ABSTRACT

Title of dissertation: TWIST-BULGE DERIVATIVES AND DEFORMATIONS OF CONVEX REAL PROJECTIVE STRUCTURES ON SURFACES

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Let S be a closed orientable surface with genus $g > 1$ equipped with a convex $\mathbb{RP}^2$ structure. A basic example of such a convex $\mathbb{RP}^2$ structure on a surface S is the one associated to a hyperbolic structure on S, and in this special case Wolpert proved formulas for computing the Lie derivatives $t_\alpha l_\beta$ and $t_\gamma t_\alpha l_\beta$, where t_α is the Fenchel-Nielsen twist vector field associated to the twist along a geodesic α, and l_* is the hyperbolic geodesic length function. In this dissertation, we extend Wolpert's calculation of $t_\alpha l_\beta$ and $t_\gamma t_\alpha l_\beta$ in the hyperbolic setting to the case of convex real projective surfaces; in particular, our t_α is the twist-bulge vector field along geodesic α coming from the parametrization of the deformation space of convex $\mathbb{RP}^2$ structures on a surface due to Goldman, and our geodesic length function l_* is in terms of a generalized cross-ratio in the sense of Labourie. To this end, we use results due to Labourie and Fock-Goncharov on the existence of an equivariant flag curve associated to Hitchin representations, of which convex real projective surfaces are an example. This flag curve allows us to extend the notions arising in the hyperbolic case to that of convex real projective structures and to complete our generalization of Wolpert's formulas.

TWIST-BULGE DERIVATIVES AND DEFORMATIONS OF CONVEX REAL PROJECTIVE STRUCTURES ON SURFACES

by

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To Islem, my patient and long-suffering wife,

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Contents

Chapter 1

Introduction

1.1 Context

By a *convex* $\mathbb{RP}^2$*-manifold* we mean a representation of a surface S as a quotient Ω/Γ with $\Omega \subset \mathbb{RP}^2$ a convex domain and $\Gamma \subset$ PSL(3,$\mathbb{R}$) is discrete and acts properly on Ω. Arising from this definition is the identification of homotopy marked projective equivalence classes of convex $\mathbb{RP}^2$ structures on such a surface S with a preferred open subspace of Hom($\pi_1(S)$,PSL(3,$\mathbb{R}$))/PSL(3,$\mathbb{R}$), or conjugacy classes of representations of the fundamental group of S into PSL(3,$\mathbb{R}$). We call this the *deformation space of convex* $\mathbb{RP}^2$ *structures on* S and denote it by $\mathfrak{P}(S)$.

A basic example of a convex $\mathbb{RP}^2$ structure on a surface S is the one associated to a hyperbolic structure on S. Given a hyperbolic structure on S, the convex $\mathbb{RP}^2$ structure associated to it is the one obtained via the Klein projective model from hyperbolic geometry. In this way we see that the Teichmüller space $\mathcal{T}(S)$ of S embeds in $\mathfrak{P}(S)$. In fact, Goldman [2] provides an explicit parametrization of $\mathfrak{P}(S)$ by generalizing the classical Fenchel-Nielsen coordinates on Teichmüller space. In particular, a component of this parametrization is the *twist-bulge deformations* which are generalizations of the Fenchel-Nielsen twists in the Teichmüller case. These form a central object of investigation for this paper.

In [11], Wolpert proves formulas for calculating the Lie derivatives $t_\alpha l_\beta$ and $t_\gamma t_\alpha l_\beta$, where t_α is the Fenchel-Nielsen twist vector field associated to the twist along a geodesic

α, and l_* is the geodesic length function. These Lie derivative formulas are basic ingredients for the symplectic geometry of Teichmüller space. Since this geodesic length function is given in terms of the classical cross-ratio of four points, a key part of Wolpert's calculation is a lemma that gives a formula for determining the twist derivative of the cross-ratio. The main achievement of the current paper is to generalize these formulas of Wolpert to the case of convex $\mathbb{RP}^2$ structures. We carry out the analogous computation in this case, where our t_α is now the twist-bulge vector field along geodesic α coming from Goldman's parametrization of $\mathfrak{P}(S)$, and the geodesic length function l_* is in terms of a *generalized cross-ratio* in the sense of Labourie [6],[7]. Defining the necessary quantities requires the use of a powerful property of more general *Hitchin representations* $\rho : \pi_1(S) \to$PSL(n,$\mathbb{R}$), of which the representation associated to a convex $\mathbb{RP}^2$ structure is a special case. In particular, due to independent work of Labourie [5] and Fock-Goncharov [4] there exists an invariant *flag curve* that encodes a great deal of geometric data for an individual representation, and it is precisely this data that is necessary to define the relevant quantities.

1.2 Results

Let S be a convex real projective surface, and let α, β be simple closed geodesics on S, let t_α be the vector field associated to the twist-bulge deformation along α, and let l_β be the geodesic length function.

Theorem 1. *The twist-bulge derivative $t_\alpha l_\beta$ is given by*

$$t_\alpha l_\beta = \sum_{C \in \mathfrak{I}} [I_{\widehat{c_1 c_2}}(r_B, a_B) - I_{\widehat{c_1 c_2}}(a_B, r_B)]$$

where the sum is over representatives C in a particular double coset, $\widehat{c_1 c_2}$ is the geodesic associated to the representative C, and r_B, a_B are the repelling and attracting fixed points of matrix B associated to geodesic β, respectively.

The $I_{\widehat{c_1 c_2}}(\cdot, \cdot)$ terms arise from the calculation of the twist-bulge derivative of a generalized cross-ratio of four points on the boundary $\partial\Omega$. The aforementioned flag curve associates to each of these boundary points a flag with various special properties, and it is this data encoded by the flag curve that goes into all of our calculations.

For the second order twist-bulge derivative $t_\gamma t_\alpha l_\beta$, we have

Theorem 2. *Let A, B, C be matrix representatives associated to geodesics α, β, γ, respectively. Then $t_\gamma t_\alpha l_\beta$ breaks up as sums over $\langle A\rangle$ and $\langle B\rangle$ orbits, where the total contribution coming from a $\langle B\rangle$-orbit is*

$$(C_1^{\alpha,\gamma}\frac{\lambda_B}{\mu_B}+C_2^{\alpha,\gamma})\frac{1}{1-\frac{\lambda_B}{\mu_B}}+(C_3^{\alpha,\gamma}\frac{\lambda_B}{\nu_B}+C_4^{\alpha,\gamma})\frac{1}{1-\frac{\lambda_B}{\nu_B}}+(C_5^{\alpha,\gamma}\frac{\mu_B}{\nu_B}+C_6^{\alpha,\gamma})\frac{1}{1-\frac{\mu_B}{\nu_B}}$$

where the $C_i^{\alpha,\gamma}$ are constants and λ_B, μ_B, ν_B are the eigenvalues of B as in (2.5). The contribution coming from the $\langle A\rangle$-orbit is similar.

The paper proceeds as follows: We begin with a review of the relevant basic properties of convex $\mathbb{RP}^2$ structures on a surface S, culminating in a description of Goldman's parametrization of the deformation space of such structures. Of particular importance are the twist-bulge deformations. We then review the basics of higher Teichmüller theory, and state results due to Labourie and Fock-Goncharov concerning the aforementioned flag curves. This allows us to define the quantities necessary to carry out our calculation of $t_\alpha l_\beta$ and $t_\gamma t_\alpha l_\beta$ in the convex $\mathbb{RP}^2$ structures case. Our approach, as in Wolpert's original calculation, is to prove a preliminary lemma for computing the twist-bulge derivative of a generalized cross-ratio of four points on $\partial\Omega$ and use this to show that the sum arising in the calculation of the first twist-bulge derivative telescopes, yielding the result in Theorem 1. We then prove a second lemma which allows us to find the twist-bulge derivative of the terms showing up in the sum in Theorem 1, and use this to complete the second order calculation which culminates in Theorem 2.

Chapter 2

Background

2.1 Convex $\mathbb{RP}^2$ structures on a surface

The starting point in our discussion is Goldman's [2] Fenchel-Nielsen type parametrization of the deformation space of convex $\mathbb{RP}^2$ structures on a surface. Of particular interest is the component that generalizes the Fenchel-Nielsen twists, which we will refer to as *twist-bulges*.

2.1.1 Parametrization of the deformation space $\mathfrak{P}(S)$

Definition 2.1. A *convex* $\mathbb{RP}^2$*-manifold* is a quotient $M = \Omega/\Gamma$, where $\Omega \subset \mathbb{RP}^2$ is a convex domain, $\Gamma \subset$ PSL(3,$\mathbb{R}$) is discrete and acts properly on Ω. We can identify Ω with the universal covering of M and Γ with the fundamental group $\pi_1(M)$ of M. Two homotopy marked convex $\mathbb{RP}^2$-manifolds $M_1 = \Omega_1/\Gamma_1, M_2 = \Omega_2/\Gamma_2$ are *projectively equivalent* if there exists $h \in$PSL(3,$\mathbb{R}$) such that $h\Omega_1 = \Omega_2$ and $h\Gamma_1 h^{-1} = \Gamma_2$.

A *convex* $\mathbb{RP}^2$ *structure* on a surface S is a diffeomorphism $f : S \to M$ where M is a convex $\mathbb{RP}^2$-manifold. Two such structures $(f_1, M_1), (f_2, M_2)$ are considered equivalent if there is a projective equivalence $h : M_1 \to M_2$ such that $h \circ f_1$ is isotopic to f_2.

Remark: Kuiper [10] classified convex $\mathbb{RP}^2$-manifolds S with $\chi(S) \geq 0$ in the 1950's, and so hereafter we will assume that our surface S is a closed orientable surface of genus $g > 1$.

A basic example of a convex $\mathbb{RP}^2$ structure on a surface S is the one inherited from a hyperbolic structure on S. This structure is the one coming from the Klein model. In fact, if $\partial\Omega$ is a conic, a convex $\mathbb{RP}^2$ structure on S reduces to a hyperbolic structure. This is included in the list of the following fundamental facts about convex $\mathbb{RP}^2$ structures due to Kuiper [10] and Benzécri [9]:

Theorem. (Kuiper, Benzécri) *Let $S = \Omega/\Gamma$ be a closed surface of genus $g > 1$ with a convex $\mathbb{RP}^2$ structure. Then we have the following facts:*

i) $\Omega \subset \mathbb{RP}^2$ is strictly convex.

ii) Either $\partial\Omega$ is a conic in $\mathbb{RP}^2$ or is not $C^{1+\epsilon}$ for some $0 < \epsilon < 1$.

iii)If $\gamma \in \Gamma$ is nontrivial, then γ has positive distinct real eigenvalues.

Furthermore, the set of projective equivalence classes of convex $\mathbb{RP}^2$ structures on a surface S can be identified with an open subspace of Hom($\pi_1(S)$,PSL(3,$\mathbb{R}$))/PSL(3,$\mathbb{R}$) of conjugacy classes of representations $\rho : \pi_1(S) \to$PSL(3,$\mathbb{R}$), which we denote by $\mathfrak{P}(S)$ and call this space the *deformation space of convex $\mathbb{RP}^2$ structures on S*. Goldman provided an explicit parametrization of $\mathfrak{P}(S)$ via an extension of the Fenchel-Nielsen parametrization of Teichmüller space $\mathcal{T}(S)$. Our main results involve calculating the effect of the vector field t_α associated to Goldman's generalized twist-bulge deformation on the Hilbert length functional l_β for geodesics α and β, and so this parametrization is of key importance.

