 1$ under the condition $1 + \rho \ln \Gamma(0) < 0$, while Lemma 2.5 (with $C_n = \mu(0)$ for all $n = 1, 2, \ldots$) yields

$$\lim_{n\to\infty} n\mu(0)^{L_n} = \infty \tag{3.40}$$

under (3.1). The desired conclusion $\lim_{n\to\infty} \mathbb{E}\left[I_n(L_n)\right] = \infty$ follows from the bound (3.20).

■

Upon inspecting the proof of Proposition 3.5 we see (with the help of (3.39)) that we have also shown the following result to be used shortly.

Proposition 3.6. *Assume* $\Gamma(0) < \Gamma(1)$*. With* $\rho > 0$*, assume also that (3.1) holds. For any* ρ*-admissble scaling* $\mathtt{L} : \mathbb{N}_0 \to \mathbb{N}_0$*, we have*

$$\lim_{n\to\infty} \mathbb{E}\left[I_n^{(0)}(L_n)\right] = \begin{cases} \infty & \text{if } 1 + \rho \ln \Gamma(0) < 0 \\ \\ 0 & \text{if } 1 + \rho \ln \Gamma(0) > 0. \end{cases} \tag{3.41}$$

The reason for this additional "infinity-zero" law will soon become apparent.

3.5 A proof of Theorem 3.1

Let $\mathtt{L} : \mathbb{N}_0 \to \mathbb{N}_0$ denote a ρ-admissible scaling. Under the condition $1 + \rho \ln \Gamma(0) > 0$, Proposition 3.5 yields $\lim_{n\to\infty} \mathbb{E}\left[I_n(L_n)\right] = 0$, whence $\lim_{n\to\infty} \mathbb{P}\left[I_n(L_n) = 0\right] = 1$ by the method of first moment, and this establishes the one-law part of Theorem 3.1.

The proof of the zero-law part of Theorem 3.1 is more involved. Indeed, in view of the second moment results of Section 2.4.4, a straightforward application of the method of second moment to the count rvs

$$Z_n = I_n(L_n) \quad n = 2, 3, \ldots \tag{3.42}$$

appears problematic. Instead we focus on the related count variables

$$Z_n = I_n^{(0)}(L_n), \quad n = 2, 3, \ldots. \tag{3.43}$$

Under the condition $1 + \rho \ln \Gamma(0) < 0$, Proposition 3.6 already gives the convergence $\lim_{n\to\infty} \mathbb{E}\left[I_n^{(0)}(L_n)\right] = \infty$. If we were able to establish the appropriate version of (2.49), namely

$$\limsup_{n\to\infty} \frac{\mathbb{E}\left[\xi_{n,L_n}^{(0)}(1) \cdot \xi_{n,2}^{(0)}(L_n)\right]}{\left(\mathbb{E}\left[\xi_{n,L_n}^{(0)}(1)\right]\right)^2} \leqslant 1, \tag{3.44}$$

we would then be in a position to conclude

$$\lim_{n\to\infty} \mathbb{P}\left[I_n^{(0)}(L_n) = 0\right] = 0 \tag{3.45}$$

by the method of second moment applied to the rvs (3.43). Using the bound (3.20) (with $\ell = 0$) we would then obtain $\lim_{n\to\infty} \mathbb{P}\left[I_n(L_n) = 0\right] = 0$, and this completes the proof of the zero-law part of Theorem 3.1.

To establish (3.44) we proceed as follows: Fix $n = 2, 3, \ldots$ and $L = 1, \ldots$. Applying (3.21) (with $\ell = 0$) gives

$$\mathbb{E}\left[\xi_{n,L}^{(0)}(1)\right] = \left(1 - \Gamma(0)^L\right)^{n-1} \cdot \mathbb{P}\left[S_L(1) = 0\right] = \left(1 - \Gamma(0)^L\right)^{n-1} \cdot \mu(0)^L.$$

On the other hand, specializing (3.33) to $k = \ell = 0$ we obtain the bound

$$\mathbb{E}\left[\xi_{n,L}^{(0)}(1) \cdot \xi_{n,L}^{(0)}(2)\right] \leqslant \mathbb{P}\left[S_L(1) = 0\right] \mathbb{P}\left[S_L(2) = 0\right] = \mu(0)^{2L},$$

whence

$$\begin{aligned}
\frac{\mathbb{E}\left[\xi^{(0)}_{n,L}(1)\cdot\xi^{(0)}_{n,L}(2)\right]}{\left(\mathbb{E}\left[\xi^{(0)}_{n,L}(1)\right]\right)^2} &\leqslant \frac{\mu(0)^{2L}}{\left((1-\Gamma(0)^L)^{n-1}\cdot\mu(0)^L\right)^2} \\
&= \frac{1}{(1-\Gamma(0)^L)^{2(n-1)}}.
\end{aligned}$$

As we substitute according to the ρ-admissible scaling $\mathtt{L} : \mathbb{N}_0 \to \mathbb{N}_0$ in this last inequality we obtain

$$\frac{\mathbb{E}\left[\xi^{(0)}_{n,L_n}(1)\cdot\xi^{(0)}_{n,L_n}(2)\right]}{\left(\mathbb{E}\left[\xi^{(0)}_{n,L_n}(1)\right]\right)^2} \leqslant \frac{1}{(1-\Gamma(0)^{L_n})^{2(n-1)}}, \quad n = 2, 3, \ldots.$$

Let n go infinity in this resulting inequality: We readily get $\lim_{n\to\infty}\left(1-\Gamma(0)^{L_n}\right)^n = 1$ by virtue of Lemma 2.4 (with $x_n = \Gamma(0)^{L_n}$) under the condition $1 + \rho\ln\Gamma(0) < 0$ and (3.44) follows. This concludes the proof of Theorem 3.1. ■

3.6 A zero-infinity laws when $1 + \rho\ln\mu(0) < 0$

Although the arguments for proving Theorem 3.2 are similar to the ones used in the proof of Theorem 3.1, they differ in some major ways as will become clear from the proof the analog of Proposition 3.6.

Here as well, we begin with the appropriate first moment conditions (2.42) and (2.48). This is contained in the following "zero-infinity" law for the first moment; note the analogy with Theorem 3.2.

Proposition 3.7. *Assume* $\Gamma(0) < \Gamma(1)$. *With* $\rho > 0$, *assume also that (3.6) holds.*

For any ρ-admissble scaling $\mathtt{L} : \mathbb{N}_0 \to \mathbb{N}_0$, we have

$$\lim_{n\to\infty} \mathbb{E}\left[I_n(L_n)\right] = \begin{cases} \infty & \text{if } 1+\rho\ln\left(\Gamma(1)^{\nu_\star(\rho)}\Gamma(0)^{1-\nu_\star(\rho)}\right) < 0 \\ \\ 0 & \text{if } 1+\rho\ln\left(\Gamma(1)^{\nu_\star(\rho)}\Gamma(0)^{1-\nu_\star(\rho)}\right) > 0 \end{cases} \tag{3.46}$$

where $\nu_\star(\rho)$ is the unique solution in the interval $(0, \mu(1))$ to the equation (3.8).

As in the proof Theorem 3.1 we need to complement the "zero-infinity" law of Proposition 3.7. This time, however, the needed result assumes a more complicated form than the one taken in Proposition 3.6.

We prove the zero-law first. It follows from (3.24) that

$$\begin{aligned} \mathbb{E}\left[I_n(L)\right] &= n\mathbb{E}\left[\left(1-\Gamma(1)^{S_L}\Gamma(0)^{L-S_L}\right)^{n-1}\right] \\ &= n\sum_{\ell=0}^{L}\binom{L}{\ell}\mu(1)^{\ell}\mu(0)^{L-\ell}\left(1-\Gamma(1)^{\ell}\Gamma(0)^{L-\ell}\right)^{n-1} \end{aligned} \tag{3.47}$$

for $n = 2, 3, \ldots$. We will split this sum into two parts and show that each part converges to 0 when n grows unboundedly large under the condition (3.12).

Proof. Based on the arguments at the end of Section 3.1, when the condition (3.12) holds, there exists $\varepsilon > 0$ such that $1+\rho\ln\left(\Gamma(1)^{\nu_\star(\rho)-\varepsilon}\Gamma(0)^{1-\nu_\star(\rho)+\varepsilon}\right) > 0$.

Fix $n = 2, 3, \ldots$. For any $\rho-$admissible scaling $\mathtt{L} : \mathbb{N}_0 \to \mathbb{N}_0$ with $\rho > 0$, (3.47) takes the from

$$
\begin{aligned}
&\mathbb{E}\left[I_n(L_n)\right] \\
=&n\sum_{\ell=0}^{L_n}\binom{L_n}{\ell}\mu(1)^{\ell}\mu(0)^{L_n-\ell}\left(1-\Gamma(1)^{\ell}\Gamma(0)^{L_n-\ell}\right)^{n-1} \\
=&\sum_{\ell=0}^{\lfloor(\nu_*(\rho)-\varepsilon)L_n\rfloor} n\binom{L_n}{\ell}\mu(1)^{\ell}\mu(0)^{L_n-\ell}\left(1-\Gamma(1)^{\ell}\Gamma(0)^{L_n-\ell}\right)^{n-1} \qquad (3.48a)\\
&+\sum_{\ell=\lfloor(\nu_*(\rho)-\varepsilon)L_n\rfloor+1}^{L_n} n\binom{L_n}{\ell}\mu(1)^{\ell}\mu(0)^{L_n-\ell}\left(1-\Gamma(1)^{\ell}\Gamma(0)^{L_n-\ell}\right)^{n-1} \qquad (3.48b)
\end{aligned}
$$

For $\ell = 0, 1, \ldots, L_n$, it holds that

$$
\begin{aligned}
& n\binom{L_n}{\ell}\mu(1)^{\ell}\mu(0)^{L_n-\ell}\left(1-\Gamma(1)^{\ell}\Gamma(0)^{L_n-\ell}\right)^{n-1} \\
\leqslant\ & n\binom{L_n}{\ell}\mu(1)^{\ell}\mu(0)^{L_n-\ell}. \qquad (3.49)
\end{aligned}
$$

Since $\nu_*(\rho)-\varepsilon$ lies in $(0,\mu(1))$, $\binom{L_n}{\ell}\mu(1)^{\ell}\mu(0)^{L_n-\ell}$ increases with respect to ℓ when $0 \leqslant \ell \leqslant (\nu_*(\rho)-\varepsilon)L_n$, and an upper bound for (3.48a) is given by

$$
\begin{aligned}
& \sum_{\ell=0}^{\lfloor(\nu_*(\rho)-\varepsilon)L_n\rfloor} n\binom{L_n}{\ell}\mu(1)^{\ell}\mu(0)^{L_n-\ell}\left(1-\Gamma(1)^{\ell}\Gamma(0)^{L_n-\ell}\right)^{n-1} \\
\leqslant\ & L_n\cdot n\binom{L_n}{\lfloor(\nu_*(\rho)-\varepsilon)L_n\rfloor}\mu(1)^{\lfloor(\nu_*(\rho)-\varepsilon)L_n\rfloor}\mu(0)^{L_n-\lfloor(\nu_*(\rho)-\varepsilon)L_n\rfloor}. \qquad (3.50)
\end{aligned}
$$

By virtue of Lemma 2.6 (with $t_n = L_n$ and $\tau_n = \lfloor(\nu_*(\rho)-\varepsilon)L_n\rfloor$), it holds that

$$\begin{aligned}
& \binom{L_n}{\lfloor(\nu_*(\rho)-\varepsilon)L_n\rfloor}\mu(1)^{\lfloor(\nu_*(\rho)-\varepsilon)L_n\rfloor}\mu(0)^{L_n-\lfloor(\nu_*(\rho)-\varepsilon)L_n\rfloor} \\
\sim\ & \frac{\sqrt{L_n}}{\sqrt{2\pi\lfloor(\nu_*(\rho)-\varepsilon)L_n\rfloor(L_n-\lfloor(\nu_*(\rho)-\varepsilon)L_n\rfloor)}} \\
& \times\left(\frac{\mu(1)L_n}{\lfloor(\nu_*(\rho)-\varepsilon)L_n\rfloor}\right)^{\lfloor(\nu_*(\rho)-\varepsilon)L_n\rfloor}\left(\frac{\mu(0)L_n}{L_n-\lfloor(\nu_*(\rho)-\varepsilon)L_n\rfloor}\right)^{L_n-\lfloor(\nu_*(\rho)-\varepsilon)L_n\rfloor} \\
<\ & \left(\frac{\mu(1)L_n}{\lfloor(\nu_*(\rho)-\varepsilon)L_n\rfloor}\right)^{\lfloor(\nu_*(\rho)-\varepsilon)L_n\rfloor}\left(\frac{\mu(0)L_n}{L_n-\lfloor(\nu_*(\rho)-\varepsilon)L_n\rfloor}\right)^{L_n-\lfloor(\nu_*(\rho)-\varepsilon)L_n\rfloor} \\
=\ & G\left(\frac{\lfloor(\nu_*(\rho)-\varepsilon)L_n\rfloor}{L_n},\mu(1)\right)^{L_n} \qquad (3.51)
\end{aligned}$$

where second to the last step was based on the fact that

$$\lim_{n\to\infty}\frac{\sqrt{L_n}}{\sqrt{2\pi\lfloor(\nu_*(\rho)-\varepsilon)L_n\rfloor(L_n-\lfloor(\nu_*(\rho)-\varepsilon)L_n\rfloor)}}=0,$$

and the last step used the definition of $G(\cdot,\cdot)$ in (3.3). As a result, the upper bound in (3.50) becomes

$$\begin{aligned}
& \sum_{\ell=0}^{\lfloor(\nu_*(\rho)-\varepsilon)L_n\rfloor} n\binom{L_n}{\ell}\mu(1)^{\ell}\mu(0)^{L_n-\ell}\left(1-\Gamma(1)^{\ell}\Gamma(0)^{L_n-\ell}\right)^{n-1} \\
\leqslant\ & L_n\cdot nG\left(\frac{\lfloor(\nu_*(\rho)-\varepsilon)L_n\rfloor}{L_n},\mu(1)\right)^{L_n} \qquad (3.52)
\end{aligned}$$

for sufficiently large n.

The definition of $\nu_*(\rho)$ gives $1+\rho\ln G(\nu_*(\rho)-\varepsilon,\mu(1))<0$. Let n go to infinity in (3.52): Because $\lim_{n\to\infty}\frac{\lfloor(\nu_*(\rho)-\varepsilon)L_n\rfloor}{L_n}=\nu_*(\rho)-\varepsilon$, Lemma 2.5 $\left(\text{with } C_n = G\left(\frac{\lfloor(\nu_*(\rho)-\varepsilon)L_n\rfloor}{L_n},\mu(1)\right)\right)$ yields

$$\lim_{n\to\infty}L_n\cdot nG\left(\frac{\lfloor(\nu_*(\rho)-\varepsilon)L_n\rfloor}{L_n},\mu(1)\right)^{L_n}=0,$$

whence

$$\lim_{n\to\infty}\sum_{\ell=0}^{\lfloor(\nu_*(\rho)-\varepsilon)L_n\rfloor} n\binom{L_n}{\ell}\mu(1)^{\ell}\mu(0)^{L_n-\ell}\left(1-\Gamma(1)^{\ell}\Gamma(0)^{L_n-\ell}\right)^{n-1}=0. \qquad (3.53)$$

Now we derive an upper bound for (3.48b). It is plain that

$$
\begin{aligned}
&\sum_{\ell=\lfloor(\nu_*(\rho)-\varepsilon)L_n\rfloor+1}^{L_n} n\binom{L_n}{\ell}\mu(1)^\ell\mu(0)^{L_n-\ell}\left(1-\Gamma(1)^\ell\Gamma(0)^{L_n-\ell}\right)^{n-1} \\
\leqslant\ & n\left(1-\Gamma(1)^{\lceil(\nu_*(\rho)-\varepsilon)L_n\rceil}\Gamma(0)^{L_n-\lceil(\nu_*(\rho)-\varepsilon)L_n\rceil}\right)^{n-1} \\
\leqslant\ & n\left(1-\Gamma(1)^{(\nu_*(\rho)-\varepsilon)L_n}\Gamma(0)^{L_n-(\nu_*(\rho)-\varepsilon)L_n}\right)^{n-1} \qquad (3.54)
\end{aligned}
$$

since $\left(1-\Gamma(1)^\ell\Gamma(0)^{L_n-\ell}\right)^{n-1}$ is monotonically decreasing in ℓ under the assumption $\Gamma(1)>\Gamma(0)$, and the bound

$$
\sum_{\ell=\lfloor(\nu_*(\rho)-\varepsilon)L_n\rfloor+1}^{L_n}\binom{L_n}{\ell}\mu(1)^\ell\mu(0)^{L_n-\ell}\leqslant 1
$$

holds. With arguments similar to (3.35)-(3.37), we conclude that

$$
\lim_{n\to\infty} n\left(1-\Gamma(1)^{(\nu_*(\rho)-\varepsilon)L_n}\Gamma(0)^{L_n-(\nu_*(\rho)-\varepsilon)L_n}\right)^{n-1}=0,
$$

whence

$$
\lim_{n\to\infty}\sum_{\ell=\lfloor(\nu_*(\rho)-\varepsilon)L_n\rfloor+1}^{L_n} n\binom{L_n}{\ell}\mu(1)^\ell\mu(0)^{L_n-\ell}\left(1-\Gamma(1)^\ell\Gamma(0)^{L_n-\ell}\right)^{n-1}=0 \qquad (3.55)
$$

under the condition $1+\rho\ln\left(\Gamma(1)^{\nu_\star(\rho)-\varepsilon}\Gamma(0)^{1-\nu_\star(\rho)+\varepsilon}\right)>0$.

Combining the two partial sums (3.53) and (3.55), the desired zero-law is readily established.

■

3.7 An alternative approach to the infinity-law in Proposition 3.7

For the infinity-law part, we need to rely on the proposition given next. As in the proof of Theorem 3.1 we need to complement the "zero-infinity" law of Proposition 3.7. This time, however, the needed result assumes a more complicated form than the one taken in Proposition 3.6. First we need to set the stage: Pick ν in $(0,1)$, and consider any sequence $\ell : \mathbb{N}_0 \to \mathbb{N}$ such that

$$\ell_n \leqslant L_n, \quad n = 1, 2, \ldots \tag{3.56}$$

under the additional property

$$\lim_{n\to\infty} \frac{\ell_n}{L_n} = \nu. \tag{3.57}$$

Any sequence $\ell : \mathbb{N}_0 \to \mathbb{N}$ satisfying (3.56) is said to be a sequence *associated* with the scaling $\mathtt{L} : \mathbb{N}_0 \to \mathbb{N}_0$. An associated sequence satisfying (3.57) can be easily generated through the formula

$$\ell_n = \lfloor \nu L_n \rfloor, \quad n = 1, 2, \ldots.$$

Any associated sequence $\ell : \mathbb{N}_0 \to \mathbb{N}$ induces the $[0,1]$-valued sequence $\nu : \mathbb{N}_0 \to \mathbb{N}$ defined by

$$\nu_n = \frac{\ell_n}{L_n}, \quad n = 1, 2, \ldots.$$

In this notation the constraints (3.56) and (3.57) can now be expressed as

$$\ell_n = \nu_n L_n, \quad n = 1, 2, \ldots \tag{3.58}$$

and

$$\lim_{n\to\infty} \nu_n = \nu. \tag{3.59}$$

Proposition 3.8. *Assume $\Gamma(0) < \Gamma(1)$. With $\rho > 0$, assume also that (3.6) holds. Consider a ρ-admissible scaling $\mathtt{L} : \mathbb{N}_0 \to \mathbb{N}_0$, and an associated sequence $\ell : \mathbb{N}_0 \to \mathbb{N}$ which satisfies both (3.56) and (3.57) for some ν in $(0,1)$. Under the condition (3.11), ν can be selected in the interval $(\nu_\star(\rho), \mu(1))$ such that*

$$\lim_{n\to\infty} \mathbb{E}\left[I_n^{(\ell_n)}(L_n)\right] = \infty. \tag{3.60}$$

Proof. Fix $n = 2, 3, \ldots$ and $L = 1, 2, \ldots$. Our point of departure is the expression (3.23), namely

$$\begin{aligned}
\mathbb{E}\left[I_n^{(\ell)}(L)\right] &= n\left(1 - \Gamma(1)^\ell \Gamma(0)^{L-\ell}\right)^{n-1} \cdot \mathbb{P}\left[S_L(1) = \ell\right] \\
&= n\left(1 - \Gamma(1)^\ell \Gamma(0)^{L-\ell}\right)^{n-1} \cdot \binom{L}{\ell} \mu(1)^\ell \mu(0)^{L-\ell} \qquad (3.61)
\end{aligned}$$

with $\ell = 0, 1, \ldots, L$.

Pick ν in $(0,1)$. Substituting L and ℓ according to the scaling $\mathtt{L} : \mathbb{N}_0 \to \mathbb{N}_0$ and any associated sequence $\ell : \mathbb{N}_0 \to \mathbb{N}$ satisfying (3.56) (or equivalently, (3.58)) and (3.57) for the selected ν, we get

$$\begin{aligned}
\mathbb{E}\left[I_n^{(\ell_n)}(L_n)\right] &= n\left(1 - \Gamma(1)^{\ell_n} \Gamma(0)^{L_n-\ell_n}\right)^{n-1} \cdot \binom{L_n}{\ell_n} \mu(1)^{\ell_n} \mu(0)^{L_n-\ell_n} \\
&= n\binom{L_n}{\nu_n L_n} \left(\mu(1)^{\nu_n} \mu(0)^{1-\nu_n}\right)^{L_n} \cdot \left(1 - \left(\Gamma(1)^{\nu_n}\Gamma(0)^{1-\nu_n}\right)^{L_n}\right)^{n-1}
\end{aligned}$$

where we note that $\nu_n L_n$ and $L_n - \nu_n L_n = (1-\nu_n)L_n$ are integers by construction.

Lemma 2.6 gives

$$\begin{aligned}
\binom{L_n}{\nu_n L_n} &\sim \frac{\sqrt{2\pi L_n}\left(\frac{L_n}{e}\right)^{L_n}}{\sqrt{2\pi \nu_n L_n}\left(\frac{\nu_n L_n}{e}\right)^{\nu_n L_n} \cdot \sqrt{2\pi(1-\nu_n)L_n}\left(\frac{(1-\nu_n)L_n}{e}\right)^{(1-\nu_n)L_n}} \\
&= \frac{1}{\sqrt{2\pi\nu_n(1-\nu_n)L_n}} \cdot \frac{1}{\left(\nu_n^{\nu_n}(1-\nu_n)^{1-\nu_n}\right)^{L_n}}
\end{aligned}$$

so that

$$\begin{aligned}
n\binom{L_n}{\nu_n L_n}\left(\mu(1)^{\nu_n}\mu(0)^{1-\nu_n}\right)^{L_n} &\sim \frac{n}{\sqrt{2\pi\nu_n(1-\nu_n)L_n}} \cdot \left(\frac{\mu(1)^{\nu_n}\mu(0)^{1-\nu_n}}{\nu_n^{\nu_n}(1-\nu_n)^{1-\nu_n}}\right)^{L_n} \\
&= \frac{n}{\sqrt{2\pi\nu_n(1-\nu_n)L_n}} \cdot G(\nu_n, \mu(1))^{L_n}. \qquad (3.62)
\end{aligned}$$

Collecting we obtain

$$\begin{aligned}
\mathbb{E}\left[I_n^{(\ell_n)}(L_n)\right] &\sim \frac{n}{\sqrt{2\pi\nu_n(1-\nu_n)L_n}} \cdot G(\nu_n,\mu(1))^{L_n} \cdot \left(1-\left(\Gamma(1)^{\nu_n}\Gamma(0)^{1-\nu_n}\right)^{L_n}\right)^{n-1} \\
&\sim \frac{1}{\sqrt{2\pi\nu(1-\nu)}} \cdot \frac{n \cdot G(\nu_n,\mu(1))^{L_n}}{\sqrt{L_n}} \cdot \left(1-\left(\Gamma(1)^{\nu_n}\Gamma(0)^{1-\nu_n}\right)^{L_n}\right)^{n-1}
\end{aligned}$$

as we make use of (3.59) in the last step.

Recall now that both conditions (3.6) and (3.11) are enforced. Therefore, as discussed at the end of Section 3.1, condition (3.9) holds on the interval $I_-(\rho) = (\alpha_-(\rho), \beta_-(\rho)) \subseteq (0, \mu(1))$, said interval containing $\nu_\star(\rho)$. As we *restrict* ν to be an element of $(\nu_\star(\rho), \beta_-(\rho))$, we conclude by Lemma 2.4 that

$$\lim_{n\to\infty}\left(1-\left(\Gamma(1)^{\nu_n}\Gamma(0)^{1-\nu_n}\right)^{L_n}\right)^{n-1} = 1, \tag{3.63}$$

and the desired conclusion $\lim_{n\to\infty}\mathbb{E}\left[I_n^{(\ell_n)}(L_n)\right] = \infty$ follows provided we show

$$\liminf_{n\to\infty}\frac{n \cdot G(\nu_n,\mu(1))^{L_n}}{\sqrt{L_n}} > 0. \tag{3.64}$$

It is always possible to find $\varepsilon > 0$ so that the interval $(\nu-\varepsilon, \nu+\varepsilon)$ is contained in the interval $(\nu_\star(\rho), \beta_-(\rho))$. By virtue of (3.57) there exists a finite integer $n(\varepsilon)$ such that

$$\nu - \varepsilon < \nu_n < \nu + \varepsilon, \quad n \geqslant n(\varepsilon)$$

and on that range, the monotonicity of the mapping $\nu \to 1 + \rho \ln G(\nu, \mu(1))$ on $(0, \mu(1))$ yields

$$0 < 1 + \rho \ln G(\nu-\varepsilon, \mu(1)) \leqslant 1 + \rho \ln G(\nu_n, \mu(1))$$

because $1 + \rho \ln G(\nu, \mu(1)) > 0$ on the interval $(\nu_\star(\rho), \beta_-(\rho))$. It is plain that

$$\begin{aligned}
n \cdot G(\nu_n,\mu(1))^{L_n} &= n^{1+\rho_n \ln G(\nu_n,\mu(1))} \\
&\geqslant n^{1+\rho_n \ln G(\nu-\varepsilon,\mu(1))}, \quad n \geqslant n(\varepsilon)
\end{aligned} \tag{3.65}$$

and the conclusion

$$\liminf_{n\to\infty} \frac{n\cdot G(\nu_n,\mu(1))^{L_n}}{\sqrt{L_n}} \geqslant \liminf_{n\to\infty} \frac{n^{1+\rho_n \ln G(\nu-\varepsilon,\mu(1))}}{\sqrt{\rho_n \ln n}} = \infty \tag{3.66}$$

follows immediately as we use the aforementioned fact that $1+\rho\ln G(\nu-\varepsilon,\mu(1)) > 0$. This establishes (3.64), and the proof of Proposition 3.8 is now completed. The infinity-law in Proposition 3.7 is now established as we use (3.20). ■

An alternative proof of Proposition 3.7 is given in Section 3.10 and Section 3.11, and relies on a change of measure argument introduced in Section 3.9.

3.8 A proof of Theorem 3.2

Let $\mathtt{L} : \mathbb{N}_0 \to \mathbb{N}_0$ denote a ρ-admissible scaling for some $\rho > 0$. Under the condition $1+\rho\ln\Gamma(1)^{\nu_\star(\rho)}\Gamma(0)^{1-\nu_\star(\rho)} > 0$, Proposition 3.7 yields $\lim_{n\to\infty}\mathbb{E}\left[I_n(L_n)\right] = 0$, whence $\lim_{n\to\infty}\mathbb{P}\left[I_n(L_n) = 0\right] = 1$ by the method of first moments, and this establishes the one-law part of Theorem 3.2.

Assume now that $1+\rho\ln\Gamma(1)^{\nu_\star(\rho)}\Gamma(0)^{1-\nu_\star(\rho)} < 0$. Here as well, we will not attempt to apply the method of second moment directly to the count variables (3.42) in order to establish the zero-law part of Theorem 3.2. Under the enforced assumptions, we shall show instead that ν can be selected in $(\nu_\star(\rho),\mu(1)))$ in such a manner that the method of second moment applies to the count variables

$$Z_n = I_n^{(\ell_n)}(L_n), \quad n = 1,2,\ldots \tag{3.67}$$

where the sequence $\ell : \mathbb{N}_0 \to \mathbb{N}$ associated with the scaling $\mathtt{L} : \mathbb{N}_0 \to \mathbb{N}_0$ satisfies (3.59) with the selected value.

This will require showing the validity of both

$$\lim_{n\to\infty} \mathbb{E}\left[I_n^{(\ell_n)}(L_n)\right] = \infty \tag{3.68}$$

and

$$\limsup_{n\to\infty} \frac{\mathbb{E}\left[\xi_{n,L_n}^{(\ell_n)}(1)\cdot\xi_{n,L_n}^{(\ell_n)}(2)\right]}{\left(\mathbb{E}\left[\xi_{n,L_n}^{(\ell_n)}(1)\right]\right)^2} \leqslant 1. \tag{3.69}$$

Once this is done, it will follow from the method of second moment applied to the rvs (3.67) that

$$\lim_{n\to\infty} \mathbb{P}\left[I_n^{(\ell_n)}(L_n) = 0\right] = 1. \tag{3.70}$$

Using the bound (3.20) (with $L = L_n$ and $\ell = \ell_n$ for each $n = 2, 3, \ldots$) we immediately obtain $\lim_{n\to\infty} \mathbb{P}\left[I_n(L_n) = 0\right] = 1$, and the zero-law part of Theorem 3.2 will then be established.

To establish the convergence statements (3.68) and (3.69), we proceed as follows: By Proposition 3.8 we already know that there exists some ν in the interval $(\nu_\star(\rho), \mu(1))$ such that (3.60), namely (3.68), holds – In fact the proof shows that it happens for every ν in the interval $(\nu_\star(\rho), \beta_-(\rho))$. It remains only to establish (3.69) for any ν selected in the interval $(\nu_\star(\rho), \beta_-(\rho))$. To that end, fix $n = 2, 3, \ldots$ and $L = 1, 2, \ldots$. Using the expression (3.21) we can write

$$\begin{aligned} \mathbb{E}\left[\xi_{n,L}^{(\ell)}(1)\right] &= \left(1 - \Gamma(1)^\ell \Gamma(0)^{L-\ell}\right)^{n-1} \cdot \mathbb{P}\left[S_L(1) = \ell\right] \\ &= \left(1 - \Gamma(1)^\ell \Gamma(0)^{L-\ell}\right)^{n-1} \cdot \binom{L}{\ell} \mu(1)^\ell \mu(0)^{L-\ell} \end{aligned} \tag{3.71}$$

with $\ell = 0, 1, \ldots, L$, On the other hand, specializing (3.33) to $k = \ell$ yields

$$\mathbb{E}\left[\xi_{n,L}^{(\ell)}(1)\cdot\xi_{n,L}^{(\ell)}(2)\right] \leqslant \mathbb{P}\left[S_L(1) = \ell\right]\mathbb{P}\left[S_L(2) = \ell\right] = \left(\binom{L}{\ell}\mu(1)^\ell \mu(0)^{L-\ell}\right)^2, \tag{3.72}$$

whence

$$\begin{aligned}\frac{\mathbb{E}\left[\xi_{n,L}^{(\ell)}(1)\cdot\xi_{n,L}^{(\ell)}(2)\right]}{\left(\mathbb{E}\left[\xi_{n,L}^{(\ell)}(1)\right]\right)^2} &\leqslant \frac{\left(\binom{L}{\ell}\mu(1)^\ell\mu(0)^{L-\ell}\right)^2}{\left((1-\Gamma(1)^\ell\Gamma(0)^{L-\ell})^{n-1}\cdot\binom{L}{\ell}\mu(1)^\ell\mu(0)^{L-\ell}.\right)^2}\\ &= \frac{1}{(1-\Gamma(1)^\ell\Gamma(0)^{L-\ell})^{2(n-1)}}.\end{aligned}$$

Now, substitute in this last inequality according to the given ρ-admissible scaling $\mathtt{L}:\mathbb{N}_0\to\mathbb{N}_0$ and the sequence $\ell:\mathbb{N}_0\to\mathbb{N}$ associated with it where ν appearing in (3.59) is the one selected earlier in the interval $(\nu_\star(\rho),\beta_-(\rho))$. This yields

$$\begin{aligned}&\frac{\mathbb{E}\left[\xi_{n,L_n}^{(\ell_n)}(1)\cdot\xi_{n,L_n}^{(\ell_n)}(2)\right]}{\left(\mathbb{E}\left[\xi_{n,L_n}^{(\ell_n)}(1)\right]\right)^2}\\ \leqslant\ &\frac{1}{(1-\Gamma(1)^{\ell_n}\Gamma(\ell_n)^{L_n-\ell_n})^{2(n-1)}}\\ =\ &\frac{1}{\left(1-(\Gamma(1)^{\nu_n}\Gamma(0)^{(1-\nu_n)})^{L_n}\right)^{2(n-1)}},\quad n=2,3,\ldots\end{aligned}\tag{3.73}$$

Letting n go infinity in (3.73) we get $\lim_{n\to\infty}\left(1-\left(\Gamma(1)^{\nu_n}\Gamma(0)^{(1-\nu_n)}\right)^{L_n}\right)^{n-1}=1$ by virtue of Lemma 2.4 since the condition $1+\rho\ln\Gamma(1)^\nu\Gamma(0)^{1-\nu}<0$ holds for the value ν selected in the interval $(\nu_\star(\rho),\beta_-(\rho))$. This establishes (3.69) and the proof of Theorem 3.2 is now complete. ■

3.9 A change of measure

Int the last three section of this chapter, we present an alternative approach to establish the zero-infinity law stated in Proposition 3.7 based on the idea of a change of measure.

