L. Landweber, Doctor of Philosophy, 1951

An Iteration Formula for Fredholm Integral Equations of the First Kind with Application to the Axially Symmetric Potential Flow About Elongated Bodies of Revolution

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May 1951

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AN ITERATION FORMULA FOR FREDHOLM INTEGRAL
EQUATIONS OF THE FIRST KIND WITH APPLICATION
TO THE AXIALLY SYMMETRIC POTENTIAL FLOW
ABOUT ELONGATED BODIES OF REVOLUTION

by
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Thesis submitted to the Faculty of the Graduate
School of the University of Maryland in partial
fulfillment of the requirements for the
degree of Doctor of Philosophy

1951

ACKNOWLEDGEMENTS

The author is indebted to Professor D. C. Lewis for his valuable suggestions and continuing encouragement in the development of the integral equation theory of Part I, and to Dr. E. H. Kennard for his stimulating criticism of the entire work. The author gratefully acknowledges the capable assistance of Mr. J. L. Powers and Mr. J. S. Maybee in the performance of the calculations and of Mrs. J. D. Truex in the preparation of the manuscript.

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PART I

AN ITERATION FORMULA FOR FREDHOLM INTEGRAL EQUATIONS OF THE FIRST KIND

INTRODUCTION

Neumann's method of solving Fredholm integral equations of the second kind by iteration is of great practical and theoretical value. For Fredholm integral equations of the first kind, on the other hand, Hellinger and Toeplitz[3] remark that a method of solution by iteration is not available.

Physical problems often lead to an integral equation of the first kind to which a good first approximation may be derived by physical reasoning. An example of this is the problem of determining an axial source-sink or doublet distribution which would yield the axially-symmetric potential flow about a body of revolution in a uniform stream. This problem leads to an integral equation of the first kind

$$\int_0^1 m(t)\left[(x-t)^2+y(x)^2\right]^{-3/2}dt = \frac{1}{2}$$

where the axis of the body coincides with the x-axis from $x = 0$ to $x = 1$, $y(x)$ is a known function, representing the ordinates of the intersection of the given surface with a meridian plane and $m(x)$ is an unknown function, representing the distribution of the doublet strength per unit length along the axis. A well-known, excellent, first approximation to the source distribution for elongated bodies of revolution is[4]

$$m_1(x) = \frac{1}{4}\left[y(x)\right]^2$$

In cases such as this it would be highly desirable to have a method of successive approximations for improving upon this approximation.

The theories of Schmidt and Picard furnish expressions for solutions to integral equations of the first kind. However, these expressions are of little practical value since they involve the characteristic numbers and functions of an arbitrary kernel, and the methods for obtaining these are both tedious and approximate.

It is proposed to present an iteration formula for obtaining successive approximations to the solution of Fredholm integral equations of the first kind, and to prove the convergence of the successive approximations under various conditions.

REVIEW OF THEORY

We are concerned with solutions and approximations to solutions of the integral equation of the first kind

$$f(x) = \int_a^b k(x,y)g(y)dy \qquad (1)$$

where f(x) and $f(x) = \int_a^\infty k(x,y)g(y)dy$ nuous real functions in $a \leq x,y \leq b$, and g(y) is an unknown function. As is well known, (1) may be transformed into the integral equation with a symmetric kernel,

$$F(x) = \int_a^b K(x,y)g(y)dy, \qquad (2)$$

where $$K(x,y) = \int_a^b k(t,x)k(t,y)dt, \qquad (3)$$

and hence $$F(x) = \int_a^b k(y,x)f(y)dy, \qquad (4)$$

Schmidt Theory. A theory due to E. Schmidt[6] shows that there exists a set $\{\lambda_i\}$ of positive characteristic numbers, which may be supposed arranged in increasing order of magnitude, and corresponding adjoint sets $\varphi_i(x)$ and $\psi_i(x)$ of real, continuous, orthonormalized characteristic functions, $(i = 1,2\ldots.)$, such that

$$\varphi_i(x) = \lambda_i \int_a^b k(x,y)\psi_i(y)dy, \tag{5}$$

$$\psi_i(x) = \lambda_i \int_a^b k(y,x)\varphi_i(y)dy. \tag{6}$$

It will be convenient, hereafter, to employ the customary operator notation for integral transforms, viz

$$kg \equiv \int_a^b k(x,y)g(y)dy, \quad K_g \equiv \int_a^b K(x,y)g(y)dy;$$

furthermore, since the range of variation and the integration limits will always be from a to b, specific reference to these limits will be omitted and we will frequently write integrals in an abbreviated form, viz

$$\int_a^b f(x)\varphi_i(x)dx \equiv \int f\varphi_i$$

If the kernel k(x,y) is degenerate, the number of characteristic functions is finite and they can be found by a well known procedure[1]. If f(x) is expressible in the form

$$f(x) = \sum_{i=1}^{n} a_i\varphi_i(x)$$

the solution of (1) is

$$g(x) = \sum_{i=1}^{n} \lambda_i a_i \psi_i(x), \quad a_i = \int f\varphi_i \tag{7}$$

If f(x) is not of the above form, then (7) gives the best approximate solution of (1) in the least square sense, as can easily be shown. If the kernel $k(x,y)$ is non-degenerate, the sets $\{\lambda_i\}$, $\{\varphi_i(x)\}$ and $\{\psi_i(x)\}$ are infinite. Since the degenerate case is readily disposed of, only the non-degenerate case will be considered hereafter.

These characteristic numbers and adjoint functions have several properties which will be required in the following:

a) λ_i^2 and $\psi_i(x)$ are characteristic numbers and functions of $K(x,y)^2$, i.e.

$$\psi_i = \lambda_i^2 K \psi_i \tag{8}$$

b) A positive lower bound for the set $\{\lambda_i\}$ is given by the inequality[6]

$$\frac{1}{\lambda_1^2} < \iint k^2(x,y)\,dx\,dy \tag{9}$$

c) Expansion theorems: Every function $f(x)$ of the form (1), where $g(y)$ is any piecewise-continuous function, can be expanded in the absolutely and uniformly convergent series[2]

$$f(x) = \sum_{i=1}^{\infty} a_i \varphi_i(x); \quad a_i = \int f\varphi_i = \frac{1}{\lambda_i}\int g\psi_i \tag{10}$$

Every function $F(x)$ of the form (4), where $f(x)$ is any piecewise-continuous function, can be expanded in the absolutely and uniformly convergent series

$$F(x) = \sum_{i=1}^{\infty} c_i \psi_i(x); \quad c_i = F\psi_i = \frac{1}{\lambda_i}\int f\varphi_i \tag{11}$$

If f is the same function in (10) and (11), the relations between the "Fourier" coefficients may be written

$$c_i = \int F\psi_i = \frac{1}{\lambda_i}\int f\varphi_i = \frac{1}{\lambda_i^2}\int g\psi_i \tag{12}$$

Picard Theory. In general a solution of (1) does not exist. A theorem due to E. Picard[5] states that, if the orthogonal set φ_i is complete, a solution of the integral equation (1) exists if and only if the series

$$\sum_{i=1}^{\infty} \lambda_i^2 a_i^2, \quad a_i = \int f\varphi_i \tag{13}$$

is convergent.

In the Schmidt-Picard theory, the solution of (1) is intimately related to the sequence

$$\bar{g}_n \equiv \sum_{i=1}^{n} \lambda_i a_i \psi_i(x), \; n = 1,2,\ldots \tag{14}$$

as is expressed in the following theorems:

THEOREM 1: The sequence $\{k\bar{g}_n\}$ converges in the mean to f(x) if and only if the set $\{\varphi_i\}$ is complete relative to f(x). The sequence converges uniformly to f(x), if a piecewise-continuous solution of the integral equation (1) exists.

THEOREM 2: If a piecewise-continuous solution g(x) of (1) exists, the sequence $\{\bar{g}_n\}$ converges in the mean to g(x) if and only if the set $\{\psi_i\}$ is complete relative to g(x). If g(x) is of the form k(y,x)h(y)dy, where h(y) is any piecewise-continuous function, then the sequence $\bar{g}_n$ converges uniformly to g(x).

The completeness conditions on the sequences $\{\varphi_i\}$ and $\{\psi_i\}$ in Theorems 1 and 2 refer to the so-called completeness relations

$$\int f^2 = \sum_{i=1}^{\infty} a_i^2, \quad a_i = \int f\varphi_i \tag{15}$$

and $$\int g^2 = \sum_{i=1}^{\infty} b_i^2, \quad b_i = \int g\psi_i \tag{16}$$

The phrase "complete relative to f(x)" in Theorem 1 signifies that (15) need be satisfied only by the particular function f(x), a condition which is considerably weaker than the assumption that the set $\{\varphi_i\}$ is complete relative to a class of functions. Similarly (16) is assumed to apply only to the particular function g(x) in Theorem 2.

The first part of Theorem 1 is of especial interest since it indicates that with increasing n, the error due to the assumption of $\overline{g}_n(x)$ as an approximate solution of (1) diminishes in a least square sense, even if a solution of (1) does not exist. However the disagreeable possibility exists that, beyond some value of n, the error may accumulate and increase at some values of x. Nevertheless, even in this case, such a sequence may give useful successive approximations in a particular problem, if the errors are observed at each step, and the approximations stopped when the error exceeds an acceptable value at any point.

The second part of Theorem 1 asserts that, for sufficiently large n, $\overline{g}_n$ satisfies the integral equation (1) as closely as desired. It is noteworthy that no assumptions are made with regard to the convergence of the sequence $\{\overline{g}_n\}$. Indeed, Theorem 2 shows that an additional condition is necessary to assure even convergence in the mean.

The expression (14) for $\overline{g}_n$, however, is of little practical value since it is expressed in terms of the characteristic numbers and functions of the kernel k(x,y). Principally for these reasons the Fredholm integral equation of the first kind has been considered to be of little value[7]. On the other hand another readily calculable sequence of functions $\{g_n(x)\}$ will be defined, which, it will be shown, has properties relative to a solution of the integral equation (1) identical to those of $\bar{g}_n(x)$.

THE ITERATION FORMULA

Let us now extend the operator notation, denoting
$K^r g \equiv \int \ldots \int K(x,y_r)K(y_r,y_{r-1})\ldots K(y_2,y_1)g(y_1)dy_r dy_{r-1}\ldots dy_1.$
This notation is appropriate since the relation $K^r(K^s g) \equiv K^{r+s}g$ is satisfied, as is easily verified.

Let $g_o(x)$ be an assumed, approximate, piecewise-continuous solution of the integral equation (1). Then a set of continuous functions $g_1(x)$, $g_2(x)$,... is defined by the iteration formula

$$g_n = g_{n-1} + F - Kg_{n-1} \tag{17}$$

where K and F are the functions defined in equations (3) and (4). The convergence of this sequence of functions and the applicability of its members as successive approximations to a solution of the integral equation (1) is the subject of the subsequent discussion.

The recurrence formula (17) can be readily solved for g_n in terms of g_o. First put

$$\gamma_n = g_n - g_{n-1} \tag{18}$$

Then

$$g_n = g_o + \sum_{i=1}^{n} \gamma_i \tag{19}$$

and also (17) may be written as

$$\gamma_n = F - Kg_{n-1} \qquad (20)$$

Thus the γ_n's are not only the differences between successive g_n's but also serve as measures of the errors corresponding to the g_n's as approximate solutions of the iterated integral equation (2). Now from (20), we have

$$\gamma_n - \gamma_{n-1} = -K\gamma_{n-1}$$

or, in operation notation,

$$\gamma_n = (1-K)\gamma_{n-1}$$

Hence, since the operator K satisfies the associative laws of multiplication, we obtain

$$\gamma_n = (1-K)^{n-1}\gamma_1 \qquad (21)$$

where $(1-K)^{n-1}$ is to be formally expanded by the binomial theorem before operating on γ_1. Substituting for the γ_i in equation (19) from equation (21), and performing the indicated summation, we obtain

$$g_n = g_o + \frac{1-(1-K)^n}{K}(F-Kg_o) \qquad (22)$$

where, in the fractional operator, $(1-K)^n$ is to be expanded by the binomial theorem and a factor K in the numerator cancelled with the denominator before operating on $(F-Kg_o)$.

If the sequence $\{g_n(x)\}$ converges uniformly, it is clear from (17), that $\lim_{n\to\infty} g_n$ is a solution of the iterated integral equation (2). However, since an integral equation of the first kind has a solution only under special circumstances,

$\{g_n(x)\}$ may not converge uniformly, and indeed may not converge at all. Nevertheless the g_n's may serve as useful approximations to a solution of (1) and (2) as will be evident on the basis of the convergence theorems in the next section.

CONVERGENCE THEOREMS

It will be assumed hereafter that

$$\int_a^b \int_a^b k^2(x,y)dxdy \leq 2 \qquad (23)$$

This is no restriction since the kernel k(x,y) can always be modified, so as to satisfy the condition (23), by multiplying the integral equation (1) by a suitable factor and, in the right member of the equation, incorporating the factor into the kernel.

Statement of Convergence Theorems. The convergence theorems will be stated and discussed before their proofs are presented.

THEOREM 3: The sequence $\{Kg_n\}$ converges uniformly to F(x).

Theorem 3 is very strong. Without any restrictive assumptions about completeness, the existence of a solution, or the convergence of the sequence $\{g_n\}$, it asserts that, for sufficiently large n, g_n satisfies the iterated integral equation (2) as closely as desired. Basically, however, our interest is in the integral equation (1), rather than with (2). Concerning the suitability of the g_n's as approximate solutions of (1) we have the weaker theorems:

THEOREM 4: The sequence $\{kg_n\}$ converges in the mean to f(x) if and only if the set $\{\varphi_i\}$ is complete relative to f(x). The

<u>sequence converges uniformly to $f(x)$ if a piecewise-continuous solution of the integral equation (1) exists</u>.

It will now be supposed that the zero-th approximation $g_o(x)$ is chosen of the form

$$g_o(x) = \int k(y,x)h(y)dy \qquad (24)$$

where $h(y)$ is any piecewise-continuous function. The special case $h(y) \equiv 0$ is also allowed. Concerning the convergence of the sequence $\{g_n\}$ we then have

THEOREM 5: <u>If a piecewise-continuous solution $g(x)$ of (1) exists, the sequence $\{g_n\}$ converges in the mean to $g(x)$ if and only if the set $\{\psi_i\}$ is complete relative to $g(x)$. If $g(x)$ is of the form $\int k(y,x)h(y)dy$, where $h(y)$ is any piecewise continuous function, then the sequence $\{g_n\}$ converges uniformly to $g(x)$.</u>

It should be noted that Theorems 4 and 5 are identical, word for word, with Theorems 1 and 2 except for the substitution of g_n for $\bar{g}_n$. Hence the remarks concerning the suitability of the $\bar{g}_n$'s as approximations to a solution of the integral equation (1) are applicable to the g_n's as well.

<u>Proof of Lemmas</u>. In order to prove the foregoing theorems it is first convenient to establish several lemmas. Put

$$F_n(x) \equiv Kg_n \qquad (25)$$

$$f_n(x) \equiv kg_n \qquad (26)$$

The "Fourier" coefficients of F_n, f_n and g_n then satisfy the relations

$$c_{in} = \int F_n\psi_i = \frac{1}{\lambda_i}\int f_n\varphi_i = \frac{1}{\lambda_i^2}\int g_n\psi_i \qquad (27)$$

We then have

LEMMA 1: <u>$F_n(x)$ and $f_n(x)$ can be expanded in the absolutely and uniformly convergent series</u>

$$F_n(x) = \sum_{i=1}^{\infty} c_{in}\psi_i(x), \quad n = 0,1,2\ldots \tag{28}$$

$$f_n(x) = \sum_{i=1}^{\infty} \lambda_i c_{in}\varphi_i(x), \quad n = 0,1,2\ldots \tag{29}$$

<u>If $g_o(x)$ is chosen of the form (24), then also $g_n(x)$ may be expanded in the absolutely and uniformly convergent series</u>

$$g_n(x) = \sum_{i=1}^{\infty} \lambda_i^2 c_{in}\psi_i(x), \quad n = 0,1,2\ldots \tag{30}$$

Proof: It is clear, from their definitions in (25) and (26), that the expansion theorems apply to $F_n(x)$ and $f_n(x)$ and consequently the series (28) and (29) converge as stated in the lemma. In the case of the g_n's, it can readily be shown, successively, from the iteration formula (17), that $g_1(x)$ $g_2(x), \ldots$ are of the same form as $g_o(x)$. Thus, we have

$$g_1 = g_o + F - Kg_o \tag{31}$$

But $g_o = \int k(y,x)h(y)dy$; from (4), $F = \int k(y,x)f(y)dy$; and from (3)(26), $Kg_o = \int k(y,x)f_o(y)dy$. Hence (31) becomes

$$g_1 = \int k(y,x)[h(y) + f(y) - f_o(y)]\,dy.$$

Hence the expansion theorem is applicable to $g_n(x)$ and the series (30) also converge, as stated.

LEMMA 2:

$$c_{in} - c_i = \mu_i^n(c_{io} - c_i) \tag{32}$$

<u>where $c_i = \int F\psi_i$, and the sequence μ_i is such that</u>

$$|\mu_i| < 1, \ \mu_{i+1} \geq \mu_i \text{ and } \lim_{i\to\infty}\mu_i = 1, \quad i = 1,2,\ldots \tag{33}$$

<u>Proof</u>: We obtain, from (17) and (8),

$$\int g_n\psi_i = (1-\frac{1}{\lambda_i^2})\int g_{n-1}\psi_i + \int F\psi_i$$

Put $\mu_i = 1-1/\lambda_i^2$. Then, by successive reduction, we obtain

$$\int g_n\psi_i = \mu_i^n \int g_o\psi_i + \lambda_i^2(1-\mu_i^n)\int F\psi_i$$

which, by (12) and (27), is seen to be equivalent to (32). Furthermore, from (9) and (23), we obtain

$$0<\frac{1}{\lambda_i^2}<\iint k^2(x,y)\,dx\,dy \leqq 2$$

or $-1<\mu_i<1$. Thus, since the sequence $\{\lambda_i\}$ increases monotonically to infinity, it is seen that (33) is also satisfied. This completes the proof of Lemma 2.

LEMMA 3:

$$\lim_{n\to\infty}\sum_{i=1}^{\infty}(c_{in}-c_i)^2 = \lim_{n\to\infty}\sum_{i=1}^{\infty}\lambda_i^2(c_{in}-c_i)^2 = 0 \qquad (34)$$

<u>If a solution g(x) of (1) or (2) exists, then also</u>

$$\lim_{n\to\infty}\sum_{i=1}^{\infty}\lambda_i^4(c_{in}-c_i)^2 = 0 \qquad (35)$$

<u>Proof</u>: We first note that the series $\sum_{i=1}^{\infty}(c_{io}-c_i)^2$ converges since we have, from Bessel's inequality,

$$\sum_{i=1}^{\infty}(c_{io}-c_i)^2 \leqq \int(F_o-F)^2$$

Hence, by (32) and the comparison test,

$$\sum_{i=1}^{\infty}(c_{in}-c_i)^2 = \sum_{i=1}^{\infty}\mu_i^{2n}(c_{io}-c_i)^2$$

is uniformly convergent in n, and consequently

$$\lim_{n\to\infty}\sum_{i=1}^{\infty}(c_{in}-c_i)^2 = \sum_{i=1}^{\infty}\lim_{n\to\infty}\mu_i^{2n}(c_{io}-c_i)^2 = 0$$

Similarly, applying Bessel's inequality to $f_o - f$, and then to $g_o - g$, when $g(x)$ is assumed to exist, we obtain (34) and (35), as desired.

LEMMA 4: <u>If the series $\Gamma_o(x) = \sum_{i=1}^{\infty} w_i(x)$, where the $w_i(x)$ are continuous functions, is absolutely and uniformly convergent, and if $\Gamma_n(x) = \sum_{i=1}^{\infty} \mu_i^n w_i(x)$, $n = 0,1,2,\ldots$, where μ_i satisfies condition (33), then the sequence $\Gamma_n(x)$ converges uniformly to zero.</u>

<u>Proof:</u> From the hypotheses on μ_i we have, for some sufficiently large r, $\mu_r \geqq |\mu_i|$, $r > i$. Also, considering the series for $\Gamma_o(x)$, given an $\epsilon > 0$, r can be chosen so large, and independent of x, that $\sum_{i=r-1}^{\infty} |w_i| < \epsilon/2$. Let r be chosen so that both conditions are satisfied. Further, we have $\sum_{i=1}^{r} |w_i| \leqq \sum_{i=1}^{\infty} |w_i| < M$, where M is an upper bound independent of x. Choose N sufficiently large so that $\mu_r^n < \epsilon/(2M)$ for $n > N$. Then

$$|\Gamma_n| \leqq \sum_{i=1}^{r} |\mu_i^n w_i| + \sum_{i=r-1}^{\infty} |\mu_i^n w_i| < \mu_r^n M + \frac{\epsilon}{2} < \epsilon ,$$

when $n > N(\epsilon)$, as we wished to prove.