Theorem. (Goldman)[2] *Let S be a closed oriented surface of genus $g > 1$. Then the deformation space $\mathfrak{P}(S)$ of convex $\mathbb{RP}^2$ structures on S is diffeomorphic to an open cell of dimension $16g - 16$.*

Essentially the parametrization proceeds in the following way: associated to a pants decomposition of the surface S, the coordinates are of two types. The first component of the coordinates are the twist-bulge parameters describing a deformation along the geodesic boundary components of the pairs of pants. The complete parametrization is provided via these twist-bulge parameters together with "internal parameters" associated to each pair of pants. A dimension count then yields the result.

2.1.2 Twist-bulge deformations and the geometry of convex real projective structures

In order to eventually carry out the calculations necessary to prove our main results, we now give a detailed description of the twist-bulge deformations. For the most part we follow the exposition in Goldman [2] and Zocca [12]

Recall that the classical cross-ratio of four points z_1, z_2, z_3, z_4 on a projective line is given by

$$(z_1, z_2; z_3, z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)} \tag{2.1}$$

The cross-ratio is a well-known projective invariant for any four such distinct points.

Let $\Omega \subset \mathbb{RP}^2$ be convex. The *Hilbert distance* on Ω is defined to be

$$h_\Omega(a, b) := |\log(a, b; x, y)| \tag{2.2}$$

where $a, b \in \Omega$, x, y are the points of intersection of the line (ab) and the boundary $\partial\Omega$, and $(a, b; x, y)$ is the classical cross-ratio of the quadruple (a, b, x, y). This defines a metric on convex subsets $\Omega \subset \mathbb{RP}^2$. (See [2]). In the case that $\partial\Omega$ is a conic, the Hilbert metric is the hyperbolic metric, and as mentioned above the convex real projective structure in this case reduces to a hyperbolic structure. In fact, a choice of a conic in $\mathbb{RP}^2$ is the geometric equivalent of choosing an algebraic embedding of PSL(2,$\mathbb{R}$) into PGL(3,$\mathbb{R}$). Furthermore, we can explicitly write the embedding of

$$\mathrm{Hom}(\pi_1(S), \mathrm{PSL}(2, \mathbb{R})) \subset \mathrm{Hom}(\pi_1(S), \mathrm{PGL}(3, \mathbb{R})) \tag{2.3}$$

by identifying PSL(2,$\mathbb{R}$) with a connected component of SO(2,1) $\subset$ GL(3,$\mathbb{R}$).

Let $\beta \in \pi_1(S)$ be nontrivial. Using the identification of $\pi_1(S)$ with $\Gamma \subset$ PGL(3,$\mathbb{R}$) as before, together with the isomorphism PGL(3,$\mathbb{R}$) $\cong$ SL(3,$\mathbb{R}$) and the theorem above, then if $B \in$ SL(3,$\mathbb{R}$) is the matrix associated to β, then in particular B is conjugate in SL(3,$\mathbb{R}$) to a diagonal matrix with positive eigenvalues:

$$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{bmatrix} \tag{2.4}$$

where

$$\lambda\mu\nu = 1, \quad 0 < \lambda < \mu < \nu \tag{2.5}$$

Now, let $p_1 = [1, 0, 0]$, $p_2 = [0, 1, 0]$, $p_3 = [0, 0, 1]$ be the points corresponding to the coordinate axes in $\mathbb{R}^3$. The three lines joining these points separate $\mathbb{RP}^2$ into four triangular regions. A projective transformation $A \in$ SL(3,$\mathbb{R}$) that fixes the points p_1, p_2, p_3 is represented by a unique diagonal matrix in SL(3,$\mathbb{R}$) and in fact A leaves invariant each of the four triangular regions if and only if it is represented by a diagonal matrix with positive eigenvalues as above in (2.4).

Let $B \in$ SL(3,$\mathbb{R}$) be as above, represented by a diagonal matrix (2.4). Then the set Fix(B) of fixed points consists of three noncollinear points, and we define $\text{Fix}_-(B)$ to be the repelling fixed point associated to the smallest eigenvalue λ, $\text{Fix}_+(B)$ to be the attracting fixed point associated to the largest eigenvalue ν, and $\text{Fix}_0(B)$ to be the saddle fixed point associated to the middle eigenvalue μ. We define $l(B) \subset \mathbb{RP}^2$ to be the line joining the attracting and repelling fixed points of B, and we call this the *principal line* for B. Note that by the remarks above, in the case that B is diagonal, $\text{Fix}_+(B) = [1, 0, 0]$, $\text{Fix}_0(B) = [0, 1, 0]$, and $\text{Fix}_-(B) = [0, 0, 1]$. In fact, the basic picture of the fundamental triangle (see Figure 1) described in this way is important in visualizing the effect of the twist-bulge deformations, and we will use the associated terminology and notation throughout the rest of the paper.

We now consider the effect of the PGL(3,$\mathbb{R}$) action via calculating its effect on the displacement of points relative to the Hilbert metric. Let $B \in$ SL(3,$\mathbb{R}$) be represented by diagonal matrix (2.4) as above. We consider two cases: that of a point lying on the principal line $l(B)$ and that of a point not on the principal line. In the first case, if a is on the principal line $l(B)$, then $a = [a_1, 0, a_3]$, whence $B(a) = [\lambda a_1, 0, \nu a_3]$, and the Hilbert distance $h_\Omega(a, B(a))$ is

$$h_\Omega(a, B(a)) = \log(\frac{\nu}{\lambda}) \tag{2.6}$$

Thus the action of a matrix $B \in$ SL(3,$\mathbb{R}$) on a point lying on the principal line $l(B)$ displaces the point by $\log(\frac{\nu}{\lambda})$ relative to the Hilbert metric. Note again that in the PSL(2,$\mathbb{R}$) case, i.e. $\partial\Omega$ is a conic, then the Hilbert metric is the hyperbolic metric, and this displacement is simply the hyperbolic distance between a and $B(a)$.

Next, we choose a point a not lying on the principal line $l(B)$. We will see that in this case, there are two components to the action, which we refer to as the *horizontal* and *vertical* components. Indeed, the action of PSL(2,$\mathbb{R}$) moves points lying off the principal line $l(B)$ from the repelling fixed point to the attracting fixed point "horizontally": If $B \in$ PSL(2,$\mathbb{R}$), then a point $a = [a_1, a_2, a_3]$ maps to $B(a) = [\lambda a_1, a_2, \frac{1}{\lambda}a_3]$.

To see the vertical component of the action, note that the Lie algebra of the group of diagonal matrices of PSL(2,$\mathbb{R}$) in SL(3,$\mathbb{R}$) is given by

$$\left\{\begin{bmatrix} -t & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & t \end{bmatrix}\right\} \in \left\{\begin{bmatrix} -t-s & 0 & 0 \\ 0 & 2s & 0 \\ 0 & 0 & t-s \end{bmatrix}\right\}$$

Furthermore, corresponding to the Lie algebra generated by

$$\left\{\begin{bmatrix} -s & 0 & 0 \\ 0 & 2s & 0 \\ 0 & 0 & -s \end{bmatrix}\right\}$$

is the group H defined by

$$H = \left\{\begin{bmatrix} \frac{1}{\sqrt{\mu}} & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \frac{1}{\sqrt{\mu}} \end{bmatrix}\right\}$$

and H is orthogonal to PSL(2,$\mathbb{R}$) in SL(3,$\mathbb{R}$): in particular, if $a = [a_1, a_2, a_3]$ is a point not lying on the principal line as above, then H moves the point a along the line through a connecting the saddle fixed point $\mathrm{Fix}_0(B) = [0, 1, 0]$ with the principal line

$l(B)$. To see this, note that the image of a point $a = [a_1, a_2, a_3]$ under H is given by $H(a) = [\frac{1}{\sqrt{\mu}}a_1, \mu a_2, \frac{1}{\sqrt{\mu}}a_3]$ and the Hilbert distance is

$$\begin{aligned} h_\Omega(a, H(a)) &= \log|([a_1, a_2, a_3], [\frac{1}{\sqrt{\mu}}a_1, \mu a_2, \frac{1}{\sqrt{\mu}}a_3]; [0, 1, 0], [\frac{a_1}{1-a_2}, 0, \frac{a_3}{1-a_2}])| \\ &= \log|([a_1, a_2, a_3], [a_1, \sqrt{\mu^3}a_2, a_3]; [0, 1, 0], [a_1, 0, a_3])| \\ &= \log(\sqrt{\mu^3}) = \frac{3}{2}\log(\mu) \end{aligned}$$

This represents the "vertical" displacement by H.

In summary, if the vertical line passing through point $a = [a_1, a_2, a_3]$ is given as $[a_1, ta_2, a_3]$(i.e. it is the pencil based at $[0, 1, 0]$) and the line representing the horizontal direction is given by $[\frac{2}{t+1}a_1, a_2, \frac{2t}{t+1}a_3]$ (i.e. it is the tangent line at $a = [a_1, a_2, a_3]$ of the orbit $[a_1, ta_2, t^2a_3]$ through the point), then $B \in$ SL(3,$\mathbb{R}$) as above moves the point a horizontally by $\log(\frac{\nu}{\lambda})$ and vertically by $\frac{3}{2}\log(\mu)$. These are Goldman's (l, m) parameters, respectively, where $l = \log(\frac{\nu}{\lambda})$ represents the translation by the PSL(2,$\mathbb{R}$) component of SL(3,$\mathbb{R}$), and $m = \frac{3}{2}\log(\mu)$ represents its orthogonal translation (see Figure 1).

Finally, we are ready to define the twist-bulge deformation in the form of an $\mathbb{R}^2$ action on $\mathfrak{P}(S)$. Let $(u, v) \in \mathbb{R}^2$ and let a point $x \in \mathfrak{P}(S)$ be represented by a convex real projective manifold M. From this we wish to construct a new convex real projective manifold $\Psi_{(u,v)}(M) \in \mathfrak{P}(S)$. Let $p : \tilde{M} \to M$ be the universal covering of M and let (dev,h) be a developing pair. Let γ be a simple closed geodesic on M and suppose we have chosen a representative element $\gamma \in \pi_1$. Then by the above theorem, this can be done so that $h(\gamma)$ can be represented by matrix (2.4). Now the centralizer of $h(\gamma)$ in SL(3,$\mathbb{R}$) is the full group of diagonal matrices in SL(3,$\mathbb{R}$). This has identity component that is the direct product of the following two one-parameter groups

$$T^u = \begin{bmatrix} e^{-u} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^u \end{bmatrix}, \quad U^v = \begin{bmatrix} e^{-v} & 0 & 0 \\ 0 & e^{2v} & 0 \\ 0 & 0 & e^{-v} \end{bmatrix}$$

where $(u, v) \in \mathbb{R}^2$ as above.

Now, define $M|\gamma$ to be the split real projective manifold formed by cutting along the geodesic γ, and let $c_1, c_2 \subset \partial(M|\gamma)$ be the two boundary components corresponding to γ. For any $(u, v) \in \mathbb{R}^2$, there are principal collar neighborhoods of c_i which we denote by $N(c_i) \subset M|\gamma$ for $i = 1, 2$. Furthermore, there is a projective isomorphism $f : N(c_1) \to N(c_2)$ such that f is related via the developing map dev to the projective transformation $g_{(u,v)}$, where

$$g_{(u,v)} = T^u U^v = \begin{bmatrix} e^{-u-v} & 0 & 0 \\ 0 & e^{2v} & 0 \\ 0 & 0 & e^{u-v} \end{bmatrix}$$

We call $g_{(u,v)}$ the *twist-bulge deformation* along the geodesic γ (Compare this with our previous discussion of the SL(3,$\mathbb{R}$) action on a point lying in the interior of a fundamental triangle region).

