ON THE STRUCTURE OF LOCALLY CONNECTED PLANE CONTINUA ON WHICH IT IS POSSIBLE TO DEFINE A POINTWISE PERIODIC HOMEOMORPHISM WHICH IS NOT ALMOST PERIODIC

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A favorite field of mathematical investigation concerns itself with the structure of sets. The method employed consists of imposing certain properties upon a set and then seeing what other properties, if any, the set has. After one obtains various necessary conditions in this way, one usually also investigates the sufficiency of these conditions. The results are given as theorems on the structure of the set. These theorems are of some importance, inasmuch as the behavior of a set, when it appears in applications, usually will depend on its structure.

In this paper we shall make some investigations into the structure of a locally connected, plane continuum on which is defined a pointwise periodic homeomorphism. We start by giving a few definitions and quoting some known results. All sets considered lie in a separable metric space.

We shall denote the Euclidean plane by Π. By M we shall mean a locally connected continuum (Whyburn [5] p. 18) in Π. The set-theoretic boundary of M defined as $M\cdot\overline{(\Pi - M)}$ will be called B. The letter f will denote a single-valued continuous transformation which is pointwise periodic.

The mapping $T(X)\subset X$, where X is a set, is said to be <u>pointwise periodic</u> if for every point $x\in X$ there is an integer N_x such that $T^{N_x}(x)=x$. The least such integer N_x is called the <u>period</u> of x, which we shall denote by p(x). If there is an integer N such that $T^N(x)=x$ for <u>all</u> $x\in X$, then T is said to be <u>periodic</u>. The least such integer N is called the period of T. The mapping T will be said to be <u>almost periodic</u> if for every $\epsilon > 0$ there is an integer N_ϵ such that

$\rho\left[x, T^{N_\epsilon}(x)\right] < \epsilon$ for all $x \in X$.

If $T(X) \subset X$ is continuous, a subset Y of X is said to be invariant provided $T(Y) = Y$.

If $T(X) \subset X$ is a pointwise periodic mapping, then the set $O_T(x) = \bigcup_n T^n(x)$ consisting of x and all (a finite number, namely $p(x)$) of its images under T will be called the orbit of x under T.

An immediate consequence of these definitions is that a mapping $T(X) \subset X$ which is pointwise periodic and continuous on a compact set X is actually a homeomorphism $T(X) = X$. (Whyburn [5] p. 240) We now give some known results of a less trivial nature.

THEOREM A: Let X be a closed and compact metric space, and $f(X) = X$ a pointwise periodic homeomorphism. Let A be those points $x \in X$ such that in every neighborhood of x there is a point z with $p(z) \neq p(x)$. Let $C = X - A$. Then C is open in X, C is dense in X, $p(x)$ is constant on every component of C, and $p(x)$ is continuous on C. (Montgomery [3] p. 118)

THEOREM B: If $f(M) = M$ is apointwise periodic homeomorphism, where M is any locally connected continuum in the plane π such that no two points of M separate M, then f is periodic. (Hall & Kelley [2] p. 630)

A separable metric space X is said to be a 2-dimensional manifold provided that for any $x \in X$ there exists a neighborhood U of x such that $\overline{U}$ is a 2-cell. (Whyburn [5] p. 193)

THEOREM C: If X is a connected metric space which is locally Euclidean, that is each point is in an open set homeomorphic to the interior of a solid n-dimensional sphere,

<u>and $T(X)=X$ is pointwise periodic, then T is periodic.</u> (Montgomery [3] p. 118)

COROLLARY: <u>Any pointwise periodic mapping $f(X)=X$ on a connected 2-dimensional manifold X is periodic.</u> (Whyburn [5] p. 262)

Next we shall establish several results of a general nature about almost periodic mappings.