LEMMA 5: <u>If $G_n(x)$ can be expanded in a uniformly convergent series</u>

$$G_n(x) = \sum_{i=1}^{\infty} e_{in}\theta_i(x), \quad n = 0,1,2,\ldots \tag{36}$$

<u>in terms of the real, continuous, orthonormalized functions $\theta_i(x)$, $i = 1,2,\ldots$ and if $G(x)$ is piecewise-continuous, with $e_i = \int G\theta_i$, then necessary and sufficient conditions for the sequence $G_n(x)$ to converge in the mean to $G(x)$ are that</u>

$$\int G^2 dx = \sum_{i=1}^{\infty} e_i^2 \quad \text{and} \quad \lim_{n\to\infty} \sum_{i=1}^{\infty} (e_{in} - e_i)^2 = 0.$$

<u>Proof</u>: Since the series (36) is uniformly convergent, we have

$$\int GG_n = \sum_{i=1}^{\infty} e_{in} \int G\theta_i = \sum_{i=1}^{\infty} e_{in} e_i ,$$

and similarly $\int G_n^2 = \sum_{i=1}^{\infty} e_{in}^2$. Hence

$$\int (G_n - G)^2 = \int G^2 + \sum_{i=1}^{\infty} (e_{in} - e_i)^2 - \sum_{i=1}^{\infty} e_i^2 \qquad (37)$$

Now suppose the conditions of the lemma to be satisfied. Then $\int (G_n - G)^2 = \sum_{i=1}^{\infty} (e_{in} - e_i)^2$, and consequently by hypothesis, $\lim_{n\to\infty} \int (G_n - G)^2 = 0$. This proves the first part of the lemma.

Now suppose that $\lim_{n\to\infty} \int (G_n - G)^2 = 0$. From (37), we have,

$$\int G^2 dx \leq \sum_{i=1}^{\infty} e_i^2 + \int (G_n - G)^2$$

for all n. Hence $\int G^2 \leq \sum_{i=1}^{\infty} e_i^2$. But, by Bessel's inequality, $\int G^2 \geq \sum_{i=1}^{\infty} e_i^2$. Hence $\int G^2 = \sum_{i=1}^{\infty} e_i^2$. Then, from (37),

$$\sum_{i=1}^{\infty} (e_{in} - e_i)^2 = \int (G_n - G)^2$$

whence we obtain $\lim_{n\to\infty} \sum_{i=0}^{\infty} (e_{in} - e_i)^2 = 0$, also. This completes the proof.

<u>Proofs of Convergence Theorems.</u> We can now proceed to the proof of the convergence theorems.

<u>Proof of Theorem 3:</u> By the expansion theorem and (12) and (27), the series $F_n - F = \sum_{i=1}^{\infty} (c_{in} - c_i)\, \psi_i$, $n = 0,1,2,\ldots$ are absolutely and uniformly convergent in x. Hence, by Lemma 2, the series $\sum_{i=1}^{\infty} \mu_i^n (c_{io} - c_i) \psi_i$ are also absolutely and uniformly convergent in x. Hence the conditions of Lemma 4 are satisfied and the sequence $\{F_n - F\}$ converges uniformly to zero; or by (25), $\{Kg_n\}$ converges uniformly to F, as we wished to prove.

Proof of Theorem 4: By Lemmas 1 and 3 all the conditions of Lemma 5 are satisfied by the functions $f_n(x)$ and $f(x)$. Hence by (26) the first part of the theorem, concerning the convergence in the mean of $\{kg_n\}$ to $f(x)$, is proved.

In the second part of the theorem, since $g(x)$ exists by hypothesis, the expansion theorem may be applied to $f(x)$ as well as to $f_n(x)$. Hence, by (12) and Lemmas 1 and 2, the series

$$f_n - f = \sum_{i=1}^{\infty} \mu_i{}^n \lambda_i (c_{i0} - c_i) \varphi_i(x), \quad n = 0,1,2,\ldots$$

are absolutely and uniformly convergent in x, and the conditions of Lemma 4 are satisfied. Hence the sequence $\{f_n - f\}$ converges uniformly to zero, or, by (26), $\{kg_n\}$ converges uniformly to $f(x)$. This completes the proof.

Proof of Theorem 5: Since $g_0(x)$ is of the form (24), Lemmas 1 and 3 indicate that the conditions of Lemma 5 are satisfied by the functions $g_n(x)$ and $g(x)$. Hence the first part of the theorem, concerning convergence in the mean of $\{g_n\}$ to $g(x)$, is proved.

In the second part of the theorem, the expansion theorem is applicable to $g(x)$, by hypothesis. Hence, by (12) and Lemmas 1 and 2, the series

$$g_n - g = \sum_{i=1}^{\infty} \mu_i{}^n \lambda_i{}^2 (c_{i0} - c_i) \psi_i(x), \quad n = 0,1,2,\ldots$$

are absolutely and uniformly convergent in x, and the conditions of Lemma 4 are satisfied. Hence the sequence $\{g_n\}$ converges uniformly to $g(x)$, as we wished to prove.

SUMMARY

A method of solving the Fredholm integral equation of the first kind

$$f(x) = \int_a^b k(x,y)g(y)dy$$

by means of the iteration formula

$$g_n(x) = g_{n-1}(x) + F(x) - \int_a^b K(x,y)g_{n-1}(y)dy$$

where

$$F(x) = \int_a^b k(y,x)f(y)dy$$

$$K(x,y) = \int_a^b k(t,x)k(t,y)dt$$

is discussed. Several theorems concerning the convergence of the sequence of functions $g_n(x)$ to $g(x)$ under various conditions are stated and proved. It is shown that this sequence bears the identical relations to a solution of the integral equation as a sequence consisting of finite sums of orthogonal functions associated with the kernel $k(x,y)$, given by the classical Schmidt-Picard theory of integral equations. The latter sequence is of little practical value, however because of the difficulty of obtaining the characteristic numbers and functions of the kernel. In contrast with this, the successive members of the sequence given by the present iteration formula are obtained by simple quadratures.

REFERENCES

1. Courant, R. and D. Hilbert. Methoden der Mathematischen Physik. Berlin: Julius Springer, 1931, Vol. 1, Chap. III, §2.

2. Ibid, §10.

3. Hellinger, E. and O. Toeplitz. Integralgleichungen und Gleichungen mit unendlichvielen Unbekannten. Encyklopädie der Mathematischen Wissenschaften II C 13. Berlin: B. G. Teubner, 1928, p. 1349.

4. Munk, M. M. Fluid Mechanics. Vol. 1, Part II of Aerodynamic Theory, Edited by W. F. Durand, Berlin: Julius Springer, 1935, Division C, p. 269.

5. Picard, E. Sur un théorème général relatif aux équations intégrales de première espèce et sur quelques problèmes de physique mathématique. Rend. Circ. Mat. Palermo, 1910, Vol. 29, pp. 79-97.

6. Schmidt, E. Zur Theorie der linearen und nichtlinearen Integralgleichungen. Math. Annalen, Vol. 63, 1907, pp. 433-476.

7. Weinstein, A. On Axially Symmetric Flows. Quarterly of Applied Mathematics, Vol. 5, No. 4, 1948, p. 433.

PART II

THE AXIALLY SYMMETRIC POTENTIAL FLOW ABOUT ELONGATED BODIES OF REVOLUTION

INTRODUCTION

History. The determination of the flow about elongated bodies of revolution is of great practical and theoretical importance in aero- and hydrodynamics. Such knowledge is required in connection with bodies such as airships, torpedoes, projectiles, airplane fuselages, pitot tubes, etc. Since it is well-known that for a streamlined body, moving in the direction of the axis of symmetry, the actual flow is very closely approximated by the potential (inviscid) flow about the body[2], numerous attempts have been made to find a convenient theoretical method for obtaining numerical solutions of the potential flow problem.

At first the problem was attacked by indirect means. In 1871 Rankine[16] showed how one could obtain families of bodies of revolution of known potential flow, generated by placing several point sources and sinks of various strengths on the axis. This method was extended and used by D. W. Taylor[20] in 1894 and by G. Fuhrmann[1] in 1911. The latter also constructed models of the computed forms and showed that the measured distributions of the pressures over them agreed very well with the computed values except for a small region at the downstream ends. More recently, in 1944, the Rankine method was employed by Munzer and Reichardt[15] to obtain bodies with flat pressure distribution curves, and a further refinement of the technique was published by Riegels and Brandt[17]. Most recently the indirect method has been employed to obtain bodies generated by axisymmetric source-sink distributions on circumferences, rings, discs and cylinders. This development, which enabled bodies with much blunter noses to be generated, was initiated by Weinstein[25] in 1948 and continued by Van Tuyl[22] and by Sadowsky and Sternberg[18] in 1950.

A method of solving the direct problem, i.e. to determine the flow over a given body of revolution, appears to have been first published by von Karman[5] in 1927. Karman reduced the problem to that of solving a Fredholm integral equation of the first kind for the axial source-sink distribution which would generate the given body, and solved the integral equation approximately by replacing it by a set of simultaneous linear equations. Although this method is of limited accuracy and becomes very laborious when, for greater refinement, a large number of linear equations is employed, nevertheless it is the best known and most frequently used of the direct methods. A modification of the von Karman method was published by Wijngaarden[26] in 1948.

An interesting attempt to solve the direct problem was made by Weinig[24] in 1928. He also formulated the problem in terms of an integral equation for an axial doublet distribution which would generate the given body and employed an iteration formula to obtain successive approximations. Since the successive approximations diverged, the recommended procedure was to extrapolate one step backwards to obtain a solution.

In 1935 an entirely different approach, in which a solution for the velocity potential was assumed in the form of an infinite linear sum of orthogonal functions, was made by Kaplan[3] and independently by Smith[19]. The coefficients of this series are given as the solution of a set of linear equations, infinite in number. In practice a finite number of these equations is solved for a finite number of coefficients, and Kaplan has shown that the approximate solution thus obtained is that due to an axial source-sink distribution which is also determined. A simplification of Kaplan's method by means of additional approximations was proposed by Young and Owen[27] in 1943.

It appears to be generally agreed, by those who have tried them, that the aforementioned methods are both laborious and approximate. Thus, according to Young and Owen[27]:

> In every case, however, the methods proposed are laborious to apply, and the labour and heaviness of the computations increase rapidly with the rigour and accuracy of the process. Inevitably, a compromise is necessary between the accuracy aimed at and the difficulties of computation. All the methods reduce, ultimately, to finding in one way or another the equivalent sink-source distribution, and it is this part of the process which in general involves the heaviest computing.

Furthermore a fundamental objection is that only a special class of bodies of revolution can be represented by a distribution of sources and sinks on the axis of symmetry. According to von Karman[5]:

> This (representability by an axial source-sink distribution) is possible only in the exceptional case when the analytical continuation of the potential function, free from singularities in the space outside the body, can be extended to the axis of symmetry without encountering singular spots.

The dissatisfaction with these methods is reflected by the continuing attempts to devise other procedures.

A new method published by Kaplan[4] in 1943 is free of the assumption of axial singularities and appears to be exact in the sense that the solution can be made as accurate as desired; but the labor required for the same accuracy appears to be much greater than by other methods. The application of the method requires that first the conformal transformation which transforms the given meridian profile into a circle be determined. The velocity potential is then expressed as an infinite series whose terms are universal functions involving the coefficients of the conformal transformation. Kaplan[4] has derived only the first three of these universal functions.

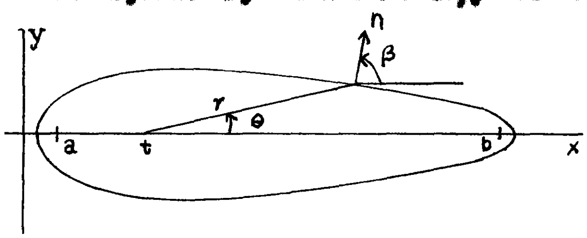
Cummins of the David Taylor Model Basin is developing a method based on a distribution of sources and sinks on the surface of the given body. This method is also exact, but the labor involved in its application has not yet been evaluated.

Another exact method, based on a distribution of vorticity over the surface of the body, is being developed by Dr. Vandry of the Admiralty Research Laboratory, Teddington, England. The methods of Cummins and Vandry both lead to Fredholm integral equations of the second kind, which can be solved by iteration.

The present writer has developed two new methods, an approximate one in which an axial doublet distribution is assumed, and an exact one based on a general application of Green's theorem of potential theory. Both methods lead to

Fredholm integral equations of the first kind for which a solution by iteration has been discussed in Part I. Indeed the consideration of this iteration formula was initiated in an attempt to find more satisfactory solutions of the integral equations of von Karman[5] and Weinig[24]. These new methods will be presented, and, by application to a particular body, compared with other methods from the point of view of accuracy and convenience of application.

<u>Formulation of the Problem</u>. We will consider the steady, irrotational, axially symmetric flow of a perfect incompressible fluid about a body of revolution. Take the x-axis as the axis of symmetry and let x,y be the coordinates in a meridian plane.



Denote the equation of the body profile by

$$y^2 = f(x) \tag{1}$$

FIGURE 1 - The Meridian Plane

Since the flow is irrotational there exists a velocity potential which, for axisymmetric flows, depends only on the cylindrical coordinates x,y and satisfies Laplace's equation in cylindrical coordinates

$$\frac{\partial}{\partial x}\left(y\frac{\partial \varphi}{\partial x}\right) + \frac{\partial}{\partial y}\left(y\frac{\partial \varphi}{\partial y}\right) = 0 \tag{2}$$

Also, since the flow is axisymmetric, there exists a Stokes stream function (x,y) which is related to the velocity potential by the equations

$$\frac{\partial \Psi}{\partial x} = -y\frac{\partial \varphi}{\partial y}, \quad \frac{\partial \Psi}{\partial y} = y\frac{\partial \varphi}{\partial x} \tag{3}$$

It is seen that equation (2) may be interpreted as the necessary and sufficient condition insuring the existence of the function ψ. As is well known, ψ is constant along a streamline and, considering the surface of revolution generated by rotation of a streamline about the axis of symmetry, $2\pi\psi$ may be considered as the flux bounded by this surface. On the surface of the given body and along the axis of symmetry outside the body we have $\psi = 0$. ψ satisfies the equation

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = \frac{1}{y}\frac{\partial\psi}{\partial y} \tag{4}$$

which is obtained by eliminating φ between equations (3).

The velocity will be taken as the negative gradient of the velocity potential. Let u,v be the velocity components in the x,y directions. Then, by (3), we have

$$u = -\frac{\partial\varphi}{\partial x} = -\frac{1}{y}\frac{\partial\psi}{\partial y} \tag{5}$$

$$v = -\frac{\partial\varphi}{\partial y} = \frac{1}{y}\frac{\partial\psi}{\partial x} \tag{6}$$

For a uniform flow of velocity U parallel to the x-axis we have

$$\varphi = -Ux, \quad \psi = -\tfrac{1}{2}Uy^2. \tag{7}$$

The boundary condition for the body to be a stream surface may be written in various ways. If the body is stationary the boundary condition is

$$\psi(x, \sqrt{f(x)}) = 0 \tag{8a}$$

or, equivalently,

$$\left(\frac{\partial\varphi}{\partial n}\right)_s = 0 \tag{8b}$$

where the derivative in (8b) is evaluated on the surface of the body in the direction of the outward normal to the body. If the body is moving with velocity V parallel to the x-axis the boundary condition becomes

$$\left(\frac{\partial\varphi}{\partial n}\right)_s = - V \cos\beta \tag{9}$$

where β is the angle between the outward normal to the body and the x-axis.

It is desired to obtain a solution of (2) or (4) which satisfies the boundary conditions (7) at infinity and (8) or (9) on the body.

METHOD OF AXIAL DISTRIBUTIONS

Sources and Sinks. The potential and stream functions for a point source of strength M situated on the x-axis at $x = t$ are

$$\varphi = \frac{M}{r}, \quad \psi = M\left(-1 + \frac{x-t}{r}\right) \tag{10}$$

where

$$r^2 = (x-t)^2 + y^2 \tag{11}$$

If the sources are distributed along the x-axis between the points a and b (see Figure 1) with a strength (x) per unit length, the potential and stream functions are

$$\varphi = \int_a^b \frac{\mu(t)}{r}\, dt \tag{12}$$

$$\psi = \int_a^b \mu(t)\left(-1 + \frac{x-t}{r}\right)dt \tag{13}$$

As is well-known, Rankine bodies are obtained by superposition of these flows with a uniform stream so as to obtain a dividing streamline beginning at a stagnation point. Without loss of generality we may suppose this uniform stream to be of unit magnitude. This dividing streamline is the profile of the Rankine body for which, by (7), the stream function is

$$\psi = -\tfrac{1}{2}y^2 + \int_a^b \mu(t)\left(-1 + \frac{x-t}{r}\right)dt \qquad (14)$$

The boundary condition (8a) then gives as the implicit equation for the body

$$\int_a^b \mu(t)\left(-1 + \frac{x-t}{r}\right)dt = \tfrac{1}{2}y^2 \qquad (15)$$

where now $y^2 = f(x)$ and $r^2 = (x-t)^2 + f(x)$. In order to obtain a closed body the total strengt of sources and sinks must be zero, i.e.

$$\int_a^b \mu(t)dt = 0.$$

In this case (15) becomes

$$\int_a^b \mu(t)\frac{x-t}{r}\,dt = \tfrac{1}{2}y^2 \qquad (15a)$$

In general (15a) cannot be solved explicitly for f(x) when $\mu(t)$ is given. A practical procedure for obtaining f(x) for a given x is to evaluate the integral numerically for various assumed values of f(x) and to determine the value which satisfies (15a) by graphical means.

When f(x) is prescribed (15a) may be considered as a Fredholm integral equation of the first kind for determining the unknown function $\mu(t)$. This equation will not be treated, however, since, as will be shown it is a special case of the more general equation for doublet distributions which will now be derived.

Doublet Distributions. Let m(x) be the strength per unit length of a distribution of doublets along the x-axis between the points a and b; (see Figure 1). The potential and stream functions may be taken as

$$\varphi = \int_a^b m(t)\, \frac{t-x}{r^3}\, dt \tag{16}$$

and

$$\psi = y^2 \int_a^b \frac{m(t)}{r^3}\, dt \tag{17}$$

The stream function for a Rankine flow now becomes

$$\psi = -\tfrac{1}{2}y^2 + y^2 \int_a^b \frac{m(t)}{r^3}\, dt \tag{18}$$

Hence the boundary condition (8a) gives

$$\int_a^b \frac{m(t)}{r^3}\, dt = \tfrac{1}{2} \tag{19}$$

Here again equation (19) may be considered as an implicit equation for the Rankine body when m(t) is given, or as a Fredholm integral equation of the first kind when the body profile $y^2 = f(x)$ is prescribed.

In order to show the relation between the source and doublet distributions in equations (15a) and (19), integrate by parts in (19). We have

$$\int_a^b m(t)\, \frac{y^2}{r^3}\, dt = m(t)\, \frac{t-x}{r}\Big|_a^b + \int_a^b \frac{dm}{dt}\, \frac{x-t}{r}\, dt$$

Hence (19) may be written as

$$m(t)\, \frac{t-x}{r}\Big|_a^b + \int_a^b \frac{dm}{dt}\, \frac{x-t}{r}\, dt = \tfrac{1}{2}y^2 \tag{20}$$

The interpretation of equation (20) is that a doublet distribution of strength m is equivalent to a source-sink distribution of strength $\frac{dm}{dt}$ together with point sources of strength m(a) and -m(b) at the end points. Hence source-sink distributions are completely equivalent only to those doublet distributions which vanish at the end points. This justifies the remark in the previous section that the integral equation for the doublet distributions is more general than that for the source-sink distributions.

Munk's Approximate Distribution. Munk[12] has given an approximate solution of (19) for elongated bodies. His formula may be derived as follows. For a very elongated body at a great distance from the ends, the integrand of (19), $m(t)/r^3$, will peak sharply in the neighborhood of $t = x$. In the range of the peak, in which the value of the integral is principally determined, $m(t)$ will vary little from $m(x)$. Also only a small error will be introduced by replacing the limits of integration by $-\infty$ and $+\infty$. Hence, as a first approximation to a solution of (19), try

$$m_1(x) \int_{-\infty}^{\infty} \frac{dt}{r^3} = \frac{1}{2} \tag{21}$$

We obtain

$$m_1(x) = \tfrac{1}{4}y^2 \tag{22}$$

a distribution proportional to the section-area curve of the body. This approximation was independently derived by Weinig[24] who employed it as the first step in a divergent iteration procedure. It has also been rediscovered by Young and Owen[27] and Laitone[8] who have shown the accuracy of the approximation for elongated bodies by several examples.

It is apparent from its derivation that (22) also gives the asymptotic radius of the half-body generated by a constant axial dipole distribution extending from a point on the axis to infinity. It is readily seen that this distribution is equivalent to a point source at the initial point.

As a refinement to Munk's formula, Weinblum[23] has used the approximation

$$m_1(x) = Cy^2 \tag{23}$$

where C is a factor obtained by comparison of the distributions and section area curves of several bodies. Weinblum's factor bears an interesting relation to the virtual mass of the body. This is seen by considering the expression for the virtual mass $k_1\Delta$ in terms of the mass of the displaced fluid Δ and the totality of the doublets, $\int_a^b m\,dx$,[10,14,21]

$$k_1\Delta = 4\pi\rho\int_a^b m\,dx - \Delta \tag{24}$$

where k_1 is designated the longitudinal virtual mass coefficient, and ρ is the density of the fluid. But, from (23),

$$4\pi\rho\int_a^b m_1\,dx = 4\rho C\int_a^b \pi y^2\,dx \doteq 4C\Delta$$

since, for elongated bodies, a and b very nearly coincide with the body ends. Hence

$$C = \tfrac{1}{4}(1+k_1) \tag{25}$$

In practice an approximate value of k_1 may be taken as that of the prolate spheroid having the same length-diameter ratio as the given body. The values of k_1 for a prolate spheroid may be computed from the formula[9]

$$k_1 = \frac{\lambda\ln(\lambda+\sqrt{\lambda^2-1})-\sqrt{\lambda^2-1}}{\lambda^2\sqrt{\lambda^2-1}-\lambda\ln(\lambda+\sqrt{\lambda^2-1})}$$

where λ is the length-diameter ratio. Hence

$$C = \frac{(\lambda^2-1)^{3/2}}{\lambda^2\sqrt{\lambda^2-1}-\lambda\ln(\lambda+\sqrt{\lambda^2-1})}$$

The values of k_1 versus λ have also been tabulated by Lamb[9] and graphed by Munk[13].