As stated earlier, all rvs are defined on the measurable space $(\Omega, \mathcal{F})$ and their statistics computed under the given probability measure $\mathbb{P}$ as stipulated by Assumptions (i)-(iii). To proceed we will find it convenient to embed $\mathbb{P}$ into a collection of probability measures $\{\mathbb{P}_\nu,\ \nu \in (0,1)\}$ defined on the σ-field $\mathcal{F}$ with the following properties: For each ν in $(0,1)$, under the probability measure $\mathbb{P}_\nu$, Assumptions (i) and (ii) remain unchanged but Assumption (iii) is replaced by the following assumption:

(iii-ν) The rvs $\{A, A_\ell, A_\ell(u),\ \ell = 1, 2, \ldots L;\ u \in V_n\}$ form a collection of *i.i.d.* $\{0,1\}$-valued rvs with pmf $\boldsymbol{\nu} = (\nu, 1-\nu)$ where

$$\mathbb{P}_\nu[A = 0] = 1 - \nu \quad \text{and} \quad \mathbb{P}_\nu[A = 1] = \nu.$$

Let $\mathbb{E}_\nu$ denote the expectation operator associated with $\mathbb{P}_\nu$.

Obviously, we have $\mathbb{P} \equiv \mathbb{P}_\nu$ when selecting $\nu = \mu(1)$. It is always possible to construct a measurable space $(\Omega, \mathcal{F})$, the appropriate collections of rvs on it and a collection $\{\mathbb{P}_\nu,\ \nu \in (0,1)\}$ of probability measures defined on the σ-field $\mathcal{F}$ with the requisite properties; details are well known and omitted here for the sake of brevity.

In fact, given ν in $(0,1)$, for each $L = 1, \ldots$, the probability measures $\mathbb{P}$ and $\mathbb{P}_\nu$ are mutually absolutely continuous when restricted to the σ-field $\sigma\{A_1, \ldots, A_L\}$ with Radon-Nikodym derivative given by

$$\left(\frac{d\mathbb{P}}{d\mathbb{P}_\nu}\right)_L = \prod_{\ell=1}^{L} \left(\frac{\mu(1)}{\nu}\right)^{A_\ell} \left(\frac{1-\mu(1)}{1-\nu}\right)^{1-A_\ell} = \left(\frac{\mu(1)}{\nu}\right)^{S_L} \left(\frac{1-\mu(1)}{1-\nu}\right)^{L-S_L}.$$

It is worth noting that the probability measures $\mathbb{P}$ and $\mathbb{P}_\nu$ are not mutually absolutely continuous on the entire σ-field $\mathcal{F}$.

To take advantage of this change of measure we proceed as follows: Fix ν in $(0,1)$, $n=2,3,\ldots$ and $L=1,2,\ldots$. The expression (3.24) can be written

$$\begin{aligned}
\mathbb{E}\left[I_n(L)\right] &= n\mathbb{E}\left[\left(1-\Gamma(1)^{S_L}\Gamma(0)^{L-S_L}\right)^{n-1}\right] \\
&= n\cdot\mathbb{E}_\nu\left[\left(1-\Gamma(1)^{S_L}\Gamma(0)^{L-S_L}\right)^{n-1}\cdot\left(\frac{\mu(1)}{\nu}\right)^{S_L}\left(\frac{1-\mu(1)}{1-\nu}\right)^{L-S_L}\right] \\
&= n\left(\left(\frac{\mu(1)}{\nu}\right)^{\nu}\left(\frac{1-\mu(1)}{1-\nu}\right)^{1-\nu}\right)^{L}\cdot E_n(\nu,L) \\
&= nG(\nu,\mu(1))^L\cdot E_n(\nu,L) \qquad (3.74)
\end{aligned}$$

where we have set

$$E_n(\nu,L)=\mathbb{E}_\nu\left[\left(1-\Gamma(1)^{S_L}\Gamma(0)^{L-S_L}\right)^{n-1}\cdot\left(\frac{\mu(1)}{\nu}\cdot\frac{1-\nu}{1-\mu(1)}\right)^{S_L-L\nu}\right] \qquad (3.75)$$

with the definition (3.3) used in the last step. For future reference we note the decomposition

$$E_n(\nu,L)=E_n^+(\nu,L)+E_n^-(\nu,L) \qquad (3.76)$$

with $E_n^+(\nu,L)$ and $E_n^-(\nu,L)$ given by

$$E_n^+(\nu,L)=\mathbb{E}_\nu\left[\left(1-\Gamma(1)^{S_L}\Gamma(0)^{L-S_L}\right)^{n-1}\cdot\left(\frac{\mu(1)}{\nu}\cdot\frac{1-\nu}{1-\mu(1)}\right)^{S_L-L\nu}\mathbf{1}\left[S_L-\nu L>0\right]\right]$$

and

$$E_n^-(\nu,L)=\mathbb{E}_\nu\left[\left(1-\Gamma(1)^{S_L}\Gamma(0)^{L-S_L}\right)^{n-1}\cdot\left(\frac{\mu(1)}{\nu}\cdot\frac{1-\nu}{1-\mu(1)}\right)^{S_L-L\nu}\mathbf{1}\left[S_L-\nu L\leqslant 0\right]\right].$$

It is plain that

$$\frac{\mu(1)}{\nu}\cdot\frac{1-\nu}{1-\mu(1)}>1 \text{ if and only if } \nu<\mu(1). \qquad (3.77)$$

We shall also use the simple fact that

$$\Gamma(1)^{S_L}\Gamma(0)^{L-S_L} = \left(\Gamma(1)^{\nu}\Gamma(0)^{1-\nu}\right)^L \cdot \left(\frac{\Gamma(1)}{\Gamma(0)}\right)^{S_L - L\nu}. \tag{3.78}$$

These observations form the basis for the arguments given next.

3.10 A proof of Proposition 3.7 – The zero-law

Consider a ρ-admissible scaling $\mathtt{L} : \mathbb{N}_0 \to \mathbb{N}_0$ such that (3.6) holds, or equivalently,

$$1 + \rho \ln(1 - \mu(1)) < 0. \tag{3.79}$$

By the discussion preceding the statement of Theorem 3.2, the non-linear equation (3.8) admits a single solution $\nu_\star(\rho)$ in the interval $(0, \mu(1))$ and

$$1 + \rho \ln G(\nu, \mu(1)) < 0, \quad \nu \in (0, \nu_\star(\rho)).$$

It follows that

$$\lim_{n\to\infty} nG(\nu, \mu(1))^{L_n} = 0, \quad \nu \in (0, \nu_\star(\rho)).$$

Therefore, by virtue of (3.74) the desired result $\lim_{n\to\infty} \mathbb{E}\left[I_n(L_n)\right] = 0$ will be established if we show that

$$\limsup_{n\to\infty} E_n(\nu, L_n) < \infty \tag{3.80}$$

for *some* ν in $(0, \nu_\star(\rho))$. This issue is explored with the help of the decomposition (3.76): Fix $n = 2, 3, \ldots$ and pick ν in the interval $(0, \nu_\star(\rho))$. Thus, (3.77) holds, and we have

$$\left(\frac{\mu(1)}{\nu} \cdot \frac{1-\nu}{1-\mu(1)}\right)^{S_{L_n} - L_n\nu} \leqslant \left(\frac{\mu(1)}{\nu} \cdot \frac{1-\nu}{1-\mu(1)}\right)^{(1-\nu)L_n}$$

since $S_{L_n} \leqslant L_n$. Using $\Gamma(0) < \Gamma(1)$ in (3.78) we then conclude that

$$\big(\Gamma(1)^\nu \Gamma(0)^{1-\nu}\big)^{L_n} \leqslant \Gamma(1)^{S_{L_n}} \Gamma(0)^{L_n - S_{L_n}} \text{ on } [S_{L_n} - L_n\nu > 0],$$

whence

$$\big(1 - \Gamma(1)^{S_{L_n}} \Gamma(0)^{L_n - S_{L_n}}\big)^{n-1} \leqslant \Big(1 - \big(\Gamma(1)^\nu \Gamma(0)^{1-\nu}\big)^{L_n}\Big)^{n-1} \text{ on } [S_{L_n} - L_n\nu > 0].$$

Using these bounds in the definition of $E_n^+(\nu, L_n)$, we obtain

$$\begin{aligned} & E_n^+(\nu, L_n) \\ \leqslant & \Big(1 - \big(\Gamma(1)^\nu \Gamma(0)^{1-\nu}\big)^{L_n}\Big)^{n-1} \cdot \left(\frac{\mu(1)}{\nu} \cdot \frac{1-\nu}{1-\mu(1)}\right)^{(1-\nu)L_n} \mathbb{P}_\nu \left[S_{L_n} - \nu L_n > 0\right] \\ \leqslant & \Big(1 - \big(\Gamma(1)^\nu \Gamma(0)^{1-\nu}\big)^{L_n}\Big)^{n-1} \cdot \left(\frac{\mu(1)}{\nu} \cdot \frac{1-\nu}{1-\mu(1)}\right)^{(1-\nu)L_n}. \end{aligned} \tag{3.81}$$

Next we turn to bounding $E_n^-(\nu, L_n)$. Because $\Gamma(0) < \Gamma(1) < 1$, we always have

$$\big(1 - \Gamma(1)^{S_{L_n}} \Gamma(0)^{L_n - S_{L_n}}\big)^{n-1} \leqslant 1$$

and exploiting the bound (3.77) gives

$$\left(\frac{\mu(1)}{\nu} \cdot \frac{1-\nu}{1-\mu(1)}\right)^{S_{L_n} - L_n\nu} \leqslant 1 \text{ on } [S_{L_n} - L_n\nu \leqslant 0].$$

As we apply these two bounds to the expression of $E_n^-(\nu, L_n)$ we find

$$E_n^-(\nu, L_n) \leqslant \mathbb{P}_\nu \left[S_{L_n} - L_n\nu \leqslant 0\right] \leqslant 1. \tag{3.82}$$

Thus, in order to establish (3.80) we need only show that

$$\limsup_{n\to\infty} E_n^+(\nu, L_n) < \infty \tag{3.83}$$

for *some* ν in $(0, \nu_\star(\rho))$, possibly under additional conditions which ensure that the constraint (3.10) also holds. As per the discussion following Theorem 3.2, the condition (3.12) guarantees (3.10) when ν is selected in the interval $(\alpha_+(\rho), \nu_\star(\rho))$, as we do from now on.

First, for each $n = 2, 3, \ldots$ consider each of the factors in the bound at (3.81). We find that

$$\begin{aligned}
\left(1 - \left(\Gamma(1)^\nu \Gamma(0)^{1-\nu}\right)^{L_n}\right)^{n-1} &= \left(1 - \left(\Gamma(1)^\nu \Gamma(0)^{1-\nu}\right)^{\rho_n \ln n}\right)^{n-1} \\
&\leqslant e^{-(n-1)\left(\Gamma(1)^\nu \Gamma(0)^{1-\nu}\right)^{\rho_n \ln n}} \\
&= e^{-\frac{(n-1)}{n} \cdot n^{1+\rho_n \ln\left(\Gamma(1)^\nu \Gamma(0)^{1-\nu}\right)}} \qquad (3.84)
\end{aligned}$$

and

$$\begin{aligned}
\left(\frac{\mu(1)}{\nu} \cdot \frac{1-\nu}{1-\mu(1)}\right)^{(1-\nu)L_n} &= \left(\frac{\mu(1)}{\nu} \cdot \frac{1-\nu}{1-\mu(1)}\right)^{(1-\nu)\rho_n \ln n} \\
&= n^{(1-\nu)\rho_n \ln\left(\frac{\mu(1)}{\nu} \cdot \frac{1-\nu}{1-\mu(1)}\right)}. \qquad (3.85)
\end{aligned}$$

By the ρ-admissibility of the scaling $\mathtt{L} : \mathbb{N}_0 \to \mathbb{N}_0$, for every $\varepsilon > 0$ there exists a positive integer $n_\star(\varepsilon)$ such that

$$\rho - \varepsilon < \rho_n < \rho + \varepsilon, \quad n \geqslant n_\star(\varepsilon).$$

On that range the bounds (3.84) and (3.85) imply

$$\left(1 - \left(\Gamma(1)^\nu \Gamma(0)^{1-\nu}\right)^{L_n}\right)^{n-1} \leqslant e^{-\frac{(n-1)}{n} \cdot n^{1+(\rho+\varepsilon) \ln\left(\Gamma(1)^\nu \Gamma(0)^{1-\nu}\right)}} \qquad (3.86)$$

and

$$\left(\frac{\mu(1)}{\nu} \cdot \frac{1-\nu}{1-\mu(1)}\right)^{(1-\nu)L_n} \leqslant n^{(1-\nu)(\rho+\varepsilon) \ln\left(\frac{\mu(1)}{\nu} \cdot \frac{1-\nu}{1-\mu(1)}\right)} \qquad (3.87)$$

as we recall that $\Gamma(0)$ and $\Gamma(1)$ both live in $(0, 1)$ and the inequality (3.77) holds. Given that (3.10) holds for the choice of ν, then it is also the case that

$$1 + (\rho + \varepsilon) \ln\left(\Gamma(1)^\nu \Gamma(0)^{1-\nu}\right) > 0 \qquad (3.88)$$

provided $\varepsilon > 0$ is selected small enough (as we do from now on).

Let n go to infinity in (3.81). It is plain from (3.84) that

$$\lim_{n\to\infty} e^{-(n-1)\left(\Gamma(1)^{\nu}\Gamma(0)^{1-\nu}\right)^{\rho_n \ln n}} = 0$$

by virtue of condition (3.88), while (3.85) implies

$$\lim_{n\to\infty} \left(\frac{\mu}{\nu}\cdot\frac{1-\nu}{1-\mu}\right)^{(1-\nu)\rho_n \ln n} = \infty$$

under (3.77). Nevertheless, appealing to the bounds (3.86) and (3.87), $\lim_{n\to\infty} E_n^+(\nu, L_n) = 0$ in view of the fact that

$$\lim_{n\to\infty} \left(e^{-\frac{(n-1)}{n}\cdot n^{1+(\rho+\varepsilon)\ln\left(\Gamma(1)^{\nu}\Gamma(0)^{1-\nu}\right)}} \cdot n^{(1-\nu)(\rho+\varepsilon)\ln\left(\frac{\mu(1)}{\nu}\cdot\frac{1-\nu}{1-\mu(1)}\right)} \right) = 0.$$

This is because the first factor goes to zero like $e^{-n^{\delta}}$ (with $\delta > 0$) while the second factor explodes to infinity like n^{β} (with $\beta > 0$). Obviously, $\limsup_{n\to\infty} E_n^-(\nu, L_n) \leqslant 1$ and the conclusion $\limsup_{n\to\infty} E_n(\nu, L_n) \leqslant 1$ follows. This concludes the proof of the zero-law in Theorem 3.2. ■

3.11 A proof of Proposition 3.7 – The infinity-law

Consider a ρ-admissible scaling $\mathtt{L} : \mathbb{N}_0 \to \mathbb{N}_0$ such that (3.6) holds, or equivalently, (3.79). We already know that

$$1 + \rho \ln G(\nu, \mu(1)) > 0, \quad \nu \in (\nu_\star(\rho), \mu(1)), \tag{3.89}$$

and the convergence

$$\lim_{n\to\infty} nG(\nu, \mu(1))^{L_n} = \infty, \quad \nu \in (\nu_\star(\rho), \mu(1))$$

follows by Lemma 2.4 (with $C_n = G(\nu, \mu(1))$ for all $n = 1, 2, \ldots$). By virtue of (3.74) the desired result $\lim_{n\to\infty} \mathbb{E}\left[I_n(L_n)\right] = \infty$ will be established if we show that

$$\liminf_{n\to\infty} E_n^+(\nu, L_n) > 0 \tag{3.90}$$

for *some* ν in $(\nu_\star(\rho), \mu(1))$ possibly constrained by some additional condition.

Pick ν still in $(\nu_\star(\rho), \mu(1))$ for the time being, and fix $n = 2, 3, \ldots$. Because (3.77) holds here, we have

$$\left(\frac{\mu(1)}{\nu} \cdot \frac{1-\nu}{1-\mu(1)}\right)^{S_{L_n} - L_n \nu} \geqslant 1 \text{ on } [S_{L_n} - L_n \nu > 0] \tag{3.91}$$

so that

$$E_n^+(\nu, L_n) \geqslant \mathbb{E}_\nu\left[\left(1 - \Gamma(1)^{S_{L_n}} \Gamma(0)^{L_n - S_{L_n}}\right)^{n-1} \mathbf{1}\left[S_{L_n} - \nu L_n > 0\right]\right]. \tag{3.92}$$

Next, we write

$$\left(1 - \Gamma(1)^{S_{L_n}} \Gamma(0)^{L_n - S_{L_n}}\right)^{n-1} = \left(1 - \left(\Gamma(1)^{\frac{S_{L_n}}{L_n}} \Gamma(0)^{1 - \frac{S_{L_n}}{L_n}}\right)^{L_n}\right)^{n-1} \tag{3.93}$$

and note that

$$\left|\left(1 - \Gamma(1)^{S_{L_n}} \Gamma(0)^{L_n - S_{L_n}}\right)^{n-1}\right| \leqslant 1. \tag{3.94}$$

Now further restrict the value of ν to the interval $(\nu_\star(\rho), \beta_-(\rho))$ discussed at the end of Section 3.1. Condition (3.11) ensures that (3.9) holds, and by Lemma 2.4 (with $\nu_n = \frac{S_{L_n}}{L_n}$ for all $n = 1, 2, \ldots$, with the help of (3.93)), we have the convergence

$$\lim_{n\to\infty} \left(1 - \Gamma(1)^{S_{L_n}} \Gamma(0)^{L_n - S_{L_n}}\right)^{n-1} = 1 \quad \mathbb{P}_\nu - \text{a.s.}. \tag{3.95}$$

Indeed, the Strong Law of Large Numbers (under $\mathbb{P}_\nu$) yields the convergence

$$\lim_{n\to\infty} \frac{S_{L_n}}{L_n} = \nu \quad \mathbb{P}_\nu - \text{a.s.}$$

and this leads to the needed conclusion

$$\lim_{n\to\infty}\left(1+\rho_n \ln\left(\Gamma(1)^{\frac{S_{L_n}}{L_n}}\Gamma(0)^{1-\frac{S_{L_n}}{L_n}}\right)\right) = 1+\rho\ln\left(\Gamma(0)^{1-\nu}\Gamma(1)^{\nu}\right) < 0 \quad \mathbb{P}_\nu - \text{a.s.}$$

under (3.9).

Pick ε in $(0,1)$. It follows from the bound (3.92) that

$$E_n^+(\nu, L_n) \geqslant (1-\varepsilon)\mathbb{P}_\nu\left[A_n(\varepsilon)\cap[S_{L_n}-\nu L_n>0]\right], \quad n=2,3,\ldots \tag{3.96}$$

where for notational simplicity we have introduced the event

$$A_n(\varepsilon) = \left[\left(1-\Gamma(1)^{S_{L_n}}\Gamma(0)^{L_n-S_{L_n}}\right)^{n-1} > 1-\varepsilon\right].$$

Since a.s. convergence implies convergence in probability (under $\mathbb{P}_\nu$), it is plain from (3.95) that $\lim_{n\to\infty}\mathbb{P}_\nu\left[A_n(\varepsilon)\right]=1$. On the other hand we also have

$$\lim_{n\to\infty}\mathbb{P}_\nu[S_{L_n}-L_n\nu>0] = \frac{1}{2}$$

by the Central Limit Theorem (under $\mathbb{P}_\nu$), whence

$$\lim_{n\to\infty}\mathbb{P}_\nu\left[A_n(\varepsilon)\cap[S_{L_n}-\nu L_n>0]\right] = \frac{1}{2}$$

by standard arguments. Therefore, $\liminf_{n\to\infty} E_n^+(\nu, L_n) \geqslant (1-\varepsilon)/2$ and the desired conclusion $\liminf_{n\to\infty} E_n^+(\nu, L_n) \geqslant 1$ follows since ε is arbitrary in $(0,1)$. This concludes the proof of the infinity-law in Theorem 3.2. ■

Chapter 4

The Zero-one Law for Connectivity

In the precious chapter, we have established the zero-one law for the absence of isolated nodes. A very related property to the absence of isolated nodes is connectivity since the one law for the absence of isolated nodes serves as a necessary condition for connectivity while the zero law serves as a sufficient condition for a graph not being connected. For instance, for Erdös-Rényi graphs [6], random geometric graphs [3] and random key graphs [27], the zero-one law for the absence of isolated nodes coincide with the zero-one law for connectivity.

In this chapter, we are interested in establishing the zero-one law for connectivity in MAGs. Recall that an undirected graph G is said to be connected if there is at least one path between every (unordered) pair of distinct nodes in G. It is of interest to know whether the two zero-one laws coincide in MAGs. If not, then it is natural to consider what is the zero-one law for connectivity in MAGs.

Unfortunately, the two zero-one laws are not identical in MAGs. In other words, $\Gamma(1) > \Gamma(0)$ is too weak for the zero-one law for connectivity to hold. To convince the reader, we give a counter example to show that the graph, with high probability, is not connected while there is no isolated nodes for n large:

Consider the setting where $\alpha = \gamma = 1$, $\beta = 0$ and $\mu(1) > \mu(0)$, in which case $\Gamma(1) = \mu(1) > \Gamma(0) = \mu(0)$. With the conditions $1 + \rho \ln \mu(0) > 0$ and $1+\rho \ln \Gamma(0) > 0$ enforced, with high probability, no isolated nodes are guaranteed by the one law in Theorem 3.1. However, in this case, nodes *only* connect to other nodes which have exactly the same attribute vectors as themselves. As the probability of all nodes having identical attribute vectors converges to 0 when n and L_n grow unboundedly large, the graph is not connected with high probability.

As a result, in this chapter, we will establish the zero-one law for connectivity in MAGs under some additional assumptions.

4.1 The theorem

Instead of assuming $0 \leqslant \Gamma(0) \leqslant \Gamma(1) \leqslant 1$, we will need the stronger condition

$$\alpha > \beta > \gamma \tag{4.1}$$

to be enforced. The condition (4.1) implies $\Gamma(1) \geqslant \Gamma(0)$, but rules out the case $\alpha = \gamma = 1$ and $\beta = 0$.

Theorem 4.1. *Assume $\alpha > \beta > \gamma$. For any ρ-admissible scaling $\mathtt{L} : \mathbb{N}_0 \to \mathbb{N}_0$ with $\rho > 0$, we further assume $1 + \rho \ln \mu(0) > 0$. We have*

$$\lim_{n\to\infty} \mathbb{P}\left[\ \mathbb{M}(n; L_n) \text{ is connected } \right] = \begin{cases} 0 & \text{if } 1 + \rho \ln \Gamma(0) < 0 \\ \\ 1 & \text{if } 1 + \rho \ln \Gamma(0) > 0. \end{cases} \tag{4.2}$$

As the zero law for the absence of isolated nodes in Theorem 3.1 automatically implies the zero law for connectivity in Theorem 4.1, we only need to show that the one law for connectivity holds in order to complete the proof of Theorem 4.1.

4.2 A general idea

A graph G is connected iff no cut of any sizes exists in G. A well-known proof [8], which makes use of this equivalence, establishes the zero-one law for connectivity for Erdös-Rényi graphs. However, unlike in Erdös-Rényi graphs, the link variables

$$\{\chi_L(u, v),\ 1 < u < v < n\}$$

in $\mathbb{M}(n; L)$ are not mutually independent, so that it may not be easy to make use of this equivalence directly. Alternatively, we will first look at some properties of nodal attributes since the link variables are conditionally mutually independent given all nodal attributes.

Fix $n = 2, 3, \ldots$ and $L = 1, 2, \ldots$. We say that a node u in V_n has the attribute property $\mathcal{P}_L \subseteq \{0, 1\}^L$ in $\mathbb{M}(n; L)$ iff $\mathbf{A}_L(u)$ is an element of $\mathcal{P}_L$. The (random) collection of nodes which have the property $\mathcal{P}_L$ is denoted by $T_n(\mathcal{P}_L)$. We are interested in finding a property $\mathcal{P}_L$ such that the random subset $T_n(\mathcal{P}_L)$ can serve as a core component in $\mathbb{M}(n; L)$ in a sense to be specified shortly. When n and L grow unboundedly large, where the latter quantity scales with the former, we seek to balance the following two criteria:

1. The subgraph induced by $T_n(\mathcal{P}_L)$ in $\mathbb{M}(n; L)$ is a.a.s. connected.

2. All nodes outside $T_n(\mathcal{P}_L)$ (i.e. in $V_n \backslash T_n(\mathcal{P}_L)$) connect a.a.s. to $T_n(\mathcal{P}_L)$ in one hop.

Obviously, the random subset $T_n(\mathcal{P}_L)$ cannot be too large for otherwise Criterion 1 would be hard to prove. As a matter of fact, if the one law for connectivity holds, then V_n itself is a set (i.e. $\mathcal{P}_L = \{0,1\}^L$) that trivially satisfies Criterion 1. But the random subset $T_n(\mathcal{P}_L)$ cannot be too small either for otherwise it may not be connected in one hop to the nodes in $V_n \backslash T_n(\mathcal{P}_L)$.

Moreover, it is worth pointing out that for any attribute property $\mathcal{P}_L$, with $\{\mathbf{A}_L(u),\ u \in V_n\}$ given, the random subset $T_n(\mathcal{P}_L)$ can be thought as deterministic.

4.3 The second criterion

Before additional conditions are imposed, we explore the requirements on the random subset $T_n(\mathcal{P}_L)$ to ensure that all the nodes outside the random subset $T_n(\mathcal{P}_L)$ connect a.a.s. in one hop to $T_n(\mathcal{P}_L)$. The following lemma gives a useful bound to the probability of the desired event.

Lemma 4.2. *For $n = 2, 3, \ldots,$ $L = 1, 2, \ldots$ and a fixed attribute property $\mathcal{P}_L$, it holds that*

$$\begin{aligned} &\mathbb{P}\left[\bigcap_{u \in V_n \backslash T_n(\mathcal{P}_L)} \left[\begin{array}{l} \text{Node } u \text{ connects in one hop} \\ \text{to } T_n(\mathcal{P}_L) \end{array} \right] \middle| \mathbf{A}_L(w),\ w \in V_n \right] \\ &\geqslant\ 1 - \sum_{u \in V_n} \mathbf{1}[u \in V_n \backslash T_n(\mathcal{P}_L)] \left(\prod_{v \in T_n(\mathcal{P}_L)} \big(1 - Q_L(\mathbf{A}_L(u), \mathbf{A}_L(v))\big) \right). \end{aligned} \tag{4.3}$$

Proof. Fix $n = 2, 3, \ldots$ and $L = 1, 2, \ldots$. For each node u in V_n, let $\mathcal{N}_L(u)$ denote the set of nodes who are $L-$adjacent to u. For any node u in $V_n \backslash T_n(\mathcal{P}_L)$, u connects in one hop to $T_n(\mathcal{P}_L)$ iff the intersection of $T_n(\mathcal{P}_L)$ and $\mathcal{N}_L(u)$ is not empty. Moreover, recall that the random subset $T_n(\mathcal{P}_L)$ is determined once $\{\mathbf{A}_L(w),\ w \in V_n\}$ is given, and so is $V_n \backslash T_n(\mathcal{P}_L)$. As a result,

$$
\begin{aligned}
& \mathbb{P}\left[\bigcap_{u \in V_n \backslash T_n(\mathcal{P}_L)} \left[\begin{array}{l} \text{Node } u \text{ connects in one hop} \\ \text{to } T_n(\mathcal{P}_L) \end{array} \right] \middle| \mathbf{A}_L(w),\ w \in V_n \right] \\
= \ & \mathbb{P}\left[\bigcap_{u \in V_n \backslash T_n(\mathcal{P}_L)} \left[\mathcal{N}_L(u) \cap T_n(\mathcal{P}_L) \neq \varnothing \right] \middle| \mathbf{A}_L(w),\ w \in V_n \right] \\
= \ & 1 - \mathbb{P}\left[\bigcup_{u \in V_n \backslash T_n(\mathcal{P}_L)} \left[\mathcal{N}_L(u) \cap T_n(\mathcal{P}_L) = \varnothing \right] \middle| \mathbf{A}_L(w),\ w \in V_n \right] \\
\geqslant \ & 1 - \sum_{u \in V_n \backslash T_n(\mathcal{P}_L)} \mathbb{P}\left[\mathcal{N}_L(u) \cap T_n(\mathcal{P}_L) = \varnothing \middle| \mathbf{A}_L(w),\ w \in V_n \right] \qquad (4.4)
\end{aligned}
$$

where a standard union bound was applied in the second to last step. Further simplifications are possible for (4.4) by using the fact that $\mathbf{1}[u \in V_n \backslash T_n(\mathcal{P}_L)] = 0$ if u is in $T_n(\mathcal{P}_L)$. Indeed, we get

$$
\begin{aligned}
& \sum_{u \in V_n \backslash T_n(\mathcal{P}_L)} \mathbb{P}\left[\mathcal{N}_L(u) \cap T_n(\mathcal{P}_L) = \varnothing \middle| \mathbf{A}_L(w),\ w \in V_n \right] \\
= \ & \sum_{u \in V_n} \mathbf{1}[u \in V_n \backslash T_n(\mathcal{P}_L)] \mathbb{P}\left[\mathcal{N}_L(u) \cap T_n(\mathcal{P}_L) = \varnothing \middle| \mathbf{A}_L(w),\ w \in V_n \right]. \qquad (4.5)
\end{aligned}
$$

Because $\mathcal{N}_L(u) \cap T_n(\mathcal{P}_L) = \varnothing$ iff u does not connect to any nodes in $T_n(\mathcal{P}_L)$, we get

$$
\begin{aligned}
& \mathbb{P}\left[\mathcal{N}_L(u) \cap T_n(\mathcal{P}_L) = \varnothing \middle| \mathbf{A}_L(w),\ w \in V_n\right] \\
= \ & \mathbb{P}\left[\bigcap_{v \in T_n(\mathcal{P}_L),\ v \neq u} [u \sim_L v]^c \middle| \mathbf{A}_L(w),\ w \in V_n\right] \\
= \ & \mathbb{P}\left[\bigcap_{v \in T_n(\mathcal{P}_L),\ v \neq u} \left[U(u,v) > Q_L(\mathbf{A}_L(u), \mathbf{A}_L(v))\right] \middle| \mathbf{A}_L(w),\ w \in V_n\right] \\
= \ & \prod_{v \in T_n(\mathcal{P}_L),\ v \neq u} \mathbb{P}\left[U(u,v) > Q_L(\mathbf{A}_L(u), \mathbf{A}_L(v)) \middle| \mathbf{A}_L(w),\ w \in V_n\right] \\
= \ & \prod_{v \in T_n(\mathcal{P}_L),\ v \neq u} (1 - Q_L(\mathbf{A}_L(u), \mathbf{A}_L(v))), \qquad u \in V_n
\end{aligned}
\tag{4.6}
$$

upon using the fact that the rvs $\{U(u,v),\ u \in V_n,\ v \in T_n(\mathcal{P}_L),\ v \neq u\}$ are mutually independent.

Lemma 4.2 is now straightforward once we substitute the expression (4.6) for $\mathbb{P}\left[\mathcal{N}_L(u) \cap T_n(\mathcal{P}_L) = \varnothing \middle| \mathbf{A}_L(w),\ w \in V_n\right]$ into (4.5). ■

Upon taking expectations on both sides of (4.3), we get the following corollary.

Corollary 4.3. *Fix $L = 1, 2, \ldots$. For a given attribute property $\mathcal{P}_L$, we have*

$$
\lim_{n \to \infty} \mathbb{P}\left[\bigcap_{u \in V_n \setminus T_n(\mathcal{P}_L)} \left[\begin{array}{l} \textit{Node } u \textit{ connects in one hop} \\ \textit{to } T_n(\mathcal{P}_L) \end{array}\right]\right] = 1 \tag{4.7}
$$

if

$$
\lim_{n \to \infty} \sum_{u \in V_n} \mathbb{E}\left[\mathbf{1}[u \in V_n \setminus T_n(\mathcal{P}_L)] \prod_{\substack{v \in T_n(\mathcal{P}_L), \\ v \neq u}} (1 - Q_L(\mathbf{A}_L(u), \mathbf{A}_L(v)))\right] = 0. \tag{4.8}
$$

4.4 A core component of MAG

With conditions for Criterion 2 settled, we now seek to identify the candidate property $\mathcal{P}_L$, so that the (random) subset $T_n(\mathcal{P}_L)$ can serve as a core component in the sense of Criterion 1.

4.4.1 A connectivity criterion

We first explore the connectivity conditions when the random subset $T_n(\mathcal{P}_L)$ is fixed. This is the content of the following lemma.