To finish the construction of the new convex real projective manifold, we identify $\Psi_{(u,v)}(M)$ with the real projective manifold $(M|\gamma)/f$, i.e. the manifold obtained from $(M|\gamma)$ by identifying the principal collar neighborhoods $N(c_i)$ of the boundary components c_i by the projective isomorphism $f : N(c_1) \to N(c_2)$. This constructon of $(M|\gamma)/f$ is independent of the choices of collar neighborhoods, and so given $(u, v) \in \mathbb{R}^2$ and a convex real projective manifold M, we have constructed a new convex real projective manifold $\Psi_{(u,v)}(M) \in \mathfrak{P}(S)$ by way of the twist-bulge deformations $g_{(u,v)}$ defined above. The flows $\Psi_{(u,0)}$ and $\Psi_{(0,v)}$ are examples of generalized twist flows showing up in Goldman [2], and the potential functions of these twist flows are exactly the (l, m) coordinates associated to γ described previously.

2.2 Hitchin representations and equivariant flag curves

We now state results for representations into PSL(n,$\mathbb{R}$), applying them later in the case where $n = 3$. Our exposition closely follows the treatment of Bonahon-Dreyer [1].

Let S be a closed oriented surface of genus $g > 1$ as above, and let $\rho : \pi_1(S) \to$PSL(n,$\mathbb{R}$) be a representation of its fundamental group into PSL(n,$\mathbb{R}$).

Let

$$\mathcal{R}_{PSL(n,\mathbb{R})}(S) = Hom(\pi_1(S), PSL(n, \mathbb{R}))//PSL(n, \mathbb{R}) \tag{2.7}$$

where the action of PSL(n,$\mathbb{R}$) is by conjugation. In the case when $n = 2$, $\mathcal{R}_{PSL(2,\mathbb{R})}(S)$ has $4g - 3$ components [Go2], two of which correspond to all injective homomorphisms $\rho : \pi_1(S) \to$ PSL(2,$\mathbb{R}$) having discrete image in PSL(2,$\mathbb{R}$). The orientation of S picks out one of these components: the one where the map $S \to \mathbb{H}^2/\rho(\pi_1(S))$ has degree $+1$. This is the *Teichmüller component* $\mathcal{T}(S)$ of $\mathcal{R}_{PSL(2,\mathbb{R})}(S)$.

Now, the homomorphism PSL(2,$\mathbb{R}$)$\to$ PSL(n,$\mathbb{R}$) induces a map $\mathcal{R}_{PSL(2,\mathbb{R})}(S) \to \mathcal{R}_{PSL(n,\mathbb{R})}(S)$, and under this map the *Hitchin component* $\mathrm{Hit}_n(S)$ is the component of $\mathcal{R}_{PSL(n,\mathbb{R})}(S)$ containing the image of the Teichmüller component of $\mathcal{R}_{PSL(2,\mathbb{R})}(S)$. A *Hitchin representation* $\rho : \pi_1(S) \to$ PSL(n,$\mathbb{R}$) is an element of the Hitchin component $\mathrm{Hit}_n(S)$.

Hitchin [8] used the theory of Higgs bundles to prove the following:

Theorem 2.2.1. (Hitchin [8]): *When $n \geq 3$, the character variety $\mathcal{R}_{PSL(n,\mathbb{R})}(S)$ has 3 or 6 components according to whether n is odd or even, and the Hitchin component $Hit_n(S)$ is diffeomorphic to $\mathbb{R}^{-\chi(S)(n^2-1)}$.*

Our interest lies mainly in the $n = 3$ case, where Choi and Goldman [3] show:

Theorem 2.2.1. (Choi-Goldman [3]): *For $n = 3$, the Hitchin component $Hit_n(S)$ consists of holonomies of convex real projective structures on S.*

Notice in particular that when $n = 3$, by the above theorems $\mathfrak{P}(S)$ $=\mathrm{Hit}_3(S)$ is diffeomorphic to $\mathbb{R}^{-8\chi(S)}$, providing a different proof of the dimension count of $\mathfrak{P}(S)$ $=\mathrm{Hit}_3(S)$.

These facts allow us to use an important property of Hitchin representations in general in our calculations for the $n = 3$ case of convex real projective structures, namely, the existence of an equivariant flag curve associated to a Hitchin representation that contains a great deal of geometric information on individual Hitchin representations, in stark contrast to Hitchin's original proof via techniques of Higgs bundles since although Hitchin's proof also provided an explicit parametrization of the Hitchin component $\mathrm{Hit}_n(S)$, the construction provides very little geometric information about the underlying representation.

2.2.1 Flag curves

Labourie [5] and Fock-Goncharov [4] independently established the existence of an equivariant flag curve associated to a Hitchin representation satisfying certain positivity conditions, among other properties. This flag curve is central to our calculations in Chapter 3, where it is used to define a generalized cross-ratio in the sense of Labourie [6],[7]. This flag curve has also been used by Bonahon and Dreyer [1] to provide an explicit parametrization of $\mathrm{Hit}_n(S)$ that is essentially a generalization of Goldman's parametrization of $\mathfrak{P}(S)$ =$\mathrm{Hit}_3(S)$. In their construction, the *triangle invariants* of Fock-Goncharov play the role of Goldman's internal parameters.

We begin by stating the following fact which will allow us to define the quantities necessary to state the important results of Labourie and Fock-Goncharov on the existence of the flag curve for a Hitchin representation and its various properties.

Theorem 2.2.1. (Labourie): *Let $\rho : \pi_1(S) \to PSL(n,\mathbb{R})$ be a Hitchin representation. Then for every nontrivial $\gamma \in \pi_1(S)$, the element $\rho(\gamma) \in PSL(n,\mathbb{R})$ admits a lift $\rho(\gamma)' \in SL(n,\mathbb{R})$ whose eigenvalues are distinct and all positive.*

Let $\rho : \pi_1(S) \to \mathrm{PSL(n,\mathbb{R})}$ be a Hitchin representation, with $\gamma \in \pi_1(S)$ nontrivial and $\rho(\gamma)' \in \mathrm{SL(n,\mathbb{R})}$ the lift of $\rho(\gamma) \in \mathrm{PSL(n,\mathbb{R})}$ as above. Define

$$\lambda_1^\rho(\gamma) > \lambda_2^\rho(\gamma) > ... > \lambda_n^\rho(\gamma) > 0 \tag{2.8}$$

to be the eigenvalues of the lift $\rho(\gamma)'$. Since in particular these eigenvalues are distinct, $\rho(\gamma)'$ is diagonalizable, and we define L_i to be the 1-dimensional eigenspace associated to the eigenvalue $\lambda_i^\rho(\gamma)$.

This allows us to define two flags $E, F \in \mathrm{Flag}(\mathbb{R}^n)$ associated to $\rho(\gamma)$ in the following way:

$$E^{(i)} = \bigoplus_{j=1}^{i} L_j \tag{2.9}$$

and

$$F^{(i)} = \bigoplus_{j=n-i+1}^{n} L_j \tag{2.10}$$

We call $E \in$Flag($\mathbb{R}^n$) the *stable flag* associated to $\rho(\gamma) \in$PSL(n,$\mathbb{R}$) and F is its *unstable flag*.

Let $S = \Omega/\Gamma$ be a convex real projective surface. Then we identify Ω with its universal cover, and let $\partial\Omega$ be its boundary at infinity. Every nontrivial $\gamma \in \pi_1(S)$ fixes two points of $\partial\Omega$, where one of them is the attracting fixed point and the other is the repelling fixed point. We are now prepared to state the fundamental result.

Theorem. (Labourie, Fock-Goncharov): *Let* $\rho : \pi_1(S) \to$ *PSL(n,$\mathbb{R}$) be a Hitchin representation. Then there exists a unique continuous map* $\mathcal{F}_\rho : \partial\Omega \to$*Flag($\mathbb{R}^n$) such that:*

(i) if $a_\gamma \in \partial\Omega$ *is the attracting fixed point of* $\gamma \in \pi_1(S)$*, then* $\mathcal{F}_\rho(a_\gamma) \in$*Flag($\mathbb{R}^n$) is the stable flag of* $\rho(\gamma) \in$*PSL(n,$\mathbb{R}$). The similar statement is true with attracting fixed point replaced by repelling and the stable flag replaced by the unstable.*

(ii) $\mathcal{F}_\rho$ *is equivariant with respect to* $\rho : \pi_1(S) \to$ *PSL(n,$\mathbb{R}$), i.e.* $\mathcal{F}_\rho(\gamma x) = \rho(\gamma)(x)$ *for every* $\gamma \in \pi_1(S)$ *and every* $x \in \partial\Omega$.

(iii) for any two distinct points $x_1, x_2 \in \partial\Omega$*, the flag pair* $(\mathcal{F}_\rho(x_1), \mathcal{F}_\rho(x_2))$ *is generic.*

(iv)for any three distinct points $x_1, x_2, x_3 \in \partial\Omega$*, the flag triple* $(\mathcal{F}_\rho(x_1), \mathcal{F}_\rho(x_2), \mathcal{F}_\rho(x_3))$ *is positive.*

(v)for any four distinct point $x_1, x_2, x_3, x_4 \in \partial\Omega$ *occurring in this order around the boundary at infinity* $\partial\Omega$*, the flag quadruple* $(\mathcal{F}_\rho(x_1), \mathcal{F}_\rho(x_2), \mathcal{F}_\rho(x_3), \mathcal{F}_\rho(x_4))$ *is positive.*

We call $\mathcal{F}_\rho : \partial\Omega \to$Flag($\mathbb{R}^n$) the *equivariant flag curve* associated to the Hitchin representation $\rho : \pi_1(S) \to$ PSL(n,$\mathbb{R}$). Bonahon-Dreyer use the flag curve to define their parametrization of $\mathrm{Hit}_n(S)$, with the latter positivity conditions implying the non-degeneracy of certain quantities involved, namely the so-called triangle invariants associated to a triangulization of the surface S.

In the next section we use this flag curve to define a generalized cross-ratio of four points $x_1, x_2, x_3, x_4 \in \partial\Omega$ which allows us to define the geodesic length function l_* and begin our calculation of $t_\alpha l_\beta$. As for Bonahon-Dreyer, the positivity conditions of the flag curve imply the non-degeneracy of our generalized cross-ratio.