THEOREM 1: <u>Let X be a metric space and $T(X)=X$ a single valued transformation. If N is a positive integer, then T^N is almost periodic if and only if T is almost periodic.</u>

PROOF: (Sufficiency) Assume T is almost periodic. Given $\epsilon>0$, choose an integer m such that $\rho[x, T^m(x)]<\frac{\epsilon}{N}$ for all $x\in X$. Then

$$\rho[x,T^{Nm}(x)] \leq \rho[x,T^m(x)]+\rho[T^m(x),T^{2m}(x)]+\cdots+\rho[T^{(N-1)m}(x),T^{Nm}(x)]$$

or $\rho[x, T^{Nm}(x)] \leq \frac{\epsilon}{N}+\frac{\epsilon}{N}+\cdots+\frac{\epsilon}{N} = N(\frac{\epsilon}{N}) = \epsilon$ for all $x\in X$. Hence T^N is almost periodic.

(Necessity) Assume T^N is almost periodic. Given $\epsilon>0$, choose an integer m such that $\rho[x, T^{Nm}(x)]<\epsilon$ for all $x\in X$. Thus there is an integer $r=Nm$ such that $\rho[x, T^r(x)]<\epsilon$ for all $x\in X$, which means that T is almost periodic.

THEOREM 2: <u>Let X be a metric space and $T(X)=X$ a single valued continuous mapping. If T is almost periodic ON a subset $E\subset X$, then T is almost periodic on $\overline{E}\cdot X$.</u>

PROOF: Given $\epsilon>0$, choose an integer N such that $\rho[x, T^N(x)]<\frac{\epsilon}{3}$ for all $x\in E$. Take any point $x\in E'\cdot X$. There is a sequence $\{x_n\}$ of points of E such that $x_n\rightarrow x$.

Now T^N is continuous since T is continuous. Thus there exists a $\delta=\delta(x,\epsilon)>0$ such that $\rho(y,x)<\delta$ implies that $\rho\left[T^N(y),T^N(x)\right]<\frac{\epsilon}{3}$. Let $\alpha=\min(\delta,\frac{\epsilon}{3})$ and choose n large enough so that $\rho(x_n, x)<\alpha$. Then

$$\rho\left[x,T^N(x)\right]\leq\rho(x,x_n)+\rho\left[x_n,T^N(x_n)\right]+\rho\left[T^N(x_n),T^N(x)\right]<\alpha+\frac{\epsilon}{3}+\frac{\epsilon}{3}.$$

Hence $\rho\left[x, T^N(x)\right]<\epsilon$ for all $x\in\overline{E}\cdot X$.

THEOREM 3: <u>Let X be a metric space and</u> $T(X)=X$ <u>a single valued transformation. If E and F are subsets of X such that T is periodic on E and almost periodic on F, then T is almost periodic on</u> $E+F$.

PROOF: Since T is periodic on E there is an integer n such that $T^n(x)=x$ for all $x\in E$. By theorem 1, T^n is almost periodic on F. Thus, given $\epsilon>0$, there is an integer m such that $\rho\left[x, T^{nm}(x)\right]<\epsilon$ for all $x\in F$. Furthermore, $T^{nm}(x)=x$ for all $x\in E$. Hence $\rho\left[x, T^{nm}(x)\right]<\epsilon$ for all $x\in E+F$. This shows that T is almost periodic on $E+F$.

We return now to a consideration of locally connected continua. For some time it was conjectured that a pointwise periodic homeomorphism of such a continuum onto itself necessarily would be almost periodic. This now is known to be false. A counter example has been constructed by Ralph Phillips which settles the problem except in the case of a 2-dimensional locally connected continuum. (Ayres [1] p.95) It is the latter case which we shall consider. What must be the structure of a locally connected continuum M in the plane Π such that $f(M)=M$ can be a pointwise periodic homeomorphism which is not almost periodic? It is known that M cannot be a dendrite. (Whyburn [5] p. 252) We shall establish several other properties which, it is hoped, will prove useful in determining ultimately if such a set M exists.