Lemma 4.4. *For $n = 2, 3, \ldots,$ $L = 1, 2, \ldots,$ and a deterministic subset T of V_n, we have*

$$\mathbb{P}\left[\ T_n(\mathcal{P}_L) \text{ is connected } \ |T_n(\mathcal{P}_L) = T\right] \geqslant 1 - \sum_{r=1}^{\frac{|T|}{2}} \left(\frac{|T|}{2} e^{-B(\mathcal{P}_L)\frac{|T|}{2}} \right)^r \tag{4.9}$$

provided there exits a scalar $B(\mathcal{P}_L)$ in $(0, 1)$, such that

$$Q_L(\mathbf{A}_L(u), \mathbf{A}_L(v)) \geqslant B(\mathcal{P}_L) \tag{4.10}$$

whenever both $\mathbf{A}_L(u)$ and $\mathbf{A}_L(v)$ are in $\mathcal{P}_L$.

Proof. Fix $n = 2, 3, \ldots$ and $L = 1, 2, \ldots$. With $T_n(\mathcal{P}_L) = T \subset V_n$ given, the probability that $T_n(\mathcal{P}_L) = T$ is connected is equal to the probability that no cut of any sizes exists in the subgraph of $\mathbb{M}(n;L)$ induced by T, namely

$$
\begin{aligned}
& \mathbb{P}\left[\, T_n(\mathcal{P}_L) \text{ is connected } \;\middle|\, T_n(\mathcal{P}_L) = T\right] \\
= \; & \mathbb{P}\left[\bigcap_{R \subset T_n(\mathcal{P}_L)} \left[R \text{ connects to } T_n(\mathcal{P}_L)\backslash R\right] \middle| T_n(\mathcal{P}_L) = T\right] \\
= \; & 1 - \mathbb{P}\left[\bigcup_{R\subset T,\ 1 \leqslant |R| \leqslant \frac{|T|}{2}} \left[R \text{ does not connect to } T\backslash R\right] \middle| T_n(\mathcal{P}_L) = T\right]. \qquad (4.11)
\end{aligned}
$$

A standard union bound leads to

$$
\begin{aligned}
& \mathbb{P}\left[\bigcup_{R\subset T,\ 1 \leqslant |R| \leqslant \frac{|T|}{2}} \left[R \text{ does not connect to } T\backslash R\right] \middle| T_n(\mathcal{P}_L) = T\right] \\
\leqslant \; & \sum_{r=1}^{\frac{|T|}{2}} \sum_{R\subset T,\ |R|=r} \mathbb{P}\left[\bigcap_{u\in R,\ v\in T\backslash R} [u \sim_L v]^c \middle| T_n(\mathcal{P}_L) = T\right]. \qquad (4.12)
\end{aligned}
$$

Pre-conditioning on $\{\mathbf{A}_L(w), w \in V_n\}$, the events $\{[u \sim_L v],\ u, v \in V_n,\ u < v\}$ are now mutually independent. The probability that R does not connect to $T\backslash R$ becomes

$$
\begin{aligned}
& \mathbb{P}\left[\bigcap_{u\in R,\ v\in T\backslash R} [u \sim_L v]^c \middle| T_n(\mathcal{P}_L) = T\right] \\
= \; & \frac{\mathbb{P}\left[\left(\bigcap_{u\in R,\ v\in T\backslash R}[u \sim_L v]^c\right) \bigcap [T_n(\mathcal{P}_L) = T]\right]}{\mathbb{P}\left[T_n(\mathcal{P}_L) = T\right]} \\
= \; & \frac{\mathbb{E}\left[\mathbb{P}\left[\left(\bigcap_{u\in R,\ v\in V\backslash R}\left[U(u,v) > Q_L(\mathbf{A}_L(u), \mathbf{A}_L(v))\right]\right) \bigcap [T_n(\mathcal{P}_L) = T] \middle| \mathbf{A}_L(w), w \in V_n\right]\right]}{\mathbb{P}\left[T_n(\mathcal{P}_L) = T\right]} \\
= \; & \frac{\mathbb{E}\left[\mathbb{P}\left[\bigcap_{u\in R,\ v\in V_n\backslash R}\left[U(u,v) > Q_L\left(\mathbf{A}_L(u), \mathbf{A}_L(v)\right)\right] \middle| \mathbf{A}_L(w), w \in V_n\right] \mathbf{1}[T_n(\mathcal{P}_L) = T]\right]}{\mathbb{P}\left[T_n(\mathcal{P}_L) = T\right]} \\
= \; & \frac{\mathbb{E}\left[\left(\prod_{u\in R,\ v\in T\backslash R}\left(1 - Q_L(\mathbf{A}_L(u), \mathbf{A}_L(v))\right)\right) \cdot \mathbf{1}[T_n(\mathcal{P}_L) = T]\right]}{\mathbb{P}\left[T_n(\mathcal{P}_L) = T\right]} \qquad (4.13)
\end{aligned}
$$

where the second to last step used the fact that $T_n(\mathcal{P}_L)$ is $\sigma\{\mathbf{A}_L(w),\ w \in V_n\}$ measurable.

The condition $T_n(\mathcal{P}_L) = T$ amounts to $\mathbf{A}_L(w)$ is an element of $\mathcal{P}_L$ for all w in T. As a result, condition (4.10) implies

$$\begin{aligned}
&\left(\prod_{u\in R\ ,v\in T\backslash R}(1-Q_L(\mathbf{A}_L(u),\mathbf{A}_L(v)))\right)\cdot\mathbf{1}[T_n(\mathcal{P}_L)=T] \\
\leqslant\ &\left(\prod_{u\in R\ ,v\in T\backslash R}(1-B(\mathcal{P}_L))\right)\cdot\mathbf{1}[T_n(\mathcal{P}_L=T] \\
=\ &(1-B(\mathcal{P}_L))^{|R|(|T|-|R|)}\mathbf{1}[T_n(\mathcal{P}_L)=T]. \qquad (4.14)
\end{aligned}$$

Using this bound in (4.13), we find

$$\begin{aligned}
&\mathbb{P}\left[\bigcap_{u\in R,\ v\in T\backslash R}[u\sim_L v]^c\middle|T_n(\mathcal{P}_L)=T\right] \\
\leqslant\ &\frac{\mathbb{E}\left[(1-B(\mathcal{P}_L))^{|R|(|T|-|R|)}\mathbf{1}[T_n(\mathcal{P}_L)=T]\right]}{\mathbb{P}\left[T_n(\mathcal{P}_L)=T\right]} \\
=\ &\frac{(1-B(\mathcal{P}_L))^{|R|(|T|-|R|)}\mathbb{E}\left[\mathbf{1}[T_n(\mathcal{P}_L)=T]\right]}{\mathbb{P}\left[T_n(\mathcal{P}_L)=T\right]} \\
=\ &(1-B(\mathcal{P}_L))^{|R|(|T|-|R|)}. \qquad (4.15)
\end{aligned}$$

Collecting (4.12), (4.13) and (4.15), we obtain

$$\begin{aligned}
&\mathbb{P}\left[\bigcup_{R\subset T,\ 1\leqslant|R|\leqslant\frac{|T|}{2}}\left[R\text{ does not connect to }T\backslash R\right]\middle|T_n(\mathcal{P}_L)=T\right] \\
\leqslant\ &\sum_{r=1}^{\frac{|T|}{2}}\sum_{R\subset T,\ |R|=r}(1-B(\mathcal{P}_L))^{|R|(|T|-|R|)} \\
\leqslant\ &\sum_{r=1}^{\frac{|T|}{2}}\binom{|T|}{r}(1-B(\mathcal{P}_L))^{\frac{r|T|}{2}}. \qquad (4.16)
\end{aligned}$$

This bound depends on the deterministic set T only through its cardinality $|T|$. By Lemma 2.4 and the trivial bound

$$\binom{|T|}{r}\leqslant(|T|)^r\ ,\ r=0,1,\ldots,\frac{|T|}{2},$$

further simplifications are possible, namely,

$$\sum_{r=1}^{\frac{|T|}{2}} \binom{|T|}{r} (1 - B(\mathcal{P}_L))^{\frac{r|T|}{2}} \leqslant \sum_{r=1}^{\frac{|T|}{2}} (|T|)^r \, e^{-B(\mathcal{P}_L)\frac{r|T|}{2}}$$

$$= \sum_{r=1}^{\frac{|T|}{2}} \left(|T| e^{-B(\mathcal{P}_L)\frac{|T|}{2}} \right)^r . \tag{4.17}$$

Combining (4.16) and (4.17), we readily obtain (4.9) with the help of (4.11).

■

Lemma 4.4 tells us that if the sum in (4.17) is small, then there is a high probability that $T_n(\mathcal{P}_L) = T$ is connected when n becomes large. As the sum depends on only the two quantities $B(\mathcal{P}_L)$ and $|T|$, we seek to balance these two quantities so that the sum in (4.17) can be made sufficiently small.

4.4.2 Constructing $T_n(\mathcal{P}_L)$

For $n = 2, 3, \ldots$ and $L = 1, 2, \ldots$, we define two (random) subsets of V_n as follow:

$$W_n(L;\ell) = \{u \in V_n : S_L(u) = \ell\}, \quad \ell = 0, 1, 2, \ldots, L$$

and

$$Z_n(L;\ell) = \{u \in V_n : S_L(u) \geqslant \ell\}, \quad \ell = 0, 1, 2, \ldots, L$$

where the rvs $\{S_L(u),\ u \in V_n\}$ were defined in (2.2). From the inclusion $W_n(L;\ell) \subseteq Z_n(L;\ell)$, we get the elementary bound

$$|W_n(L;\ell)| \leqslant |Z_n(L;\ell)|, \qquad \ell = 0, 1, \ldots, L. \tag{4.18}$$

For any ρ-admissible scaling $\mathtt{L} : \mathbb{N}_0 \to \mathbb{N}_0$, we want to argue that the cardinality of the random set $Z_n(L_n; \lceil \lambda L_n \rceil)$ with λ selected as $\frac{\mu(1)\beta}{\mu(1)\beta + \mu(0)\gamma}$ is large with high probability and the probabilities of link establishments between nodes in this set are not too small as n grows large, so that it can be an ideal (random) subset to serve as the core component in the sense of Criterion 1. In the first step, we will show that the cardinality of $Z_n(L_n; \lceil \lambda L_n \rceil)$ is sufficiently (unboundedly) large a.a.s..

Lemma 4.5. *Assume $\alpha > \beta > \gamma$. For any ρ-admissible scaling $\mathtt{L} : \mathbb{N}_0 \to \mathbb{N}_0$ with $\rho > 0$, assume that*

$$1 + \rho \ln \Gamma(0) > 0. \tag{4.19}$$

For any constant c in $(0, 1)$, it holds that

$$\lim_{n\to\infty} \mathbb{P}\left[\frac{|Z_n(L_n; \lceil \lambda L_n \rceil)|}{\mathbb{E}\left[|W_n(L_n; \lceil \lambda L_n \rceil)|\right]} > c \right] = 1 \tag{4.20}$$

with λ selected as

$$\lambda = \frac{\mu(1)\beta}{\mu(1)\beta + \mu(0)\gamma}. \tag{4.21}$$

Before establishing Lemma 4.5, we state another useful fact that will facilitate the forthcoming analysis.

Lemma 4.6. *For any ρ-admissible scaling $\mathtt{L} : \mathbb{N}_0 \to \mathbb{N}_0$ with $\rho > 0$, there exists a positive integer N such that*

$$\begin{aligned} &\binom{L_n}{\lceil \lambda L_n \rceil} \\ \geqslant\ & \frac{k(\lambda)}{2\sqrt{2\pi\lambda(1-\lambda)L_n}} \left(\frac{1}{\lambda}\right)^{\lambda L_n} \left(\frac{1}{1-\lambda}\right)^{(1-\lambda)L_n}, \qquad n \geqslant N \end{aligned} \tag{4.22}$$

with

$$\lambda \in (0, 1) \text{ and } k(\lambda) = \min\left(1, \frac{1-\lambda}{\lambda}\right).$$

Details of Lemma 4.6 can be found in Appendix B.

Fix c in $(0,1)$. We know from (4.18) that

$$\mathbb{P}\left[\frac{|Z_n(L_n;\lceil\lambda L_n\rceil)|}{\mathbb{E}\left[|W_n(L_n;\lceil\lambda L_n\rceil)|\right]} > c\right] \geqslant \mathbb{P}\left[\frac{|W_n(L_n;\lceil\lambda L_n\rceil)|}{\mathbb{E}\left[|W_n(L_n;\lceil\lambda L_n\rceil)|\right]} > c\right], \qquad n = 2,3,\ldots. \tag{4.23}$$

Lemma 4.5 will be established if we can show that

$$\lim_{n\to\infty} \mathbb{P}\left[\frac{|W_n(L_n;\lceil\lambda L_n\rceil)|}{\mathbb{E}\left[|W_n(L_n;\lceil\lambda L_n\rceil)|\right]} > c\right] = 1.$$

Now we proceed to prove Lemma 4.5.

Proof. For $n = 2,3,\ldots$ and $L = 1,2,\ldots$, the expression

$$|W_n(L;\ell)| = \sum_{u=1}^{n} \mathbf{1}[S_L(u) = \ell], \qquad \ell = 0,1,\ldots,L \tag{4.24}$$

shows that the quantity $|W_n(L;\ell)|$ is the sum of i.i.d. $0,1-$valued indicator rvs $\left\{\mathbf{1}\left[S_L(u) = \ell\right],\ u \in V_n\right\}$. The expected cardinality of the random set $W_n(L;\ell)$ is therefore given by

$$\begin{aligned} \mathbb{E}\left[|W_n(L;\ell)|\right] &= \mathbb{E}\left[\sum_{u=1}^{n} \mathbf{1}[S_L(u) = \ell]\right] \\ &= n\mathbb{P}\left[S_L(1) = \ell\right] \\ &= n\binom{L}{\ell}\mu(1)^{\ell}\mu(0)^{L-\ell}, \qquad \ell = 0,1,\ldots,L \end{aligned} \tag{4.25}$$

upon using the fact that the rvs $\{S_L(u),\ u \in V_n\}$ are a collection of i.i.d. binomial rvs with parameters $(\mu(1), L)$.

For any ρ-admissible scaling $\mathtt{L} : \mathbb{N}_0 \to \mathbb{N}_0$ with $\rho > 0$, by Lemma 4.6, with sufficiently large n and

$$\delta(n;L_n) = \lceil\lambda L_n\rceil - \lambda L_n,$$

we conclude that

$$
\begin{aligned}
& \mathbb{E}\left[|W_n(L_n;\lceil\lambda L_n\rceil)|\right] \\
= \;& n\binom{L_n}{\lceil\lambda L_n\rceil}\mu(1)^{\lceil\lambda L_n\rceil}\mu(0)^{L_n-\lceil\lambda L_n\rceil} \\
\geqslant \;& n\frac{k(\lambda)}{2\sqrt{2\pi\lambda(1-\lambda)L_n}}\left(\frac{\mu(1)}{\lambda}\right)^{\lambda L_n}\left(\frac{\mu(0)}{1-\lambda}\right)^{(1-\lambda)L_n}\left(\frac{\mu(1)}{\mu(0)}\right)^{\delta(n;L_n)} \\
\geqslant \;& n\frac{k^*(\lambda,\mu(1))}{\sqrt{2\pi\lambda(1-\lambda)L_n}}\left(\frac{\mu(1)}{\lambda}\right)^{\lambda L_n}\left(\frac{\mu(0)}{1-\lambda}\right)^{(1-\lambda)L_n} \qquad (4.26)
\end{aligned}
$$

where we have set

$$k^*(\lambda;\mu(1)) = \frac{k(\lambda)}{2}\times \min\left(1,\frac{\mu(1)}{\mu(0)}\right).$$

Indeed, the last step was based on the facts that

$$\left(\frac{\mu(1)}{\mu(0)}\right)^{\delta(n;L_n)} \geqslant 1, \qquad \text{if } \frac{\mu(1)}{\mu(0)} \geqslant 1$$

and

$$\left(\frac{\mu(1)}{\mu(0)}\right)^{\delta(n;L_n)} \geqslant \frac{\mu(1)}{\mu(0)}, \quad \text{if } \frac{\mu(1)}{\mu(0)} < 1.$$

Consider the quantity

$$\beta^{\lambda L_n}\gamma^{(1-\lambda)L_n} = n^{\rho_n\tau} < 1 \qquad (4.27)$$

where τ is given by

$$\tau = \tau(\lambda,\beta,\gamma) = \lambda\ln\beta + (1-\lambda)\ln\gamma < 0.$$

Multiplying (4.26) by (4.27), we get

$$
\begin{aligned}
& \mathbb{E}[|W_n(L_n;\lceil\lambda L_n\rceil)|]\beta^{\lambda L_n}\gamma^{(1-\lambda)L_n} \\
\geqslant\ & n\frac{k^*(\lambda;\mu(1))}{\sqrt{2\pi\lambda(1-\lambda)L_n}}\left(\frac{\mu(1)\beta}{\lambda}\right)^{\lambda L_n}\left(\frac{\mu(0)\gamma}{1-\lambda}\right)^{(1-\lambda)L_n} \\
=\ & n\frac{k^*(\lambda;\mu(1))}{\sqrt{2\pi\lambda(1-\lambda)L_n}}\Gamma(0)^{L_n} \\
=\ & n^{1+\rho_n\ln\Gamma(0)}\frac{k^*(\lambda;\mu(1))}{\sqrt{2\pi\lambda(1-\lambda)L_n}}, \qquad (4.28)
\end{aligned}
$$

as elementary calculations yield

$$\frac{\mu(1)\beta}{\lambda}=\frac{\mu(0)\gamma}{1-\lambda}=\Gamma(0). \qquad (4.29)$$

Under the condition $1+\rho\ln\Gamma(0)>0$, we obtain

$$\lim_{n\to\infty}\mathbb{E}[|W_n(L_n;\lceil\lambda L_n\rceil)|]\beta^{\lambda L_n}\gamma^{(1-\lambda)L_n}=\infty$$

and the result

$$\lim_{n\to\infty}\mathbb{E}[|W_n(L_n;\lceil\lambda L_n\rceil)|]=\infty \qquad (4.30)$$

follows by virtue of (4.27). Moreover, when n is large, it is plain that

$$\mathbb{E}[|W_n(L_n;\lceil\lambda L_n\rceil)|]\geqslant n^{1+\rho_n\ln\Gamma(0)-\rho_n\tau}\frac{k^*(\lambda;\mu(1))}{\sqrt{2\pi\lambda(1-\lambda)L_n}}.$$

As a result, for any constant c in $(0,1)$, a lower bound to the probability of the event $|W_n(L_n;\lceil\lambda L_n\rceil)|>c\mathbb{E}\left[|W_n(L_n;\lceil\lambda L_n\rceil)|\right]$ can be obtained by the Chernoff-Hoeffding inequality [23, Prop. 2.4]. This takes the form

$$
\begin{aligned}
& \mathbb{P}\left[\left|W_n(L_n;\lceil\lambda L_n\rceil)\right|>c\mathbb{E}\left[\left|W_n(L_n;\lceil\lambda L_n\rceil)\right|\right]\right] \\
=\ & 1-\mathbb{P}\left[\left|W_n(L_n;\lceil\lambda L_n\rceil)\right|\leqslant c\mathbb{E}\left[\left|W_n(L_n;\lceil\lambda L_n\rceil)\right|\right]\right] \\
\geqslant\ & 1-e^{-\frac{(1-c)^2\mathbb{E}[|W_n(L_n;\lceil\lambda L_n\rceil)|]}{2}}. \qquad (4.31)
\end{aligned}
$$

By virtue of (4.23) and (4.30), we get

$$\begin{aligned}
& \liminf_{n\to\infty} \mathbb{P}\left[\frac{|Z_n(L_n;\lceil\lambda L_n\rceil)|}{\mathbb{E}\left[|W_n(L_n;\lceil\lambda L_n\rceil)|\right]} > c\right] \\
\geqslant & \liminf_{n\to\infty} \mathbb{P}\left[\frac{\left|W_n(L_n;\lceil\lambda L_n\rceil)\right|}{\mathbb{E}\left[|W_n(L_n;\lceil\lambda L_n\rceil)|\right]} > c\right] \\
= & \ 1
\end{aligned}$$

which completes the proof of the lemma.

■

Fix $n = 2, 3, \ldots$ and $L = 1, 2, \ldots$, if

$$\mathcal{P}_L = \left\{ \mathbf{a}_L \in \{0,1\}^L : \sum_{\ell=1}^{L} a_\ell \geqslant \lceil \lambda L \rceil \right\},$$

then

$$T_n(\mathcal{P}_L) = Z_n(L;\lceil\lambda L\rceil).$$

The bound $B(\mathcal{P}_L)$ can be constructed as follows. Under the assumption $\alpha > \beta > \gamma$, we now show that

$$\mathbb{P}\left[u \sim_L v | \mathbf{A}_L(u), \mathbf{A}_L(v)\right] \geqslant \beta^{\lambda L}\gamma^{(1-\lambda)L}, \qquad \begin{array}{l} u, v \in Z_n(L;\lceil\lambda L\rceil), \\ u \neq v. \end{array} \tag{4.32}$$

Indeed, we know

$$\begin{aligned}
& \mathbb{P}\left[u \sim_L v | \mathbf{A}_L(u), \mathbf{A}_L(v)\right] \\
= & \ \alpha^{\sum_{\ell=1}^{L} A_\ell(u)A_\ell(v)} \beta^{\sum_{\ell=1}^{L}(1-A_\ell(u))A_\ell(v) + A_\ell(u)(1-A_\ell(v))} \gamma^{\sum_{\ell=1}^{L}(1-A_\ell(u))(1-A_\ell(v))}
\end{aligned} \tag{4.33}$$

for distinct u, v in V_n. For distinct u and v in $Z_n(L;\lceil\lambda L\rceil)$, we can split (4.33) into two parts, namely

$$
\begin{aligned}
\alpha^{\sum_{\ell=1}^{L} A_\ell(u)A_\ell(v)}\beta^{\sum_{\ell=1}^{L}(1-A_\ell(u))A_\ell(v)} &\geqslant \beta^{\sum_{\ell=1}^{L} A_\ell(vu)} \\
&= \beta^{\lambda L+\left(\sum_{\ell=1}^{L} A_\ell(v)-\lambda L\right)} \\
&\geqslant \beta^{\lambda L}\gamma^{\left(\sum_{\ell=1}^{L} A_\ell(v)-\lambda L\right)} \qquad (4.34)
\end{aligned}
$$

and

$$
\begin{aligned}
\beta^{A_\ell(u)(1-A_\ell(v))}\gamma^{\sum_{\ell=1}^{L}(1-A_\ell(u))(1-A_\ell(v))} &\geqslant \gamma^{\sum_{\ell=1}^{L}(1-A_\ell(v))} \\
&= \gamma^{(1-\lambda)L-\left(\sum_{\ell=1}^{L} A_\ell(v)-\lambda L\right)}. \qquad (4.35)
\end{aligned}
$$

The bound in (4.34) follows from the fact that $\sum_{\ell=1}^{L} A_\ell(u)-\lambda L > 0$ since $\sum_{\ell=1}^{L} A_\ell(w) \geqslant \lceil \lambda L \rceil \geqslant \lambda L$ for all w in $Z_n(L;\lceil \lambda L \rceil)$. Now, multiplying (4.34) to (4.35), we get the bound in (4.32). The quantity $\beta^{\lambda L}\gamma^{(1-\lambda)L}$ is therefore a qualified candidate for $B(\mathcal{P}_L)$.

Eventually, for any $\rho-$admissible scaling $\mathtt{L} : \mathbb{N}_0 \to \mathbb{N}_0$ with $\rho > 0$, we are ready to show that $Z_n(L_n;\lceil \lambda L_n \rceil)$ is a.a.s. connected, and therefore can serve as a core component in the sense of Criterion 1.

Lemma 4.7. *Assume $\alpha > \beta > \gamma$. For any $\rho-$admissible scaling $\mathtt{L} : \mathbb{N}_0 \to \mathbb{N}_0$ with $\rho > 0$, we further assume that $1 + \rho \ln \Gamma(0) > 0$. With*

$$\lambda = \frac{\mu(1)\beta}{\mu(1)\beta + \mu(0)\gamma},$$

the subgraph induced by nodes in $Z_n(L_n;\lceil \lambda L_n \rceil)$ of $\mathbb{M}(n;L_n)$ is connected a.a.s., namely

$$\lim_{n\to\infty} \mathbb{P}\left[\ Z_n(L_n;\lceil \lambda L_n \rceil) \textit{ is connected } \right] = 1. \qquad (4.36)$$

Proof. Fix $n = 2, 3, \dots$. For any $\rho-$admissible scaling $\mathtt{L} : \mathbb{N}_0 \to \mathbb{N}_0$ with $\rho > 0$, let

$$\mathcal{P}_{L_n} = \left\{ \mathbf{a}_{L_n} \in \{0,1\}^{L_n} : \sum_{\ell=1}^{L_n} a_\ell \geqslant \lceil \lambda L_n \rceil \right\}, \quad n = 2, 3, \dots.$$

Therefore we have

$$T_n(\mathcal{P}_{L_n}) = Z_n(L_n; \lceil \lambda L_n \rceil).$$

With $Z_n(L_n; \lceil \lambda L_n \rceil) = T \subset V_n$ given, Lemma 4.4 yields

$$\begin{aligned} & \mathbb{P}\left[\, Z_n(L_n; \lceil \lambda L_n \rceil) \text{ is connected } \; | Z_n(L_n; \lceil \lambda L_n \rceil) = T \right] \\ \geqslant & \; 1 - \sum_{r=1}^{\frac{|T|}{2}} \left(|T| e^{-B(\mathcal{P}_{L_n}) \frac{|T|}{2}} \right)^r . \end{aligned} \tag{4.37}$$

Now, assume $|T| \geqslant c\mathbb{E}\left[|W_n(L_n; \lceil \lambda L_n \rceil)| \right]$ for some c in $(0,1)$. As a result, for n sufficiently large and with

$$B(\mathcal{P}_{L_n}) = \beta^{\lambda L_n} \gamma^{(1-\lambda) L_n},$$

the inequality (4.28) yields

$$\begin{aligned} \frac{|T|}{2} B(\mathcal{P}_{L_n}) & \geqslant \frac{c}{2} \mathbb{E}\left[|W_n(L_n; \lceil \lambda L_n \rceil)| \right] \beta^{\lambda L_n} \gamma^{(1-\lambda) L_n} \\ & \geqslant cn^{(1+\rho_n \ln \Gamma(0))} \frac{k^*(\lambda; \mu(1))}{2\sqrt{2\pi\lambda(1-\lambda)L_n}}. \end{aligned} \tag{4.38}$$

For any δ in $(0, 1 + \rho \ln \Gamma(0))$, there exists a positive integer $N = N(\delta)$ such that

$$\begin{aligned} \frac{|T|}{2} B(\mathcal{P}_{L_n}) & = \frac{|T|}{2} \beta^{\lambda L_n} \gamma^{(1-\lambda) L_n} \\ & \geqslant n^\delta, \qquad n \geqslant N \end{aligned}$$

Fix δ in $(0, 1 + \rho \ln \Gamma(0))$. For sufficiently large n, noting that

$$|T| e^{-B(\mathcal{P}_{L_n}) \frac{|T|}{2}} \leqslant e^{\ln n - n^\delta} < 1,$$

we get

$$\begin{aligned}\sum_{r=1}^{\frac{|T|}{2}}\left(|T|e^{-B(\mathcal{P}_{L_n})\frac{|T|}{2}}\right)^r &\leqslant \sum_{r=1}^{\frac{|T|}{2}}\left(e^{\ln n - n^\delta}\right)^r \\ &= \frac{e^{\ln n-n^\delta}\left(1-e^{\frac{|T|}{2}\left(\ln n-n^\delta\right)}\right)}{1-e^{\ln n-n^\delta}} \\ &\leqslant \frac{e^{\ln n-n^\delta}}{1-e^{\ln n-n^\delta}}. \end{aligned} \tag{4.39}$$

It is now plain from (4.37) and (4.39) that there exists N in $\mathbb{N}_0$ such that

$$\begin{aligned}&\mathbb{P}\Big[\ Z_n(L_n;\lceil\lambda L_n\rceil) \text{ is connected } \Big| Z_n(L_n;\lceil\lambda L_n\rceil) = T\Big] \\ \geqslant\ & 1-\frac{e^{\ln n-n^\delta}}{1-e^{\ln n-n^\delta}}, \qquad |T|\geqslant c\mathbb{E}\left[|W_n(L_n;\lceil\lambda L_n\rceil)|\right],\ n\geqslant N.\end{aligned}$$

But we have

$$\begin{aligned}&\mathbb{P}\Big[\,[\ Z_n(L_n;\lceil\lambda L_n\rceil) \text{ is connected }]\cap[\ |Z_n(L_n;\lceil\lambda L_n\rceil)|\geqslant c\mathbb{E}\left[|W_n(L_n;\lceil\lambda L_n\rceil)|\right]\ ]\Big] \\ =\ & \mathbb{P}\Big[\ |Z_n(L_n;\lceil\lambda L_n\rceil)|\geqslant c\mathbb{E}\left[|W_n(L_n;\lceil\lambda L_n\rceil)|\right]\ \Big] \\ & \times\mathbb{P}\Big[\ Z_n(L_n;\lceil\lambda L_n\rceil) \text{ is connected } \Big| Z_n(L_n;\lceil\lambda L_n\rceil)\ \geqslant c\mathbb{E}\left[|W_n(L_n;\lceil\lambda L_n\rceil)|\right]\Big] \\ \geqslant\ & \left(1-\frac{e^{\ln n-n^\delta}}{1-e^{\ln n-n^\delta}}\right)\mathbb{P}\left[|Z_n(L_n;\lceil\lambda L_n\rceil)|\geqslant c\mathbb{E}\left[|W_n(L_n;\lceil\lambda L_n\rceil)|\right]\right]\end{aligned} \tag{4.40}$$

with

$$\lim_{n\to\infty}\mathbb{P}\left[|Z_n(L_n;\lceil\lambda L_n\rceil)|\geqslant c\mathbb{E}\left[|W_n(L_n;\lceil\lambda L_n\rceil)|\right]\right] = 1$$

from Lemma 4.5 and

$$\lim_{n\to\infty}\left(1-\frac{e^{\ln n-n^\delta}}{1-e^{\ln n-n^\delta}}\right) = 1.$$

Therefore

$$\begin{aligned}&\liminf_{n\to\infty}\mathbb{P}\left[\begin{array}{l}[\ Z_n(L_n;\lceil\lambda L_n\rceil) \text{ is connected }] \\ \cap\Big[|Z_n(L_n;\lceil\lambda L_n\rceil)|\geqslant c\mathbb{E}\left[|W_n(L_n;\lceil\lambda L_n\rceil)|\right]\Big]\end{array}\right] \\ \geqslant\ & 1.\end{aligned}$$

The desire result readily follows by leveraging the trivial bound that

$$
\begin{aligned}
& \mathbb{P}\left[\left[\ Z_n(L_n; \lceil \lambda L_n \rceil) \text{ is connected }\right] \cap \left[|Z_n(L_n; \lceil \lambda L_n \rceil)| \geqslant c\mathbb{E}\left[|W_n(L_n; \lceil \lambda L_n \rceil)|\right]\right]\right] \\
\leqslant\ & \mathbb{P}\left[\ Z_n(L_n; \lceil \lambda L_n \rceil) \text{ is connected }\right] \\
\leqslant\ & 1.
\end{aligned}
$$

■

4.5 A proof of Theorem 4.1

From Lemma 4.7, we know that the subgraph induced by the random node set

$Z_n(L_n; \lceil \lambda L_n \rceil) \subseteq V_n$ is a.a.s. connected. This random subset serves as a core component in $\mathbb{M}(n; L_n)$ in the sense of Criterion 1. To complete the proof for the one law of connectivity, we only need to show that the random node set $Z_n(L_n; \lceil \lambda L_n \rceil)$ satisfies the condition in Corollary 4.3, namely

$$
\lim_{n\to\infty} \sum_{u \in V_n} \mathbb{E}\left[\mathbf{1}[u \in V_n \backslash Z_n(L_n; \lceil \lambda L_n \rceil)] \prod_{\substack{v \in Z_n(L_n; \lceil \lambda L_n \rceil), \\ v \neq u}} \left(1 - Q_{L_n}(\mathbf{A}_{L_n}(u), \mathbf{A}_{L_n}(v))\right) \right] = 0. \tag{4.41}
$$