Chapter 3

Main Results

3.1 Setup and a preliminary lemma

Let S be a closed oriented surface of genus $g > 1$ with a convex real projective structure, so that $S = \Omega/\Gamma$ as above. Let $\rho : \pi_1(S) \to$ PSL(3,$\mathbb{R}$) be the associated Hitchin representation. Thus by the theorem of Labourie-Fock-Goncharov, there exists the equivariant flag curve $\mathcal{F}_\rho : \partial\Omega \to$Flag($\mathbb{R}^3$) satisfying the aforementioned properties. We define a *generalized cross-ratio* of four distinct boundary points $x_1, x_2, x_3, x_4 \in \partial\Omega$ in the sense of Labourie in the following way:

Definition 1. Let $x_1, x_2, x_3, x_4 \in \partial\Omega$ be four distinct points on the boundary at infinity. Associated to each x_i is the flag $\mathcal{F}_\rho(x_i) \in$Flag($\mathbb{R}^3$), which we write as a covector-vector pair $\mathcal{F}_\rho(x_i) = (\phi_i, v_i)$. The *generalized cross-ratio* of the quadruple (x_1, x_2, x_3, x_4) is defined to be

$$b(x_1, x_2, x_3, x_4) = \frac{\phi_1(v_3)\phi_2(v_4)}{\phi_1(v_4)\phi_2(v_3)} \tag{3.1}$$

where the pairing $\phi_i(v_j)$ is the inner product. The *period* of an element $B \in \pi_1(S)$ associated to the cross-ratio b is

$$l_b(B) := \log|b(Bt, t, r_B, a_B)| \tag{3.2}$$

where a_B, r_B are the attracting and repelling fixed points associated to the element $B \in \pi_1(S)$ and t is any other element of the boundary $\partial\Omega$. The period is independent of the choice of t by invariance under the action of B and properties of generalized cross-ratios (see Labourie [7]).

Remark. For the $n = 2$ case, notice that the analogously-defined cross-ratio reduces exactly to the classical cross-ratio.

We wish to compute the twist-bulge derivative $t_\alpha l_\beta$, where α and β are simple closed geodesics on S, t_α is the twist-bulge vector field associated to the twist-bulge deformation along geodesic α, and l_β is the period defined above.

To this end, as in the calculation in Wolpert [11], we wish to first compute the twist-bulge derivative of a generalized cross-ratio as defined previously:

Lemma 1. *Let S be a closed oriented surface of genus $g > 1$ with a convex real projective structure (so identify S with quotient Ω/Γ with $\Omega \subset \mathbb{RP}^2$ convex and $\Gamma \subset SL(3,\mathbb{R})$ in the usual way). Let x_1, x_2, x_3, x_4 be distinct points on the boundary $\partial\Omega$, and let $\widehat{s_1 s_2}$ be an oriented geodesic with $s_1, s_2 \in \partial\Omega$ its endpoints. Then the* twist-bulge derivative *of the generalized cross-ratio of the boundary points x_1, x_2, x_3, x_4 with respect to the twist-bulge deformation along geodesic $\widehat{s_1 s_2}$ is given by:*

$$t_{\widehat{s_1 s_2}} b(x_1,x_2,x_3,x_4) = b(x_1,x_2,x_3,x_4)\left[\sum_{i=1}^{2} \chi_L(x_i)\left(\frac{\phi_i^T L_{\widehat{s_1 s_2}} v_{\sigma(i)}}{\phi_i^T \cdot v_{\sigma(i)}} - \frac{\phi_i^T L_{\widehat{s_1 s_2}} v_{\tau(i)}}{\phi_i^T \cdot v_{\tau(i)}}\right) - \sum_{i=3}^{4} \chi_L(x_i)\left(\frac{\phi_{\sigma(i)}^T L_{\widehat{s_1 s_2}} v_i}{\phi_{\sigma(i)}^T \cdot v_i} - \frac{\phi_{\tau(i)}^T L_{\widehat{s_1 s_2}} v_i}{\phi_{\tau(i)}^T \cdot v_i}\right)\right]$$

where χ_L is the characteristic function on the left half of Ω (see Figure 2) as defined by $\widehat{s_1 s_2}$, (ϕ_i, v_i) is the covector-vector pair associated to x_i by the flag curve $\mathcal{F}_\rho : \partial\Omega \to$Flag$(\mathbb{R}^3)$, $\sigma = (13)(24)$ and $\tau = (14)(23)$ are permutations coming from the labeling of our generalized cross-ratio, and $L_{\widehat{s_1 s_2}} \in \mathfrak{sl}(3,\mathbb{R})$ is the infinitesimal generator associated to the twist-bulge deformation along geodesic $\widehat{s_1 s_2}$.

Remark. The formula for $t_{\widehat{s_1s_2}}b(x_1, x_2, x_3, x_4)$ holds if we replace χ_L by $-\chi_R$, where χ_R is the characteristic function on the right half of Ω. Without loss of generality we can orient our picture so that the twist-bulge deformation along $\widehat{s_1s_2}$ is bulging to the left.

Proof. Let S be as above, and let x_1, x_2, x_3, x_4 be distinct points on the boundary $\partial\Omega$. The equivariant flag curve $\mathcal{F}_\rho : \partial\Omega \to \text{Flag}(\mathbb{R}^3)$ associates to each of these points x_i a flag $\mathcal{F}_\rho(x_i) \in \text{Flag}(\mathbb{R}^3)$, so in particular it associates to each x_i a pair (L_i, P_i), where L_i is a line and P_i is a plane containing it. Equivalently, we can write this as a covector-vector pair $\mathcal{F}_\rho(x_i) = (\phi_i, v_i)$, with $\phi_i(v_i) := \phi_i^T \cdot v_i = 0$, and we write

$$v_i = \begin{pmatrix} a_i \\ b_i \\ c_i \end{pmatrix}, \qquad \phi_i = \begin{pmatrix} a_i' & b_i' & c_i' \end{pmatrix}$$

where all of the entries are some real numbers, with $a_i a_i' + b_i b_i' + c_i c_i' = 0$. Under this formulation, recall from Definition 1 that that generalized cross-ratio of the points x_1, x_2, x_3, x_4 is

$$b(x_1, x_2, x_3, x_4) = \frac{\phi_1(v_3)\phi_2(v_4)}{\phi_1(v_4)\phi_2(v_3)}$$

Now, let $\widehat{s_1s_2}$ be an oriented geodesic with endpoints s_1, s_2 on the boundary. Recall from chapter 2 that the twist-bulge deformation $g_{(u,v)}$ along $\widehat{s_1s_2}$ is given by

$$g_{(u,v)} = \begin{bmatrix} e^{-u-v} & 0 & 0 \\ 0 & e^{2v} & 0 \\ 0 & 0 & e^{u-v} \end{bmatrix}$$

We consider the effect of the twist-bulge deformation $g_{(u,v)}$ along $\widehat{s_1s_2}$ on the generalized cross-ratio $b(x_1, x_2, x_3, x_4)$. As mentioned earlier, without loss of generality we orient our picture so that the twist-bulging is occuring on the left half of the domain Ω as defined by $\widehat{s_1s_2}$ (see Figure 2). For a point x_i lying on the affected (left) half of the domain, the action of $g_{(u,v)}$ is given by

$$g_{(u,v)}(x_i) = g_{(u,v)} \cdot (\phi_i, v_i) = (\phi_i g^*_{(u,v)}, g_{(u,v)} v_i) = (\phi_i g_{(-u,-v)}, g_{(u,v)} v_i)$$

where $g^*_{(u,v)}$ is the adjoint.

Define $b_{(u,v)}(x_1, x_2, x_3, x_4)$ to be the deformed generalized cross-ratio under this action. In order to compute

$$t_{\widehat{s_1 s_2}} b(x_1, x_2, x_3, x_4)$$

we differentiate $b_{(u,v)}(x_1, x_2, x_3, x_4)$ with respect to the twist-bulge parameters (u, v), and in order to do so we must consider separately cases depending on the the relative positions of the points x_1, x_2, x_3, x_4 on the boundary.

Firstly, note that if x_1, x_2, x_3, x_4 all lie in a common half of $\partial\Omega$, then

$$b_{(u,v)}(x_1, x_2, x_3, x_4) = b(x_1, x_2, x_3, x_4),$$

and so the derivative is zero.

Let us first consider the case where a single point x_1 lies on the left, i.e. on the side where the twist-bulge is occurring. In this case, we have

$$b_{(u,v)}(x_1, x_2, x_3, x_4) = \frac{(\phi_1 g^*_{(u,v)})(v_3)\phi_2(v_4)}{(\phi_1 g^*_{(u,v)})(v_4)\phi_2(v_3)},$$

We now compute the derivative $\frac{\partial}{\partial u}$ of this quantity (the calculation for $\frac{\partial}{\partial v}$ is similar): Let

$$C_{(u,v)} = e^{u+v} a_1' a_3 + e^{-2v} b_1' b_3 + e^{-u+v} c_1' c_3$$

and

$$E_{(u,v)} = e^{u+v} a_1' a_4 + e^{-2v} b_1' b_4 + e^{-u+v} c_1' c_4$$

Then in this case

$$b_{(u,v)}(x_1, x_2, x_3, x_4) = \frac{C_{(u,v)}\phi_2(v_4)}{E_{(u,v)}\phi_2(v_3)}$$

and so

$$\begin{aligned}
\frac{\partial}{\partial u} b_{(u,v)}(x_1, x_2, x_3, x_4) &= \left(\frac{\partial}{\partial u} \frac{C_{(u,v)}}{E_{(u,v)}} \right) \frac{\phi_2(v_4)}{\phi_2(v_3)} \\
&= \left[\frac{\left(\frac{\partial}{\partial u} C_{(u,v)}\right) E_{(u,v)} - C_{(u,v)} \left(\frac{\partial}{\partial u} E_{(u,v)}\right)}{E^2_{(u,v)}} \right] \frac{\phi_2(v_4)}{\phi_2(v_3)} \\
&= \left[\frac{(e^{u+v} a'_1 a_3 - e^{-u+v} c'_1 c_3)\, E_{(u,v)} - C_{(u,v)}\, (e^{u+v} a'_1 a_4 - e^{-u+v} c'_1 c_4)}{E^2_{(u,v)}} \right] \frac{\phi_2(v_4)}{\phi_2(v_3)}
\end{aligned}$$

and thus upon noting that

$$C_{(0,0)} = \phi_1(v_3), \qquad E_{(0,0)} = \phi_1(v_4)$$

we have

$$\begin{aligned}
\left.\frac{\partial}{\partial u} b_{(u,v)}(x_1, x_2, x_3, x_4)\right|_{(u,v)=(0,0)} &= \left[\frac{(a'_1 a_3 - c'_1 c_3)\, \phi_1(v_4) - \phi_1(v_3)\, (a'_1 a_4 - c'_1 c_4)}{(\phi_1(v_4))^2} \right] \frac{\phi_2(v_4)}{\phi_2(v_3)} \\
&= \frac{a'_1 a_3 - c'_1 c_3}{\phi_1(v_4)} \frac{\phi_2(v_4)}{\phi_2(v_3)} - b(x_1, x_2, x_3, x_4) \frac{a'_1 a_4 - c'_1 c_4}{\phi_1(v_4)} \\
&= b(x_1, x_2, x_3, x_4) \left[\frac{a'_1 a_3 - c'_1 c_3}{\phi_1(v_3)} - \frac{a'_1 a_4 - c'_1 c_4}{\phi_1(v_4)} \right] \\
&= b(x_1, x_2, x_3, x_4) \left[\frac{\phi_1^T L_{\widehat{s_1 s_2}} v_3}{\phi_1^T \cdot v_3} - \frac{\phi_1^T L_{\widehat{s_1 s_2}} v_4}{\phi_1^T \cdot v_4} \right]
\end{aligned}$$

where $L_{\widehat{s_1 s_2}} \in \mathfrak{sl}(3, \mathbb{R})$ is the Lie algebra element that is the infinitesimal generator associated to the twist-bulge deformation along geodesic $\widehat{s_1 s_2}$.