We assume that $f(M)=M$ is a pointwise periodic homeomorphism which is not almost periodic. Define the set L to be those points x of M such that the period function $p(y)$ is unbounded in every neighborhood of x. It is clear that L is closed and contained in the set A of theorem A.

THEOREM 4: <u>If R is any component of $M-B$, then $p(x)$ is bounded on R. Thus $L\subset B$.</u>

PROOF: The set $M-B$ is open in π. Any component R of $M-B$ is an open connected subset of π since π is locally connected. Hence R is a connected 2-dimensional manifold. Choose $x\in R$ and let $n=p(x)$. Define $T=f^n$. Inasmuch as $T(M-B)=M-B$, then $T(R)=R$. But T is pointwise periodic. By the corollary to theorem C we find that T, and hence f, is periodic on R. Thus $p(y)$ is bounded on R. Assume that $L\cdot(M-B)\neq 0$ and choose a point $x\in L\cdot(M-B)$. Let R be the component of $M-B$ containing x. Since R is open there is a neighborhood U of x such that $U\subset R$. But $p(y)$ is bounded on R and hence on U. This contradicts the definition of L. Consequently $L\cdot(M-B)=0$, or $L\subset B$.

THEOREM 5: <u>If M has no cut point, then the period function $p(x)$ is unbounded on B.</u>

PROOF: The boundary of every domain of $\pi-M$ is a simple closed curve. (Moore [4] p. 212) The union of these boundaries is contained in B. Now if $p(x)$ were bounded on B, then there would be an integer n such that $f^n(x)=x$ for all $x \in B$. In particular, the points on the boundaries of the complementary domains would be fixed under this power of f. Hence the mapping f^n could be extended to the whole plane by the definition $f^n(x) = x$ for all $x \in \pi - M$. This mapping f^n then is known to be periodic on π by virtue of theorem C. Consequently f^n is periodic on M. This means f is periodic on M and, a fortiori, almost periodic on M. This contradiction to our hypothesis on f shows that p(x) is unbounded on B.

THEOREM 6: <u>There exists an integer m and a non-degenerate continuum $H \subset M$ such that, under f^m,</u>

a) <u>H is the limit of a sequence of point orbits,</u>

b) <u>H is invariant, and</u>

c) <u>H contains uncountably many fixed points.</u>

PROOF: Since f is not almost periodic there exists an $\eta > 0$ and a sequence $\{x_n\}$ of points of M such that $\rho[x_n, f^n(x_n)] \geqslant \eta$. Define $y_n = x_{n!}$. Suppose $\{p(y_n)\}$ is bounded by N. Then N! is divisible by $p(y_N)$, so that $f^{N!}(y_N) = y_N$. Hence $0 = \rho[y_N, f^{N!}(y_N)] = \rho[x_{N!}, f^{N!}(x_{N!})] \geqslant \eta$. This is a contradiction; thus $\{p(y_n)\}$ is unbounded. We may choose a subsequence $\{w_i\} = \{y_{n_i}\}$ such that $p(w_i) < p(w_{i+1})$ and $w_i \rightarrow w$. Let $p = p(w)$.

Define $T = f^p$. Then $T(w) = w$. Let $z_n = f^{n!}(x_{n!}) = f^{n!}(y_n)$ and let $v_i = z_{n_i}$. We may assume that $v_i \rightarrow v$. Since $\rho(y_n, z_n) = \rho[x_{n!}, f^{n!}(x_{n!})] \geqslant \eta$, we have $\rho(v,w) \geqslant \eta$, so that $v \neq w$. Inasmuch as T is continuous and M is compact, $T(v_i) \rightarrow T(v) \neq w$.