Fix $n = 2, 3, \ldots$. For any node u in V_n, under the assumption $\alpha > \beta > \gamma$, it follows that

$$
\begin{aligned}
& \prod_{v \in Z_n(L_n;\lceil \lambda L_n \rceil),\ v \neq u} \left(1 - Q_{L_n}(\mathbf{A}_{L_n}(u), \mathbf{A}_{L_n}(v))\right) \\
= & \prod_{v \in Z_n(L_n;\lceil \lambda L_n \rceil),\ v \neq u} \Big(1 - \alpha^{\sum_{\ell=1}^{L_n} A_\ell(u) A_\ell(v)} \beta^{\sum_{\ell=1}^{L_n} ((1-A_\ell(u)) A_\ell(v)} \\
& \times \beta^{(1-A_\ell(v)) A_\ell(u))} \gamma^{\sum_{\ell=1}^{L_n} (1-A_\ell(u))(1-A_\ell(v))}\Big) \\
= & \prod_{v \in Z_n(L_n;\lceil \lambda L_n \rceil),\ v \neq u} \left(1 - \left(\alpha^{\sum_{\ell=1}^{L_n} A_\ell(u)} \beta^{\sum_{\ell=1}^{L_n} (1-A_\ell(u))}\right)^{\sum_{\ell=1}^{L_n} A_\ell(v)}\right. \\
& \left. \times \left(\beta^{\sum_{\ell=1}^{L_n} A_\ell(u)} \gamma^{\sum_{\ell=1}^{L_n} (1-A_\ell(u))}\right)^{\sum_{\ell=1}^{L_n} (1-A_\ell(v))}\right) \\
\leqslant & \prod_{v \in Z_n(L_n;\lceil \lambda L_n \rceil),\ v \neq u} \left(1 - \beta^{\sum_{\ell=1}^{L_n} A_\ell(v)} \gamma^{\sum_{\ell=1}^{L_n} (1-A_\ell(v))}\right) \\
\leqslant & \prod_{v \in Z_n(L_n;\lceil \lambda L_n \rceil),\ v \neq u} \left(1 - \beta^{\lambda L_n} \gamma^{(1-\lambda) L_n} \left(\frac{\beta}{\gamma}\right)^{\sum_{\ell=1}^{L} A_\ell(v) - \lambda L_n}\right) \\
\leqslant & \prod_{v \in Z_n(L_n;\lceil \lambda L_n \rceil),\ v \neq u} \left(1 - \beta^{\lambda L_n} \gamma^{(1-\lambda) L_n}\right) \qquad (4.42)
\end{aligned}
$$

since v in $Z_n(L_n; \lceil \lambda L_n \rceil)$ implies $\sum_{\ell=1}^{L} A_\ell(v) - \lambda L_n > 0$. Substituting (4.42) into the condition (4.41) gives

$$
\begin{aligned}
& \sum_{u=1}^{n} \mathbb{E}\left[\mathbf{1}[u \in V_n \backslash Z_n(L_n; \lceil \lambda L_n \rceil)] \prod_{v \in Z_n(L_n;\lceil \lambda L_n \rceil),\ v \neq u} \left(1 - Q_{L_n}(\mathbf{A}_{L_n}(u), \mathbf{A}_{L_n}(v))\right)\right] \\
\leqslant & \sum_{u=1}^{n} \mathbb{E}\left[\mathbf{1}[u \in V_n \backslash Z_n(L_n; \lceil \lambda L_n \rceil)] \prod_{v \in Z_n(L_n;\lceil \lambda L_n \rceil),\ v \neq u} (1 - \beta^{\lambda L_n} \gamma^{(1-\lambda) L_n})\right] \\
\leqslant & \sum_{u=1}^{n} \mathbb{E}\left[\prod_{v \in V_n, v \neq u} \left(1 - \beta^{\lambda L_n} \gamma^{(1-\lambda) L_n} \mathbf{1}[v \in Z_n(L_n; \lceil \lambda L_n \rceil)]\right)\right] \\
= & \ n \prod_{v=2}^{n} \mathbb{E}\left[\left(1 - \beta^{\lambda L_n} \gamma^{(1-\lambda) L_n} \mathbf{1}[v \in Z_n(L_n; \lceil \lambda L_n \rceil)]\right)\right] \\
= & \ n \left(1 - \beta^{\lambda L_n} \gamma^{(1-\lambda) L_n} \mathbb{P}\left[1 \in Z_n(L_n; \lceil \lambda L_n \rceil)\right]\right)^{n-1}. \qquad (4.43)
\end{aligned}
$$

By virtue of Lemma 2.4, it is plain that

$$\begin{aligned}
& n\left(1-\beta^{\lambda L_n}\gamma^{(1-\lambda)L_n}\mathbb{P}\left[1\in Z_n(L_n;\lceil\lambda L_n\rceil)\right]\right)^{n-1} \\
\leqslant\; & ne^{-(n-1)\beta^{\lambda L_n}\gamma^{(1-\lambda)L_n}\mathbb{P}[1\in Z_n(L_n;\lceil\lambda L_n\rceil)]} \\
=\; & e^{\ln n-\frac{n-1}{n}\beta^{\lambda L_n}\gamma^{(1-\lambda)L_n}\mathbb{E}[|Z_n(L_n;\lceil\lambda L_n\rceil)|]} \qquad (4.44)
\end{aligned}$$

since $n\mathbb{P}\left[1\in Z_n(L_n;\lceil\lambda L_n\rceil)\right]=\mathbb{E}\left[|Z_n(L_n;\lceil\lambda L_n\rceil)|\right]$. Recall from (4.18) and (4.28) that

$$\begin{aligned}
\beta^{\lambda L_n}\gamma^{(1-\lambda)L_n}\mathbb{E}\left[|Z_n(L_n;\lceil\lambda L_n\rceil)|\right] &\geqslant \beta^{\lambda L_n}\gamma^{(1-\lambda)L_n}\mathbb{E}\left[|W_n(L_n;\lceil\lambda L_n\rceil)|\right] \\
&\geqslant n^{(1+\rho_n\ln\Gamma(0))}\frac{k^*(\lambda;\mu(1))}{\sqrt{2\pi\lambda(1-\lambda)L_n}}
\end{aligned}$$

for sufficiently large n. Under the condition $1+\rho\ln\Gamma(0)>0$, fix some δ in $(0,1+\rho\ln\Gamma(0))$. There exists a positive integer $N=N(\delta)$ such that

$$n^{(1+\rho_n\ln\Gamma(0))}\frac{k^*(\lambda;\mu(1))}{\sqrt{2\pi\lambda(1-\lambda)L_n}}>n^\delta,\quad n\geqslant N.$$

It is plain that

$$\lim_{n\to\infty}e^{\ln n-\frac{n-1}{n}n^\delta}=0,\qquad \delta>0,$$

and therefore

$$\begin{aligned}
& \lim_{n\to\infty}e^{\ln n-\frac{n-1}{n}\beta^{\lambda L_n}\gamma^{(1-\lambda)L_n}\mathbb{E}[|Z_n(L_n;\lceil\lambda L_n\rceil)|]} \\
\leqslant\; & \lim_{n\to\infty}e^{\ln n-\frac{n-1}{n}n^\delta} \\
=\; & 0.
\end{aligned}$$

As a result, the left hand side of (4.44) converges to 0 when n grows unboundedly large, and the condition in Corollary 4.3 is therefore satisfied. This completes the proof of Theorem 4.1 . ■

Chapter 5

Triadic Closure

In the previous two chapters, the zero-one law for the absence of isolated nodes and the zero-one law for connectivity for MAG were established. Now we turn our attention to two other important properties of MAGs, the number of triangles and triadic closure.

Triangles, or cliques formed by three nodes are one of the most important building blocks of social networks. They form the basic structure that reflects transitivity, where the number of triangles is closely related to the clustering properties of the graph. In this chapter, we discuss triadic closure, the emergence of triangles and the limiting behavior of the total clustering coefficient in MAGs.

5.1 The property of triadic closure

Our discussion starts with a widely studied phenomenon, known as triadic closure, which depicts the tendency of closing a triad to form a triangle. In the context of social networks, if A is familiar with both B and C, then B and C are more likely to know each other. This can be formalized by requiring

$$\mathbb{P}[w \sim v | u \sim w, u \sim v] > \mathbb{P}[w \sim v], \qquad \text{distinct } u, v, w \in V_n,$$

i.e. by sharing a common neighbor (acquaintance), two nodes are more likely to be themselves adjacent.

We claim that $\mathbb{M}(n; L)$ has the property of triadic closure. This is the content of Theorem 5.1.

Theorem 5.1. *Assume $\alpha > \beta > \gamma$. The following two strict inequalities*

$$\mathbb{P}[u \sim_L v | u \sim_L w, v \sim_L w] > \mathbb{P}[u \sim_L v], \quad \textit{distinct } u, v, w \in V_n \tag{5.1}$$

and

$$\mathbb{P}[u \sim_L v, v \sim_L w, u \sim_L w,] > \mathbb{P}[u \sim_L v]^3, \quad \textit{distinct } u, v, w \in V_n. \tag{5.2}$$

hold for $n = 3, 4, \ldots$ and $L = 1, 2, \ldots$.

As stated in (2.10), the link variables have the form

$$\chi_L(u, v) = \mathbf{1}\big[U(u, v) \leqslant Q_L(\boldsymbol{A}_L(u), \boldsymbol{A}_L(v))\big], \qquad \begin{array}{c} u, v \in V_n \\ u \neq v. \end{array} \tag{5.3}$$

For each pair of distinct nodes u and v in V_n, u being L-adjacent to v ($u \sim_L v$) is equivalent to the event $\chi_L(u, v) = 1$, whence we have

$$\mathbb{P}[u \sim_L v] = \mathbb{E}\left[\chi_L(u, v)\right],$$

$$\mathbb{P}[u \sim_L v,\ v \sim_L w] = \mathbb{E}\left[\chi_L(u, v)\chi_L(v, w)\right]$$

and

$$\mathbb{P}[u \sim_L v, v \sim_L w, u \sim_L w] = \mathbb{E}\left[\chi_L(u,v)\chi_L(v,w)\chi_L(u,w)\right]$$

for distinct u, v and w in V_n.

Therefore, establishing (5.1) and (5.2) is equivalent to showing

$$\mathbb{E}\left[\chi_L(u,v), \chi_L(v,w)\right] > \mathbb{E}\left[\chi_L(u,v)\right]\mathbb{E}\left[\chi_L(v,w)\right]$$

and

$$\mathbb{E}\left[\chi_L(u,v)\chi_L(v,w)\chi_L(u,w)\right] > \mathbb{E}\left[\chi_L(u,v)\right]\mathbb{E}\left[\chi_L(v,w)\chi_L(u,w)\right],$$

respectively. As these two inequalities are reminiscent of the notion of association of rvs [21], it is not surprising that we rely on the following technical facts to prove Theorem 5.1.

According to the definition of association, the $\mathbb{R}-$valued rvs $X_1, \ldots, X_m$ for some positive integer m are associated iff

$$\mathbb{E}\left[f(\mathbf{X})g(\mathbf{X})\right] \geqslant \mathbb{E}\left[f(\mathbf{X})\right]\mathbb{E}\left[g(\mathbf{X})\right]$$

for all nondecreasing mappings $f, g : \mathbb{R}^m \to \mathbb{R}$ for which $\mathbb{E}\left[f(\mathbf{X})g(\mathbf{X})\right]$, $\mathbb{E}\left[f(\mathbf{X})\right]$ and $\mathbb{E}\left[g(\mathbf{X})\right]$ exist where we use the notation $\mathbf{X} = (X_1, \ldots, X_m)$.

Lemma 5.2. *For positive integers n and L, if the rvs in the triangular array*

$$\{\chi_L(u,v),\ u, v \in V_n,\ u < v\}$$

are associated, then the two inequalities

$$\mathbb{E}\left[\chi_L(1,2)\chi_L(2,3)\chi_L(1,3)\right] \geqslant \mathbb{E}\left[\chi_L(1,2)\right]\mathbb{E}\left[\chi_L(2,3)\chi_L(1,3)\right] \tag{5.4}$$

and

$$\mathbb{E}\left[\chi_L(1,2)\chi_L(2,3)\right] \geqslant \mathbb{E}\left[\chi_L(1,2)\right]\mathbb{E}\left[\chi_L(2,3)\right] \tag{5.5}$$

hold.

Proof. Here we use the definition of association with $\mathbf{x} = \{x_{u,v},\ u,v \in V_n,\ u < v\}$ so that $m = \frac{n(n-1)}{2}$ for some integer n. We define two mappings $h_1, h_2, h_3 : \mathbb{R}^{\frac{n(n-1)}{2}} \to \mathbb{R}$ by

$$h_1(\mathbf{x}) \equiv x_{1,2},$$

$$h_2(\mathbf{x}) \equiv x_{1,3}^+ x_{2,3}^+$$

and

$$h_3(\mathbf{x}) \equiv x_{2,3}.$$

respectively, where $x^+ = \max(0,x)$ for x in $\mathbb{R}$.

If we set $\boldsymbol{\chi}_L = \{\chi_L(u,v),\ u,v \in V_n,\ u < v\}$, then it is plain that

$$h_1(\boldsymbol{\chi}_L) = \chi_L(1,2),$$

$$h_2(\boldsymbol{\chi}_L) = \chi_L(1,3)^+\chi_L(2,3)^+ = \chi_L(1,3)\chi_L(2,3)$$

and

$$h_3(\boldsymbol{\chi}_L) = \chi_L(2,3),$$

respectively, since all rvs in $\boldsymbol{\chi}_L$ are $\{0,1\}$-valued. By the association of the rvs in $\boldsymbol{\chi}$, we get the two inequalities

$$\mathbb{E}\left[\chi_L(1,2)\chi_L(2,3)\chi_L(1,3)\right] \geqslant \mathbb{E}\left[\chi_L(1,2)\right]\mathbb{E}\left[\chi_L(1,3)\chi_L(2,3)\right]$$

and

$$\mathbb{E}\left[\chi_L(1,2)\chi_L(2,3)\right] \geqslant \mathbb{E}\left[\chi_L(1,2)\right]\mathbb{E}\left[\chi_L(2,3)\right].$$

■

5.2 Association of rvs

Thus, in order to make use of Lemma 5.2, we need to prove that the rvs in the triangular array $\{\chi_L(u,v),\ u,v \in V_n,\ u < v\}$ form a collection of associated rvs.

Lemma 5.3. *For each $n = 2, 3, \ldots$ and $L = 1, 2, \ldots,$ the rvs*

$$\{\chi_L(u,v),\ u,v \in V_n,\ u < v\}$$

are associated if $\alpha > \beta > \gamma$ or $\alpha < \beta < \gamma$.

This lemma will be established with the help of the following two lemmas.

Lemma 5.4. *For some positive integer m, let $U_1, \ldots, U_m$ and $P_1, \ldots, P_m$ be two independent collections of rvs. It is further assumed that*

1. *The rvs $U_1, \ldots, U_m$ are* i.i.d. *rvs, each of which is uniformly distributed on the interval $(0,1)$.*

2. *The rvs $P_1, \ldots, P_m$ are $[0,1]-$valued rvs.*

We set

$$X_k = \mathbf{1}\big[U_k \leqslant P_k\big],\ k = 1, \ldots, m. \tag{5.6}$$

If the rvs $P_1, \ldots, P_m$ form a collection of associated rvs, then the rvs $X_1, \ldots, X_m$ also form a collection of associated rvs.

Proof. Set

$$\mathbf{X} = (X_1, \ldots, X_m) \text{ and } \mathbf{P} = (P_1, \ldots, P_m).$$

When we need to explicitly address the relationship between $\mathbf{X}$ and $\mathbf{P}$, we write

$$\mathbf{X} = \mathbf{X}(\mathbf{P}) = \big(X_1(P_1), \ldots, X_m(P_m)\big),$$

where

$$X_k(p_k) = \mathbf{1}\big[U_k \leqslant p_k\big], \quad p_k \in [0,1],\ k = 1, \ldots, m.$$

With non-decreasing mappings $f, g : \mathbb{R}^m \to \mathbb{R}$, we need to show that

$$\mathbb{E}\left[f(\mathbf{X})g(\mathbf{X})\right] \geqslant \mathbb{E}\left[f(\mathbf{X})\right]\mathbb{E}\left[g(\mathbf{X})\right]. \tag{5.7}$$

Under the enforced assumptions, we have

$$\begin{aligned}\mathbb{E}\left[f(\mathbf{X})g(\mathbf{X})\right] &= \mathbb{E}\left[\mathbb{E}\left[f(\mathbf{X}(\mathbf{P}))g(\mathbf{X}(\mathbf{P}))|\mathbf{P}\right]\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[f(\mathbf{X}(\mathbf{p}))g(\mathbf{X}(\mathbf{p}))\right]_{\mathbf{p}=\mathbf{P}}\right], \quad \mathbf{p} \in [0,1]^m. \end{aligned} \tag{5.8}$$

With $\mathbf{p}$ in $[0,1]^m$ fixed, the rvs $X_1(p_1), \ldots, X_m(p_m)$ are mutually independent and independent of $\mathbf{P}$, hence associated [21, Thm 2.1]. Therefore

$$\mathbb{E}\left[f(\mathbf{X}(\mathbf{p}))g(\mathbf{X}(\mathbf{p}))\right] \geqslant \mathbb{E}\left[f(\mathbf{X}(\mathbf{p}))\right]\mathbb{E}\left[g(\mathbf{X}(\mathbf{p}))\right]. \tag{5.9}$$

Now consider the two mappings $\widehat{f} : [0,1]^m \to \mathbb{R}$ and $\widehat{g} : [0,1]^m \to \mathbb{R}$ given by

$$\widehat{f}(\mathbf{p}) = \mathbb{E}\left[f(\mathbf{X}(\mathbf{p}))\right], \quad \mathbf{p} \in [0,1]^m \tag{5.10}$$

and

$$\widehat{g}(\mathbf{p}) = \mathbb{E}\left[g(\mathbf{X}(\mathbf{p}))\right], \quad \mathbf{p} \in [0,1]^m. \tag{5.11}$$

With this notation, the inequality (5.9) becomes

$$\mathbb{E}\left[f(\mathbf{X}(\mathbf{p}))g(\mathbf{X}(\mathbf{p}))\right] \geqslant \widehat{f}(\mathbf{p})\widehat{g}(\mathbf{p}). \tag{5.12}$$

It now follows from (5.8) that

$$\begin{aligned} \mathbb{E}\left[f(\mathbf{X})g(\mathbf{X})\right] &= \mathbb{E}\left[\mathbb{E}\left[f(\mathbf{X}(\mathbf{P}))g(\mathbf{X}(\mathbf{P}))|\mathbf{P}\right]\right] \\ &\geqslant \mathbb{E}\left[\widehat{f}(\mathbf{P})\widehat{g}(\mathbf{P})\right]. \end{aligned} \tag{5.13}$$

The mappings $\mathbf{p} \to \widehat{f}(\mathbf{p})$ and $\mathbf{p} \to \widehat{g}(\mathbf{p})$ are non-decreasing in $\mathbf{p}$, a property inherited from the fact that the mappings $f, g : \mathbb{R}^m \to \mathbb{R}$ are themselves non-decreasing. The fact that the rvs $P_1, \ldots, P_m$ are associated immediately implies

$$\mathbb{E}\left[\widehat{f}(\mathbf{P})\widehat{g}(\mathbf{P})\right] \geqslant \mathbb{E}\left[\widehat{f}(\mathbf{P})\right]\mathbb{E}\left[\widehat{g}(\mathbf{P})\right] \tag{5.14}$$

with

$$\mathbb{E}\left[\widehat{f}(\mathbf{P})\right] = \mathbb{E}\left[\mathbb{E}\left[f(\mathbf{X}(\mathbf{p}))\right]_{\mathbf{p}=\mathbf{P}}\right] = \mathbb{E}\left[f(\mathbf{X})\right]$$

and

$$\mathbb{E}\left[\widehat{g}(\mathbf{P})\right] = \mathbb{E}\left[\mathbb{E}\left[g(\mathbf{X}(\mathbf{p}))\right]_{\mathbf{p}=\mathbf{P}}\right] = \mathbb{E}\left[g(\mathbf{X})\right].$$

The desired result (5.7) now follows. ■

It is easy to see that (5.3) is of the form (5.6) with $m = \frac{n(n-1)}{2}$ and

$$\mathbf{P} = \{Q_L(\boldsymbol{A}_L(u), \boldsymbol{A}_L(v)),\ u, v \in V_n,\ u < v\}.$$

By virtue of Lemma 5.4, the rvs $\{\chi_L(u, v),\ u, v \in V_n,\ u < v\}$ are therefore associated if we show that the rvs $\{Q_L(\boldsymbol{A}_L(u), \boldsymbol{A}_L(v)),\ u, v \in V_n,\ u < v\}$ are associated.

Lemma 5.5. *For $n = 3, 4, \ldots$ and $L = 1, 2, \ldots$, if either $\alpha > \beta > \gamma$ or $\alpha < \beta < \gamma$, then the rvs*

$$\{Q_L(\boldsymbol{A}_L(u), \boldsymbol{A}_L(v)),\ u, v \in V_n,\ u < v\}$$

are associated.

Proof. Fix $n = 3, 4, \ldots$ and $L = 1, 2, \ldots$. Recall that, for $u, v \in V_n,\ u < v$, we have

$$Q_L(\boldsymbol{A}_L(u), \boldsymbol{A}_L(v)) = \prod_{\ell=1}^{L} q(A_\ell(u), A_\ell(v)). \tag{5.15}$$

For each pair of (u, v) in V_n where $u < v$, the rv $Q_L(\boldsymbol{A}_L(u), \boldsymbol{A}_L(v))$ is non-decreasing in the non-negative rvs $\{q(A_\ell(u), A_\ell(v)),\ \ell = 1, \ldots, L\}$. If the rvs $\{q(A_\ell(u), A_\ell(v)),\ \ell = 1, \ldots, L,\ u, v \in V_n,\ u < v\}$ are associated, then the target rvs $\{Q_L(\boldsymbol{A}_L(u), \boldsymbol{A}_L(v)),\ u, v \in V_n,\ u < v\}$ are associated since association is preserved under this non-decreasing transformation [21, Property 4].

Furthermore, because the rvs $\{A_\ell(u),\ \ell = 1, \ldots, L,\ u \in V_n\}$ are mutually independent, the collections of rvs $\{q(A_\ell(u), A_\ell(v)),\ u, v \in V_n,\ u < v\}$ $\ell = 1, 2, \ldots, L$ form L mutually independent sets of rvs.

Now, it suffices to show that, for each $\ell = 1, \ldots, L$, the non-negative rvs

$$\{q(A_\ell(u), A_\ell(v)),\ u, v \in V_n,\ u < v\}$$

are associated since the union of independent sets of associated rvs is a set of associated rvs [21, Property 2].

For each $\ell = 1, \ldots, L$, the rvs $\{A_\ell(u),\ u \in V_n\}$ being mutually independent, hence associated [21, Thm 2.1], whence the rvs $\{q(A_\ell(u), A_\ell(v)),\ u, v \in V_n,\ u < v\}$

are associated since

$$q(a,b) = ab\alpha + (a(1-b) + b(1-a))\beta + (1-a)(1-b)\gamma, \ (a,b) \in [0,1]^2$$

is monotone.

Indeed, partial differentiation with respect to a and b give

$$\begin{aligned}\frac{\partial q(a,b)}{\partial a} &= b\alpha + ((1-b)-b)\beta - (1-b)\gamma \\ &= b(\alpha-\beta) + (1-b)(\beta-\gamma)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial q(a,b)}{\partial b} &= a\alpha + ((1-a)-a)\beta - (1-a)\gamma \\ &= a(\alpha-\beta) + (1-a)(\beta-\gamma),\end{aligned}$$

respectively. Both partial derivatives obviously being non-negative (resp. non-positive) when $\alpha > \beta > \gamma$ (resp $\alpha < \beta < \gamma$). It follows that $q(a,b)$ is monotonically increasing (resp. decreasing) on $[0,1]^2$ Lemma 5.5 is now readily established, so is Lemma 5.3. ■

5.3 Probability of forming a triangle

For $n = 2, 3, \ldots$ and $L = 1, 2, \ldots$, the rvs $\{\chi_L(u,v), \ u,v \in V_n, \ u < v\}$ being associated, it follows that both the inequalities

$$\mathbb{E}\left[\chi_L(1,2)\chi_L(2,3)\chi_L(1,3)\right] \geqslant \mathbb{E}\left[\chi_L(1,2)\right]\mathbb{E}\left[\chi_L(2,3)\chi_L(1,3)\right]$$

and

$$\mathbb{E}\left[\chi_L(1,2)\chi_L(2,3)\right] \geqslant \mathbb{E}\left[\chi_L(1,2)\right]\mathbb{E}\left[\chi_L(2,3)\right]$$

hold, or equivalently,

$$\mathbb{P}\left[1 \sim_L 2, 2 \sim_L 3, 1 \sim_L 3\right] \geqslant \mathbb{P}\left[1 \sim_L 2\right] \mathbb{P}\left[1 \sim_L 3, 2 \sim_L 3\right]$$

and

$$\mathbb{P}\left[1 \sim_L 2, 2 \sim_L 3\right] \geqslant \mathbb{P}\left[1 \sim_L 2\right] \mathbb{P}\left[2 \sim_L 3\right]$$

by virtue of Lemma 5.2.

To establish the two strict inequalities of (5.1) and (5.2), we need to show that

$$\mathbb{P}\left[1 \sim_L 2, 1 \sim_L 3, 2 \sim_L 3\right] \neq \mathbb{P}\left[1 \sim_L 2\right] \mathbb{P}\left[1 \sim_L 3, 2 \sim_L 3\right] \tag{5.16}$$

and

$$\mathbb{P}\left[1 \sim_L 2, 1 \sim_L 3, 2 \sim_L 3\right] \neq \mathbb{P}\left[1 \sim_L 2\right]^3 . \tag{5.17}$$

Since both inequalities involve the probability of forming a triangle, we proceed by computing this quantity.

Lemma 5.6. *For $n = 3, 4, \ldots,$ $L = 1, 2, \ldots$ and distinct u, v and w in V_n, we have*

$$\begin{aligned} & \mathbb{P}\left[u \sim_L v, u \sim_L w, v \sim_L w\right] \\ = \quad & \left(\mu(1)^3\alpha^3 + 3\mu(1)^2\mu(0)\alpha\beta^2 + 3\mu(1)\mu(0)^2\beta^2\gamma + \mu(0)^3\gamma^3\right)^L . \end{aligned} \tag{5.18}$$

Proof. Fix $n = 3, 4, \ldots,$ $L = 1, 2, \ldots, L$ and pick distinct $u,\ v,\ w$ from V_n. From the definition of the link variables in (5.3), we have

$$\mathbb{P}\left[u \sim_L v, u \sim_L w, v \sim_L w\right] = \mathbb{P}\begin{bmatrix} U(u,v) \leqslant Q_L(\mathbf{A}_L(u), \mathbf{A}_L(v)), \\ U(u,w) \leqslant Q_L(\mathbf{A}_L(u), \mathbf{A}_L(w)), \\ U(w,v) \leqslant Q_L(\mathbf{A}_L(w), \mathbf{A}_L(v)) \end{bmatrix}. \tag{5.19}$$

Since the rvs $\{U(u,v),\ u,v \in V_n,\ u < v\}$ are mutually independent and independent of the rvs $\{\mathbf{A}_L(u),\ u \in V_n\}$, the right hand side of (5.19) becomes

$$\begin{aligned}
&\mathbb{P}\big[u \sim_L v, u \sim_L w, v \sim_L w\big] \\
&= \mathbb{E}\big[Q_L(\mathbf{A}_L(u), \mathbf{A}_L(v))Q_L(\mathbf{A}_L(u), \mathbf{A}_L(w))Q_L(\mathbf{A}_L(w), \mathbf{A}_L(v))\big] \\
&= \mathbb{E}\left[\prod_{\ell=1}^{L} q(A_\ell(u), A_\ell(v)) \prod_{\ell=1}^{L} q(A_\ell(u), A_\ell(w)) \prod_{\ell=1}^{L} q(A_\ell(w), A_\ell(v))\right] \\
&= \prod_{\ell=1}^{L} \mathbb{E}\big[q(A_\ell(u), A_\ell(v))q(A_\ell(u), A_\ell(w))q(A_\ell(w), A_\ell(v))\big] \\
&= \prod_{\ell=1}^{L} \left(\mu(1)^3\alpha^3 + 3\mu(1)^2\mu(0)\alpha\beta^2 + 3\mu(1)\mu(0)^2\beta^2\gamma + \mu(0)^3\gamma^3\right) \\
&= \left(\mu(1)^3\alpha^3 + 3\mu(1)^2\mu(0)\alpha\beta^2 + 3\mu(1)\mu(0)^2\beta^2\gamma + \mu(0)^3\gamma^3\right)^L
\end{aligned} \tag{5.20}$$

as we make use of (5.15) and of the mutual independence of the rvs $\{A_\ell(u),\ \ell = 1, \ldots, L,\ u \in V_n\}$.

■

Noting that

$$\mathbb{P}\left[u \sim_L v\right] = \left(\mu(1)\Gamma(1) + \mu(0)\Gamma(0)\right)^L$$

and

$$\mathbb{P}\left[u \sim_L v, u \sim_L w\right] = \left(\mu(1)\Gamma(1)^2 + \mu(0)\Gamma(0)^2\right)^L$$

from (2.18) and (2.26), the desired results (5.16) and (5.17) now follow.

The proof of Theorem 5.1 is completed as we note that none of the quantities $\mathbb{P}\left[u \sim_L v\right]$, $\mathbb{P}\left[u \sim_L w, v \sim_L w\right]$ or $\mathbb{P}\left[u \sim_L v, u \sim_L w, v \sim_L w\right]$ depend on the particular choice of nodes in V_n.

■

5.4 The zero-one law for the existence of triangles

Having the expression for the probability of forming a triangle, it is natural to investigate the zero-one law for the emergence of triangles. To facilitate the analysis, with

$$\theta = (\mu(1), \alpha, \beta, \gamma),$$

we write

$$\begin{aligned} K(\theta) &= \mathbb{E}\left[q(A_\ell(1), A_\ell(2))q(A_\ell(1), A_\ell(3))q(A_\ell(2), A_\ell(3))\right] \\ &= \mu(1)^3\alpha^3 + 3\mu(1)^2\mu(0)\alpha\beta^2 + 3\mu(1)\mu(0)^2\beta^2\gamma + \mu(0)^3\gamma^3. \end{aligned} \tag{5.21}$$

Obviously, the quantity $K(\theta)$ is in $(0, 1)$. Moreover, the probability of forming a triangle can now be rewritten as

$$\mathbb{P}\left[u \sim_L v, u \sim_L w, v \sim_L w\right] = K(\theta)^L$$

for $n = 3, 4, \ldots,$, $L = 1, 2, \ldots$. and distinct u, v, w in V_n.

The zero-one law for the emergence of triangles is given in the following two theorems. We start with the zero law.

Theorem 5.7. *Assume $\alpha > \beta > \gamma$. For any $\rho-$admissible scaling $\mathtt{L} : \mathbb{N}_0 \to \mathbb{N}_0$ with $\rho > 0$, the zero law*

$$\lim_{n\to\infty} \mathbb{P}[\ \mathbb{M}(n, L_n) \textit{ contains triangles }] = 0$$

holds if

$$3 + \rho \ln K(\theta) < 0. \tag{5.22}$$

The one law takes a very similar form.

Theorem 5.8. *Assume $\alpha > \beta > \gamma$. For any $\rho-$admissible scaling $\mathtt{L} : \mathbb{N}_0 \to \mathbb{N}_0$ with $\rho > 0$, the one law*

$$\lim_{n\to\infty} \mathbb{P}[\ \mathbb{M}(n; L_n) \textit{ contains triangles }] = 1$$

holds if

$$3 + \rho \ln K(\theta) > 0. \tag{5.23}$$

We start by counting triangles. Pick positive integers $n = 3, 4, \ldots$ and $L = 1, 2, \ldots$. For distinct u, v and w in V_n, let $\xi_{n,L}(u, v, w)$ denote the indicator that a triangle having end points (u, v, w) exists in $\mathbb{M}(n; L)$, it is given by

$$\xi_{n,L}(u, v, w) = \mathbf{1}[u \sim_L v, u \sim_L w, v \sim_L w] = \chi_L(u, v)\chi_L(v, w)\chi_L(u, w). \tag{5.24}$$

It is obvious that the order of the triple does not matter, (e.g. $\xi_{n,L}(u, v, w) = \xi_{n,L}(v, u, w)$, etc.,) under the enforced symmetric condition for the link variables, namely

$$\chi_L(u,v) = \chi_L(v,u), \qquad \begin{array}{c} u,v \in V_n \\ \\ u \neq v. \end{array}$$

The number of distinct triangles in $\mathbb{M}(n;L)$ is given by the counting rv $T_n(L)$ defined by

$$T_n(L) = \sum_{1 \leqslant u < v < w \leqslant n} \xi_{n,L}(u,v,w). \tag{5.25}$$

Interest in this count variable stems from the observation that $\mathbb{M}(n;L)$ contains at least one triangle iff $T_n(L) > 0$, leading to the key relation

$$\mathbb{P}\left[\ \mathbb{M}(n;L) \text{ contains triangles }\right] = \mathbb{P}\left[T_n(L) > 0\right].$$

This fact will be used to establish the zero-one law for the emergence of triangles by leveraging easy bounds on the probability $\mathbb{P}\left[T_n(L) > 0\right]$ and $\mathbb{P}\left[T_n(L) = 0\right]$ in terms of the first and second moments of the rv $T_n(L)$ (as discussed in Section 2.4.4).

The following proposition is an immediate consequence from Lemma 5.6.

Proposition 5.9. *For $n = 3, 4, \ldots$ and $L = 1, 2, \ldots$, we have*

$$\mathbb{E}\left[T_n(L)\right] = \binom{n}{3} K(\theta)^L \tag{5.26}$$

with $K(\theta)$ defined at (5.21).

Proof. Fix $n = 2, 3, \ldots$ and $L = 1, 2 \ldots$. With the notation defined in (5.21), we have

$$\begin{aligned} \mathbb{E}\left[\xi_{n,L}(u,v,w)\right] &= \mathbb{E}\left[\mathbf{1}[u \sim_L v, u \sim_L w, v \sim_L w]\right] \\ &= \mathbb{P}\left[u \sim_L v, u \sim_L w, v \sim_L w\right] \\ &= K(\theta)^L \end{aligned} \tag{5.27}$$

for distinct $u,\ v,\ w$ in V_n. As a result, the expected number of triangles in $\mathbb{M}(n;L)$ is given by

$$\begin{aligned}\mathbb{E}\left[T_n(L)\right] &= \mathbb{E}\left[\sum_{1\leqslant u<v<w\leqslant n} \xi_{n,L}(u,v,w)\right] \\ &= \sum_{1\leqslant u<v<w\leqslant n} \mathbb{P}\left[u \sim_L v, u \sim_L w, v \sim_L w\right] \\ &= \sum_{1\leqslant u<v<w\leqslant n} K(\theta)^L \\ &= \binom{n}{3} K(\theta)^L \qquad (5.28)\end{aligned}$$

since there are $\binom{n}{3}$ distinct unordered subsets of V_n. ■

The second moment of the count variable (5.25) can also be evaluated in the usual manner.