End Points of a Distribution. A difficulty in determining the doublet distribution from equation (19) is that the limits of integration, a and b, are also unknown. In the method of von Karman[5] the end points are arbitrarily chosen; Kaplan[4] takes the end point of the distribution midway between the end of the body and the center of curvature at that end.

Kaplan based his choice on a consideration of the prolate spheroid. Thus the equation of the spheroid of unit length and length-diameter ratio λ, extending from $x = 0$ to $x = 1$, is

$$y^2 = \frac{1}{\lambda^2}(x-x^2) \tag{28}$$

The radius of curvature at $x = 0$ is then $\frac{1}{2\lambda^2}$. The exact doublet distribution, however, extends between the foci of the spheroid which are situated at distances $(\lambda-\sqrt{\lambda^2-1})/(2\lambda)$ from the end points. Hence the error in Kaplan's assumptio

$$\frac{\lambda-\sqrt{\lambda^2-1}}{4\lambda} - \frac{1}{4\lambda^2} = \frac{1}{16\lambda^4}\left(1+\frac{1}{2\lambda^2}+\cdots\right)$$

diminishes rapidly with increasing λ.

For the half-body generated by a constant doublet
tion (a point source) Kaplan's assumption gives a p
mation. Let a^2 be the strength of the distributic
can easily be shown from (19) that the source is
a from the end of the body (stagnation point),

the origin is chosen at the latter point, the equation of the half-body is

$$\left(\frac{y}{a}\right)^2 = \frac{8}{3}\frac{x}{a} - \frac{20}{27}\left(\frac{x}{a}\right)^2 + \frac{16}{243}\left(\frac{x}{a}\right)^3 + \dots \qquad (29)$$

Hence the radius of curvature at the end is $\frac{4}{3}$ a, so that Kaplan's assumption for the start of the distribution gives $\frac{2}{3}$a. This is in error by $\frac{1}{3}$a.

An approximate method for determining the end points of a distribution and its trends at the ends will now be described. Let $y^2 = f(x)$ be the equation of the given profile extending from $x = 0$ to $x = 1$; let $m(x)$ be the corresponding doublet distribution, extending from $x = a$ to $x = b$. It will be assumed that $0 < a << b < 1$ and that a is near 0, b is near 1.

Various conditions on $m(x)$ may now be obtained by differentiating (19) repeatedly with respect to x. We get

$$\int_a^b \frac{m(t)}{r^5}\left[2x - 2t + f'(x)\right] dt = 0$$

$$\int_a^b m(t)\left[-\frac{5}{2r^7}(2x - 2t + f')^2 + \frac{1}{r^5}(2 + f'')\right] dt = 0 \qquad (30)$$

$$\int_a^b m(t)\left[\frac{35}{4r^9}(2x-2t+f')^3 - \frac{15}{2r^7}(2+f'')(2x-2t+f') + \frac{f'''(x)}{r^5}\right] dt = 0$$

When $x = 0$, $r = t$ and, writing $f(x)$ as a Taylor expansion

$$f(x) = a_1 x + a_2 x^2 + a_3 x^3 + \dots \qquad (31)$$

then also $f'(0) = a_1$, $f''(0) = 2a_2$, $f'''(0) = 6a_3$. Now, setting $x = 0$ in equations (19) and (30), we obtain

$$\int_a^b \frac{m(t)}{t^3}\, dt = \tfrac{1}{2} \tag{32a}$$

$$\int_a^b \frac{m(t)}{t^5}(a_1-2t)dt = 0 \tag{32b}$$

$$\int_a^b \frac{m(t)}{t^7}\left[5a_1{}^2-20a_1t+4(4-a_2)t^2\right]dt = 0 \tag{32c}$$

$$\int_a^b \frac{m(t)}{t^9}\left[35a_1{}^3-210a_1{}^2t+60a_1(6-a_2)t^2+40(3a_2-4)t^3+24a_3t^4\right]dt = 0 \tag{32d}$$

Also assume that m(x) may be expressed as a power series

$$m(x) = c_0 + c_1x + c_2x^2 + \cdots \tag{33}$$

Then the first of equations (32) gives

$$\frac{c_0}{2}\left(\frac{1}{a^2} - \frac{1}{b^2}\right) + c_1\left(\frac{1}{a} - \frac{1}{b}\right) + c_2 \log \frac{b}{a} + \cdots = \tfrac{1}{2};$$

or, neglecting $1/b^2$ in comparison with $1/a^2$ and setting $b = 1$ in comparison with $1/a$,

$$c_0 + 2c_1a(1-a) + 2c_2a^2 \log \frac{1}{a} + \cdots = a^2 \tag{34a}$$

Similarly the other equations (32) give, approximately,

$$c_0(3a_1-8a) + 4c_1a(a_1-3a) + 6c_2a^2(a_1-4a+4a^2) = 0 \tag{34b}$$

$$2c_0\left[5a_1{}^2-24a_1a+6(4-a_2)a^2\right] + 4c_1a\left[3a_1{}^2-15a_1a+4(4-a_2)a^2\right] + c_2a^2\left[15a_1{}^2-80a_1a+24(4-a_2)a^2\right] = 0 \tag{34c}$$

$$\begin{aligned}&3c_0\left[35a_1{}^3-240a_1{}^2a+80a_1(6-a_2)a^2+64(3a_2-4)a^3+48a_3a^4\right]\\ &+24c_1\left[5a_1{}^3a-35a_1{}^2a^2+12a_1(6-a_2)a^3+10(3a_2-4)a^4+8a_3a^5\right]\\ &+4c_2\left[35a_1{}^3a^2-252a_1{}^2a^3+90a_1(6-a_2)a^4+80(3a_2-4)a^5+72a_3a^6\right] = 0\end{aligned} \tag{34d}$$

Equations (34) are sufficient in number to determine the unknowns a, c_0, c_1, c_2. Since the latter 3 equations are linear and homogeneous in c_0, c_1 and c_2, a can be determined from the condition that the determinant of their coefficients must vanish. In this way the following equation of the 7th degree in $\alpha = a_1/a$ was obtained:

$$\begin{aligned}&\alpha(\alpha-4)^2(5\alpha^4-83\alpha^3+288\alpha^2-368\alpha+128)-96a_2{}^2\alpha(3\alpha-4)\\&+4a_2\alpha(\alpha-4)(53\alpha^2-148\alpha+128)+1152a_1a_2{}^2(2\alpha-3)\\&+72a_1(\alpha-4)^2(5\alpha^3-25\alpha^2+40\alpha-16)+48a_1a_3\ (3\alpha-8)\\&-288a_1a_2(\alpha-4)(5\alpha^2-16\alpha+16)-1152a_1{}^2a_3(\alpha-3)=0\qquad(35)\end{aligned}$$

Corresponding to a solution α of (35), c_0, c_1 and c_2 can be obtained from equations (34a, b, c). The solution of the latter equations gives

$$\begin{aligned}c_0D&=-4a^2\left[3\alpha^3-37\alpha^2+120\alpha-96+24a_2+24a(3\alpha^2-15\alpha+16-4a_2)\right]\\c_1D&=a\left[15\alpha^3-168\alpha^2+512\alpha-384+96a_2+48a(5\alpha^2-24\alpha+24-6a_2)\right]\\c_2D&=-4\left[(\alpha-4)^2(\alpha-1)+4a_2\right]\end{aligned}$$

where

$$\begin{aligned}D=\ &2(9\alpha^3-94\alpha^2+272\alpha-192)+8\left[(\alpha-4)^2(\alpha-1)+4a_2\right]\log\ a+96a_2\\&-2a(15\alpha^3-264\alpha^2+944\alpha-768)-384aa_2-96a^2(5\alpha^2-24\alpha+24)\\&+576a^2a_2.\qquad(36)\end{aligned}$$

The initial doublet strength at $x = a$ is

$$m(a) = c_0 + c_1a + c_2a^2 + \ldots,$$

or, from equations (36),

$$m(a) = -\frac{a^2}{D}\left[(\alpha-4)(\alpha^2-12\alpha+16)+48a(\alpha-4)(\alpha-2)+16a_2-96aa_2\right]\qquad(37)$$

Equations (35), (36), and (37) determine the end points of the distribution and its initial trends. In general equation (35) will have more than one real root. In this case the initial trends corresponding to each of the roots should be examined, and that root chosen which appears to give the "simplest" trend.

The equations can be solved explicitly in the case of a very elongated body for which a_1, a_2, a_3,... in (31) are all very small. First let us suppose that they are so small that all the terms in (35) containing them are negligible, so that the first product term alone may be equated to zero, i.e.

$$\alpha(\alpha-4)^2(5\alpha^4-83\alpha^3+288\alpha^2-368\alpha+128) = 0 \tag{38}$$

whose real roots are $\alpha = 0$, 0.547, 4.0, 4.0, and 12.429.

Let us consider the solution $\alpha = 4$; i.e. $a = a_1/4$. Since the radius of curvature at $x = 0$ is $a_1/2$, this solution is seen to be in accord with Kaplan's assumption for the end points of the distribution. Furthermore, substituting $\alpha = 4$ into equations (36) and (37), we obtain, to the same order of approximation,

$$D = 64, \; c_0 = -a_1^2/16, \; c_1 = a_1/4, \; c_2 = 0$$

whence

$$\begin{aligned} m(x) &= -\frac{a_1^2}{16} + \frac{a_1}{4}x \\ m(a) &= 0 \end{aligned} \tag{39}$$

In order to obtain a second approximation it will be assumed that not only a_1, a_2, a_3... but also $(\alpha-4)$ are small to the first order, equation (35) becomes

$$-3072(\alpha-4)^2 + 6144a_2(\alpha-4) - 3072a_2^2 + 768a_1a_3 = 0 \tag{40}$$

whence

$$\alpha = 4 + a_2 \pm \tfrac{1}{2}\sqrt{a_1a_3}$$

Provided $\quad a_3 \geq 0 \quad$ (41)

Corresponding to this value of α we obtain from equations

(36), to the same order of approximation,

$$m(x) = C(-\frac{a_1^2}{4} + a_1x + a_2x^2 + \ldots)$$

where $$C = \tfrac{1}{4}(1+\frac{a_1}{2}+\frac{a_2}{2}\log\frac{a_1}{4}) \tag{42}$$

and $$m(a) = \pm \tfrac{1}{2}Ca^2\sqrt{a_1a_3}$$

The expression for m(x) in (42) may also be written as

$$m(x) = C(-\frac{a_1^2}{4} - y^2) \tag{42a}$$

This form immediately suggests a modification and refinement of the Munk-Weinblum approximation (23) which will be considered in the next section.

When $a_3 < 0$ the solution for α in (41) indicates that there would be no real roots near $\alpha = 4$. In this case a graph of the complete polynomial in (35) should be examined either for the possibility that more complete calculations would show that there are real roots near $\alpha = 4$ nevertheless, or that the maximum value of the complete polynomial in the neighborhood of $\alpha = 4$ is so nearly zero, that the value of α corresponding to this maximum may be taken as an approximate solution. On this assumption, the second order analysis would give

$$\left.\begin{aligned} \alpha &= 4 + a_2 \\ a_3 &< 0 \end{aligned}\right\} \tag{41a}$$

Since a_3 does not occur explicitly in equations (42), it is seen that they would also be obtained, to the same order of approximation, if the value of α in (41a) were substituted into equation (36).

If it is determined that not even an approximate solution can be assumed near $\alpha = 4$ it would be necessary to consider solutions in the neighborhood of the other roots of equation (38).

In order to facilitate the computations for graphing the polynomial in (35), the functions $A(\alpha)$, $B(\alpha)$,...$H(\alpha)$, where

$$
\begin{aligned}
A(\alpha) &= \alpha(\alpha-4)^2(5\alpha^4-83\alpha^3+288\alpha^2-368\alpha+128) \\
B(\alpha) &= 72(\alpha-4)^2(5\alpha^3-25\alpha^2+40\alpha-16) \\
C(\alpha) &= 4\alpha(\alpha-4)(53\alpha^2-148\alpha+128) \\
D(\alpha) &= -288(\alpha-4)(5\alpha^2-16\alpha+16) \\
E(\alpha) &= -96\alpha(3\alpha-4) \\
F(\alpha) &= 1152(2\alpha-3) \\
G(\alpha) &= 48\alpha(3\alpha-8) \\
H(\alpha) &= -1152(\alpha-3)
\end{aligned}
\tag{43}
$$

have been tabulated in Table 1. In terms of these functions, equation (35) becomes

$$A+a_1B+a_2C+a_1a_2D+a_2{}^2E+a_1a_2{}^2F+a_1a_3G+a_1{}^2a_3H = 0 \tag{44}$$

It is of interest to compare the approximate value for from equation (41) with the exact value for the prolate spheroid, equation (28). In this case we have

$$a_1 = -a_2 = 1/\lambda^2,\ a_3 = 0$$

and the exact value of α is

$$\alpha = 2 + 2\sqrt{1 - \frac{1}{\lambda^2}} = 4 - \frac{1}{\lambda^2} - \frac{1}{4\lambda^4} - \ldots$$

But when the length-diameter ratio λ is large, equation (41) gives the approximate value $\alpha = 4-1/\lambda^2$, which is seen to consist of the first two terms of the series expansion of the exact value of α. The following table shows that the approximate formula gives excellent agreement with the exact values even for very thick sections. Both the exact and the

TABLE 1

FUNCTIONS FOR DETERMINING LIMITS OF DOUBLET DISTRIBUTIONS

α	A(α)	B(α)	C(α)	D(α)	E(α)	F(α)	G(α)	H(α)
0	0	-18432.0	0	18432.0	0	-3456.0	0	3456.0
.1	143.0	-13409.7	-177.4	16230.2	35.5	-3225.6	-37.0	3340.8
.2	188.5	-9315.5	-305.6	14227.2	65.3	-2995.2	-71.0	3225.6
.3	169.7	-6027.4	-392.4	12414.2	89.3	-2764.8	-102.2	3110.4
.4	112.5	-3433.9	-445.1	10782.7	107.5	-2534.4	-130.6	2995.2
.5	36.4	-1433.3	-470.8	9324.0	120.0	-2304.0	-156.0	2880.0
.6	-44.4	66.6	-475.6	8029.4	126.7	-2073.6	-178.6	2764.8
.7	-120.1	1148.7	-465.4	6890.4	127.7	-1843.2	-198.2	2649.6
.8	-184.5	1887.4	-445.6	5898.2	122.9	-1612.8	-215.0	2534.4
.9	-234.8	2349.1	-421.1	5044.3	112.3	-1382.4	-229.0	2419.2
1.0	-270.0	2592.0	-396.0	4320.0	96.0	-1152.0	-240.0	2304.0
1.1	-291.2	2667.3	-374.3	3716.6	73.9	-921.6	-248.2	2188.8
1.2	-300.5	2619.2	-359.1	3225.6	46.1	-691.2	-253.4	2073.6
1.3	-300.9	2485.3	-353.4	2838.2	12.5	-460.8	-255.8	1958.4
1.4	-295.9	2297.3	-359.3	2545.9	-26.9	-230.4	-255.4	1843.2
1.5	-288.9	2081.3	-378.8	2340.0	-72.0	0	-252.0	1728.0
1.6	-283.1	1857.9	-412.9	2211.8	-122.9	230.4	-245.8	1612.8
1.7	-281.5	1643.5	-462.5	2152.8	-179.5	460.8	-236.6	1497.6
1.8	-286.2	1449.7	-527.8	2154.2	-241.9	691.2	-224.6	1382.4
1.9	-298.8	1284.4	-608.6	2207.5	-310.1	921.6	-209.8	1267.2
2.0	-320.0	1152.0	-704.0	2304.0	-384.0	1152.0	-192.0	1152.0
2.1	-349.8	1054.0	-812.8	2435.0	-463.7	1382.4	-171.4	1036.8
2.2	-387.3	989.1	-933.3	2592.0	-549.1	1612.8	-147.8	921.6
2.3	-430.9	954.0	-1063.1	2766.2	-640.3	1843.2	-121.4	806.4
2.4	-478.2	943.7	-1199.3	2949.1	-737.3	2073.6	-92.2	691.2
2.5	-526.3	951.8	-1338.8	3132.0	-840.0	2304.0	-60.0	576.0
2.6	-572.0	970.9	-1477.5	3306.2	-948.5	2534.4	-25.0	460.8
2.7	-611.7	993.5	-1611.4	3463.2	-1062.7	2764.8	13.0	345.6
2.8	-641.8	1011.9	-1735.4	3594.2	-1182.7	2995.2	53.8	230.4
2.9	-658.9	1018.9	-1844.2	3690.7	-1308.5	3225.6	97.4	115.2
3.0	-660.0	1008.0	-1932.0	3744.0	-1440.0	3456.0	144.0	0
3.1	-642.8	974.2	-1992.4	3745.4	-1577.3	3686.4	193.6	-115.2
3.2	-606.1	914.2	-2018.5	3686.4	-1720.3	3916.8	245.8	-230.4
3.3	-549.6	826.8	-2003.0	3558.2	-1869.1	4147.2	301.0	-345.6
3.4	-474.9	713.3	-1937.8	3352.3	-2023.7	4377.6	359.0	-460.8
3.5	-385.3	578.3	-1814.8	3060.0	-2184.0	4608.0	420.0	-576.0
3.6	-286.2	429.5	-1624.8	2672.6	-2350.1	4838.4	483.8	-691.2
3.7	-185.8	278.7	-1358.5	2181.6	-2521.9	5068.8	550.6	-806.4
3.8	-94.8	142.2	-1006.0	1578.2	-2699.5	5299.2	620.2	-921.6
3.9	-27.0	40.6	-556.8	853.9	-2882.9	5529.6	692.6	-1036.8

α	$A(\alpha)$	$B(\alpha)$	$C(\alpha)$	$D(\alpha)$	$E(\alpha)$	$F(\alpha)$	$G(\alpha)$	$H(\alpha)$
4.0	0	0	0	0	-3072.0	5760.0	768.0	-1152.0
4.1	-34.7	52.1	675.9	-992.2	-3266.9	5990.4	846.2	-1267.2
4.2	-156.4	234.5	1482.8	-2131.2	-3467.5	6220.8	927.4	-1382.4
4.3	-394.3	591.5	2433.3	-3425.8	-3673.9	6451.2	1011.4	-1497.6
4.4	-782.7	1174.1	3540.3	-4884.5	-3886.1	6681.6	1098.2	-1612.8
4.5	-1360.2	2040.8	4817.3	-6516.0	-4104.0	6912.0	1188.0	-1728.0
4.6	-2170.8	3257.6	6278.2	-8329.0	-4327.7	7142.4	1280.6	-1843.2
4.7	-3263.7	4899.2	7937.7	-10332.0	-4557.1	7372.8	1376.2	-1958.4
4.8	-4693.4	7048.4	9810.7	-12533.8	-4792.3	7603.2	1474.6	-2073.6
4.9	-6520.2	9797.5	11912.8	-14942.9	-5033.3	7833.6	1575.8	-2188.8
5.0	-8810.0	13248.0	14260.0	-17568.0	-5280.0	8064.0	1680.0	-2304.0

0	0	-18432	0	18432	0	-3456	0	3456
1	-270	2592	-396	4320	96	-1152	-240	2304
2	-320	1152	-704	2304	-384	1152	-192	1152
3	-660	1008	-1932	3744	-1440	3456	144	0
4	0	0	0	0	-3072	5760	768	-1152
5	-8810	13248	14260	-17568	-5280	8064	1680	-2304
6	-75840	116352	55104	-57600	-8064	10368	2880	-3456
7	-302400	488592	141876	-128736	-11424	12672	4368	-4608
8	-819200	1456128	299008	-239616	-15360	14976	6144	-5760
9	-1700550	3535200	556020	-398880	-19872	17280	8208	-6912
10	-2790720	7475328	947520	-615168	-24960	19584	10560	-8064
11	-3417260	14306040	1513204	-897120	-30624	21888	13200	-9216
12	-1966080	25362432	2297856	-1253376	-36864	24192	16128	-10368
13	4706910	42363648	3351348	-1692576	-43680	26496	19344	-11520
14	22052800	67420800	4728640	-2223360	-51072	28800	22848	-12672
15	58820520	103097808	6489780	-2854368	-59040	31104	26640	-13824

$$A+a_1B+a_2C+a_1a_2D+a_2^2E+a_1a_2^2F+a_1a_3G+a_1^2a_3H = 0$$

$$A = \alpha(\alpha-4)^2(5\alpha^4-83\alpha^3+288\alpha^2-368\alpha+128)$$

$$B = +72(\alpha-4)^2(5\alpha^3-25\alpha^2+40\alpha-16)$$

$$C = 4\alpha(\alpha-4)(53\alpha^2-148\alpha+128)$$

$$D = -288(\alpha-4)(5\alpha^2-16\alpha+16)$$

$$E = -96\alpha(3\alpha-4) \qquad F = 1152(2\alpha-3)$$

$$G = 48(3\alpha-8)\alpha \qquad H = -1152(\alpha-3)$$

TABLE 2

COMPARISON OF EXACT AND COMPUTED VALUES

OF $\alpha = a_1/a$ FOR A PROLATE SPHEROID

λ	2	3	4	5	6
Exact α	3.732	3.886	3.936	3.960	3.972
Approx. α	3.750	3.889	3.937	3.960	3.972

approximate formulas give $m(a) = 0$. Thus the present approximate methods work very well for the prolate spheroid.

