ABSTRACT

Title of dissertation: THE BOREL COMPLEXITY OF ISOMORPHISM FOR SOME FIRST-ORDER THEORIES

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In this work we consider several instances of the following problem: "how complicated can the isomorphism relation for countable models be?" Using the Borel reducibility framework from [4], we investigate this question with regard to the space of countable models of particular complete first-order theories. We also investigate to what extent this complexity is mirrored in the number of back-and-forth inequivalent models of the theory, denoted $I_{\infty\omega}(T)$. We consider this question for two large and related classes of theories.

First, we consider o-minimal theories, showing that if T is o-minimal, then $\cong_T$ is either Borel complete or Borel. Further, if it is Borel, then it is exactly equivalent to one of the following: $\cong_1$, $\cong_2$, or $(3^a 6^b, =)$, with $a, b \in \omega$. All values are possible, and we characterize exactly when each possibility occurs. Further, in all cases Borel completeness implies λ-Borel completeness for all λ. Much of this portion appeared in [21] and extends work from [25], which itself builds upon [15].

Second, we consider colored linear orders, which are (complete theories of) a linear order expanded by countably many unary predicates. We discover the

same characterization as with o-minimal theories, taking the same values, with the exception that all finite values are possible except two. We characterize exactly when each possibility occurs, which is similar to the o-minimal case. Additionally, we extend Schirrman's theorem from [26], showing that if the language is finite, then T is $\aleph_0$-categorical or Borel complete. As before, in all cases Borel completeness implies λ-Borel completeness for all λ. This work appeared in [20] and builds heavily on [24].

The Borel Complexity of Isomorphism for Some First-Order Theories

by

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Dedication

For my wife, Ran Cui.

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Table of Contents

List of Abbreviations and Notations

ZFC	Zermelo-Frankel set theory (with Choice)
$\mathrm{Mod}(\Phi, \omega)$	The Polish space of models of Φ with universe ω
$\mathrm{Mod}(\Phi)$	Synonymous with $\mathrm{Mod}(\Phi, \omega)$
$\mathrm{CSS}(\Phi)$	The class of all canonical Scott sentences extending Φ
$\|\Phi\|$	The potential cardinality of Φ, equal to $\lvert\mathrm{CSS}(\Phi)\rvert$
$I_{\infty\omega}(\Phi)$	The number of back-and-forth inequivalent models of Φ
cl	Definable closure
otp	Order Type
IT	Interval (or convex) Type, and the associated type-space

Chapter 1: The Complexity of First-Order Theories

Given a first-order theory T, how can we measure its complexity? We would like such a measure to be representative of some important features of the theory and to take on enough different values to distinguish between many different theories.

A classical answer to this question is to consider the *spectrum* function of T – the function taking uncountable cardinals λ to the number $I(T, \lambda)$ of pairwise non-isomorphic models of T. By important work of Shelah, Hart, Hrushovski, and Laskowski (see [27] and [3] for example), we know that these spectrum functions are always comparable (indeed there is essentially a list of them), and that they represent important properties of the theory, such as stability, superstability, NDOP, depth, and so on. Thus this is an excellent indicator of complexity; if T_1 has fewer models than T_2 (eventually) then T_1 is "less complex" than T_2, and so on.

However, this completely ignores the situation for countable model theory. For example, the theory of dense linear orders (without endpoints) has only one countable model, yet is "maximally complex" with regard to the spectrum function above. This sort of behavior is extremely common – one can be quite complicated at the countable level, but be quite simple at the uncountable level, or vice-versa. One fix seems to be to consider simply the number of countable models, but this turns out to be quite a coarse invariant (regardless of how Vaught's conjecture turns out), and we can do better.

One better way turns out to be Borel reducibility. We begin this chapter by defining Borel reductions. We will then build up the general theory around these, including basic results and several benchmarks. Additionally, we discuss back-and-forth equivalence, potential cardinality, and the connection this gives between countable model theory and uncountable model theory.

We hope to justify our assertion that (among other things) $\aleph_0$-categorical theories are "minimally complex," despite the possibility that they could be quite complex in terms of classical stability theory.

1.1 Borel Reductions

We are interested in measuring the complexity of the isomorphism problem for countable models of T, which is a finer measurement than just counting the number of countable models. A common way to do this is through the idea of Borel reducibility – establishing a natural way to see if one relation is "more difficult" than another to compute. That is: given two sentences Φ and Ψ, we say that "Φ is at least as complex as Ψ" if $(\mathrm{Mod}(\Psi, \omega), \cong) \leq_B (\mathrm{Mod}(\Phi, \omega), \cong)$. We will now define these terms.

The notion of a Borel reduction as a way to compare complexity of classes was introduced by Friedman and Stanley in [4]. Consider pairs of the form (X, E), where X is a Borel subset of a Polish space, and $E \subseteq X^2$ is an equivalence relation. Given two such pairs, we say $(X_1, E_1) \leq_B (X_2, E_2)$ (sometimes written as $E_1 \leq_B E_2$) if there is a Borel function $f : X_1 \to X_2$ where, for all $a, b \in X$, aE_1b holds if and only if $f(a)E_2f(b)$. It is clear that $\leq_B$ forms a preorder, and the requirement that f be Borel provides a reasonable analogue to saying "E_1 is effectively computable from E_2" when (as is usual) X_1 and X_2 are uncountable. We will use $<_B$ when the relation is strict – that is, $(X, E) <_B (Y, F)$ precisely when $(X, E) \leq_B (Y, F)$, but $(Y, F) \not\leq_B (X, E)$. Similarly we use $\sim_B$ when $(X, E) \leq_B (Y, F)$ and $(Y, F) \leq_B (X, E)$.

For a countable language L, define X_L^ω (for the purpose of this section) to be the set of L-structures with universe ω. This is a Polish space using the formula topology: for any formula $\phi(\overline{x})$ and any tuple $\overline{n}$ from ω, the set $\{\mathcal{M} \in X_L^\omega : \mathcal{M} \models \phi(\overline{n})\}$ is open. L-isomorphism is a natural equivalence relation on this space.

It is well-known that for any isomorphism-invariant Borel subset B of X_L^ω, there is a corresponding $L_{\omega_1,\omega}$-sentence Φ_B, such that B is the set of models of Φ_B – see, for example, Theorem 16.8 in [8]. We refer to B as $\mathrm{Mod}(\Phi_B, \omega)$.

This preorder notion is well-defined and independently beautiful, but it has a shortcoming: given a specific Φ, one might ask "how complex is Φ?" The most precise answer to this question – it is Borel bireducible to $(\mathrm{Mod}(\Phi, \omega), \cong)$ – is less than illuminating.

Less precisely but more usefully, we might ask how Φ compares to some specific "test" relation, the significance of which is understood to the writer and the reader. Toward this end, we have assembled a useful class of benchmark relations, each of which has some independent significance.

1.1.1 Smooth Relations

Formally, (X, E) is *smooth* if, for some standard Borel space Y, $(X, E) \leq_B (Y, =)$. It turns out that all standard Borel spaces (such as X_L^ω, $\mathbb{R}$, or 2^ω) are in Borel bijection with one another, so the Y above can be chosen to be whatever you like (see [8] for this and similar results). Intuitively, (X, E) is smooth if there is a "mechanical" assignment of real invariants to the E-classes of X.

The smooth relations are linearly ordered by $<_B$, and are completely classified by their size. In fact, they are as follows:

$$(1, =) <_B (2, =) <_B \cdots (\omega, =) <_B (2^\omega, =)$$

It is straightforward to show that all of these reductions are strict: if $(X, E) \leq_B (Y, F)$, then $|X/E| \leq |Y/F|$. That these are all the possible smooth relations is a theorem of Silver, related to the fact that the continuum hypothesis holds for analytic sets. The point for us is that if (X, E) is smooth, then it is completely described by the number of E-classes.

Some of the smooth classes can be seen as *minimal*, in the following sense:

Proposition 1.1.1. *Suppose $1 \leq \kappa \leq \aleph_0$ and $\Psi \in L_{\omega_1\omega}$. Then either $\Psi \leq_B (\kappa, =)$ or $(\kappa, =) \leq_B \Psi$.*

Indeed $\Psi \leq_B (\kappa, =)$ if and only if Ψ has at most κ countable models, and symmetrically with the latter.

Note that here and throughout, we will use $\Phi \leq_B \Psi$ as shorthand for

$$(\mathrm{Mod}(\Phi, \omega), \cong) \leq_B (\mathrm{Mod}(\Psi, \omega), \cong)$$

Similarly we will say "Φ is smooth" when we really mean $(\mathrm{Mod}(\Phi, \omega), \cong)$ is smooth.

The proof of this Proposition requires Scott sentences which we discuss in detail in the next section. The reader may prefer to skip ahead to there, but only the standard facts from any beginning model theory course are used.

Proof of Proposition 1.1.1. Let Ψ be an $L_{\omega_1\omega}$-sentence. First suppose that Ψ has exactly $\kappa \leq \aleph_0$ countable models (up to isomorphism); let $\{\Psi_i : i \in \kappa\}$ be the Scott sentences of these models. We show $(\mathrm{Mod}(\Psi, \omega), \cong) \leq_B (\kappa, =)$. Given $M \in \mathrm{Mod}(\Psi, \omega)$, say $f(M) = i$ if and only if $M \models \Psi_i$. This is a well-defined bijection. Additionally, if $M, N \in \mathrm{Mod}(\Psi, \omega)$, then $M \cong N$ if and only if their Scott sentences are equal, if and only if $f(M) = f(N)$. Finally, this is a Borel reduction: $(\kappa, =)$ has the discrete topology, so it's enough to show that $f^{-1}(i)$ is Borel for each i. This is true, since $f^{-1}(i) = \mathrm{Mod}(\Psi_i, \omega)$, which is Borel by a standard argument. So f is a Borel reduction.

On the other hand, suppose Ψ has at least κ models; let $\{M_i : i \in \kappa\}$ be pairwise nonisomorphic elements of $\mathrm{Mod}(\Psi)$. Define $f : \kappa \to \mathrm{Mod}(\Psi, \omega)$ where $f(i) = M_i$. This is an injection, meaning $i = j$ if and only if $f(i) = f(j)$. Since κ has the discrete topology, f is continuous and thus Borel, completing the proof. $\square$

It is noteworthy that we did not say "if Φ is smooth and Ψ has at least as many countable models as Φ, then $\Phi \leq_B \Psi$;" that is, the smooth relations are not necessarily minimal in this sense. The only exception is $(2^\omega, =)$, the minimality of which would imply Vaught's conjecture (among other things).

1.1.2 Borel Relations

Suppose (X, E) is some equivalence relation, and X is a standard Borel space. We say E is *Borel* if it is a Borel subset of $X \times X$. As usual, say Φ is Borel if the graph of the isomorphism relation is Borel as a subset of $\mathrm{Mod}(\Phi, \omega)^2$. In particular, we say that E is Π^0_α if it a Π^0_α subset of $X \times X$, and say Φ is Π^0_α likewise. The following theorem motivates the definition:

Theorem 1.1.2. *Let Φ be an $L_{\omega_1\omega}$-sentence. The following are equivalent:*

- *Φ is Borel.*
- *There is a countable ordinal β where for all $M, N \in Mod(\Phi, \omega)$, $M \cong N$ if and only if $M \equiv_\beta N$.*[1]

This result is folklore with no clear origin point, but a proof appears in [2]. As might be expected, if Φ is Π^0_α, there is a β which is a linear polynomial in α (with coefficients in ω_1) such that $\equiv_\beta$ is sufficient for isomorphism among countable models of Φ. Likewise, given such a β, one can similarly compute an α from β where Φ is Π^0_α. The exact computations are inexact (that is, they provide non-sharp upper bounds) and depend on one's specific formulation of back-and-forth equivalence, so we will not describe them here.

[1] The relation $\equiv_\beta$ is defined by the usual back-and-forth game of ordinal length. See for instance [13] or [5].

One could conclude from the above theorem that there is "no hope" of "understanding" the isomorphism relation for Φ if Φ is not Borel; certainly there is not a bounded computation which can determine isomorphism. However, understanding an equivalence relation often involves assigning invariants to the classes, and that ship sailed when we left smooth, so discarding the non-Borel relations may be premature.

The Borel relations are "simpler" than non-Borel relations in another sense: if $\Phi \leq_B \Psi$ and Ψ is Borel, then Φ is also Borel. It is thus easy to imagine the Borel relations as minimal, but this is incorrect (for some notions of minimal):

Theorem 1.1.3 (Friedman, Stanley)**.** *There are $L_{\omega_1\omega}$-sentences Φ and Ψ where Φ is Borel, Ψ is not Borel, and $\Phi \not\leq_B \Psi$ (indeed they are incomparable).*

We will give several such examples later.

It turns out that the smooth relations really are minimal among the Borel relations: if Φ is smooth, then Φ is Borel. Additionally, if Ψ is Borel but not smooth, then $\Phi <_B \Psi$.

Unlike the smooth relations, there is no nice characterization of the Borel relations – even those of the form $(\mathrm{Mod}(\Phi, \omega), \cong)$. Despite this, it is a theorem of Hjorth, Kechris, and Louveau that the Borel equivalence relations are stratified into ω_1 distinct classes, each of which has a maximal element and which corresponds to being "potentially Π^0_α" in a particular sense. We describe this result now.

We define $\cong_0$ as $(\omega, =)$, which can be seen as the isomorphism relation of some complete first-order theory T_0 in a relational language L_0.

Given $\cong_\alpha$ as the isomorphism relation of some theory T_α in a relational language L_α, we define $L_{\alpha+1} = \{E_\alpha\} \cup L_\alpha$, where E_α is some binary relation not appearing in L_α. Then $T_{\alpha+1}$ states that E_α is an equivalence relation with infinitely many classes, each of which is a model of T_α, and the models do not interact (say, the relations in L_α are always false on mixed tuples). The resulting theory $T_{\alpha+1}$ is complete by a standard argument.

Finally for limit λ, given $\cong_\alpha$, T_α, and L_α for all $\alpha < \lambda$, let L_λ be $\{U_\alpha : \alpha < \lambda\}$ along with the disjoint union of the L_α. Let T_λ state that the unary predicates U_α are disjoint, and the set of realizations of U_α form a model of T_α. Additionally, if $R \in L_\alpha$ and $\overline{a}$ is a tuple which contains some element not of U_α, then $R(\overline{a})$ is taken to be false. The resulting theory T_λ is complete; the "unsorted elements" which exist by compactness are unstructured and turn out not to matter.

The reader is encouraged to check that these $\cong_\alpha$ are Borel bi-reducible to the $=_\alpha$ described in [7]. Intuitively, we should think of $T_{\alpha+1}$ as coding sets of models of T_α, while T_0 codes natural numbers. Thus T_α codes hereditarily countable sets of rank α, where we consider the natural numbers as urelements.

Theorem 1.1.4 (Hjorth, Kechris, Louveau)**.** *Let α be a countable ordinal. Then there is a countable ordinal β such that:*

For all $L_{\omega_1\omega}$-sentences Φ, $\Phi \leq_B \cong_\alpha$ if and only if Φ is "potentially Π^0_β:" for some equivalent topology[2] τ on Mod(Φ, ω), the graph of isomorphism for Φ is a Π^0_β subset of $\tau \times \tau$.

[2]That is, yielding the same Borel sets

For example, one can easily see that if we add relations for countably many (infinitary) formulas, we gain a refined topology on $\mathrm{Mod}(\Phi, \omega)$ which is equivalent to the original, although not all equivalent topologies are so easily described. Thus the idea of being potentially Π^0_β is a sort of "language-free" measure of complexity of the sentence.

Therefore the relations $\cong_\alpha$ are of independent interest. The following extension explains why they are useful for us:

Theorem 1.1.5. *For all $\alpha < \beta$, $\cong_\alpha <_B \cong_\beta$.*

If Φ is Borel, then $\Phi \leq_B \cong_\alpha$ for some α.

The second clause of Theorem 1.1.5 follows from Theorem 1.1.4, but the first (showing the strictness of the embedding) needs an additional argument. A highly technical argument appears in [4] which relies on Borel determinacy; a much simpler argument appears in [28] which uses potential cardinality, and is sketched in Proposition 1.2.16 later in this chapter.

Thus the $\cong_\alpha$ form a strictly increasing sequence which is cofinal in the Borel sentences. Thus we can measure the complexity of a sentence Φ with some precision by proving results of the form $\cong_\alpha \leq_B \Phi <_B \cong_{\alpha+1}$. Indeed, even if Φ is not Borel, it makes sense to find the minimal α where $\cong_\alpha \not\leq_B \Phi$, and this is a good measure of complexity of Φ.

The first three of the $\cong_\alpha$ will be of particular interest to us. $\cong_0$ and $\cong_1$ are smooth: $\cong_0$ is $(\omega, =)$, and $\cong_1$ is $(\mathbb{R}, =)$. Less trivially, we have the following theorem of Marker from [14]:

Theorem 1.1.6 (Marker)**.** *Let T be a complete first-order theory which is non-small – that is, where $S(T)$ is uncountable. Then $\cong_2 \leq_B T$.*

That is, $\cong_2$ is the minimal element among isomorphism relations for non-small theories. Thus, if T is non-small and $T \sim_B \cong_2$, then T is $\leq_B$-minimal among all non-small theories.

This concludes our survey of the Borel equivalence relations.

1.1.3 Borel Complete Relations

We have now discussed various notions of "minimal" isomorphism relations; from $\aleph_0$-categorical theories which are actually minimal, to various objects which are minimal among those which satisfy certain constraints. There is also a *maximal* relation.

Say Φ is **Borel complete** if, for all Ψ, $\Psi \leq_B \Phi$. The word is due to Friedman and Stanley in [4], and is somewhat unfortunate. It means "complete with regard to Borel reductions" and not "complete among Borel relations" or "maximal among Borel relations." In fact:

Theorem 1.1.7 (Friedman, Stanley)**.** *If Φ is Borel complete, then Φ is not Borel; in fact its isomorphism relation is complete analytic.*

It is easy to see that if such a relation *were* Borel, of some height Π^0_α, then every Ψ would be potentially Π^0_α as well by Theorem 1.1.5, collapsing the hierarchy (which doesn't happen; see [7] or [5]). The result stated above is somewhat stronger, although it should be noted that the converse does not hold (also proven in that paper).

The really surprising thing about Borel complete relations is that they exist. Indeed, they are common:

Theorem 1.1.8 (Friedman, Stanley)**.** *The following classes are all Borel complete: graphs, trees, groups, fields, linear orders, ...*

Indeed many of these results can be sharpened, in the sense that much smaller subclasses can be shown to be Borel complete (which implies Borel completeness of the larger class). For example, the class of discrete linear orders without endpoints is Borel complete (see Chapter 2). For any prime p, the class of nilpotent class 2 groups with exponent p are Borel complete [4]. The class of fields of characteristic p is Borel complete, for any prime p [4]. The class of real-closed fields is Borel complete [25]. Many other results along these lines have been proven as well.

It seems that any reasonably expressive class without a "depth limit" is Borel complete. In a way it is almost more surprising that a natural class along these lines would not be Borel complete, but this can happen:

Theorem 1.1.9 (Friedman, Stanley)**.** *The class of abelian groups is neither Borel nor Borel complete.*

Indeed, for any prime p, the class of abelian p-groups is neither Borel nor Borel complete. Indeed, $\cong_2$ is not Borel reducible to any of these classes. Despite this, the graph of isomorphism for this class is complete analytic.

This example is not first-order, but in [28], we demonstrated several complete first-order theories with similar behavior.

1.2 Back-and-Forth Equivalence

Let us now consider the question of back-and-forth equivalence. Given two L-structures A and B, we say a set $\mathcal{F}$ is a *back-and-forth system* from A to B if all the following are satisfied:

1. The elements of $\mathcal{F}$ are partial functions from A to B.
2. If $f \in \mathcal{F}$, $\overline{a} \subset \mathrm{dom}(f)$, and $R \in L$, then $A \models R(\overline{a})$ if and only if $B \models R(f\overline{a})$.
3. If $f \in \mathcal{F}$ and $a \in A$, there is $g \in \mathcal{F}$ where $f \subset g$ and $a \in \mathrm{dom}(g)$
4. If $f \in \mathcal{F}$ and $b \in B$, there is $g \in \mathcal{F}$ where $f \subset g$ and $b \in \mathrm{im}(g)$.

We say A and B are *back-and-forth equivalent*, denoted $A \equiv_{\infty\omega} B$, if there is a back-and-forth system from A to B. It is immediate that if $A \cong B$, then $A \equiv_{\infty\omega} B$ – if $f : A \to B$ is an isomorphism, then $\{f\}$ is a back-and-forth system. Despite this, we often think of $\mathcal{F}$ as consisting of *finite* partial functions. With this restriction, we could instead let $\mathcal{F}$ be the set of finite sub-functions of f.

The converse holds when A and B are countable; this is the origin of the "back and forth argument" and the proof is immediate:

Proposition 1.2.1. *Suppose $A \equiv_{\infty\omega} B$. If A and B are countable, then $A \cong B$.*

The property "$\mathcal{F}$ is a back-and-forth system from A to B" is absolute between transitive models of ZFC; the following is an immediate consequence:

Corollary 1.2.2. *Regardless of the cardinalities of A and B, if $A \equiv_{\infty\omega} B$, then $A \cong B$ in any forcing extension $\mathbb{V}[G]$ in which both A and B are countable.*

In fact, the converse is true, although we will save the proof for the next section. Thus, if $A \equiv_{\infty\omega} B$, then A and B are "potentially isomorphic." Consequently, they are "indistinguishable" from the point of view of certain logics:

Corollary 1.2.3. *Suppose $A \equiv_{\infty\omega} B$. If $\phi \in L_{\infty\omega}$ is any infinitary sentence, then $A \models \phi$ if and only if $B \models \phi$.*

The proof of this fact follows from an observation, immediate by standard absoluteness results, that for any structure M and any $\phi \in L_{\infty\omega}$, the expression "$M \models \phi$" is absolute.

With this in mind, one could reasonably ask the following question. Suppose A and B are structures which are non-isomorphic, but nevertheless they are back-and-forth equivalent. In what sense are they distinguishable?

It is our opinion that in many interesting ways, they are *not* distinguishable, as evidenced by Corollary 1.2.3. Thus, we consider the following a meaningful invariant of a theory T:

Definition 1.2.4. Let T be a first-order theory (or more generally, a sentence of $L_{\omega_1\omega}$). Let $I_{\infty\omega}(T)$ be the number of back-and-forth inequivalent models of T, of any cardinality; let $I_{\infty\omega}(T) = \infty$ if there are class-many such.

It is more interesting to give a few examples:

Example 1.2.5. Suppose T is $\aleph_0$-categorical. Then $I_{\infty\omega}(T) = 1$.

Proof. Let A and B be models of T. Let $\mathbb{V}[G]$ be a forcing extension in which A and B are both countable. Being $\aleph_0$-categorical is absolute, so $A \cong B$ in the forcing extension. Thus $A \equiv_{\infty\omega} B$ in the ground model $\mathbb{V}$, witnessing $I_{\infty\omega}(T) = 1$. $\square$

Example 1.2.6. Let T be the theory of algebraically closed fields of characteristic p (p prime or zero). Then $I_{\infty\omega}(T) = \aleph_0$. The models are given by the transcendence degree, where the options are n, for any $n \in \omega$, or "infinite."

Proof. If two countable models are back-and-forth equivalent, they are isomorphic; thus the usual models of finite transcendence degree are all back-and-forth inequivalent, and inequivalent to the (countable) model with infinite transcendence degree.

But now let A and B be models with infinite transcendence degree. Let $\mathbb{V}[G]$ be a forcing extension in which both are countable. Then it is easy to see they still have infinite transcendence degree in the forcing extension, but are countable, so isomorphic. Hence they are back-and-forth equivalent in $\mathbb{V}$, as desired. □

Before developing this theory further, we need to introduce Scott sentences.

1.2.1 Scott Sentences

The concept of a Scott sentence is standard, but we will be doing some nonstandard things with it, so we review. First, to clear up any misconceptions, we define *canonical Scott sentences* for all infinite L-structures, regardless of cardinality. The definition below is in both Barwise [1] and Marker [13].

Definition 1.2.7. Suppose L is countable and M is an L-structure of cardinality κ. For each $\alpha < \kappa^+$, define an $L_{\kappa^+,\omega}$ formula $\phi_\alpha^{\overline{a}}(\overline{x})$ for each finite $\overline{a} \in M^{<\omega}$ as follows:

- $\phi_0^{\overline{a}}(\overline{x}) := \bigwedge\{\theta(\overline{x}) : \theta \text{ atomic or negated atomic and } M \models \theta(\overline{a})\}$;
- $\phi_{\alpha+1}^{\overline{a}}(\overline{x}) := \phi_\alpha^{\overline{a}}(\overline{x}) \ \wedge\ \bigwedge\left\{\exists y\, \phi_\alpha^{\overline{a},b}(\overline{x}, y) : b \in M\right\} \ \wedge \forall y \bigvee\left\{\phi_\alpha^{\overline{a},b}(\overline{x}, y) : b \in M\right\}$;

- For α a non-zero limit, $\phi_\alpha^{\overline{a}}(\overline{x}) := \bigwedge \left\{\phi_\beta^{\overline{a}}(\overline{x}) : \beta < \alpha\right\}$.

Next, let $\alpha^*(M) < \kappa^+$ be least such that for all finite $\overline{a}, \overline{a}'$ from M,

$$\forall\overline{x}[\phi^{\overline{a}}_{\alpha^*(M)}(\overline{x}) \leftrightarrow \phi^{\overline{a}'}_{\alpha^*(M)}(\overline{x})] \Longrightarrow \forall\overline{x}[\phi^{\overline{a}}_{\alpha^*(M)+1}(\overline{x}) \leftrightarrow \phi^{\overline{a}'}_{\alpha^*(M)+1}(\overline{x})]$$

Finally, put $\mathrm{css}(M) := \phi^{\emptyset}_{\alpha^*(M)} \wedge \bigwedge \left\{\forall\,\overline{x}[\phi^{\overline{a}}_{\alpha^*(M)}(\overline{x}) \to \phi^{\overline{a}}_{\alpha^*(M)+1}(\overline{x})] : \overline{a} \in M^{<\omega}\right\}$.