Let $d_i = \delta[O_T(w_i)]$, and assume that $d_i \rightarrow 0$. Then for any $d > 0$ there is an i_1 depending on d such that $i > i_1$ implies that $0 < d_i < d$. That is, $\rho\left[T^r(w_i), T^s(w_i)\right] < d$ for all $i > i_1$ and all r and s. Furthermore, since $\{p(w_i)\}$ is unbounded, there is an i_2 such that $i > i_2$ implies $d_i > 0$. Finally there is an i_3 such that $i > i_3$ implies that $n_i!$ is divisible by p. Let $i_0 = \max(i_1, i_2, i_3)$. Now for each $i > i_0$ let $r = p(w_i)$ and $s = 1 + \frac{n_i!}{p}$ in the above expression. Then

$\rho\left[T^r(w_i), T^s(w_i)\right] = \rho\left[w_i, f^{p+n_i!}(y_{n_i})\right] = \rho\left[w_i, T(v_i)\right] < d$ for all $i > i_0$. Hence $T(v_i) \rightarrowtail w$. This contradiction shows that our assumption was incorrect. Consequently there is a subsequence of $\{d_i\}$ which is bounded away from zero. Thus there is a subsequence of $\{O_T(w_i)\}$ which converges to a non-degenerate set H^*. We may as well assume $O_T(w_i) \rightarrow H^*$. The limit set H^* contains the fixed point w, and thus is connected. (Whyburn [5] p. 260) Thus H^* consists of a single non-degenerate component. It follows that H^* is a continuum, and that $T(H^*) = H^*$. (Whyburn [5] p. 259)

Since H^* satisfies the conditions on X in theorem A, there is a set C which is open and dense in H^*. Let k be the common period under T of the points of some component of C. These points, of which there are uncountably many, are all fixed under T^k. For $j = 1, 2, \ldots, k$ define $p_{ij} = T^j(w_i)$ and let $G_{ij} = O_{T^k}(p_{ij})$. For each value of j we may assume $G_{ij} \rightarrow G_j$. In view of the fact that the range of j is finite, it appears that each G_j contains the fixed point w. Consequently, as in the case of H^* itself, each G_j is a continuum such that $T^k(G_j) = G_j$. Furthermore the relationship $T(G_j) = G_{j+1}$ is valid. It is clear that $H^* = \cup G_j$. Thus each G_j is non-degenerate, and one of them must contain uncountably many of the fixed points described above. We denote this particular G_j by H. We have found a sequence of points $\{p_{ij}\}$, an integer $m = pk$, and a non-degenerate continuum H which satisfy the conditions in the statement of the theorem.

Despite the fact that we have said the set M is in the plane π, the preceeding theorem is valid without this restriction.

THEOREM 7: <u>If F is a finite subset of M, then</u> $H-F$ <u>is contained in a single component of</u> $M-F$.

PROOF: Let G be the union of the orbits under f^m of the points of F. Then G is a finite set, so that H contains a fixed point p not in G. There is a neighborhood U_p of p which contains no point of G. Take any point x in $H-F$. There is a neighborhood U_x of x which does not intersect F. Now M is locally connected. Thus points sufficiently close to p can be joined to p by an arc which is contained in U_p. A similar statement holds for x and its neighborhood U_x. From the orbits which converge to H we may choose a point q which is "sufficiently close" to p such that one of its images $f^{mr}(q)$ is likewise "sufficiently close" to x. Join p and q by an arc K contained in U_p. Then $f^{mr}(K)$ is an arc from p to $f^{mr}(q)$ which does not intersect F since K does not intersect G. Now join x and $f^{mr}(q)$ by an arc lying in U_x. This arc does not intersect F. We now have a connected set (which is the sum of two arcs with at least the point $f^{mr}(q)$ in common) joining x and p and not intersecting F. We see finally that x and p lie together in a connected subset of

$M-F$. Since x is an arbitrary point of $H-F$, the assertion of the theorem is proved.

THEOREM 8: Let X be a locally connected continuum without cutpoints. If two points p and q cut X, then

a) $X-(p+q)$ has finitely many components,

b) if K is any component of $X-(p+q)$, then $p+q=\overline{K}-K$,

c) if $\overline{K}$ has a cut point r, then $\overline{K}-r$ has exactly two components, and

d) $X-(p+r)$ and $X-(q+r)$ are separated.