An Improved First Approximation. According to its derivation the Munk approximation could be expected to be useful only at a distance from the end points of a distribution. It was seen, however, (42a), that under certain circumstances a distribution which was a suitable approximation for the nose and tail of a body also appeared as a generalization of the Munk-Weinblum approximation (23). This suggests a procedure for obtaining an improved approximate distribution.

It is desired to obtain a distribution m(x) which satisfies the following conditions:

a. m(x) assumes known values m_a and m_b at the distribution limits a and b, i.e.

$$m(a) = m_a, \quad m(b) = m_b \tag{45}$$

b. m(x) is nearly equivalent to the Munk-Weinblum approximation (23) at a distance from the distribution limits, i.e.

$$m(a) \cong Cy^2 \quad \text{for} \quad a \ll x \ll b$$

c. m(x) satisfies the virtual mass relation (24) which may be written in the convenient form

$$\int_a^b m(x)dx = \tfrac{1}{4}(1+k_1)\int_0^1 y^2 dx \tag{46}$$

It is readily verified that condition (a) is satisfied by the distribution

$$m(x) = Cy^2 + e_0 + e_1 x \tag{47}$$

where
$$e_0 = \frac{1}{b-a}\left[bm_a - am_b + C(af_b - bf_a)\right] \tag{48}$$

$$e_1 = \frac{1}{b-a}\left[m_b - m_a + C(f_a - f_b)\right] \tag{49}$$

If the linear term e_0+e_1x in (47) is small in comparison with $m(x)$ at a distance from the ends then condition (b) is also satisfied. Finally condition (c) can be satisfied by a proper choice of C in (47). This is accomplished by writing $m(x)$ in the form $m(x) = C(y^2 - \frac{b-x}{b-a} f_a - \frac{x-a}{b-a} f_b) + \frac{b-x}{b-a} m_a + \frac{x-a}{b-a} m_b$

substituting it into equation (46), and solving for C. We obtain

$$C = \frac{\frac{1}{4}(1+k_1)\int_0^1 y^2 dx - \frac{1}{2}(b-a)(m_a+m_b)}{\int_a^b y^2 dx - \frac{1}{2}(b-a)(f_a+f_b)} \tag{50}$$

<u>Solution of Integral Equation by Iteration</u>. Now that we have derived a good first approximation to the doublet distribution function in the integral equation (19), it would be very desirable to apply it to obtain a second, closer approximation. This can be accomplished by means of the iteration formula which we will now derive.

Let $m_1(x)$ be a known first approximation and $\psi_1(x)$ the corresponding values of the stream function ψ on the given profile $y^2 = f(x)$. Then, from Equation (18),

$$\psi_1(x) = -\tfrac{1}{2}f(x) + f(x)\int_a^b \frac{m_1(t)}{r^3}dt \tag{51}$$

Thus $\psi_1(x)$ is a measure of the error when $m_1(t)$ is tried as a solution of the integral equation (19). If $m(t)$ is a solution of (19), (51) may be written in the form

$$\psi_1(x) = f(x)\int_a^b \frac{m_1(t) - m(t)}{r^3}dt \tag{52}$$

But, on the same assumptions as were used to derive Munk's approximate distribution (22), we obtain as an approximate solution of the integral equation (52)

$$m_1(x) - m(x) \doteq \tfrac{1}{2}\psi_1(x) \tag{53}$$

or, denoting the new approximation to m(x) by $m_2(x)$,

$$m_2(x) = m_1(x) - \tfrac{1}{2}\psi_1(x) \tag{54}$$

Hence, from (51)

$$m_2(x) = m_1(x) + \tfrac{1}{2}f(x)\left[\tfrac{1}{2} - \int_a^b \frac{m_1(t)}{r^3}dt\right] \tag{55}$$

Since the foregoing procedure can be repeated successively, we obtain the iteration formula

$$m_{i+1}(x) = m_i(x) + \tfrac{1}{2}f(x)\left[\tfrac{1}{2} - \int_a^b \frac{m_i(t)}{r^3}dt\right] \tag{56}$$

and

$$m_{i+1}(x) - m_i(x) = -\tfrac{1}{2}\psi_i(x) \tag{57}$$

It is seen that ψ_i is the value of the stream function on the given profile corresponding to the i th approximation $m_i(x)$ and hence serves as a measure of the error when $m_i(t)$ is tried as a solution of the integral equation (19).

Although successive approximations to m(x) may be computed directly from (56), an alternative form, which is both more convenient and more significant, will now be derived. From (56) we may write

$$m_i(x) = m_{i-1}(x) + \tfrac{1}{2}f(x)\left[\tfrac{1}{2} - \int_a^b \frac{m_{i-1}(t)}{r^3}dt\right] \tag{56a}$$

Hence, deducting (56a) from (56) and making use of (57), we get

$$\psi_i(x) = \psi_{i-1}(x) - \tfrac{1}{2}f(x)\int_a^b \frac{\psi_{i-1}(t)}{r^3}dt \tag{58}$$

Also, from (57), we obtain

$$m_{i+1}(x) = m_i(x) - \tfrac{1}{2} \sum_{j=1}^{i} \psi_j(x) \qquad (59)$$

Thus, in order to obtain $m_{i+1}(x)$, we first assume an $m_i(x)$ then determine $\psi_1(x)$ from (51). $\psi_2(x)$, $\psi_3(x)$,... can then be successively obtained from (58), and finally $m_{i+1}(x)$ from (59).

It has been stated that the magnitude of $\psi_i(x)$ is a measure of the approximateness of $m_i(x)$. This property of $\psi_i(x)$ can be given a geometrical interpretation. Corresponding to the distribution $m_i(x)$ there is an exact stream surface on which the stream function $\psi_i(x,y) = 0$. Let Δn_i be the distance from a point (x,y) on the given body to this exact stream surface, measured along the normal to the given body, positive outwards. Let u_s be the tangential component of the flow along the body. Then we have

$$u_s = -\frac{1}{y}\,\frac{\partial \psi_i(x,y)}{\partial n} \doteq -\frac{\Delta \psi_i(x,y)}{y \Delta n_i}$$

But $\Delta\psi = -\psi_r(x)$, since $\psi_i(x,y) = 0$ on the exact stream surface. Hence

$$\Delta n_i = \frac{\psi_i(x)}{y u_s} \qquad (60)$$

Since, for an elongated body $u_s \doteq 1$, except in the neighborhood of the stagnation points, it is seen that $\psi_i(x)$ enables a rapid estimate to be made of the variation from the desired profile of the exact stream surface corresponding to $m_i(x)$. This is an important property because it can be used to monitor the successive approximations. Thus, the sequence $\psi_i(x)$ can be terminated when Δn_i becomes uniformly less than some specified tolerance; or, since there is no assurance that the infinite sequence $\psi_i(x)$ converges, the sequence can conceivably give useful results even without convergence if it is continued as long as Δn_i decreases on the average, and is terminated when the error begins to increase and grows to an unacceptable magnitude at some point along the body. The strong similarity between these remarks and the discussion following Theorem 2 of Part I should be noted.

There is also a strong similarity between the iteration formulas, equation (17) of Part I, whose convergence was thoroughly discussed, and the present equation (56). An essential difference between the iteration formulas is that the former employs the iterated kernel of the integral equation, the latter does not, so that the convergence theorems of Part I are not applicable. Nevertheless it is proposed to use the form in (56) (or the equivalent iteration formula (58) for the following reasons:

a. The labor of numerical calculations would be greatly increased by iterating the kernel, and even then only convergence in the mean would be guaranteed (Theorem 4 of Part I).

b. The physical derivation of equation (56) indicates that at least the first few approximations should be successively improving.

c. The successive approximations are monitored so that the sequence can be stopped when the error is as small as desired or, in the case of initial convergence and then divergence, when the errors begin to grow.

<u>Velocity and Pressure Distribution on the Surface</u>. When an approximate doublet distribution $m_1(x)$ has been obtained, the velocity components u,v can be computed from the corresponding stream function (18)

$$\psi_1(x,y) = y^2\left[\int_a^b \frac{m_1(t)}{r^3}dt - \tfrac{1}{2}\right] \tag{61}$$

from which, in accordance with equations (5) and (6),

$$u = 1 + \int_a^b\left(\frac{3y^2}{r^5} - \frac{2}{r^3}\right) m_1(t)dt \tag{62}$$

and

$$v = 3y\int_a^b \frac{t-x}{r^5}\, m_1(t)dt \tag{63}$$

On the given surface we have, from (61),

$$\int_a^b \frac{m_1(t)}{r^3}dt = \tfrac{1}{2} + \frac{\psi_1(x)}{y^2(x)} \tag{64}$$

where now

$$r^2 = (x-t)^2 + f(x) \tag{65}$$

Differentiating (64) with respect to x gives

$$3\int_a^b \frac{t-x-yy'}{r^5}\, m_1(t)dt = \frac{\psi_1{}'(x)}{y^2(x)} - \frac{2\psi_1(x)y'(x)}{y^3(x)} \tag{66}$$

Hence, from (62) and (64) we obtain

$$u = 3y^2\int_a^b \frac{m_1(t)}{r^5}\, dt - \frac{2\psi_1(x)}{f(x)} \tag{67}$$

and, from (63), (66) and (67),

$$v = uy'(x) + \frac{\psi_1{}'(x)}{y(x)} \tag{68}$$

where the primes denote differentiation with respect to x. Equations (67) and (68) are the desired expressions for u and v. If the approximation $m_1(t)$ is very good, the contributions of the error function $\psi_1(x)$ should be very small.

It is interesting to note that the form of equation (68) shows the deviation of the resultant velocity from the tangent to the given body.

Bernoulli's equation for steady, incompressible, irrotational flow now gives the pressure distribution p,

$$\frac{p}{q} = 1 - (u^2+v^2) \tag{69}$$

where q is the stagnation pressure.

Numerical Evaluation of Integrals. In order to perform the iterations in equations (56) and (58) and to compute the velocity distribution it will be frequently necessary to evaluate integrals of the form

$$\int_a^b \frac{m(t)}{r^3}\,dt \quad \text{and} \quad \int_a^b \frac{m(t)}{r^5}\,dt$$

where

$$r^2 = (x-t)^2 + f(x)$$

Because, in this form, these integrals peak sharply in the neighborhood of $t = x$, especially when the body is elongated, they are consequently unsuited for numerical evaluation.

A more suitable form can be obtained by means of the following transformation. Let (x,y) be the coordinates of a point on the body, t the abscissa of a point on the axis, θ the angle between line joining these two points and the x-axis; see Figure 1. Then

$$x - t = y(x)\cot\theta \tag{70}$$

We may now transform the integrals so that θ becomes the variable of integration. Then

$$\int_a^b \frac{y^2}{r^3}\,m(t)dt = \int_\alpha^\beta m(t)\,\sin\theta d\theta \tag{71}$$

and $$\int_a^b \frac{y^4}{r^5} m(t) = \int_\alpha^\beta m(t) \sin^3\theta d\theta \tag{72}$$

where $$\alpha = \arctan \frac{y}{x-a}, \qquad \beta = \arctan \frac{y}{x-b} \tag{73}$$

An alternate procedure, which eliminates the peak without a transformation of variables, is the following. We have

$$\int_a^b \frac{y^2}{r^3} m(t)dt \equiv \int_a^b \frac{y^2}{r^3}\left[m(t)-m(x)\right] dt + m(x)\int_a^b \frac{y^2}{r^3}dt$$

and $$\int_a^b \frac{y^4}{r^5} m(t)dt \equiv \int_a^b \frac{y^4}{r^5}\left[m(t)-m(x)\right] dt + m(x)\int_a^b \frac{y^4}{r^5}dt$$

Hence $$\int_a^b \frac{y^2}{r^3} m(t)dt = \int_a^b \frac{y^2}{r^3}\left[m(t)-m(x)\right] dt + m(x)(\cos\alpha-\cos\beta) \tag{71a}$$

$$\int_a^b \frac{y^4}{r^5} m(t)dt = \int_a^b \frac{y^4}{r^5}\left[m(t)-m(x)\right] dt$$
$$+ m(x)\left[\cos\alpha-\cos\beta-\frac{1}{3}(\cos^3\alpha-\cos^3\beta)\right] \tag{72a}$$

Gauss' quadrature formula is a convenient and accurate method of evaluating these integrals. The formula may be expressed in the form

$$\int_{-1}^{1} F(\xi)d\xi = \sum_{i=1}^{n} R_{ni}F(\xi_{ni}) \tag{74}$$

where the ξ_i are the zeros of Legendre's polynomial of degree n and the R_{ni} are weighting factors. These have been tabulated[11] for values of n from 1 to 16. These numbers have the properties

$$R_{ni} = R_{n,n-i+1} \text{ and } \xi_{ni} = -\xi_{n,n-i+1} \tag{75}$$

The value of the integral given by the formula (74) is the same as could be obtained by fitting a polynomial of degree 2n-1 to F(x). The values of R_{ni} and ξ_{ni} are tabulated below for n = 7, 11, and 16.

When the limits of integration are α and β, as in equations (71) and (72), Gauss' formula becomes

$$\int_{\alpha}^{\beta} F(\theta)d\theta = \frac{\beta-\alpha}{2} \sum_{i=1}^{n} R_{ni}F(\theta_i) \tag{76}$$

where

$$\theta_i = \frac{\beta-\alpha}{2}\xi_{ni} + \frac{\alpha+\beta}{2} \tag{77}$$

TABLE 3

ABSCISSAE AND WEIGHTING FACTORS FOR

GAUSS' QUADRATURE FORMULA

	n = 7		n = 11		n = 16	
i	ξ_i	R_i	ξ_i	R_i	ξ_i	R_i
1	-.949108	.129485	-.978229	.055669	-.989401	.027152
2	-.741531	.279705	-.887063	.125580	-.944575	.062254
3	-.405845	.381830	-.730152	.186290	-.865631	.095159
4	0	.417959	-.519096	.233194	-.755404	.124629
5	$\xi_i = -\xi_{n-i+1}$	$R_i = R_{n-i+1}$	-.269543	.262805	-.617876	.149596
6			0	.272925	-.458017	.169157
7			$\xi_i = -\xi_{n-i+1}$	$R_i = R_{n-i+1}$	-.281604	.182603
8					-.095013	.189451
					$\xi_i = -\xi_{n-i+1}$	$R_i = R_{n-i+1}$

<u>Illustrative Example</u>. The foregoing considerations will now be applied to a body of revolution whose meridian profile is given, for $-1 \leqq x \leqq 1$, by

$$y^2 = f(x) = 0.04\ (1-x^4) \tag{78}$$

The body is symmetric fore and aft, has a length-diameter ratio $\lambda = 5$, and a prismatic coefficient

$$\varphi = \int_0^1 (1-x^4)dx = 0.80 \tag{79}$$

By applying to (78) the transformation

$$x = 2\xi - 1,\ y = 2\eta \tag{80}$$

We obtain the equation for the geometrically similar body of unit length, for $0 \leqq \xi \leqq 1$,

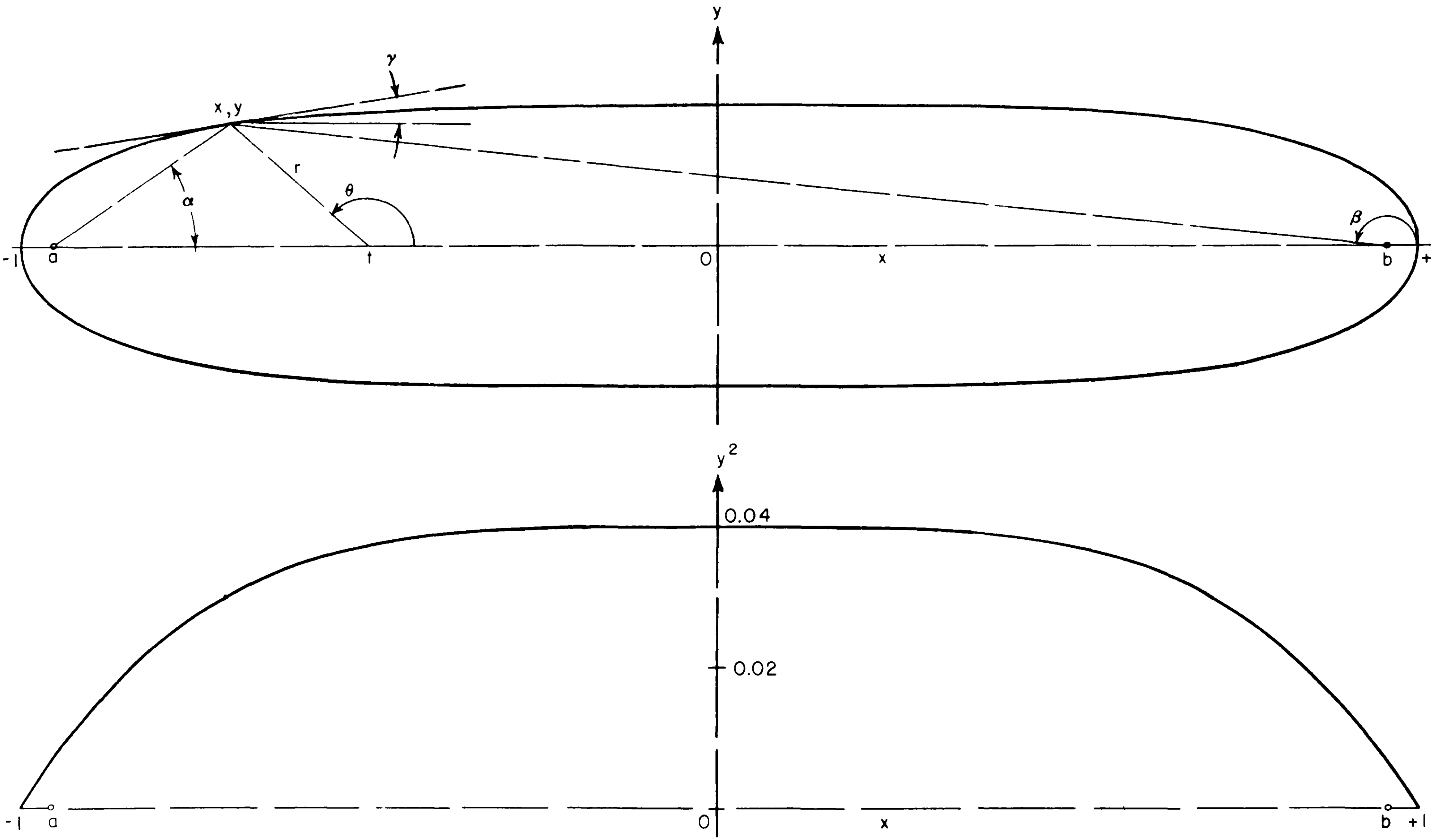


Figure 2 - Graphs of $y(x)$ and $y^2(x)$ for $y^2(x) = 0.04(1 - x^4)$

$$\eta^2 = 0.88(\xi-3\xi^2+4\xi^3-2\xi^4) = 0.08\,(1-\xi)(2\xi^2-2\xi+1) \qquad (81)$$

We will also need the slope of the profile which, from (78) is

$$y' = \frac{f'(x)}{2y} \simeq -\frac{0.4x^3}{(1-x^4)^{\frac{1}{2}}} \qquad (82)$$

The profile and f(x) are graphed in Figure 2.

First let us find the end points of the distribution. We have, from (81), $a_1 = 0.08$, $a_2 = -0.24$, $a_3 = 0.32$. The approximate formula (41) then gives $\alpha = 3.68$ or 3.84, whence $a = a_1/\alpha = 0.0217$ or 0.0208. An examination of the complete polynomial (35) with the aid of Table 1 shows that its zeros occur at $\alpha = 3.65$, 3.85, 12.1. In the application of Table 1 to determine these roots the regions of possible zeros should be determined by inspection, the values of the polynomial in these regions calculated from equation (44) and Table 1, and then graphed to obtain the zeros. It is seen that in the present case the approximate formula (41) would have been sufficiently accurate for the determination of the roots near $\alpha = 4$. The solution of the complete polynomial equation will always yield an additional large root, corresponding to the large root of equation (38); in general, however, this root should be rejected since as will be shown, the initial doublet distribution corresponding to it is less simple than for the roots near $\alpha = 4$.

The initial behavior of the distributions corresponding to each of the three roots, as determined from equations (36) and (37), is shown in the following table. It is seen from the table that the distribution for $\alpha = 12.1$ begins with practically a zero value for m(a), with a small negative slope and with up curvature. Since the distribution curve cannot

TABLE 4

CHARACTERISTICS OF INITIAL DISTRIBUTION

α	a	m(a)	c_1	c_2
3.65	.0219	.0000216	.0375	-0.103
3.85	.0208	-.0000191	.0376	-0.109
12.1	.0066	.0000008	-.0064	0.35

continue very far with up curvature, there must be an inflection point nearby. In contrast, the distribution corresponding to the other two roots begin with positive slopes and down curvatures and hence must be considered simpler. Furthermore the distribution for the first root is considered simpler than for the second since the distribution curves are practically identical except that, for the second root, the curve is extended a distance $\Delta a = .0011$, in the course of which m(a) changes from a positive to almost a numerically equal negative value. If we take the point of view that the positive and negative values of this extension counterbalance each other, the curve without the extension, corresponding to the first root, must be considered the simplest.