It can easily be seen that $\mathrm{css}(M) \in L_{|M|^+\omega}$. Also, it turns out the choice of $\mathrm{css}(M)$ is highly canonical, assuming one codes formulas properly as sets. We avoid these details and simply assert the following:

Fact 1.2.8. *Let M and N be infinite L-structures. The following are equivalent:*

- $M \equiv_{\infty\omega} N$
- $M \models css(N)$
- $N \models css(M)$
- $css(M) = css(N)$

Of course if both M and N are countable, then all of these are equivalent to $M \cong N$ as well. The proofs are standard; although the reader may only be familiar with them in the countable case, the proofs in (e.g.) [5] apply equally well to uncountable structures if one is willing to conclude back-and-forth equivalence instead of isomorphism. Of particular note, to us, is the following:

Proposition 1.2.9. *The function css is absolute; that is, for any ϕ and M, the relation "$css(M) = \phi$" is absolute between models of ZFC.*

These two facts are why we refer to $\mathrm{css}(M)$ as the "canonical" Scott sentence; other sentences (conceivably simpler sentences) could capture M up to back-and-forth equivalence, but we want a specific sentence with useful properties.

Fact 1.2.8 and Prop 1.2.9 together give a proof of the following:

Proposition 1.2.10. *Let M and N be infinite L-structures. The following are equivalent:*

1. $M \equiv_{\infty\omega} N$,
2. *in some forcing extension* $\mathbb{V}[G]$, $M \equiv_{\infty\omega} N$, *and*
3. *in some forcing extension* $\mathbb{V}[G]$, $M \cong N$.

Proof. If $M \equiv_{\infty\omega} N$, then let $\mathcal{F}$ be a back-and-forth system. In any forcing extension $\mathbb{V}[G]$, $\mathcal{F}$ is still a back-and-forth system, so $M \equiv_{\infty\omega} N$ in $\mathbb{V}[G]$ as well, establishing (1) $\Rightarrow$ (2).

If $M \equiv_{\infty\omega} N$ in $\mathbb{V}[G]$, then in any forcing extension $\mathbb{V}[G][H]$, $M \equiv_{\infty\omega} N$ still. So let $\mathbb{V}[G][H]$ collapse $|M|$ and $|N|$ to $\aleph_0$. Then $M \cong N$ in $\mathbb{V}[G][H]$, which is a forcing extension of $\mathbb{V}$, establishing (2) $\Rightarrow$ (3).

Finally, suppose $M \cong N$ in $\mathbb{V}[G]$. Let $\phi = \mathrm{css}(M)$ in $\mathbb{V}$. Then still $\phi = \mathrm{css}(M)$ in $\mathbb{V}[G]$ and $M \cong N$ (hence $M \equiv_{\infty\omega} N$) in $\mathbb{V}[G]$, so $N \models \phi$ in $\mathbb{V}[G]$. Sentence satisfaction is absolute, so $N \models \phi$ in $\mathbb{V}$. By Fact 1.2.8, $N \equiv_{\infty\omega} M$ in $\mathbb{V}$, establishing (3) $\Rightarrow$ (1). $\square$

We believe that Proposition 1.2.10 is the real justification for studying $\equiv_{\infty\omega}$ and $I_{\infty\omega}(T)$. If two models are back-and-forth equivalent, but not isomorphic, it is not because of any intrinsic difference between the two, but instead because of a characteristic of the surrounding universe of set theory, related to the existence or nonexistence of certain functions which *could* exist, but happen to not exist.

We end this section with a nice application of Scott sentences to counting $I_{\infty\omega}(T)$:

Proposition 1.2.11. *Suppose T is a theory, or possibly a countable set of $L_{\omega_1\omega}$ sentences. Suppose λ is an infinite cardinal and T has exactly $\kappa \leq \lambda$ back-and-forth inequivalent models of size at most λ.*

Then $I_{\infty\omega}(T) = \kappa$.

Indeed, every model of T is back-and-forth equivalent to one of size at most λ.

Proof. Let $\{M_i : i \in \kappa\}$ be an exhaustive (up to equivalence) list of models of T of size at most λ, and let $\phi_i = \mathrm{css}(M_i)$ for all i. Then suppose the result is false. That is, for some model $M \models T$, M is not back-and-forth equivalent to any M_i.

Let ψ be $\bigwedge T \wedge \bigwedge_{i<\kappa} \neg\phi_i$. Since $\kappa \leq \lambda$, ψ is a $L_{\lambda^+\omega}$-sentence, so there is a fragment $\mathcal{F}$ of $L_{\lambda^+\omega}$ of size λ which contains it. Observe that $M \models \psi$. Since $\mathcal{F}$ has size λ, there is an $\mathcal{F}$-elementary substructure of M of size at most λ by a standard downward Löwenheim-Skolem argument.

Let N be this structure, observing that $N \models \psi$. Then $N \models T$. Since $|N| \leq \lambda$, $N \equiv_{\infty\omega} M_i$ for some i, hence $N \models \mathrm{css}(M_i) = \phi_i$, a contradiction of the fact that $N \models \psi$. $\square$

1.2.2 Potential Cardinality

Our next background concept is the notion of potential cardinality. Unlike the previous topics discussed here, this is fairly new – at time of writing, the definition exists only in preprint form. The interested reader should check [28] for details; we present a digested form of the exposition there which is useful for our work here.

By the original completeness theorem, if a first-order theory T is formally consistent, then it has a model. We can define a proof system for sentences of $L_{\infty\omega}$ as well. The proofs are now well-founded trees, rather than finite sequences, but the system is otherwise predictable (see e.g. [29] for details). With this in mind, we can say a sentence $\phi \in L_{\infty\omega}$ is *consistent* if it does not prove its own negation, in this sense. After these definitions are settled, one can show that if $\phi \in L_{\infty\omega}$ is countable – that is, $\phi \in L_{\omega_1\omega}$ – then consistency of ϕ implies the existence of a model for ϕ. This is a theorem of Karp in her thesis, but for a more modern treatment, see for example [9] or [29].

Unfortunately, this ceases to be true for uncountable sentences. For example, let $L = \{<\} \cup \{c_n : n \in \omega\}$. Let ψ be the Scott sentence of $(\omega_1, <)$ (making no mention of the c_n) and let ϕ be $\psi \wedge \forall x \bigvee_n x = c_n$. Then ψ has no countable models (ordinals are characterized up to isomorphism by back-and-forth equivalence) but ϕ has no uncountable models, so ϕ simply has no models.

Despite this, ϕ is consistent – it does not prove its own negation. More to the point, ϕ *potentially* has a model – if $\mathbb{V}[G]$ collapses the ordinal we think of as ω_1, then ϕ has a model there, which is any expansion of $(\omega_1, <)$ by an exhaustive countable set of constants. Indeed, those who wish to avoid mention of an infinitary proof system could take this as the definition of consistency of an infinitary sentence, although it robs Karp's theorem of some of its weight.

With this in mind, we make the following definition:

Definition 1.2.12. Let $\phi \in L_{\infty\omega}$. We say ϕ is a *potential Scott sentence* if, in some forcing extension $\mathbb{V}[G]$, there is an infinite L-structure M where $\mathrm{css}(M) = \phi$.

This is a well-defined notion; indeed if ϕ is a potential Scott sentence then ϕ is a canonical Scott sentence in *every* forcing extension in which is it countable. We sketch the proof: if ϕ is inconsistent in a forcing extension, then a proof of that inconsistency is contained in the least admissible fragment containing ϕ, and thus in any transitive model of ZFC which contains ϕ, including $\mathbb{V}$. Therefore, if ϕ is consistent in $\mathbb{V}$, it remains so in any $\mathbb{V}[G]$ where it is countable, and then by Karp's theorem, it has a model there.

We can now make our final definition:

Definition 1.2.13. Let T be a theory (or $L_{\omega_1\omega}$-sentence). Let $\mathrm{CSS}(T)$ denote the set of all *potential* canonical Scott sentences which (formally) imply T.

If $\mathrm{CSS}(T)$ is a set, let $\|T\| = |\mathrm{CSS}(T)|$; otherwise let $\|T\| = \infty$, which we consider strictly greater than any cardinal. Refer to $\|T\|$ as the *potential cardinality* of T.

The following is immediate from the definitions:

Remark 1.2.14. Let T be a theory. Let $I(T, \aleph_0)$ be the number of countable models of T, up to isomorphism (or back-and-forth equivalence).

Then $I(T, \aleph_0) \leq I_{\infty\omega}(T) \leq \|T\|$.

All these inequalities can be strict; in particular $I_{\infty\omega}(T)$ can be strictly less than $\|T\|$ if there are potential Scott sentences of T which do not have models. In any case, the real point of potential cardinality is the following, which is proved in [28]:

Theorem 1.2.15. *Let T_1 and T_2 be theories. If $T_1 \leq_B T_2$, then $\|T_1\| \leq \|T_2\|$.*

We sketch the proof; a completely rigorous treatment is in [28]. Fix a Borel reduction $f : \mathrm{Mod}(T_1) \to \mathrm{Mod}(T_2)$ and suppose $\Phi \in \mathrm{CSS}(T_1)$. In any forcing extension $\mathbb{V}[G]$ making Φ countable, f is still a Borel reduction from $\mathrm{Mod}(T_1)$ to $\mathrm{Mod}(T_2)$, so takes (the unique countable model of) Φ to some (countable model which has) canonical Scott sentence Ψ. Since Borel reductions are Borel, we can apply Schoenfield absoluteness to the appropriate statement about its codes, and discover that f always takes Φ to Ψ, in any universe extending the ground model. By a forcing argument, one can show that Ψ is actually in $\mathbb{V}$, so that this induced function $\overline{f} : \mathrm{CSS}(T_1) \to \mathrm{CSS}(T_2)$ is well-defined. It is clearly an injection, so $\overline{f}$ witnesses $\|T_1\| \leq \|T_2\|$.

Since $\leq_B$ does not give rise to a linear order, there is no converse to this theorem, even in "normal" circumstances (although it will be true for the smooth relations). Also note that this is *not* true for the number of back-and-forth inequivalent models. Indeed there are examples of theories T_1 and T_2 which are Borel equivalent and where $I_{\infty\omega}(T_1) = \infty$, while $I_{\infty\omega}(T_2) = \beth_2$.

The following fact is easily shown; see [28]:

Proposition 1.2.16. *Let α be a countable ordinal.*

If α is finite, then $\| \cong_\alpha \| = \beth_\alpha$. If α is infinite, then $\| \cong_\alpha \| = \beth_{\alpha+1}$.

For conciseness, observe that this is equivalent to the expression: for all countable α, $\| \cong_\alpha \| = \beth_{-1+\alpha+1}$. We can now prove our first significant implication showing that the countable model theory of T "controls" the uncountable model theory, in the form of $I_{\infty\omega}(T)$:

Corollary 1.2.17. *If the isomorphism relation for T is Borel, then $I_{\infty\omega}(T) < \beth_{\omega_1}$.*

In particular, if the isomorphism relation for T is Π^0_α, then $I_{\infty\omega}(T) \leq \beth_{-1+\alpha+1}$.

This is striking, but the only direct consequence of this we will need is the following:

Proposition 1.2.18. *Let T be a theory. If the isomorphism relation for T is smooth, then $I(T, \aleph_0) = I_{\infty\omega}(T) = \|T\|$.*

Proof. Since T is smooth, $T \leq_B \cong_1$. Since $\cong_1$ is the unique successor of $\cong_0$, this means that either $T \leq_B \cong_0$ or $T \sim_B \cong_1$.

If $T \leq_B \cong_0$, then $I(T, \aleph_0) = I_{\infty\omega}(T)$ by 1.2.11. Additionally, $T \sim_B (\kappa, =)$ where $\kappa = I(T, \aleph_0)$, and it is easily seen that $\|(\kappa, =)\| = \kappa$ and potential cardinality is preserved under Borel equivalence. So $\|T\| = \kappa$, as desired.

On the other hand, if $T \sim_B \cong_1$, then $\|T\| = \| \cong_1 \| = \beth_1$. Additionally, since $\cong_1 \leq_B T$, there are continuum-many countable models of T, so $I(T, \aleph_0) = \beth_1$. Since $\beth_1 = I(T, \aleph_0) \leq I_{\infty\omega}(T) \leq \|T\| = \beth_1$, all the inequalities are equalities, completing the proof. □

Beyond what we will need for this thesis, the aim of potential cardinality is to show that a particular theory is *not* that complex, in terms of Borel reducibility. Assuming $\|T\|$ can be shown to have some cardinality which is not ∞, then T is not Borel complete, and assuming it is less than $\beth_{\omega_1}$, many Borel relations cannot be reduced to it. This is the engine by which one can see theories which are neither Borel nor Borel complete; see [28] for details.

1.2.3 λ-Borel Completeness

The final topic of this chapter is the idea of λ-Borel completeness. Since we are discussing $I_{\infty\omega}(T)$ and the uncountable model theory of T generally, one might wonder "how maximal" T can be. Borel completeness is the right notion for the countable models, and in [11], Laskowski and Shelah defined a corresponding notion for uncountable models. The idea is for the models of a theory T to have maximally complicated $\equiv_{\infty\omega}$ relation among the models of size λ, for every λ. In particular this implies Borel completeness, $I_{\infty\omega}(T) = \infty$, and $\|T\| = \infty$, and simultaneously strengthens all of them.

To make this definition, for any infinite cardinal λ and any $\Phi \in L_{\lambda^+\omega}$, let $\mathrm{Mod}_\lambda(\Phi)$ be the space of L-structures with universe λ which model Φ. We make this a topological space using atomic formulas to form a subbasis, as with $\mathrm{Mod}_\omega(\Phi)$. A function $f : \mathrm{Mod}_\lambda(\Phi) \to \mathrm{Mod}_\lambda(\Psi)$ is said to be λ-*Borel* if the preimage of any subbasic open set is λ-Borel, meaning it can be formed as a usual Borel set, but with conjunctions and disjunctions of size at most λ. Because of the presence of parameters from λ, it can easily be seen that the λ-Borel subsets of $\mathrm{Mod}_\lambda(\Phi)$ are precisely (infinite) Boolean combinations of subbasic open sets, so there is no incongruity with [11].

A λ-Borel function $f : \mathrm{Mod}_\lambda(\Phi) \to \mathrm{Mod}_\lambda(\Psi)$ is a *λ-Borel reduction* when for all $M, N \in \mathrm{Mod}_\lambda(\Phi)$, $M \equiv_{\infty\omega} N$ if and only if $f(M) \equiv_{\infty\omega} f(N)$. We denote the existence of such a function by saying $(\mathrm{Mod}_\lambda(\Phi), \equiv_{\infty\omega}) \leq_B^\lambda (\mathrm{Mod}_\lambda(\Psi), \equiv_{\infty\omega})$, often shortened to $\Phi \leq_B^\lambda \Psi$. We say Φ is *λ-Borel complete* if, for all $\Psi \in L_{\lambda^+\omega}$, $\Psi \leq_B^\lambda \Phi$. Observe that in the case that $\lambda = \aleph_0$, we recover the original notion of Borel reductions, Borel completeness, and so on, since back-and-forth equivalence is the same as isomorphism for countable structures; thus examples exist in that case. But actually such sentences exist for all λ:

Theorem 1.2.19 (Laskowski, Shelah)**.** *For any infinite cardinal λ, the class of (downward closed) subtrees of $\lambda^{<\omega}$ is λ-Borel complete.*

To make this completely precise, we fix a bijection $\lambda^{<\omega} \to \lambda$ so that λ has a tree structure on it. Then a "subtree of $\lambda^{<\omega}$" is formed by expanding this structure by a unary predicate whose realizations are downward-closed with regard to the tree order, and outside of which we forget the tree order, along with some standard tricks so that the complement of the "subtree" is always infinite, and thus irrelevant to the back-and-forth equivalence structure. In [11], Laskowski and Shelah introduce the notion of "λ-Borel complete for all λ" as a kind of maximal level of complexity of a theory, and using Theorem 1.2.19 as a "test class," they also produce a large class of examples. For our purposes we will need a different test class:

Theorem 1.2.20. *Let LO be the sentence "$<$ is a linear order" in the language $\{<\}$. Then LO is λ-Borel complete for all λ. In particular, for all infinite λ, there are 2^λ pairwise back-and-forth inequivalent linear orders of size λ, so $I_{\infty\omega}(LO) = \infty$.*

Proof. The "in particular" is a corollary, as follows. Trivially there are at most 2^λ orders of size λ, up to back-and-forth equivalence (or isomorphism). For the other direction, it is enough to show a finite language where there are 2^λ back-and-forth inequivalent structures of size λ in that language.

To see this, recall the classical result that distinct ordinals are back-and-forth inequivalent. Therefore, there are at least λ^+ linear orders of size λ, indexed by the interval $[\lambda, \lambda^+)$. Then consider the language $\{E, <\}$, and the incomplete theory which states that E is an equivalence relation and $<$ is a linear order on each class (but not well-defined between classes). Then for any $X \subset [\lambda, \lambda^+)$ of size at most λ, let M_X have E-classes indexed by X, where the class corresponding to $\alpha \in X$ has order type $(\alpha, <)$. If $X \neq Y$ then $M_X \not\equiv_{\infty\omega} M_Y$, so there are at least $[\lambda^+]^{\leq\lambda} = 2^\lambda$ inequivalent structures – and therefore linear orders, by the reduction – of size λ. This holds for all λ, proving that $I_{\infty\omega}(\mathrm{LO}) = \infty$.

The main result is an extension of Friedman and Stanley's proof that linear orders are Borel complete.

Let λ be any infinite cardinal. It follows from Theorem 1.2.19 that there is a finite language L where $\mathrm{Mod}_\lambda(L)$ – the space of L-structures with universe λ – is λ-Borel complete. To imitate the original proof we need a notion of a λ-*dense linear order*: a structure of size λ in the language $\{<\} \cup \{P_\alpha : \alpha \in \lambda\}$ where $<$ is a dense linear order without endpoints, the P_i are disjoint unary predicates, and they are dense, codense, and exhaustive in the order.

Such a model can be constructed directly. Start with $M_0 = (\mathbb{Q}, <)$, where each element has color P_0. Given M_n, construct M_{n+1} by adding new elements of color α between every two elements of M_n, for every $\alpha < \lambda$. Let $M_\omega = \bigcup_n M_n$. It is clear that M_ω is λ-dense.

From now we follow [4] quite closely; we have imitated the notation to assist the reader.

Next, we need to define a particular linear order $I_{<\omega}$ as a directed union $\bigcup_n I_n$. We say I_{-1} is empty. For each $n \in \omega$, we say I_n is $I_{n-1} \times (-\infty \frown I)$, where we identify I_{n-1} with $I_{n-1} \times \{-\infty\}$ inside I_n. For any $x \in I_{<\omega}$, define $\ell(x)$ as the least n where $x \in I_n$.

We give a labeling f of $I_{<\omega}$ by $\lambda^{<\omega}$ satisfying the following conditions:

- If $\ell(x) = n$, then $f(x) \in \lambda^n$.
- If $x \in I_n$, then f maps $\{x\} \times I$ *onto* $\{f(x) \frown \alpha : \alpha \in \lambda\}$.
- For any $x \in I_n$ and any $\alpha \in \lambda$, $f^{-1}(\{f(x) \frown \alpha : \alpha \in \lambda\}$ is dense in $\{x\} \times I$.

We define f by induction. If $\ell(x) = 0$, $f(x)$ is the empty sequence $() \in \lambda^0$. If $\ell(x) = n + 1$, then $x = (y, i)$ for some $y \in I_n$ and some $i \in I$, and there is a unique $\alpha \in \lambda$ where $I \models P_\alpha(i)$. So let $f(x) = f(y) \frown (\alpha)$. Visibly this function has the desired properties, using λ-density of I.

Next, for each $n \in \omega$, let TY_n be the set of all complete atomic L-types in variables $x_1, \ldots, x_n$; since L is finite, so is TY_n. Let $e(0) = 0$, and for each $n \in \omega$, let $e(n+1) = e(n) + |\mathrm{TY}_n|$. Let $\mathrm{TY} = \bigcup_n \mathrm{TY}_n$. We fix some bijection $k : \mathrm{TY} \to \omega$, so that if $p \in \mathrm{TY}_n$, then $e(n) \leq k(p) < e(n+1)$.

We can now finally produce our λ-Borel reduction. Let A be an L-structure with universe λ; we construct a linear order M_A with universe λ in a λ-Borel way, such that for any L-structures A and B on λ, $A \equiv_{\infty\omega} B$ if and only if $M_A \equiv_{\infty\omega} M_B$. We construct M_A from A by expanding $I_{<\omega}$ according to A.

So for any $x \in I_{<\omega}$ with $\ell(x) = n$, there is a corresponding tuple $f(x) \in \lambda^n$, and this tuple has an atomic type $\mathrm{otp}^A(f(x))$, which has a corresponding index $k(\mathrm{otp}^A(f(x)))$. So let J_x be the linear order $\mathbb{Q} \frown 2 + k(\mathrm{otp}^A(f(x))) \frown \mathbb{Q}$; this is a dense piece, followed by a long enough finite piece not to disappear but which uniquely captures the type of $f(x)$, followed by a dense piece to separate this information from others. So let M_A be the sum $\sum_x J_x$. The map $A \mapsto M_A$ can easily be made a λ-Borel function from $\mathrm{Mod}_\lambda(L)$ to $\mathrm{Mod}_\lambda(\mathrm{LO})$; the detail to check is that each J_x is countable and $I_{<\omega}$ is a fixed set of size λ, so $|\sum_x J_x|$ can be put into (more or less) canonical bijection with λ.

To show it is a reduction, let $\mathbb{V}[G]$ be a forcing extension in which λ is countable (e.g. a Levy collapse of λ^+ to ω_1 will do). Observe that $A \equiv_{\infty\omega} B$ if and only if $A \cong B$ in $\mathbb{V}[G]$, and likewise with M_A and M_B. So pass to $\mathbb{V}[G]$. Once there, observe that I is isomorphic to any $\aleph_0$-dense partition of $(\mathbb{Q}, <)$, and A and B are (up to isomorphism) just elements of $\mathrm{Mod}_\omega(L)$. Therefore, this collapses to the exact construction showing $\mathrm{Mod}_\omega(L) \leq_B \mathrm{Mod}_\omega(\mathrm{LO})$ from [4], so $A \cong B$ (in $\mathbb{V}[G]$) if and only if $M_A \cong M_B$ (in $\mathbb{V}[G]$). This completes the proof. $\square$

We will use this result in the following chapters to show λ-Borel completeness for several first-order theories.

Chapter 2: O-Minimal Theories

In 1988, Laura Mayer proved Vaught's Conjecture for o-minimal theories in a surprising way- an o-minimal T either has finitely many countable models or continuum many. This was accomplished through a sharp dichotomy she introduced: whether or not T admits a "nonsimple type." If T admits a nonsimple type, T must have continuum-many models. If not, the isomorphism relation can be simply characterized, and T has continuum-many countable models if and only if there are infinitely many nonisolated types.

In this paper we sharpen this divide, completely characterizing where $\cong_T$ lies in the Borel complexity hierarchy. Most prominently we show that, given a nonsimple type, T is Borel complete, and indeed λ-Borel complete for all λ. This is proved by reducing the isomorphism problem for linear orders into the one for models of T; by Theorem 3 in [4] and Theorem 1.2.20, every isomorphism problem is reducible to this one, so this is sufficient. Note that this implies $I_{\infty\omega}(T) = \infty$ trivially.

If there is no such type, one of two things happens. If T is non-small – that is, $S(T)$ is uncountable – then $\cong_T$ is $\cong_2$, the "equality relation on countable sets of reals." Correspondingly, $I_{\infty\omega}(T) = \beth_2$. If T is small – that is, $S(T)$ is countable – then $\cong_T$ is smooth, and equivalent to the equality relation on κ, the number of countable models of T, which can also be computed directly by a type counting argument. Curiously, κ can be 1, any finite number of the form $3^a 6^b$, or $2^{\aleph_0}$, but cannot be $\aleph_0$. In all of these cases, $I_{\infty\omega}(T) = \kappa$.

These values are significant. In case T is small, then $\cong_T$ is the minimal value among all theories with at least κ countable models. In case T is not small, $\cong_T$ is the minimal value among all non-small theorems (see Theorem 1.3 in [14]). Thus in some important sense, $\cong_T$ is either maximal or minimal. Also, among o-minimal theories, $I_{\infty\omega}(T)$ is always the "expected" value from the Borel complexity of $\cong_T$.

Therefore the divide is as sharp as it could be- $(\mathrm{Mod}(T), \cong)$ is either maximal among all isomorphism problems, or minimal among all problems with which it can be reasonably compared.

We continue with two interesting corollaries to the main theorem. The first states that any nontrivial o-minimal theory (in particular, any theory which defines an infinite group) is λ-Borel complete. The second states that any discretely o-minimal theory (or even one with an infinite discrete part) is λ-Borel complete. Together, these imply and strengthen virtually all known Borel completeness results for concrete examples of o-minimal theories.

We end with a section filled with examples to demonstrate the different kinds of behavior which can occur, as well as settle a few easy questions one might ask when comparing this case to colored linear orders (see Chapter 3) which admits a very similar "main theorem" to this one.