Similarly, in the case when we have two points x_1, x_2 in the left half, then

$$b_{(u,v)}(x_1, x_2, x_3, x_4) = \frac{(\phi_1 g^*_{(u,v)})(v_3)(\phi_2 g^*_{(u,v)})(v_4)}{(\phi_1 g^*_{(u,v)})(v_4)(\phi_2 g^*_{(u,v)})(v_3)}$$

and so the derivative is given by

$$\left.\frac{\partial}{\partial u} b_{(u,v)}\right|_{(u,v)=(0,0)} = b(x_1,x_2,x_3,x_4)\left[\frac{a_1'a_3 - c_1'c_3}{\phi_1(v_3)} - \frac{a_1'a_4 - c_1'c_4}{\phi_1(v_4)} + \frac{a_2'a_4 - c_2'c_4}{\phi_2(v_4)} - \frac{a_2'a_3 - c_2'c_3}{\phi_2(v_3)}\right]$$
$$= b(x_1,x_2,x_3,x_4)\left[\frac{\phi_1^T L_{\widehat{s_1s_2}} v_3}{\phi_1^T \cdot v_3} - \frac{\phi_1^T L_{\widehat{s_1s_2}} v_4}{\phi_1^T \cdot v_4} + \frac{\phi_2^T L_{\widehat{s_1s_2}} v_4}{\phi_2^T \cdot v_4} - \frac{\phi_2^T L_{\widehat{s_1s_2}} v_3}{\phi_2^T \cdot v_3}\right]$$

In the final case where we have three points x_1, x_2, x_3 on the left, a similar calculation shows that

$$\left.\frac{\partial}{\partial u} b_{(u,v)}\right|_{(u,v)=(0,0)} = b(x_1,x_2,x_3,x_4)\left[\frac{\phi_1^T L_{\widehat{s_1s_2}} v_3}{\phi_1^T \cdot v_3} - \frac{\phi_1^T L_{\widehat{s_1s_2}} v_4}{\phi_1^T \cdot v_4} + \frac{\phi_2^T L_{\widehat{s_1s_2}} v_4}{\phi_2^T \cdot v_4} - \frac{\phi_2^T L_{\widehat{s_1s_2}} v_3}{\phi_2^T \cdot v_3} - \frac{\phi_1^T L_{\widehat{s_1s_2}} v_3}{\phi_1^T \cdot v_3} + \frac{\phi_2^T L_{\widehat{s_1s_2}} v_3}{\phi_2^T \cdot v_3}\right]$$

so that

$$\left.\frac{\partial}{\partial u} b_{(u,v)}\right|_{(u,v)=(0,0)} = b(x_1,x_2,x_3,x_4)\left[-\frac{\phi_1^T L_{\widehat{s_1s_2}} v_4}{\phi_1^T \cdot v_4} + \frac{\phi_2^T L_{\widehat{s_1s_2}} v_4}{\phi_2^T \cdot v_4}\right]$$

Note that in this last case, we see that the formula holds if we replace χ_L with$-\chi_R$ (in fact we could have seen this directly from $b_{(u,v)}(x_1,x_2,x_3,x_4)$ in this case), and so the proof of the lemma is complete. □

For distinct points x_i, x_j on the boundary $\partial\Omega$, the flag curve $\mathcal{F}_\rho : \partial\Omega \to$Flag($\mathbb{R}^3$) associates covector-vector pairs $\mathcal{F}_\rho(x_i) = (\phi_i, v_i)$ and $\mathcal{F}_\rho(x_j) = (\phi_j, v_j)$, and let s_1, s_2 be endpoints of a geodesic $\widehat{s_1s_2}$. For such points we define

$$I_{\widehat{s_1s_2}}(x_i,x_j) := \frac{\phi_i^T L_{\widehat{s_1s_2}} v_j}{\phi_i^T \cdot v_j} \tag{3.3}$$

Then the formula in lemma 3.1 becomes

$$t_{\widehat{s_1s_2}}b(x_1,x_2,x_3,x_4) = b(x_1,x_2,x_3,x_4)\left[\sum_{i=1}^{2}\chi_L(x_i)\left(I_{\widehat{s_1s_2}}(x_i,x_{\sigma(i)}) - I_{\widehat{s_1s_2}}(x_i,x_{\tau(i)})\right)\right.$$
$$\left. - \sum_{i=3}^{4}\chi_L(x_i)\left(I_{\widehat{s_1s_2}}(x_{\sigma(i)},x_i) - I_{\widehat{s_1s_2}}(x_{\tau(i)},x_i)\right)\right]$$

This will simplify the notation in the upcoming calculations.

3.2 Main results

We are now ready to state and prove the formulas for $t_\alpha l_\beta$ and $t_\alpha t_\beta l_\gamma$.

Theorem 1. *Let S be as in Lemma 1, and let α, β be simple closed geodesics on S. Let t_α be the vector field associated to the twist-bulge deformation along α, and let l_β be the geodesic length function defined in Definition 1. Then*

$$t_\alpha l_\beta = \sum_{C\in\mathfrak{J}}\left[I_{\widehat{c_1c_2}}(r_B,a_B) - I_{\widehat{c_1c_2}}(a_B,r_B)\right]$$

where the sum is over representatives C in a particular double coset, $\widehat{c_1c_2}$ is the geodesic associated to the representative C, and r_B, a_B are the repelling and attracting fixed points of matrix B associated to geodesic β, respectively.

Proof. Choose t on the boundary such that t is not fixed by an element of Γ, and let $B\in$SL(3,$\mathbb{R}$) be the matrix associated to the geodesic β, with r_B and a_B its repelling and attracting fixed points, respectively. By Definition 1 (3.2), the quantity

$$l_\beta = \log b(Bt,t,r_B,a_B)$$

is independent of the choice of t. Let A be a matrix associated to the geodesic α, and let $\widehat{a_1a_2}$ be the axis of A. Assume that $\widehat{a_1a_2}$ separates r_B and a_B, with r_B to its left. Cosets $\langle A\rangle\backslash\Gamma$ are identified with distinct translates of $\widehat{a_1a_2}$ in the following way: $C^{-1}(\widehat{a_1a_2})$ is identified with $\langle A\rangle$C$\in\langle A\rangle\backslash\Gamma$. Define $\mathfrak{J}$ to be the subset of elements $C\in\Gamma$ such that

$C^{-1}(\widehat{a_1a_2})$ separates r_B and a_B. In particular, $\langle B\rangle$ acts on $\mathfrak{J}$ on the right by multiplication: For an element $C \in \mathfrak{J}$, $CB^n \in \mathfrak{J}$ for $n \in \mathbb{Z}$. Now, the double cosets $\langle A\rangle\backslash\mathfrak{J}/\langle B\rangle$ are identified with the intersection locus for the geodesics α and β.

We now compute the contribution to the derivative $t_\alpha l_\beta$ coming from the axes of $\langle A\rangle\backslash\mathfrak{J}$, and we will consider every $\langle B\rangle$-orbit separately (In fact, it is sufficient to consider the $\langle B\rangle$-orbit of A). The contribution of the $\langle B\rangle$-orbit of A to $t_\alpha l_\beta$ is

$$\sum_{n=-\infty}^{\infty} t_{B^{-n}(\widehat{a_1a_2})} \log b(Bt, t, r_B, a_B) \tag{3.4}$$

We divide this sum into three parts: $n \leq -1$, $n = 0$, and $n \geq 1$. In each case we consider the location of the four points $\{Bt, t, r_B, a_B\}$ relative to $B^{-n}(\widehat{a_1a_2})$. By replacing A with $B^{-k}AB^k$ if necessary, we can assume that t lies in the strip bounded by $\widehat{a_1a_2}$ and $B^{-n}(\widehat{a_1a_2})$. As in the proof of Lemma 1, without loss of generality we can arrange the picture so that the twist-bulge deformation is bulging to the left.

For $n \geq 1$, the only point of $\{Bt, t, r_B, a_B\}$ to the left of $B^{-n}(\widehat{a_1a_2})$ is r_B. Thus, applying Lemma 1 we have that

$$\begin{aligned}
&\sum_{n=1}^{\infty} t_{B^{-n}(\widehat{a_1a_2})} \log b(Bt, t, r_B, a_B) \\
&\qquad = -\frac{1}{b(Bt, t, r_B, a_B)} b(Bt, t, r_B, a_B) \sum_{n=1}^{\infty} \left(I_{B^{-n}(\widehat{a_1a_2})}(Bt, r_B) - I_{B^{-n}(\widehat{a_1a_2})}(t, r_B) \right) \\
&\qquad = -\sum_{n=1}^{\infty} \left(I_{B^{-n}(\widehat{a_1a_2})}(Bt, r_B) - I_{B^{-n}(\widehat{a_1a_2})}(t, r_B) \right) \\
&\qquad = -\sum_{n=1}^{\infty} \left(I_{\widehat{a_1a_2}}(B^{n+1}t, r_B) - I_{\widehat{a_1a_2}}(B^n t, r_B) \right)
\end{aligned}$$

by the invariance of the cross-ratio $b(Bt, t, r_B, a_B)$ with respect to the action of B. Note that the last sum telescopes, and we have

$$
\begin{aligned}
\sum_{n=1}^{\infty} t_{B^{-n}(\widehat{a_1a_2})} \log b(Bt,t,r_B,a_B) &= -\sum_{n=1}^{\infty}\left(I_{\widehat{a_1a_2}}(B^{n+1}t,r_B) - I_{\widehat{a_1a_2}}(B^n t,r_B)\right)\\
&= -\left(I_{\widehat{a_1a_2}}(\lim_{n\to+\infty} B^{n+1}t,r_B) - I_{\widehat{a_1a_2}}(Bt,r_B)\right)\\
&= I_{\widehat{a_1a_2}}(Bt,r_B) - I_{\widehat{a_1a_2}}(a_B,r_B)
\end{aligned}
$$

For the $n=0$ case, both t and r_B lie to the left of $B^0(\widehat{a_1a_2}) = \widehat{a_1a_2}$, and so again by Lemma 1 the contribution to $t_\alpha l_\beta$ is single term in the sum given by

$$
\begin{aligned}
t_{\widehat{a_1a_2}} \log b(Bt,t,r_B,a_B) &= I_{\widehat{a_1a_2}}(t,a_B) - I_{\widehat{a_1a_2}}(t,r_B) - (I_{\widehat{a_1a_2}}(Bt,r_B) - I_{\widehat{a_1a_2}}(t,r_B))\\
&= I_{\widehat{a_1a_2}}(t,a_B) - I_{\widehat{a_1a_2}}(Bt,r_B)
\end{aligned}
$$

Lastly, in the $n \le 1$ case, a_B is the only point of $\{Bt,t,r_B,a_B\}$ to the right of $B^{-n}(\widehat{a_1a_2})$, and so again by Lemma 1 we have:

$$
\begin{aligned}
&\sum_{n=-\infty}^{-1} t_{B^{-n}(\widehat{a_1a_2})} \log b(Bt,t,r_B,a_B)\\
&\qquad= \frac{1}{b(Bt,t,r_B,a_B)} b(Bt,t,r_B,a_B) \sum_{n=-\infty}^{-1}\left(I_{B^{-n}(\widehat{a_1a_2})}(t,a_B) - I_{B^{-n}(\widehat{a_1a_2})}(Bt,a_B)\right)\\
&\qquad= \sum_{n=-\infty}^{-1}\left(I_{B^{-n}(\widehat{a_1a_2})}(t,a_B) - I_{B^{-n}(\widehat{a_1a_2})}(Bt,a_B)\right)\\
&\qquad= \sum_{n=-\infty}^{-1}\left(I_{\widehat{a_1a_2}}(B^n t,a_B) - I_{\widehat{a_1a_2}}(B^{n+1}t,a_B)\right)
\end{aligned}
$$