Hence, for the purpose of obtaining a first approximation, we will assume $\alpha = 3.65$ and, correspondingly, $a = 0.022$, $m(a) = 0.000022$. Often, as in this case, the labor of obtaining a and m(a) can be considerably reduced by using the less exact equations (41) and (42) instead of (35), (36) and (37). Since, as will be seen, the iteration formulas rapidly improve upon the first approximation, great effort should not be expended to determine an initial value for $m(a)$.

The values $a = 0.022$ and $m(a) = 0.000022$ have been derived for the profile in the ξ, η-plane. The corresponding values in the x,y-plane are $a = -0.956$ and $m_a = 0.000088$. By symmetry we also have $b = -a$, $m_b = m_a$.

A first approximation can now be obtained from (47), (48), (49) and (50). Since $\lambda = 5.0$, we have $k_1 = 0.059$. Also, from (78): $f_a = 0.00659$, $\int_{-1}^{1} y^2 dx = 0.0640$, $\int_a^b y^2 dx = 0.0637$. Hence from (50), $C = 0.328$. Then, from (48), $e_0 = m_a - Cf_a = -0.00207$; from (49), $e_1 = 0$. Finally we obtain from (47)

$$m_1(x) = 0.328y^2 - 0.00207 \qquad (83)$$

We can now apply equation (51) and the iteration formula (58) to obtain the sequence of functions $\psi_i(x)$. Let us suppose that it is desired to obtain a distribution $m_i(x)$ whose exact stream surface deviates from the given surface by less than one percent of the maximum radius, i.e. $\Delta n < 0.002$. Then, by (60), the sequence $\psi_i(x)$ should be continued until $\psi_i(x) < 0.002\sqrt{f(x)}$ for $a \leqq x \leqq b$, unless the error, as represented by $\psi(x)$, begins to grow before the desired degree of approximation is attained. In the latter case the best approximation attainable would fall short of the specified accuracy.

The integrations in (50) and (51) may be carried out in the form (71) in terms of θ defined in (70). For a fixed (x,y) on the given profile, α and β are first computed from (73). Then, to apply Gauss' quadrature formula (76), the interval is subdivided at the points θ_j given by (77) and the integrands evaluated at these points. The corresponding values of t at which $m_1(t)$ in (51) or $\psi_{i-1}(t)$ in (58) is to be read are, from (70),

$$t_j = x - y \cot \theta_j \tag{70a}$$

Since the values t_j and $\sin \theta_j$ are used repeatedly in the successive iterations at a given (x,y), these should be stored in a form convenient for application.

The calculations for obtaining the integration limits α and β for several values of x are given in Table 5. The values of θ_j from (77), and the corresponding values of $R_j \sin\theta_j$ for application of the Gauss 11 ordinate formula, and the values of t_j from (70a) for each x are entered as the first three columns in Tables 7a through 7h, in which are given the calculations for $\psi_i(x)$.

In order to compute $\psi_1(x)$, $m_1(t)$ is computed from (83), then $m_1 R \sin\theta$ is obtained. These are tabulated in Table 7. Gauss' formula then gives $\int m_1 \sin\theta d\theta$. $\psi_1(x)$ is then obtained from (51); its graph is given in Figure 3. It is important to note that $m_1(t)$ is obtained by calculation, rather than graphically, in this operation. This procedure is recommended since it gives greater accuracy in a critical step. In the subsequent operations on the ψ's considerably less percentage accuracy is required, since the ψ's are of the nature of first differences between the m's, so that graphical operations are sufficiently accurate.

As a check on the accuracy of the integration, $\psi_1(0)$ was also evaluated by two other means, with the following results:

from Gauss 7 ordinate formula $\quad \psi_1(0) = 0.001258$

from Gauss 11 ordinate formula $\quad \psi_1(0) = 0.001243$

from exact integration $\quad \psi_1(0) = 0.001243$

It is seen that the 7 ordinate formula introduces an error in the fifth decimal place.

The first step in the determination of $\psi_2(x)$ is to read the values of $\psi_1(t)$ from the graph, Figure 3. $\psi_1 R \sin\theta$ and $\int \psi_1 \sin\theta d\theta$ are then obtained. $\psi_2(x)$ is then given by (58) and graphed in Figure 3. Repeated application of this procedure gives $\psi_3(x)$ and $\psi_4(x)$ which are also graphed in Figure 3. The sequence is stopped at $\psi_4(x)$ since ψ_4 has increased appreciably over ψ_3 at $x = -0.956$.

Hence, from (59), we have the approximate distribution

$$m_4(x) = m_1(x) - \tfrac{1}{2}\left[\psi_1(x) - \psi_2(x) - \psi_3(x)\right] \qquad (84)$$

to which $\psi_4(x)$ is the corresponding error function. The distance Δn between the stream surface for $m_4(x)$ and the given profile is seen to be very small; the largest error, $\psi_4 = -0.00007$ at $x = -.956$, gives a Δn of about one per cent of the maximum ordinate. A graph of $m_4(x)$ is given in Figure 4. For the sake of comparison the curves for $m_1(x)$ and the original Munk approximation $\frac{1}{2}f(x)$ are also shown.

Table 6 shows the calculations for obtaining the velocity components u,v from (67) and (68), in which the integrals have been evaluated in terms of the polar angle θ, according to equations (71), (72) and (73). Here also Gauss' eleven ordinate formula is used. The values of θ and t are again taken from Table 7; the values of $m_4(t)$ are given by (84), in which the ψ's are read from Figure 3 and $m_1(t)$ is given in Table 7.

The pressure distribution can now be obtained from (69). Graphs of p/q are shown in Figure 5.

Error in Determination of p/q. Let $\Delta(p/q)$, Δu, Δv and Δm denote errors in p/q, u, v, and m. Then, from (69), we have

$$\Delta(p/q) = -2(u\Delta u+v\Delta v);$$

from (68),

$$\Delta v = y'\Delta u$$

and from (67) and (72), except near the stagnation points,

$$\Delta u \doteq \frac{3\Delta m}{y^2}\int_0^{\pi}\sin^3\theta d\theta = \frac{4\Delta m}{y^2}$$

Hence

$$\Delta(p/q) \doteq -\frac{8u\Delta m}{y^2}(1+y'^2)$$

If now we assume $u \cong 1$, $y' \cong 0$, $y^2 \cong 4m$ (Munk's approximation), we obtain

$$\Delta(p/q) = -2\Delta m/m.$$

Thus an error of one percent in the determination of m would introduce an error of 0.02 in p/q.

In the foregoing example the minimum value of p/q was about -0.20. Hence an error of one percent in m would have produced an error of ten percent in the minimum value of p/q. It was found, in fact, that the application of Gauss' seven ordinate rule introduced deviations in the values of p/q given by the 11 point rule of less than 0.003 for the entire body. For this reason Gauss' eleven point rule was used in the example, although the seven point rule would have sufficed if an accuracy of only .003 in p/q were required; see Figure 5.

If greater accuracy is desired the integrals can be evaluated in the forms (71a) and (72a). If the latter forms are used in conjunction with the Gauss quadrature formula the values of x should be chosen identical with the t's required by the Gauss formula. This enables the entire calculations, including the iterations and the velocity determinations, to be made arithmetically, without resort to graphical operations, so that the method becomes suitable for processing on an automatic-sequence computing machine. In order to obtain sufficient accuracy in the integrations and to obtain the velocities and pressures at a sufficient number of points along the body a Gauss formula of high order should be used, say $n = 16$. For this reason the procedure that has been illustrated in detail may be less tedious for manual application.

Comparison with Karman and Kaplan Methods. In order to compare the accuracy of the Karman method with the present one, the error function $\psi_k(x)$ was computed for a distribution derived by the Karman method, employing 14 intervals extending from $-0.98 \leq x \leq 0.98$. $\psi_k(x)$ is graphed in Figure 3. It is seen that the errors are much greater than for the present method, especially near the ends of the body. The oscillatory character of $\psi_k(x)$ is imposed by the condition that the stream function should vanish at the center of each interval. It is conceivable that the amplitude of the oscillations in $\psi_k(x)$ may remain large even when the number of intervals is greatly increased; i.e. the Karman method may give a poorer approximation when the number of source-sink segments is greatly increased. The pressure distribution obtained by the Karman method is graphed in Figure 5.

Kaplan's first method[3] was also applied to obtain the pressure distribution. Kaplan expresses the potential function φ in the form $\varphi = \sum A_n Q_n(\lambda) P_n(\mu)$ where λ and μ are confocal elliptic coordinates, $P_n(\mu)$ and $Q_n(\lambda)$ the nth Legendre and associated Legendre polynomials, and the A_n's are coefficients to be determined from a set of linear equations which express the condition that the given profile is a stream function. In the present case it was assumed that φ was expressed in terms of the first 9 Legendre functions and the A_n's determined from the conditions that the stream function should vanish at 9 prescribed points (including the stagnation points) on the body. The resulting pressure distribution is also shown in Figure 5.

TABLE 5

CALCULATIONS FOR INTEGRATION LIMITS α, β

x	x-a	x-b	y	$\tan\alpha$	$\tan\beta$	α	β	$\frac{1}{2}(\beta-\alpha)$	$\frac{1}{2}(\alpha+\beta)$
0	0.956	-0.956	0.20000	0.20921	-0.20921	0.2062	2.9354	1.3646	1.5708
-0.20	0.756	-1.156	0.19984	0.26434	-0.17287	0.2584	2.9704	1.3560	1.6144
-0.40	0.556	-1.356	0.19742	0.35507	-0.14559	0.3412	2.9970	1.3279	1.6691
-0.60	0.356	-1.556	0.18659	0.52413	-0.11992	0.4828	3.0222	1.2697	1.7525
-0.70	0.256	-1.656	0.17435	0.68105	-0.10528	0.5979	3.0367	1.2194	1.8173
-0.80	0.156	-1.756	0.15368	0.98513	-0.08752	0.7779	3.0543	1.1382	1.9161
-0.90	0.056	-1.856	0.11729	2.09446	-0.06320	1.1254	3.0785	0.9766	2.1020
-0.956	0	-1.912	0.08117	∞	-0.04245	1.5708	3.0992	0.7642	2.3350

TABLE 6

CALCULATIONS FOR PRESSURE DISTRIBUTION p/q

x	y^2	y	y'	ψ'_4	u	uy'	ψ'_4/y	v	u^2+v^2	p/q
0	.040000	.20000	.0000	.000000	1.02640	.00000	.00000	.00000	1.0535	-.0535
-0.20	.039936	.19984	.0032	-.000082	1.03441	.00331	-.00041	.00290	1.0700	-.0700
-0.40	.038976	.19742	.0259	.000060	1.05618	.02739	.00030	.02769	1.1163	-.1163
-0.60	.034816	.18659	.0926	.000306	1.07907	.09993	.00164	.10157	1.1747	-.1747
-0.70	.030396	.17435	.1574	.000317	1.07866	.16978	.00182	.17160	1.1930	-.1930
-0.80	.023616	.15368	.2665	-.000129	1.04917	.27960	-.00084	.27876	1.1785	-.1785
-0.90	.013756	.11729	.4972		.92425	.45954		.4489*	1.0557	-.0557
-0.956	.006588	.08117	.8611		.68161	.58693		.5768*	.7973	.2027

*v obtained from equation $v = \frac{3}{y}\int_{\alpha}^{\beta} m(t)\sin^2\theta\cos\theta\,d\theta$.

TABLE 7

CALCULATIONS FOR $\psi_1(x)$ AND $u(x)$

(a) $x = 0$: $\frac{1}{2}(\beta-\alpha) = 1.3646$, $y^2 = 0.0400$

θ	t	$R\sin\theta$	$m_1(t)$	$m_1(t)R\sin\theta$	$\psi_1(t)$	$\psi_1(t)R\sin\theta$	$\psi_2(t)$	$\psi_2(t)R\sin\theta$	$\psi_3(t)$	$\psi_3(t)R\sin\theta$	$m_4(t)$	$m_4R\sin^3\theta$
.2359	-.8320	.01301	.004763	.0000620	-.001307	-.0000170	-.000428	-.0000056	-.000151	-.0000020	.005706	.0000041
.3603	-.5309	.04428	.010008	.0004432	-.000370	-.0000164	-.000107	-.0000047	-.000019	-.0000008	.010256	.0000565
.5744	-.3090	.10121	.010930	.0011062	.000652	.0000660	.000188	.0000190	.000063	.0000064	.010478	.0003130
.8624	-.1713	.17708	.011039	.0019548	.001058	.0001874	.000281	.0000498	.000075	.0000133	.010332	.0010552
1.2030	-.0771	.24522	.011050	.0027097	.001198	.0002938	.000307	.0000753	.000075	.0000184	.010260	.0021907
1.5708	.0000	.27293	.011050	.0030159	.001244	.0003395	.000311	.0000849	.000071	.0000194	.010237	.0027940
1.9386	.0771	.24522	.011050	.0027097	.001198	.0002938	.000307	.0000753	.000075	.0000184	.010260	.0021907
2.2792	.1713	.17708	.011039	.0019548	.001058	.0001874	.000281	.0000498	.000075	.0000133	.010332	.0010552
2.5672	.3090	.10121	.010930	.0011062	.000652	.0000660	.000188	.0000190	.000063	.0000064	.010478	.0003130
2.7813	.5309	.04428	.010008	.0004432	-.000370	-.0000164	-.000107	-.0000047	-.000019	-.0000008	.010256	.0000565
2.9057	.8320	.01301	.004763	.0000620	-.001307	-.0000170	-.000428	-.0000056	-.000151	-.0000020	.005706	.0000041
			$\sum m_1R\sin\theta$	= .0155677	$\sum\psi_1R\sin\theta$	= .0013671	$\sum\psi_2R\sin\theta$	= .0003525	$\sum\psi_3R\sin\theta$	= .0000900	$\sum Rm\ \sin^3\theta$	= .0100330
			$\int m_1\sin\theta$	= .0212437	$\int\psi_1\sin\theta$	= .001866	$\int\psi_2\sin\theta d\theta$	= .0004810	$\int\psi_3\sin\theta d\theta$	= .0001228	$\int m\ \sin^3\theta$	= .013691
			ψ_1	= .001244	ψ_2	= .000311	ψ_3	= .000071	ψ_4	= .000010	u	= 1.0264

(b) $x = -0.20$: $\frac{1}{2}(\beta-\alpha) = 1.3560$, $y^2 = .039936$

θ	t	$R\sin\theta$	$m_1(t)$	$m_1(t)R\sin\theta$	$\psi_1(t)$	$\psi_1(t)R\sin\theta$	$\psi_2(t)$	$\psi_2(t)R\sin\theta$	$\psi_3(t)$	$\psi_3(t)R\sin\theta$	$m_4(t)$	$m_4R\sin^3\theta$
.2879	-.8748	.01580	.003366	.0000532	-.001189	-.0000188	-.000361	-.0000057	-.000112	-.0000018	.004197	.0000053
.4115	-.6579	.05023	.008592	.0004316	-.000997	-.0000501	-.000329	-.0000165	-.000109	-.0000055	.009310	.0000748
.6243	-.4774	.10889	.010369	.0011291	-.000098	-.0000107	-.000018	-.0000020	.000013	.0000014	.010421	.0003877
.9105	-.3552	.18417	.010841	.0019966	.000469	.0000864	.000141	.0000260	.000055	.0000101	.010508	.0012072
1.2489	-.2666	.24929	.010984	.0027382	.000799	.0001992	.000221	.0000551	.000070	.0000175	.010439	.0023418
1.6144	-.1913	.27266	.011032	.0030080	.001017	.0002773	.000271	.0000739	.000074	.0000202	.010351	.0028166
1.9799	-.1133	.24112	.011048	.0026639	.001152	.0002778	.000300	.0000723	.000076	.0000183	.010284	.0020874
2.3183	-.0148	.17102	.011050	.0018898	.001240	.0002121	.000310	.0000530	.000072	.0000123	.010239	.0009419
2.6045	.1356	.09531	.011046	.0010528	.001121	.0001068	.000296	.0000282	.000076	.0000072	.010299	.0002569
2.8173	.3945	.04001	.010732	.0004294	.000300	.0000120	.000101	.0000040	.000048	.0000019	.010507	.0000427
2.9409	.7823	.01110	.006136	.0000681	-.001336	-.0000148	-.000445	-.0000049	-.000159	-.0000018	.007106	.0000031
			$\sum m_1R\sin\theta$	= .0154607	$\sum\psi_1R\sin\theta$	= .0010772	$\sum\psi_2R\sin\theta$	= .0002834	$\sum\psi_3R\sin\theta$	= .0000798	$\sum Rm_4\sin^3\theta$	= .0101654
			$\int m_1\sin\theta$	= .0209647	$\int\psi_1\sin\theta d\theta$	= .001460	$\int\psi_2\sin\theta d\theta$	= .0003843	$\int\psi_3\sin\theta d\theta$	= .0001082	$\int m_4\sin^3\theta$	= .013784
			ψ_1	= .000997	ψ_2	= .000267	ψ_3	= .000075	ψ_4	= .000021	u	= 1.0344

(c) $x = -0.40$: $\frac{1}{2}(\beta-\alpha) = 1.3279$, $y^2 = 0.038976$

θ	t	$R\sin\theta$	$m_1(t)$	$m(t)R\sin\theta$	$\psi_1(t)$	$\psi_1(t)R\sin\theta$	$\psi_2(t)$	$\psi_2(t)R\sin\theta$	$\psi_3(t)$	$\psi_3(t)R\sin\theta$	$m_4(t)$	$m_4R\sin^3\theta$
.3701	-.9089	.02014	.002096	.0000422	-.000982	-.0000198	-.000272	-.0000055	-.000060	-.0000012	.002753	.0000072
.4912	-.7691	.05924	.006460	.0003827	-.001328	-.0000787	-.000440	-.0000261	-.000158	-.0000094	.007423	.0000978
.6995	-.6346	.11993	.008922	.0010700	-.000895	-.0001073	-.000292	-.0000350	-.000092	-.0000110	.009562	.0004754
.9798	-.5325	.19364	.009995	.0019354	-.000376	-.0000728	-.000109	-.0000211	-.000020	-.0000039	.010248	.0013684
1.3112	-.4524	.25400	.010500	.0026670	.000023	.0000058	.000021	.0000053	.000027	.0000069	.010464	.0024827
1.6691	-.3805	.27162	.010775	.0029267	.000360	.0000978	.000115	.0000312	.000050	.0000136	.010512	.0028278
2.0270	-.3031	.23592	.010939	.0025807	.000673	.0001588	.000191	.0000451	.000065	.0000153	.010474	.0019912
2.3584	-.2017	.16454	.011028	.0018145	.000991	.0001631	.000265	.0000436	.000075	.0000123	.010362	.0008488
2.6387	-.0411	.08979	.011050	.0009922	.001228	.0001103	.000310	.0000278	.000073	.0000066	.010244	.0002137
2.8470	.2506	.03647	.010998	.0004011	.000851	.0000310	.000233	.0000085	.000071	.0000026	.010420	.0000320
2.9681	.7264	.00961	.007397	.0000711	-.001259	-.0000121	-.000412	-.0000040	-.000147	-.0000014	.008306	.0000024
			$\Sigma m_1R\sin\theta$	= .0148836	$\Sigma\psi_1R\sin\theta$	= .0002761	$\Sigma\psi_2R\sin\theta$	= .0000698	$\Sigma\psi_3R\sin\theta$	= .0000304	$\Sigma Rm_4\sin^3\theta$	= .0103474
			$\int m_1\sin\theta$	= .0197639	$\int\psi_1\sin\theta$	= .0003666	$\int\psi_2\sin\theta$	= .0000926	$\int\psi_3\sin\theta$	= .0000404	$\int m_4\sin^3\theta$	= .013740
			ψ_1	= .000276	ψ_2	= .000093	ψ_3	= .000047	ψ_4	= .000027	u	= 1.0562

(d) $x = -0.60$: $\frac{1}{2}(\beta-\alpha) = 1.2697$, $y^2 = 0.034816$

θ	t	$R\sin\theta$	$m_1(t)$	$m_1(t)R\sin\theta$	$\psi_1(t)$	$\psi_1(t)R\sin\theta$	$\psi_2(t)$	$\psi_2(t)R\sin\theta$	$\psi_3(t)$	$\psi_3(t)R\sin\theta$	$m_4(t)$	$m_4R\sin^3\theta$
.5104	-.9333	.02719	.001095	.0000298	-.000798	-.0000217	-.000189	-.0000051	-.000011	-.0000003	.001594	.0000104
.6262	-.8580	.07360	.003940	.0002900	-.001248	-.0000919	-.000392	-.0000289	-.000132	-.0000097	.004826	.0001221
.8254	-.7722	.13689	.006385	.0008740	-.001330	-.0001821	-.000441	-.0000604	-.000159	-.0000218	.007350	.0005432
1.0934	-.6965	.20712	.007962	.0016491	-.001151	-.0002384	-.000381	-.0000789	-.000131	-.0000271	.008794	.0014369
1.4103	-.6302	.25941	.008981	.0023298	-.000873	-.0002265	-.000287	-.0000745	-.000090	-.0000233	.009606	.0024280
1.7525	-.5657	.26843	.009706	.0026054	-.000551	-.0001479	-.000168	-.0000451	-.000042	-.0000113	.010087	.0026190
2.0947	-.4922	.22756	.010280	.0023393	-.000170	-.0000387	-.000041	-.0000093	.000005	.0000011	.010383	.0017715
2.4116	-.3915	.15551	.010742	.0016705	.000311	.0000484	.000102	.0000159	.000049	.0000076	.010511	.0007271
2.6796	-.2253	.08303	.011016	.0009147	.000928	.0000771	.000251	.0000208	.000072	.0000060	.010390	.0001714
2.8788	.0936	.03263	.011049	.0003605	.001179	.0000385	.000302	.0000099	.000075	.0000024	.010271	.0000226
2.9946	.6602	.00816	.008558	.0000698	-.001005	-.0000082	-.000332	-.0000027	-.000110	-.0000009	.009282	.0000016
			$\Sigma m_1R\sin\theta$	= .0131329	$\Sigma\psi_1R\sin\theta$	= -.0007914	$\Sigma\psi_2R\sin\theta$	= -.0002583	$\Sigma\psi_3R\sin\theta$	= -.0000773	$\Sigma Rm_4\sin^3\theta$	= .0098538
			$\int m_1\sin\theta$	= .0166748	$\int\psi_1\sin\theta$	= -.0010048	$\int\psi_2\sin\theta$	= -.0003280	$\int\psi_3\sin\theta$	= -.0000981	$\int m_4\sin^3\theta$	= .012511
			ψ_1	= .000733	ψ_2	= -.000231	ψ_3	= -.000067	ψ_4	= -.000018	u	= 1.0791