These questions were originally explored in Dave Sahota's PhD thesis [25], where a partial form of the main result was shown and where many of the techniques here were first employed. In particular, he introduced the concept of faithfulness and showed that the existence of a faithful type is sufficient for Borel completeness. From there, given a nonsimple type over finitely many parameters, he added finitely many more parameters to produce a faithful non-cut, yielding a Borel completeness result for the extended theory. It is a major open problem whether this implies Borel completeness for the original theory, however, and the thesis stopped there.

The main addition of this chapter, aside from extending the scope to $I_{\infty\omega}(T)$ and to λ-Borel completeness, is to get around this issue of parameters. This requires more refined analysis of cuts and substantially new ideas to deal with atomic intervals (the canonical tail). Although Sahota finished his thesis before I was aware of the problem, it was never published; thus, to recognize the contributions of both authors, much of this work was published in [21] under both names.

2.1 Background

Throughout this chapter, T will refer to an o-minimal theory in a countable language. We will *not* assume the underlying order is dense. However, as we will see in Theorem 2.4.2, if the order has an infinite non-dense part, then there is a very simple answer to our main question, which shows that T is λ-Borel complete for all λ. Where relevant, we will work in a sufficiently saturated monster model from which all parameters will be drawn; all models will be elementary substructures of this monster model. When we say a set A or a tuple of elements $\bar{b}$, we mean these to be subsets or elements of this monster model.

Since our models will be linearly ordered, for any set A, $\mathrm{dcl}(A) = \mathrm{acl}(A)$. We will refer to this shared value as $\mathrm{cl}(A)$. Since closure will be used so heavily in this chapter, we define two extended notations. If A is any set, then $\mathrm{cl}_A(B)$ will be $\mathrm{cl}(A \cup B)$, where the intention is that A acts like a set of parameters. If p is a partial type over A, then $\mathrm{cl}^p_A(B)$ will be those elements of $\mathrm{cl}_A(B)$ which satisfy p.

Much of the basic theory of o-minimality was developed in [18] by Pillay and Steinhorn and in [10] by Knight, Pillay, and Steinhorn. In particular, they showed the cell decomposition theorem, the continuity-monotonicity theorem, and the existence of (unique) constructible models over sets. Readers unfamiliar with o-minimal theories are directed to [30] for a modern introduction to o-minimality.

For definitions – a structure $\mathcal{M}$ is *o-minimal* if $<$ is a linear order of the structure and, for all parameter-definable sets $\phi(x, \overline{a})$, $\phi(\mathcal{M}, \overline{a})$ is a finite union of points and open intervals. It is a nontrivial fact, shown in [18] and [10], that if $\mathcal{M} \equiv \mathcal{N}$ and $\mathcal{M}$ is o-minimal, then $\mathcal{N}$ is o-minimal as well. Thus we may say a complete theory T is o-minimal if some (all) of its models are.

First, for cell decomposition. A 1-*cell* (over A) is either a single point in $\mathrm{cl}(A)$ or an open interval whose endpoints are in $\mathrm{cl}(A)$. An $n+1$-*cell* $\mathcal{V}$ (over A) consists of an $n-1$-cell $\mathcal{U}$ over A and one of two things:

1. An A-definable partial function f, defined on $\mathcal{U}$. Then $\mathcal{V} = \{(\overline{x}, y) : \overline{x} \in \mathcal{U}, f(\overline{x}) = y\}$.

2. Two A-definable partial functions f and g, defined on $\mathcal{U}$, where $f(\overline{x}) < g(\overline{x})$ on $\mathcal{U}$. Then $\mathcal{V} = \{(\overline{x}, y) : \overline{x} \in \mathcal{U}, f(\overline{x}) < y < g(\overline{x})\}$.

An n-cell is *open* if the underlying 1-cell is an open interval, and in higher cases, we always take case (2). The cell-decomposition theorem states that for any A-formulas $\phi_1(\overline{x}), \ldots, \phi_k(\overline{x})$, $\mathcal{M}^n$ can be A-definably divided into n-cells such that $\phi_i(\mathcal{M}^n)$ is a disjoint union of cells. The only usage of this theorem directly in this chapter is the following easy consequence:

Proposition 2.1.1. *Suppose $\overline{a}$ is a tuple, $\phi(\overline{x})$ is a formula over A, and $\phi(\overline{a})$ is true. Then there is a n-cell $\mathcal{U}$, definable over A, where $\overline{a} \in \mathcal{U}$ and ϕ holds for all elements of $\mathcal{U}$.*

Along with the cell-decomposition theorem is a corresponding statement for functions. For general A-definable n-ary partial functions $f_1, \ldots, f_k$, we can definably decompose $\mathcal{M}^n$ into n-cells over A such that for all i, $\mathrm{dom}(f_i)$ is a disjoint union of cells and where f_i is continuous on each cell. When $n = 1$, we can do even better and insist that on each cell, f_i is not only continuous, but either constant, strictly increasing, or strictly decreasing; this is call the continuity-monotonicity theorem. The following is an easy consequence we will use frequently:

Proposition 2.1.2. *Suppose A is a set and a and b are single elements outside of $cl(A)$. If f is an A-definable partial function and $f(a) = b$, then f is a strictly monotone homeomorphism from the realizations of $tp(a/A)$ to the realizations of $tp(b/A)$. We will refer to this condition as "f is a bijection from $tp(a/A)$ to $tp(b/A)$."*

It follows immediately that cl_A satisfies the Steinitz exchange axiom: for any set A and any single elements a and b, if $a \in \mathrm{cl}(Ab) \setminus \mathrm{cl}(A)$, then $b \in \mathrm{cl}(Aa)$. Thus for any sets A and B, $\dim_A(B)$ is well-defined to be the cardinality of the largest cl_A-independent subset of B.

A model $\mathcal{M}$ is *constructible over* A if $A \subset \mathcal{M}$ and there is an enumeration $(a_\alpha : \alpha < \delta)$ of $\mathcal{M}$ such that for all α, $\mathrm{tp}(a_\alpha/A_\alpha)$ is isolated, where $A_\alpha = A \cup \{a_\beta : \beta < \alpha\}$. If there is a constructible model over A, then that model is the unique prime model over A, up to isomorphism fixing A. When the underlying theory is o-minimal, for all sets A (from some model), there is a constructible model over A which we will denote $\Pr(A)$.

In [12], Marker identified the three kinds of complete nonalgebraic 1-types which can arise. A complete 1-type p which has both a definable infimum L and a supremum R is atomic, and is either algebraic (if $L = R$) or is generated by the atomic interval (L, R); note that L or R may be among $\pm\infty$. If p has a definable infimum *or* a definable supremum, but not both, then p is nonisolated and is called a "non-cut." Finally, if p has neither endpoint, then p is nonisolated and is called a "cut." It is easy to see that there are no definable bijections between types of different "kinds." From this observation Marker showed that if $\mathrm{tp}(a/A)$ is a non-cut and $q \in S_1(A)$ is a cut, then the prime model over Aa omits q, and likewise when exchanging cuts for non-cuts.

2.1.1 Nonsimplicity

Less well-known than the general theory but fundamental to our work here, is the notion of *nonsimplicity*. This definition is due to Mayer and is central to her solution of Vaught's conjecture for o-minimal theories in [15].

Definition 2.1.3. A type $p \in S_1(A)$ is *simple* if, for every set B of realizations of p, $\mathrm{cl}^p_A(B)$ is B.

Say p is *nonsimple* if p is not simple; that is, for some set B of realizations of p, there is a $b \notin B$ which realizes p and which is B-definable.

By compactness, if p is nonsimple, there is a *finite* set B satisfying the above. In particular, we will say p is n-nonsimple if there is some B as above with $|B| \leq n$. We will say p is n-simple if there is no such set B. The following remark makes the *minimal* nonsimplicity index very interesting:

Remark / Definition 2.1.4. If p is k-simple, then the type $p^{k+1}(x_0, \ldots, x_k)$ generated by $\{x_0 < \cdots < x_k\} \cup \bigcup_{i=0}^{k} p(x_i)$ is complete.

If p is n-nonsimple, then there is some ascending n-tuple $\overline{a}$ of realizations of p, and some element b which realizes p and is $\overline{a}$-definable but is not in $\overline{a}$. If n is *minimal* such that p is n-nonsimple, then by the remark, p^n is a complete type. By combining these two facts, we get a definable function $f : p^n \to p$ such that $f(\overline{a}) = b$.

While we will be very interested in particular nonsimple types, we are using the existence of a nonsimple type as a property of the theory which forms the most important dividing line for complexity. Since the use of parameters can be a significant obstacle to descriptive set theoretic analysis, the following lemma is extremely helpful:

Lemma 2.1.5. *Let A be a finite set, and suppose $p \in S_1(A)$ is n-nonsimple. Then the restriction p_0 of p to $S_1(\emptyset)$ is is $n + |A|$-nonsimple.*

Proof. By an obvious inductive argument, we assume A is a singleton a, and $p \in S_1(a)$. Assume n is minimal such that p is n-nonsimple. By way of contradiction, suppose that p_0 is $n+1$-simple. By nonsimplicity and the remark, there is an a-definable function $f : p^n \to p$. In fact, we may take f to be $f(\overline{x}; y)$ such that $f(\overline{x}; a)$ is a nontrivial function $p^n \to p$, and by exchange over a, may assume that $f(\overline{x}; a)$ is defined on ascending tuples $x_1 < \cdots < x_n$ and satisfies $f(\overline{x}; a) > x_n$ everywhere.

Suppose that $\mathrm{cl}^{p_0}(a)$ is nonempty; that is, there is an $a' \in \mathrm{cl}(a)$ which realizes p_0. But then by exchange, $a \in \mathrm{cl}(a')$, so $f(\overline{x}; a)$ is a'-definable, witnessing $n+1$-nonsimplicity of p_0. So it must be that p is equivalent to p_0; therefore we take $p = p_0$ for the remainder of the proof. Let $q(y) = \mathrm{tp}(a)$.

But now the type $p_0^n \times q$ is complete, and f is a function $p_0^n \times q \to p_0$. By exchange and the preparation above, we may replace f with a function $g : p_0^{n+1} \to q$. Assume g is of minimal arity with this property. Then for any $b_1 < \cdots < b_n$ from p_0, the function $g(\overline{b}; x_{n+1})$ is a bijection from the complete $\overline{b}$-type $p_0(x) \cup \{x > b_n\}$ to the complete $\overline{b}$-type q.

Therefore, define the function $h : p_0^{n+1} \to p_0$ by $h(x_1, \ldots, x_{n+1})$ to be the unique $y > x_{n+1}$ from p_0 where $g(x_1, \ldots, x_{n+1}) = g(x_2, \ldots, x_{n+1}, y)$; such a y must exist and be greater than x_{n+1} by the above proof, yielding $n+1$-nonsimplicity of p_0, as desired. □

In fact, since functions require only finitely many parameters to be defined, if T admits a nonsimple type over *any* set, then T has a nonsimple type over the empty set. Although we will not use it, the converse is also true – any nonsimple type over any set A admits a nonsimple extension to any $B \supset A$. Therefore:

Corollary 2.1.6. *For a complete o-minimal T, the following are equivalent:*

- *T admits a nonsimple type over $\emptyset$.*
- *T admits a nonsimple type over A, for* some *set A.*
- *T admits a nonsimple type over A, for* every *set A.*

We will refer to any of the above conditions on T as *admitting a nonsimple type.*

2.1.2 Outline

Most of the content of this chapter is in proving the following Theorem:

Theorem 2.1.7. *Let T be a complete o-minimal theory in a countable language.*

1. *If T has no nonsimple types and $S_1(T)$ is countable, then $\cong_T$ is $(3^a 6^b, =)$ and $I_{\infty\omega}(T) = \|T\| = 3^a 6^b$, where a and b are the number of independent non-cuts and cuts, respectively. Note that either or both could be infinite.*
2. *If T has no nonsimple types and $S_1(T)$ is uncountable, then $\cong_T$ is $\cong_2$ and $I_{\infty\omega}(T) = \|T\| = \beth_2$.*
3. *If T admits a nonsimple type, then $(Mod(T), \cong)$ is Borel complete and indeed λ-Borel complete for all λ.*

First, we consider the case where T has no nonsimple types. We state and reprove Mayer's characterization of isomorphism for such T. Then we go on to prove the exact place in the Borel hierarchy for T by giving explicit Borel reductions into the appropriate spaces.

The more complicated case is when T has a nonsimple type. In all cases, we will give a λ-Borel reduction from $(\mathrm{LO}, \cong)$ into $(\mathrm{Mod}(T), \cong)$. In essence, we will give a λ-Borel function $\mathrm{LO} \to \mathrm{Mod}(T)$ where L appears as the Archimedean ladder of some nonsimple type in $\mathcal{M}_L$. In actuality, this only works in the presence of a *faithful* nonsimple type. The notion of faithfulness applies in different ways depending on the 'kind' of nonsimple type we have, so we divide into cases based on whether our nonsimple type is isolated or not.

If p is a nonsimple, nonisolated type, then either p is a non-cut or a cut. We show that all nonsimple non-cuts are faithful, and that every cut is either faithful or can be used to produce a nonsimple non-cut (which is necessarily faithful). When p is isolated, there may be no faithful types anywhere. We exploit the idea that we can add parameters to produce a non-cut, so that we can embed a linear order as the ladder of this type. This will not be preserved under isomorphism of models, but we show that such an embedding has a *canonical tail* which is preserved.

We then show that this is enough – there is a λ-Borel complete class of linear orders where tail isomorphism is equivalent to actual isomorphism, so we can still produce a Borel reduction from linear orders to T. Therefore, given a nonsimple type, T must be λ-Borel complete for all λ.

We follow up with two corollaries which provide sufficient conditions for λ-

Borel completeness. First, if the theory itself is nontrivial (regardless of whether this happens in a single type), we can use exchange to produce a nonsimple type over finitely many parameters. Second, if the underlying order is not almost dense (that is, there are infinitely many non-dense points), we will be able to generate a faithful nonsimple type over $\emptyset$, just using the successor function.

We end with a section of examples of o-minimal theories which exhibit several types of behavior discussed in the main body of this chapter.

2.2 No Nonsimple Types

The aim of this section is to completely characterize the complexity of $\cong_T$ in the case that T does not admit a nonsimple type. Therefore, **for the rest of this section, T is a countable o-minimal theory with no nonsimple types.** Our characterization will depend entirely on the size of $S_1(T)$ and the number of independent cuts and non-cuts. To do this, consider the following definition, which is implicit in [15]:

Definition 2.2.1. Let $\mathcal{M}$ and $\mathcal{N}$ be countable models of T. We say that $\mathcal{M}$ and $\mathcal{N}$ are *apparently isomorphic* if, for every $p \in S_1(\emptyset)$, $p(\mathcal{M}) \cong p(\mathcal{N})$ as linear orders.

Our characterization relies on two major facts; that "apparent" isomorphism is equivalent to "actual" isomorphism, and that apparent isomorphism is a relatively simple thing to compute.

2.2.1 Apparent Isomorphism is Equivalent to Isomorphism

We begin by summarizing the part of Mayer's work which is relevant to us. All results and definitions in this subsection are due to her and proved in [15], although the exposition is new.

Lemma 2.2.2. *Given countable models $\mathcal{M}$ and $\mathcal{N}$ of T, $\mathcal{M} \cong \mathcal{N}$ if and only if $\mathcal{M}$ and $\mathcal{N}$ are apparently isomorphic.*

This lemma follows from a back-and-forth argument using the following lemma as an inductive step:

Lemma 2.2.3 (Mayer)**.** *Suppose $\mathcal{M}$ and $\mathcal{N}$ are countable models of T and A is a finite set of parameters in $M \cap N$ where, for all $p \in S_1(A)$, $p(\mathcal{M}) \cong p(\mathcal{N})$ as linear orders. Then, for any $a \in \mathcal{M}$, there is a $b \in \mathcal{N}$ such that $tp(a) = tp(b)$ and for all $q(x; y) \in S_2(A)$, $q(\mathcal{M}; a) \cong q(\mathcal{N}; b)$ as linear orders.*

Proof. Let $\mathcal{M}$ and $\mathcal{N}$ be as described; clearly we may assume $A = \emptyset$ by adding it into the language. Let $a \in \mathcal{M}$ be arbitrary, and for every $p \in S_1(\emptyset)$, let $f_p : p(\mathcal{M}) \to p(\mathcal{N})$ be an order isomorphism as guaranteed by hypothesis.

First, note that any $\emptyset$-definable function between complete 1-types must be a continuous, strictly monotone bijection, either order-preserving or order-reversing; this follows from the continuity-monotonicity theorem and the fact that both types are complete. Next, note that since all types are simple, there is at most one $\emptyset$-definable function between any two 1-types, since if $f, g : p \to q$ are distinct, then $g^{-1} \circ f : p \to p$ makes p nonsimple. As a consequence, for every type p, there is at most one element $a' \in \mathrm{cl}(a)$ which realizes p. The same holds for any b in $\mathcal{N}$.

With all this said, fix $p = \mathrm{tp}(a)$ and let $b = f_p(a) \in \mathcal{N}$. Observe that $\mathrm{tp}(a) = \mathrm{tp}(b) = p$. We argue that this choice of b works; that for any $q(x; y) \in S_2(\emptyset)$, $q(\mathcal{M}; a) \cong q(\mathcal{N}; b)$. By the previous paragraph, every type over $\emptyset$ either stays the same or splits into two convex pieces. If $q(x; a)$ is equivalent to its restriction q_0 to $\emptyset$, then so is $q(x; b)$, and they are already isomorphic under f_q. If it does split, then there is an $a' \in \mathrm{cl}(a)$ which realizes q_0, so there is a unique $\emptyset$-definable homeomorphism $f : p \to q_0$ where $f(a) = a'$. Observe that f works in $\mathcal{N}$ as well, and $q(x; b)$ splits into two pieces over $b' = f(b)$.

Assume that $f : p \to q_0$ is strictly decreasing (the strictly increasing case is similar). Then f is a strictly decreasing bijection $p \cup \{x > a\} \to q_0 \cup \{x < a'\}$ and $p \cup \{x < a\} \to q_0 \cup \{x > a'\}$, and similarly for b and b' in $\mathcal{N}$. So $f \circ f_p \circ f^{-1}$ is an order-preserving bijection $q_0 \cup \{x < a'\} \to q_0 \cup \{x < b'\}$ and $q_0 \cup \{x > a'\} \to q_0 \cup \{x > b'\}$. Since $q(x; a)$ is either $q_0 \cup \{x < a'\}$ or $q_0 \cup \{x > a'\}$, and $q(x; b)$ similarly, the function $f \circ f_p \circ f^{-1}$ is the desired order-isomorphism $q(\mathcal{M}; a) \to q(\mathcal{N}; b)$, completing the proof. □

It only remains to prove that "apparent isomorphism" is a comparatively simple notion to compute. To that end, consider the following lemma:

Lemma 2.2.4. *[Mayer] For any simple $p \in S_1(\emptyset)$ and any countable $\mathcal{M} \models T$, if $a, b \in p(\mathcal{M})$ and $a < b$, then there is a $c \in p(\mathcal{M})$ with $a < c < b$.*

Therefore, $p(\mathcal{M})$ is order-isomorphic to one of six countable linear orders.

Proof. First, suppose $p(\mathcal{M})$ has at least two elements, $a < b$. If a has an immediate successor, then at most b, and by convexity of p, *every* realization of p has an immediate successor which realizes p. Therefore, the successor function is a $\emptyset$-definable witness to p being 1-nonsimple (against hypothesis). Thus this cannot happen, so no element of $p(\mathcal{M})$ has an immediate successor or predecessor.

Therefore, either $|p(\mathcal{M})| \leq 1$ (yielding two possible isomorphism types) or $p(\mathcal{M})$ is a dense linear order. In this case, two choices remain with respect to endpoints, and therefore four more possible options for the isomorphism type of $p(\mathcal{M})$, for a total of six. $\square$

As a consequence, two models $\mathcal{M}$ and $\mathcal{N}$ are apparently isomorphic if and only if, for every type $p \in S_1(\emptyset)$ which is realized in $\mathcal{M}$ or $\mathcal{N}$, if $p(\mathcal{M})$ has a first element (or last element, or sole element), then so does $p(\mathcal{N})$, and vice-versa.

2.2.2 The Complexity of Isomorphism for Theories With No Nonsimple Types

With both models in hand, computing apparent isomorphism is not especially difficult. Determining precisely how difficult leads to the following characterization:

Theorem 2.2.5. *Suppose T is o-minimal with no nonsimple types.*

- *If there are c pairwise-independent cuts and n pairwise-independent non-cuts over $\emptyset$, both finite, then $\cong_T$ is Borel equivalent to $(3^n 6^c, =)$. Additionally, $I_{\infty\omega}(T) = 3^n 6^c$.*

- *If there are an infinite but countable number of pairwise-independent non-*

isolated 1-types over $\emptyset$, then $\cong_T$ is Borel equivalent to $\cong_1$. Additionally, $I_{\infty\omega}(T) = \beth_1$.

- *If $S_1(T)$ is uncountable, then $\cong_T$ is Borel equivalent to $\cong_2$. Additionally, $I_{\infty\omega}(T) = \beth_2$.*

To be precise, we should define independence. Two complete, nonalgebraic types p and q (over A) are *dependent* if there is an A-definable function f which takes realizations of p to realizations of q. A set $\Gamma \subset S_1(T)$ is *independent* if no pair of types in Γ is dependent. In [15], a more complicated notion of dependence was given so that a type could depend on finitely many others. This will never be necessary because of Theorem 2.4.1, which states that if T is nontrivial then T admits a nonsimple type (this is not circular). Consequently dependence is an equivalence relation on complete types over a set A, and the equivalence classes have size at most $\aleph_0 + |A|$. Additionally, though we will not need it specifically, cuts can only depend on cuts, and likewise with non-cuts and atomic intervals.

For the first and second points, we need only show that if T is small, then T is smooth. In this case the Borel equivalence class of T is defined exactly by the number of nonisomorphic countable models of T, a count which has already been done in [15], where the notion of "pairwise-independent" is also made precise. Moreover, the computation of $I_{\infty\omega}(T)$ follows immediately from Proposition 1.2.18.

Lemma 2.2.6. *If $S_1(T)$ is countable and has no nonsimple types, then T is smooth.*

Proof. We need a Borel function $F : \mathrm{Mod}(\omega, T) \to X$, for some Polish space X, where $\mathcal{M}_1 \cong \mathcal{M}_2$ iff $F(\mathcal{M}_1) = F(\mathcal{M}_2)$. So let $X = 6^{S_1(T)}$, which is a countable product of Polish spaces and is therefore Polish. Fix an enumeration of the six possible countable dense linear orders, and for any $\mathcal{M} \models T$ and $p \in S_1(T)$, let $F(\mathcal{M})(p)$ be the index of the order-type of $p(\mathcal{M})$. This function is clearly Borel and satisfies the requirements. $\square$

For the third point, it's enough to show that if $S_1(T)$ is uncountable, then $T \leq_B \cong_2$. For as has already been mentioned, $\cong_2$ embeds into $\cong_T$ whenever T is not small.

Lemma 2.2.7. *If $S_1(T)$ is uncountable, then $\cong_T \leq_B \cong_2$.*

Proof. Since T is not small, $X = S_1(T) \times 6$ is an uncountable Polish space. We will produce a Borel function $F : \mathrm{Mod}(\omega, T) \to X^\omega$ such that $\mathcal{M}_1 \cong \mathcal{M}_2$ iff $\{F(\mathcal{M}_1)_n : n \in \omega\}$ and $\{F(\mathcal{M}_2)_n : n \in \omega\}$ are equal as sets.

To that end, fix an enumeration of the six possible countable dense linear orders, and define $F(\mathcal{M})(n)$ be $(\mathrm{tp}_\mathcal{M}(n), k)$, where k is the index of the isomorphism type of $\mathrm{tp}_\mathcal{M}(n)(\mathcal{M})$. This function is again Borel, and two models yield the same set of sequence values if and only if they are apparently isomorphic. $\square$

Unfortunately, Marker's theorem does not give any information about $I_{\infty\omega}(T)$. We can do this ourselves without much extra effort:

Lemma 2.2.8. *If $S_1(T)$ is uncountable, then $I_{\infty\omega}(T) = \beth_2$.*

Proof. First, observe that almost isomorphism is absolute. Thus, if two models (of any cardinality) are almost isomorphic, then in any forcing extension in which both are countable, they are still almost isomorphic, and thus isomorphic (in the forcing extension), so back-and-forth equivalent (in the forcing extension), and thus back-and-forth equivalent (in the ground model) since back-and-forth equivalence is absolute. It is easy to see that there are only $\beth_2$ possible almost-isomorphism classes of models of T, indexed by functions from $S_1(T)$ to 6, so $I_{\infty\omega}(T) \leq \beth_2$. The other direction is less trivial.

Dependence among types is an equivalence relation with countably many classes. Since $S_1(T)$ is uncountable, it has size continuum, and thus there is a set $X \subset S_1(T)$ of size continuum consisting of nonalgebraic, nonisolated, mutually independent types. For each $Y \subset X$, let C_Y be a set of constants from the monster model consisting of one realization of each type in Y, and nothing else. The prime model M_Y over C_Y will realize each type in Y. If $p \in X \backslash Y$ is realized in M_Y by some element a, then for some finite $\overline{c}$ from C_Y, $\mathrm{tp}(a\overline{c})$ is isolated. Since p is nonisolated and $\mathrm{tp}(a/\overline{c})$ extends it, this means there is a $b \in \mathrm{cl}(\overline{c})$ which realizes p. By triviality of T (see Theorem 2.4.1; this is not circular), this means p is mutually dependent with some type in Y, against construction of X. This contradiction shows that p is omitted in M_Y.