Proposition 5.10. *For $n = 3, 4, \dots$ and $L = 1, 2, \dots,$ we have*

$$\begin{aligned}\mathbb{E}\left[T_n(L)^2\right] &= \binom{n}{3}\mathbb{P}[1 \sim_L 2, 2 \sim_L 3, 1 \sim_L 3] \\ &\quad + 3(n-3)\binom{n}{3}\mathbb{P}[1 \sim_L 2, 2 \sim_L 3, 1 \sim_L 3, 1 \sim_L 4, 2 \sim_L 4] \\ &\quad + 3\binom{n}{3}\binom{n-3}{2}\mathbb{P}[1 \sim_L 2, 2 \sim_L 3, 1 \sim_L 3, 1 \sim 4, 5 \sim_L 4, 1 \sim_L 5] \\ &\quad + \binom{n}{3}\binom{n-3}{3}\mathbb{P}[1 \sim_L 2, 2 \sim_L 3, 1 \sim_L 3]^2. \qquad (5.29)\end{aligned}$$

Additional details about Proposition 5.10 are given in Section C.

5.5 A proof of Theorem 5.7

The first step deals with the first moment of the count variable $T_n(L_n)$. A zero-infinity law is given as a straightforward consequence of Proposition 5.9.

Proposition 5.11. *Assume $\alpha > \beta > \gamma$. For any $\rho-$admissible scaling* $\mathtt{L} : \mathbb{N}_0 \to \mathbb{N}_0$ *with $\rho > 0$, we have*

$$\lim_{n\to\infty} \mathbb{E}[T_n(L_n)] = \begin{cases} \infty & \textit{if } 3 + \rho \ln K(\theta) > 0 \\ \\ 0 & \textit{if } 3 + \rho \ln K(\theta) < 0. \end{cases} \tag{5.30}$$

Proof. Fix $n = 3, 4, \ldots$. For any $\rho-$admissible scaling $\mathtt{L} : \mathbb{N}_0 \to \mathbb{N}_0$ with $\rho > 0$, recall from Proposition 5.9 that

$$\mathbb{E}\left[T_n(L_n)\right] = \binom{n}{3} K(\theta)^{L_n}.$$

We have

$$\begin{aligned} \mathbb{E}\left[T_n(L_n)\right] &= \frac{n(n-1)(n-2)}{3} K(\theta)^{L_n} \\ &= \frac{n(n-1)(n-2)}{n^3} \frac{1}{3} n^{3+\rho_n \ln K(\theta)} \end{aligned} \tag{5.31}$$

which diverges to ∞ (resp. converges to 0) if $3+\rho \ln K(\theta) > 0$ (resp. $3+\rho \ln K(\theta) < 0$) as we note that

$$\lim_{n\to\infty} \frac{n(n-1)(n-2)}{n^3} = 1.$$

■

Under the condition (5.22), Theorem 5.7 follows immediately upon applying the bound in (2.39) with $Z_n = T_n(L_n)$ here. ■

5.6 A proof of Theorem 5.8

To prove Theorem 5.8, we use the method of second moment already discussed in Section 2.4.4 through the bound

$$\frac{\mathbb{E}\left[T_n(L_n)\right]^2}{\mathbb{E}\left[T_n(L_n)^2\right]} \leqslant \mathbb{P}\left[T_n(L_n) > 0\right], \quad n = 3, 4, \ldots. \tag{5.32}$$

Theorem 5.8 will be established if we show that condition (5.23) implies

$$\limsup_{n\to\infty} \frac{\mathbb{E}\left[T_n(L_n)^2\right]}{\mathbb{E}\left[T_n(L_n)\right]^2} = 1. \tag{5.33}$$

Fix $n = 3, 4, \ldots$. For any $\rho-$admissible scaling $\mathtt{L} : \mathbb{N}_0 \to \mathbb{N}_0$ with $\rho > 0$, Proposition 5.10 yields

$$\begin{aligned}
\frac{\mathbb{E}[T_n(L_n)^2]}{\mathbb{E}[T_n(L_n)]^2} &= \frac{\left(1 + \binom{n-3}{3}\mathbb{E}\left[T_n(L_n)\right]\right)\mathbb{E}\left[T_n(L_n)\right]}{\mathbb{E}[T_n(L_n)]^2} \\
&\quad + \frac{3(n-3)\binom{n}{3}\mathbb{E}\left[\xi_{n,L_n}(1,2,3)\xi_{n,L_n}(1,2,4)\right]}{\mathbb{E}[T_n(L_n)]^2} \\
&\quad + \frac{3\binom{n}{3}\binom{n-3}{2}\mathbb{E}\left[\xi_{n,L_n}(1,2,3)\xi_{n,L_n}(1,4,5)\right]}{\mathbb{E}[T_n(L_n)]^2} \\
&= \frac{\binom{n-3}{3}}{\binom{n}{3}} + \frac{1}{\binom{n}{3}K(\theta)^{L_n}} \\
&\quad + \frac{3(n-3)\mathbb{P}[1 \sim_{L_n} 2, 2 \sim_{L_n} 3, 3 \sim_{L_n} 1, 4 \sim_{L_n} 1, 4 \sim_{L_n} 2]}{\binom{n}{3}K(\theta)^{2L_n}} \\
&\quad + \frac{3\binom{n-3}{2}\mathbb{P}[1 \sim_{L_n} 2, 2 \sim_{L_n} 3, 3 \sim_{L_n} 1, 4 \sim_{L_n} 1, 4 \sim_{L_n} 5, 1 \sim_{L_n} 5]}{\binom{n}{3}K(\theta)^{2L_n}}.
\end{aligned}$$

Under condition (5.23), we have $\lim_{n\to\infty} \binom{n}{3}K(\theta)^{L_n} = \infty$ by Proposition 5.11, implying

$$\lim_{n\to\infty} \frac{1}{\binom{n}{3}K(\theta)^{L_n}} = 0.$$

Since

$$\lim_{n\to\infty} \frac{\binom{n-3}{3}}{\binom{n}{3}} = \lim_{n\to\infty} \frac{(n-3)(n-4)(n-5)}{n(n-1)(n-2)} = 1,$$

Theorem 5.8 holds if we show that both ratios

$$\frac{3(n-3)\binom{n}{3}\mathbb{P}[1 \sim_{L_n} 2, 2 \sim_{L_n} 3, 3 \sim_{L_n} 1, 4 \sim_{L_n} 1, 4 \sim_{L_n} 2]}{\left(\binom{n}{3}\mathbb{P}[1 \sim_{L_n} 2, 2 \sim_{L_n} 3, 3 \sim_{L_n} 1]\right)^2} \tag{5.34}$$

and

$$\frac{3\binom{n}{3}\binom{n-3}{2}\mathbb{P}[1 \sim_{L_n} 2, 2 \sim_{L_n} 3, 3 \sim_{L_n} 1, 4 \sim_{L_n} 1, 4 \sim_{L_n} 5, 1 \sim_{L_n} 5]}{\left(\binom{n}{3}\mathbb{P}[1 \sim_{L_n} 2, 2 \sim_{L_n} 3, 3 \sim_{L_n} 1]\right)^2} \tag{5.35}$$

converge to 0 when n grows unboundedly large.

By arguments similar to those given in the proof of Lemma 5.6, we have

$$\begin{aligned}
& \mathbb{P}[1 \sim_{L_n} 2, 2 \sim_{L_n} 3, 3 \sim_{L_n} 1, 4 \sim_{L_n} 1, 4 \sim_{L_n} 2] \\
= \; & \mathbb{E}\big[Q_{L_n}(\mathbf{A}_{L_n}(1), \mathbf{A}_{L_n}(2)) \cdot Q_{L_n}(\mathbf{A}_{L_n}(1), \mathbf{A}_{L_n}(3)) \cdot Q_{L_n}(\mathbf{A}_{L_n}(2), \mathbf{A}_{L_n}(3)) \\
& \cdot Q_{L_n}(\mathbf{A}_{L_n}(1), \mathbf{A}_{L_n}(4)) \cdot Q_{L_n}(\mathbf{A}_{L_n}(2), \mathbf{A}_{L_n}(4))\big] \\
= \; & \mathbb{E}\big[q(A_1(1), A_1(2)) \cdot q(A_1(1), A_1(3)) \cdot q(A_1(2), A_1(3)) \\
& \cdot q(A_1(1), A_1(4)) \cdot q(A_1(2), A_1(4))\big]^{L_n} \\
= \; & \Phi(\theta)^{L_n} \qquad (5.36)
\end{aligned}$$

where we have set

$$\begin{aligned}
\Phi(\theta) \; = \; & \mathbb{E}\big[q(A_1(1), A_1(2)) \cdot q(A_1(1), A_1(3)) \cdot q(A_1(2), A_1(3)) \\
& \cdot q(A_1(1), A_1(4)) \cdot q(A_1(2), A_1(4))\big] \\
= \; & \mu(1)^4\alpha^5 + \mu(1)^3\mu(0)(2\alpha^3\beta^2 + 2\alpha^2\beta^3) + \mu(1)^2\mu(0)^2(\alpha\beta^4 + \beta^4\gamma + 4\alpha\beta^3\gamma) \\
& + \mu(1)\mu(0)^3(2\beta^3\gamma^2 + 2\beta^2\gamma^3) + \mu(0)^4\gamma^5. \qquad (5.37)
\end{aligned}$$

In the same manner, we obtain

$$
\begin{aligned}
& \mathbb{P}[1 \sim_{L_n} 2, 2 \sim_{L_n} 3, 3 \sim_{L_n} 1, 1 \sim_{L_n} 4, 1 \sim_{L_n} 5, 4 \sim_{L_n} 5] \\
= \;& \mathbb{E}\big[Q_{L_n}(\mathbf{A}_{L_n}(1), \mathbf{A}_{L_n}(2)) \cdot Q_{L_n}(\mathbf{A}_{L_n}(1), \mathbf{A}_{L_n}(3)) \cdot Q_{L_n}(\mathbf{A}_{L_n}(2), \mathbf{A}_{L_n}(3)) \\
& \cdot Q_{L_n}(\mathbf{A}_{L_n}(1), \mathbf{A}_{L_n}(4)) \cdot Q_{L_n}(\mathbf{A}_{L_n}(1), \mathbf{A}_{L_n}(5)) \cdot Q_{L_n}(\mathbf{A}_{L_n}(4), \mathbf{A}_{L_n}(5)\big] \\
= \;& \mathbb{E}\big[q(A_1(1), A_1(2)) \cdot q(A_1(1), A_1(3)) \cdot q(A_1(2), A_1(3)) \\
& \cdot q(A_1(1), A_1(4)) \cdot q(A_1(1), A_1(5)) \cdot q(A_1(4), A_1(5))\big]^{L_n} \\
= \;& \Psi(\theta)^{L_n} \qquad (5.38)
\end{aligned}
$$

where we have defined

$$
\begin{aligned}
\Psi(\theta) \;=\; & \mathbb{E}\big[q(A_1(1), A_1(2)) \cdot q(A_1(1), A_1(3)) \cdot q(A_1(2), A_1(3)) \\
& \cdot q(A_1(1), A_1(4)) \cdot q(A_1(1), A_1(5)) \cdot q(A_1(4), A_1(5))\big] \\
= \;& \mu(1)^5\alpha^6 + \mu(1)^4\mu(0)(4\alpha^4\beta^2 + \alpha^2\beta^4) \\
& + \mu(1)^3\mu(0)^2(2\alpha^3\beta^2\gamma + 4\alpha^2\beta^4 + 4\alpha\beta^4\gamma) \\
& + \mu(1)^2\mu(0)^3(4\alpha\beta^4\gamma + 2\alpha\beta^2\gamma^3 + 4\beta^4\gamma^2) \\
& + \mu(1)\mu(0)^4(4\beta^2\gamma^4 + \beta^4\gamma^2) + \mu(0)^5\gamma^6. \qquad (5.39)
\end{aligned}
$$

As a result, for $n = 3, 4, \ldots$, the ratio (5.34) becomes

$$
\begin{aligned}
& \frac{3(n-3)\Phi(\theta)^{L_n}}{\binom{n}{3}K(\theta)^{2L_n}} \\
= \;& \frac{3(n-3)n^{-4+4\left(1+\frac{1}{4}\rho_n \ln \Psi(\theta)\right)}}{\binom{n}{3}n^{-6+6\left(1+\frac{1}{3}\rho_n \ln K(\theta)\right)}} \\
= \;& \frac{3(n-3)n^{-4}}{\binom{n}{3}n^{-6}} n^{4\left(1+\frac{1}{4}\rho_n \ln \Phi(\theta)\right)-6\left(1+\frac{1}{3}\rho_n \ln K(\theta)\right)}. \qquad (5.40)
\end{aligned}
$$

Since

$$
\lim_{n\to\infty} \frac{3(n-3)n^{-4}}{\binom{n}{3}n^{-6}} = 18,
$$

the ratio (5.34) converges to 0 if

$$\limsup_{n\to\infty}\left(4\left(1+\frac{1}{4}\rho_n\ln\Phi(\theta)\right)-6\left(1+\frac{1}{3}\rho_n\ln K(\theta)\right)\right)<0. \tag{5.41}$$

Observe that

$$\begin{aligned}
& 4\left(1+\frac{1}{4}\rho_n\ln\Phi(\theta)\right)-6\left(1+\frac{1}{3}\rho_n\ln K(\theta)\right)\\
\leqslant\ & 4\left(1+\frac{1}{4}\rho_n\ln\Phi(\theta)\right)-4\left(1+\frac{1}{3}\rho_n\ln K(\theta)\right)\\
=\ & 4\left(\frac{1}{4}\rho_n\ln\Phi(\theta)-\frac{1}{3}\rho_n\ln K(\theta)\right)
\end{aligned}$$

for sufficiently large n, since $\lim\limits_{n\to\infty} 1+\frac{1}{3}\rho_n\ln K(\theta)=1+\frac{1}{3}\rho\ln K(\theta)>0$ under condition (5.23). It is plain that the inequality (5.41) holds if

$$\frac{1}{4}\ln\Phi(\theta)<\frac{1}{3}\ln K(\theta),$$

or equivalently,

$$\Phi(\theta)^3<K(\theta)^4.$$

However, it is plain that

$$\Phi(\theta)^3-K(\theta)^4<0$$

by direct (and painstaking) calculations.

Similarly, for $n=3,4,\ldots,$ the ratio (5.35) can be rewritten as

$$\begin{aligned}
& \frac{3\binom{n-3}{2}\Psi(\theta)^{L_n}}{\binom{n}{3}K(\theta)^{2L_n}}\\
=\ & \frac{3\binom{n-3}{2}n^{-5+5\left(1+\frac{1}{5}\rho_n\ln\Psi(\theta)\right)}}{\binom{n}{3}n^{-6+6(1+\frac{1}{3}\rho_n\ln K(\theta))}}\\
=\ & \frac{3\binom{n-3}{2}n^{-5}}{\binom{n}{3}n^{-6}}n^{5\left(1+\frac{1}{5}\rho_n\ln\Psi(\theta)\right)-6\left(1+\frac{1}{3}\rho_n\ln K(\theta)\right)}.
\end{aligned} \tag{5.42}$$

Since

$$\lim_{n\to\infty} \frac{3\binom{n-3}{2}n^{-5}}{\binom{n}{3}n^{-6}} = 9,$$

we have (5.35) converge to 0 if

$$\limsup_{n\to\infty} \left(5\left(1+\frac{1}{5}\rho_n \ln \Psi(\theta)\right) - 6\left(1+\frac{1}{3}\rho_n \ln K(\theta)\right)\right) < 0. \tag{5.43}$$

Observe that

$$\begin{aligned}
& 5\left(1+\frac{1}{5}\rho_n \ln \Psi(\theta)\right) - 6\left(1+\frac{1}{3}\rho_n \ln K(\theta)\right) \\
\leqslant\ & 5\left(1+\frac{1}{5}\rho_n \ln \Psi(\theta)\right) - 5\left(1+\frac{1}{3}\rho_n \ln K(\theta)\right) \\
=\ & 5\left(\frac{1}{5}\rho_n \ln \Psi(\theta) - \frac{1}{3}\rho_n \ln K(\theta)\right)
\end{aligned}$$

for sufficiently large n, since $\lim\limits_{n\to\infty} 1+\frac{1}{3}\rho_n \ln K(\theta) = 1+\frac{1}{3}\rho \ln K(\theta) > 0$ under condition (5.23). Simple calculations show that (5.43) holds if

$$\frac{1}{5}\ln \Psi(\theta) - \frac{1}{3}\ln K(\theta) < 0,$$

or equivalently,

$$\Psi(\theta)^3 < K(\theta)^5.$$

Indeed, a direct, though tedious, subtraction yields

$$\Psi(\theta)^3 - K(\theta)^5 < 0.$$

Collecting, we have that both ratios in (5.34) and (5.35) converge to 0 when n grows unboundedly large, implying

$$\lim_{n\to\infty} \frac{\mathbb{E}\left[T_n(L_n)\right]^2}{\mathbb{E}\left[T_n(L_n)^2\right]} = 1,$$

which serves as a lower bound to $\lim_{n\to\infty} \mathbb{P}[T_n(L_n) > 0]$. This completes the proof of the one law for the emergence of triangles by virtue of **??**eq:ZeroLawZ)-2.44 with $Z_n = T_n(L_n)$. ■

5.7 Convergence of the total clustering coefficient

Many real world social networks are known to exhibit high clustering (or transitivity) [4]. This phenomenon is informally characterized by the propensity of a node's neighbors to also be neighbors of each other. The clustering properties of a network are closely related to the property of triadic closure discussed in Section 5.1 and to the emergence of triangles discussed in Section 5.4. To that end, we find it interesting to look at clustering coefficients in MAGs and to investigate their limiting behavior when the number of nodes n grows unboundedly large.

While there are several clustering coefficients, we are specifically interested in the version defined by

$$C^*(\mathbb{M}(n; L)) = \begin{cases} \frac{T_n(L)}{\frac{1}{3}T_n^*(L)} & \text{if } T_n^*(L) > 0 \\ \\ 0 & \text{if } T_n^*(L) = 0 \end{cases} \tag{5.44}$$

where

$$T_n^*(L) = \sum_{u=1}^{n} \sum_{1 \leqslant v < w \leqslant n; v,w \neq u} \mathbf{1}[u \sim_L v, u \sim_L w]$$

is the number of triads (i.e., spanning trees consisting of three nodes) in $\mathbb{M}(n; L)$. The quantity $C^*(\mathbb{M}(n; L))$ is known as the total clustering coefficient and is expected to give a good approximation to the left hand side of (5.1). The next theorem formalizes this idea.

Theorem 5.12. *Assume $\alpha > \beta > \gamma$. For each $L = 1, 2, \ldots,$ we have*

$$C^*(\mathbb{M}(n; L)) \xrightarrow{P}_n \mathbb{P}[1 \sim_L 2 | 1 \sim_L 3, 2 \sim_L 3]. \tag{5.45}$$

We will prove Theorem 5.12 with the help of the following two auxiliary lemmas. The first one concerns the convergence of the number of triangles.

Lemma 5.13. *Assume $\alpha > \beta > \gamma$. For each $L = 1, 2, \ldots,$ we have*

$$\frac{1}{\binom{n}{3}} T_n(L) \xrightarrow{P}_n \mathbb{P}[1 \sim_L 2, 2 \sim_L 3, 1 \sim_L 3] > 0. \tag{5.46}$$

The second lemma deals with the convergence of the number of triads.

Lemma 5.14. *Assume $\alpha > \beta > \gamma$. For each $L = 1, 2, \ldots,$ we have*

$$\frac{1}{3\binom{n}{3}} T_n^*(L) \xrightarrow{P}_n \mathbb{P}[1 \sim_L 2, 2 \sim_L 3] > 0. \tag{5.47}$$

Theorem 5.12 can be established once these two lemmas are proved. The basic argument relies on the following continuity result for convergence in probability [22, Cor 2, p. 31]. For a sequence of $\mathbb{R}^d$-valued rvs $\{X_n,\ n = 1, 2, \ldots\}$ such that $X_n \xrightarrow{P}_n a$ for some a in $\mathbb{R}^d$, we have

$$h(X_n) \xrightarrow{P}_n h(a)$$

if $h : \mathbb{R}^d \to \mathbb{R}$ is a mapping that continuous at a.

From (5.44), we know that $C^*(\mathbb{M}(n;L))$ can be expressed as the ratio

$$C^*(\mathbb{M}(n;L)) = \begin{cases} \dfrac{\frac{1}{\binom{n}{3}}T_n(L)}{\frac{1}{3\binom{n}{3}}T_n^*(L)} & \text{if } T_n^*(L) > 0 \\ \\ 0 & \text{if } T_n^*(L) = 0. \end{cases} \tag{5.48}$$

We define the mapping $\mathbb{R} \times \mathbb{R}_+ \to \mathbb{R} : (x,y) \to h(x,y)$ by

$$h(x,y) = \begin{cases} \frac{x}{y} & \text{if } y > 0 \\ \\ 0 & \text{if } y = 0. \end{cases}$$

It is obvious that

$$C^*(\mathbb{M}(n;L)) = h\left(\frac{1}{\binom{n}{3}}T_n(L), \frac{1}{3\binom{n}{3}}T_n^*(L)\right), \quad n = 3, 4, \ldots.$$

Having the rvs $\left(\frac{1}{\binom{n}{3}}T_n(L), \frac{1}{3\binom{n}{3}}T_n^*(L)\right)$ converge in probability to $(\mathbb{P}[1 \sim_L 2, 2 \sim_L 3, 1 \sim_L 3], \mathbb{P}[1 \sim_L 2, 2 \sim_L 3])$ and $h(x,y)$ being continuous at $(\mathbb{P}[1 \sim_L 2, 2 \sim_L 3, 1 \sim_L 3], \mathbb{P}[1 \sim_L 2, 2 \sim_L 3])$, the convergence in Theorem 5.12 is now straightforward.

5.8 A proof of Lemma 5.13 and Lemma 5.14

The proof of Lemma 5.13 is given first:

Proof. Fix $n = 3, 4, \ldots$ and $L = 1, 2, \ldots$. Recall that the count variable $T_n(L)$ takes the form stated in (5.25). By Proposition 5.9, we have

$$\begin{aligned} \mathbb{E}\left[\frac{1}{\binom{n}{3}}T_n(L)\right] &= \frac{1}{\binom{n}{3}}\binom{n}{3}\mathbb{P}[1 \sim_L 2, 2 \sim_L 3, 1 \sim_L 3] \\ &= \mathbb{P}[1 \sim_L 2, 2 \sim_L 3, 1 \sim_L 3]. \end{aligned} \tag{5.49}$$

For arbitrary $\varepsilon > 0$, Chebyshev's inequality gives

$$\mathbb{P}\left[\left|\frac{1}{\binom{n}{3}}T_n(L) - \mathbb{P}[1 \sim_L 2, 2 \sim_L 3, 1 \sim_L 3]\right| > \varepsilon\right] \leqslant \frac{\operatorname{Var}\left(\frac{1}{\binom{n}{3}}T_n(L)\right)}{\varepsilon^2}.$$

The lemma is established if we show that $\lim_{n\to\infty} \operatorname{Var}\left(\frac{1}{\binom{n}{3}}T_n(L)\right) = 0$.

For each $n = 2, 3, \ldots$ we have

$$\operatorname{Var}\left(\frac{1}{\binom{n}{3}}T_n(L)\right) = \frac{1}{\binom{n}{3}^2}\left(\mathbb{E}[T_n(L)^2] - \mathbb{E}^2[T_n(L)]\right)$$

with

$$\begin{aligned}
\frac{1}{\binom{n}{3}^2}\mathbb{E}[T_n(L)^2] \quad = \quad & \frac{1}{\binom{n}{3}}\mathbb{P}[1 \sim_L 2, 2 \sim_L 3, 1 \sim_L 3] \\
& + \frac{3(n-3)}{\binom{n}{3}}\mathbb{P}[1 \sim_L 2, 2 \sim_L 3, 1 \sim_L 3, 1 \sim_L 4, 2 \sim_L 4] \\
& + \frac{3\binom{n-3}{2}}{\binom{n}{3}}\mathbb{P}[1 \sim_L 2, 2 \sim_L 3, 1 \sim_L 3, 1 \sim_L 4, 5 \sim_L 4, 1 \sim_L 5] \\
& + \frac{\binom{n-3}{3}}{\binom{n}{3}}\mathbb{P}[1 \sim_L 2, 2 \sim_L 3, 1 \sim_L 3]^2
\end{aligned} \tag{5.50}$$

by Proposition 5.10. As n goes to infinity, we have

$$\lim_{n\to\infty} \frac{\binom{n-3}{3}}{\binom{n}{3}} = \lim_{n\to\infty} \frac{(n-3)(n-4)(n-5)}{n(n-1)(n-2)} = 1$$

while all other coefficients in (5.50) converge to 0.

As a result, it follows that

$$\lim_{n\to\infty} \frac{1}{\binom{n}{3}^2}\mathbb{E}[T_n(L)^2] = \mathbb{P}[1 \sim_L 2, 2 \sim_L 3, 1 \sim_L 3]^2.$$

While it is plain from (5.49) that

$$\frac{1}{\binom{n}{3}^2}\big(\mathbb{E}[T_n(L)]\big)^2 = \mathbb{P}[1 \sim_L 2, 2 \sim_L 3, 1 \sim_L 3]^2$$

it follows that

$$\lim_{n\to\infty} \operatorname{Var}\left(\frac{1}{\binom{n}{3}} T_n(L)\right) = 0.$$

Thus, for $L = 1, 2, \ldots$ and arbitrary $\varepsilon > 0$, we have

$$\lim_{n\to\infty} \mathbb{P}\left[\left|\frac{1}{\binom{n}{3}} T_n(L) - \mathbb{P}[1 \sim_L 2, 2 \sim_L 3, 1 \sim_L 3]\right| > \varepsilon\right] = 0,$$

and the convergence (5.46) holds

■

The proof of Lemma 5.14 proceeds in a similar way. Before we proceed, we first give an expression for the second moment of the number of triads.

Proposition 5.15. *For $n = 3, 4, \ldots$ and $L = 1, 2, \ldots$, we have*

$$\begin{aligned}
\mathbb{E}\left[T_n^*(L)^2\right] \quad = \quad & 3\binom{n}{3}\mathbb{P}[1 \sim_L 2, 2 \sim_L 3] \\
& +6\binom{n}{3}\mathbb{P}[1 \sim_L 2, 2 \sim_L 3, 1 \sim_L 3] \\
& +3\binom{n}{3}\binom{n-3}{1}\mathbb{P}[1 \sim_L 2, 2 \sim_L 3, 2 \sim_L 4] \\
& +6\binom{n}{3}\binom{n-3}{1}\mathbb{P}[1 \sim_L 2, 2 \sim_L 3, 1 \sim_L 4, 2 \sim_L 4] \\
& +3\binom{n}{3}\binom{n-3}{1}\mathbb{P}[1 \sim_L 2, 2 \sim_L 3, 1 \sim_L 4, 3 \sim_L 4] \\
& +6\binom{n}{3}\binom{n-3}{1}\mathbb{P}[1 \sim_L 2, 2 \sim_L 3, 3 \sim_L 4] \\
& +3\binom{n}{3}\binom{n-3}{2}\mathbb{P}[1 \sim_L 2, 2 \sim_L 3, 2 \sim_L 4, 2 \sim_L 5] \\
& +12\binom{n}{3}\binom{n-3}{2}\mathbb{P}[1 \sim_L 2, 2 \sim_L 3, 1 \sim_L 4, 1 \sim_L 5] \\
& +12\binom{n}{3}\binom{n-3}{2}\mathbb{P}[1 \sim_L 2, 2 \sim_L 3, 3 \sim_L 4, 4 \sim_L 5] \\
& +9\binom{n}{3}\binom{n-3}{3}\mathbb{P}[1 \sim_L 2, 2 \sim_L 3]^2.
\end{aligned} \tag{5.51}$$

Now we proceed with establishing Lemma 5.14.

Proof. Fix each $n = 3, 4, \ldots,$ and $L = 1, 2, \ldots.$ We have

$$\begin{aligned}
\mathbb{E}\left[\frac{1}{3\binom{n}{3}} T_n^*(L)\right] &= \frac{1}{3\binom{n}{3}} \mathbb{E}\left[\sum_{u=1}^{n} \sum_{1 \leqslant v < w \leqslant n,\ v,w \neq u} \mathbf{1}[u \sim_L v, u \sim_L w]\right] \\
&= \frac{1}{3\binom{n}{3}} \sum_{u=1}^{n} \sum_{1 \leqslant v < w \leqslant n,\ v,w \neq u} \mathbb{P}\left[u \sim_L v, u \sim_L w\right] \\
&= \mathbb{P}[1 \sim_L 2, 1 \sim_L 3]. \qquad (5.52)
\end{aligned}$$

For arbitrary $\varepsilon > 0$, Chebyshev's inequality gives

$$\mathbb{P}\left[\left|\frac{1}{3\binom{n}{3}} T_n^*(L) - \mathbb{P}[1 \sim_L 2, 1 \sim_L 3]\right| > \varepsilon\right] \leqslant \frac{\operatorname{Var}\left(\frac{1}{3\binom{n}{3}} T_n^*(L)\right)}{\varepsilon^2}.$$

The lemma will be established once we show that $\lim_{n \to \infty} \operatorname{Var}\left(\frac{1}{3\binom{n}{3}} T_n^*(L)\right) = 0.$

For each $n = 2, 3, \ldots,$ we have

$$\operatorname{Var}\left(\frac{1}{3\binom{n}{3}} T_n^*(L)\right) = \frac{1}{9\binom{n}{3}^2} \left(\mathbb{E}[T_n^*(L)^2] - \mathbb{E}^2[T_n^*(L)]\right)$$

with

$$
\begin{aligned}
\frac{1}{9\binom{n}{3}^2}\mathbb{E}\left[T_n^*(L)^2\right] &= \frac{1}{3\binom{n}{3}}\mathbb{P}[1 \sim_L 2, 2 \sim_L 3] \\
&+\frac{2}{3\binom{n}{3}}\mathbb{P}[1 \sim_L 2, 2 \sim_L 3, 1 \sim_L 3] \\
&+\frac{\binom{n-3}{1}}{3\binom{n}{3}}\mathbb{P}[1 \sim_L 2, 2 \sim_L 3, 2 \sim_L 4] \\
&+\frac{2\binom{n-3}{1}}{3\binom{n}{3}}\mathbb{P}[1 \sim_L 2, 2 \sim_L 3, 1 \sim_L 4, 2 \sim_L 4] \\
&+\frac{\binom{n-3}{1}}{3\binom{n}{3}}\mathbb{P}[1 \sim_L 2, 2 \sim_L 3, 1 \sim_L 4, 3 \sim_L 4] \\
&+\frac{2\binom{n-3}{1}}{3\binom{n}{3}}\mathbb{P}[1 \sim_L 2, 2 \sim_L 3, 3 \sim_L 4] \\
&+\frac{\binom{n-3}{2}}{3\binom{n}{3}}\mathbb{P}[1 \sim_L 2, 2 \sim_L 3, 2 \sim_L 4, 2 \sim_L 5] \\
&+\frac{4\binom{n-3}{2}}{3\binom{n}{3}}\mathbb{P}[1 \sim_L 2, 2 \sim_L 3, 1 \sim_L 4, 1 \sim_L 5] \\
&+\frac{4\binom{n-3}{2}}{3\binom{n}{3}}\mathbb{P}[1 \sim_L 2, 2 \sim_L 3, 3 \sim_L 4, 4 \sim_L 5] \\
&+\frac{\binom{n-3}{3}}{\binom{n}{3}}\mathbb{P}[1 \sim_L 2, 2 \sim_L 3]^2
\end{aligned}
$$

by Proposition 5.15.

While the last coefficient converges to 1, the other ones converge to 0 when n goes to infinity. As a result, we have

$$
\lim_{n\to\infty} \frac{1}{9\binom{n}{3}^2}\mathbb{E}\left[T_n^*(L)^2\right] = \mathbb{P}[1 \sim_L 2, 2 \sim_L 3]^2 \tag{5.53}
$$

which implies

$$
\lim_{n\to\infty} \operatorname{Var}\left(\frac{1}{3\binom{n}{3}}T_n^*(L)\right) = 0
$$

as we recall that

$$
\mathbb{E}\left[\frac{1}{3\binom{n}{3}}T_n^*(L)\right] = \mathbb{P}[1 \sim_L 2, 1 \sim_L 3]
$$

for $L = 1, 2, \ldots$. This result leads us to conclude to the validity of 5.47.

$\blacksquare$

Chapter 6

Degree Distribution

The last property to be discussed in this dissertation is the nodal degree distribution in the MAG model. In this chapter, we discuss the limiting behavior of the nodal degree distribution.