(e) $x = -0.70$: $\frac{1}{2}(\beta-\alpha) = 1.2194$, $y^2 = 0.030396$

θ	t	$R\sin\theta$	$m_1(t)$	$m_1(t)R\sin\theta$	$\psi_1(t)$	$\psi_1(t)R\sin\theta$	$\psi_2(t)$	$\psi_2(t)R\sin\theta$	$\psi_3(t)$	$\psi_3(t)R\sin\theta$	$m_4(t)$	$m_4R\sin^3\theta$
.6244	-.9420	.03254	.000719	.0000234	-.000711	-.0000231	-.000154	-.0000050	.000008	.0000003	.001148	.0000127
.7356	-.8926	.08426	.002722	.0002294	-.001093	-.0000921	-.000318	-.0000268	-.000086	-.0000072	.003471	.0001317
.9270	-.8308	.14899	.004799	.0007150	-.001309	-.0001950	-.000428	-.0000638	-.000152	-.0000226	.005744	.0005475
1.1843	-.7710	.21598	.006414	.0013853	-.001329	-.0002870	-.000441	-.0000952	-.000159	-.0000343	.007379	.0013672
1.4886	-.7144	.26191	.007633	.0019992	-.001219	-.0003193	-.000400	-.0001048	-.000140	-.0000367	.008513	.0022144
1.8173	-.6561	.26469	.008619	.0022814	-.000989	-.0002618	-.000327	-.0000866	-.000108	-.0000286	.009331	.0023229
2.1460	-.5870	.22052	.009492	.0020932	-.000663	-.0001462	-.000206	-.0000454	-.000058	-.0000128	.009956	.0015458
2.4503	-.4893	.14866	.010298	.0015309	-.000155	-.0000230	-.000035	-.0000052	.000008	.0000012	.010389	.0006277
2.7076	-.3238	.07833	.010906	.0008543	.000559	.0000438	.000172	.0000135	.000062	.0000049	.010509	.0001455
2.8990	.0045	.03016	.011050	.0003333	.001242	.0000375	.000310	.0000093	.000073	.0000022	.010237	.0000178
3.0102	.6192	.00729	.009121	.0000665	-.000821	-.0000060	-.000266	-.0000019	-.000080	-.0000006	.009705	.0000012
			$\Sigma m_1R\sin\theta = .0115119$		$\Sigma\psi_1R\sin\theta = -.0012722$		$\Sigma\psi_2R\sin\theta = -.0004119$		$\Sigma\psi_3R\sin\theta = -.0001342$		$\Sigma Rm_4\sin^3\theta = .0089344$	
			$\int m_1\sin\theta = .0140376$		$\int\psi_1\sin\theta = -.0015513$		$\int\psi_2\sin\theta = -.0005023$		$\int\psi_3\sin\theta = -.0001636$		$\int m_4\sin^3\theta = .010895$	
			$\psi_1 = -.001160$		$\psi_2 = -.000384$		$\psi_3 = -.000133$		$\psi_4 = -.000051$		$u = 1.0787$	

(f) $x = -0.80$: $\frac{1}{2}(\beta-\alpha) = 1.1382$, $y^2 = 0.023616$

θ	t	$R\sin\theta$	$m_1(t)$	$m_1(t)R\sin\theta$	$\psi_1(t)$	$\psi_1(t)R\sin\theta$	$\psi_2(t)$	$\psi_2(t)R\sin\theta$	$\psi_3(t)$	$\psi_3(t)R\sin\theta$	$m_4(t)$	$m_4R\sin^3\theta$
.8027	-.9485	.04004	.000431	.0000173	-.000655	-.0000262	-.000127	-.0000051	.000021	.0000008	.000812	.0000168
.9064	-.9204	.09887	.001634	.0001616	-.000896	-.0000886	-.000233	-.0000230	-.000037	-.0000037	.002217	.0001359
1.0850	-.8811	.16474	.003143	.0005178	-.001158	-.0001908	-.000347	-.0000572	-.000102	-.0000168	.003947	.0005084
1.3253	-.8385	.22619	.004565	.0010326	-.001297	-.0002934	-.000421	-.0000952	-.000149	-.0000337	.005499	.0011704
1.6093	-.7941	.26262	.005833	.0015319	-.001339	-.0003516	-.000446	-.0001171	-.000158	-.0000415	.006805	.0017847
1.9161	-.7447	.25683	.007015	.0018017	-.001299	-.0003336	-.000427	-.0001097	-.000151	-.0000388	.007954	.0018090
2.2229	-.6827	.20887	.008200	.0017127	-.001099	-.0002295	-.000363	-.0000760	-.000123	-.0000257	.008993	.0011865
2.5069	-.5913	.13826	.009446	.0013060	-.000690	-.0000954	-.000215	-.0000297	-.000061	-.0000084	.009929	.0004825
2.7472	-.4308	.07159	.010598	.0007587	.000128	.0000092	.000051	.0000037	.000037	.0000026	.010490	.0001108
2.9258	-.0989	.02689	.011049	.0002971	.001172	.0000315	.000302	.0000081	.000076	.0000020	.010274	.0000127
3.0295	.5652	.00623	.009711	.0000605	-.000548	-.0000034	-.000169	-.0000011	-.000041	-.0000003	.010090	.0000008
			$\Sigma m_1R\sin\theta = .0091979$		$\Sigma\psi_1R\sin\theta = -.0015718$		$\Sigma\psi_2R\sin\theta = -.0005023$		$\Sigma\psi_3R\sin\theta = -.0001635$		$\Sigma Rm_4\sin^3\theta = -.0072185$	
			$\int m_1\sin\theta = .0104690$		$\int\psi_1\sin\theta$ -.0017890		$\int\psi_2\sin\theta = -.0005717$		$\int\psi_3\sin\theta = -.0001861$		$\int m_4\sin^3\theta$ = .008216	
			$\psi_1 = -.001339$		ψ_2 -.000444		$\psi_3 = -.000158$		$\psi_4 = -.000065$		$u = 1.0492$	

(g) $x = -0.90$: $\frac{1}{2}(\beta-\alpha) = 0.9766$, $y^2 = 0.013756$

θ	t	$R\sin\theta$	$m_1(t)$	$m_1(t)R\sin\theta$	$\Psi_1(t)$	$\Psi_1(t)R\sin\theta$	$\Psi_2(t)$	$\Psi_2(t)R\sin\theta$	$\Psi_3(t)$	$\Psi_3(t)R\sin\theta$	$m_4(t)$	$m_4R\sin^3\theta$
1.1467	-.9530	.05074	.000228	.0000116	-.000600	-.0000304	-.000104	-.0000053	.000035	.0000018	.000563	.0000237
1.2357	-.9408	.11860	.000772	.0000916	-.000720	-.0000854	-.000154	-.0000183	.000006	.0000007	.001206	.0001276
1.3889	-.9216	.18322	.001585	.0002904	-.000888	-.0001627	-.000230	-.0000421	-.000034	-.0000062	.002161	.0003830
1.5950	-.8972	.23312	.002549	.0005942	-.001060	-.0002471	-.000305	-.0000711	-.000080	-.0000186	.003272	.0007623
1.8388	-.8678	.25342	.003609	.0009146	-.001215	-.0003079	-.000375	-.0000950	-.000121	-.0000307	.004465	.0010522
2.1020	-.8311	.23532	.004790	.0011272	-.001307	-.0003076	-.000428	-.0001007	-.000151	-.0000355	.005733	.0010030
2.3652	-.7806	.18414	.006179	.0011378	-.001335	-.0002458	-.000444	-.0000818	-.000159	-.0000293	.007148	.0006462
2.6090	-.7010	.11841	.007882	.0009333	-.001168	-.0001383	-.000388	-.0000459	-.000134	-.0000159	.008727	.0002665
2.8151	-.5536	.05974	.009818	.0005865	-.000486	-.0000290	-.000145	-.0000087	-.000034	-.0000020	.010151	.0000624
2.9683	-.2300	.02165	.011013	.0002384	.000915	.0000198	.000248	.0000054	.000072	.0000016	.010395	.0000067
3.0573	.4880	.00469	.010306	.0000483	-.000149	-.0000007	-.000033	-.0000002	.000008	.0000000	.010393	.0000003
			$\Sigma m_1R\sin\theta$ =	.0059739	$\Sigma\Psi_1R\sin\theta$ =	-.0015351	$\Sigma\Psi_2R\sin\theta$ =	-.0004637	$\Sigma\Psi_3R\sin\theta$ =	-.0001341	$\Sigma Rm_4\sin^3\theta$ =	.0043339
			$\int m_1\sin\theta$ =	.0058341	$\int\Psi_1\sin\theta$ =	-.001499	$\int\Psi_2\sin\theta$ =	-.0004528	$\int\Psi_3\sin\theta$ =	-.0001309	$\int m_4\sin^3\theta$ =	.004233
			Ψ_1 =	-.001044	Ψ_2	-.000299	Ψ_3 =	-.000073	Ψ_4 =	-.000008	u =	.9243

(h) $x = -0.956$: $\frac{1}{2}(\beta-\alpha) = 0.7642$, $y^2 = 0.006588$

θ	t	$R\sin\theta$	$m_1(t)$	$m_1(t)R\sin\theta$	$\Psi_1(t)$	$\Psi_1(t)R\sin\theta$	$\Psi_2(t)$	$\Psi_2(t)R\sin\theta$	$\Psi_3(t)$	$\Psi_3(t)R\sin\theta$	$m_4(t)$	$m_4R\sin^3\theta$
1.5872	-.9547	.05566	.000151	.0000084	-.000581	-.0000323	-.000099	-.0000055	.000039	.0000022	.000472	.0000263
1.6572	-.9490	.12512	.000409	.0000512	-.000652	-.0000816	-.000126	-.0000158	.000022	.0000028	.000787	.0000977
1.7773	-.9390	.18234	.000850	.0001550	-.000745	-.0001358	-.000167	-.0000305	.000000	.0000000	.001306	.0002282
1.9387	-.9247	.21761	.001457	.0003171	-.000866	-.0001885	-.000221	-.0000481	-.000029	-.0000063	.002015	.0003820
2.1297	-.9053	.22291	.002237	.0004986	-.001005	-.0002240	-.000281	-.0000626	-.000064	-.0000143	.002912	.0004670
2.3359	-.8782	.19703	.003246	.0006396	-.001173	-.0002311	-.000353	-.0000696	-.000107	-.0000211	.004063	.0004173
2.5421	-.8375	.14851	.004595	.0006824	-.001300	-.0001931	-.000421	-.0000625	-.000150	-.0000223	.005531	.0002623
2.7331	-.7692	.09293	.006457	.0006000	-.001328	-.0001234	-.000440	-.0000409	-.000159	-.0000148	.007421	.0001096
2.8945	-.6362	.04583	.008901	.0004079	-.000889	-.0000407	-.000297	-.0000136	-.000094	-.0000043	.009541	.0000265
3.0146	-.3288	.01611	.010897	.0001756	.000609	.0000098	.000175	.0000028	.000061	.0000010	.010474	.0000028
3.0843	.4182	.00328	.010649	.0000349	.000191	.0000006	.000070	.0000002	.000040	.0000001	.010498	.0000001
			$\Sigma m_1R\sin\theta$ =	.0035707	$\Sigma\Psi_1R\sin\theta$ =	-.0012401	$\Sigma\Psi_2R\sin\theta$ =	-.0003461	$\Sigma\Psi_3R\sin\theta$ =	-.0000770	$\Sigma Rm_4\sin^3\theta$ =	.0020198
			$\int m_1\sin\theta$ =	.0027287	$\int\Psi_1\sin\theta$ =	-.0009477	$\int\Psi_2\sin\theta$ =	-.0002645	$\int\Psi_3\sin\theta$ =	-.0000588	$\int m_4\sin^3\theta$ =	.001544
			Ψ_1 =	-.000565	Ψ_2 =	-.000091	Ψ_3 =	.000041	Ψ_4 =	.000070	u =	.6816

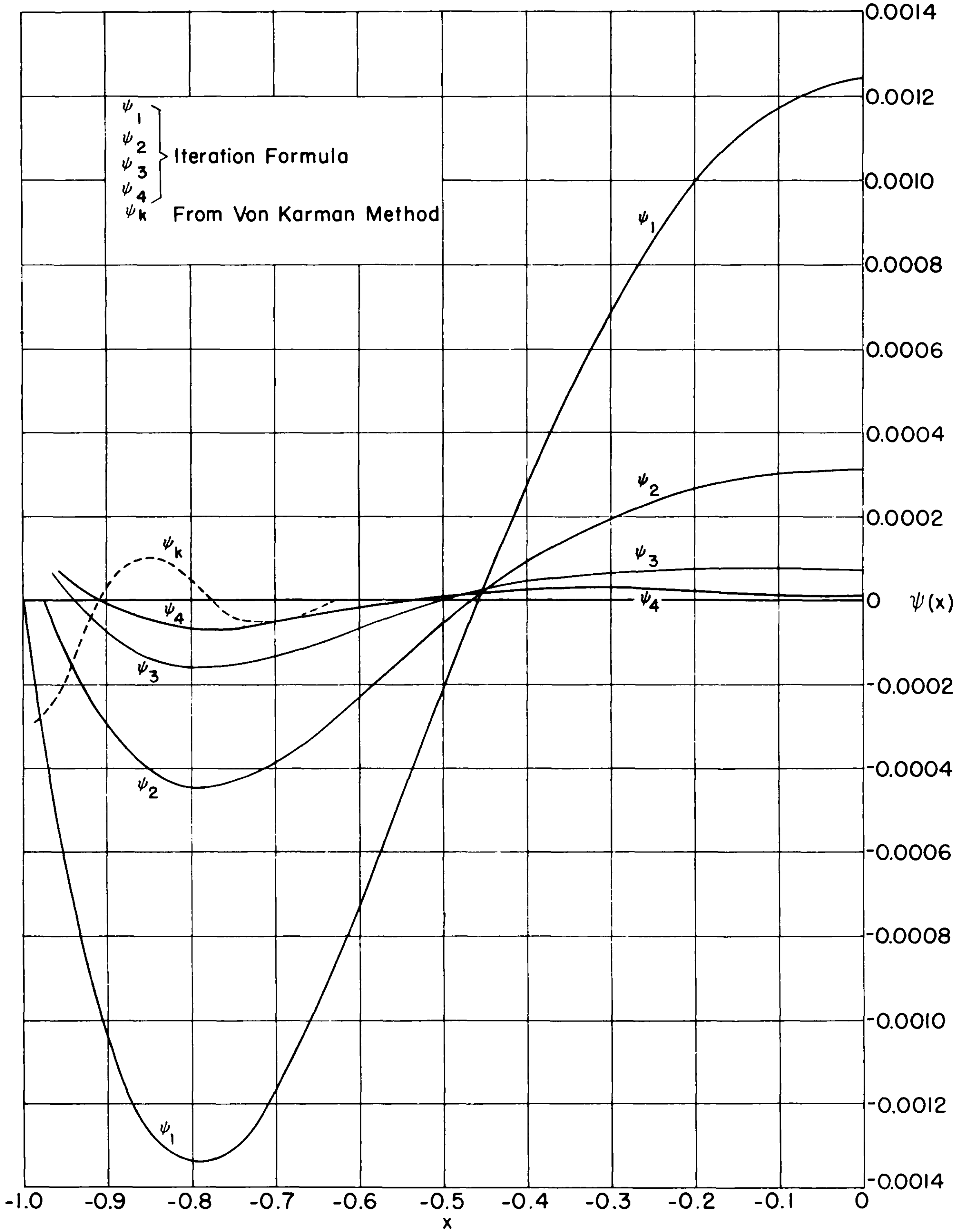


Figure 3 - Comparison of Error Functions $\psi(x)$ from Iteration Formula and Von Karman Method

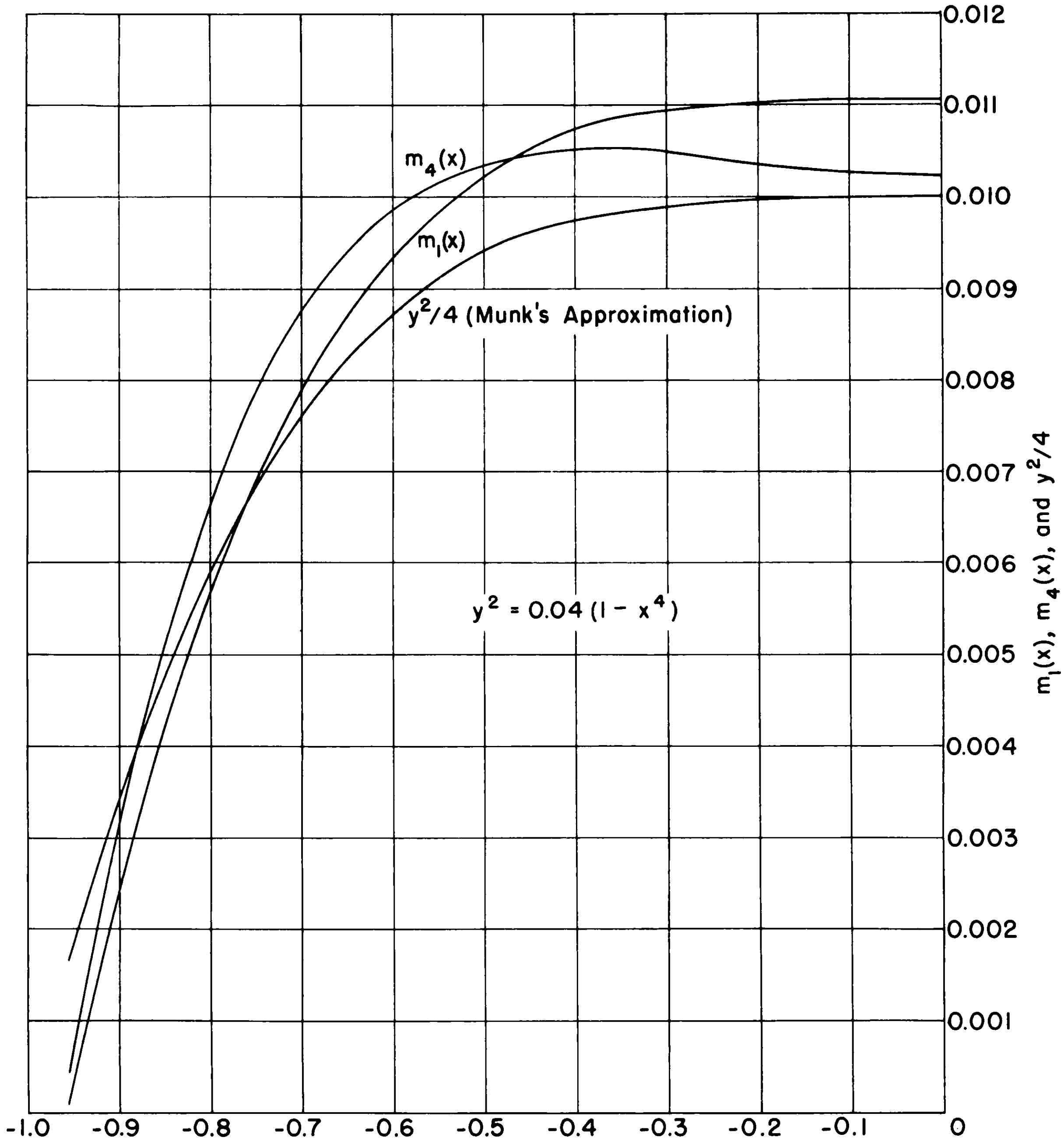


Figure 4 - Comparison of Doublet Distributions $m_1(x)$, $m_4(x)$, and Munk's Approximation $y^2/4$