Thus, if Y_1 and Y_2 are distinct subsets of X, then M_{Y_1} and M_{Y_2} realize different types, so are not back-and-forth equivalent. Since $|X| = \beth_1$, $|\mathcal{P}(X)| = \beth_2$, so $I_{\infty\omega}(T) \geq \beth_2$, as desired. Note that a more delicate form of this argument would show that $\cong_2 \leq_B T$, along essentially the same lines as the proof of Marker's theorem in [14]. $\square$

This proves the main results for this section, as well as the following unexpected corollary, which gives another way in which these theories are dominated by their 1-types:

Corollary 2.2.9. *Let T be o-minimal with no nonsimple types. T is small if and only if $S_1(T)$ is countable.*

Proof. If T is small, then $S(T)$ is countable, so $S_1(T) \subseteq S(T)$ is countable. If T is not small, then $F_2 \leq_B T$, so T is not smooth, so $S_1(T)$ is uncountable. $\square$

2.3 A Nonsimple Type

Our goal for this section is to show that if T is a countable o-minimal theory which admits a nonsimple type, then the $\cong_T$ is λ-Borel complete. Therefore, **for the rest of this section, T is a countable o-minimal theory which admits a nonsimple type.** Since the isomorphism relation on linear orders (of size λ) is known to be λ-Borel complete, our goal will be to show a λ-Borel reduction from linear orders to T.

Given a complete type p over some set A, and for any set $B \supset A$, define an Archimedean equivalence relation on realizations of p as follows: given a and b realizing p, say $a \sim_B b$ if there are $a_1, a_2 \in \mathrm{cl}^p_B(a)$ and $b_1, b_2 \in \mathrm{cl}^p_B(b)$ such that $a_1 \leq b \leq a_2$ and $b_1 \leq a \leq b_2$. For our purposes, A will usually be $\emptyset$. In the quite common case that $A = B = \emptyset$, we will omit the subscript on $\sim$.

This is easily seen to be an equivalence relation. Moreover, the equivalence classes are *convex*, and thus they are totally ordered. As a result, given any model $\mathcal{M}$ of T which contains B, the quotient $p(\mathcal{M})/\sim_B$ is a linear order. We call this the *Archimedean ladder.* If $A = B = \emptyset$, this is an invariant of the model which is preserved under isomorphism. Assuming we can construct models with arbitrary countable ladders, and do this in a λ-Borel fashion, we can give a λ-Borel reduction from linear orders to T and show λ-Borel completeness.

The next step toward this is the notion of faithfulness:

Definition 2.3.1. A nonsimple type $p \in S_1(A)$ is *faithful* if, for any set B of realizations of p which are pairwise $\sim_A$-inequivalent, and any $c \in \mathrm{cl}^p_A(B)$, $c \sim_A b$ for some $b \in B$.

Approximately, "faithfulness" says that given some realizations of p, you can't access anything too fundamentally different. In particular, you can't access any new Archimedean classes. Since o-minimal theories have constructible models over sets, this gives a technique: given a countable linear order L, pick a *faithful* type $p \in S_1(\emptyset)$ and a set of $\sim$-inequivalent constants which realize p, and which are indexed and ordered by L. The constructible model over this set of constants will have ladder exactly isomorphic to L, and we're done. The details will be shown later, but there is no hidden difficulty. The problem is finding a faithful type at all.

Our first stage is to show that if there is a *nonisolated* nonsimple type over $\emptyset$, then there is a *faithful* nonsimple type over $\emptyset$. It turns out all nonsimple non-cuts are 1-nonsimple, and all 1-nonsimple non-cuts are faithful, so if there is a nonsimple non-cut, there is a faithful type. Neither of these properties are true for cuts, but if there is an *unfaithful* cut over $\emptyset$, then we can use it to produce a nonsimple non-cut over $\emptyset$. So if T admits any nonisolated nonsimple type, we can produce a faithful type and conclude that T is λ-Borel complete.

This does not completely resolve the question, however. Consider the theory of ordered, affinized divisible abelian groups (Example 2.5.9). In this case, the only 1-type over $\emptyset$ is given by the atomic formula $x = x$, which is 1-simple but 2-nonsimple. No such type can be faithful, so this theory admits no faithful types over $\emptyset$. However, if we add two parameters (call them 0 and 1), there is a resulting non-cut "at infinity," which is faithful by the work above. We can build a ladder in this non-cut by faithfulness, but because the definition of the type relies on parameters, it will not be preserved under isomorphism. To deal with this, we introduce the notion of a canonical tail:

Definition 2.3.2. Let $p \in S_1(T)$ be an atomic nonsimple type, and let n be minimal where p is n-nonsimple. Say p *has a canonical tail* if, for all sets A and B from p of size n, $\sim_A$ and $\sim_B$ coincide above $\mathrm{cl}^p(AB)$. That is, for all elements c, d from p, if $c, d > \mathrm{cl}^p(AB)$, then $c \sim_A d$ if and only if $c \sim_B d$.

The problem from before is that if we use parameters to construct a ladder, it will not be preserved under isomorphism. However, if the atomic type has a canonical tail, then any isomorphism between suitably chosen models will preserve a *tail* of the intended linear order. With this in mind, we will first show that every nonsimple atomic type has a canonical tail. Next, we will construct a λ-Borel complete class of linear orders on which isomorphism and sharing a tail are the same notion, and use this to show that an atomic type with a canonical tail provides λ-Borel completeness.

Combining these two ideas shows that if T is a countable, o-minimal theory which admits a nonsimple type, then T is λ-Borel complete for all λ.

2.3.1 A Nonsimple Nonisolated Type

Our goal in this section is to show that if T admits a nonsimple nonisolated type over $\emptyset$, then T admits a faithful nonisolated type over $\emptyset$. There are two distinct cases – non-cuts and cuts. Before we prove our needed results, we will need one lemma which is used frequently and without explicit mention. Note that in this lemma and all that follow, we can also work with types over parameters with no change in the argument.

Lemma 2.3.3. *If $p \in S_1(T)$ is n-nonsimple, then for any set B of realizations of p with $|B| \geq n$, $cl^p(B)$ has no first or last element. Further, if p is 1-simple, then $cl^p(B)$ is a dense linear order.*

Proof. Let n be minimal where p is n-nonsimple, and let $\overline{a} = a_1 < \cdots < a_n$ be realizations of p. By n-nonsimplicity, there is a $b \in \mathrm{cl}^p(\overline{a})$ which is not in $\overline{a}$. By exchange, any of the elements of $\overline{a}b$ is definable over the other n. By minimality of n, this yields functions $f_i : p^n \to p$ for $i = 0, 1, \ldots, n$ where $f_0(\overline{a}) < a_1 < f_1(\overline{a}) < \cdots < a_n < f_n(\overline{a})$.

In particular, $f_0(\overline{a}) < a_1$ and $a_n < f_n(\overline{a})$, so no sufficiently large set's closure has a first or last element. Furthermore, if $n \geq 2$, then we can use f_1 to get between a_1 and a_2, establishing density. □

Lemma 2.3.4. *If $p \in S_1(T)$ is a nonsimple non-cut, then p is 1-nonsimple.*

Proof. We assume p is a left non-cut, with definable supremum L; the infimum case is symmetric. Let n be minimal such that p is n-nonsimple. By exchange and minimality of n, construct $f(\overline{x}) : p^n \to p$ such that if $\overline{x} = x_1 < \cdots < x_n$ are all from p, then $x_n < f(\overline{x}) < L$. Then f is defined and has this property on a convex set below L, so there is a $b \in \mathrm{cl}(\emptyset)$ where if $b < x_1 < \cdots < x_n < L$, then $f(\overline{x})$ is defined and $x_n < f(\overline{x}) < L$.

But since p is a non-cut, $\mathrm{cl}(\emptyset)$ approaches L from the left, so there are elements $a_1 < \cdots < a_{n-1}$ from $\mathrm{cl}(\emptyset)$ satisfying $b < a_1 < \cdots < a_{n-1} < L$. So the function $g(x) = f(a_1, \ldots, a_{n-1}, x)$ is a nonsimple function from p to p such that $x < g(x)$, establishing 1-nonsimplicity. $\square$

Lemma 2.3.5. *If $p \in S_1(T)$ is a nonsimple non-cut, then p is faithful.*

Proof. Suppose p is unfaithful. We may assume p has a supremum L; the infimum case is symmetric. By unfaithfulness, there is a tuple $a_1 < \cdots < a_n$ of realizations of p where $[a_1] < \cdots < [a_n]$ and where there is a $b \in \mathrm{cl}^p(\overline{a})$ such that $b \not\sim a_i$ for any $i = 1, \ldots, n$. We assume n is minimal with this property. Clearly $n > 1$. By exchange, we may assume $b < a_1$. Let $A = \{a_1, \ldots, a_{n-1}\}$.

By minimality of n, b cannot be defined over a proper subset of $\overline{a}$; thus, every point of $\overline{a}b$ is definable over the other n. Also, observe that $\mathrm{tp}(b/\overline{c})$ is a cut (p is nonsimple, so the closure of a nonempty set has no first or last element), while $\mathrm{tp}(a_n/Ab)$ is a non-cut. Both are nonisolated over A, so neither is realized in $\mathrm{Pr}(A)$. Yet $\mathrm{tp}(a_n/A)$ is realized in $\mathrm{Pr}(Ab)$, indicating that $\mathrm{tp}(a_n/A)$ is isolated over Ab, so there is some element of $\mathrm{cl}(Ab)$ which realizes $\mathrm{tp}(a_n/A)$. This means there is an

A-definable function from a cut over A to a non-cut over A, which is impossible. $\square$

Thus, if there is a nonsimple non-cut, we are done. We will now address the issue of nonsimple cuts, which are not as convenient as non-cuts:

Lemma 2.3.6. *If a cut $p \in S_1(T)$ is nonsimple, then it is 2-nonsimple.*

Proof. Suppose not. That is, let $p \in S_1(T)$ be a nonsimple cut, and let n be minimal such that p is n-nonsimple, and such that $n \geq 3$.

By exchange and minimality of n, there is a $\emptyset$-definable function $f : p^n \to p$, defined on ascending n-tuples from p, such that if $x_1 < \cdots < x_n$ are realizations of p, then $x_1 < f(\overline{x}) < x_2 < \cdots < x_n$. Then these properties hold on a convex set, so hold on a $\emptyset$-definable open interval $I = (a, b)$ containing p. Since $\mathrm{cl}(\emptyset)$ approaches p from the right, we can choose elements $c_3 < \cdots < c_n$ from $\mathrm{cl}(\emptyset)$ where $c_n < b$ and $p(x)$ implies $x < c_3$.

Then the function $g(x_1, x_2) = f(x_1, x_2, c_3, \ldots, c_n)$ is $\emptyset$-definable and defined on the interval (a, c_3). Further, if $x_1 < x_2$ realize p, then $x_1 < g(x_1, x_2) < x_2$, so by convexity of p, $g(x_1, x_2)$ realizes p as well, establishing 2-nonsimplicity of p. $\square$

Note that this lemma is best-possible; Example 2.5.6 gives a nonsimple cut which is 1-simple. The binary function making this cut nonsimple is an averaging function, which "spills over" to nearby non-cuts and makes them nonsimple instead (and thus a faithful type is exhibited). This behavior turns out to be completely general:

Lemma 2.3.7. *If $p \in S_1(T)$ is a nonsimple, 1-simple cut, then there is a faithful non-cut $q \in S_1(T)$.*

Proof. Let n be minimal such that p is n-nonsimple. By hypothesis and Lemma 2.3.6 we may conclude that $n = 2$. By exchange, there is a function $g : p^2 \to p$ where if $x < y$ and both realize p, then $x < g(x, y) < y$. But then this property holds on an $\emptyset$-definable interval I containing p. Since $\mathrm{cl}(\emptyset)$ approaches p from the right, let $c \in \mathrm{cl}(\emptyset)$ be some element such that $c \in I$ and $p(x)$ implies $x < c$. Then the function $g(x, c)$ is $\emptyset$-definable, defined on the open interval (a, c) which includes p, and if $a < x < c$, then $x < g(x, c) < c$.

Since $\mathrm{cl}(\emptyset)$ approaches p from the right, let $c' \in \mathrm{cl}(\emptyset)$ be such that $p(x) < c' < c$. Then the set $\{y \in \mathrm{cl}(\emptyset) : a < y < c\}$ is nonempty, and because $g(x, c)$ is $\emptyset$-definable, it must therefore approach c from the left. So the type $q(x) = \{x < c\} \cup \{x > y : y \in \mathrm{cl}(\emptyset) \wedge y < c\}$ is a non-cut, and is nonsimple under the function $g(x, c)$. By Lemma 2.3.5, q is faithful, completing the proof. $\square$

Of course a cut need not be faithful; the example of a 1-simple, 2-nonsimple cut exhibits this. However, Example 2.5.8 shows that even if a cut is 1-nonsimple, it may be unfaithful due to the presence of a function of larger arity. In that example, the binary function "overspills" and makes nearby non-cuts nonsimple, as before. This is again completely general, but the proof is more delicate than before. We will temporarily require the notion of *n-unfaithfulness*; the property of a type which says that it is unfaithful, and there is a witness of length at most n.

Lemma 2.3.8. *If a cut $p \in S_1(T)$ is 1-nonsimple but 2-unfaithful, then for any b realizing p, the non-cut below b is 1-nonsimple as a type over b.*

Proof. Suppose that p is 2-unfaithful, and pick a witnessing pair. That is, there is some pair $[a] < [b]$ from p and a $\emptyset$-definable $g(x, y)$ where $[a] < [g(a, b)] < [b]$. We will show that the non-cut $(b)^-$ is 1-nonsimple as a b-type.

Since the type $r(x, y) = p(x) \cup p(y) \cup \{[x] < [y]\}$ is complete, g witnesses unfaithfulness for every sufficiently spread pair. In particular, g is defined, continuous, strictly increasing in both of its arguments, and satisfies $x < g(x, y) < y$ everywhere on this type. We may therefore consider a $\emptyset$-definable open 2-cell $\mathcal{U}(x, y)$ containing the descending pair (b, a) (order intentional) on which these properties are satisfied. Its underlying interval must contain all of $p(x)$, and its boundary functions $L(x)$ and $R(x)$ are everywhere defined on $p(x)$.

Then $L(b) < a < R(b)$; since $[a] < [b]$ in $p(x)$, this means $L(b)$ is beyond the left edge of $p(x)$, so there is an element of $\mathrm{cl}(\emptyset)$ which is strictly between p and every value that $L(x)$ can take on p. Thus we may assume that $L(x)$ is a constant function whose value lies below p. In particular, if (x, y) are from p, then the pair is in $\mathcal{U}$ if and only if $y < R(x)$. Since $R(b) \leq b$, $R(y) \leq y$ for all y realizing p, so R is a strictly increasing function $p \to p$.

Consider the function $g(R(x), y)$. If y realizes p and $x \in (y)^-$, then $x < y$, so $R(x) < R(y)$, so $g(R(x), y)$ is defined. Since g is strictly increasing in both arguments and R is strictly increasing, the composition will be strictly increasing in both of its arguments. If we fix y (as b, for example), then since the function is strictly increasing in x, it must be a bijection from $(y)^-$ to some other non-cut $(f(y))^-$ for some $\emptyset$-definable function $f : p \to p$. Therefore $h(x, y) = f^{-1}(g(R(x), y))$ is a function $(y)^- \to (y)^-$; it only remains to show it's not equal to the identity function x.

But if it is – that is, if $h(x,y) = x$ for all $x \in (y)^-$ – then $f(x) = g(R(x), y)$ for all y and all $x \in (y)^-$. This is impossible, since $g(R(x), y)$ is strictly increasing in y, but the equality would imply $g(R(x), y)$ is locally constant in y, a contradiction [note that $g(R(x), y)$ has an open domain, so the notion of being locally strictly increasing or locally constant in y does make sense]. In particular, $h(x, b)$ is a nontrivial b-definable function from $(b)^-$ to itself, completing the proof. $\square$

By repeatedly applying the previous lemma, we can produce a faithful non-cut from any unfaithful cut:

Lemma 2.3.9. *If $p \in S_1(T)$ is an unfaithful cut, then there is a faithful non-cut $q(x) \in S_1(T)$.*

Proof. Let $p \in S_1(T)$ be unfaithful. We may assume p is 1-nonsimple. Fix a tuple from p of minimal length which witnesses unfaithfulness; this length must be at least two, so we label it $[a] < [b] < [c_1] < \cdots < [c_k]$ where $k \geq 0$, such that for some $\emptyset$-definable $f(x, y, \overline{z})$, $[a] < [f(a, b, \overline{c})] < [b]$. The type $q(x) = p(x) \cup \{[x] < [c_1]\}$ is a complete $\overline{c}$-type which is 2-unfaithful under the function $f(x, y, \overline{c})$, so by Lemma 2.3.8, there is a $b\overline{c}$-definable function $g(x, b, \overline{c})$ where if $x \in (b)^-$, then $x < g(x, b, \overline{c}) < b$.

This is a definable property, so pick a $k+1$-cell $\mathcal{U}$ containing the tuple $(b, c_1, \ldots, c_k)$ such that if $(y, z_1, \ldots, z_k)$ is in $\mathcal{U}$, and if $x \in (y)^-$, then $x < g(x, y, \overline{z}) < y$. Thus there are $\emptyset$-definable functions $L(y, z_1, \ldots, z_{k-1})$ and $R(y, z_1, \ldots, z_{k-1})$ such that $L(b, c_1, \ldots, c_{k-1}) < c_k < R(b, c_1, \ldots, c_{k-1})$. By minimality of the length of the unfaithful tuple and the fact that $[c_{k-1}] < [c_k]$, R is above p entirely. Thus there is an element $d_k \in \mathrm{cl}(\emptyset)$ such that $(b, c_1, \ldots, c_{k-1}, d_k) \in \mathcal{U}$, and therefore, that if $x \in (b)^-$, then $x < g(x, b, c_1, \ldots, c_{k-1}, d_k) < b$.

Continuing in this way, we see that the boundary functions must always jump over the end of the type, and therefore can be replaced by constant functions. That is, we can replace all the c_i in $g(x, y, \bar{c})$ with elements of $\mathrm{cl}(\emptyset)$. So there is a $\emptyset$-definable $g(x, y)$ where for *any* y from p, and any $x \in (y)^-$, $x < g(x, y) < y$. Since this property holds on an infinite set, it holds on an interval I, which must necessarily include all of p. Since $\mathrm{cl}(\emptyset)$ approaches p from the right, there is an element $b' \in \mathrm{cl}(\emptyset)$ which is in I. But then the $\emptyset$-definable function $g(x, b')$ is a function from the $\emptyset$-definable non-cut $(b')^-$ to itself, completing the proof. □

Thus we have shown that if there is a nonsimple nonisolated type, there is a faithful nonisolated type. It is tempting to conjecture, based on all we have shown, that if there is a nonsimple cut, there must be a nonsimple non-cut. However, Example 2.5.7 shows this is not the case – the theory has a nonsimple (faithful) cut, but no nonsimple non-cuts or atomic intervals. Thus, the above seems to be the most direct path to the conclusion of this Subsection:

Lemma 2.3.10. *If T admits a nonisolated nonsimple type over $\emptyset$, then T admits a faithful type over $\emptyset$.*

We will use this fact in Subsection 2.3.4.

2.3.2 A Nonsimple Isolated Type

Our goal in this subsection is to show that if there is a nonsimple, atomic type over $\emptyset$, then that type has a canonical tail. Throughout, p will refer to a nonsimple atomic type, and I will refer to the atomic interval which generates it. We will refer to the left and right endpoints of I as $-\infty$ and ∞, respectively, though they may actually be standard elements of the structure. Throughout this section, $\mathrm{cl}^I(A)$ will refer to the closure of A within I; this is used instead of p to emphasize that p is isolated.

Because of the restrictions in Lemma 2.3.3, the cases where this type is 1-nonsimple and 1-simple are fairly different. We deal with the 1-nonsimple case first.

Lemma 2.3.11. *If $p \in S_1(T)$ is a 1-nonsimple atomic type, then p has a canonical tail.*

Proof. Suppose not. Then there are a, b, c, d in I, such that $c, d > \mathrm{cl}^I(ab)$ and $c \sim_a d$ but $c \not\sim_b d$. Clearly $c \not\sim d$, so by symmetry, we may assume that $[c] < [d]$. Then there is a definable function $f(x, y)$ such that $f(c, a) \geq d$. By completeness of the c-type $\{x \in I\} \cup \{[x] < [c]\}$, $f(c, y)$ is strictly monotone and continuous on the interval $(-\infty, c')$ for some $c' \in \mathrm{cl}^I(c)$. If $f(c, y)$ is strictly increasing in y, then pick any $c'' < c'$ in $\mathrm{cl}^I(c)$, observing that $a < c''$, so $f(c, c'') > f(c, a) \geq d$, so that $c \sim d$ (and therefore $c \sim_b d$ as well, a contradiction).

Therefore $f(c, y)$ is strictly decreasing in y. If $b < a$ then $f(c, b) > f(c, a) \geq d$ so $c \sim_b d$ again; therefore $a < b$. If $a \sim b$, then there is $b' \leq a$ in $\mathrm{cl}^I(b)$, so that $f(c, b') \geq f(c, a) \geq d$, a contradiction. So $[a] < [b]$, and in fact $[a] < [b] < [c] < [d]$.

Let $a' = f(b, a)$, defined since $[a] < [b]$. By construction of f, $[a] < [b] < [a'] < [c] < [d]$. Since $\mathrm{cl}(ab) = \mathrm{cl}(ba')$ and $c \sim_a d$, $c \sim_{ab} d$, so $c \sim_{ba'} d$. Therefore, there is a b-definable function $g(x, y)$ where $g(a', c) \geq d$. But all of the elements a', c and d come from the non-cut $q(x)$ above $\mathrm{cl}(b)$ and below ∞. Therefore, we may say that $c \sim_{a'} d$ in this non-cut $q(x)$, which is over b. Further, a' and c are inequivalent in $q(x)$: if they were equivalent, then c would be bounded by $\mathrm{cl}(ba') = \mathrm{cl}(ab)$, against hypothesis. But then by faithfulness of $q(x)$, $c \sim d$ in $q(x)$, implying $c \sim_b d$. □

We can now deal with the 1-simple case by an inductive argument, using both clauses of Lemma 2.3.3 freely.

Lemma 2.3.12. *If $p \in S_1(T)$ is nonsimple and atomic, then p has a canonical tail.*

Proof. Let n be minimal such that p is n-nonsimple. By Lemma 2.3.11, we may assume $n \geq 2$. We use the following claim as an inductive step:

Claim 1. Let a, c, d realize p, and let $|A| \geq n$ be a set of realizations of p. Suppose $c, d > \mathrm{cl}^p(Aa)$ and $c \sim_{Aa} d$. Then $c \sim_A d$ as well.

Proof of Claim 1. Let A, a, c, and d be as described, and suppose $c \not\sim_A d$. Since $c \sim_{Aa} d$, there is an A-definable function $f(x, y)$ such that $f(c, a) \geq d$. Since $c \not\sim_A d$, $[c] < [d]$ in the non-cut $(\infty)^-_A$, the type $\{x \in I\} \cup \{x > a' : a' \in \mathrm{cl}^I(A)\}$.

First, consider the case where a realizes this type. Then because $c > \mathrm{cl}^I(Aa)$, it must be that $[a] < [c] < [d]$ in $(\infty)^-_A$. Since $f(a, c) \in \mathrm{cl}^I_A(ac)$, we conclude $f(a, c) \sim_A a$ by faithfulness of non-cuts, so $d \sim_A c$ by convexity of $\sim_A$ classes. Thus it only remains to show a lies in $(\infty)^-_A$.

Second, suppose $\mathrm{tp}(a/A)$ is a non-cut at some $L \in \mathrm{cl}^I(A)$, which may be $\pm\infty$; say $a \in (L)^-$ for concreteness. Then the formula $\lim_{y \to L^-} f(x, y) = \infty$ is satisfied when $c = x$, and therefore is satisfied by *all* x which are sufficiently large over A. So pick some $c' \in \mathrm{cl}^I(A)$ where $\lim_{x \to L^-} f(c', x) = \infty$. Let $a' = f(c', a)$. Then $f(c', a) \in \mathrm{cl}(Aa) \setminus \mathrm{cl}(A)$, so $\mathrm{cl}(Aa) = \mathrm{cl}(Aa')$, so $c \sim_{Aa'} d$. Moreover, $\mathrm{tp}(a'/A)$ is $(\infty)^-$, so by the first case (using a' in place of a), $c \sim_A d$.

Third, suppose $\mathrm{tp}(a/Ac)$ is a cut. Then the function $f(c, y)$ must be strictly monotone at a, else $f(a, c) \in \mathrm{cl}(Ac)$, so $c \sim_A d$. But since $\mathrm{tp}(a/Ac)$ is a cut, $\mathrm{cl}^I(Ac)$ approaches a on both sides, in particular touching the "nice domain" of $f(c, y)$ on both sides. So if $f(c, y)$ is strictly increasing at a, then pick an $a' \in \mathrm{cl}(Ac)$ above a and in the "nice domain," noting that $f(c, a') > f(c, a) \geq d$, so $c \sim_A d$. The strictly decreasing case is similar.