Given a graph $G(V, E)$, the degree of a node u in V is the number of neighbors to which u is connected in one hop. In $\mathbb{M}(n; L)$, the degree of node u in V_n is the number of nodes that are L-adjacent to u. Let the rv $D_{n,L}(u)$ denote the degree of node u in V_n of $\mathbb{M}(n; L)$. It is easy to see

$$D_{n,L}(u) = \sum_{v \in V_n, v \neq u} \chi_L(u, v), \qquad u \in V_n. \tag{6.1}$$

6.1 The PMF and the conditional PMF of $D_{n,L}(u)$

We first obtain the closed form expression of the pmf of the degree variable $D_{n,L}(u)$ for a node u in V_n.

Proposition 6.1. *For $n = 2, 3, \ldots$, $L = 1, 2, \ldots$ and u in V_n, the pmf of the rv*

$D_{n,L}(u)$ is given by

$$\begin{aligned}
&\mathbb{P}\left[D_{n,L}(u) = d\right] \\
&= \sum_{\ell=0}^{L} \binom{L}{\ell} \mu(1)^{\ell} \mu(0)^{L-\ell} \mathbb{P}\left[D_{n,L}(u) = d | S_L(u) = \ell\right], \ d = 0, 1, \ldots, n-1,
\end{aligned} \tag{6.2}$$

where

$$\begin{aligned}
&\mathbb{P}\left[D_{n,L}(u) = d | S_L(u) = \ell\right] \\
&= \binom{n-1}{d} \left(\Gamma(1)^{\ell}\Gamma(0)^{L-\ell}\right)^{d} \left(1 - \Gamma(1)^{\ell}\Gamma(0)^{L-\ell}\right)^{n-1-d}
\end{aligned} \tag{6.3}$$

for each $\ell = 0, 1, \ldots, L$.

Proof. Fix $n = 2, 3, \ldots,$ $L = 1, 2, \ldots$ and u in V_n. The rv $D_{n,L}(u)$ is the sum of the $n-1$ link variables $\{\chi_L(u,v),\ v \neq u,\ v \in V_n\}$. Since the link variables are not mutually independent, we cannot evaluate the distribution of the rv $D_{n,L}(u)$ directly.

However, from Lemma 2.2, we know that the rvs $\{\chi_L(u,v),\ v \neq u,\ v \in V_n\}$ are conditionally mutually independent given $S_L(u)$ and that they are therefore conditional Bernoulli rvs with parameter $\Gamma(1)^{S_L(u)}\Gamma(0)^{L-S_L(u)}$. Hence, given $S_L(u)$, the rv $D_{n,L}(u)$ is a conditionally binomial rv with parameters $\left(n-1, \Gamma(1)^{S_L(u)}\Gamma(0)^{L-S_L(u)}\right)$.

The conditional pmf of the rv $D_{n,L}(u)$ is therefore given by (6.3), and the law of total probability yields

$$\begin{aligned}
&\mathbb{P}\left[D_{n,L}(u) = d\right] \\
&= \sum_{\ell=0}^{L} \mathbb{P}\left[D_{n,L}(u) = d | S_L(u) = \ell\right] \mathbb{P}\left[S_L(u) = \ell\right] \\
&= \sum_{\ell=0}^{L} \binom{L}{\ell} \mu(1)^{\ell} \mu(0)^{L-\ell} \mathbb{P}\left[D(u) = d | S_L(u) = \ell\right], \quad \begin{array}{l} u \in V_n \\ d = 0, 1, \ldots, n-1. \end{array}
\end{aligned} \tag{6.4}$$

■

The pmf of the rv $D_{n,L}(u)$ does not depend on the choice of node u in V_n. We shall write $D_{n,L}$ for $D_{n,L}(1)$, and all subsequent discussions are given in terms of the generic rv $D_{n,L}$. Throughout, let Z be a standard Gaussian rv (i.e., with mean 0 and variance 1).

6.2 The convergence theorem

The pmf of the rv $D_{n,L}$ does not have a well-known form. We seek a condition that the scaled degree rv will converge to well-known distribution. The main result of this chapter is the following convergence, the content of the next theorem.

Theorem 6.2. *Assume* $\Gamma(1) > \Gamma(0)$. *For any* $\rho-$*admissible scaling* $\mathtt{L} : \mathbb{N}_0 \to \mathbb{N}_0$ *with* $\rho > 0$, *it holds that*

$$\begin{aligned} &\sqrt{L_n}\left(\frac{1}{L_n}\left(\ln D^+_{n,L_n}\right) - \left(\frac{1}{\rho_n} + \mu(1)\ln\Gamma(1) + \mu(0)\ln\Gamma(0)\right)\right) \\ &\Rightarrow_n \left(\sqrt{\mu(1)\mu(0)}\ln\frac{\Gamma(1)}{\Gamma(0)}\right) Z \end{aligned} \tag{6.5}$$

whenever $1 + \rho\ln\Gamma(0) > 0$, *where*

$$D^+_{n,L_n} = D_{n,L_n} + \mathbf{1}[D_{n,L_n} = 0]. \tag{6.6}$$

The reason for using the rv D^+_{n,L_n} instead of the rv D_{n,L_n} is to avoid the boundary case $D_{n,L_n} = 0$. Later, it will be clear that the probability of $D_{n,L_n} = 0$ is asymptotically negligible.

We establish this convergence result in the next two sections by a proper decomposition of the left hand side of (6.5).

6.3 Applying the Central Limit Theorem

We proceed with two lemmas, which will make the convergence a straightforward consequence of the standard Central Limit Theorem.

Lemma 6.3. *For any $\rho-$admissible scaling $\mathtt{L} : \mathbb{N}_0 \to \mathbb{N}_0$ with $\rho > 0$, we have*

$$\begin{aligned}&\sqrt{L_n}\left(\frac{1}{L_n}\ln \mathbb{E}\left[D_{n,L_n}|S_{L_n}(1)\right] - \left(\frac{1}{\rho_n} + \mu(1)\ln\frac{\Gamma(1)}{\Gamma(0)} + \ln\Gamma(0)\right)\right)\\ &\Rightarrow_n \left(\sqrt{\mu(1)\mu(0)}\ln\frac{\Gamma(1)}{\Gamma(0)}\right) Z.\end{aligned} \tag{6.7}$$

Proof. Earlier in Section 6.1, we showed that for each $n = 2, 3, \ldots$ and $L = 1, 2, \ldots,$ the rv $D_{n,L}$ is a conditional binomial rv with parameters $\left(n-1, \Gamma(1)^{S_L(1)}\Gamma(0)^{L-S_L(1)}\right)$ given $S_L(1)$. The conditional expectation of the rv $D_{n,L}$ is therefore given by

$$\mathbb{E}\left[D_{n,L}|S_L(1)\right] = (n-1)\Gamma(1)^{S_L(1)}\Gamma(0)^{L-S_L(1)}. \tag{6.8}$$

It follows that

$$\ln \mathbb{E}\left[D_{n,L}|S_L(1)\right] = \ln(n-1) + S_L(1)\ln\frac{\Gamma(1)}{\Gamma(0)} + L\ln\Gamma(0).$$

Fix $n = 2, 3, \ldots$. For any $\rho-$admissible scaling $\mathtt{L} : \mathbb{N}_0 \to \mathbb{N}_0$ with $\rho > 0$, a simple subtraction yields

$$\begin{aligned}&\frac{1}{L_n}\ln \mathbb{E}\left[D_{n,L_n}|S_{L_n}(1)\right] - \left(\frac{1}{\rho_n} + \mu(1)\ln\frac{\Gamma(1)}{\Gamma(0)} + \ln\Gamma(0)\right)\\ = \quad &\left(\frac{\ln(n-1)}{L_n} - \frac{1}{\rho_n}\right) + \left(\frac{1}{L_n}S_{L_n}(1) - \mu(1)\right)\ln\frac{\Gamma(1)}{\Gamma(0)}.\end{aligned} \tag{6.9}$$

Because the rv $S_{L_n}(1)$ is a sum of L_n i.i.d. Bernoulli rvs with parameter $\mu(1)$ defined in (2.2), the Central Limit Theorem yields

$$\sqrt{L_n}\left(\frac{1}{L_n}S_{L_n}(1)-\mu(1)\right) \Rightarrow_n \sqrt{\mu(1)\mu(0)}Z$$

where Z is a standard Gaussian rv, whence

$$\sqrt{L_n}\left(\frac{1}{L_n}S_{L_n}(1)-\mu(1)\right)\ln\frac{\Gamma(1)}{\Gamma(0)} \Rightarrow_n \sqrt{\mu(1)\mu(0)}\left(\ln\frac{\Gamma(1)}{\Gamma(0)}\right)Z.$$

On the other hand, we have

$$\begin{aligned}\sqrt{L_n}\left(\frac{\ln(n-1)}{L_n}-\frac{1}{\rho_n}\right) &= \sqrt{L_n}\left(\frac{\ln(n-1)-\ln n}{L_n}\right)\\ &= \frac{1}{\sqrt{L_n}}\ln\frac{n-1}{n}\end{aligned} \tag{6.10}$$

so that

$$\lim_{n\to\infty}\sqrt{L_n}\left(\frac{\ln(n-1)}{L_n}-\frac{1}{\rho_n}\right)=0.$$

Collecting, we readily conclude the desired convergence (6.7). ■

Motivated by Lemma 6.3, for $n=2,3,\ldots$, we can always decompose the left hand side of (6.5) as

$$\begin{aligned}&\sqrt{L_n}\left(\frac{1}{L_n}\ln\left(D^+_{n,L_n}\right)-\left(\frac{1}{\rho_n}+\mu(1)\ln\Gamma(1)+\mu(0)\ln\Gamma(0)\right)\right)\\ =\ &\sqrt{L_n}\left(\frac{1}{L_n}\ln\left(D^+_{n,L_n}\right)-\frac{1}{L_n}\ln\mathbb{E}\left[D_{n,L_n}|S_{L_n}(1)\right]\right)\\ &+\sqrt{L_n}\left(\frac{1}{L_n}\ln\mathbb{E}\left[D_{n,L_n}|S_{L_n}(1)\right]-\left(\frac{1}{\rho_n}+\mu(1)\ln\frac{\Gamma(1)}{\Gamma(0)}+\ln\Gamma(0)\right)\right)\end{aligned} \tag{6.11}$$

Theorem 6.2 will then be established if we can show that

$$\sqrt{L_n}\left(\frac{1}{L_n}\ln\left(D^+_{n,L_n}\right)-\frac{1}{L_n}\ln\mathbb{E}\left[D_{n,L_n}|S_{L_n}(1)\right]\right)\xrightarrow{P}_n 0 \tag{6.12}$$

under the assumption $\Gamma(1)>\Gamma(0)$. This issue will be addressed in the next section.

6.4 A proof of Theorem 6.2

Observe that

$$\ln D^{+}_{n,L_n} - \ln \mathbb{E}\left[D_{n,L_n}|S_{L_n}(1)\right] = \ln\left(\frac{D^{+}_{n,L_n}}{\mathbb{E}\left[D_{n,L_n}|S_{L_n}(1)\right]}\right) \tag{6.13}$$

for $n = 2, 3, \ldots$. Because the function $x \to \ln x$ is continuous on $(0, \infty)$, the left hand side of (6.13) converges in probability to 0 if

$$\frac{D^{+}_{n,L_n}}{\mathbb{E}\left[D_{n,L_n}|S_{L_n}(1)\right]} \xrightarrow{P}_n 1. \tag{6.14}$$

We will first show that

$$\frac{D_{n,L_n}}{\mathbb{E}\left[D_{n,L_n}|S_{L_n}(1)\right]} \xrightarrow{P}_n 1, \tag{6.15}$$

and then argue that

$$\lim_{n\to\infty} \mathbb{P}\left[\left|\frac{D^{+}_{n,L_n}}{\mathbb{E}\left[D_{n,L_n}|S_{L_n}(1)\right]} - \frac{D_{n,L_n}}{\mathbb{E}\left[D_{n,L_n}|S_{L_n}(1)\right]}\right| > 0\right] = 0. \tag{6.16}$$

Lemma 6.4. *Assume* $\Gamma(1) > \Gamma(0)$. *For any* $\rho-$*admissible scaling* $\mathtt{L} : \mathbb{N}_0 \to \mathbb{N}_0$ *with* $\rho > 0$, *it holds that*

$$\frac{D_{n,L_n}}{\mathbb{E}\left[D_{n,L_n}|S_{L_n}(1)\right]} \xrightarrow{P}_n 1 \tag{6.17}$$

if $1 + \rho \ln \Gamma(0) > 0$.

Proof. In order to establish the convergence (6.17), we would like to show that

$$\lim_{n\to\infty} \mathbb{P}\left[\left|\frac{D_{n,L_n}}{\mathbb{E}\left[D_{n,L_n}|S_{L_n}(1)\right]} - 1\right| > \delta\right] = 0$$

for any $\delta > 0$.

For $n = 2, 3, \ldots,$ and any $\delta > 0$ fixed, the Markov inequality gives

$$\mathbb{P}\left[\left|\frac{D_{n,L_n}}{\mathbb{E}\left[D_{n,L_n}|S_{L_n}(1)\right]} - 1\right| > \delta\right] \leqslant \frac{\mathbb{E}\left[\left(\frac{D_{n,L_n}}{\mathbb{E}\left[D_{n,L_n}|S_{L_n}(1)\right]} - 1\right)^2\right]}{\delta^2}. \tag{6.18}$$

As a result, Lemma 6.4 will be established if we show

$$\lim_{n\to\infty} \mathbb{E}\left[\left(\frac{D_{n,L_n}}{\mathbb{E}\left[D_{n,L_n}|S_{L_n}(1)\right]} - 1\right)^2\right] = 0. \tag{6.19}$$

Fix $n = 2, 3, \ldots$. By iterated expectations, we get

$$\mathbb{E}\left[\left(\frac{D_{n,L_n}}{\mathbb{E}\left[D_{n,L_n}|S_{L_n}(1)\right]} - 1\right)^2\right] = \mathbb{E}\left[\mathbb{E}\left[\left(\frac{D_{n,L_n}}{\mathbb{E}\left[D_{n,L_n}|S_{L_n}(1)\right]} - 1\right)^2 \middle| S_{L_n}(1)\right]\right].$$

It is plain that

$$\begin{aligned} & \mathbb{E}\left[\left(\frac{D_{n,L_n}}{\mathbb{E}\left[D_{n,L_n}|S_{L_n}(1)\right]} - 1\right)^2 \middle| S_{L_n}(1)\right] \\ = \ & \mathbb{E}\left[\frac{D^2_{n,L_n}}{\mathbb{E}\left[D_{n,L_n}|S_{L_n}(1)\right]^2} - 2\frac{D_{n,L_n}}{\mathbb{E}\left[D_{n,L_n}|S_{L_n}(1)\right]} + 1 \middle| S_{L_n}(1)\right] \\ = \ & \frac{\mathbb{E}\left[D^2_{n,L_n}|S_{L_n}(1)\right]}{\mathbb{E}\left[D_{n,L_n}|S_{L_n}(1)\right]^2} - 1. \end{aligned} \tag{6.20}$$

Recalling that

$$D_{n,L_n} = \sum_{v=2}^{n} \chi_{L_n}(1, v),$$

we get

$$D^2_{n,L_n} = \sum_{v=2}^{n} \chi_{L_n}(1, v) + \sum_{v=2}^{n} \sum_{w=2, w\neq v}^{n} \chi_{L_n}(1, v)\chi_{L_n}(1, w)$$

upon using the fact that $\chi_{L_n}(1, v)^2 = \chi_{L_n}(1, v)$ for $v = 2, \ldots, n$.

Since the rvs $\{\chi_{L_n}(1, v),\ v = 2, \ldots, n\}$ are conditionally mutually independent given $S_{L_n}(1)$, we obtain

$$\begin{aligned}
\mathbb{E}\left[D_{n,L_n}^2|S_{L_n}(1)\right] &= \mathbb{E}\left[\sum_{v=2}^{n}\chi_{L_n}(1,v)+\sum_{v=2}^{n}\sum_{w=2,w\neq v}^{n}\chi_{L_n}(1,v)\chi(1,w)\middle|S_{L_n}(1)\right] \\
&= \sum_{v=2}^{n}\mathbb{E}\left[\chi_{L_n}(1,v)|S_{L_n}(1)\right] \\
&\quad +\sum_{v=2}^{n}\sum_{w=2,w\neq v}^{n}\mathbb{E}\left[\chi_{L_n}(1,v)|S_{L_n}(1)\right]\mathbb{E}\left[\chi_{L_n}(1,w)|S_{L_n}(1)\right] \\
&= (n-1)\Gamma(1)^{S_{L_n}}\Gamma(0)^{L_n-S_{L_n}}+(n-1)(n-2)\Gamma(1)^{2S_{L_n}}\Gamma(0)^{2(L_n-S_{L_n})}.
\end{aligned}$$

While it is plain from (6.8) that

$$\mathbb{E}\left[D_{n,L_n}|S_{L_n}(1)\right]^2=(n-1)^2\Gamma(1)^{2S_{L_n}}\Gamma(0)^{2(L_n-S_{L_n})},$$

we have

$$\begin{aligned}
&\frac{\mathbb{E}\left[D_{n,L_n}^2|S_{L_n}(1)\right]}{\mathbb{E}\left[D_{n,L_n}|S_{L_n}(1)\right]^2}-1 \\
=\;& \frac{(n-1)\Gamma(1)^{S_{L_n}}\Gamma(0)^{L_n-S_{L_n}}+(n-1)(n-2)\Gamma(1)^{2S_{L_n}}\Gamma(0)^{2(L_n-S_{L_n})}}{(n-1)^2\Gamma(1)^{2S_{L_n}}\Gamma(0)^{2(L_n-S_{L_n})}}-1 \\
=\;& \frac{1}{(n-1)\Gamma(1)^{S_{L_n}}\Gamma(0)^{L_n-S_{L_n}}}+\frac{n-2}{n-1}-1 \\
\leqslant\;& \frac{1}{(n-1)\Gamma(0)^{L_n}}-\frac{1}{n-1}. \qquad (6.21)
\end{aligned}$$

In the last step, we used the fact that $\Gamma(1)>\Gamma(0)$. Taking expectations on both side of (6.21) we get

$$\mathbb{E}\left[\left(\frac{D_{n,L_n}}{\mathbb{E}\left[D_{n,L_n}|S_{L_n}(1)\right]}-1\right)^2\right]\leqslant\frac{1}{(n-1)\Gamma(0)^{L_n}}-\frac{1}{n-1},\qquad n=2,3,\ldots. \qquad (6.22)$$

Let n go to infinity in (6.22). Under the condition $1+\rho\ln\Gamma(0)>0$, we have

$$\lim_{n\to\infty}(n-1)\Gamma(0)^{L_n}=\infty,$$

and therefore (6.19) holds. This completes the proof of Lemma 6.4. ■

To establish the convergence in (6.12), we still need to establish (6.16) which allows us to conclude (6.14).

Indeed, for each $n = 2, 3, \ldots$, it is plain from (6.6) that

$$\begin{aligned}\mathbb{P}\left[\left|\frac{D^+_{n,L_n}}{\mathbb{E}\left[D_{n,L_n}|S_{L_n}(1)\right]} - \frac{D_{n,L_n}}{\mathbb{E}\left[D_{n,L_n}|S_{L_n}(1)\right]}\right| > 0\right] &= \mathbb{P}\left[\left|\frac{\mathbf{1}[D_{n,L_n} = 0]}{\mathbb{E}\left[D_{n,L_n}|S_{L_n}(1)\right]}\right| > 0\right] \\ &= \mathbb{P}\left[D_{n,L_n} = 0\right].\end{aligned}$$

Under the condition $1 + \rho \ln \Gamma(0) > 0$, the one-law of Theorem 3.1 yields

$$\lim_{n\to\infty} \mathbb{P}\left[\ \mathbb{M}(n; L_n) \text{ has no isolated nodes }\right] = 1.$$

The easy bound

$$1 - \mathbb{P}\left[\ \mathbb{M}(n; L_n) \text{ has no isolated nodes }\right] \geqslant \mathbb{P}\left[D_{n,L_n} = 0\right]$$

therefore implies

$$\lim_{n\to\infty} \mathbb{P}\left[D_{n,L_n} = 0\right] = 0,$$

and (6.16) holds. This completes the proof of Theorem 6.2. ■

6.5 A log-normal limit

Theorem 6.2 says that the logarithm of the rv D^+_{n,L_n} converges in distribution to a Gaussian rv after properly centering and scaling. The rv D^+_{n,L_n} should then converge in distribution to a log-normal rv through the continuous mapping $\mathbb{R} \to \mathbb{R} : x \to e^x$. Because of the vanishingly probability of the event $D_{n,L_n} = 0$ as n grows unboundedly large, the rv D_{n,L_n} should also converge in distribution to a log-normal rv with a proper scaling. The following corollary formalizes this idea.

Corollary 6.5. *Assume* $\Gamma(1) > \Gamma(0)$. *For any* $\rho-$*admissible scaling* $\mathtt{L} : \mathbb{N}_0 \to \mathbb{N}_0$ *with* $\rho > 0$, *it holds that*

$$(D_{n,L_n})^{\frac{1}{\sqrt{L_n}}} e^{-\sqrt{L_n}\left(\frac{1}{\rho_n}+\mu(1)\ln\Gamma(1)+\mu(0)\ln\Gamma(0)\right)} \Rightarrow_n \ln\mathcal{N}\left(0, \mu(1)\mu(0)\ln^2\frac{\Gamma(1)}{\Gamma(0)}\right) \quad (6.23)$$

if $1 + \rho\ln\Gamma(0) > 0$.

Proof. By the Continuous Mapping Theorem for weak convergence, we easily get

$$e^{\Lambda^+_{n,L_n}} \Rightarrow_n \ln\mathcal{N}\left(0, \mu(1)\mu(0)\ln^2\frac{\Gamma(1)}{\Gamma(0)}\right) \quad (6.24)$$

from Theorem 6.2, where have we used the notation

$$\Lambda^+_{n,L_n} = \sqrt{L_n}\left(\frac{1}{L_n}\ln\left(D^+_{n,L_n}\right) - \left(\frac{1}{\rho_n} + \mu(1)\ln\Gamma(1) + \mu(0)\ln\Gamma(0)\right)\right), \quad n = 2, 3, \ldots.$$

In other words, the sequence

$$\left\{\left(D^+_{n,L_n}\right)^{\frac{1}{\sqrt{L_n}}} e^{-\sqrt{L_n}\left(\frac{1}{\rho_n}+\mu(1)\ln\Gamma(1)+\mu(0)\ln\Gamma(0)\right)}, \ n = 2, 3, \ldots\right\}$$

converges in distribution to a log-Normal rv with parameters $\left(0, \mu(1)\mu(0)\ln^2\frac{\Gamma(1)}{\Gamma(0)}\right)$. Moreover, we have

$$\begin{aligned} & \mathbb{P}\left[\left|(D^+_{n,L_n})^{\frac{1}{\sqrt{L_n}}} - (D_{n,L_n})^{\frac{1}{\sqrt{L_n}}}\right| e^{-\sqrt{L_n}\left(\frac{1}{\rho_n}+\mu(1)\ln\Gamma(1)+\mu(0)\ln\Gamma(0)\right)} > 0\right] \\ = \ & \mathbb{P}[D_{n,L_n} = 0] \quad n = 2, 3, \ldots \end{aligned} \quad (6.25)$$

with

$$\lim_{n\to\infty}\mathbb{P}[D_{n,L_n} = 0] = 0,$$

and the desired convergence (6.23) follows. ■

This result is significantly different from the one given by Kim and Leskovec in [17], which argued that the *tail* of the pmf of the degree rv is approximately log-normal distribution. It was not a limiting result and it is unclear in their argument that how a discreet distribution is approximated by a continuous distribution. Here, we have a simple and concise approach which leads to an exact convergence result.

In the next chapter, we are going to approximate the CDF of the rv D_{n,L_n} based on the convergence results we have just obtained; the performance of this approximation will then be evaluated.

Chapter 7

Approximating the Degree Distribution

From Section 6.1, we know that, for $n = 2, 3, \dots$ and $L = 1, 2, \dots$ fixed, the rv $D_{n,L}$ is a compounded binomial rv and its CDF is given by the sum of its pmf available (6.2), namely

$$F_{D_{n,L}}(x) = \sum_{d=0}^{\min(\lfloor x \rfloor, n-1)} \sum_{\ell=0}^{L} \binom{L}{\ell} \mu(1)^{\ell} \mu(0)^{L-\ell} \mathbb{P}\left[D_{n,L} = d | S_L(1) = \ell\right], \qquad x \geqslant 0,$$

with

$$\begin{aligned} &\mathbb{P}\left[D_{n,L} = d | S_L(1) = \ell\right] \\ &= \binom{n-1}{d} \left(\Gamma(1)^{\ell}\Gamma(0)^{L-\ell}\right)^{d} \left(1 - \Gamma(1)^{\ell}\Gamma(0)^{L-\ell}\right)^{n-1-d}. \end{aligned}$$

Calculating this CDF for any $x > 0$ is no easy task. Therefore, we need to find a good approximation as an efficient alternative. Below, we present such an approximation and then evaluate its performance when n is large.

7.1 The approximation

Theorem 7.1. *Assume* $\Gamma(1) > \Gamma(0)$*. For any* $\rho-$*admissible scaling* $\mathtt{L} : \mathbb{N}_0 \to \mathbb{N}_0$ *with* $\rho > 0$*, under the condition* $1 + \rho \ln \Gamma(0) > 0$*, it holds for each* t *in* $\mathbb{R}$ *that*

$$\lim_{n\to\infty} \left| \mathbb{P}\left[D_{n,L_n} \leqslant x_{n,L_n}(t)\right] - \mathbb{P}\left[\sigma_0 Z \leqslant \frac{1}{\sqrt{L_n}} \ln(x_{n,L_n}(t)) - \mathcal{E}_{n,L_n}\right]\right| = 0 \quad (7.1)$$

with

$$x_{n,L_n}(t) = e^{\sqrt{L_n}\left(t+\mathcal{E}_{n,L_n}\right)}, \quad (7.2)$$

where we have set

$$\sigma_0 = \sqrt{\mu(1)\mu(0)} \left(\ln \frac{\Gamma(1)}{\Gamma(0)} \right) \quad (7.3)$$

and

$$\mathcal{E}_{n,L_n} = \sqrt{L_n} \left(\frac{1}{\rho_n} + \mu(1) \ln \frac{\Gamma(1)}{\Gamma(0)} + \ln \Gamma(0) \right), \qquad n = 2, 3, \ldots. \quad (7.4)$$

Proof. Consider any $\rho-$admissible scaling $\mathtt{L} : \mathbb{N}_0 \to \mathbb{N}_0$ with $\rho > 0$. For $n = 2, 3, \ldots$ and t in $\mathbb{R}$, we substitute (7.2) into (7.1) to get

$$\left| \mathbb{P}\left[D_{n,L_n} \leqslant e^{\sqrt{L_n}\left(t+\mathcal{E}_{n,L_n}\right)}\right] - \mathbb{P}\left[\sigma_0 Z \leqslant t\right]\right|. \quad (7.5)$$

If D_{n,L_n} in (7.5) is replaced by D^+_{n,L_n}, then it is plain that

$$\begin{aligned}
& \lim_{n\to\infty} \left| \mathbb{P}\left[D^+_{n,L_n} \leqslant e^{\sqrt{L_n}\left(t+\mathcal{E}_{n,L_n}\right)}\right] - \mathbb{P}\left[\sigma_0 Z \leqslant t\right]\right| \\
= & \lim_{n\to\infty} \left| \mathbb{P}\left[\frac{1}{\sqrt{L_n}} \ln\left(D^+_{n,L_n}\right) - \mathcal{E}_{n,L_n} \leqslant t\right] - \mathbb{P}\left[\sigma_0 Z \leqslant t\right]\right| \\
= & \ 0
\end{aligned}$$

by Theorem 6.2. To establish Theorem 7.1, we only need to show that

$$\lim_{n\to\infty}\left|\mathbb{P}\left[D^{+}_{n,L_n}\leqslant e^{\sqrt{L_n}\left(t+\mathcal{E}_{n,L_n}\right)}\right]-\mathbb{P}\left[D_{n,L_n}\leqslant e^{\sqrt{L_n}\left(t+\mathcal{E}_{n,L_n}\right)}\right]\right|=0. \tag{7.6}$$

However, for each $n = 2, 3, \ldots$, we have

$$\mathbb{P}\left[D^{+}_{n,L_n}\leqslant x\right]=\mathbb{P}\left[D_{n,L_n}\leqslant x\right],\quad x\geqslant 1, \tag{7.7}$$

while

$$\mathbb{P}\left[D_{n,L_n}\leqslant x\right]-\mathbb{P}\left[D^{+}_{n,L_n}\leqslant x\right]=\mathbb{P}[D_{n,L_n}=0],\quad x\in[0,1) \tag{7.8}$$

from (6.6). With $\lim_{n\to\infty}\mathbb{P}[D_{n,L_n}=0]=0$, we readily conclude that (7.6) holds, and this completes the proof of Theorem 7.1. ■

As a result, we can use $\mathbb{P}\left[\sigma_0 Z\leqslant\frac{1}{\sqrt{L_n}}\ln(x_{n,L_n}(t))-\mathcal{E}_{n,L_n}\right]$ to approximate $\mathbb{P}\left[D_{n,L_n}\leqslant x_{n,L_n}(t)\right]$ for each t in $\mathbb{R}$ when n is large. We are interested in the performance of this approximation as n is growing large. An upper bound to the absolute difference

$$\Delta_n(t)=\left|\mathbb{P}\left[D_{n,L_n}\leqslant x_{n,L_n}(t)\right]-\mathbb{P}\left[\sigma_0 Z\leqslant\frac{1}{\sqrt{L_n}}\ln\left(x_{n,L_n}(t)\right)-\mathcal{E}_{n,L_n}\right]\right| \tag{7.9}$$

is given in the next section.

7.2 A Berry-Esseen type result

Theorem 7.2. *Assume* $\Gamma(1)>\Gamma(0)$. *For any* $\rho-$*admissible scaling* $\mathtt{L}:\mathbb{N}_0\to\mathbb{N}_0$ *with* $\rho>0$, *under the condition* $1+\rho\ln\Gamma(0)>0$, *there exist a positive constant* C *and a positive integer* N *such that*

$$\sup_{t\in\mathbb{R}}\Delta_n(t)\ \leqslant\ \frac{C}{\sqrt{L_n}} \tag{7.10}$$

whenever $n \geqslant N$, where $\Delta_n(t)$ was defined in (7.9).

To establish Theorem 7.2, we need the CDF of the rv D^+_{n,L_n} to bridge the gap between the CDF of the rv D_{n,L_n} and the CDF of the rv $\sigma_0 Z$ so that Theorem 6.2 can be applied. A natural approach of proving Theorem 7.2 is to decompose the left hand side of (7.10).

7.3 A decomposition of (7.10)

Consider a $\rho-$admissible scaling $\mathtt{L} : \mathbb{N}_0 \to \mathbb{N}_0$ with $\rho > 0$. For $n = 2, 3, \ldots$, the triangular inequality gives

$$\begin{aligned}
&\sup_{t\in\mathbb{R}} \left| \mathbb{P}\left[D_{n,L_n} \leqslant x_{n,L_n}(t)\right] - \mathbb{P}\left[\sigma_0 Z \leqslant \frac{1}{\sqrt{L_n}} \ln\left(x_{n,L_n}(t)\right) - \mathcal{E}_{n,L_n}\right]\right| \\
&\leqslant \sup_{t\in\mathbb{R}} \left| \mathbb{P}\left[D_{n,L_n} \leqslant x_{n,L_n}(t)\right] - \mathbb{P}\left[D^+_{n,L_n} \leqslant x_{n,L_n}(t)\right]\right| && (7.11a)\\
&\quad + \sup_{t\in\mathbb{R}} \left| \mathbb{P}\left[D^+_{n,L_n} \leqslant x_{n,L_n}(t)\right] - \mathbb{P}\left[\sigma_0 Z \leqslant \frac{1}{\sqrt{L_n}} \ln\left(x_{n,L_n}(t)\right) - \mathcal{E}_{n,L_n}\right]\right|. && (7.11b)
\end{aligned}$$

Given the relationship between the rvs D^+_{n,L_n} and D_{n,L_n} in (6.6), an upper bound to (7.11a) is easy to obtain; Lemma 7.4 given later will address this issue. Before presenting this upper bound, we first proceed to further decompose (7.11b).