PROOF: Part b) is trivial. Part a) follows immediately from the local connectivity of X. We proceed to prove parts c) and d).

Let K_1 be a component of $\overline{K}-r$, and assume neither p nor q is in K_1. Then any connected set joining a point of K_1 to a point not in K_1 must contain r. This means, contrary to hypothesis, that r cuts X. Hence we may assume $p \in K_1$. On the other hand we cannot also have $q \in K_1$, for then another component K_2 (there are at least two components since r cuts $\overline{K}$) would contain neither p nor q. Thus $q \in K_2$, and these can be the only two components. Therefore $\overline{K}-r=K_1+K_2$ with $p \in K_1$ and $q \in K_2$.

Finally, any connected set joining a point of K_1-p to a point not in K_1-p must contain either p or r. Consequently $X-(p+r)$ is separated. Similarly $X-(q+r)$ is separated. This completes the proof of the theorem.

We consider next certain ways in which the set H can be cut. Before we proceed we note that, in view of theorem one, we may assume $m = 1$ in theorem 6 without any loss in generality.

Consider the case when M has no cut points. Since $f(M) = M$ is <u>not</u> periodic it follows from theorem B that some two points cut M. Consider all cuttings of M of the type $M-(p_\alpha + q_\alpha)$. For each α let K_α be the component of $M-(p_\alpha + q_\alpha)$ such that $H \subset \overline{K}_\alpha$. That this is possible follows from theorem 7.

LEMMA 1: <u>If K_1 and K_2 are elements of the class $\{K_\alpha\}$ such that $M-(\overline{K}_1 + \overline{K}_2) \neq 0$, then there is a set K_3 in $\{K_\alpha\}$ such that</u> $K_3 \subset K_1 \cdot K_2$.

PROOF: Choose a point y in $M-(\overline{K}_1 + \overline{K}_2)$ and a point z in $H-(p_1 + q_1 + p_2 + q_2)$. Since M has no cut point, then there is a simple closed curve J in M which passes through y and z. It follows that J contains $p_1 + q_1 + p_2 + q_2$. We may name the points so that one arc $\widehat{yz}$ of J contains $p_1 + p_2$ and the other contains $q_1 + q_2$. If $K_1 = K_2$ then we take $K_3 = K_1$. Accordingly we assume $K_1 \neq K_2$. We may assume then that $p_1 \neq p_2$. Since it is immaterial which set is called K_1 and which K_2 we may assume that p_2 is closer to z than is p_1 along the arc $\widehat{yp_2z}$ of J. In the other arc of J let q_3 be the q_i which is nearer to z along the arc. (This admits even the possibility $q_1 = q_2 = q_3$.)

We assert that $M-(p_2 + q_3)$ is separated. This is trivial if $q_3 = q_2$; accordingly we take $q_3 = q_1 \neq q_2$. Suppose that $M-(p_2 + q_3)$ is connected. Both y and z are in this set. Any arc joining y and z in this set must contain both p_1 and q_2. Consider any such arc. If we start at z and proceed along the arc towards y we must come first, say, to p_1. But then we may proceed along J to y without passing through q_2, which is impossible. We reach a similar contradiction if we meet q_2 before p_1. Hence $M-(p_2+q_1)$ is separated. Let K_3 be the component of $M-(p_2+q_1)$ which belongs to the class $\{K_\alpha\}$. Then $K_3 \subset M-(p_1+q_1+p_2+q_2)$ which implies $K_3 \subset K_1$ and $K_3 \subset K_2$, so that $K_3 \subset K_1 \cdot K_2$.