Since $n \geq 2$, by Lemma 2.3.3, $\mathrm{cl}^I(A)$ and $\mathrm{cl}^I(Ac)$ are dense, so neither $\mathrm{tp}(a/A)$ nor $\mathrm{tp}(a/Ac)$ is an atomic interval. Clearly neither is algebraic, so by exhaustion of cases, we may assume $\mathrm{tp}(a/A)$ is a cut and $\mathrm{tp}(a/Ac)$ is a non-cut. This means there is an element $L \in \mathrm{cl}^I(Ac)$ where (we may assume) $a \in (L)^-$, but $L \notin \mathrm{cl}^I(A)$ (the case $a \in (L)^+$ is similar). Then there is a function g over A which sends c to L and which is locally strictly monotone at c. However, $\mathrm{tp}(c/A)$ is $(\infty)^-_A$, a non-cut, while $\mathrm{tp}(L/A) = \mathrm{tp}(a/A)$ is a cut, so no such function exists. This contradiction completes our proof. $\square$ (Claim 1)

The lemma follows immediately from the claim. Let A and B be n-element sets of realizations of p. Let c and d realize p and satisfy $c, d > \mathrm{cl}^p(AB)$. If $c \sim_A d$, then trivially $c \sim_{AB} d$ as well. By applying the claim n times, we can remove all elements of A from consideration and conclude $c \sim_B d$, establishing the canonical tail. $\square$

2.3.3 A Useful Class of Linear Orders

This subsection is a temporary departure from model theory. We need to produce a subclass of the class of linear orders on λ such that $(\mathrm{LO}_\lambda, \equiv_{\infty\omega})$ is λ-Borel reducible to it, and where for any L_1 and L_2, if L_1 and L_2 are back-and-forth equivalent on a tail, then L_1 and L_2 are back-and-forth equivalent. We do this by giving two λ-Borel maps f and g from LO to itself, so that the class will be the image of $g \circ f$.

We define a *tail* of a linear order L to be any *interval* of the form $[a, \infty)$, interpreted in L, where a is in L.[1] Two orders L_1 and L_2 are *tail-equivalent*, or back-and-forth equivalent on a tail, if there are tails E_1 of L_1 and E_2 of L_2 such that $E_1 \equiv_{\infty\omega} E_2$ as linear orders.

To define the maps, first define the order $X = \{0\} \cup \{x \in \mathbb{Q} : 1 \leq x \leq 2\} \cup \{3\}$, with the inherited order from $\mathbb{Q}$. Then define $f : \mathrm{LO} \to \mathrm{LO}$ by $L \mapsto L \times X$, with the lexicographic order. That is, f expands every point of L to a copy of X.

Lemma 2.3.13. *For any linear orders L_1 and L_2, $L_1 \cong L_2$ if and only if $f(L_1) \cong f(L_2)$.*

Proof. The left-to-right direction is obvious. For the right-to-left direction, observe that the set $\{(x, 1)\} \subset f(L)$ of "1-points" is uniformly definable by the formula

[1] The reason we use this notion of tail, rather than the more-convenient "upwards closed set," is that this way, tail-equivalence is more obviously absolute. This will be useful almost immediately.

which expresses "there is a unique predecessor, but there is an interval to the right which is pure dense." Further, this is order-isomorphic to L itself under the map $(x, 1) \mapsto x$. Therefore, if $f(L_1) \cong f(L_2)$, then the "1-points" of $f(L_1)$ are isomorphic to the "1-points" of $f(L_2)$, so $L_1 \cong L_2$. $\square$

Next, define $g : \text{LO} \to \text{LO}$ by $L \mapsto \omega \times (L \cup \{\infty\})$, where ∞ is some point not in L which is above every point in L. That is, g stacks up ω copies of L, with a separating "∞-point" between each one; in particular each ∞-point has an immediate "next" ∞-point. We will show that these ∞-points are (eventually) definable, even on tails. Therefore, if $g(f(L_1))$ and $g(f(L_2))$ are isomorphic on a tail, then we can match up consecutive ∞-points between the tails, and get an isomorphism between $f(L_1)$ and $f(L_2)$.

Lemma 2.3.14. *For any linear orders L_1 and L_2, the following are equivalent:*

1. $L_1 \cong L_2$,
2. $g(f(L_1))$ *and* $g(f(L_2))$ *are isomorphic, and*
3. $g(f(L_1))$ *and* $g(f(L_2))$ *are isomorphic on a tail.*

Proof. Two of the implications are obvious; it remains to show that if $g(f(L_1))$ and $g(f(L_2))$ are isomorphic on a tail, then $L_1 \cong L_2$. We will need a claim:

Claim 1. Let L be any countable linear order. There is a $\{<\}$-formula $\phi(v)$ such that for any tail E of $g(f(L))$, there is a point $b \in E$ such that, $\phi(E) \cap (b, \infty)$ is exactly the set of "∞-points" above b.

Proof of Claim 1. Let $E = [a, \infty)$ be some tail of $g(f(L))$. Let b be the first ∞-point satisfying $b > a$. On $[b, \infty)$, every point which is *not* an ∞-point has a neighborhood which is isomorphic to $f(L)$; therefore, any formula which gave its "class" before – as a 0-point, a 1-point, a 2-point, a 3-point, or a "pure dense" point – will still apply here. More precisely:

The pure dense points are exactly those satisfying the formula stating "there is an open neighborhood around v which is pure dense." The 1-points are exactly those stating "v is not pure dense, but there is a right-neighborhood which consists entirely of pure dense points," and the 2-points are defined symmetrically to the 1-points. The 0-points are exactly those stating "v has an immediate successor which is a 1-point," and the 3-points are defined symmetrically to the 0-points.

Let c be any ∞-point above b. Then every left-neighborhood c contains infinitely many 0-points, and thus is neither pure dense nor empty, so c does not satisfy the defining formulas for pure dense points, 2-points, or 3-points. If L has no first element, then the right neighborhoods of c will have the same properties. Otherwise, if L does have a first element, then the immediate successor of c will be a 0-point. Either way, c does not satisfy the defining formulas of 0-points or 1-points.

So let $\phi(v)$ be the negation of all the above defining formulas. Then for all $x > b$, ϕ holds on x if and only if x is an ∞-point. $\phi(v)$ is defined independent of everything, completing the proof. $\square$ (Claim 1)

With this in mind, suppose $g(f(L_1))$ and $g(f(L_2))$ are isomorphic on a tail, say $[a_1, \infty) \cong [a_2, \infty)$. Fix an isomorphism $\sigma : [a_1, \infty) \to [a_2, \infty)$. Let $b_1 \in g(f(L_1))$ and

$b_2 \in g(f(L_2))$ be as in the claim. Since σ is an order-isomorphism, it preserves ϕ. Let $c > \max(b_1, \sigma^{-1}(b_2))$ be some ∞-point, and let c' be the next ∞-point after c. Then the interval (c, c') is order-isomorphic to $f(L_1)$. Also, $\sigma(c)$ and $\sigma(c')$ are consecutive ∞-points in $g(f(L_2))$ by construction, so $(\sigma(c), \sigma(c'))$ is order-isomorphic to $f(L_2)$. Since σ is an isomorphism $(c, c') \to (\sigma(c), \sigma(c'))$, this shows $f(L_1) \cong f(L_2)$, so $L_1 \cong L_2$, completing the proof. $\square$

The preceding has only concerned isomorphism, but concluding results about back-and-forth equivalence is straightforward:

Theorem 2.3.15. *Let L_1 and L_2 be linear orders on λ. The following are equivalent:*

1. $L_1 \equiv_{\infty\omega} L_2$
2. $g(f(L_1)) \equiv_{\infty\omega} g(f(L_2))$
3. $g(f(L_1))$ *and* $g(f(L_2))$ *are tail-equivalent.*

Proof. The maps f and g are absolute, as are back-and-forth equivalence and tail-equivalence; note this last one is absolute because tails must be of the form $[a, \infty)$, and forcing does not add new elements to a specific set. So let $\mathbb{V}[G]$ be a forcing extension collapsing λ. The truth values of (1), (2), and (3) are invariant between $\mathbb{V}$ and $\mathbb{V}[G]$, and because of countability, are equivalent (in $\mathbb{V}[G]$) to the equivalent expressions where $\equiv_{\infty\omega}$ is replaced by $\cong$. But the conditions (1), (2), and (3) are equivalent in this form by Lemma 2.3.14, completing the proof. $\square$

2.3.4 λ-Borel Completeness

In this section, our goal is to show that if T admits a nonsimple type, then T is λ-Borel complete for all λ. We have already shown that if T admits a nonsimple type, then T admits a nonsimple type over $\emptyset$. So we have two cases, in line with our previous work: either this type is nonisolated or atomic. The first case is straightforward:

Lemma 2.3.16. *If T admits a nonsimple nonisolated type over the empty set, then T is λ-Borel complete for all λ.*

Proof. If T admits a nonsimple, nonisolated type over $\emptyset$, then by Lemma 2.3.10, T also admits a faithful nonsimple type p over the empty set. Fix such a p.

Our main concern is to show that given any countable linear order L, there is a countable model $\mathcal{M}_L \models T$ such that $p(\mathcal{M}_L)/\sim$ is isomorphic to L as a linear order. A close examination of the proof will show that this can be made a λ-Borel function from LO to Mod(T). Since isomorphism of models implies isomorphism of the ladders, this establishes a λ-Borel reduction from the λ-Borel complete class LO, establishing λ-Borel completeness.

So fix a linear order L with universe λ, and let $X_L = \{a_\alpha : \alpha \in L\}$ be a set of realizations of p, such that if $\alpha < \beta$ in L, then $[a_\alpha] < [a_\beta]$ – by faithfulness, this is a complete specification of $\mathrm{tp}(X_L)$. Let $\mathcal{M}_L$ be a constructible model over X_L. Clearly $\mathcal{M}_L$ has cardinality λ, and this construction is predictable enough to be made λ-Borel.

Claim 1. For any L, $p(\mathcal{M}_L)/\sim$ is isomorphic to L as a linear order.

Proof of Claim 1. Define the function $f : L \to p(\mathcal{M}_L)/\sim$ by $f(\alpha) = [a_\alpha]$. By construction of X_L, f is injective and order-preserving. So it only remains to show surjectivity.

Let $c \in p(\mathcal{M}_L)$. Since $\mathcal{M}_L$ is atomic over X_L, $\mathrm{tp}(c/X_L)$ is either algebraic or an atomic interval. If $c \in \mathrm{cl}(X_L)$, then for some sequence $[a_{\alpha_1}] < \cdots < [a_{\alpha_n}]$, $c \in \mathrm{cl}(\overline{a})$. By faithfulness, this means $c \sim a_{\alpha_i}$ for some i, so $[c] = f(\alpha_i)$.

Alternately, suppose $\mathrm{tp}(c/X_L)$ is an atomic interval. Let (a, b) be an X_L-atomic interval in p where $a < c < b$. By faithfulness, p is 1-nonsimple, so there is an $a' \in \mathrm{cl}^p(a)$ where $a' > a$. Since $a \in \mathrm{cl}(X_L)$, we also have $a' \in \mathrm{cl}(X_L)$, so by X_L-atomicity of (a, b), we have $a' \geq b$. Clearly $a \sim a'$, so by convexity, $a \sim c$. By the previous paragraph, $a \sim x_\alpha$ for some α, so by transitivity, $c \sim x_\alpha$ as well, so $[c] = f(\alpha)$.

Therefore f is surjective, so is an isomorphism. Thus $p(\mathcal{M}_L)/\sim$ is isomorphic to L. □ (Claim 1)

Now we want to show that for all linear orders L_1 and L_2 on λ, $L_1 \equiv_{\infty\omega} L_2$ if and only if $\mathcal{M}_{L_1} \equiv_{\infty\omega} \mathcal{M}_{L_2}$. So let $\mathbb{V}[G]$ be any forcing extension in which λ is countable. The conditions $L_1 \equiv_{\infty\omega} L_2$ and $\mathcal{M}_{L_1} \equiv_{\infty\omega}\equiv_{\infty\omega} \mathcal{M}_{L_2}$ are both absolute and (in the forcing extension) are equivalent to isomorphism, since the structures are countable. Additionally, the construction $L \mapsto \mathcal{M}_L$ is absolute, so our claim still holds in $\mathbb{V}[G]$. It is clear that if $L_1 \cong L_2$, then $\mathrm{tp}(X_{L_1}) = \mathrm{tp}(X_{L_2})$, so by uniqueness of constructible models, $\mathcal{M}_{L_1} \cong \mathcal{M}_{L_2}$. On the other hand, if $\mathcal{M}_{L_1} \cong \mathcal{M}_{L_2}$, then

$p(\mathcal{M}_{L_1})/\sim\cong p(\mathcal{M}_{L_2})/\sim$ as linear orders, so by the claim, $L_1 \cong L_2$, as desired. □

Lemma 2.3.17. *If T admits a nonsimple isolated type over the empty set, then T is λ-Borel complete for all λ.*

Proof. Let λ be an infinite cardinal. Recall that in Theorem 2.3.15, we constructed λ-Borel reductions g and f from LO to itself such that for all L_1 and L_2, $L_1 \equiv_{\infty\omega} L_2$ if and only if $g(f(L_1)) \equiv_{\infty\omega} g(f(L_2))$, if and only if $g(f(L_1))$ and $g(f(L_2))$ are equivalent on a tail. To simplify notation, we will assume that all linear orders used in this proof are in the image of $g \circ f$, and we will have no particular use for the preimage of these orders under $g \circ f$.

So let p be a nonsimple atomic type, and fix the minimal n where p is n-nonsimple. For any linear order L on λ, let $L^* = \{1, \ldots, n\} \cup g(f(L))$, where $1 < 2 < \cdots < n$ and $n < \alpha$ for all $\alpha \in g(f(L))$. Let X_L be $\{x_\alpha : \alpha \in L^*\}$ where for all $\alpha \in L^*$, x_α realizes p and $x_\alpha > \mathrm{cl}^p(\{x_\beta : \beta < \alpha\})$. Evidently this condition completely specifies $\mathrm{tp}(x_\alpha/X_{<\alpha})$, so by a standard argument this completely specifies $\mathrm{tp}(X_L)$. Let $\mathcal{M}_L$ be constructible over X_L. The function $L \mapsto \mathcal{M}_L$ can be made λ-Borel, and the isomorphism type of $\mathcal{M}_L$ is completely determined by $\mathrm{tp}(X_L)$, which is completely determined by the isomorphism type of L. By a standard forcing argument, this can be extended to show that if $L_1 \equiv_{\infty\omega} L_2$ then $\mathcal{M}_{L_1} \equiv_{\infty\omega} \mathcal{M}_{L_2}$. The remainder of this proof is to show the converse of this fact.

For any n-element set B from $\mathcal{M}_L$, let $p_B(x)$ be the nonsimple non-cut $p(x) \cup \{x > \mathrm{cl}^p(B)\}$. The primary claim in this proof is to show that for *any* n-element set B from $p(\mathcal{M}_L)$, we recover a tail of $g(f(L))$ in p_B. That is, $p_B(\mathcal{M}_L)/\sim_B$ is isomorphic on a tail to $g(f(L))$. As before, we must divide into two cases, based on whether p is 1-nonsimple, because of the restrictions in Lemma 2.3.3.

Claim 1. If p is 1-simple, then for any set $B \subseteq p(\mathcal{M}_L)$ with $|B| = n$, $p_B(\mathcal{M}_L)/\sim_B$ is isomorphic on a tail to $g(f(L))$.

Proof of Claim 1. Let $A = \{1, \ldots, n\}$; then by construction of $\mathcal{M}_L$ and the fact that non-cuts are faithful, $p_{\overline{a}}(\mathcal{M}_L)/\sim_A$ is isomorphic to $g(f(L))$. It is therefore sufficient to show that for any B, $p_B(\mathcal{M}_L)/\sim_B$ and $p_A(\mathcal{M}_L)/\sim_A$ are isomorphic on a tail.

So fix an n-element set B from $p(\mathcal{M}_L)$. Since p is 1-simple, by Lemma 2.3.3, $\mathrm{cl}^p(X_L)$ is a dense linear order without endpoints. Since $\Pr(X_L)$ is atomic over X_L, $p(\mathcal{M}_L)$ *is* $\mathrm{cl}^p(X_L)$. So by compactness, there is a finite subset $L_0 \subset L^*$ containing $\{1, \ldots, n\}$ such that $AB \subset \mathrm{cl}^p(\{x_\alpha : \alpha \in L_0\})$. Let X_0 be the tail of X_L above X_{L_0}; that is, the set of all x_α such that for all $\beta \in L_0$, $\alpha > \beta$. Since L has no largest element, X_0 is nonempty. We will show it forms a common tail of $p_A(\mathcal{M}_L)/\sim_A$ and $p_B(\mathcal{M}_L)/\sim_B$, which is sufficient to prove the claim.

X_0 forms a tail of $p_A(\mathcal{M}_L)/\sim_A$ under the function $x_\alpha \mapsto [x_\alpha]$, by the characterization of $p_A(\mathcal{M}_L)/\sim_A$ at the beginning of this proof. As for p_B, by construction of L_0, if $x_\alpha \in X_0$, then x_α realizes p_B. Each of the $x_\alpha \in X_0$ is $\sim_{L_0}$-inequivalent by construction of X_L, so must be $\sim_B$-inequivalent as well; it only remains to show that the set $\{[x] : x \in X_0\}$ is right-closed in $p_B(\mathcal{M}_L)/\sim_B$.

So suppose $x_\alpha \in X_0$ and $c > x_\alpha$ realizes p. By the characterization of $p_A(\mathcal{M}_L)/\sim_A$, $c \sim_A x_\beta$ for some $\beta \geq \alpha$. Since c and x_β are both greater than or equal to x_α, which is above $\mathrm{cl}^p(\{x_\gamma : \gamma < \alpha\}) \supset \mathrm{cl}^p(AB)$, we can use the canonical tail condition to conclude that $c \sim_B x_\beta$ as well. Therefore, $[c] \in \{[x] : x \in X_0\}$, so X_0 forms a tail of $p_B(\mathcal{M}_L)/\sim_B$. This completes the proof. $\square$ (Claim 1)

The difference between the preceding claim and the next is that if $n = 1$, we cannot necessarily assume that $p(\mathcal{M}_L)$ is equal to $\mathrm{cl}^p(X_L)$: the latter may not be a dense linear order, so $\mathrm{tp}(a/X_L)$ being isolated may not imply it being algebraic.

Claim 2. If p is 1-nonsimple, then for any set B from $p(\mathcal{M}_L)$ with $|B| = n = 1$, $p_B(\mathcal{M}_L)/\sim_B$ is isomorphic on a tail to $g(f(L))$.

Proof of Claim 2. Let $a = x_1$. As before, we can conclude that $p_a(\mathcal{M}_L)/\sim_a$ is isomorphic to $g(f(L))$, and therefore that we need to show for every $b \in p(\mathcal{M}_L)$, $p_b(\mathcal{M}_L)/\sim_b$ and $p_a(\mathcal{M}_L)/\sim_a$ agree on a tail. So, fix such a b. Since $\mathrm{tp}(b/X_L)$ is atomic, either $b \in \mathrm{cl}(X_L)$, or $\mathrm{tp}(b/X_L)$ is generated by an atomic interval. If $b \in \mathrm{cl}(X_L)$, then the previous proof applies without change. Therefore, assume $\mathrm{tp}(b/X_L)$ is an atomic interval (C, D) where $C, D \in \mathrm{cl}(X_L)$.

Let L_0 be a finite subset of L^* which contains 1 and such that $C, D \in \mathrm{cl}(\{x_\alpha : \alpha \in L_0\})$. Let X_0 be the elements of X_L which are above L_0. This is a right-closed subset of $g(f(L))$, so it forms a tail of $p_a(\mathcal{M}_L)/\sim_a$; it remains to show it forms a tail of $p_b(\mathcal{M}_L)/\sim_b$. As before, the function $x \mapsto [x]$ is a well-defined, order-preserving injection from X_0 to $p_b(\mathcal{M}_L)/\sim_b$. It remains to show surjectivity.

So pick a c from $p_b(\mathcal{M}_L)$ such that for some $x_\alpha \in X_0$, $c > x_\alpha$. For some $\beta \geq \alpha$, $c \sim_a x_\beta$; we want to show $c \sim_b x_\beta$ as well. Since p has a canonical tail, it is enough to show that $x_\alpha > \mathrm{cl}^p(ab)$, so *suppose not.* Then there is an a-definable function $f(x)$ where $f(b) \geq x_\alpha$. Then $f(x)$ is defined and strictly monotone on the atomic interval (C, D); we may assume strict increasing. Since $x_\alpha > \mathrm{cl}^p(X_0)$, it must be that $\lim_{x \to D^-} f(x) = \infty$. We will use this limit to prove that (C, D) is not X_0-atomic, yielding a contradiction.

The image of (C, D) under the function f must also be an interval, since f is continuous and strictly increasing, and so by the argument above, it must be of the form (E, ∞) for some $E \in \mathrm{cl}^p(X_0)$. By 1-nonsimplicity, there is a $E' > E$ in p which is E-definable; since the interval is right-infinite, $E' \in \mathrm{im}(f)$. But then $f^{-1}(E') \in (C, D)$ and is X_0-definable, a contradiction of atomicity of (C, D). $\square$ (Claim 2)

Having performed these two claims, the result follows immediately. Suppose $\mathcal{M}_{L_1} \equiv_{\infty\omega} \mathcal{M}_{L_2}$. Let $\mathbb{V}[G]$ be a forcing extension which collapses λ, so that (in the extension) $\mathcal{M}_{L_1} \cong \mathcal{M}_{L_2}$. Let $f : \mathcal{M}_{L_1} \to \mathcal{M}_{L_2}$ be an isomorphism, and let $\overline{b}$ be $f(\overline{a})$, where $\overline{a} = \{x_1, \ldots, x_n\}$. Then f is an isomorphism between the expanded structures $(\mathcal{M}_{L_1}, \overline{a})$ and $(\mathcal{M}_{L_2}, \overline{b})$, so in particular $p_(\overline{a})(\mathcal{M}_{L_1})/\sim_{\overline{a}}$ and $p_{\overline{b}}(\mathcal{M}_{L_2})/\sim_{\overline{b}}$ are isomorphic as linear orders.

The former is isomorphic to $g(f(L_1))$. By the claims, the latter is isomorphic to a tail of $g(f(L_2))$, so by construction of our linear orders, $g(f(L_1)) \cong g(f(L_2))$ (in $\mathbb{V}[G]$). Thus $g(f(L_1)) \equiv_{\infty\omega} g(f(L_2))$ in $\mathbb{V}[G]$ and in the ground model, completing the proof. $\square$

Combining this with results from the above, we have proved the main theorem of the section.

Theorem 2.3.18. *Let T be a countable o-minimal theory. If T admits a nonsimple type, then T is λ-Borel complete for all λ.*

2.4 Corollaries

Most interesting o-minimal theories admit nonsimple types, and are therefore Borel complete. Our aim for this section is to establish two broad classes of o-minimal theories which are λ-Borel complete; these yield very general sufficient conditions for λ-Borel completeness.

The first such class of such theories is the class of nontrivial theories - those where it is possible for a point to be definable over a set without being definable over any single point inside that set. For example, any theory with an infinite definable group would satisfy this property. We show that any nontrivial o-minimal theory is λ-Borel complete by using nontriviality to construct a nonsimple type over a finite set, then appealing to Theorem 2.1.7.

The other broad class is the discretely o-minimal theories, or even those which have a significant discrete part. Although it was shown in [19] that the discrete part of an o-minimal theory is completely trivial (in the above sense), the successor function still provides an interesting (unary) function on that part of the structure where it is defined, which is enough to construct a nonsimple type and show λ-Borel completeness.

We remark that these two results, while quite general, do not suggest a characterization of all nonsimple types. Example 2.5.1 and its variations give theories which are trivial and completely dense, but still admit nonsimple types.

2.4.1 Nontrivial Theories

Recall that a theory T *nontrivial* if there is some point b and some set A where $b \in \mathrm{cl}(A)$ but $b \notin \bigcup_{a \in A} \mathrm{cl}(a)$. We use exchange and nontriviality to produce a nonsimple type over finitely many parameters, and therefore conclude with λ-Borel completeness.

Theorem 2.4.1. *If T is a nontrivial o-minimal theory then T admits a nonsimple type.*

Proof. Suppose T is nontrivial. We will produce a nonsimple type $p(x)$ over finitely many parameters, establishing λ-Borel completeness by Theorem 2.1.7 and Corollary 2.1.6. By nontriviality, there is a set A and $b \in \mathrm{cl}(A)$ where $b \notin \mathrm{cl}(a)$ for any $a \in A$. We may assume A is finite, and that A has minimal cardinality among all "nontrivial sets." Enumerate A in an ascending way as $a_1 < \cdots < a_n$, remarking that $n \geq 2$. The set $B = \{a_3, \ldots, a_n\}$ will be the first part of our parameter set.

Then $b \in \mathrm{cl}_B(a_1, a_2)$ but $b \notin \mathrm{cl}_B(a_i)$ for either i. Let $p(x) = \mathrm{tp}(a_1/B)$, $q(x) = \mathrm{tp}(a_2/B)$, and $r(x) = \mathrm{tp}(b/B)$. Each of these types is nonalgebraic. Suppose (for example) that $\mathrm{cl}_B^p(a_2)$ is nonempty; then we may replace a_2 with some realization of $p(x)$, bidefinable with a_2 over B, without affecting the dependence relation $b \in \mathrm{cl}(a_1 a_2) \setminus (\mathrm{cl}(a_1) \cup \mathrm{cl}(a_2))$. Using this idea and exchange, we may assume the following cases are exhaustive:

First: One of the types p, q, or r is a nonsimple B-type, in which case the theorem is proved.

Second: $p = q = r$; then p is a 2-nonsimple B-type under whatever function takes the pair (a_1, a_2) to b.

Third: $p = q$ and $p \neq r$. Then there is a B-definable binary function $f : p^2 \to r$ taking ascending pairs from p into single elements of r. Then for any a modeling p, there is a unique extension of r to a Ba-type (or else we're actually in the previous case) and the function $f(a, y)$ must be a bijection from the complete Ba-type $p(x) \cup \{x > a\}$ to r. But then the function $g : p \to p$ where $g(x)$ is the unique $y > x$ such that $f(x, y) = b$ is well-defined and nonsimple, so that $p(x)$ is a complete, nonsimple Bb-type.