7.4 A decomposition of (7.11b)

Fix $n = 2, 3, \ldots$. We know from (7.2) that (7.11b) is equivalent to

$$\sup_{t\in\mathbb{R}} \left| \mathbb{P}\left[\frac{1}{\sqrt{L_n}} \ln\left(D^+_{n,L_n}\right) - \mathcal{E}_{n,L_n} \leqslant t\right] - \mathbb{P}\left[\sigma_0 Z \leqslant t\right]\right|. \tag{7.12}$$

From (6.11) and (6.9) we also know that the rv $\frac{1}{\sqrt{L_n}} \ln\left(D^+_{n,L_n}\right) - \mathcal{E}_{n,L_n}$ can be decomposed as

$$
\begin{aligned}
& \sqrt{L_n}\left(\frac{1}{L_n}\ln\left(D^+_{n,L_n}\right) - \left(\frac{1}{\rho_n} + \mu(1)\ln\Gamma(1) + \mu(0)\ln\Gamma(0)\right)\right) \\
= \; & \sqrt{L_n}\left(\frac{1}{L_n}\ln\left(D^+_{n,L_n}\right) - \frac{1}{L_n}\ln\mathbb{E}\left[D_{n,L_n}|S_{L_n}(1)\right]\right) \\
& + \sqrt{L_n}\left(\frac{1}{L_n}\ln\mathbb{E}\left[D_{n,L_n}|S_{L_n}(1)\right] - \left(\frac{1}{\rho_n} + \mu(1)\ln\frac{\Gamma(1)}{\Gamma(0)} + \ln\Gamma(0)\right)\right) \\
= \; & \frac{1}{\sqrt{L_n}}\left(\ln\left(D^+_{n,L_n}\right) - \ln\mathbb{E}\left[D_{n,L_n}|S_{L_n}(1)\right] + \ln\frac{n-1}{n}\right) \\
& + \sqrt{L_n}\left(\frac{1}{L_n}S_{L_n}(1) - \mu(1)\right)\ln\frac{\Gamma(1)}{\Gamma(0)} \\
= \; & X_{n,L_n} + Y_{L_n}
\end{aligned}
$$

where we have set

$$
X_{n,L_n} = \frac{1}{\sqrt{L_n}}\left(\ln\left(D^+_{n,L_n}\right) - \ln\mathbb{E}\left[D_{n,L_n}|S_{L_n}(1)\right] + \ln\frac{n-1}{n}\right) \tag{7.13}
$$

and

$$
Y_{L_n} = \frac{1}{\sqrt{L_n}}\left(S_{L_n}(1) - L_n\mu(1)\right)\ln\frac{\Gamma(1)}{\Gamma(0)}. \tag{7.14}
$$

Bounding (7.11b) is therefore equivalent to bounding

$$
\sup_{t\in\mathbb{R}}\left|\mathbb{P}\left[X_{n,L_n} + Y_{L_n} \leqslant t\right] - \mathbb{P}\left[\sigma_0 Z \leqslant t\right]\right|. \tag{7.15}
$$

The next lemma gives an upper bound to (7.15).

Lemma 7.3. *For $n = 2, 3, \ldots,$ $L = 1, 2, \ldots$ and any $\varepsilon > 0$, with the notation used* *in (7.13) and (7.14), we have*

$$\begin{aligned}
&\sup_{y\in\mathbb{R}} \left|\mathbb{P}\left[X_{n,L}+Y_L \leqslant t\right]-\mathbb{P}\left[\sigma_0 Z \leqslant t\right]\right| \\
\leqslant\ &\sup_{y\in\mathbb{R}} \left|\mathbb{P}\left[Y_L \leqslant y\right]-\mathbb{P}\left[\sigma_0 Z \leqslant y\right]\right| && \text{(7.16a)}\\
&+\sup_{y\in\mathbb{R}} \mathbb{P}\left[y<\sigma_0 Z \leqslant y+\varepsilon\right] && \text{(7.16b)}\\
&+2\mathbb{P}\left[\left|X_{n,L}\right|>\varepsilon\right]. && \text{(7.16c)}
\end{aligned}$$

The proof of Lemma 7.3 is given at the end of this section. We are going to develop four lemmas to deal with bounds to each of the terms (7.11a), (7.16a), (7.16b) and (7.16c), respectively, when L is substituted by a $\rho-$admissible L_n.

7.5 Four building blocks for Theorem 7.2

Recall that

$$D^+_{n,L_n} = D_{n,L_n} + \mathbf{1}[D_{n,L_n}=0], \qquad n=2,3,\ldots.$$

The following lemma bridges the gap between the CDFs of the two rvs D_{n,L_n} and D^+_{n,L_n}.

Lemma 7.4. *Assume* $\Gamma(1)>\Gamma(0)$. *For any* $\rho-$*admissible scaling* $\mathtt{L}:\mathbb{N}_0\to\mathbb{N}_0$ *with* $\rho>0$, *there exists* $\delta>0$ *such that*

$$\sup_{x\in\mathbb{R}_+}\left|\mathbb{P}\left[D^+_{n,L_n}\leqslant x\right]-\mathbb{P}\left[D_{n,L_n}\leqslant x\right]\right| = \mathbb{P}\left[D_{n,L_n}=0\right] = o(e^{-n^\delta})$$

if $1+\rho\ln\Gamma(0)>0$.

Proof. Consider a $\rho-$admissible scaling $\mathtt{L} : \mathbb{N}_0 \to \mathbb{N}_0$ with $\rho > 0$. For $n = 2, 3, \ldots,$ it is plain from (7.7) and (7.8) that

$$\sup_{x \in [0,+\infty)} \left| \mathbb{P}\left[D^+_{n,L_n} \leqslant x \right] - \mathbb{P}\left[D_{n,L_n} \leqslant x \right] \right| = \mathbb{P}\left[D_{n,L_n} = 0 \right].$$

Now, recall the easy bound

$$\mathbb{P}\left[D_{n,L_n} = 0 \right] \leqslant \mathbb{P}\left[I_n(L_n) > 0 \right] \leqslant \mathbb{E}\left[I_n(L_n) \right]$$

where $I_n(L_n)$ is the number of isolated nodes in $\mathbb{M}(n; L_n)$ defined in (3.14) with (3.36) stating that

$$\mathbb{E}\left[I_n(L_n) \right] \leqslant e^{\ln n - (n-1)\Gamma(0)^{L_n}} = e^{\ln n - \frac{n-1}{n} n^{1+\rho_n \ln \Gamma(0)}}.$$

Under the condition $1 + \rho \ln \Gamma(0) > 0$, it is plain that there exists a constant δ^* in $(0, 1 + \rho \ln \Gamma(0))$ such that

$$e^{\ln n - \frac{n-1}{n} n^{1+\rho_n \ln \Gamma(0)}} = o\left(e^{-n^{\delta^*}} \right),$$

whence

$$\mathbb{P}\left[D_{n,L_n} = 0 \right] = o\left(e^{-n^{\delta}} \right), \quad \delta \in (0, \delta^*].$$

This readily concludes Lemma 7.4. ■

The second lemma deals with the upper bound to (7.16a) and relates to the convergence rate of the standard Central Limit Theorem. It is a direct application of the Berry-Esseen Theorem [24, Thm 3, p. 299] applied to the sum of a collection of i.i.d. Bernoulli rvs.

Lemma 7.5. *For any $\rho-$admissible scaling $\mathtt{L}:\mathbb{N}_0\to\mathbb{N}_0$ with $\rho>0$, there exists a universal constant $C^*>0$ such that*

$$\sup_{y\in\mathbb{R}}|\mathbb{P}\left[Y_{L_n}\leqslant y\right]-\mathbb{P}\left[\sigma_0 Z\leqslant y\right]|\leqslant\frac{1}{\sqrt{L_n}}\frac{C^*\left(\mu(1)^2+\mu(0)^2\right)}{\sqrt{\mu(1)\mu(0)}}\tag{7.17}$$

for $n=2,3,\ldots,$ where Y_{L_n} and σ_0 is defined in (7.14) and (7.3), respectively.

Proof. Consider a $\rho-$admissible scaling $\mathtt{L}:\mathbb{N}_0\to\mathbb{N}_0$ with $\rho>0$. Fix $n=2,3,\ldots$. We note that

$$\begin{aligned}(S_{L_n}(1)-L_n\mu(1))\ln\frac{\Gamma(1)}{\Gamma(0)} &= \sqrt{L_n}Y_{L_n}\\ &= \sum_{\ell=1}^{L_n}\left(\big(A_\ell(1)-\mu(1)\big)\cdot\ln\frac{\Gamma(1)}{\Gamma(0)}\right)\end{aligned}$$

where the rvs $\left\{\big(A_\ell(1)-\mu(1)\big)\cdot\ln\frac{\Gamma(1)}{\Gamma(0)},\ \ell=1,\ldots,L_n\right\}$ are i.i.d. zero-mean rvs since the rvs $\{A_\ell(1),\ \ell=1,\ldots,L_n\}$ form a collection of i.i.d. Bernoulli rvs with parameter $\mu(1)$. It is easy to check that

$$\Gamma_{L_n}^3=\sum_{\ell=1}^{L_n}\mathbb{E}\left[\left|\big(A_\ell(1)-\mu(1)\big)\cdot\ln\frac{\Gamma(1)}{\Gamma(0)}\right|^3\right]=L_n\mu(1)\mu(0)\left(\mu(1)^2+\mu(0)^2\right)\cdot\left|\ln\frac{\Gamma(1)}{\Gamma(0)}\right|^3$$

and

$$s_{L_n}^2=\sum_{\ell=1}^{L_n}\mathbb{E}\left[\left(\big(A_\ell(1)-\mu(1)\big)\cdot\ln\frac{\Gamma(1)}{\Gamma(0)}\right)^2\right]=L_n\mu(1)\mu(0)\left(\ln\frac{\Gamma(1)}{\Gamma(0)}\right)^2=L_n\sigma_0^2.$$

Applying the Berry-Esseen Theorem gives

$$\begin{aligned}\sup_{-\infty<w<+\infty}\left|\mathbb{P}\left[\sqrt{L_n}Y_n<s_{L_n}w\right]-\mathbb{P}\left[Z\leqslant w\right]\right| &\leqslant C^*\left(\frac{\Gamma_{L_n}}{s_{L_n}}\right)^3\\ &= \frac{1}{\sqrt{L_n}}\frac{C^*\left(\mu(1)^2+\mu(0)^2\right)}{\sqrt{\mu(1)\mu(0)}}\end{aligned}\tag{7.18}$$

for some $C^*>0$ which does not depend on n. It is now plain that (7.17) follows by letting $y=\sigma_0 w$. ■

In the next lemma, we are interested in the convergence rate of the rv X_{n,L_n} defined in (7.13).

Lemma 7.6. *Assume* $\Gamma(1) > \Gamma(0)$. *For any* $\rho-$*admissible scaling* $\mathtt{L} : \mathbb{N}_0 \to \mathbb{N}_0$ *with* $\rho > 0$, *and any* $\varepsilon > 0$, *there exists a positive integer* N *such that*

$$\mathbb{P}\left[|X_{n,L_n}| > \varepsilon\right] \leqslant \frac{2}{(n-1)\Gamma(0)^{L_n}} \frac{1}{\left(1 - e^{-\sqrt{L_n}\varepsilon + \ln \frac{n}{n-1}}\right)^2}, \qquad n \geqslant N$$

if $1 + \rho \ln \Gamma(0) > 0$.

Details of Lemma 7.6 can be found in Appendix D. The upper bound to (7.16b) is based on the uniform continuity of the CDF of a Gaussian rv with parameters $(0, \sigma^2)$.

Lemma 7.7. *Let* σ *be any positive number. For any positive* $\Delta > 0$, *we have*

$$\sup_{x\in\mathbb{R}} \mathbb{P}\left[x < \sigma Z \leqslant x + \Delta\right] \leqslant \frac{1}{\sqrt{2\pi\sigma^2}}\Delta + \frac{1}{2\sigma^2\sqrt{2\pi}} e^{-\frac{1}{2}}\Delta^2.$$

Proof. Let $F_{\sigma Z}(x)$ and $f_{\sigma Z}(x)$ denote the CDF and the pdf of a Gaussian rv with parameters $(0, \sigma^2)$, respectively. By definition, we know

$$f_{\sigma Z}(x) = F'_{\sigma Z}(x), \quad x \in \mathbb{R}.$$

For any x_0 in $\mathbb{R}$, Taylor series expansion yields

$$F_{\sigma Z}(x_0 + \Delta) = F_{\sigma Z}(x_0) + \Delta f_{\sigma Z}(x_0) + R_1\left(x_0; \Delta\right), \quad n = 1, 2, \ldots$$

where $R_1(x_0; \Delta)$ is given by the Lagrange Mean Value Theorem, namely

$$R_1(x_0; \Delta) = \frac{f'_{\sigma Z}(x_0^*)}{2}\Delta^2 \tag{7.19}$$

for some x_0^* in $[x_0, x_0 + \Delta]$.

Simple calculations yield

$$f'_{\sigma Z}(x) = -\frac{x}{\sigma^3\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$$

which has bound

$$|f'_{\sigma Z}(x)| \leqslant \frac{1}{\sigma^2\sqrt{2\pi}} e^{-\frac{1}{2}}, \qquad x \in \mathbb{R}.$$

The remainder term (7.19) is therefore uniformly bounded with respect to x, namely

$$-\frac{1}{2\sigma^2\sqrt{2\pi}} e^{-\frac{1}{2}} \Delta^2 \leqslant R_1(x; \Delta) \leqslant \frac{1}{2\sigma^2\sqrt{2\pi}} e^{-\frac{1}{2}} \Delta^2, \quad x \in \mathbb{R}.$$

As a result, it is plain that, for any x_0 in $\mathbb{R}$,

$$\begin{aligned} \mathbb{P}\left[x_0 < \sigma Z \leqslant x_0 + \Delta\right] &= F_{\sigma Z}(x_0 + \Delta) - F_{\sigma Z}(x_0) \\ &\leqslant \Delta \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x_0^2}{2}} + \frac{1}{2\sigma^2\sqrt{2\pi}} e^{-\frac{1}{2}} \Delta^2. \end{aligned}$$

The lemma is established upon noting that $\sup_{x \in \mathbb{R}} e^{-\frac{x^2}{2}} = 1$. ■

7.6 A proof of Theorem 7.2

By Lemma 7.4, we know that there exist a constant $C_1 > 0$ and a positive integer N_1 such that (7.11a) is bounded above by $\frac{C_1}{\sqrt{L_n}}$ whenever $n \geqslant N_1$. Next we proceed to give an upper bound to (7.11b). We need to find a constant $C_2 > 0$ and a positive integer N_2 such that (7.11b) is bounded above by $\frac{C_2}{\sqrt{L_n}}$ whenever $n \geqslant N_2$, in which case we are able to conclude Theorem 7.2 by letting $C = C_1 + C_2$ and $N = \max(N_1, N_2)$.

According to Lemma 7.5, there exists a constant $C_{21}^{*} > 0$ such that (7.16a) can be bounded above by

$$\sup_{y\in\mathbb{R}} \left| \mathbb{P}\left[Y_{L_n} \leqslant y\right] - \mathbb{P}\left[\sigma_0 Z \leqslant y\right] \right| \quad \leqslant \quad \frac{C_{21}}{\sqrt{L_n}}, \qquad n = 2, 3, \ldots.$$

where

$$C_{21} = \frac{C_{21}^{*}\left(\mu(1)^2 + \mu(0)^2\right)}{\sqrt{\mu(1)\mu(0)}}.$$

In a similar manner, an upper bound to (7.16b) is given by

$$\sup_{x\in\mathbb{R}} \mathbb{P}\left[x < \sigma_0 Z \leqslant x + \varepsilon\right] \quad \leqslant \quad \frac{1}{\sqrt{2\pi\sigma_0^2}}\varepsilon + \frac{1}{2\sigma^2\sqrt{2\pi}} e^{-\frac{1}{2}}\varepsilon^2, \qquad \varepsilon > 0$$

by virtue of Lemma 7.7. Let $\varepsilon = \frac{1}{\sqrt{L_n}}$, then there exists a positive integer N_{22} such that

$$\frac{1}{2\sigma^2\sqrt{2\pi}} e^{-\frac{1}{2}} \frac{1}{L_n} < \frac{1}{\sqrt{2\pi\sigma_0^2}} \frac{1}{\sqrt{L_n}}, \qquad n \geqslant N_{22},$$

which implies

$$\frac{1}{\sqrt{2\pi\sigma_0^2}} \frac{1}{\sqrt{L_n}} + \frac{1}{2\sigma^2\sqrt{2\pi}} e^{-\frac{1}{2}} \frac{1}{L_n} < \frac{C_{22}}{\sqrt{L_n}}, \qquad n \geqslant N_{22}$$

with

$$C_{22} = \frac{2}{\sqrt{2\pi\sigma_0^2}}.$$

With $\varepsilon = \frac{1}{\sqrt{L_n}}$, an upper bound to (7.16c) is given by Lemma 7.6, and takes the form

$$\mathbb{P}\left[|X_{n,L_n}| > \frac{1}{\sqrt{L_n}}\right] \leqslant \frac{2}{(n-1)\Gamma(0)^{L_n}} \frac{1}{\left(1 - e^{-1+\ln\frac{n}{n-1}}\right)^2}, \qquad n \geqslant N_{23}$$

for some positive integer N_{23}. Under the condition $1 + \rho \ln \Gamma(0) > 0$, there exists a positive integer N_{24} such that

$$\frac{2}{(n-1)\Gamma(0)^{L_n}} \frac{1}{\left(1 - e^{-1+\ln\frac{n}{n-1}}\right)^2} < \frac{1}{2\sqrt{L_n}}, \qquad n \geqslant N_{24}.$$

Collecting these bounds, we see that (7.11b) is bounded above by $\frac{C_2}{\sqrt{L_n}}$ whenever $n \geqslant N_2$ with $C_2 = C_{21} + C_{22} + 1$ and $N_2 = \max(N_{22}, N_{23}, N_{24})$. The proof of Theorem 7.2 is now completed. ■

7.7 A proof of Lemma 7.3

Fix $n = 2, 3, \ldots,$ and $L = 1, 2, \ldots$. For any $\varepsilon > 0$ and t in $\mathbb{R}$, we have

$$\mathbb{P}\left[X_{n,L} + Y_L \leqslant t\right] = \mathbb{P}\left[X_{n,L} + Y_L \leqslant t, |X_{n,L}| \leqslant \varepsilon\right] + \mathbb{P}\left[X_{n,L} + Y_L \leqslant t, |X_{n,L}| > \varepsilon\right].$$

For each t in $\mathbb{R}$, we can write

$$\begin{aligned} & \left|\mathbb{P}\left[X_{n,L} + Y_L \leqslant t\right] - \mathbb{P}\left[\sigma_0 Z \leqslant t\right]\right| \\ = & \left|\mathbb{P}\left[X_{n,L} + Y_L \leqslant t, |X_{n,L}| \leqslant \varepsilon\right] + \mathbb{P}\left[X_{n,L} + Y_L \leqslant t, |X_{n,L}| > \varepsilon\right] - \mathbb{P}\left[\sigma_0 Z \leqslant t\right]\right| \\ \leqslant & \left|\mathbb{P}\left[X_{n,L} + Y_L \leqslant t, |X_{n,L}| \leqslant \varepsilon\right] - \mathbb{P}\left[\sigma_0 Z \leqslant t\right]\right| + \mathbb{P}\left[X_{n,L} + Y_L \leqslant t, |X_{n,L}| > \varepsilon\right] \end{aligned}$$

by the triangular inequality.

Further simplifications are possible, namely

$$\begin{aligned} & \left|\mathbb{P}\left[X_{n,L} + Y_L \leqslant t\right] - \mathbb{P}\left[\sigma_0 Z \leqslant t\right]\right| \\ \leqslant & \left|\mathbb{P}\left[X_{n,L} + Y_L \leqslant t, |X_{n,L}| \leqslant \varepsilon\right] - \mathbb{P}\left[\sigma_0 Z \leqslant t\right]\right| + \mathbb{P}\left[|X_{n,L}| > \varepsilon\right] \end{aligned} \quad (7.20)$$

since

$$\mathbb{P}\left[X_{n,L} + Y_L \leqslant t, |X_{n,L}| > \varepsilon\right] \leqslant \mathbb{P}\left[|X_{n,L}| > \varepsilon\right].$$

The following lemma will simplify our analysis.

Lemma 7.8. *Let a_1, a_2, a_3, a_4, b_1 and b_2 be scalars in $\mathbb{R}$.*

1. If $a_3 \leqslant a_1 \leqslant a_4$, then

$$|a_1 + b_1| \leqslant \max(|a_3 + b_1|, |a_4 + b_1|).$$

2. If $b_2 > 0$, then

$$\max(|a_1 + b_1|, |a_2 + b_1 \pm b_2|) \leqslant \max(|a_1 + b_1|, |a_2 + b_1|) + b_2.$$

3. It holds that

$$\max(a_1 + b_1, a_2 + b_2) \leqslant \max(a_1, a_2) + \max(b_1, b_2).$$

We can easily obtain an upper bound to $\mathbb{P}\left[X_{n,L} + Y_L \leqslant t, |X_{n,L}| \leqslant \varepsilon\right]$, namely

$$\begin{aligned} \mathbb{P}\left[X_{n,L} + Y_L \leqslant t, |X_{n,L}| \leqslant \varepsilon\right] &\leqslant \mathbb{P}\left[Y_L \leqslant t + \varepsilon, |X_{n,L}| \leqslant \varepsilon\right] \\ &\leqslant \mathbb{P}\left[Y_L \leqslant t + \varepsilon\right] \end{aligned}$$

while a lower bound is given by

$$\begin{aligned} \mathbb{P}\left[X_{n,L} + Y_L \leqslant t, |X_{n,L}| \leqslant \varepsilon\right] &\geqslant \mathbb{P}\left[Y_L \leqslant t - \varepsilon, |X_{n,L}| \leqslant \varepsilon\right] \\ &= \mathbb{P}\left[Y_L \leqslant t - \varepsilon\right] - \mathbb{P}\left[Y_L \leqslant t - \varepsilon, |X_{n,L}| > \varepsilon\right] \\ &\geqslant \mathbb{P}\left[Y_L \leqslant t - \varepsilon\right] - \mathbb{P}\left[|X_{n,L}| > \varepsilon\right]. \end{aligned}$$

By virtue of Part 1 and Part 2 of Lemma 7.8, the bound to (7.15) can be refined as follows: Starting with (7.20), we get

$$
\begin{aligned}
& \Big|\mathbb{P}\left[X_{n,L}+Y_L \leqslant t\right]-\mathbb{P}\left[\sigma_0 Z \leqslant t\right]\Big| \\
\leqslant\ & \Big|\mathbb{P}\left[X_{n,L}+Y_L \leqslant t, |X_{n,L}| \leqslant \varepsilon\right]-\mathbb{P}\left[\sigma_0 Z \leqslant t\right]\Big| + \mathbb{P}\left[|X_{n,L}|>\varepsilon\right] \\
\leqslant\ & \max\left(\begin{array}{l} |\mathbb{P}\left[Y_L \leqslant t+\varepsilon\right]-\mathbb{P}\left[\sigma_0 Z \leqslant t\right]\Big|, \\ \Big|\mathbb{P}\left[Y_L \leqslant t-\varepsilon\right]-\mathbb{P}\left[|X_{n,L}|>\varepsilon\right]-\mathbb{P}\left[\sigma_0 Z \leqslant t\right]\Big| \end{array}\right) \\
& +\mathbb{P}\left[|X_{n,L}|>\varepsilon\right] \\
\leqslant\ & \max\left(\begin{array}{l} |\mathbb{P}\left[Y_L \leqslant t+\varepsilon\right]-\mathbb{P}\left[\sigma_0 Z \leqslant t\right]\Big|, \\ \Big|\mathbb{P}\left[Y_L \leqslant t-\varepsilon\right]-\mathbb{P}\left[\sigma_0 Z \leqslant t\right]\Big| + \mathbb{P}\left[|X_{n,L}|>\varepsilon\right] \end{array}\right) \\
& +\mathbb{P}\left[|X_{n,L}|>\varepsilon\right] \\
\leqslant\ & \max\left(\Big|\mathbb{P}\left[Y_L \leqslant t+\varepsilon\right]-\mathbb{P}\left[\sigma_0 Z \leqslant t\right]\Big|, \Big|\mathbb{P}\left[Y_L \leqslant t-\varepsilon\right]-\mathbb{P}\left[\sigma_0 Z \leqslant t\right]\Big|\right) \\
& +2\mathbb{P}\left[|X_{n,L}|>\varepsilon\right]. \qquad (7.21)
\end{aligned}
$$

Further considering of the two items in the maximum in (7.21) yields

$$
\begin{aligned}
& \Big|\mathbb{P}\left[Y_L \leqslant t+\varepsilon\right]-\mathbb{P}\left[\sigma_0 Z \leqslant t\right]\Big| \\
=\ & \Big|\mathbb{P}\left[Y_L \leqslant t+\varepsilon\right]-\mathbb{P}\left[\sigma_0 Z \leqslant t+\varepsilon\right]+\mathbb{P}\left[\sigma_0 Z \leqslant t+\varepsilon\right]-\mathbb{P}\left[\sigma_0 Z \leqslant t\right]\Big| \\
\leqslant\ & \Big|\mathbb{P}\left[Y_L \leqslant t+\varepsilon\right]-\mathbb{P}\left[\sigma_0 Z \leqslant t+\varepsilon\right]\Big| + \mathbb{P}\left[t<\sigma_0 Z \leqslant t+\varepsilon\right]
\end{aligned}
$$

and

$$
\begin{aligned}
& \Big|\mathbb{P}\left[Y_L \leqslant t-\varepsilon\right]-\mathbb{P}\left[\sigma_0 Z \leqslant t\right]\Big| \\
=\ & \Big|\mathbb{P}\left[Y_L \leqslant t-\varepsilon\right]-\mathbb{P}\left[\sigma_0 Z \leqslant t-\varepsilon\right]+\mathbb{P}\left[\sigma_0 Z \leqslant t-\varepsilon\right]-\mathbb{P}\left[\sigma_0 Z \leqslant t\right]\Big| \\
\leqslant\ & \Big|\mathbb{P}\left[Y_L \leqslant t-\varepsilon\right]-\mathbb{P}\left[\sigma_0 Z \leqslant t-\varepsilon\right]\Big| + \mathbb{P}\left[t-\varepsilon<\sigma_0 Z \leqslant t\right].
\end{aligned}
$$

By Part 3 of Lemma 7.8 , the upper bound to (7.15) becomes

$$\begin{aligned}
&\Big|\mathbb{P}\left[X_{n,L}+Y_L\leqslant t\right]-\mathbb{P}\left[\sigma_0 Z\leqslant t\right]\Big| \\
\leqslant\ & \max\left(\Big|\mathbb{P}\left[Y_L\leqslant t+\varepsilon\right]-\mathbb{P}\left[\sigma_0 Z\leqslant t+\varepsilon\right]\Big|,\Big|\mathbb{P}\left[Y_L\leqslant t-\varepsilon\right]-\mathbb{P}\left[\sigma_0 Z\leqslant t-\varepsilon\right]\Big|\right) \\
&+\max\Big(\mathbb{P}\left[t<\sigma_0 Z\leqslant t+\varepsilon\right],\mathbb{P}\left[t-\varepsilon<\sigma_0 Z\leqslant t\right]\Big)+2\mathbb{P}\left[|X_{n,L}|>\varepsilon\right] \\
\leqslant\ & \sup_{y\in\mathbb{R}}\Big|\mathbb{P}\left[Y_L\leqslant y\right]-\mathbb{P}\left[\sigma_0 Z\leqslant y\right]\Big| \\
&+\sup_{y\in\mathbb{R}}\mathbb{P}\left[y<\sigma_0 Z\leqslant y+\varepsilon\right]+2\mathbb{P}\left[|X_{n,L}|>\varepsilon\right]. \qquad (7.22)
\end{aligned}$$

The proof of Lemma 7.3 is completed by noting that the bound in (7.22) does not depend on t. ■

7.8 Simulations

To visualize the approximation to the CDF of the rv D_{n,L_n}, we plot the empirical CDF of the rv D_{n,L_n} where the data is collected from simulations, and compare it with the CDF of the rv $e^{\sqrt{L_n}(\sigma_0 Z+\mathcal{E}_{n,L_n})}$. The comparison is based on the approximation in (7.1), namely

$$\mathbb{P}\left[D_{n,L_n}\leqslant x\right]=_{\text{Approx}}\mathbb{P}\left[\sigma_0 Z\leqslant\frac{1}{\sqrt{L_n}}\ln(x)-\mathcal{E}_{n,L_n}\right]=\mathbb{P}\left[e^{\sqrt{L_n}(\sigma_0 Z+\mathcal{E}_{n,L_n})}\leqslant x\right]$$

for $x=1,2,\ldots$.

Fix $n=2,3,\ldots$ and a $\rho-$admissible scaling $\mathtt{L}:\mathbb{N}_0\to\mathbb{N}_0$ with $\rho>0$. By Theorem 7.2, the quantity

$$\Delta_{n,L_n}=\max_{x=0,1,\ldots,n-1}\left|\mathbb{P}\left[D_{n,L_n}<x\right]-\mathbb{P}\left[e^{\sqrt{L_n}(\sigma_0 Z+\mathcal{E}_{n,L_n})}\leqslant x\right]\right|$$

is bounded above by $\frac{C}{L_n}$ for some $C>0$ when n is sufficiently large.

Let $K = 1, 2, \ldots$ be the sample size which does not depend on the number of nodes n. Fix α, β, γ and $\mu(1)$ in $(0, 1)$. For each $n = 2, 3, \ldots$ and the corresponding L_n, we generate K independent MAGs. The degree of node 1 is collected from each of these K MAGs to form the collection of i.i.d. degree samples $\{D_{n,L_n,i},\ i = 1, \ldots, K\}$. We use

$$\widehat{\Delta}_{n,L_n,K} = \max_{x=0,1,\ldots,n-1} \left| \frac{1}{K} \sum_{i=1}^{K} \mathbf{1}[D_{n,L_n,i} \leqslant x] - \mathbb{P}\left[e^{\sqrt{L_n}(\sigma_0 Z + \mathcal{E}_{n,L_n})} \leqslant x \right] \right|$$

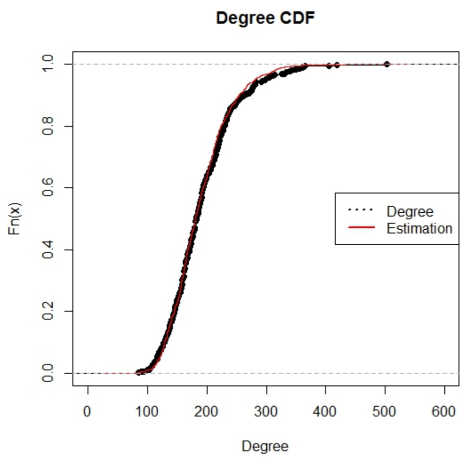
to evaluate the performance of the approximation instead of Δ_{n,L_n} since the empirical CDF $\frac{1}{K} \sum_{i=1}^{K} \mathbf{1}[D_{n,L,i} \leqslant x]$ is a strongly consistent point estimator of $\mathbb{P}[D_{n,L} \leqslant x]$ for each $n = 2, 3, \ldots,\ L = 1, 2, \ldots$ and $x = 0, 1, \ldots, n-1$.

Selected results are plotted in Figure 7.1. This figure is generated as follows: We have set $\alpha = 0.9$, $\beta = 0.7$, $\gamma = 0.6$, $\mu(1) = 0.35$ and the $\rho-$admissible scaling to be

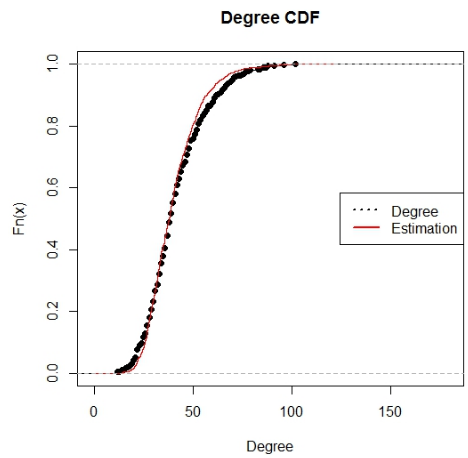
$$L_n = \lfloor \rho \ln n \rfloor.$$

We varied n and ρ while keeping $1 + \rho \ln \Gamma(0) > 0$. Four cases are considered (i.e., $(n = 4000, \rho = 1)$, $(n = 4000, \rho = 1.5)$, $(n = 8500, \rho = 1)$ and $(n = 8500, \rho = 1.5)$). For each case, we generated 1000 graphs (i.e., $K = 1000$), and collected the degree value of node 1 from each graph so that all samples are mutually independent. The black dots form the empirical CDF of D_{n,L_n} while the red line reflects the function $F(x) = \mathbb{P}\left[e^{\sqrt{L_n}(\sigma_0 Z + \mathcal{E}_{n,L_n})} \leqslant x \right]$. We find that $\widehat{\Delta}_{\max}$s are very small in all cases, especially comparing with $\frac{1}{\sqrt{L}}$. Due to computational limitations, we cannot make L_n (or $\rho_n \ln n$) very large in order to see an obvious trend of convergence. But nevertheless, the maximum difference between the two CDFs being less than 0.05

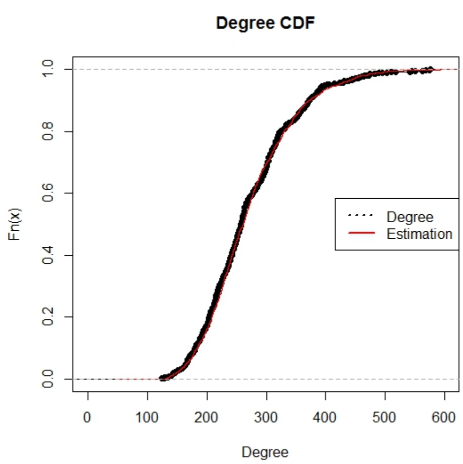
suggested that our approximation is very accurate.