As before, this last sum telescopes and we have

$$\sum_{n=-\infty}^{-1} t_{B^{-n}(\widehat{a_1a_2})}\log b(Bt,t,r_B,a_B) = \sum_{n=-\infty}^{-1}\left(I_{\widehat{a_1a_2}}(B^n t,a_B) - I_{\widehat{a_1a_2}}(B^{n+1}t,a_B)\right)$$
$$= I_{\widehat{a_1a_2}}(\lim_{n\to-\infty} B^n t, a_B) - I_{\widehat{a_1a_2}}(t,a_B)$$
$$= I_{\widehat{a_1a_2}}(r_B,a_B) - I_{\widehat{a_1a_2}}(t,a_B)$$

Thus the total contribution of all three parts is given by

$$\sum_{n=-\infty}^{\infty} t_{B^{-n}(\widehat{a_1a_2})}\log b(Bt,t,r_B,a_B)$$
$$= I_{\widehat{a_1a_2}}(Bt,r_B) - I_{\widehat{a_1a_2}}(a_B,r_B) + I_{\widehat{a_1a_2}}(t,a_B) - I_{\widehat{a_1a_2}}(Bt,r_B) + I_{\widehat{a_1a_2}}(r_B,a_B) - I_{\widehat{a_1a_2}}(t,a_B)$$
$$= I_{\widehat{a_1a_2}}(r_B,a_B) - I_{\widehat{a_1a_2}}(a_B,r_B)$$

and each $\langle B\rangle$-orbit in $\langle A\rangle\backslash\mathfrak{J}$ contributes such a term, and thus we have

$$\sum_{C\in\langle A\rangle\backslash\mathfrak{J}} t_{C^{-1}(\widehat{a_1a_2})}\log b(Bt,t,r_B,a_B) = \sum_{C\in\langle A\rangle\backslash\mathfrak{J}/\langle B\rangle}\left[I_{\widehat{c_1c_2}}(r_B,a_B) - I_{\widehat{c_1c_2}}(a_B,r_B)\right]$$

where $\widehat{c_1c_2}$ is the axis of $C^{-1}AC$ and has r_B to its left.

Finally, if we show that

$$\sum_{C\in\langle A\rangle\backslash\Gamma-\mathfrak{J}} t_{C^{-1}(\widehat{a_1a_2})}\log b(Bt,t,r_B,a_B)$$

vanishes, then the proof of Theorem 1 is complete. We again wish to consider the relative positions of the points $\{Bt,t,r_B,a_B\}$. Note that, as discussed in the proof of Lemma 1, if these points all lie in a common half-plane of $C^{-1}(\widehat{a_1a_2})$, then the terms in the twist-bulge derivative formula all vanish. Thus we now consider the cosets $\langle A\rangle C\in\langle A\rangle\backslash\Gamma$ such that

$C^{-1}(\widehat{a_1a_2})$ separates either Bt or t from $\widehat{r_Ba_B}$. Note in particular that $(CB)^{-1}(\widehat{a_1a_2})$ separates t from $\widehat{r_Ba_B}$ exactly when $C^{-1}(\widehat{a_1a_2})$ separates Bt from $\widehat{r_Ba_B}$. The idea is to group the remaining terms of the sum in ordered pairs $(\langle A\rangle C, \langle A\rangle CB)$, where $C^{-1}(\widehat{a_1a_2})$ separates t from $\widehat{r_Ba_B}$, and in the case when $C^{-1}(\widehat{a_1a_2})$ separates both t and Bt from $\widehat{r_Ba_B}$, the coset $\langle A\rangle C$ will occur twice in two distinct pairs.

Grouping the terms for the pair $(\langle A\rangle C, \langle A\rangle CB)$,

$$(t_{C^{-1}(\widehat{a_1a_2})} + t_{(CB)^{-1}(\widehat{a_1a_2})})\log b(Bt, t, r_B, a_B)$$

and applying Lemma 1 tells us that the $C^{-1}(\widehat{a_1a_2})$ term is

$$I_{C^{-1}(\widehat{a_1a_2})}(r_B, Bt) - I_{C^{-1}(\widehat{a_1a_2})}(a_B, Bt)$$

and the $(CB)^{-1}(\widehat{a_1a_2})$ term is

$$I_{(CB)^{-1}(\widehat{a_1a_2})}(a_B, t) - I_{(CB)^{-1}(\widehat{a_1a_2})}(r_B, t) = I_{C^{-1}(\widehat{a_1a_2})}(a_B, Bt) - I_{C^{-1}(\widehat{a_1a_2})}(r_B, Bt)$$

These sum to zero, whence grouping the remaining terms of the sum

$$\sum_{C\in\langle A\rangle\backslash\Gamma-\mathfrak{J}} t_{C^{-1}(\widehat{a_1a_2})}\log b(Bt, t, r_B, a_B)$$

implies that it vanishes. This completes the proof. □

We now wish to compute $t_\gamma t_\alpha l_\beta$. As before, a preliminary lemma is needed. By Theorem 1 and linearity, we see that the second order computation relies on finding the twist-bulge derivative $t_{\widehat{s_1s_2}} I_{\widehat{a_1a_2}}(x_1, x_2)$ for the relevant geodesics. The following lemma is concerned with precisely this matter:

Lemma 2. *Let x_1, x_2 be distinct points on the boundary $\partial\Omega$, and let $\widehat{s_1s_2}$ and $\widehat{a_1a_2}$ be simple closed non-intersecting geodesics with the indicated endpoints on $\partial\Omega$ (see Figure 3). Then*

$$t_{\widehat{s_1s_2}} I_{\widehat{a_1a_2}}(x_1, x_2) = I_{\widehat{a_1a_2}}(x_1, x_2)\left[\frac{\phi_1^T(L_{\widehat{s_1s_2}}L_{\widehat{a_1a_2}})v_2}{\phi_1^T L_{\widehat{a_1a_2}}v_1} - I_{\widehat{s_1s_2}}(x_1, x_2)\right]$$

and

$$t_{\widehat{s_1s_2}} I_{\widehat{a_1a_2}}(x_2, x_1) = -I_{\widehat{a_1a_2}}(x_2, x_1)\left[\frac{\phi_2^T(L_{\widehat{a_1a_2}}L_{\widehat{s_1s_2}})v_1}{\phi_2^T L_{\widehat{a_1a_2}}v_1} - I_{\widehat{s_1s_2}}(x_2, x_1)\right]$$

Proof. We use the same notation as in the proof of Lemma 1. Let $\widehat{s_1s_2}$ and $\widehat{a_1a_2}$ be as in Figure 3. Recall definition 3.3:

$$I_{\widehat{a_1a_2}}(x_i, x_j) := \frac{\phi_i^T L_{\widehat{a_1a_2}} v_j}{\phi_i^T \cdot v_j}$$

where $L_{\widehat{a_1a_2}}$ is the infinitesimal generator for the twist-bulge along $\widehat{a_1a_2}$ arising in the calculation of $\frac{\partial}{\partial u} b_{(u,v)}$ from Lemma 1. Define

$$I_{\widehat{a_1a_2}}(x_i, x_j)_{(u,v)}$$

to be the deformation of $I_{\widehat{a_1a_2}}(x_i, x_j)$ by the action of the twist-bulge deformation $g_{(u,v)}$ along $\widehat{s_1s_2}$ as in Lemma 1. As before, we wish to calculate the derivative of this deformed I term with respect to the twist-bulge along $\widehat{s_1s_2}$. Finding

$$\frac{\partial}{\partial u} I_{\widehat{a_1a_2}}(x_1, x_2)_{(u,v)}$$

is sufficient since the calculation for the other derivative is similar.

As before, we arrange the picture so that the twist-bulge along $\widehat{s_1s_2}$ is happening on the left. Note that if both points x_1, x_2 lie on one side of $\widehat{s_1s_2}$, then

$$I_{\widehat{a_1a_2}}(x_1, x_2)_{(u,v)} = I_{\widehat{a_1a_2}}(x_1, x_2)$$

and so the derivative in this case is zero. Thus we suppose x_1 lies to the left of $\widehat{s_1s_2}$. In this case,

$$I_{\widehat{a_1a_2}}(x_1, x_2)_{(u,v)} = \frac{(g^*_{(u,v)}\phi_1^T) \cdot (L_{\widehat{a_1a_2}} v_2)}{(g^*_{(u,v)}\phi_1^T) \cdot v_2}$$

and so if we let

$$v_2 = \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}, \qquad \phi_1 = \begin{pmatrix} a_1' & b_1' & c_1' \end{pmatrix},$$

$$L_{\widehat{a_1 a_2}} = \begin{bmatrix} a' & b' & c' \\ d' & e' & f' \\ g' & h' & i' \end{bmatrix}$$

as well as

$$A(u,v) = e^{u+v} a_1'(a'a_2 + b'b_2 + c'c_2) + e^{-2v} b_1'(d'a_2 + e'b_2 + f'c_2) + e^{-u+v} c_1'(g'a_2 + h'b_2 + i'c_2)$$

and

$$B(u,v) = e^{u+v} a_1' a_2 + e^{-2v} b_1' b_2 + e^{-u+v} c_1' c_2$$

then we have

$$I_{\widehat{a_1 a_2}}(x_1, x_2)_{(u,v)} = \frac{(g^*_{(u,v)} \phi_1^T) \cdot (L_{\widehat{a_1 a_2}} v_2)}{(g^*_{(u,v)} \phi_1^T) \cdot v_2} = \frac{A(u,v)}{B(u,v)}$$

whence

$$\begin{aligned} \frac{\partial}{\partial u} I_{\widehat{a_1 a_2}}(x_1, x_2)_{(u,v)} &= \frac{\left(\frac{\partial}{\partial u} A(u,v)\right) B(u,v) - A(u,v) \left(\frac{\partial}{\partial u} B(u,v)\right)}{B(u,v)^2} \\ &= \frac{(\frac{\partial}{\partial u} A(u,v)) B(u,v) - (e^{u+v} a_1' a_2 - e^{-u+v} c_1' c_2) A(u,v)}{B(u,v)^2} \end{aligned}$$

where

$$\frac{\partial}{\partial u} A(u,v) = e^{u+v} a_1'(a'a_2 + b'b_2 + c'c_2) - e^{-u+v} c_1'(g'a_2 + h'b_2 + i'c_2).$$

Noticing that

$$A(0,0) = a_1'(a'a_2 + b'b_2 + c'c_2) + b_1'(d'a_2 + e'b_2 + f'c_2) + c_1'(g'a_2 + h'b_2 + i'c_2),$$

$$B(0,0) = a_1'a_2 + b_1'b_2 + c_1'c_2$$

and thus

$$\frac{A(0,0)}{B(0,0)} = I_{\widehat{a_1a_2}}(x_1, x_2),$$

we have that

$$\begin{aligned}
&\frac{\partial}{\partial u} I_{\widehat{a_1a_2}}(x_1, x_2)_{(u,v)}\Bigg|_{(u,v)=(0,0)} \\
&= \frac{a_1'(a'a_2 + b'b_2 + c'c_2) - c_1'(g'a_2 + h'b_2 + i'c_2)}{B(0,0)} - \frac{a_1'a_2 - c_1'c_2}{B(0,0)} \frac{A(0,0)}{B(0,0)} \\
&= I_{\widehat{a_1a_2}}(x_1, x_2) \left[\frac{a_1'(a'a_2 + b'b_2 + c'c_2) - c_1'(g'a_2 + h'b_2 + i'c_2)}{a_1'a_2 - c_1'c_2} - \frac{a_1'a_2 - c_1'c_2}{a_1'a_2 + b_1'b_2 + c_1'c_2} \right] \\
&= I_{\widehat{a_1a_2}}(x_1, x_2) \left[\frac{a_1'(a'a_2 + b'b_2 + c'c_2) - c_1'(g'a_2 + h'b_2 + i'c_2)}{\phi_1^T L_{\widehat{a_1a_2}} v_2} - I_{\widehat{s_1s_2}}(x_1, x_2) \right] \\
&= I_{\widehat{a_1a_2}}(x_1, x_2) \left[\frac{\phi_1^T (L_{\widehat{s_1s_2}} L_{\widehat{a_1a_2}}) v_2}{\phi_1^T L_{\widehat{a_1a_2}} v_2} - I_{\widehat{s_1s_2}}(x_1, x_2) \right]
\end{aligned}$$