Fourth: p, q, and r are all distinct. Let f be such that $f(a_1, a_2) = b$. Then for any c modeling r, the types p and q are completely described over Bc and the function $f(a, y)$ is a bijection from q to r. So for any c realizing r, we have a bijection $h_c : p \to q$ taking x to the unique y where $f(x, y) = c$.

Therefore, fix $c_1 < c_2$ realizing r. If p or q does *not* extend uniquely to a complete Ac_1c_2-type, then we have a function $r^2 \to p$ or $r^2 \to q$, and T is λ-Borel complete by a previous case. But otherwise, p and q are complete over Ac_1c_2, and the functions h_{c_1} and h_{c_2} are *distinct* bijections $p \to q$. Therefore $h_{c_1}^{-1} \circ h_{c_2}$ is a nontrivial bijection $p \to p$, so p is a nonsimple type over Bc_1c_2. $\square$

By Corollary 2.1.6, such a theory admits a nonsimple type. By Theorem 2.1.7, this implies λ-Borel completeness and more. Note that this implies the λ-Borel completeness of any o-minimal theory which defines a group, such as the theory of

$(\mathbb{R}, +, \cdot)$.

2.4.2 Non-Dense Theories

Given an o-minimal theory T, a model $\mathcal{M} \models T$, say a point $a \in \mathcal{M}$ is *non-dense* if a has either an immediate successor or an immediate predecessor (which may be among $\pm\infty$). If T has only finitely many such points, they play no role in the countable model theory of T; we can canonically fit a copy of $(\mathbb{Q}, <)$ between any non-dense point and its successor or predecessor, resulting in a theory which is essentially identical (for our purposes) to T, but which is everywhere dense. Our theorem for this section is the following:

Theorem 2.4.2. *If T is an o-minimal theory with infinitely many non-dense points, then T admits a nonsimple type.*

Proof. We construct a nonsimple type over the empty set. Since there are infinitely many non-dense points, there is an infinite interval I_0 over $\emptyset$ which consists entirely of non-dense points. Therefore, there is a subinterval I of I_0 of points which all have immediate successors and predecessors. Let $S(x)$ denote the immediate successor function, where it is defined. We will construct a complete type extending I which is nonsimple under the function $x \mapsto S(x)$.

Let $I = (a, b)$, noting $a, b \in \mathrm{cl}(\emptyset)$. We have several cases:

First: If a has no immediate successor, then define $p(x)$ by

$$p(x) = \{a < x\} \cup \{x < c : c > a, c \in \mathrm{cl}(\emptyset)\}$$

By o-minimality, $p(x)$ is a complete type, and clearly extends I. It may be either an atomic interval (if $\mathrm{cl}^I(\emptyset) = \emptyset$) or a non-cut $(a)^+$ (if not), but either way, it must be closed under S. For if not, there is an x realizing $p(x)$ such that $S(x) \geq c$ for some $c \in \mathrm{cl}^I(\emptyset)$. But then $S(x) = c$, so $S^{-1}(c)$ is well-defined, in $\mathrm{cl}(\emptyset)$, and equal to x, so that x does not model p after all.

Thus $p(x)$ is complete and nonsimple under the function S.

Second: If b has no immediate predecessor, then define $q(x)$ by

$$q(x) = \{x < b\} \cup \{x > c : c < b, c \in \mathrm{cl}(\emptyset)\}$$

By the same logic as above, $q(x)$ is complete and closed under the function $x \mapsto S^{-1}(x)$, so is nonsimple.

Finally: If $S(a)$ and $S^{-1}(b)$ both exist, then define $r(x)$ by

$$r(x) = \{x > S^n(a) : n \in \omega\} \cup \{x < c : c \in \mathrm{cl}(\emptyset) \wedge c > S^n(a) \text{ for all } n \in \omega\}$$

Then $r(x)$ is a complete type over $\emptyset$ as before, and is a cut. But as before, if x realizes r, then $S(x)$ is defined and must still realize r. Thus r is nonsimple. $\square$

By Theorem 2.1.7, such a theory must be λ-Borel complete.

2.5 Examples

We begin with several basic example of o-minimal theories. We assume the reader is aware of classical quantifier elimination results for the theory of real-closed fields and for ordered divisible abelian groups; if not see [13]. Most of our examples are definable reducts of an expansion of these theories by constants, so that o-minimality follows automatically. We have omitted many of the verifications of these examples when they seem similar to previous proofs; the interested reader is encouraged to fill them in as s/he desires.

Example 2.5.1. Let $\mathcal{M}$ have universe $\mathbb{Q}$ and have a unique unary function f given by $x \mapsto x+1$.

Then $S_1(T)$ has a single element, the atomic interval $x = x$, and this is 1-nonsimple and faithful.

Proof. $\mathcal{M}$ is o-minimal as a reduct of $(\mathbb{Q}, +, <)$. The function $x \mapsto x + q$ is an automorphism of $\mathcal{M}$ for any $q \in \mathbb{Q}$, and thus every element has the same 1-type, so $x = x$ is a complete formula. This atomic interval is 1-nonsimple under the function f.

To see faithfulness, we will need to see that the following statements are a complete axiomatization of T, and that the resulting theory has quantifier elimination:

- $<$ is a dense linear order without endpoints.
- f is a strictly increasing bijection on the universe.
- For all x, $x < f(x)$.

The proof of quantifier elimination of the above axioms is standard; completeness follows from the fact that $\mathcal{M}$ satisfies these axioms and embeds into every model of them.

To see faithfulness, let $\mathcal{N}$ be some elementary extension of $\mathcal{M}$. It is enough to see that if $[x_1] < \cdots < [x_n]$ and $z \in \mathrm{cl}(\overline{x})$, then $z \in [x_i]$ for some i. For each i, let $Q_i = \{y : \exists n \in \mathbb{Z}(s^n(x) \leq y < s^{n+1}(x))\}$. Evidently $Q_i \subset [x_i]$, so $Q_1 < Q_2 < \cdots < Q_n$. Also, by completeness of the above axioms, the set $Q = Q_1 \cup \cdots \cup Q_n$ is a model of T; by quantifier elimination, $Q \prec \mathcal{N}$. Since $z \in \mathrm{cl}(\overline{x})$, $z \in Q$, so $z \in Q_i$ for some i, so $z \in [x_i]$. □

Example 2.5.2. Let $\mathcal{M}$ have universe $\mathbb{Q}$, have a single function $f(x) = x + 1$, and have constant symbols for $n \in \mathbb{Z}$.

Then $S_1(T)$ has infinitely many atomic intervals, all dependent on one another and simple, and two non-cuts, which are independent, nonsimple, and faithful.

The above properties still hold if we only have a single constant symbol.

Proof. The types $n < x < n + 1$ are atomic – any order-preserving bijection from $(n, n+1)$ to itself induces an automorphism of the structure, witnessing completeness of the intervals. They must be simple since all countable dense linear orders are n-transitive for all n. Of course the non-cuts at ∞ and $-\infty$ are nonsimple under the function f, and non-cuts are faithful, as desired. □

Example 2.5.3. Let $\mathcal{M}$ have universe $\mathbb{Q}$, constant symbols for each $n \in \mathbb{Z}$, and unary functions f_n for $n \in \mathbb{Z}$ where $f_n(x) = x+1$ and f_n is only defined on $[n, n+1)$.

Then $S_1(T)$ has infinitely many atomic intervals, all dependent on one another and simple, and two non-cuts, which are independent and simple.

Example 2.5.4. Let $\mathcal{M}$ have universe $\mathbb{Q}$, constant symbols for each $q \in \mathbb{Q}$, and unary functions f_n for $n \in \mathbb{Z}$ where $f_n(x) = x+1$ and f_n is only defined on $[n, n+1)$.

Then $S_1(T)$ has no atomic intervals, infinitely many independent non-cuts (all of which have an infinite dependence class), and uncountably many independent cuts (all of which have an infinite dependence class). Additionally T has two non-cuts which are independent from each other and everything else. All these types are simple.

Example 2.5.5. Let $\mathcal{M}$ have universe $\mathbb{Q}$, have a single function $f(x) = x + 1$, and have constant symbols for $q \in \mathbb{Q}$.

Then $S_1(T)$ has infinitely many independent non-cuts, each of which has an infinite dependence class. $S_1(T)$ also has uncountably many cuts, each of which has an infinite dependence class and is simple. $S_1(T)$ has no atomic intervals.

Additionally, $S_1(T)$ has two independent non-cuts which are nonsimple.

The above properties still hold if we give $\mathcal{M}$ symbols for $f_q(x) = x + q$ for all $q \in \mathbb{Q}$.

Example 2.5.6. Let $\mathcal{M}$ be $(\mathbb{Q}, <, +, 0, 1)$, and let p be the cut corresponding to any irrational number.

Then the cut p is 2-nonsimple and 1-simple, hence unfaithful. There are no faithful cuts, but there are nonsimple non-cuts near p.

Proof. Clearly p is 2-nonsimple under the function $(x, y) \mapsto \frac{x+y}{2}$.

Let $\mathcal{N}$ be $(\mathbb{R}, <, +, 0, 1)$. Since $T = \mathrm{Th}(\mathcal{M})$ has quantifier elimination, $\mathcal{M} \prec \mathcal{N}$. The type p is realized by a single element (the irrational number used to create p); call it a. If p were 1-nonsimple under some function f, then $f(a)$ would be a realization of p which is not equal to a, which is impossible. This proves p is 1-simple.

Since every cut is of this form, we have proven the result. The nonsimple non-cuts are *any* non-cut, including those at ∞. The non-cuts at $(q)^+$ and $(q)^-$ for rational q are nonsimple under the function $x \mapsto \frac{x+q}{2}$. $\square$

Example 2.5.7. Let $\mathcal{M}$ have underlying set $\mathbb{Q}_1 + \mathbb{Q}_2$, where both $\mathbb{Q}_i$ are copies of $(\mathbb{Q}, <)$ and $\mathbb{Q}_1 < \mathbb{Q}_2$. Give $\mathcal{M}$ symbols for all n in both copies of $\mathbb{Z}$, and a unary function $f(x)$ where $f(x) = x + 1_i$, where 1_i is the copy of 1 in the $\mathbb{Q}_i$ containing x. Let p be the type $\{m < x < n : m \in \mathbb{Z}_i, n \in \mathbb{Z}_i\}$.

Then the cut p is 1-nonsimple and faithful, but there are no nonsimple non-cuts or atomic intervals anywhere. $\mathcal{M}$ has infinitely many atomic intervals (two dependence classes) and two non-cuts (independent of one another).

If we add constants for all rationals, then we have no atomic intervals, many non-cuts, and uncountably many cuts, but the only nonsimple type is still p.

Example 2.5.8. Let $\mathcal{M}$ be an $\aleph_0$-saturated elementary extension of $(\mathbb{Q}, <, +, 0, 1)$, let a be some realization of the non-cut $(0)^+$, and let T be the theory of $\mathcal{M}$ with a symbol for a.

Let π be some irrational number in $\mathbb{R}$, and let $p(x)$ be the type given by $\{x > q + na : q \in \mathbb{Q}^{<\pi}, n \in \mathbb{Z}\} \cup \{x < q + na : q \in \mathbb{Q}^{>\pi}, n \in \mathbb{Z}\}$.

Then the cut p is 1-nonsimple yet unfaithful. There are no faithful cuts, but there are nonsimple non-cuts near p.

Proof. First, by the usual quantifier elimination for the theory of ordered divisible abelian groups, for any set X, $\mathrm{cl}(X) = q + ra + sx$, where $q, r, s \in \mathbb{Q}$, a is our fixed infinitesimal, and $x \in X$. In particular this shows $p(x)$ is a complete type and a cut. Clearly p is closed under the function $x \mapsto x + a$, witnessing 1-nonsimplicity.

To see the unfaithfulness, suppose $[x] < [y]$; we will show $[x] < [\frac{x+y}{2}] < [y]$. This follows quickly from the following claim:

Claim 1. Suppose $b \in \mathrm{cl}^p(x)$. Then $b = x + ra$ for some rational r.

Proof of Claim 1. If $b \in \mathrm{cl}^p(x)$, then $b = q + ra + sx$ for some rational q, r, s. We will show $q = 0$ and $s = 1$. Since p is fixed under $x \mapsto x + a$ and is convex, it is also closed under $x \mapsto x - ra$, so $b - ra = q + sx$ realizes p. If $s = 0$ then $b = q + ra \in \mathrm{cl}(\emptyset)$, which is not a realization of p, so $s \neq 0$. Also, sx realizes the (partial) cut $\pi - q$, so x realizes the (partial) cut $\frac{\pi - s}{q}$. But of course x realizes the (partial) cut π, so $\frac{\pi-s}{q} = \pi$, so $(1 - q)\pi = s$. If $q \neq 1$, then π is irrational (a contradiction); if $q = 1$, then $s = (1 - q)\pi = 0$, proving the claim. □ (Claim 1)

Since $[y] > [x]$, $y > \mathrm{cl}^p(x)$, so in particular $y > 2ra + x$ for all rational r. Equivalently, $y - x > 2ra$ for all r. Then $\frac{x+y}{2} - x = \frac{y-x}{2} > ra$ for all r, so $\frac{x+y}{2} > x + ra$ for all rational r. Thus $[x] < [\frac{x+y}{2}]$. Similarly $[\frac{x+y}{2}] < [y]$, as desired. □

Example 2.5.9. Let $\mathcal{M}$ have universe $\mathbb{Q}$ and have a single function $f(x, y, z) = x + y - z$.

Then $S_1(T)$ has a single element, the atomic interval $x = x$, and this is 1-simple, 2-nonsimple, and unfaithful.

Proof. T is a definable reduct of the theory of $(\mathbb{Q}, +, 0, <)$, so is o-minimal. If $a < b$ and $c < d$, then the function $x \mapsto (x - a) \cdot \frac{d-c}{b-a}$ preserves f, is a bijection, and takes a to c and b to d. Thus the formula $x = x$ defines a complete type which is 1-simple. Visibly this is 2-nonsimple under the function $(x, y) \mapsto x + x - y$, so the type must be unfaithful as well. □

Example 2.5.10. Let $\mathcal{M}$ have universe $\mathbb{Q}$, a unary function $s(x) = x + 1$, and a ternary function $f(x, y, z) = x + y - z$.

Then $S_1(T)$ has a single element, the atomic interval $x = x$, and this is 1-nonsimple and unfaithful.

Proof. T is a definable reduct of an o-minimal theory, so is o-minimal. For any x and y in $\mathbb{Q}$, the function $z \mapsto z + (y - x)$ is readily seen to be an automorphism of the structure taking x to y, so $x = x$ is a complete type. It is visibly 1-nonsimple under the function s. To see the unfaithfulness, we must construct a nonstandard model, since $\mathcal{M}$ has only one Archimedean class.

So let $\mathcal{M}'$ be the expansion of $\mathcal{M}$ by a constant symbol for zero. Then $\mathcal{M}'$ is a definable expansion of $(\mathbb{Q}, +, 0, 1, <)$, so the theory T' of $\mathcal{M}'$ implies that the definable reduct to $s(x) = x + 1$ and $f(x, y, z) = x + y - z$ is a model of our original theory T, and this will hold for any model of T'. So let $\mathcal{M}_2'$ be a model of T' with infinite elements x and y such that $[x] < [y]$ in $\mathcal{M}_2'$.

Let $\mathcal{M}_2$ be the reduct of $\mathcal{M}_2'$ to our original language. Since Archimedean equivalence is finer in a reduct, it is still true that $[x] < [y]$ in $\mathcal{M}_2$. The function $z \mapsto z + (y - x)$ is an automorphism of $\mathcal{M}_2$ taking x to y and y to $2y - x$; thus $[x] < [y] < [2y - x]$ in $\mathcal{M}_2$. Yet $2y - x = f(y, y, x)$, so is in the closure of $\{x, y\}$, witnessing 2-unfaithfulness of the type. $\square$

In Chapter 3, we will see that if T is a colored linear order in a finite language, then either T is λ-Borel complete or $\aleph_0$-categorical. It is natural to ask if the same is true for o-minimal theories, and the answer is no:

Example 2.5.11. Let $\mathcal{M}$ have universe $\mathbb{Q} \cup \mathbb{Q}\sqrt{2}$. Give $\mathcal{M}$ two unary functions f and g:

- f has domain $[0, 2]$, and where defined, $f(x) = x + 1$.
- g has domain $[0, 2]$, and where defined, $g(x) = x + \sqrt{2}$.

Then $T = \mathrm{Th}(\mathcal{M})$ is o-minimal in a finite language, and $\cong_T$ is Borel equivalent to $\cong_2$.

Proof. The theory of "ordered divisible abelian groups" is o-minimal, and $\mathbb{Q} \cup \mathbb{Q}\sqrt{2}$ is a model of this theory. If we expand it by constants for 0, 1, 2, and $\sqrt{2}$, it is still o-minimal, and f and g are definable on this structure. So $\mathcal{M}$ is a definable reduct of this o-minimal structure, so is o-minimal. It remains to show that $S_1(T)$ is uncountable and that it contains no nonsimple types.

First we must show $S_1(T)$ is uncountable. To see this, we first show $\mathrm{cl}(\emptyset)$ is dense in some interval. Toward this end, note that the function $h(x)$ taking $x \in [0, 1]$ to $(x+\sqrt{2}) \bmod 1$ is definable – $h(x)$ is the unique $y \in [0, 1]$ such that $x+\sqrt{2}$ is equal to $y + 1$ or $y + 2$. It is a classical fact if r is irrational, the orbits of $x \mapsto x + r$ in S^1 are dense in S^1, implying our desired result. Then $\mathrm{cl}(\emptyset) \cap (0, 1)$ is a countable dense linear order, so is order-isomorphic to $(\mathbb{Q}, <)$. Thus $(\mathbb{R}, <)$ has an order-preserving injection into $S_1(T)$, sending each rational r to the corresponding algebraic type and sending each irrational r to the corresponding cut.

Second we show that $S_1(T)$ has no nonsimple types. Consider first the intervals $(-\infty, 0)$ and $(2 + \sqrt{2}, \infty)$. Any order-preserving bijection of either set induces an isomorphism of the structure. Thus these intervals are complete types. Additionally, for any n, $(\mathbb{Q}, <)$ is n-transitive, so these intervals must be n-simple. Since this holds for all n, they are simple types.

For the rest of the types, first consider the theory $\overline{T}$ where f and g have unrestricted domain; this theory has an easy quantifier elimination. If $h(x_1, \ldots, x_n)$ is definable in T, it is also definable in $\overline{T}$, so locally $h(x_1, \ldots, x_n)$ is equal to $x_i + a$ for some $a \in \mathbb{Q} + \mathbb{Q}\sqrt{2}$. So suppose $p \in S_1(T)$ is nonsimple. Then p must be 1-nonsimple, so some function $x \mapsto x + a$ maps p to p. Yet since p implies $0 \leq x \leq 2 + \sqrt{2}$, so p has "finite width;" therefore, after enough applications of $x \mapsto x + a$, x will no longer satisfy p. This contradiction shows that p cannot be nonsimple, as desired. □

We end this chapter with an annoying open question:

Question 2.5.12. Let T be o-minimal and let p be a type. Is it possible for p to be nonsimple but 2-simple?

If so, are both cuts and atomic intervals of this form possible?

Similarly, is it possible for p to be 1-nonsimple, 2-faithful, and 3-unfaithful?

My intuition is no for several reasons. First and weakest, by analogy with colored linear orders, it an old result (see for example [23]) that any 2-transitive linear order is n-transitive for all n. Since being n-simple corresponds to p^{n+1} being complete, which has a straightforward phrasing in terms of automorphisms taking $n+1$-tuples to $n+1$-tuples, one might wonder whether his argument generalizes here. Unfortunately it does not, but the suggestion is there. Second and equally unconvincing, the examples simply refuse to present themselves, despite quite a bit of trying.

Third and finally, by appeal to the trichotomy theorem in [17]. If a type is 1-simple and 2-nonsimple, then the theory interprets a group (interval) near that type. If a type is 2-simple and 3-nonsimple, then the theory interprets a field (or rather, a portion of it). Both of these involve parameters, but if we ignore this for a moment, then either of the binary functions of the field seem to make the type 2-nonsimple, a contradiction. This would revolve around a serious examination of their proof to identify sources of parameters, but seems promising. It is well outside the scope of this work, though, so we will not address it further.

Chapter 3: Colored Linear Orders

In 1973, Matatyahu Rubin published his master's thesis on the model theory of complete theories of linear orders, possibly with countably many unary predicates added. Most prominently, he proved in [24] that such a theory has either finitely many or continuum-many countable models, up to isomorphism. This was part of a larger set of results in his master's thesis, wherein he investigated a huge variety of model-theoretic properties of such theories, such as the size of type spaces, finite axiomatizability, and characterizing saturation of models. We continue his investigation here, examining what we will call colored linear orders.

Definition 3.0.13. Say $0 \leq \kappa \leq \aleph_0$, and let L_κ be the language $\{<\} \cup \{P_i : i \leq \kappa\}$ where $<$ is a binary relation and each P_i is a unary relation. A *colored linear order*, or CLO, is a complete L_κ-theory for some κ making $<$ a linear order.

If T is a CLO (theory) and $A \models T$, we will also refer to A as a CLO. This will cause no confusion.

Note that the term "colored linear orders" can be misleading; we allow the possibility of elements having multiple "colors," or to have no "colors" at all. A CLO is merely an expansion of a linear order by up to $\aleph_0$ unary predicates.

We look into the complexity of such theories from two perspectives – the

Borel complexity of isomorphism for countable models of CLOs, and the number of models (of any cardinality) up to back-and-forth equivalence. It turns out that there are essentially five classes of such theories. First is the $\aleph_0$-categorical theories; then those with finitely many countable models; then those whose complexity corresponds exactly to $\cong_1$; then those whose complexity corresponds exactly to $\cong_2$; then those with unbounded complexity. With the exception of "finite," each of these classes contains exactly one element up to reducibility, and the Borel complexity lines up exactly with the corresponding count of back-and-forth inequivalent models. This theorem is finally stated and proved precisely in Theorem 3.3.13. It is worth noting that these five complexity classes are essentially identical to those appearing for o-minimal theories, as shown in Chapter 2, and for essentially the same reasons – a divide on local simplicity or nonsimplicity, then a type-counting argument in the simple case.

The outline of this chapter is as follows. We begin by highlighting what is the core of Rubin's work in [24], since this paper relies heavily on his work there. We then introduce other background the reader will need, such as notions of sum and shuffle and Rosenstein's characterization of $\aleph_0$-categorical linear orders.

In Section 3.2 we re-introduce the notion of self-additive CLOs (approximately those which cannot be definably divided into proper convex pieces) and show they are either minimally or maximally complex. In Section 3.3, we show that CLOs can be definably decomposed into essentially self-additive pieces, and that if any of these are maximally complicated, so is the whole theory. If not, we characterize back-and-forth equivalence for such theories as fairly simple, showing a strong dichotomy. We then fine-tune this analysis to give the exact cases which a CLO can fall into, and prove our characterization.

We end with a special case, showing that none of the "middle cases" can happen if the language is finite. This generalizes a theorem of Schirmann in [26], where a similar result was shown for complete theories of linear orders.

3.1 Background

For this section we cover several classical topics which are essential to the study of linear orders, such as convex sums, shuffles, and Rosenstein's characterization of $\aleph_0$-categorical linear orders. But first and foremost, we want to highlight the following "technical lemma" of Rubin, which appears as Corollary 2.3 in [24]:

Lemma 3.1.1. *Let A be a CLO, and let $B \subset A$ be convex. Let $\phi(\overline{x})$ be a formula, possibly with parameters from $A \setminus B$. Then there is a formula $\phi^{\#}(\overline{x})$ with no parameters where, for all $\overline{b}$ from B, $B \models \phi^{\#}(\overline{b})$ if and only if $A \models \phi(\overline{b})$.*

This is the reason that CLOs are so nice from a logical perspective. Because B is convex, the order type of some $b \in B$ and some $a \in A$ is determined by a and the fact that $b \in B$; that is, for any $b, b' \in B$, $b < a$ if and only if $b < a'$. The rest of the atoms are unary, so hold in B exactly as they would in A. So by an inductive argument, we get the above lemma.

This is used to tremendous effect throughout [24], primarily to prove that given some CLOs $A \subset B$, actually $A \prec B$. We will cite numerous lemmas from [24] which are of this form, and their proofs are all essentially of this form. We do not reproduce these arguments here, though we do need to produce one ourselves for Lemma 3.3.8, so that the reader can get some of the flavor. It is our opinion that all of our results on CLOs hinge on two points: the ease of constructing models through sums, and some form of Lemma 3.1.1.

3.1.1 Sums and Shuffles

We now introduce two classical operations, the sum and the shuffle, which go back at least to Hausdorff. Unlike the situation with o-minimal theories or $\aleph_0$-stable theories, CLOs lack prime models over sets in general. Consequently we will rely on these operations to construct new models of our theories.