Let K_o be any set in the collection $\{K_\alpha\}$. Define $\{K^*_\alpha\}$ to be the subclass of $\{K_\alpha\}$ consisting of all K_β such that $K_\beta \subset K_o$. Let y be a point <u>not</u> in $\overline{K}_o$ and z a point of H. In M there is a simple closed curve J which contains both y and z. It follows that $\bigcup(p^*_\alpha + q^*_\alpha) \subset J$, where p^*_α and q^*_α are the boundary points of K^*_α in M. The curve J contains two arcs $\widehat{yaz}$ and $\widehat{ybz}$ from y to z, and we may name the points p^*_α and q^*_α in such a way that, for every α, p^*_α is in $\widehat{yaz}$ and q^*_α is in $\widehat{ybz}$. We define an ordering of the points p^*_α by the relation $p^*_\alpha < p^*_\beta$ if p^*_α is nearer to z than is p^*_β along the arc $\widehat{yaz}$; and similarly $q^*_\alpha < q^*_\beta$ if q^*_α is nearer to z than is q^*_β along the arc $\widehat{ybz}$.

LEMMA 2: $K^*_\alpha \subset K^*_\beta$ <u>if and only if</u> $p^*_\alpha \leq p^*_\beta$ <u>and</u> $q^*_\alpha \leq q^*_\beta$.

PROOF: (Sufficiency) Assume $p^*_\alpha \leq p^*_\beta$ and $q^*_\alpha \leq q^*_\beta$. Then K^*_α is contained in $M-(p^*_\alpha+q^*_\alpha+p^*_\beta+q^*_\beta)$, and thus $K^*_\alpha \subset K^*_\beta$.

(Necessity) Assume $K^*_\alpha \subset K^*_\beta$. If $p^*_\beta \leq p^*_\alpha$ and $q^*_\beta \leq q^*_\alpha$, then, from the proof of the sufficiency, it follows that $K^*_\beta \subset K^*_\alpha$, so that $K^*_\alpha = K^*_\beta$, which means $p^*_\alpha = p^*_\beta$ and $q^*_\alpha = q^*_\beta$. Hence we assume $K^*_\alpha \neq K^*_\beta$ and take a point x in $K^*_\beta - K^*_\alpha$. Since K^*_β is a connected set containing both x and z, then either p^*_α or q^*_α is in K^*_β. We may assume $p^*_\alpha \in K^*_\beta$, so that $p^*_\alpha < p^*_\beta$.

Assume $q^*_\beta \leq q^*_\alpha$ and choose a point u in the arc $\widehat{ybz}$ such that $q^*_\beta < u < q^*_\alpha$. But then $u \in K^*_\alpha - K^*_\beta = 0$. This contradiction shows that $q^*_\alpha \leq q^*_\beta$ and completes the proof of the lemma.

COROLLARY: <u>If p_o and q_o are the boundary points of K_o in M, then $p^* \leq p_o$ and $q^* \leq q_o$ for all α.</u>

LEMMA 3: <u>$\bigcup p^*_\alpha$ is closed.</u>

PROOF: Let $F = \bigcup p^*_\alpha$ and take any $p \in F'$. Let $\{p^*_{\alpha_i}\}$ and $\{q^*_{\alpha_i}\}$ be sequences such that $p^*_{\alpha_i} \xrightarrow[i]{} p$ and $q^*_{\alpha_i} \xrightarrow[i]{} q$. In view of the preceeding corollary we have $p \leq p_o$ and $q \leq q_o$. Assume $M-(p+q)$ is connected. Then y and z can be joined by an arc N in $M-(p+q)$. There exist neighborhoods U_p of p and U_q of q which do not intersect N. Hence there exist a $p^*_{\alpha_i} \in U_p$ and a $q^*_{\alpha_i} \in U_q$ which are not on N, contrary to the fact that any arc joining y and z must pass through at least one of the points p^*_α or q^*_α for every α. Thus $M-(p+q)$ is separated. Let K be the component of $M-(p+q)$ which is in the class $\{K_\alpha\}$. Then $K \subset M-(p+q+p_o+q_o)$ since y is not in $\overline{K}$. Hence $K \subset K_o$ which means K is an element of $\{K^*_\alpha\}$. Consequently p is in F, and F is closed. Similarly $\bigcup q^*_\alpha$ is closed.