The other calculation is similar, and the lemma follows. □

The previous lemma allows us to immediately compute $t_\gamma t_\alpha l_\beta$:

Theorem 2. *Let A, B, C be the matrix representatives associated to geodesics α, β, γ, respectively. Then $t_\gamma t_\alpha l_\beta$ breaks up as sums over $\langle A \rangle$ and $\langle B \rangle$ orbits, where the total contribution coming from a $\langle B \rangle$-orbit is*

$$(C_1^{\alpha,\gamma} \frac{\lambda_B}{\mu_B} + C_2^{\alpha,\gamma}) \frac{1}{1 - \frac{\lambda_B}{\mu_B}} + (C_3^{\alpha,\gamma} \frac{\lambda_B}{\nu_B} + C_4^{\alpha,\gamma}) \frac{1}{1 - \frac{\lambda_B}{\nu_B}} + (C_5^{\alpha,\gamma} \frac{\mu_B}{\nu_B} + C_6^{\alpha,\gamma}) \frac{1}{1 - \frac{\mu_B}{\nu_B}}$$

where the $C_i^{\alpha,\gamma}$ are constants and λ_B, μ_B, ν_B are the eigenvalues of B as in (2.5). *The contribution coming from the $\langle A \rangle$-orbit is similar.*

Proof. We use the notation as in the proof of Theorem 1: Let $\widehat{a_1a_2}$, $\widehat{r_Ba_B}$, $\widehat{c_1c_2}$ be the axes associated to A, B, and C, matrix representatives of α,β, and γ, respectively. By the comments preceding Lemma 2, it suffices to consider the twist-bulge derivative of a single term of the sum in Theorem 1; in particular, we consider $t_\gamma(I_*(r_B, a_B) - I_*(a_B, r_B))$. Lemma 2 allows us to compute this, and so in order to calculate $t_\gamma t_\alpha l_\beta$ we must keep track of the positions of r_B, a_B, a_1, a_2 relative to the axes over which we are summing. As in Lemma 1 and Theorem 1, note that the twist-bulge derivative of a generalized cross-ratio of four points lying in a common half of the domain is zero, and so in this situation there is nothing to compute. Otherwise, let $\mathfrak{D}$ denote the set of axes in the orbit of $\widehat{c_1c_2}$ that separate r_B, a_B, a_1 and a_2. Note that $\mathfrak{D}$ is partitioned into three components $\mathfrak{D}_B$, $\mathfrak{D}_A$, and $\mathfrak{D}_{AB}$ in the following way:

$$\mathfrak{D}_B = \left\{\widehat{s_1s_2} \in \mathfrak{D} \mid \widehat{s_1s_2} \text{ intersects } \widehat{r_Ba_B} \text{ and for all } n \in \mathbb{Z}, B^{-n}(\widehat{s_1s_2}) \text{ does not separate } a_1, a_2\right\}$$

$$\mathfrak{D}_A = \left\{\widehat{s_1s_2} \in \mathfrak{D} \mid \widehat{s_1s_2} \text{ intersects } \widehat{a_1a_2} \text{ and for all } n \in \mathbb{Z}, A^{-n}(\widehat{s_1s_2}) \text{ does not separate } r_B, a_B\right\}$$

$$\mathfrak{D}_{AB} = \left\{\widehat{s_1s_2} \in \mathfrak{D} \mid \text{ there exists } D \in \langle A\rangle \cup \langle B\rangle \text{ with } D^{-1}(\widehat{s_1s_2}) \text{ intersecting } \widehat{a_1a_2} \text{ and } \widehat{r_Ba_B}\right\}$$

As before in the proof of Theorem 1, we sum over the $\langle A\rangle$ and $\langle B\rangle$ orbits in order to compute the total contribution to the twist-bulge derivative. We begin the calculation by considering $\widehat{s_1s_2} \in \mathfrak{D}_B$. As always, we can orient the picture so that r_B is to the left of $\widehat{s_1s_2}$, and we can suppose that $\widehat{s_1s_2}$ is in the strip bounded by $\widehat{a_1a_2}$ and $B^{-1}(\widehat{s_1s_2})$ by replacing $\widehat{s_1s_2}$ with $B^{-n}(\widehat{s_1s_2})$ if necessary. Now, the contribution to the derivative coming from the $\langle B\rangle$-orbit of $\widehat{s_1s_2}$ is

$$\sum_{n=-\infty}^{\infty} t_{B^{-n}(\widehat{s_1s_2})}\left(I_{\widehat{a_1a_2}}(r_B, a_B) - I_{\widehat{a_1a_2}}(a_B, r_B)\right)$$

Again, as in the case of Theorem 1, we split the sum into two parts: $n \geq 0$ and $n \leq -1$.

For $n \geq 0$, r_B is the only element of r_B, a_1, a_2, a_B to the left of $B^{-n}(\widehat{s_1s_2})$. Hence by Lemma 2, we have that

$$
\begin{aligned}
&\sum_{n=0}^{\infty} t_{B^{-n}(\widehat{s_1 s_2})}\left(I_{\widehat{a_1 a_2}}(r_B, a_B) - I_{\widehat{a_1 a_2}}(a_B, r_B)\right) \\
&\quad = \sum_{n=0}^{\infty}\left[I_{\widehat{a_1 a_2}}(r_B, a_B)\left(\frac{\phi_{r_B}^T (L_{\widehat{a_1 a_2}} L_{B^{-n}(\widehat{s_1 s_2})}) v_{a_B}}{\phi_{r_B}^T L_{\widehat{a_1 a_2}} v_{a_B}} - I_{B^{-n}(\widehat{s_1 s_2})}(r_B, a_B)\right)\right. \\
&\qquad \left. + I_{\widehat{a_1 a_2}}(a_B, r_B)\left(\frac{\phi_{a_B}^T (L_{B^{-n}(\widehat{s_1 s_2})} L_{\widehat{a_1 a_2}}) v_{r_B}}{\phi_{a_B}^T L_{\widehat{a_1 a_2}} v_{r_B}} - I_{B^{-n}(\widehat{s_1 s_2})}(a_B, r_B)\right)\right] \\
&\quad = I_{\widehat{a_1 a_2}}(r_B, a_B) \sum_{n=0}^{\infty}\left[\frac{\phi_{r_B}^T (L_{\widehat{a_1 a_2}} L_{B^{-n}(\widehat{s_1 s_2})}) v_{a_B}}{\phi_{r_B}^T L_{\widehat{a_1 a_2}} v_{a_B}} - I_{B^{-n}(\widehat{s_1 s_2})}(r_B, a_B)\right] \\
&\qquad + I_{\widehat{a_1 a_2}}(a_B, r_B) \sum_{n=0}^{\infty}\left[\frac{\phi_{a_B}^T (L_{B^{-n}(\widehat{s_1 s_2})} L_{\widehat{a_1 a_2}}) v_{r_B}}{\phi_{a_B}^T L_{\widehat{a_1 a_2}} v_{r_B}} - I_{B^{-n}(\widehat{s_1 s_2})}(a_B, r_B)\right]
\end{aligned}
$$

By invariance we may specialize to the case for the fundamental triangle as discussed in Chapter 2, where a_B=[1,0,0] and r_B=[0,0,1]. Recall that in this case B is a diagonal matrix given by (2.4) with eigenvalues λ_B, μ_B, ν_B given as in (2.5), i.e.

$$
B = \begin{bmatrix} \lambda_B & 0 & 0 \\ 0 & \mu_B & 0 \\ 0 & 0 & \nu_B \end{bmatrix}
$$

where

$$
\lambda_B \mu_B \nu_B = 1, \quad 0 < \lambda_B < \mu_B < \nu_B
$$

Furthermore, let

$$
L_{\widehat{s_1 s_2}} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}
$$

and

$$L_{\widehat{a_1a_2}} = \begin{bmatrix} a' & b' & c' \\ d' & e' & f' \\ g' & h' & i' \end{bmatrix}$$

be the usual infinitesimal generators associated to the twist-bulge deformation along the indicated geodesics. Also, note that since a_B=[1,0,0] and r_B=[0,0,1], we have

$$v_{a_B} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \qquad \phi_{a_B} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

and

$$v_{r_B} = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}, \qquad \phi_{r_B} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

We now use this specialization to explicitly evaluate the terms in the above sum:

$$\begin{aligned}
I_{\widehat{a_1a_2}}(r_B, a_B) &= \frac{\phi_{r_B}^T L_{\widehat{a_1a_2}} v_{a_B}}{\phi_{r_B}^T \cdot v_{a_B}} = a' \\
I_{\widehat{a_1a_2}}(a_B, r_B) &= \frac{\phi_{a_B}^T L_{\widehat{a_1a_2}} v_{r_B}}{\phi_{a_B}^T \cdot v_{r_B}} = i' \\
I_{B^{-n}(\widehat{s_1s_2})}(r_B, a_B) &= \frac{\phi_{r_B}^T L_{B^{-n}(\widehat{s_1s_2})} v_{a_B}}{\phi_{r_B}^T \cdot v_{a_B}} = \frac{\phi_{r_B}^T B^{-n} L_{\widehat{s_1s_2}} B^n v_{a_B}}{\phi_{r_B}^T \cdot v_{a_B}} = a \\
I_{B^{-n}(\widehat{s_1s_2})}(a_B, r_B) &= \frac{\phi_{a_B}^T L_{B^{-n}(\widehat{s_1s_2})} v_{r_B}}{\phi_{a_B}^T \cdot v_{r_B}} = \frac{\phi_{a_B}^T B^{-n} L_{\widehat{s_1s_2}} B^n v_{r_B}}{\phi_{a_B}^T \cdot v_{r_B}} = i \\
\frac{\phi_{r_B}^T (L_{\widehat{a_1a_2}} L_{B^{-n}(\widehat{s_1s_2})}) v_{a_B}}{\phi_{r_B}^T L_{\widehat{a_1a_2}} v_{a_B}} &= \frac{a'a + bd'(\frac{\lambda_B}{\mu_B})^n + cg'(\frac{\lambda_B}{\nu_B})^n}{a'} \\
\frac{\phi_{a_B}^T (L_{B^{-n}(\widehat{s_1s_2})} L_{\widehat{a_1a_2}}) v_{r_B}}{\phi_{a_B}^T L_{\widehat{a_1a_2}} v_{r_B}} &= \frac{cg'(\frac{\lambda_B}{\nu_B})^n + fh'(\frac{\mu_B}{\nu_B})^n + i'i}{i'}
\end{aligned}$$