We first examine the notion of a *sum*; if $(I, <)$ is a linear order and for each i, A_i is a CLO in the language L, we can define $\sum_i A_i$ in the natural way. It has universe $\{(a, i) : a \in A_i, i \in I\}$. We say $(a, i) < (b, j)$ if $i < j$, or if $i = j$ and $a < b$ in A_i. For any color P in L, we say $P(a, i)$ holds in the sum if $P(a)$ holds in A_i. This is an extremely well-behaved operation, and the following properties can be verified immediately (or see [23]):

Proposition 3.1.2. *Let $(I, <)$ be a linear order and let $(A_i : i \in I)$ be CLOs in the same language L.*

1. *If $A_i \cong B_i$ for all i, then $\sum_i A_i \cong \sum_i B_i$.*
2. *If $A_i \equiv B_i$ for all i, then$\sum_i A_i \equiv \sum_i B_i$.*

We use the familiar notation $A_1 + \cdots + A_n$ for finite sums. If $C \subset A$ is convex, then A decomposes as a sum $B_1 + C + B_2$, where B_1 is the set of elements below every element of C, and likewise with B_2. The following gives one way we will use sums to construct new models:

Proposition 3.1.3. *Suppose $\Phi(x)$ is a partial type with no parameters, A is a CLO, and $C = \Phi(A)$ is convex. Decompose A as $B_1 + C + B_2$. For any CLO D, define A_D as $B_1 + D + B_2$. If $C \equiv D$, then $A_D \equiv A$ and $\Phi(A_D) = D$.*

Proof. First, add a new predicate P to the language, and let $P(a)$ hold for some $a \in A$ if and only if $a \in C$. Expand D to the new language to let P hold everywhere. We show that for all tuples $\bar{b}_1$ and $\bar{b}_2$ from B_1 and B_2 respectively, $(A, \bar{b}_1, \bar{b}_2) \equiv (A_D, \bar{b}_1, \bar{b}_2)$. This is done by an Ehrenfeucht-Fraïsse game argument. So as usual we may assume the language is finite, fix an $n \in \omega$, and describe a strategy for the second player to win the game of length n. Since $C \equiv D$, fix a winning strategy for the second player in the game of length n between C and D. Then for any play, if the first player plays an element of B_1 or B_2 from one model, the second player plays the same element in the other model. If the first player plays within C or D, the second player follows the winning strategy for those two. This is well-defined and clearly preserves colors and $<$ within components. Since the components are convex and we stay within them, this preserves $<$ generally, so proves the result.

That $A \equiv A_D$ follows immediately. To see that $\Phi(A_D) = D$, first note that $A \models \forall x(P(x) \to \phi(x))$ for all $\phi \in \Phi$, so $D \subset \Phi(A_D)$. On other hand, for any $b \in A \setminus C$, there is a $\phi \in \Phi$ where $A \models \neg\phi(b)$. Since $(A, b) \equiv (A_D, b)$, $A_D \models \neg\phi(b)$, so $\Phi(A_D) \subset D$, proving the proposition. □

Next we define the shuffle. To do this, fix a natural number n, and form a countable structure $\mathcal{D}_n$ in the language $L_n = \{<, P_1, \ldots, P_n\}$ satisfying the following axioms:

- $<$ is a linear order which is dense and without endpoints.
- The P_i are disjoint, dense, codense, and exhaustive.

It is easy to see that these axioms are consistent, complete, and $\aleph_0$-categorical, so $\mathcal{D}_n$ is defined up to isomorphism. Now for any language L and any CLOs $A_1, \ldots, A_n$, we form the *shuffle* $\sigma(A_1, \ldots, A_n)$ as follows. For each $i \in \mathcal{D}_n$, define D_i as A_j if and only if $P_j(i)$ holds. Then $\sigma(A_1, \ldots, A_n)$ is the sum $\sum_i D_i$. The following facts are easily verified:

Proposition 3.1.4. *Let $A_1, \ldots, A_n$ be countable CLOs in the same language L. Then all the following hold:*

1. *If τ is a permutation of $\{1, \ldots, n\}$, then $\sigma(A_1, \ldots, A_n) \cong \sigma(A_{\tau(1)}, \ldots, A_{\tau(n)})$.*
2. *If for all i, $A_i \equiv B_i$, then $\sigma(A_1, \ldots, A_n) \equiv \sigma(B_1, \ldots, B_n)$.*
3. *If for all i, $A_i \cong B_i$, then $\sigma(A_1, \ldots, A_n) \cong \sigma(B_1, \ldots, B_n)$.*

While the shuffle may seem somewhat arbitrary, it is important in Rosenstein's characterization of $\aleph_0$-categorical CLOs, and will come up in a natural way in Section 3.2.

3.1.2 $\aleph_0$-categorical Theories

By convention, we will refer to a *structure* (of any size) as $\aleph_0$-categorical if and only if its complete theory has a unique countable model up to isomorphism. In particular, following [22], we will consider *finite* structures (and their complete theories) to be $\aleph_0$-categorical.

In Section 3.3, we will make important use of Rosenstein's characterization of $\aleph_0$-categorical linear orders in [22], which was extended to CLOs by Mwesigye and Truss in [16]. One begins by defining several classes, which we call $\mathcal{M}_n$.

- $\mathcal{M}_0$ is the set of all one-point CLOs; the colors can be arbitrary.
- $\mathcal{M}_{n+1}$ is the smallest class of CLOs such that all the following are satisfied:
  - If $A \in \mathcal{M}_n$, then $A \in \mathcal{M}_{n+1}$.
  - If $A, B \in \mathcal{M}_n$, then $A + B \in \mathcal{M}_{n+1}$.
  - If $A_1, \ldots, A_k \in \mathcal{M}_n$, then $\sigma(A_1, \ldots, A_k) \in \mathcal{M}_{n+1}$.
- $\mathcal{M}$ is the union $\bigcup_n \mathcal{M}_n$.

The characterization is:

Theorem 3.1.5 (Rosenstein; Mwesigye, Truss)**.** *Let T be a CLO. Then T is $\aleph_0$-categorical if and only if $T = Th(A)$ for some $A \in \mathcal{M}$.*

Note that the above makes sense and is true even if the language is infinite, and we will take advantage of that. However, this "generalization" is almost vacuous – if a CLO is $\aleph_0$-categorical, only finitely many of its colors are inequivalent.

We can also define a rank: if A is an $\aleph_0$-categorical CLO, let $r(A)$ be the least n where there is some $B \in \mathcal{M}_n$ such that $A \equiv B$. This turns out to be a useful inductive tool, allowing us to prove all the following facts:

Proposition 3.1.6. *Let A be a CLO in a language L.*

1. *If A is $\aleph_0$-categorical, then every convex subset $B \subset A$ is also $\aleph_0$-categorical. Indeed, $r(B) \leq 2 \cdot r(A) + 1$.*

   If L is finite, then we also get the following:

2. *For any $n \in \omega$, there are only finitely many $\aleph_0$-categorical CLOs in L of rank n.*

3. *For any $\aleph_0$-categorical A, there are only finitely many convex subsets of A, up to back-and-forth equivalence. This bound is uniform in $r(A)$.*

Proof. (1) First, assume A is countable; we will generalize in a moment. We show this by induction on rank. It is trivially true for $r(A) = 0$. So let $r(A) = n+1$. Then either $A = B_1 + B_2$ for some $B_i \in \mathcal{M}_n$, or $A = \sigma(B_1, \ldots, B_k)$ for some $B_i \in \mathcal{M}_n$. In the first (sum) case, if $C \subset B_1 + B_2$ is convex, then $C = (B_1 \cap C) + (B_2 \cap C)$, where each $B_i \cap C$ is a convex subset of the B_i. By induction, $r(B_i \cap C) \leq r(B_i) + 1 \leq n+1$, so C is the sum of two CLOs with rank at most $n+1$, so $r(C) \leq n+2 \leq 2(n+1)+1$, as desired.

In the other (shuffle) case, if $C \subset \sigma(B_1, \ldots, B_k)$ is convex, then C is either $B_i \cap C$ for some i, or $(B_{i_1} \cap C) + \sigma(B_1, \ldots, B_k) + (B_{i_2} \cap C)$, where either of the B_{i_j} could be empty. This is because the left "edge" of C either slips exactly between B_i components or cuts into one (corresponding to B_{i_1} being empty or some B_i, respectively). Similarly with the right "edge." If these cut into the same B_i component, there is no shuffle and C is a convex subset of B_i, so has rank at most

$2 \cdot r(B_i) + 1 \le 2n+1$. If they cut into different components, there is an isomorphic copy of the shuffle between the B_i. The shuffle has rank $n+1$, while each of the sides has at most $2n+1$, so the sum has rank at most $2n+3 = 2(n+1)+1$, as desired.

For the case when A may be uncountable, let $C \subset A$ be convex, and let (A, C) be the structure with an unary predicate for C. Let $(A_0, C_0) \prec (A, C)$ be countable, noting that $C \equiv C_0$, $A \equiv A_0$, and C_0 is a convex subset of A_0. Then the preceding special case applies to (A_0, C_0), and by elementary equivalence, the result for A_0 and C_0 implies it for A and C, as desired.

(2) If there are k distinct unary predicates in L, there are 2^k one-point CLOs, so there are 2^k elements of $\mathcal{M}_0$. If $\mathcal{M}_n$ has m elements, then $\mathcal{M}_{n+1}$ has m elements from $\mathcal{M}_n$, m^2 elements as sums from $\mathcal{M}_n$, and $\sum_{i=1}^{m} \binom{m}{i}$ elements as shuffles from $\mathcal{M}_n$. So $\mathcal{M}_{n+1}$ is finite, as desired.

(3) If B is a convex subset of some A with $r(A) \le n$, then $r(B) \le 2n+1$ by (1). By (2), there is a finite number of $\aleph_0$-categorical CLOs of rank at most $2n+1$, and this depends only on n. □

Finally, we include Corollary 5.11 of [24]:

Theorem 3.1.7 (Rubin). *If T is a CLO in a finite language and $S_1(T)$ is finite, then T is finitely axiomatizable.*

In particular, if T is $\aleph_0$-categorical in a finite language, then T is finitely axiomatizable. This special case can be proven by induction, showing that if $A \in \mathcal{M}_n$, then $\mathrm{Th}(A)$ is finitely axiomatizable. The proof of Rubin's theorem above requires a characterization of those CLOs which have $S_1(T)$ finite, which has a similar induction construction (in addition to the above rules, insist that if $A \in \mathcal{M}_n$, then $Z \times A \in \mathcal{M}_{n+1}$); see Theorem 5.9 of [24] for the details.

3.2 Self-Additive Linear Orders

The crux of the characterization of CLOs is a clever definition due to Rubin – the notion of self-additivity. We summarize their basic properties, following from Theorem 3.2 and Lemma 3.4 in [24]:

Theorem / Definition 3.2.1. Let A be a CLO with more than one point. The following properties are equivalent:

- The only $\emptyset$-definable convex subsets of A are $\emptyset$ and A.
- If $A_j \equiv A$ for all $j \in (J, \leq)$, then each canonical embedding $A_j \to \sum_{j \in J} A_j$ is elementary.

If A satisfies either of these properties, call A *self-additive*.

For example, each of $(\mathbb{Z}, \leq)$, $(\mathbb{Q}, <)$, and $(\mathbb{R}, <, \mathbb{Q})$ are self-additive, but neither $(\mathbb{N}, <)$ nor $(\mathbb{Z} + 1 + \mathbb{Z}, <)$ is. Self-additive structures are extremely useful for us because we can easily construct models using the sum operation – property (2) implies that if T is self-additive, then any sum of models of T is again a model of T. They also have another important property, namely, a nice condensation on the models.

Definition 3.2.2. Let A be a self-additive CLO, and let a and b be from A. Say $a \sim b$ if there is a formula $\phi(x, a)$ where $\phi(A, a)$ is convex, bounded, and contains both a and b.

The fact that $\sim$ is an equivalence relation is not obvious; indeed both symmetry and transitivity require a signficant argument which we do not reproduce here. That $\sim$ is an equivalence relation is a theorem of Rubin in [24], but is spelled out more plainly in Theorem 13.99 of [23].

Note that we consider a set bounded if there are elements strictly above and strictly below the entire set. Since self-additive orders cannot have first or last elements, this is equivalent to any other reasonable definition.

The following is the main way we will show complexity of CLOs:

Lemma 3.2.3. *Suppose A is a self-additive CLO, $T = Th(A)$, and $p \in S_1(T)$ is such that there is exactly one $\sim$-class in A containing a realization of p. Then T is λ-Borel complete for all λ.*

Proof. Let $A_0 \prec A$ be countable and contain a realization of p. If $a, b \in A_0$, then $a \sim b$ in A_0 if and only if $a \sim b$ in A, so A_0 still satisfies the hypotheses of the theorem. This is to say, we may assume A is countable, and in fact that A has universe ω. Fix an infinite cardinal λ and a canonical bijection $\lambda \times \omega \to \lambda$. We aim to show that $(\mathrm{LO}_\lambda, \equiv_{\infty\omega}) \leq_B^\lambda (\mathrm{Mod}_\lambda(T), \equiv_{\infty\omega})$; by Theorem 1.2.20, this shows that T is λ-Borel complete.

For any linear order $(I, <)$ with universe λ, let $A_I = \sum_{i \in I} A$ which has universe $\lambda \times \omega$; under the bijection we may assume A_I has universe λ. Let $(J, <)$ be another linear order with universe λ, and let A_J be formed like A_I. Let $\mathbb{V}[G]$ be a forcing extension in which λ is countable, so that I, J, A_I, and A_J are all countable in $\mathbb{V}[G]$. Then $(I,<) \equiv_{\infty\omega} (J,<)$ if and only if $(I,<) \cong (J,<)$ in $\mathbb{V}[G]$, and likewise with A_I and A_J. This is all to say we may work solely in the countable case, with isomorphism.

Now clearly if $I \cong J$, then $A_I \cong A_J$. On the other hand, consider the set of $\sim$-classes $E_I = \{a/\sim : A_I \models p(a)\}$. These are naturally ordered by $<$, and if $a \sim b$ in A_I, then they come from the same A_i, and are equivalent in A_I if and only if they're equivalent in A_i. Since each A_i contains exactly one $\sim$-class containing a realization of p, E_I has order type $(I,<)$. Clearly if $A_I \cong A_J$, then $(E_I,<) \cong (E_J,<)$, so $I \cong J$, completing the proof. $\square$

For example, this shows that $\mathrm{Th}(\mathbb{Z},<)$ is λ-Borel complete for all λ, since $(\mathbb{Z},<)$ has a unique $\sim$-class. But it can be used much more generally than that. We borrow Lemma 6.1 of [24]:

Lemma 3.2.4 (Rubin)**.** *Let $A \equiv B$ be self-additive, $T = Th(A)$. Let $b \in B$ be arbitrary. Then the canonical embedding from $A + (b/\sim) + A$ to $A + B + A$ is elementary.*

Lemma 3.2.5. *Let T be a theory of a self-additive CLO such that $S_1(T)$ is infinite. Then T is λ-Borel complete for all λ.*

Proof. Let $p \in S_1(T)$ be nonisolated. Let $A, B \models T$ be countable such that A omits p and B realizes p at b. Let $B_0 = b/\sim$ as computed in B, and let $C = A + B_0 + A$. By Lemma 3.2.4, $C \prec A + B + A$ is elementary. Since $A, B \models T$ and T is self-additive,

$A+B+A \models T$, so $C \models T$. Also, both embeddings $A \to A+B+A$ are elementary, so in particular, no element of A is $\sim$-equivalent to any element of B_0. Similarly, since $B \prec A+B+A$ and every element of B_0 is $\sim$-equivalent in B, they are still $\sim$-equivalent in $A+B+A$, and thus in C. Finally, $c \in C$ realizes the same type it does in $A+B+A$, and thus C contains a unique $\sim$-class containing a realization of p. So C and T satisfy the hypotheses of Lemma 3.2.3, so T is λ-Borel complete for all λ. $\square$

Of course we cannot say the same when $S_1(T)$ is finite – $(\mathbb{Q},<)$ is $\aleph_0$-categorical and thus as far from Borel complete as one could be. Our aim is to show that these are the only two cases which can occur, but we need to move slightly beyond the self-additive case to do so. We borrow Lemma 5.4 of [24]:

Lemma 3.2.6 (Rubin)**.** *Let T be self-additive with $S_1(T)$ finite. If $A \models T$ and $a \in A$, let $T_a = Th(a/\sim)$. Then $|S_1(T_a)| \leq |S_1(T)|$. Also, one of the following alternatives holds:*

1. *For every $a \in A$, $(a/\sim) \prec A$.*

2. *For every $a \in A$, the set $(a/\sim)$ is definable over a. There is no first or last element in the quotient order $A/\sim$, and if $a/\sim < b/\sim$ and $p \in S_1(T)$, there is a c realizing p where $a/\sim < c/\sim < b/\sim$.*

With this lemma in hand, we can finish our work with self-additive structures:

Lemma 3.2.7. *Let T be a theory of a self-additive CLO such that $S_1(T)$ is finite. Then either T is $\aleph_0$-categorical or T is λ-Borel complete for all λ.*

Proof. Let $n = |S_1(T)|$; we go by induction on n, beginning with $n = 1$. If $n = 1$ then every element has the same type, and thus the same color, so essentially T is the theory of a linear order. If T says the order has a first element, every element is first, so T is the theory of a singleton, so is $\aleph_0$-categorical (and indeed totally categorical); the same happens with a last element. Assume this does not happen, so there is no first or last element. If some element has a unique successor, they all do, and their successors have predecessors, so everything does. This is known to be an axiomatization of $(\mathbb{Z}, <)$, which (after expanded to an L-structure) has a unique $\sim$-class, so is λ-Borel complete for all λ by Lemma 3.2.3. Now assume these do not happen, so no element has an immediate successor or predecessor and there are no maximal or minimal elements. This is known to be an axiomatization of $(\mathbb{Q}, <)$, which is $\aleph_0$-categorical, completing the proof of the base case.

We move on to the step, where $n \geq 2$. There are several cases.

Case: T is not self-additive.

Let $\phi(x)$ be a formula with parameters such that (according to T), the realizations of ϕ form a nonempty proper initial segment of the model. Let $A \models T$, let $T_1 = \mathrm{Th}(\phi(A))$, and let $T_2 = \mathrm{Th}(\neg\phi(A))$. Note that T_1 and T_2 depend only on T, not on A. If both are $\aleph_0$-categorical, so is T, by Proposition 3.1.5. On the other hand, given any model $A \models T$ and $B \models T_1$, we can construct a structure A_B where we replace $\phi(A)$ with B. By Proposition 3.1.3, $A_B \models T$ and $\phi(A_B) = B$. Evidently this gives a λ-Borel reduction $\mathrm{Mod}(T_1) \leq_B^\lambda \mathrm{Mod}(T)$, so if T_1 is λ-Borel complete, so is T. The same goes for T_2. Since $|S_1(T)| = |S_1(T_1)| + |S_1(T_2)|$, the inductive hypothesis applies to both T_i. Thus, either both T_i are $\aleph_0$-categorical or one of them is λ-Borel complete for all λ. So the lemma holds for T.

Case: T is self-additive and case (1) of Lemma 3.2.6 applies.

Let $A \models T$, let $a \in \mathcal{A}$ be arbitrary, and let $B = a/\sim$. Then $B \prec A$, so $B \models T$, so T has a model with a single $\sim$-class. Then Lemma 3.2.3 applies, so T is λ-Borel complete for all λ.

Case: T is self-additive and case (2) of Lemma 3.2.6 applies.

Let $A \models T$ be arbitrary. Each $\sim$-class is an L-structure on its own, and the theory of $a/\sim$ is determined by $\operatorname{tp}(a)$. So there are $k \leq n$ $\sim$-classes up to elementary equivalence; enumerate their theories as $T_1, \ldots, T_k$. To simplify notation, add k unary predicates $U_1, \ldots, U_k$ to the language, and expand T to the new language by saying $U_i(a)$ holds if and only if $a/\sim \models T_i$. Since this is a definable expansion, this does not change the size of the type space, and T satisfies the lemma if and only if its expansion does.

Then T states precisely that each maximal convex piece of U_i is a model of T_i, and between any two "convex pieces" and for any $i \leq k$, there is a model of T_i as a maximal convex piece of U_i. It states that the U_i are disjoint and exhaustive. Fix particular countable models $A_i \models T_i$, and for any M fitting the preceding description, form the L-structure A by replacing each maximal convex piece of any U_i with the L-structure A_i. This can be done, and by Proposition 3.1.2, $M \equiv A_M$. However, given M and N fitting the description, $A_M \cong A_N$ by Proposition 3.1.4, so the preceding description is a complete theory, so must completely axiomatize T.

Next, see that $\sum_i |S_1(T_i)| \leq |S_1(T)|$. For if a and b come from different U_i, they have different types in T. And since the $\sim$-class of an element is formula-definable with that element, if a and b come from the same U_i but have different types in that structure, they have different types in T. Therefore, if $k \geq 2$, then $|S_1(T_i)| < |S_1(T)|$ for all i, so the inductive hypothesis applies to each of them. If each is $\aleph_0$-categorical, so is T by Proposition 3.1.5 – T is the shuffle of the T_i. If some T_i is λ-Borel complete, then so is T, as follows. Let $M \models T$ be countable and fixed. For any $A \models T_i$ of size λ, let M_A be formed by replacing each convex model of T_i with A. Then $|M_A| = \lambda$, and given a $B \models T_i$ also of size λ, form M_B in the same way. If $M_A \cong M_B$, this isomorphism preserves maximal convex pieces of U_i, so $A \cong B$. With this in mind, let $\mathbb{V}[G]$ collapse λ, so $A \equiv_{\infty\omega} B$ if and only if $A \cong B$ in $\mathbb{V}[G]$, if and only if $M_A \cong M_B$ in $\mathbb{V}[G]$, if and only if $M_A \equiv_{\infty\omega} M_B$. This shows $\mathrm{Mod}_\lambda(T_i) \leq^\lambda_B \mathrm{Mod}_\lambda(T)$, so T is also λ-Borel complete.

The only remaining case is when $k = 1$, so T is a shuffle of T_1. If T_1 is self-additive, then each $\sim$-class of any $A \models T$ is an elementary substructure of A, so $T_1 = T$ and T admits a model with a single $\sim$-class. However, since the $\sim$-class of any element is definable, T would then imply that every model has only one $\sim$-class, contradicting what we already know about T. So T_1 is not self-additive. Then a previous case applies to T_1, so T_1 is either $\aleph_0$-categorical or is λ-Borel complete for all λ. In either case, T follows T_1 by the logic in the previous paragraph, completing the proof. $\square$

While we do not care about orders with finitely many types for themselves, we do recover the following theorem which is crucial to us:

Theorem 3.2.8. *Let T be self-additive. Then either T is $\aleph_0$-categorical or is λ-Borel* *complete for all λ.*

3.3 The General Proof

Finally we consider the general case, where we break up arbitrary CLOs into what are essentially self-additive pieces. The crucial definition is the following:

Definition 3.3.1. Let T be a CLO. A *convex type* $\Phi(x)$, always in one variable, is a maximal consistent collection of convex formulas over $\emptyset$. The space $\mathrm{IT}(T)$ is the set of all convex types.

We may give $\mathrm{IT}(T)$ the usual formula topology, wherein it is compact, Hausdorff, second countable, and totally disconnected as usual. However, convex types are naturally ordered by $<$ as follows: say $\Phi < \Psi$ if there are formulas $\phi \in \Phi$ and $\psi \in \Psi$ where every realization of ϕ is strictly below every realization of Ψ (according to T). It is immediate that if $\Phi \neq \Psi$, then either $\Phi < \Psi$ or $\Psi < \Phi$, and not both. This induces the same topology as before.

For our purposes, say an L-structure A is *sufficiently saturated* if it is $\aleph_0$-saturated, and if $a, b \in A$ realize the same type, there is an automorphism of A taking a to b; we will never need a larger monster model than this. Every complete theory admits such a model, although there will not be a countable such unless the theory is small. Sufficiently saturated CLOs are "locally self-additive:"

Lemma 3.3.2. *Let T be a CLO, and let $\mathcal{S} \models T$ be sufficiently saturated. Then for all $\Phi \in IT(T)$, the set $\Phi(\mathcal{S})$ of realizations of Φ in $\mathcal{S}$ is either a singleton or self-additive as an L-structure.*

Proof. Suppose not. Then there is a formula $\phi(x)$ whose realizations are initial in $\Phi(\mathcal{S})$, and where there is are points $a, b \in \Phi(\mathcal{S})$ where $\Phi(\mathcal{S}) \models \phi(a) \wedge \neg\phi(b)$. Let $p = \mathrm{tp}(b)$ as formed in the whole of $\mathcal{S}$. Then the type $p(x) \cup \{x < a\}$ must be consistent; otherwise there would be some $\psi(x) \in p(x)$ where a lies strictly below the convex definable set $\exists y\, (y \leq x \wedge \psi(y))$ which includes b, and therefore a and b realizes different convex types. By $\aleph_0$-saturation of $\mathcal{S}$, there is $c \in \mathcal{S}$ which realizes p and has $c < a$. Clearly $\Phi(\mathcal{S}) \models \phi(c)$. But then there is an automorphism σ of $\mathcal{S}$ where $\sigma(a) = c$. But σ preserves Φ, so is an automorphism of $\Phi(\mathcal{S})$ which takes b to c, so by elementarily, $\Phi(\mathcal{S}) \models \neg\phi(c)$, a contradiction. $\square$

The *theory* of $\Phi(\mathcal{S})$ turns out to depend only on T, not on choice of sufficiently saturated model:

Lemma 3.3.3. *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be sufficiently saturated models of a CLO T. For any $\Phi \in IT(T)$, the L-structures $\Phi(\mathcal{S}_1)$ and $\Phi(\mathcal{S}_2)$ are back-and-forth equivalent, and thus elementarily equivalent.*

Let T_Φ be the theory of $\Phi(\mathcal{S})$ for any sufficiently saturated $\mathcal{S} \models T$.

Proof. Our claim is that if $\overline{a} \in \Phi(\mathcal{S}_1)^n$ and $\overline{b} \in \Phi(\mathcal{S}_2)^n$ have $(\mathcal{S}_1, \overline{a}) \equiv (\mathcal{S}_2, \overline{b})$, and if $a \in \Phi(\mathcal{S}_1)$ is arbitrary, there is a $b \in \mathcal{P}hi(\mathcal{S}_2)$ where $(\mathcal{S}_1, \overline{a}a) \equiv (\mathcal{S}_2, \overline{b}b)$. This implies that $\overline{a}a$ and $\overline{b}b$ have the same atomic type in the substructures, so together with the opposite (which follows from symmetry) gives the result.