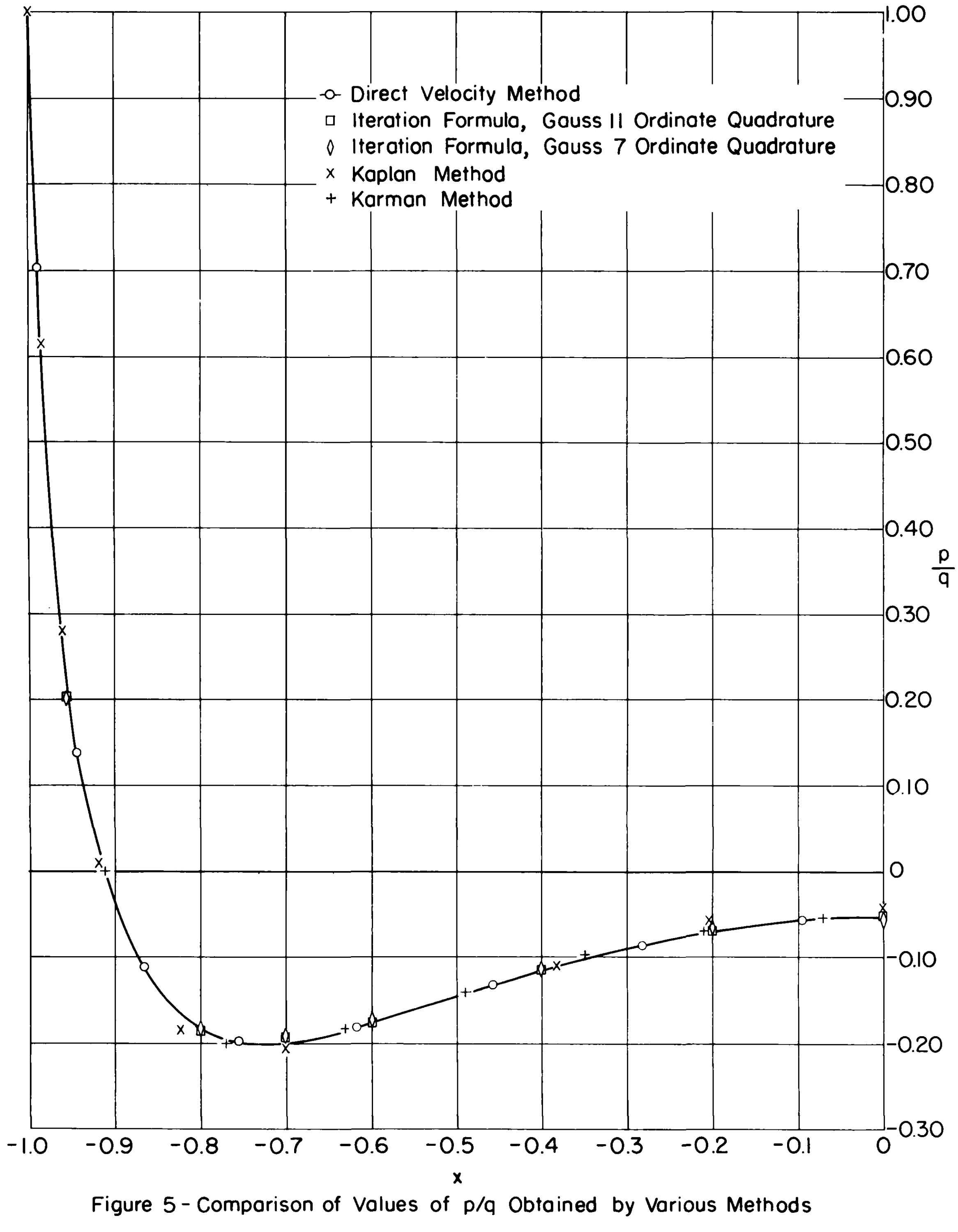


Figure 5 - Comparison of Values of p/q Obtained by Various Methods

SOLUTION BY APPLICATION OF GREEN'S THEOREM

General Application to Problems in Potential Theory. Let φ and ω be any two functions harmonic in the region exterior to a given body and vanishing at infinity. Then, a consequence of Green's second identity[6] is

$$\iint \varphi \frac{d\omega}{dn}\, dS = \iint \omega \frac{d\varphi}{dn}\, dS \tag{85}$$

where the double-integrals are taken over the boundary of the body and dn denotes an element of the outwardly-directed normal to the surface S. Also derivable from Green's formulas is the well-known expression for a potential function in terms
boundary[7]
of its values and the values of its normal derivatives on the boundary[7]

$$\varphi(Q) = \frac{1}{4\pi}\iint \left[-\frac{1}{r}\frac{d\varphi}{dn} + \varphi\frac{d}{dn}\frac{1}{r}\right] dS \tag{86}$$

where r is the distance from an arbitrary point on the body to a point Q exterior to the body.

When a distribution of φ or $\frac{d\varphi}{dn}$ over the surface of the body is given then (85) may be considered as an integral equation of the first kind for finding $\frac{d\varphi}{dn}$ or φ on the surface. If the integral equation can be solved, (86) would then give the value of φ at any point Q of the region exterior to the body.

An Integral Equation for Axisymmetric Flow. Equation (1) will now be applied to obtain an integral equation for axi-symmetric flow about a body of revolution. Let y the ordinate of a meridian section of the body and ds an element of arc length along the boundary in a meridian plane. Then we may put

$$dS = 2\pi y\, ds \tag{87}$$

It will be supposed that the body is moving with unit velocity in the negative x-direction, which is taken to coincide with the axis of symmetry. The condition that the body should be a solid boundary for the flow is that the component of the fluid velocity at the body normal to body is the same as the component of the velocity of the body normal to itself. This gives the boundary condition

$$\frac{d\varphi}{dn} = -\sin\gamma \tag{88}$$

where γ is the angle of the tangent to the body with the x-axis. Substitution of equations (87) and (88) into (85) now gives

$$\int_0^P y\varphi\frac{d\omega}{dn}\,ds = -\int_0^P y\,\omega\sin\gamma\,ds \tag{89}$$

where 2P is the perimeter of a meridian section and the arc length s is measured from the foremost point of the body.

Now let us choose for ω the potential of a doublet of unit strength situated at an arbitrary point of the axis of symmetry within the body,

$$\omega = \frac{x-t}{r^3} \tag{90}$$

where $$r^2 = (x-t)^2 + y^2.$$

Then $\dfrac{d\omega}{dn} = \dfrac{\partial}{\partial t}\dfrac{d}{dn}\dfrac{1}{r} = -\dfrac{\partial}{\partial t}\left[\dfrac{t-x}{r^3}\sin\gamma + \dfrac{y}{r^3}\cos\gamma\right]$

also $\dfrac{d}{ds}\left(\dfrac{y^2}{r^3}\right) = \dfrac{\partial}{\partial t}\dfrac{d}{ds}\dfrac{t-x}{r} = -y\dfrac{\partial}{\partial t}\left[\dfrac{t-x}{r^3}\sin\gamma + \dfrac{y}{r^3}\cos\gamma\right]$

Hence $$y\frac{d\omega}{dn} = \frac{d}{ds}\left(\frac{y^2}{r^3}\right) \tag{91}$$

The left member of (89) can now be written

$$\int_0^P y\varphi\frac{d\omega}{dn}\,ds = \int_0^P \varphi\frac{d}{ds}\left(\frac{y^2}{r^3}\right)ds = \frac{\varphi y^2}{r^3}\Big|_0^P - \int_0^P \frac{y^2}{r^3}\frac{d\varphi}{ds}ds$$

But $\varphi y^2/r^3\Big|_0^P = 0$ since y vanishes at both limits. Hence (89) becomes

$$-\int_0^P \frac{y^2}{r^3}\frac{d\varphi}{ds}\,ds = \int_0^P \frac{y(x-t)}{r^3}\sin\gamma\,ds \tag{92}$$

Equation (92) can be further simplified if we express $d\varphi/ds$ in terms of the total velocity U along the body when the flow is made steady by superposing a stream of unit velocity in the positive x-direction

$$U = -\frac{d\varphi}{ds} + \cos\gamma \tag{93}$$

Also, we have $dx = ds\cos\gamma$, $dy = ds\sin\gamma$. Then (92) may be written

$$\int_0^P U(x)\frac{y^2}{r^3}ds = \int_0^P\left[\frac{y^2}{r^3}\,dx - \frac{y(x-t)}{r^3}\,dy\right]$$

$$= \int_0^P d\left(\frac{x-t}{r}\right) = 2$$

or

$$\int_0^P \frac{U(x)y^2(x)}{2r^3}\,ds = 1 \tag{94}$$

It is seen that (94) is an integral equation of the first kind in which the unknown function is U(x) and the kernel is $y^2/(2r^3)$.

In contrast with the integral equations for source-sink or doublet distributions which can be used to obtain the potential flow about bodies of revolution, the integral equation (94) has two important advantages. The first is that a solution exists, a desirable condition which is not in general the case when a solution is attempted in terms of axial source-sink or doublet distributions. The second advantage is that (94) is expressed directly in terms of the velocity along the body so that, when U is determined, the velocity distribution along the body is immediately given by Bernoulli's equation (69). In the case of the aforementioned distributions, on the other hand, it would first be necessary to evaluate additional integrals, to obtain the velocity along the body, before the pressures could be computed.

Kennard's Derivation of the Integral Equation. A simple, physical derivation of the integral equation (94) has been given by Dr. E. H. Kennard. This will now be presented.

Imagine the body replaced by fluid at rest. Let U be the velocity on the body. Then the field of flow consists of the superposition of the uniform (unit) flow and the flow due to a vortex sheet of density U.

Now subtract the uniform flow. There remains the flow due to the vortex sheet alone, uniform inside the space originally occupied by the body, of unit magnitude.

A vortex ring of strength Uds produces at an axial point distance z from its plane the velocity

$$V = \frac{y^2 U ds}{2(y^2+z^2)^{3/2}}$$

where y is the radius of the ring. Let s be the distance of a point on the body measured along the generator from one end, in a meridian plane. The axial and radial coordinates will then be functions x(s), y(s). The velocity due to the sheet at a point t on the axis will then be

$$\int_0^P \frac{U(s)y^2(s)}{2r^3}\,ds = 1$$

where $r^2 = \left[x(s)-t\right]^2 + y^2(s)$ and P is the total length of a generator. The equivalence of this equation with (94) is evident.

A First Approximation. If we again make use of the polar transformation $x-t = y(x)\cot\theta$, (94) becomes

$$\int_0^{\pi} \frac{U(x)\ \sin^2\theta d\theta}{2\sin[\theta - \gamma(x)]} = 1 \qquad (95)$$

When $x = t$, $\theta = \pi/2$. For an elongated body the integrand in (94) peaks sharply in the neighborhood of $x = t$, so that a good approximation is obtained when $U(x)$ is replaced by $U(t)$ for the entire range of integration. Also $\gamma(x)$ will be small except near the ends of the body so that the approximation

$$\sin[\theta - \gamma(x)] \doteq \sin\theta\cos\gamma(x) \doteq \sin\theta\cos\gamma(t)$$

will also be introduced. We then obtain from (95) the approximation

$$U(t) \doteq \cos\gamma(t) \qquad (96)$$

Just as was done in the case of Munk's approximate doublet distribution we can improve upon this approximation in terms of an estimated longitudinal virtual mass coefficient for the body. For this purpose we will first derive a relation between this coefficient and the velocity distribution.

Let T be the kinetic energy of the fluid when the body is moving with unit velocity in the negative x-direction. Then

$$2T = -\rho\iint \varphi \frac{d\varphi}{dn}\, dS = 2\pi\rho\int_0^P y\varphi \sin\gamma ds$$

by (88). Integrating by parts and substituting for $d\varphi/ds$ from (93) now gives

$$2T = -\pi\rho\int_0^P y^2 \frac{d\varphi}{ds}\, ds = \pi\rho\int_0^P U(x)y^2(x)ds - \Delta$$

where Δ is the displacement of the body. But also, by definition, $2T = k_1\Delta$. Hence

$$\Delta(1+k_1) = \pi\rho\int_0^P U(x)y^2(x)ds \qquad (97)$$

This is the desired relation between k_1 and $U(x)$.

Now suppose, as a generalization of (96), that an approximate solution of the integral equation (94) is $U(x) = C\cos\gamma$.

If this value is substituted into (97), we obtain $C = 1+k_1$. Hence an improved first approximation to $U(x)$ is

$$U_1(x) = (1+k_1)\cos\gamma(x). \tag{98}$$

(98) gives an exact solution for the prolate spheroid.

Solution of Integral Equation by Iteration. In order to solve (94) by means of the iteration formula treated in Part I it would be necessary to work with the iterated kernel of this integral equation. Since this would entail considerable computational labor it is proposed to try a similar iteration formula, but employing the original kernel:

$$U_{n+1}(t) = U_n(t) + \cos\gamma(t)\left[1-\int_o^P \frac{y^2(x)}{2r^3}\,U_n(x)ds\right] \tag{99}$$

where $r^2 = (x-t)^2 + y^2(x)$ and $x = x(s)$.

Here also it is convenient to express the iterations in terms of error functions $E_n(t)$ defined by

$$E_n(t) = 1-\int_o^P \frac{U_n(x)y^2(x)}{2r^3}\,ds \tag{100}$$

or, from (99),

$$E_n(t)\cos\gamma(t) = U_{n+1}(t) - U_n(t) \tag{101}$$

Hence

$$U_{n+1}(t) = U_1(t) + \cos\gamma(t)\sum_{i=1}^{n} E_i(t) \tag{102}$$

Also, from (99),

$$E_{n+1}(t) = E_n(t) - \tfrac{1}{2}\int_{x_o}^{x_1} \frac{E_n(x)y^2(x)}{r^3} \tag{103}$$

where x_o, x_1 are the nose and tail abscissae.

Thus, to obtain $U_{n+1}(t)$, we first obtain $E_1(t)$ from $U_1(t)$ in (100), then $E_2, E_3, \ldots E_n$ from (103), and finally $U_{n+1}(t)$ from (102).

<u>Numerical Evaluation of Integrals</u>. In applying equations (100) and (103) it will frequently be necessary to evaluate integrals of the form

$$\int_{x_0}^{x_1} \frac{E(x)y^2(x)}{r^3}\,dx \text{ where } r^2 = (t-x)^2 + y^2(x).$$

This form, however, is unsuited for numerical quadrature for elongated bodies, since $y^2(x)$ peaks sharply in the neighborhood of $x = t$. Here, as in the case of the integrals for the doublet distribution, two procedures are available for avoiding this difficulty. The first employs the polar transformation (70), involves several graphical operations, but in general transforms the integrand into a slowly varying function so that the integral can be evaluated by a quadrature formula using relatively few ordinates. The second retains the original variables and eliminates the peak by subtracting from the integrand an integrable function which behaves very much like the original integrand in the neighborhood of the peak. The numerical evaluation of the resulting integral on the second method requires a quadrature formula with more ordinates than the first in order to obtain the same accuracy, but, since all graphical operations are eliminated, the second method is suitable for processing on an automatic-sequence calculating machine.

The result of the polar transformation has effectively been given in (95). We have

$$\int_{x_0}^{x_1} \frac{E(x)y^2(x)}{r^3}\,dx = \int_0^{\pi} \frac{E(x)\sin^2\theta\cos\gamma(x)}{\sin[\theta-\gamma(x)]}\,d\theta \qquad (104)$$

where $\qquad x-t = y(x)\cot\theta\,. \qquad (70)$

It is desired to evaluate this integral for a series of values of t. In general this can be done with sufficient accuracy by means of the Gauss 7 (or 11) ordinate quadrature formulas. This gives 7 (or 11) values of θ at which the integrand needs to be determined for a given t. The value of x occurring in the integrand is determined implicitly, for given values of t and θ, by the polar transformation (70). In practice the 7 (or 11) x's can be obtained graphically from the intersections with a graph of the given profile of the 7 (or 11) rays from the point $x = t$ on the axis at the angles required by the Gauss quadrature formula. If greater accuracy is desired, these graphically determined values of x can be corrected by means of the formula

$$x = x_g + \frac{t-x_g+y(x_g)\cot\theta}{1-y'(x_g)\cot\theta} \qquad (105)$$

in which x_g is the graphically determined value and y' denotes the derivative of y with respect to x.

The alternate procedure for evaluating the integral consists of expressing it in the form

$$\int_{x_0}^{x_1} \frac{y^2(x)}{r^3} E(x)dx = E(t)(\cos\alpha-\cos\beta) + \int_{x_0}^{x_1}\left[k(x,t)E(x)-k(t,x)E(t)\right]dx \qquad (106)$$

where $k(x,t) = \dfrac{y^2(x)}{[(x-t)^2+y^2(x)]^{3/2}} = \dfrac{\sin^3\theta(x,t)}{y(x)}$ (107)

and

$$\alpha = \arctan\frac{y(t)}{1+t}, \quad \beta = \pi - \arctan\frac{y(t)}{1-t} \qquad (108)$$

Then, from (98), (100) and (106) we obtain for $E_1(t)$

$$E_1(t) = 1 - \frac{1+k_1}{2}(\cos\alpha - \cos\beta) - \frac{1+k_1}{2}\int_{x_0}^{x_1}\left[k(x,t)-k(t,x)\right]dx \quad (109)$$

and from (103), (106) and (109),

$$E_{n+1}(t) = \frac{k_1+E_1(t)}{1+k_1}E_n(t) - \frac{1}{2}\int_{x_0}^{x_1}k(x,t)\left[E_n(x)-E_n(t)\right]dx \quad (110)$$

Illustrative Example. The present method will now be applied to the same profile (78) as before. By way of contrast with the semi-graphical procedures previously used, a completely arithmetical procedure will be employed.

The velocity $U(t)$ will be determined at the 16 points along the body whose abscissae are $t_i = \xi_i$, the Gaussian values for the 16 point quadrature rule, Table 3. Since the body is symmetrical fore and aft, it is necessary to determine the velocity at only half of these points. Values of $y(x)$, $\cos\gamma(x)$ and $(\cos\alpha - \cos\beta)$ for these points are given in Table 8.

In order to apply the Gauss 16 ordinate rule it is necessary to evaluate the integrands in (106) and (107) at the 16 Gaussian abscissae $x_j = \xi_j$ for each of the 8 values of t_i. Thus there are 16 x 8 = 128 values of θ to be determined from (109), which give the same number of values of the kernel

$$k(x_j,t_i) = \frac{y^2(x_j)}{\left[(x_j-t_i)^2+y^2(x_j)\right]^{3/2}}$$

This matrix of values is given in Table 9 and applied to evaluate $E_1(t)$ from (109). E_2, E_3 and E_4 are then obtained from (110). $U_5(t)$ is then given by (102) and then p/q by (69), in the form $p/q = 1 - U_5^2$. The arrangement of the calculations and the results are given in Table 10. The graph of p/q is included in Figure 5.

TABLE 8

VALUES OF y, $\cos\gamma$ AND ($\cos\alpha - \cos\beta$) FOR APPLICATION OF GAUSS 16 POINT QUADRATURE FORMULA

x	y(x)	y'(x)	γ(x)	$\cos\gamma$(x)	$\cos\alpha - \cos\beta$
-.9894009	.0408548	1.8965483	1.0856	0.4664	1.25085
-.9445750	.0903198	0.7464764	0.6412	0.8014	1.52195
-.8656312	.1324422	0.3917981	0.3734	0.9311	1.70968
-.7554044	.1642411	0.2099651	0.2070	0.9787	1.82586
-.6178762	.1848527	0.1020867	0.1017	0.9948	1.89375
-.4580168	.1955501	0.0393076	0.03932	0.9992	1.93175
-.2816036	.1993706	0.0089607	0.008961	1.0000	1.95169
-.0950125	.1999919	0.0003431	0.0003431	1.0000	1.96015

TABLE 9

MATRIX OF VALUES* OF $k_{ji} = \dfrac{y^2(x_j)}{[(x_j - t_i)^2 + y^2(x_j)]^{3/2}}$

j \ i	1	2	3	4	5	6	7	8
1	24.4769	7.4814	0.75381	0.12453	0.03197	0.01103	0.00468	0.00233
2	7.9571	11.0718	4.7258	0.88558	0.20948	0.06731	0.02723	0.01308
3	2.9448	4.7853	7.5505	3.4286	0.79113	0.22280	0.08167	0.03669
4	1.1545	1.7156	3.4856	6.0886	2.7441	0.68796	0.21392	0.08560
5	0.47818	0.64606	1.1568	2.7937	5.4097	2.3411	0.60474	0.20034
6	0.21065	0.26520	0.41384	0.84811	2.3732	5.1138	2.0933	0.54549
7	0.09997	0.11979	0.16913	0.29264	0.66530	2.10681	5.01577	1.95219
8	0.05196	0.06016	0.07926	0.12175	0.22799	0.56183	1.95461	5.00020
9	0.02983	0.03371	0.04234	0.05999	0.09854	0.19666	0.51583	1.90499
10	0.01867	0.02073	0.02518	0.03375	0.05083	0.08843	0.18638	0.51368
11	0.01227	0.01346	0.01596	0.02060	0.02924	0.04653	0.08540	0.18946
12	0.00807	0.00877	0.01023	0.01284	0.01752	0.02627	0.04413	0.08554
13	0.00501	0.00541	0.00624	0.00769	0.01020	0.01469	0.02331	0.04152
14	0.00273	0.00293	0.00335	0.00408	0.00531	0.00745	0.01139	0.01924
15	0.00112	0.00121	0.00137	0.00165	0.00213	0.00294	0.00439	0.00718
16	0.00022	0.00023	0.00026	0.00031	0.00040	0.00055	0.00081	0.00131

* For $i > 8$ use $k_{ji} = k_{17-j,\,17-i}$

TABLE 10

CALCULATIONS FOR $E_n(t)$ AND $U(t)$

Assume $K_1 = 0.06$: Put $k_{ji} = R_j k_{ji}$, $K'_{ji} = R_j k_{ij}$

$$E_n(x_j) = E_n(t_j) = E_{nj}$$

(a) $x_1 = -.989401$; $\cos\gamma = .4664$

j	K_{j1}	K'_{j1}	$K_{j1}-K'_{j1}$	$K_{j1}(E_{1j}-E_{11})$	$K_{j1}(E_{2j}-E_{21})$	$K_{j1}(E_{3j}-E_{31})$
1	.66460			0	0	0
2	.49536			-.00337	-.00104	-.00024
3	.28022			-.00504	-.00141	-.00064
4	.14389			-.00479	-.00160	-.00053
5	.07154			-.00372	-.00118	-.00038
6	.03563			-.00256	-.00078	-.00024
7	.01825			-.00164	-.00048	-.00014
8	.00984			-.00099	-.00028	-.00008
9	.00565			-.00057	-.00016	-.00005
10	.00341			-.00031	-.00009	-.00003
11	.00208			-.00015	-.00005	-.00001
12	.00121			-.00006	-.00002	-.00001
13	.00062			-.00002	-.00001	0
14	.00027			0	0	0
15	.00007			0	0	0
16	.00001			0	0	0
$\frac{k_1+E_{11}}{1+k_1}$ =	.13000		E_{11} = .07780*	$\int$ = -.02342 E_{21}=.02182	$\int$ = -.00710 E_{31}=.00639	$\int$ = -.00235 E_{41}=.00201

$U_5(x_1) = 0.5448$, $p/q = 0.7032$

(b) $x_2 = -.944575$; $\cos\gamma = .8014$

j	K_{j2}	K'_{j2}	$K_{j2}-K'_{j2}$	$K_{j2}(E_{1j}-E_{12})$	$K_{j2}(E_{2j}-E_{22})$	$K_{j2}(E_{3j}-E_{32})$
1	.20313			.00014	.00042	.00010
2	.68926			0	0	0
3	.45536			-.00510	-.00198	-.00081
4	.21382			-.00566	-.00193	-.00068
5	.09665			-.00436	-.00139	-.00046
6	.04486			-.00292	-.00088	-.00028
7	.02187			-.00181	-.00053	-.00016
8	.01140			-.00107	-.00030	-.00009
9	.00639			-.00060	-.00017	-.00005
10	.00379			-.00031	-.00009	-.00003
11	.00228			-.00015	-.00004	-.00001
12	.00131			-.00006	-.00002	-.00001
13	.00067			-.00002	-.00001	0
14	.00028			0	0	0
15	.00008			0	0	0
16	.00001			0	0	0
$\frac{k_1+E_{12}}{1+k_1}$ =	.12358		E_{12} = .0710*	$\int$ = -.02192 E_{22}=.01973	$\int$ = -.00692 E_{32}=.00590	$\int$ = -.00248 E_{42}=.00197

$U_5(x_2) = 0.9285$, $p/q = 0.1379$

* Present procedure inaccurate. E_{11} and E_{12} obtained from (104).