THEOREM 9: <u>Let M be a locally connected plane continuum without cutpoints, and $f(M)=M$ a pointwise periodic homeomorphism which is not almost periodic. Then there exists a sequence of points $\{p_i\}$ in $C \subset M$ and an integer m such that $O_T(p_i) \to H$, where $T = f^m$, and H is a non-degenerate continuum. If p,q is any pair of points which cut M, and K_o is the com-</u>

<u>ponent of $M-(p+q)$ whose closure contains H, then there exists a minimal locally connected proper subcontinuum $\overline{K}$ of M having no cut point such that $H\subset\overline{K}\subset\overline{K}_0$ and every component of $M-\overline{K}$ has exactly two boundary points in M. If $\overline{K}^*$ is a similar continuum obtained from a pair of points p^* and q^*, either $\overline{K}=\overline{K}^*$ or $\overline{K}+\overline{K}^*=M$. Finally, if x is any point in $\overline{K}$, then either $T(x)$ or $T^2(x)$ is in $\overline{K}$, or both.</u>

PROOF: The existence of H was discussed in theorem 6. That the points whose orbits converge to H can be chosen in the set C of theorem A follows at once from theorem A and theorem 2.

We return to the notation used in lemma 2 and let p be the point in $\bigcup p^*_\alpha$ such that $p\leqslant p^*_\alpha$ for all α. Similarly define q so that $q\leqslant q^*_\alpha$ for all α. Because of lemma 3 we see that there are indices α and β such that $p=p^*_\alpha$ and $q=q^*_\beta$. Corresponding to these points we have sets K^*_α and K^*_β satisfying the conditions of lemma 1. Thus there is an element K_γ of $\{K_\alpha\}$ such that $K_\gamma\subset K^*_\alpha\cdot K^*_\beta$. Furthermore, lemma 1 shows that we may choose K_γ so that $p+q=\overline{K}_\gamma-K_\gamma$. Inasmuch as $K_\gamma\subset K^*_\alpha\subset K_0$ we may say $K_\gamma=K^*_\gamma\in\{K^*_\alpha\}$.

Define $K=\bigcap K^*_\alpha$. It is clear that K is the K^*_γ of the preceeding paragraph. $\overline{K}$ is a closed proper subset of M. Take any $x\in\overline{K}-(p+q)=K$. There is a neighborhood U_x of x which contains neither p nor q. Since M is locally connected, a point of M sufficiently close to x can be joined to x by an arc N contained in U_x. Now N contains a point x of K, but contains neither p nor q. Consequently N lies in K. This shows that $\overline{K}$ is locally connected, inasmuch as it cannot fail to be locally connected only at p and q. Thus $\overline{K}$ is a locally connected proper subcontinuum of M containing H.

If $\overline{K}$ has a cut point r, theorem 8 shows that $\overline{K}-r$ consists of exactly two components. One of these components, say the one containing p, contains $H-r$. Call this component G. Now $M-(p+r)$ is separated (theorem 8) and the element of $\{K_\alpha\}$ determined by this cutting is contained in G and hence in K and K_o. Furthermore it is contained properly in K, and this contradicts the definition of K. Thus $\overline{K}$ has no cut point.

Let us suppose that p^*, q^* is a pair of points distinct from the pair p, q. Then the continuum $\overline{K^*}$ determined by the second pair of points is distinct from $\overline{K}$. Lemma 1 shows that it is necessary to have $\overline{K}+\overline{K^*}=M$.

Suppose $T(\overline{K})\neq\overline{K}$. Then, inasmuch as $T^{-1}(\overline{K})$ is a $\overline{K^*}$ as just described, $\overline{K}+T^{-1}(\overline{K})=M$. Take any $x\in\overline{K}$ such that $T(x)\notin\overline{K}$. Then $T(x)\in T^{-1}(\overline{K})$, or $T^2(x)\in\overline{K}$.

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