So fix such tuples $\overline{a}$, $\overline{b}$, and a. Then $p(\overline{x}, x) = \mathrm{tp}(\overline{a}, a)$ (evaluated in $\mathcal{S}_1$) is realized and thus consistent with T. Therefore it is realized in $\mathcal{S}_2$ by some pair $\overline{c}, c$. But then $\mathrm{tp}(\overline{c}) = \mathrm{tp}(\overline{a}) = \mathrm{tp}(\overline{b})$, so there is an automorphism σ of $\mathcal{S}_2$ taking $\overline{c}$ to $\overline{b}$;

let $b = \sigma(c)$. Clearly $\mathrm{tp}(\bar{a}a) = \mathrm{tp}(\bar{c}c) = \mathrm{tp}(\bar{b}b)$, and $b \in \Phi(\mathcal{S}_2)$ since c is, so b satisfies the conditions and proves the result. □

Therefore $\Phi(\mathcal{S}_1) \equiv \Phi(\mathcal{S}_2)$, so we may define the notation T_Φ to be the complete L-theory of $\Phi(\mathcal{S})$ for any sufficiently saturated $\mathcal{S} \models T$. We can now declare our fundamental dichotomy:

Definition 3.3.4. Let T be a CLO. Say T is *locally simple* if for all $\Phi \in \mathrm{IT}(T)$, T_Φ is $\aleph_0$-categorical. Otherwise say T is *locally nonsimple*.

Theorem 3.3.5. *If T is a locally nonsimple CLO, T is λ-Borel complete for all λ.*

Proof. Let $\mathcal{S} \models T$ be sufficiently saturated, and let $\Phi \in \mathrm{IT}(T)$ be such that T_Φ is not $\aleph_0$-categorical. Let $A \prec \mathcal{S}$ be countable such that $\Phi(A) \prec \Phi(\mathcal{S})$. Then $\Phi(A) \models T_\Phi$, which is a self-additive CLO which is λ-Borel complete for all λ by Theorem 3.2.8. Furthermore, $T_\Phi \leq^\lambda_B T$ as follows. If $B \in \mathrm{Mod}_\lambda(T_\Phi)$, construct A_B by replacing $\Phi(A)$ by B; then $A \equiv A_B$ and $\Phi(A_B) = B$ by Proposition 3.1.3, so in particular $A_B \in \mathrm{Mod}_\lambda(T)$. Clearly $B \cong B'$ if and only if $A_B \cong A_{B'}$, so by using that fact in some $\mathbb{V}[G]$ which collapses λ, $B \equiv_{\infty\omega} B'$ if and only if $A_B \equiv_{\infty\omega} A_{B'}$, completing the proof. □

Since the global behavior of a CLO is determined essentially by the structure of $\mathrm{IT}(T)$, if there is also local simplicity, there isn't much behavior left. Thus, locally simple CLOs turn out to admit a nice characterization. We borrow Lemma 2.7(1) of [24]: if B is a CLO, A is any subset of B, and C is the convex hull of A in B, then $A \prec C$. The following is the core lemma for understanding this case:

Lemma 3.3.6. *Let T be locally simple and $\Phi \in IT(T)$. Then there is a (minimal) natural number n_Φ and a set $\{T^i_\Phi : 1 \le i \le n_\Phi\}$ of distinct $\aleph_0$-categorical L-theories where for all $A \models T$, there is an i where $\Phi(A) \models T^i_\Phi$.*

Against model-theoretic convention, we include the "theory of the empty set" in the list, where we say $\emptyset \models \forall x(x \neq x)$, in case A omits Φ.

Further, $n_\Phi = 1$ if and only if Φ is isolated in $IT(T)$.

Proof. Let $A \models T$. Then $\Phi(A) \subset \Phi(\mathcal{S})$ for any sufficiently saturated $\mathcal{S} \models T$ where $A \prec \mathcal{S}$. Let C be the convex hull of $\Phi(A)$ in $\Phi(\mathcal{S})$; then $A \prec C$, so they have the same theory. Also, $\Phi(C)$ is a convex subset of an $\aleph_0$-categorical CLO, so is $\aleph_0$-categorical by Proposition 3.1.6. So A is $\aleph_0$-categorical. Also by Proposition 3.1.6, there are only finitely many pairwise inequivalent convex subsets of $\Phi(\mathcal{S})$, and this bound depends only on T_Φ. So the main text of the lemma is proven.

If Φ is nonisolated, there is a model omitting it and another realizing it, so $n_\Phi \ge 2$. If Φ is isolated by some formula ϕ, then for every sentence σ of L, the sentence "σ holds on Φ" is equivalent to "the relativization of σ to ϕ is true," which is a single L-sentence and thus decided by T. So $n_\Phi = 1$. $\square$

This allows us to give a simple characterization of back-and-forth equivalence for locally simple CLOs:

Lemma 3.3.7. *Let T be a locally simple CLO, and $A, B \models T$. The following are equivalent:*

1. $A \equiv_{\infty\omega} B$

2. *For all* $\Phi \in IT(T)$, $\Phi(A) \equiv_{\infty\omega} \Phi(B)$

3. *For all* $\Phi \in IT(T)$, $\Phi(A) \equiv \Phi(B)$.

The equivalence of (1) and (2) does not require local simplicity. If A *and* B *are countable, (1) is equivalent to* $A \cong B$*, and likewise with (2).*

Proof. Assuming (1), we can get (2) by playing the back-and-forth game within any particular Φ; we can reverse this step by patching the various solutions to the Φ together, and making sure we always match choice of interval type. Assuming (2) we get (3) immediately, since elementary equivalence is just $\equiv_{\omega\omega}$. The nontrivial step is to show that (3) implies (2), which follows from Lemma 3.3.6. For if $\Phi(A) \models T^i_\Phi$ for some i, then $\Phi(B) \models T^i_\Phi$, and since T^i_Φ is $\aleph_0$-categorical, all of its models are back-and-forth equivalent. The equivalence of back-and-forth equivalence with isomorphism when the structures are countable is standard and follows from Zorn's lemma. $\square$

It only remains to give an exhaustive list of the behaviors that a CLO can exhibit, both in terms of $I_{\infty\omega}(T)$ and of the Borel complexity of $\cong_T$. We will use the following lemma to construct many models, to the extent that the type space allows it. Approximately, we would like to choose to omit or realize whatever types we like. The problem is that if we omit too many, we don't have enough content left over to have a model. This turns out to be the only obstruction:

Lemma 3.3.8. *Suppose* $A \subset C \subset B$*, that* $A, B \models T$*, and that* $A \prec B$*. Suppose also that there is a collection* $K \subset IT(T)$ *where* $C = B \setminus \left(\bigcup_{\Phi \in K} \Phi(B)\right)$*. Then* $C \prec B$ *as well.*

Proof. Let $\phi(\overline{x}, y)$ be a formula, $\overline{c}$ a tuple from C, and b an element from B where $B \models \phi(\overline{c}, b)$. It's enough to show there is a $c \in C$ where $B \models \phi(\overline{c}, c)$. By the particular construction of C, either $b \in C$ or there is a convex formula $\psi(y)$ where $B \models \psi(b)$, and where no element of $\overline{c}$ realizes ψ. By Lemma 3.1.1, there is a formula $\phi^{\#}(y)$ where for all b' realizing ψ, $\psi(B) \models \phi^{\#}(b')$ if and only if $B \models \phi(\overline{c}, b')$. Since ψ is itself definable, there is a formula $\phi^*(y)$ where $B \models \forall y\, (\phi^*(y) \leftrightarrow \phi(\overline{c}, y))$. Of course $B \models \phi^*(b)$, and since $A \prec B$, there is an $a \in A$ where $B \models \phi^*(a)$. Since $A \subset C$, this a is the element we were looking for, which completes the proof. □

We can now give individual cases:

Proposition 3.3.9. *If T is locally simple and $IT(T)$ is finite, T is $\aleph_0$-categorical and $I_{\infty\omega}(T) = 1$.*

Proof. Since $\mathrm{IT}(T)$ is finite, every $\Phi \in \mathrm{IT}(T)$ is isolated. Thus $n_\Phi = 1$ for all Φ, so every $A, B \models T$ are back-and-forth equivalent by Lemma 3.3.7. If A and B are also countable, they are isomorphic as well. □

Proposition 3.3.10. *If T is locally simple and $IT(T)$ is infinite but with only finitely many nonisolated types, there is a natural number $n \geq 3$ where $\cong_T$ is $(n, =)$ and $I_{\infty\omega}(T) = n$.*

Proof. Let $\Phi_1, \ldots, \Phi_k \in \mathrm{IT}(T)$ be the nonisolated convex types, and let m be the product of the n_{Φ_i}. For any $A \models T$, let t_A be $(\mathrm{Th}(\Phi_i(A)) : i \leq k)$. If $A, B \models T$, then $A \equiv_{\infty\omega} B$ if and only if $t_A = t_B$. Further, there are at most m possible sequences t_A, so T has at most m countable models up to isomorphism; call the exact count

n. That $n \geq 2$ comes the fact that some type is nonisolated; that $n \geq 3$ comes from the fact that T is a complete first-order theory.

Clearly $I_{\infty\omega}(T) \geq n$. That $I_{\infty\omega}(T) \leq n$ comes as follows; if $A \models T$ is arbitrary, let $A_0 \prec A$ be countable and have $\Phi(A_0) \prec \Phi(A)$ for all $\Phi \in \mathrm{IT}(T)$. Then $A \equiv_{\infty\omega} A_0$. And for any two models $A, B \models T$, $A \equiv_{\infty\omega} B$ if and only if $A_0 \equiv_{\infty\omega} B_0$, if and only if $A_0 \cong B_0$. Since there are n isomorphism types of countable models of T, $I_{\infty\omega}(T) \leq n$, completing the proof. $\square$

Proposition 3.3.11. *If T is locally simple and $IT(T)$ is countable but with infinitely many nonisolated types, then $\cong_T$ is $\cong_1$ and $I_{\infty\omega}(T) = \beth_1$.*

Proof. We first show $\cong_T \leq_B \cong_1$ by showing a Borel reduction from $\mathrm{Mod}_\omega(T)$ to $(\omega^\omega, =)$. For each Φ, fix an indexing of $\{T^i_\Phi : 1 \leq i \leq n_\Phi\}$. Also fix an indexing $\{\Phi_n : n \in \omega\}$ of $\mathrm{IT}(T)$. Then for any model $M \models T$, let $s_M \in \omega^\omega$ take $n \in \omega$ to the unique i where $\Phi_n(A) \models T^i_\Phi$. Certainly for any $M, N \models T$, $M \equiv_{\infty\omega} N$ if and only if $s_M = s_N$, so $\cong_T \leq_B \cong_1$. Since this construction makes sense for any models of T, this also shows $I_{\infty\omega}(T) \leq \beth_1$.

For the other direction, we show $\cong_1 \leq_B \cong_T$ by giving a Borel reduction from $(2^\omega, =)$ to $\mathrm{Mod}_\omega(T)$. So let $A \models T$ be some model omitting every nonisolated type in $\mathrm{IT}(T)$, and let $B \succ A$ realize every type in $\mathrm{IT}(T)$. For $\eta \in 2^\omega$, let C_η omit $\Phi_n \in \mathrm{IT}(T)$ if and only if $\eta(n) = 0$. This is done by use of Lemma 3.3.8, so that C_η is just the elements of B which are not in $\bigcup\{\Phi(B) : \eta(n) = 0\}$. Certainly this can be made Borel and $C_\eta \cong C_\nu$ if and only if $\eta = \nu$. So $\cong_1 \leq_B \cong_T$. Since these models is countable and pairwise nonisomorphic, they are also pairwise back-and-forth inequivalent. So $I_{\infty\omega}(T) \geq \beth_1$, completing the proof. $\square$

Proposition 3.3.12. *If T is locally simple and $IT(T)$ is uncountable, then $\cong_T$ is $\cong_2$ and $I_{\infty\omega} = \beth_2$.*

Proof. We first show that $\cong_T \leq_B \cong_2$ by showing a Borel reduction from $\mathrm{Mod}_\omega(T)$ to $((X)^\omega, E)$, where X is the set of all possible L-theories and two functions are equivalent if and only if their images are equal as sets. Since X is a standard Borel space, $(X, E) \sim_B \cong_2$. So let $M \in \mathrm{Mod}_\omega(T)$, and for each $n \in \omega$, let Φ_n^M be the convex type of n in M. Then let T_n^M be the theory of $\Phi_n^M(M)$, and define our function by $M \mapsto (T_n^M : n \in \omega)$. By Lemma 3.3.7, countable models $M, N \models T$ have $M \cong N$ if and only if they realize the same convex types (necessarily a countable set), and for each realized type Φ, $\Phi(M) \equiv \Phi(N)$. This is equivalent to the *sets* $\{T_n^M : n \in \omega\}$ and $\{T_n^N : n \in \omega\}$ being equal.

The back-and-forth version of this argument is less delicate. Two models $M, N \models T$ (of any size) are back-and-forth equivalent if and only if, for all $\Phi \in \mathrm{IT}(T)$, $\Phi(M) \equiv \Phi(N)$. Since $\mathrm{IT}(T)$ is uncountable, $|\mathrm{IT}(T)| = \beth_1$, so $I_{\infty\omega}(T) \leq \omega^{\beth_1} = \beth_2$.

For the reverse, we again use Lemma 3.3.8. Fix a countable model $M \models T$ and some model $\mathcal{S} \models T$ realizing every convex type. Let X be the set of convex types *omitted* by M; since $\mathrm{IT}(T)$ is uncountable and M is countable, X is an uncountable standard Borel space using the usual topology. For any set $K \subset \mathrm{IT}(T)$, let M_K be $\mathcal{S} \setminus \bigcup_{\Phi \in X \setminus K} \Phi(\mathcal{S})$. If $K_1 \neq K_2$, M_{K_1} and M_{K_2} realize different types, so are pairwise inequivalent. Thus $I_{\infty\omega}(T) \geq \beth_2$.

For the countable version of this argument, we need to be slightly more careful. We restrict ourselves to countable K, so that M_K realizes only countably many types. We also need $\Phi(\mathcal{S})$ to be countable for each $\mathcal{S}$, which can be guaranteed by simply replacing each $\Phi(\mathcal{S})$ with a countable elementary substructure. But then we have a Borel function from (X^ω, E) to $(\mathrm{Mod}_\omega(T), \cong)$, where we take $f : \omega \to X$ to $M_{\mathrm{im}(f)}$. Certainly $M_{\mathrm{im}(f)} \cong M_{\mathrm{im}(g)}$ if and only if $\mathrm{im}(f) = \mathrm{im}(g)$, if and only if fEg. So $\cong_2 \leq_B \cong_T$, as desired. $\square$

We summarize our findings in the following compilation theorem:

Theorem 3.3.13. *Let T be a CLO. If T is locally nonsimple, T is λ-Borel complete for all λ. Otherwise T is locally simple and exactly one of the following happens:*

1. *T is $\aleph_0$-categorical.*

2. *There is some n with $3 \leq n < \omega$ where $\cong_T$ is $(n, =)$ and $I_{\infty\omega}(T) = n$.*

3. *$\cong_T \sim_B \cong_1$ and $I_{\infty\omega}(T) = \beth_1$.*

4. *$\cong_T \sim_B \cong_2$ and $I_{\infty\omega}(T) = \beth_2$.*

All five cases are possible, including every value of n with $3 \leq n < \omega$.

We end this section with a nice corollary of our findings, a special case when the language is finite and the dichotomy is very sharp. This result generalizes a result of Schirmann in [26], where a countable version of the same theorem was proven for linear orders without any colors.

Corollary 3.3.14. *If T is a CLO in a finite language L, then either T is $\aleph_0$-categorical or is λ-Borel complete for all λ.*

Proof. If T is locally nonsimple, or if $\mathrm{IT}(T)$ is finite, the corollary follows from Theorem 3.3.13. So suppose, by way of contradiction, that there is a nonisolated $\Phi \in \mathrm{IT}(T)$. Let $(\Phi_n : n \in \omega)$ be a sequence from $\mathrm{IT}(T)$ limiting to Φ. Without loss of generality, we assume $\Phi_n < \Phi_{n+1}$ for all n. Since L is finite, every $\aleph_0$-categorical CLO in L is finitely axiomatizable by Theorem 3.1.7, and there are only finitely many such theories of any particular rank by Theorem 3.1.5. Thus, for every n, there is an L-formula $\sigma(x, y)$ stating "$x < y$ and $[x, y]$ is not an $\aleph_0$-categorical CLO of rank at most n." For a moment, suppose the partial type $\Gamma(x, y) = \sigma(x, y) \cup \Phi(x) \cup \Phi(y)$ is consistent. Then any sufficiently saturated $\mathcal{S} \models T$ realizes it at some pair $[a, b]$. But then $[a, b]$ is not $\aleph_0$-categorical, despite being a dense subset of the $\aleph_0$-categorical structure $\Phi(\mathcal{S})$. This will give us our contradiction, assuming we can show Γ is consistent.

We show this by compactness. So let $\Gamma_0 \subset \Gamma$ be finite. Then $\Gamma_0(a, b)$ says at most that $a < b$, that $[a, b]$ is not $\aleph_0$-categorical, and that there is a formula $\phi(x)$, contained in cofinitely many of the Φ_n, such that both a and b satisfy ϕ. So pass to some sufficiently saturated $\mathcal{S} \models T$, and let $b \in \mathcal{S}$ realize Φ. Let m be large enough that realizing Φ_m guarantees realizing ϕ, and let $a \in \mathcal{S}$ realize a. For every $n < \omega$, there is a convex formula ϕ_n where Φ_i implies ϕ_n if and only if $i = n$. By Lemma 3.1.1, there is a formula $\phi_n^{\#}(x)$ where for all $c \in [a, b]$, $[a, b] \models \phi_n^{\#}(c)$ if and only if $\mathcal{S} \models \phi_n(c)$. But if $m < n < n' < \omega$, then $\phi_n^{\#}$ and $\phi_{n'}^{\#}$ are disjoint definable subsets of $[a, b]$, meaning $[a, b]$ admits infinitely many inequivalent formulas, so is not $\aleph_0$-categorical. Thus (a, b) realize Γ_0, completing the proof. $\square$

3.4 Examples

Here we give examples showing that all of our cases are possible. We include a basic schema of examples for each case and leave generalizations to the reader.

Example 3.4.1. Let $L = \{<\}$, and let $A = (\mathbb{Q}, <)$. Then $\mathrm{Th}(A)$ is $\aleph_0$-categorical and $I_{\infty\omega}(\mathrm{Th}(A)) = 1$.

Proof. This is classical, but we summarize it. Suppose $B \equiv A$, so in particular, B is dense without endpoints. It follows from these axioms (which turn out to be complete) that if $x_1 < \cdots < x_n$ is an ascending sequence from either model, then there is a tuple $\overline{y}$ from that mode satisfying $y_0 < x_1 < y_1 < x_2 < \cdots < x_n < y_n$. From here, showing $B \equiv_{\infty\omega} A$ is immediate. $\square$

The following examples are due to Ehrenfeucht, and give examples of every finite (non-one) value that $I_{\infty\omega}(T)$ can take. By Corollary 3.3.14, the use of an infinite language is necessary – no CLO in a finite language can fall into this class.

Example 3.4.2. Let $n \geq 1$ be a natural number. Let L be the language $\{<\} \cup \{P_k : k \in \omega\} \cup \{U_i : 1 \leq i \leq n\}$. Let A have underlying order $(\mathbb{Q}, <)$, where $P_k(q)$ holds if and only if $k = q$, and where the U_i partition the universe into dense and codense pieces; for concreteness suppose that if $P_k(q)$ holds, then $U_1(q)$ also holds.

Then $T = \mathrm{Th}(A)$ has exactly $n+2$ countable models, and $I_{\infty\omega}(\mathrm{Th}(A)) = n+2$.

Proof. As this is also classical, we only summarize. For ease of notation, let c_k refer to the unique realization of P_k in whatever model we're working with. The possible options for a countable model, up to isomorphism, are as follows:

1. The sequence (c_k) is unbounded above.

2. The sequence (c_k) is bounded above, but has no least upper bound.

3. The sequence (c_k) has a least upper bound, and this element satisfies U_i.

Since there are precisely n possible values of i in the last case, this is $n+2$ distinct cases. Clearly being in a case is preserved under isomorphism, or even back-and-forth equivalence, and all the cases are possible of models of T. So $I_{\infty\omega}(T) = n+2$ and $\cong_T$ is $(n+2, =)$. $\square$

To move up to $\cong_1$, we can simply include infinitely many copies of this idea:

Example 3.4.3. Let $S \subset \mathbb{Q}$ be the set of all rationals of the form $k + \frac{1}{n+1}$ where $k \in \mathbb{Z}$ and $n \in \omega$. Let $L = \{<\} \cup \{P_s : s \in S\}$.

Let A have underlying order type $(\mathbb{Q}, <)$, and say $P_s(q)$ holds if and only if $s = q$. If $T = \mathrm{Th}(A)$, then $\cong_T$ is $\cong_1$, and $I_{\infty\omega}(T) = \beth_1$.

Proof. This is essentially infinitely many problems of the above form. In particular, if M and N are models of T, then $M \equiv N$ if and only if, for all $k \in \mathbb{Z}$, the sequence $\{k + \frac{1}{n+1} : n \in \omega\}$ has a greatest lower bound, where we interpret these rationals as the unique realizations of the associated predicates in each model.

This shows that $\cong_T \leq_B \cong_1$ and $I_{\infty\omega}(T) \leq \beth_1$. The presence of a greatest lower bound corresponds exactly to omitting a certain interval type (the one between

k and the sequence $(k + \frac{1}{n+1})_n)$, and the model A omits all of them. Thus, by Lemma 3.3.8, one can omit or realize any set of these types, showing $\cong_1 \leq_B \cong_T$ and $I_{\infty\omega}(T) \geq \beth_1$. $\square$

The example of an $\cong_2$ example is probably the first thing you'd think of:

Example 3.4.4. Let $L = \{<\} \cup \{P_q : q \in \mathbb{Q}\}$, and let A have underlying order type $(\mathbb{Q}, <)$. Say $P_q(r)$ holds if and only if $q = r$. If $T = \mathrm{Th}(A)$, then $\cong_T = \cong_2$, and $I_{\infty\omega}(T) = \beth_2$.

Proof. Clearly $S_1(T)$ is uncountable, so $\cong_2 \leq_B T$ and $I_{\infty\omega}(T) \geq \beth_2$. It is enough to show that T is locally simple. But observe that every $\Phi \in \mathrm{IT}(T)$ is defined exactly by which rational numbers lie above or below (or in) Φ. Thus, if $\mathcal{S}$ is any sufficiently saturated model and $\Phi \in \mathrm{IT}(T)$ is nonisolated, then the set $\Phi(\mathcal{S})$ has order type which is dense and without endpoints, and all the predicates P_s will be false. So this structure is $\aleph_0$-categorical, so T is locally simple, so $\cong_T \leq_B \cong_2$ and $I_{\infty\omega}(T) \leq \beth_2$, as desired. $\square$

Local nonsimplicity can come about in a few ways. The first is simply to have non-dense order type:

Example 3.4.5. Let $L = \{<\}$ and let $A = (\mathbb{Z}, <)$. Then $T = \mathrm{Th}(A)$ is λ-Borel complete for all λ, and $I_{\infty\omega}(T) = \infty$.

Proof. This structure is self-additive by a classical argument, but $A \not\cong A + A$, so is not $\aleph_0$-categorical. The result follows. $\square$

One can also hide a non-dense order type in a predicate:

Example 3.4.6. Let $L = \{<\} \cup \{P\}$, and let $A = (\mathbb{Q}, <, \mathbb{Z})$; that is, the predicate P is said to hold only on the integers. Then $T = \mathrm{Th}(A)$ is λ-Borel complete for all λ, and $I_{\infty\omega}(T) = \infty$.

Proof. This structure is self-additive by appeal to the argument for $(\mathbb{Z}, <)$, but again, $A \ncong A + A$. $\square$

Maximal complexity can also come about without any hidden non-density, but just from too much behavior going on. The following example also gives an example of a CLO which is maximally complex, but where every reduct to a finite language is $\aleph_0$-categorical; this behavior is impossible for o-minimal theories.

Example 3.4.7. Let $L = \{<\} \cup \{P_n : n \in \omega\}$. For each n, let $L_n = \{<\} \cup \{P_k : k < n\}$, so that $L_n \subset L$. Let T_n state that the order type is dense without endpoints, and that the boolean combinations of $\{P_k : k < n\}$ are all consistent and together partition the space into dense, codense pieces.

Let $T = \bigcup_n T_n$. Then each theory T_n is $\aleph_0$-categorical, but T is locally nonsimple (hence λ-Borel complete for all λ). All the theories T_n and T are self-additive.

Proof. $\aleph_0$-categoricity and self-additivity of each T_n is immediate by a back-and-forth argument. Clearly T_{n+1} implies T_n, so $T_{n+1} \cup T_n$ is consistent, and thus T is consistent, although it is harder to construct a canonical model (indeed T has no isolated types, and hence no prime model).

To see that T is self-additive, let A and B model T, and consider the embedding $A \to A+B$ (the other case is identical). If this embedding were not elementary, there would be a formula witnessing it, and formulas use only finitely many elements of the language. Thus there is an n where the reducts A_n and B_n of A and B (respectively) to L_n make $A_n \to A_n + B_n$ not elementary. But since T implies T_n, A_n and B_n are models of T_n, contradicting self-additivity of T_n. So the original embedding must have been elementary, establishing self-additivity.

To see that T is locally nonsimple, it is enough to observe that it is self-additive and not $\aleph_0$-categorical. We have demonstrated the former; for the latter, simply observe that $S_1(T)$ is infinite (indeed, uncountable), as $\{P_n(x) : n \in X\} \cup \{\neg P_n(x) : n \notin X\}$ is consistent for all $X \subset \omega$. $\square$

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