(c) $x_3 = -.865631$, $\cos\gamma = 0.9311$

j	K_{j3}	K'_{j3}	$K_{j3}-K'_{j3}$	$K_{j3}(E_{1j}-E_{13})$	$K_{j3}(E_{2j}-E_{23})$	$K_{j3}(E_{3j}-E_{33})$
1	.02047			.00037	.00013	.00005
2	.29420			.00330	.00127	.00053
3	.71850			0	0	0
4	.43441			-.00663	-.00203	-.00061
5	.17306			-.00587	-.00174	-.00052
6	.07001			-.00377	-.00108	-.00030
7	.03088			-.00221	-.00061	-.00016
8	.01502			-.00124	-.00033	-.00009
9	.00802			-.00066	-.00018	-.00005
10	.00460			-.00033	-.00009	-.00002
11	.00270			-.00015	-.00004	-.00001
12	.00153			-.00005	-.00001	0
13	.00078			-.00001	0	0
14	.00032			0	0	0
15	.00009			0	0	0
16	.00001			0	0	0
$\frac{k_1+E_{13}}{1+k_1}$ =	.11302		E_{13} = .05980*	$\int = -.01725$ $E_{23}=.01538$	$\int = -.00471$ $E_{33}=.00410$	$\int = -.00123$ $E_{43}=.00108$

$U_5(x_3) = 1.0598$, $p/q = -0.1123$

(d) $x_4 = -.755404$, $\cos\gamma = .9787$, $1-0.53(\cos\alpha-\cos\beta) = .03229$

j	K_{j4}	K'_{j4}	$K_{j4}-K'_{j4}$	$K_{j4}(E_{1j}-E_{14})$	$K_{j4}(E_{2j}-E_{24})$	$K_{j4}(E_{3j}-E_{34})$
1	.00338	.03135	-.02797	.00011	.00004	.00001
2	.05513	.10680	-.05167	.00146	.00050	.00018
3	.32626	.33169	-.00543	.00498	.00153	.00046
4	.75882	.75882	0	0	0	0
5	.41794	.41052	.00742	-.00780	-.00226	-.00067
6	.14347	.11638	.02709	-.00554	-.00153	-.00042
7	.05344	.03906	.01438	-.00301	-.00080	-.00021
8	.02307	.01622	.00685	-.00155	-.00041	-.00010
9	.01137	.00787	.00351	-.00076	-.00020	-.00005
10	.00616	.00426	.00190	-.00035	-.00009	-.00002
11	.00348	.00249	.00099	-.00013	-.00004	-.00001
12	.00192	.00153	.00039	-.00004	-.00001	0
13	.00096	.00096	0	0	0	0
14	.00039	.00059	-.00020	.00001	0	0
15	.00010	.00034	-.00024	0	0	0
16	.00001	.00014	-.00013	0	0	0
$\frac{k_1+E_{14}}{1+k_1}$ =	.09862		$\int = -.02311$ $E_{14}=.04454$	$\int = -.01262$ $E_{24}=.01070$	$\int = -.00327$ $E_{34}=.00270$	$\int = -.00083$ $E_{44}=.00069$

$U_5(x_4) = 1.0948$, $p/q = -0.1986$

* Present procedure inaccurate. E_{13} obtained from (104).

(e) $x_5 = -.617876$, $\cos\gamma = .9948$, $1-.53(\cos\alpha-\cos\beta) = -.00369$

j	K_{j5}	K'_{j5}	$K_{j5}-K'_{j5}$	$K_{j5}(E_{1j}-E_{15})$	$K_{j5}(E_{2j}-E_{25})$	$K_{j5}(E_{3j}-E_{35})$
1	.00087	.01298	-.01211	.00005	.00001	.00006
2	.01304	.04022	-.02718	.00059	.00019	.00006
3	.07528	.11008	-.03480	.00255	.00076	.00023
4	.34200	.34818	-.00618	.00639	.00185	.00055
5	.80929	.80931	0	.00000	.00000	.00000
6	.40145	.39602	.00543	-.00800	-.00212	-.00054
7	.12148	.11043	.01105	-.00458	-.00117	-.00028
8	.04319	.03795	.00524	-.00210	-.00053	-.00012
9	.01867	.01621	.00246	-.00091	-.00023	-.00005
10	.00928	.00806	.00122	-.00035	-.00009	-.00002
11	.00495	.00444	.00051	-.00010	-.00003	-.00001
12	.00262	.00262	0	-.00000	.00000	.00000
13	.00127	.00160	-.00033	.00002	.00001	.00000
14	.00051	.00097	-.00046	.00002	.00001	.00000
15	.00013	.00055	-.00042	.00001	.00000	.00000
16	.00001	.00022	-.00021	.00000	.00000	.00000
$\frac{k_1+E_{15}}{1+k_1}$ = .08101			$\int = -.05578$ $E_{15} = .02587$	$\int = -.00641$ $E_{25} = .00530$	$\int = -.00134$ $E_{35} = .00110$	$\int = -.00018$ $E_{45} = .00018$

$U_5(x_5) = 1.0868$, $p/q = -0.1811$

(f) $x_6 = -.458017$, $\cos\gamma = .9992$, $1-.53(\cos\alpha-\cos\beta) = -.02383$

j	K_{j6}	K'_{j6}	$K_{j6}-K'_{j6}$	$K_{j6}(E_{1j}-E_{16})$	$K_{j6}(E_{2j}-E_{26})$	$K_{j6}(E_{3j}-E_{36})$
1	.00030	.00572	-.00542	.00002	.00001	.00000
2	.00419	.01651	-.01232	.00027	.00008	.00003
3	.02120	.03938	-.01818	.00114	.00033	.00009
4	.08574	.10570	-.01996	.00331	.00092	.00025
5	.35023	.35503	-.00480	.00698	.00185	.00047
6	.86505	.86505	-.00000	.00000	.00000	.00000
7	.38470	.38224	.00246	-.00683	-.00168	-.00037
8	.10644	.10334	.00310	-.00304	-.00073	-.00016
9	.03726	.03589	.00137	-.00107	-.00026	-.00005
10	.01615	.01560	.00055	-.00029	-.00007	-.00002
11	.00787	.00787	.00000	.00000	.00000	-.00000
12	.00393	.00437	-.00044	.00008	.00002	.00001
13	.00183	.00257	-.00074	.00007	.00002	.00001
14	.00071	.00152	-.00081	.00004	.00001	.00000
15	.00018	.00083	-.00066	.00001	.00000	.00000
16	.00001	.00033	-.00032	.00000	.00000	.00000
$\frac{k_1+E_{16}}{1+k_1}$ = .06221			$\int = -.05617$ $E_{16} = .00594$	$\int = .00070$ $E_{26} = .00002$	$\int = .00050$ $E_{36} = .00025$	$\int = .00026$ $E_{46} = -.00015$

$U_5(x_6) = 1.0647$, $p/q = -0.1336$

(g) $x_7 = -.281604$, $\cos\gamma = 1.000$, $1-.53(\cos\alpha - \cos\beta) = -.03440$

j	K_{j7}	K'_{j7}	$K_{j7}-K'_{j7}$	$K_{j7}(E_{1j}-E_{17})$	$K_{j7}(E_{2j}-E_{27})$	$K_{j7}(E_{3j}-E_{37})$
1	.00013	.00271	-.00258	.00001	0	0
2	.00170	.00749	-.00576	.00014	.00004	.00001
3	.00777	.01609	-.00832	.00056	.00015	.00004
4	.02666	.03647	-.00981	.00150	.00040	.00010
5	.09047	.09953	-.00906	.00341	.00087	.00021
6	.35410	.35639	-.00220	.00629	.00155	.00034
7	.91588	.91589	0	0	0	0
8	.37030	.36984	.00046	-.00401	-.00093	-.00020
9	.09772	.09732	.00040	-.00106	-.00025	-.00005
10	.03403	.03403	0	0	0	0
11	.01445	.01496	-.00051	.00026	.00006	.00001
12	.00660	.00760	-.00100	.00025	.00006	.00002
13	.00291	.00421	-.00130	.00016	.00004	.00001
14	.00108	.00240	-.00132	.00008	.00002	.00001
15	.00027	.00129	-.00102	.00002	.00001	0
16	.00002	.00051	-.00049	0	0	0
$\frac{k_1+E_{17}}{1+k_1} = .04545$			$\int = -.04260$ $E_{17} = -.01182$	$\int = .00761$ $E_{27} = -.00435$	$\int = .00202$ $E_{37} = -.00121$	$\int = .00050$ $E_{47} = -.00030$

$U_5(x_7) = 1.0423$, $p/q = -0.0864$

(h) $x_8 = -.095013$, $\cos\gamma = 1.0000$, $1-.53(\cos\alpha - \cos\beta) = -.03888$

j	K_{j8}	K'_{j8}	$K_{j8}-K'_{j8}$	$K_{j8}(E_{1j}-E_{18})$	$K_{j8}(E_{2j}-E_{28})$	$K_{j8}(E_{3j}-E_{38})$
1	.00006	.00141	-.00135	.00001	0	0
2	.00081	.00375	-.00294	.00008	.00002	.00001
3	.00349	.00754	-.00405	.00029	.00008	.00002
4	.01067	.01517	-.00450	.00072	.00019	.00005
5	.02997	.03411	-.00414	.00145	.00036	.00009
6	.09228	.09504	-.00276	.00264	.00063	.00014
7	.35647	.35691	-.00044	.00386	.00097	.00019
8	.94729	.94729	0	0	0	0
9	.36090	.36090	0	0	0	0
10	.09380	.09419	-.00039	.00102	.00025	.00005
11	.03205	.03327	-.00122	.00092	.00022	.00005
12	.01280	.01474	-.00194	.00062	.00016	.00004
13	.00518	.00748	-.00230	.00035	.00009	.00002
14	.00183	.00403	-.00220	.00015	.00004	.00001
15	.00045	.00210	-.00165	.00004	.00001	0
16	.00004	.00081	-.00077	0	0	0
$\frac{k_1+E_{18}}{1+k_1} = .03525$			$\int = -.03065$ $E_{18} = -.02264$	$\int = .01213$ $E_{28} = -.00686$	$\int = .00302$ $E_{38} = -.00171$	$\int = .00067$ $E_{48} = -.00034$

$U_5(x_8) = 1.0284$, $p/q = -.0576$

SUMMARY

Two new methods for computing the steady, irrotational, axisymmetric flow of a perfect, incompressible fluid about a body of revolution are presented.

In the first method a continuous, axial distribution of doublets which generates the prescribed body in a uniform stream is sought as a solution of the integral equation

$$\int_a^b \frac{m(t)}{r^3}\, dt = \frac{1}{2}$$

where r is the distance from a point (t,o) on the axis to a point (x,y) on the body, $r^2 = (x-t)^2 + y^2(x)$.

A method of determining the end points of the distribution and the values of the distribution at the end points is given. If the equation of the body profile, with the origin of coordinates at one end, is

$$y^2(x) = a_1x + a_2x^2 + a_3x^3 + \ldots$$

a very good approximation for the distribution limit a at that end, when the coefficients a_1, $a_2 \ldots$ are small, is given by

$$\frac{a_1}{a} = 4 + a_2 + \frac{1}{2}\sqrt{a_1a_3}$$

if $a_3 \geqq 0$. If a_3 is negative, the term containing it is neglected. The corresponding value of the doublet strength at this point is

$$m(a) = \frac{1}{8}\left(1 + \frac{a_1}{2} + \frac{a_2}{2}\log\frac{a_1}{4}\right) a^2\sqrt{a_1a_3}$$

Formulas and tables for determining a and m(a), which may be used when the above procedure is insufficiently accurate, are also given. The values a, b, $m_a = m(a)$, $m_b = m(b)$, $f_a = y^2(a)$ and $f_b = y^2(b)$ are then used to obtain the approximate

solution of the integral equation

$$m_1(x) = C\left(y^2 - \frac{b-x}{b-a}f_a - \frac{x-a}{b-a}f_b\right) + \frac{b-x}{b-a}m_a + \frac{x-a}{b-a}m_b$$

where

$$C = \frac{\frac{1+k_1}{4}\int_{x_0}^{x_1} y^2dx - \frac{1}{2}(b-a)(m_a+m_b)}{\int_a^b y^2dx - \frac{1}{2}(b-a)(f_a+f_b)}$$

and k_1 is the longitudinal virtual mass coefficient for the body.

This approximation is used to obtain a sequence of successive approximations by means of the iteration formula

$$m_{i+1}(x) = m_i(x) + \tfrac{1}{2}y^2(x)\left[\tfrac{1}{2} - \int_a^b \frac{m_i(t)}{r^3}dt\right]$$

When a doublet distribution has been assumed, the velocity components at a point (x,y) in a meridian plane are

$$u = 1 + \int_a^b \left(\frac{3y^2}{r^5} - \frac{2}{r^3}\right) m(t)dt$$

$$v = 3y \int_a^b \frac{t-x}{r^5} m(t)dt$$

and the pressure is given by

$$p/q = 1 - (u^2+v^2)$$

where q is the stagnation pressure.

The iterations are most conveniently performed in terms of the differences between successive approximations to $m(x)$, which also furnish, at each iteration, a geometric measure of the accuracy of an approximation. Simpler forms for the velocity components at the surface of the body are given in terms of this difference or error function.

Gauss' quadrature formulas are recommended for the numerical evaluation of the integrals. Two methods of carrying out the iterations are given. The first employs a polar transformation and a graphical operation between successive iterations;

the second is completely arithmetical and is suitable for processing on an automatic-sequence computing machine. All of these procedures are illustrated in detail by an example, in which the semi-graphical method is employed. The accuracy of the method is analyzed; the results are compared with those obtained by the methods of Karman and Kaplan.

In the second method the velocity U(x) on the surface of the given body is given directly as the solution of the integral equation

$$\int_0^P \frac{U(x)y^2(x)}{2r^3}\,ds = 1$$

where s is arc length along the profile, $x = x(s)$, and 2P is the perimeter of a meridian section. An approximate solution to this integral equation is

$$U_1(x) = (1+k_1)\cos\gamma(x)$$

where k_1 is the longitudinal virtual mass coefficient and $\gamma = \arctan\frac{dy}{dx}$. $U_1(x)$ is used to obtain a sequence of successive approximations by means of the iteration formula

$$U_{n+1}(t) = U_n(t) + \cos\gamma(t)\left[1 - \int_0^P \frac{y^2(x)}{r^3}\,U_n(x)\,ds\right]$$

Here also the iterations are most conveniently carried out in terms of the differences between successive approximations to U(x) which also furnish a measure of the error in the integral equation. Two methods of carrying out the iterations are again available, of which one is semi-graphical, the other completely arithmetical. The latter technique is employed on the same example as was used to illustrate the first method.

REFERENCES

1. Fuhrmann, G. Theoretical and Experimental Investigations on Balloon Models. Dissertation, Gottingen, 1912 Published in Zeitschrift fur Flugtechnik und Motorluftschiffahrt, Vol. 11, 1911.

2. Goldstein, S. Modern Developments in Fluid Dynamics. Vol. II, Oxford: Clarendon Press, 1938, p. 523.

3. Kaplan, C. Potential Flow about Elongated Bodies of Revolution. NACA Report No. 516, 1935.

4. On a New Method for Calculating the Potential Flow Past a Body of Revolution. NACA Report No. 752, 1943.

5. Karman, T. v. Calculation of Pressure Distribution on Airship Hulls. NACA Tech. Mem. No. 574, 1930; From Abhandlungen aus dem Aerodynamischen Institut an der Technischen Hochschule Aachen, No. 6, 1927.

6. Kellogg, O. D. Foundations of Potential Theory. New York: The Murray Printing Company, Frederick Unger Publishing Company, 1929, p. 215.

7. Ibid, p. 219.

8. Laitone, E. V. The Subsonic Flow about a Body of Revolution. Quarterly of Applied Mathematics, Vol. 5, No. 2, July 1947.

9. Lamb, H. Hydrodynamics. Cambridge: University Press, Sixth Edition, 1932, pp. 154-155.

10. On Wave Resistance. Proceedings of the Royal Society, A, Vol. 111, 1926.

11. Lowan, A. N., N. Davids and A. Levenson Table of the Zeros of the Legendre Polynomials of Order 1-16 and the Weight Coefficients for Gauss' Mechanical Quadrature Formula. Bulletin of the American Mathematical Society, Vol. 48, No. 10, Oct. 1942.

12. Munk, M. M. Fluid Mechanics. Vol. 1, Part II of Aerodynamic Theory, Edited by W. F. Durand. Berlin: Julius Springer, 1935, Division C, p. 269.

13. Ibid, Vol. 6, p. 36.

14. Notes on Aerodynamic Forces - I. Rectilinear Motion. NACA T.N. 104, July 1922.

15. Munzer, H. and H. Reichardt Rotationally Symmetrical Source-Sink Bodies with Predominately Constant Pressure Distribution. British Ministry of Supply, Armament Research Establishment Translation No. 1/50, 1950. (Translation of German Report U.M. 6616, 1944).

16. Rankine, W. J. M. On the Mathematical Theory of Stream Lines, especially those with Four Foci and Upwards. Philosophical Translations, 1871.

17. Riegels, F. and M. Brandt Stromfunctionen und Geschwindigkeitsfelder räumlicher Quellstrecken und ihr handlicher Gebrauch zur Bestimmung von Umriss und Druckverteilung rotationsymmetrischer Korper mit Beispielen. Deutsche Luftfahrtforschung, U. M. Bericht No. 3106.

18. Sadowsky, M. A. and E. Sternberg Elliptic Integral Representation of Axially Symmetric Flows. Quarterly of Applied Mathematics, Vol. 8, No. 2, 1950.

19. Smith, R. H. Longitudinal Potential Flow about Arbitrary Body of Revolution with Application to Airship "Akron". Jour. of the Aeronautical Sciences, Vol. 3, No. 1, Sept. 1935.

20. Taylor, D. W. On Ship-Shaped Stream Forms. Transactions of the Institution of Naval Architects, Vol. 35, 1894.

21. Taylor, G. I. The Energy of a Body Moving in an Infinite Fluid, with an Application to Airships. Proceedings of the Royal Society, A, Vol. 120, 1928.

22. Tuyl, A. Van On the Axially Symmetric Flow Around a New Family of Half Bodies. Quarterly of Applied Mathematics, Vol. 7, No. 4, 1950.

23. Weinblum, G. Rotationskörper geringsten Wellenwiderstandes. Ingenieur-Archiv, Vol. 7, 1936.

24. Weinig, F. Über schnell konvergierende graphische Lösungen von Strömungsproblemen durch Integralgleichungen. Zeitschrift fur technische Physik, Vol. 9, No. 1, 1928.

25. Weinstein, A. On Axially Symmetric Flows. Quarterly of Applied Mathematics, Vol. 5, No. 4, 1948.

26. Wijngaarden, A. v. Écoulement potentiel autour d'un corps de révolution. Colloques Internationaux du Centre National de la Recherche Scientifique. XIV, Méthodes de Calcul dans des Problèmes de Mecanique. Paris, 1948.

27. Young, A. D. and P. R. Owen A Simplified Theory for Streamline Bodies of Revolution, and its Application to the Development of High Speed Shapes. British Aeronautical Research Committee R & M 2071, 1943.