Returning to the above sum, we have

$$\sum_{n=0}^{\infty} t_{B^{-n}(\widehat{s_1s_2})}\left(I_{\widehat{a_1a_2}}(r_B,a_B) - I_{\widehat{a_1a_2}}(a_B,r_B)\right)$$

$$= I_{\widehat{a_1a_2}}(r_B,a_B)\sum_{n=0}^{\infty}\left[\frac{\phi^T_{r_B}(L_{\widehat{a_1a_2}}L_{B^{-n}(\widehat{s_1s_2})})v_{a_B}}{\phi^T_{r_B}L_{\widehat{a_1a_2}}v_{a_B}} - I_{B^{-n}(\widehat{s_1s_2})}(r_B,a_B)\right]$$

$$+ I_{\widehat{a_1a_2}}(a_B,r_B)\sum_{n=0}^{\infty}\left[\frac{\phi^T_{a_B}(L_{B^{-n}(\widehat{s_1s_2})}L_{\widehat{a_1a_2}})v_{r_B}}{\phi^T_{a_B}L_{\widehat{a_1a_2}}v_{r_B}} - I_{B^{-n}(\widehat{s_1s_2})}(a_B,r_B)\right]$$

$$= a'\sum_{n=0}^{\infty}\left[\frac{a'a + bd'(\frac{\lambda_B}{\mu_B})^n + cg'(\frac{\lambda_B}{\nu_B})^n}{a'} - a\right] + i'\sum_{n=0}^{\infty}\left[\frac{cg'(\frac{\lambda_B}{\nu_B})^n + fh'(\frac{\mu_B}{\nu_B})^n + i'i}{i'} - i\right]$$

$$= \sum_{n=0}^{\infty}\left[bd'\left(\frac{\lambda_B}{\mu_B}\right)^n + cg'\left(\frac{\lambda_B}{\nu_B}\right)^n\right] + \sum_{n=0}^{\infty}\left[cg'\left(\frac{\lambda_B}{\nu_B}\right)^n + fh'\left(\frac{\mu_B}{\nu_B}\right)^n\right]$$

So that for $n \geq 0$ the sum is

$$\sum_{n=0}^{\infty} t_{B^{-n}(\widehat{s_1s_2})}\left(I_{\widehat{a_1a_2}}(r_B,a_B) - I_{\widehat{a_1a_2}}(a_B,r_B)\right) = \sum_{n=0}^{\infty}\left[bd'\left(\frac{\lambda_B}{\mu_B}\right)^n + 2cg'\left(\frac{\lambda_B}{\nu_B}\right)^n + fh'\left(\frac{\mu_B}{\nu_B}\right)^n\right]$$

Notice that the eigenvalue ratios $\frac{\lambda_B}{\mu_B}, \frac{\lambda_B}{\nu_B}, \frac{\mu_B}{\nu_B}$ are exactly those ratios guaranteed to be < 1, and so this sum is geometric and thus

$$\sum_{n=0}^{\infty} t_{B^{-n}(\widehat{s_1s_2})}\left(I_{\widehat{a_1a_2}}(r_B,a_B) - I_{\widehat{a_1a_2}}(a_B,r_B)\right) = bd'\frac{1}{1-\frac{\lambda_B}{\mu_B}} + 2cg'\frac{1}{1-\frac{\lambda_B}{\nu_B}} + fh'\frac{1}{1-\frac{\mu_B}{\nu_B}}$$

For $n \leq -1$, a_B is the only element of r_B, a_1, a_2, a_B to the right of $B^{-n}(\widehat{s_1s_2})$. As in the $n \geq 0$ case, a similar calculation via specializing gives us the contribution as

$$\sum_{n=-\infty}^{-1} t_{B^{-n}(\widehat{s_1s_2})}\left(I_{\widehat{a_1a_2}}(r_B,a_B) - I_{\widehat{a_1a_2}}(a_B,r_B)\right) = \sum_{n=-\infty}^{-1}\left[b'd\left(\frac{\mu_B}{\lambda_B}\right)^n + 2c'g\left(\frac{\nu_B}{\lambda_B}\right)^n + fh'\left(\frac{\nu_B}{\mu_B}\right)^n\right]$$

$$= b'd\frac{\frac{\lambda_B}{\mu_B}}{1-\frac{\lambda_B}{\mu_B}} + 2c'g\frac{\frac{\lambda_B}{\nu_B}}{1-\frac{\lambda_B}{\nu_B}} + f'h\frac{\frac{\mu_B}{\nu_B}}{1-\frac{\mu_B}{\nu_B}}$$

Thus the total contribution of the $\langle B\rangle$-orbit of $\widehat{s_1 s_2}$ is

$$(b'd\frac{\lambda_B}{\mu_B} + bd')\frac{1}{1-\frac{\lambda_B}{\mu_B}} + 2(c'g\frac{\lambda_B}{\nu_B} + cg')\frac{1}{1-\frac{\lambda_B}{\nu_B}} + (f'h\frac{\mu_B}{\nu_B} + fh')\frac{1}{1-\frac{\mu_B}{\nu_B}}$$

For the other components $\widehat{s_1 s_2} \in \mathfrak{D}_A$ and $\widehat{s_1 s_2} \in \mathfrak{D}_{AB}$, the calculation reduces to the first case.

$\square$

3.3 Closing Remarks

The techniques used in the calculation of the twist-bulge derivatives in Theorems 1 and 2 readily applies to quantities formally similar to the generalized cross-ratio. Indeed, Lemma 2 is an example of this since it is simply a calculation of the twist-bulge derivatives of the quotients involving the flag terms as they arise in Lemma 1 and Theorem 1. For this reason we are optimistic about the potential for further calculations involving quantities arising from considerations of the equivariant flag curve associated to a Hitchin representation. Furthermore, the work of Bonahon-Dreyer [1] concerns deformations of Hitchin representations of higher dimension which can be thought of as a generalization of Goldman's twist-bulges. An obvious avenue of further investigation is to attempt to replicate the calculations in Theorems 1 and 2 in the more general setting of Hitchin representations into PSL(n,$\mathbb{R}$).

Appendix A

Figures

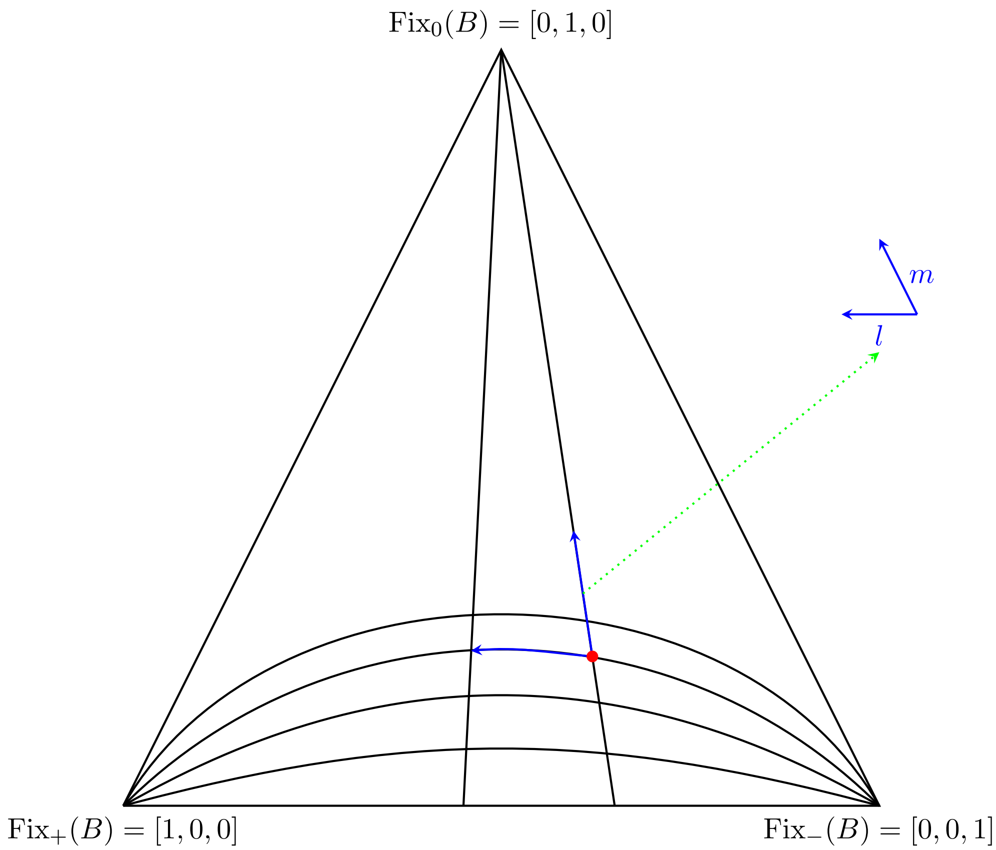

Figure 1

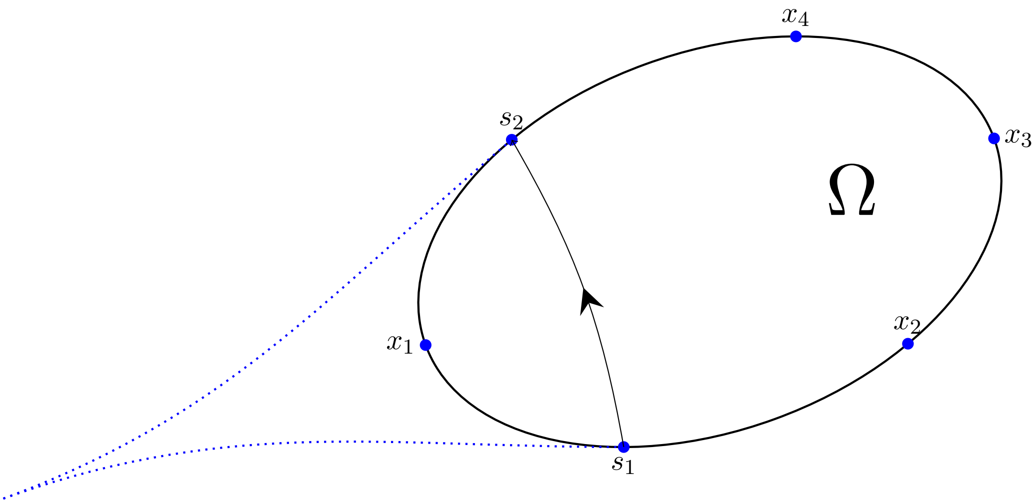

Figure 2

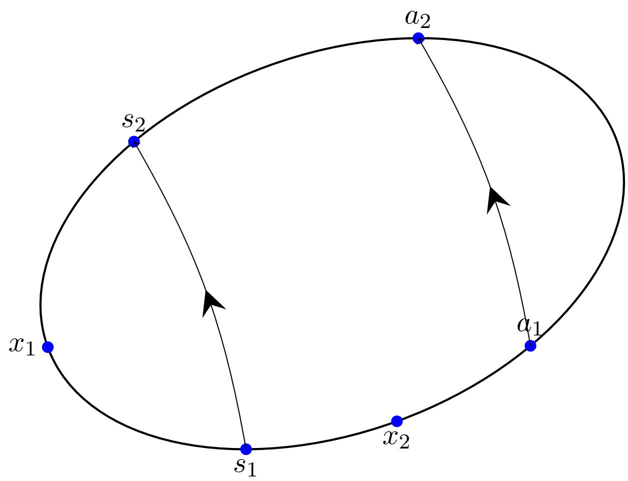

Figure 3

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