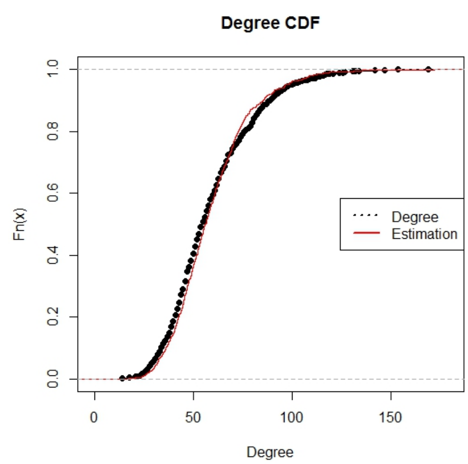
(a) $\rho = 1,\ n = 4000,\ L_n = 8,\quad \widehat{\Delta}_{max} = 0.0355$



(b) $\rho = 1.5,\ n = 4000,\ L_n = 12,\ \widehat{\Delta}_{max} = 0.0422$



(c) $\rho = 1,\ n = 8500,\ L_n = 9,\quad \widehat{\Delta}_{max} = 0.0420$



(d) $\rho = 1.5,\ n = 8500,\ L_n = 13,\ \widehat{\Delta}_{max} = 0.0451$

Figure 7.1: CDF Comparison

Appendix A:

A proof of Lemma 3.4

The arguments are very similar to the ones given in the proof of Lemma 3.3. Pick positive $n = 2, 3, \ldots$ and $L = 1, 2, \ldots$, and consider distinct nodes $u, v \in V_n$. For $k, \ell = 0, 1, \ldots, L$, not necessarily distinct, we start from the relation (3.32). Note that the factor $\xi_{n,L}(u)\xi_{n,L}(v)$ can be expressed as

$$\begin{aligned}
\xi_{n,L}(u)\xi_{n,L}(v) &= \prod_{w=1,\ w\neq u}^{n} (1 - \chi_L(u,w)) \cdot \prod_{w=1,\ w\neq v}^{n} (1 - \chi_L(v,w)) \\
&= (1 - \chi_L(u,v)) \cdot \prod_{w=1,\ w\neq u,v}^{n} (1 - \chi_L(u,w)) (1 - \chi_L(v,w))
\end{aligned}$$

with factors that can be represented as

$$1 - \chi_L(u,v) = \mathbf{1}\left[U_{u,v} > Q_L(\boldsymbol{A}_L(u), \boldsymbol{A}_L(v))\right]$$

and

$$\begin{aligned}
&\prod_{w=1,\ w\neq u,v}^{n} (1 - \chi_L(u,w)) (1 - \chi_L(v,w)) \\
= &\prod_{w=1,\ w\neq u,v}^{n} \mathbf{1}\left[U_{u,w} > Q_L(\boldsymbol{A}_L(u), \boldsymbol{A}_L(w))\right] \cdot \mathbf{1}\left[U_{v,w} > Q_L(\boldsymbol{A}_L(v), \boldsymbol{A}_L(w))\right].
\end{aligned}$$

Under the enforced independence assumptions, it is now straightforward to conclude that

$$\begin{aligned}
&\mathbb{E}\left[\xi_{n,L}(u)\xi_{n,L}(v)\middle|\boldsymbol{A}_L(1),\ldots,\boldsymbol{A}_L(n)\right] \\
= \;& (1-Q_L(\boldsymbol{A}_L(u),\boldsymbol{A}_L(v)))\cdot\prod_{w=1,\ w\neq u,v}^{n}(1-Q_L(\boldsymbol{A}_L(u),\boldsymbol{A}_L(w)))\,(1-Q_L(\boldsymbol{A}_L(v),\boldsymbol{A}_L(w)))\,.
\end{aligned}$$

The smoothing property of conditional expectations is again invoked, this time to obtain

$$\begin{aligned}
&\mathbb{E}\left[\xi_{n,L}(u)\xi_{n,L}(v)\middle|\boldsymbol{A}_L(u),\boldsymbol{A}_L(v)\right] \\
= \;& \mathbb{E}\left[\mathbb{E}\left[\xi_{n,L}(u)\xi_{n,L}(v)\middle|\boldsymbol{A}_L(1),\ldots,\boldsymbol{A}_L(n)\right]\middle|\boldsymbol{A}_L(u),\boldsymbol{A}_L(v)\right] \\
= \;& (1-Q_L(\boldsymbol{A}_L(u),\boldsymbol{A}_L(v)))\cdot\mathbb{E}\left[\prod_{w=1,\ w\neq u,v}^{n}\ldots\middle|\boldsymbol{A}_L(u),\boldsymbol{A}_L(v)\right] \qquad \text{(A.1)}
\end{aligned}$$

where

$$\begin{aligned}
&\mathbb{E}\left[\prod_{w=1,\ w\neq u,v}^{n}\ldots\middle|\boldsymbol{A}_L(u),\boldsymbol{A}_L(v)\right] \\
= \;& \mathbb{E}\left[\prod_{w=1,\ w\neq u,v}^{n}(1-Q_L(\boldsymbol{A}_L(u),\boldsymbol{A}_L(w)))\,(1-Q_L(\boldsymbol{A}_L(v),\boldsymbol{A}_L(w)))\middle|\boldsymbol{A}_L(u),\boldsymbol{A}_L(v)\right] \\
= \;& \mathbb{E}\left[\prod_{w=1,\ w\neq u,v}^{n}(1-Q_L(\boldsymbol{a}_L,\boldsymbol{A}_L(w)))\,(1-Q_L(\boldsymbol{b}_L,\boldsymbol{A}_L(w)))\middle|\boldsymbol{A}_L(u),\boldsymbol{A}_L(v)\right]_{\substack{\boldsymbol{a}_L\,=\,\boldsymbol{A}_L(u)\\ \boldsymbol{b}_L\,=\,\boldsymbol{A}_L(v)}} \\
= \;& \mathbb{E}\left[\prod_{w=1,\ w\neq u,v}^{n}(1-Q_L(\boldsymbol{a}_L,\boldsymbol{A}_L(w)))\,(1-Q_L(\boldsymbol{b},\boldsymbol{A}_L(w)))\right]_{\boldsymbol{a}_L=\boldsymbol{A}_L(u),\boldsymbol{b}_L=\boldsymbol{A}_L(v)} \\
= \;& \prod_{w=1,\ w\neq u,v}^{n}\mathbb{E}\left[(1-Q_L(\boldsymbol{a}_L,\boldsymbol{A}_L(w)))\,(1-Q_L(\boldsymbol{b}_L,\boldsymbol{A}_L(w)))\right]_{\boldsymbol{a}_L=\boldsymbol{A}_L(u),\boldsymbol{b}_L=\boldsymbol{A}_L(v)} \\
= \;& \left(\mathbb{E}\left[(1-Q_L(\boldsymbol{a}_L,\boldsymbol{A}_L))\,(1-Q_L(\boldsymbol{b}_L,\boldsymbol{A}_L))\right]_{\boldsymbol{a}_L=\boldsymbol{A}_L(u),\boldsymbol{b}_L=\boldsymbol{A}_L(v)}\right)^{n-2} \qquad \text{(A.2)}
\end{aligned}$$

under the enforced i.i.d. assumptions on the rvs $\boldsymbol{A}_L(1), \ldots, \boldsymbol{A}_L(n)$. Using the notation introduced earlier at (2.23) and (3.30) we can write

$$
\begin{aligned}
&\mathbb{E}\left[\left(1 - Q_L(\boldsymbol{a}_L, \boldsymbol{A}_L)\right)\left(1 - Q_L(\boldsymbol{b}_L, \boldsymbol{A}_L)\right)\right] \\
&\quad = \quad 1 - Q_L^\star(\boldsymbol{a}_L) - Q_L^\star(\boldsymbol{b}_L) + Q_L^{\star\star}(\boldsymbol{a}_L, \boldsymbol{b}_L), \quad \boldsymbol{a}_L, \boldsymbol{b}_L \in \{0,1\}^L.
\end{aligned}
\tag{A.3}
$$

This allows us to conclude that

$$
\begin{aligned}
&\mathbb{E}\left[\prod_{w=1,\ w\neq u,v}^{n} \left(1 - Q_L(\boldsymbol{A}_L(u), \boldsymbol{A}_L(w))\right)\left(1 - Q_L(\boldsymbol{A}_L(v), \boldsymbol{A}_L(w))\right) \,\middle|\, \boldsymbol{A}_L(u), \boldsymbol{A}_L(v)\right] \\
= \quad & \left(1 - Q_L^\star(\boldsymbol{A}_L(u)) - Q_L^\star(\boldsymbol{A}_L(v)) + Q_L^{\star\star}(\boldsymbol{A}_L(u), \boldsymbol{A}_L(v))\right)^{n-2},
\end{aligned}
\tag{A.4}
$$

and substituting into (A.1) we obtain the desired conclusion (3.29). ■

Appendix B:

A proof of Lemma 4.6

For any ρ-admissible scaling $\mathtt{L} : \mathbb{N}_0 \to \mathbb{N}_0$, Lemma 2.6 gives

$$
\begin{aligned}
&\binom{L_n}{\lceil \lambda L_n \rceil} \\
&\sim \frac{\sqrt{2\pi L_n}\left(\frac{L_n}{e}\right)^{L_n}}{\sqrt{2\pi \lceil \lambda L_n \rceil}\left(\frac{\lceil \lambda L_n \rceil}{e}\right)^{\lceil \lambda L_n \rceil}\sqrt{2\pi (L_n - \lceil \lambda L_n \rceil)}\left(\frac{L_n - \lceil \lambda L_n \rceil}{e}\right)^{L_n - \lceil \lambda L_n \rceil}} \\
&= \frac{\sqrt{2\pi L_n}\left(\frac{L_n}{e}\right)^{L_n}}{\sqrt{2\pi \lambda L_n}\left(\frac{\lambda L_n}{e}\right)^{\lambda L_n}\sqrt{2\pi (1-\lambda) L_n}\left(\frac{(1-\lambda) L_n}{e}\right)^{(1-\lambda) L_n}} && \text{(B.1a)} \\
&\quad \times \sqrt{\frac{\lambda L_n (1-\lambda) L_n}{(\lceil \lambda L_n \rceil)(L_n - \lceil \lambda L_n \rceil)}} && \text{(B.1b)} \\
&\quad \times \left(\frac{\lambda L_n}{\lceil \lambda L_n \rceil}\right)^{\lceil \lambda L_n \rceil}\left(\frac{(1-\lambda) L_n}{L_n - \lceil \lambda L_n \rceil}\right)^{L_n - \lceil \lambda L_n \rceil} && \text{(B.1c)} \\
&\quad \times \left(\frac{(1-\lambda)}{\lambda}\right)^{\delta(n; L_n)} && \text{(B.1d)}
\end{aligned}
$$

with

$$\delta(n; L_n) = \lceil \lambda L_n \rceil - \lambda L_n \tag{B.2}$$

taking values in $(0, 1)$ for $n = 1, 2, \ldots$.

The lemma will be established if we show that (B.1b) and (B.1c) converge to 1 when n goes to infinity and (B.1d) is bounded below by a carefully selected constant.

It is plain that (B.1b) converges to 1 since

$$\lim_{n\to\infty} \frac{\lambda L_n}{\lceil \lambda L_n \rceil} = 1 \text{ and } \lim_{n\to\infty} \frac{(1-\lambda)L_n}{L_n - \lceil \lambda L_n \rceil} = 1.$$

Indeed, for each $n = 2, 3, \ldots,$ we have

$$\frac{\lambda L_n}{\lceil \lambda L_n \rceil} = \frac{\lceil \lambda L_n \rceil - \delta(n; L_n)}{\lceil \lambda L_n \rceil} = 1 - \frac{\delta(n; L_n)}{\lceil \lambda L_n \rceil} \tag{B.3}$$

with $\delta(n; L_n)$ bounded between 0 and 1 and $\lceil \lambda L_n \rceil$ goes to infinity when n grows unboundedly large. Similar arguments yield

$$\lim_{n\to\infty} \frac{(1-\lambda)L_n}{L_n - \lceil \lambda L_n \rceil} = \lim_{n\to\infty} \left(1 + \frac{\delta(n; L_n)}{L_n - \lceil \lambda L_n \rceil}\right) = 1.$$

For (B.1c), it involves the technique of Taylor series expansion for $\ln(1+x)$ when x is close to 0, namely

$$\ln(1+x) = x - \frac{x^2}{2} + o(x^2)$$

From (B.3), we know

$$\begin{aligned} \left(\frac{\lambda L_n}{\lceil \lambda L_n \rceil}\right)^{\lceil \lambda L_n \rceil} &= \left(1 - \frac{\delta(n; L_n)}{\lceil \lambda L_n \rceil}\right)^{\lceil \lambda L_n \rceil} \\ &= e^{\lceil \lambda L_n \rceil \ln\left(1 - \frac{\delta(n; L_n)}{\lceil \lambda L_n \rceil}\right)}. \end{aligned} \tag{B.4}$$

Since $\frac{\delta(n;L_n)}{\lceil \lambda L_n \rceil}$ converges to 0 when n goes to infinity, Taylor series expansion of $\ln\left(1 - \frac{\delta(n;L_n)}{\lceil \lambda L_n \rceil}\right)$ gives

$$\ln\left(1 - \frac{\delta(n; L_n)}{\lceil \lambda L_n \rceil}\right) = -\frac{\delta(n; L_n)}{\lceil \lambda L_n \rceil} - \frac{\delta(n; L_n)^2}{2\lceil \lambda L_n \rceil^2} + o\left(\frac{\delta(n; L_n)^2}{\lceil \lambda L_n \rceil^2}\right).$$

As a result, (B.4) becomes

$$\left(\frac{\lambda L_n}{\lceil \lambda L_n \rceil}\right)^{\lceil \lambda L_n \rceil} = e^{-\delta(n;L_n) - \frac{\delta(n;L_n)^2}{2\lceil \lambda L_n \rceil} + o\left(\frac{\delta(n;L_n)^2}{\lceil \lambda L_n \rceil}\right)}. \tag{B.5}$$

In a similar manner, we have

$$\begin{aligned}\left(\frac{(1-\lambda)L_n}{L_n - \lceil \lambda L_n \rceil}\right)^{L_n - \lceil \lambda L_n \rceil} &= \left(1 + \frac{\delta(n;L_n)}{L_n - \lceil \lambda L_n \rceil}\right)^{L_n - \lceil \lambda L_n \rceil} \\ &= e^{(L_n - \lceil \lambda L_n \rceil)\ln\left(1 + \frac{\delta(n;L_n)}{L_n - \lceil \lambda L_n \rceil}\right)}. \end{aligned} \tag{B.6}$$

Taylor series expansion of $\ln\left(1 + \frac{\delta(n;L_n)}{L_n - \lceil \lambda L_n \rceil}\right)$ gives

$$\ln\left(1 + \frac{\delta(n;L_n)}{L_n - \lceil \lambda L_n \rceil}\right) = \frac{\delta(n;L_n)}{L_n - \lceil \lambda L_n \rceil} - \frac{\delta(n;L_n)^2}{2(L_n - \lceil \lambda L_n \rceil)^2} + o\left(\frac{\delta(n;L_n)^2}{(L_n - \lceil \lambda L_n \rceil)^2}\right).$$

since $\frac{\delta(n;L_n)}{L_n - \lceil \lambda L_n \rceil}$ converges to 0 when n goes to ∞. Then (B.6) becomes

$$\left(\frac{(1-\lambda)L_n}{L_n - \lceil \lambda L_n \rceil}\right)^{L_n - \lceil \lambda L_n \rceil} = e^{\delta(n;L_n) - \frac{\delta(n;L_n)^2}{2(L_n - \lceil \lambda L_n \rceil)} + o\left(\frac{\delta(n;L_n)^2}{L_n - \lceil \lambda L_n \rceil}\right)}. \tag{B.7}$$

Multiplying (B.5) by (B.7), we get

$$\begin{aligned}&\left(\frac{\lambda L_n}{\lceil \lambda L_n \rceil}\right)^{\lceil \lambda L_n \rceil}\left(\frac{(1-\lambda)L_n}{L_n - \lceil \lambda L_n \rceil}\right)^{L_n - \lceil \lambda L_n \rceil} \\ = \;& e^{-\frac{\delta(n;L_n)^2}{2\lceil \lambda L_n \rceil} + o\left(\frac{\delta(n;L_n)^2}{2\lceil \lambda L_n \rceil}\right) - \frac{\delta(n;L_n)^2}{2(L_n - \lceil \lambda L_n \rceil)} + o\left(\frac{\delta(n;L_n)^2}{2(L_n - \lceil \lambda L_n \rceil)}\right)}. \end{aligned} \tag{B.8}$$

It is now plain that (B.1c) converges to 1 since the exponent in (B.8) converges to 0 when n grows unboundedly large.

For (B.1d), it is plain that

$$\left(\frac{1-\lambda}{\lambda}\right)^{\delta(n;L_n)} \geqslant 1 \qquad \text{if } \frac{1-\lambda}{\lambda} \geqslant 1$$

and

$$\left(\frac{1-\lambda}{\lambda}\right)^{\delta(n;L_n)} \geqslant \frac{1-\lambda}{\lambda} \qquad \text{if } \frac{1-\lambda}{\lambda} < 1.$$

It follows that (B.1d) is bounded below by the quantity $k(\lambda)$ given by

$$k(\lambda) = \min\left(1, \frac{1-\lambda}{\lambda}\right).$$

Consequently, as (B.1b) and (B.1c) both converge to 1, we conclude that

$$\begin{aligned}
\binom{L_n}{\lceil \lambda L_n \rceil} &\sim \frac{\sqrt{2\pi L_n}\left(\frac{L_n}{e}\right)^{L_n}}{\sqrt{2\pi\lambda L_n}\left(\frac{\lambda L_n}{e}\right)^{\lambda L_n}\sqrt{2\pi(1-\lambda)L_n}\left(\frac{(1-\lambda)L_n}{e}\right)^{(1-\lambda)L_n}}\left(\frac{1-\lambda}{\lambda}\right)^{\delta(n;L_n)} \\
&= \frac{1}{\sqrt{2\pi\lambda(1-\lambda)L_n}}\left(\frac{1}{\lambda}\right)^{\lambda L_n}\left(\frac{1}{1-\lambda}\right)^{(1-\lambda)L_n}\left(\frac{1-\lambda}{\lambda}\right)^{\delta(n;L_n)}
\end{aligned}$$

Moreover, from the lower bound on (B.1d), we conclude that

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi\lambda(1-\lambda)L_n}}\left(\frac{1}{\lambda}\right)^{\lambda L_n}\left(\frac{1}{1-\lambda}\right)^{(1-\lambda)L_n}\left(\frac{1-\lambda}{\lambda}\right)^{\delta(n;L_n)} \\
\geqslant\ &\frac{k(\lambda)}{\sqrt{2\pi\lambda(1-\lambda)L_n}}\left(\frac{1}{\lambda}\right)^{\lambda L_n}\left(\frac{1}{1-\lambda}\right)^{(1-\lambda)L_n} \\
>\ &\frac{k(\lambda)}{2\sqrt{2\pi\lambda(1-\lambda)L_n}}\left(\frac{1}{\lambda}\right)^{\lambda L_n}\left(\frac{1}{1-\lambda}\right)^{(1-\lambda)L_n}, \quad n = 1, 2, \ldots. \qquad \text{(B.9)}
\end{aligned}$$

Because of the strict inequality in (B.9), there exists a positive integer N such that

$$\binom{L_n}{\lceil \lambda L_n \rceil} \geqslant \frac{k(\lambda)}{2\sqrt{2\pi\lambda(1-\lambda)L_n}}\left(\frac{1}{\lambda}\right)^{\lambda L_n}\left(\frac{1}{1-\lambda}\right)^{(1-\lambda)L_n}, \quad n \geqslant N.$$

This completes the proof of Lemma 4.6. ■

Appendix C:

A proof of Proposition 5.10

The calculation is pretty straight forward. Recall that $T_n(L)$ is the sum of indicator rvs $\{\xi_{n,L}(u,v,w), u,v,w \in V_n,\ u < v < w\}$. Expanding the square of this sum, we obtain four parts, namely

1. $\xi_{n,L}(u,v,w)\xi_{n,L}(u,v,w)$, product of components where all three end points are the same. There are $\binom{n}{3}$ of them;

2. $\xi_{n,L}(u,v,w)\xi_{n,L}(u,v,s)$, product of components which share two end points. There are $\binom{n}{3}\binom{3}{2}\binom{n-3}{1}$ different combinations;

3. $\xi_{n,L}(u,v,w)\xi_{n,L}(u,s,r)$, product of components which share only one end point. There are $\binom{n}{3}\binom{3}{1}\binom{n-3}{2}$ different combinations;

4. $\xi_{n,L}(u,v,w)\xi_{n,L}(s,r,t)$, product of distinct components which share no end points. There are $\binom{n}{3}\binom{n-3}{3}$ different combinations.

Therefore, we can write

$$
\begin{aligned}
\mathbb{E}[T_n(L)^2] &= \mathbb{E}\left[\left(\sum_{1\leqslant u<v<w\leqslant n} \xi_{n,L}(u,v,w)\right)^2\right] \\
&= \mathbb{E}\left[\sum_{1\leqslant u<v<w\leqslant n} \xi_{n,L}(u,v,w)\right] \\
&\quad +\mathbb{E}\left[\sum_{1\leqslant u<v<w\leqslant n} \sum_{\substack{u^*,\, v^* \in \{u,v,w\},\ u^* \neq v^*, \\ s \in V_n \setminus \{u,v,w\}}} \xi_{n,L}(u,v,w)\xi_{n,L}(u^*,v^*,s)\right] \\
&\quad +\mathbb{E}\left[\sum_{1\leqslant u<v<w\leqslant n} \sum_{\substack{u^* \in \{u,v,w\}, \\ r,s \in V_n \setminus \{u,v,w\},\ r<s}} \xi_{n,L}(u,v,w)\xi_{n,L}(u^*,r,s)\right] \\
&\quad +\mathbb{E}\left[\sum_{1\leqslant u<v<w\leqslant n} \sum_{\substack{r<s<t, \\ r,s,t \in V_n \setminus \{u,v,w\}}} \xi_{n,L}(u,v,w)\xi_{n,L}(r,s,t)\right],
\end{aligned}
$$

and this leads to

$$
\begin{aligned}
\mathbb{E}[T_n(L)^2] &= \binom{n}{3}\mathbb{P}[1 \sim_L 2, 2 \sim_L 3, 1 \sim_L 3] \\
&\quad +3(n-1)\binom{n}{3}\mathbb{P}[1 \sim_L 2, 2 \sim_L 3, 1 \sim_L 3, 1 \sim_L 4, 2 \sim_L 4] \\
&\quad +3\binom{n}{3}\binom{n-3}{2}\mathbb{P}[1 \sim_L 2, 2 \sim_L 3, 1 \sim_L 3, 1 \sim_L 4, 5 \sim_L 4, 1 \sim_L 5] \\
&\quad +\binom{n}{3}\binom{n-3}{3}\mathbb{P}[1 \sim_L 2, 2 \sim_L 3, 1 \sim_L 3]^2.
\end{aligned}
$$

■

Appendix D:

A proof of Lemma 7.6

Consider a $\rho-$admissible scaling $\mathtt{L} : \mathbb{N}_0 \to \mathbb{N}_0$ with $\rho > 0$. Fix $n = 2, 3, \ldots$ and $\varepsilon > 0$. The triangular inequality yields

$$\begin{aligned} & \left| \frac{1}{\sqrt{L_n}} \left(\ln D^+_{n,L_n} - \ln \mathbb{E}\left[D_{n,L_n} | S_{L_n}(1) \right] + \ln \frac{n-1}{n} \right) \right| \\ \leqslant & \left| \frac{1}{\sqrt{L_n}} \left(\ln D^+_{n,L_n} - \ln \mathbb{E}\left[D_{n,L_n} | S_{L_n}(1) \right] \right) \right| + \frac{1}{\sqrt{L_n}} \ln \frac{n}{n-1}, \end{aligned}$$

whence,

$$\mathbb{P}\left[|X_{n,L_n}| > \varepsilon \right] \leqslant \mathbb{P}\left[\left| \frac{1}{\sqrt{L_n}} \left(\ln D^+_{n,L_n} - \ln \mathbb{E}\left[D_{n,L_n} | S_{L_n}(1) \right] \right) \right| + \frac{1}{\sqrt{L_n}} \ln \frac{n}{n-1} > \varepsilon \right]. \tag{D.1}$$

For notational simplicity, we write

$$\varepsilon'_{n,L_n} = \sqrt{L_n}\varepsilon - \ln \frac{n}{n-1}, \qquad n = 2, 3, \ldots.$$

Because $\lim_{n\to\infty} \ln \frac{n}{n-1} = 0$, it is easy to see that, for any $\varepsilon > 0$, there exists a positive integer $N(\varepsilon)$ such that

$$\varepsilon'_{n,L_n} > 0, \quad n \geqslant N(\epsilon).$$

Then, when $n \geqslant N(\varepsilon)$, (D.1) can be rewritten as

$$\begin{aligned}
\mathbb{P}\left[|X_{n,L_n}| > \varepsilon\right] &\leqslant \mathbb{P}\left[\left|\ln \frac{D^+_{n,L_n}}{\mathbb{E}\left[D_{n,L_n}|S_{L_n}(1)\right]}\right| > {\varepsilon'}_{n,L_n}\right] \\
&= \mathbb{P}\left[\ln \frac{D^+_{n,L_n}}{\mathbb{E}\left[D_{n,L_n}|S_{L_n}(1)\right]} > {\varepsilon'}_{n,L_n}\right] + \mathbb{P}\left[\ln \frac{D^+_{n,L_n}}{\mathbb{E}\left[D_{n,L_n}|S_{L_n}(1)\right]} < -{\varepsilon'}_{n,L_n}\right] \\
&= \mathbb{P}\left[\frac{D^+_{n,L_n}}{\mathbb{E}\left[D_{n,L_n}|S_{L_n}(1)\right]} - 1 > e^{{\varepsilon'}_{n,L_n}} - 1\right] \\
&\quad + \mathbb{P}\left[\frac{D^+_{n,L_n}}{\mathbb{E}\left[D_{n,L_n}|S_{L_n}(1)\right]} - 1 < e^{-{\varepsilon'}_{n,L_n}} - 1\right] \\
&\leqslant \mathbb{P}\left[\left|\frac{D^+_{n,L_n}}{\mathbb{E}\left[D_{n,L_n}|S_{L_n}(1)\right]} - 1\right| > 1 - e^{-{\varepsilon'}_{n,L_n}}\right] \qquad \text{(D.2)}
\end{aligned}$$

where the last step was based on the fact that

$$1 - e^{-{\varepsilon'}_{n,L_n}} < e^{{\varepsilon'}_{n,L_n}} - 1, \quad {\varepsilon'}_{n,L_n} > 0.$$

Upon applying Markov inequality to the last step, we get

$$\mathbb{P}\left[\left(\frac{D^+_{n,L_n}}{\mathbb{E}\left[D_{n,L_n}|S_{L_n}(1)\right]} - 1\right)^2 > \left(1 - e^{{\varepsilon'}_{n,L_n}}\right)^2\right] \leqslant \frac{\mathbb{E}\left[\left(\frac{D^+_{n,L_n}}{\mathbb{E}\left[D_{n,L_n}|S_{L_n}(1)\right]} - 1\right)^2\right]}{\left(1 - e^{{\varepsilon'}_{n,L_n}}\right)^2}. \qquad \text{(D.3)}$$

With (6.6), the numerator of the left hand side of (D.3) takes the form

$$\begin{aligned}
&\mathbb{E}\left[\left(\frac{D^+_{n,L_n}}{\mathbb{E}\left[D_{n,L_n}|S_{L_n}(1)\right]} - 1\right)^2\right] \\
&= \mathbb{E}\left[\left(\frac{D_{n,L_n}}{\mathbb{E}\left[D_{n,L_n}|S_{L_n}(1)\right]} - 1 + \frac{\mathbf{1}[D_{n,L_n} = 0]}{\mathbb{E}\left[D_{n,L_n}|S_{L_n}(1)\right]}\right)^2\right] \\
&= \mathbb{E}\left[\left(\frac{D_{n,L_n}}{\mathbb{E}\left[D_{n,L_n}|S_{L_n}(1)\right]} - 1\right)^2\right] \qquad \text{(D.4a)} \\
&\quad + \mathbb{E}\left[2\left(\frac{D_{n,L_n}}{\mathbb{E}\left[D_{n,L_n}|S_{L_n}(1)\right]} - 1\right)\frac{\mathbf{1}[D_{n,L_n} = 0]}{\mathbb{E}\left[D_{n,L_n}|S_{L_n}(1)\right]}\right] \qquad \text{(D.4b)} \\
&\quad + \mathbb{E}\left[\left(\frac{\mathbf{1}[D_{n,L_n} = 0]}{\mathbb{E}\left[D_{n,L_n}|S_{L_n}(1)\right]}\right)^2\right]. \qquad \text{(D.4c)}
\end{aligned}$$

By (6.22), an upper bound to (D.4a) is given by

$$\mathbb{E}\left[\left(\frac{D_{n,L_n}}{\mathbb{E}\left[D_{n,L_n}|S_{L_n}(1)\right]}-1\right)^2\right] \leqslant \frac{1}{(n-1)\Gamma(0)^{L_n}}-\frac{1}{n-1}<\frac{1}{(n-1)\Gamma(0)^{L_n}}. \quad \text{(D.5)}$$

Because of the indicator function $\mathbf{1}[D_{n,L_n}=0]$, (D.4b) can be rewritten as

$$\begin{aligned} & \mathbb{E}\left[2\left(\frac{D_{n,L_n}}{\mathbb{E}\left[D_{n,L_n}|S_{L_n}(1)\right]}-1\right)\frac{\mathbf{1}[D_{n,L_n}=0]}{\mathbb{E}\left[D_{n,L_n}|S_{L_n}(1)\right]}\right] \\ = \quad & -2\mathbb{E}\left[\frac{\mathbf{1}[D_{n,L_n}=0]}{\mathbb{E}\left[D_{n,L_n}|S_{L_n}(1)\right]}\right] \\ < \quad & 0. \end{aligned} \quad \text{(D.6)}$$

For (D.4c), by preconditioning on $S_{L_n}(1)$, $\mathbb{E}\left[D_{n,L_n}|S_{L_n}(1)\right]$ is determined, namely

$$\begin{aligned} \mathbb{E}\left[\left(\frac{\mathbf{1}[D_{n,L_n}=0]}{\mathbb{E}\left[D_{n,L_n}|S_{L_n}(1)\right]}\right)^2\right] &= \mathbb{E}\left[\frac{\mathbb{E}\left[\mathbf{1}[D_{n,L_n}=0]|S_{L_n}(1)\right]}{\mathbb{E}\left[D_{n,L_n}|S_{L_n}(1)\right]^2}\right] \\ &= \mathbb{E}\left[\frac{\mathbb{P}\left[D_{n,L_n}=0|S_{n,L_n}(1)\right]}{(n-1)^2\Gamma(1)^{2S_{L_n}(1)}\Gamma(0)^{2L_n-2S_{L_n}(1)}}\right] \end{aligned}$$

upon using the fact that $\mathbf{1}[D_{n,L_n}=0]=\mathbf{1}[D_{n,L_n}=0]^2$.

Under the assumption $\Gamma(1)>\Gamma(0)$, it is plain that

$$(n-1)\Gamma(1)^{S_{L_n}(1)}\Gamma(0)^{L_n-S_{L_n}(1)} \geqslant (n-1)\Gamma(0)^{L_n}$$

and

$$\begin{aligned} \mathbb{P}\left[D_{n,L_n}=0|S_{L_n}(1)\right] &= \left(1-\Gamma(1)^{S_{L_n}(1)}\Gamma(0)^{L_n-S_{L_n}(1)}\right)^{n-1} \\ &\leqslant \left(1-\Gamma(0)^{L_n}\right)^{n-1} \end{aligned}$$

Using these facts, we get the bounds

$$\begin{aligned} \mathbb{E}\left[\left(\frac{\mathbf{1}[D_{n,L_n}=0]}{\mathbb{E}\left[D_{n,L_n}|S_{L_n}(1)\right]}\right)^2\right] &\leqslant \mathbb{E}\left[\frac{\left(1-\Gamma(0)^{L_n}\right)^{n-1}}{(n-1)^2\Gamma(0)^{2L_n}}\right] \\ &\leqslant \frac{e^{-(n-1)\Gamma(0)^{L_n}}}{(n-1)^2\Gamma(0)^{2L_n}}. \end{aligned} \quad \text{(D.7)}$$

Collecting (D.5), (D.6) and (D.7) we obtained, an upper bound to (D.4) is in the form

$$\mathbb{E}\left[\left(\frac{D^{+}_{n,L_n}}{\mathbb{E}\left[D_{n,L_n}|S_{L_n}(1)\right]}-1\right)^2\right] \leqslant \frac{1}{(n-1)\Gamma(0)^{L_n}}+\frac{e^{-(n-1)\Gamma(0)^{L_n}}}{(n-1)^2\Gamma(0)^{2L_n}}. \tag{D.8}$$

Under the condition $1+\rho\ln\Gamma(0)>0$, we have

$$\lim_{n\to\infty}(n-1)\Gamma(0)^{L_n}=\infty,$$

so that there exists $N_1>0$ such that for $n\geqslant N_1$, we have

$$\frac{e^{-(n-1)\Gamma(0)^{L_n}}}{(n-1)^2\Gamma(0)^{2L_n}}<\frac{1}{(n-1)\Gamma(0)^{L_n}}, \qquad n\geqslant N_1.$$

Together with (D.2) and (D.3), we readily conclude that

$$\mathbb{P}\left[|X_{n,L_n}|>\varepsilon\right]\leqslant\frac{2}{(n-1)\Gamma(0)^{L_n}}\frac{1}{\left(1-e^{-\sqrt{L_n}\varepsilon+\ln\frac{n}{n-1}}\right)^2}, \qquad n>N$$

with $N=\max(N_1,N(\varepsilon))$. ■

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